INFORMATION AND CONTROL 8, 93–110 (1965)

Identification of Linear Dynamic Systems*

YU-CHI HO

Harvard University, Pierce Hall, Cambridge, Massachusetts

AND

R. C. K. LEE†

Aeronautical Division, Honeywell Co., Minneapolis, Minnesota

A real time computational method is presented for the identification of linear discrete dynamic systems with unknown parameters. It is shown that the method is globally convergent to the true parameters in a stochastic environment. Experimental simulation of 2nd and 4th order systems further affirms the practicality of the method.

I. INTRODUCTION

The problem of identification in the current theory of control usually arises from the following consideration:
The equation of motion of a system (in the absence of control) is given as

$$\dot{y} = g(y, t) \tag{1}$$

where the functional form of g is not exactly known. The problem is then to determine g by making some form of measurement on the system. In order to reduce the complexity of the problem, one further assumption is often made. We shall assume that (1) can be written as

$$\dot{y} = g^*(y, \alpha, t) \tag{2}$$

where g^* is known and α represents a set of unknown parameters obeying the equation

$$\dot{\alpha} = \begin{cases} 0 & \text{(stationary problem)} \\ \text{known function of } \alpha,\ x, \text{ and } t & \text{(nonstationary problem)} \end{cases} \tag{3}$$

* This work is supported in part by NONR Contract 1866(16).

† Present address: Aerospace Corporation, El Segundo, California.

Equations (2) and (3) can now be combined to yield

$$\dot{x} = f(x, t) \tag{4}$$

where f is a known vector function and $x = \begin{bmatrix} y \\ \alpha \end{bmatrix}$ is the enlarged state vector. The measurements on the system can generally be represented by

$$z(t) = h(x, t) \tag{5}$$

The identification problem is then stated as:

Given the measurements $z(t)$, $t_0 \leqq t \leqq t_1 \leqq \infty$, determine the state of the dynamic system (4) at some time τ, say $\tau = t_0$. Deterministically, the above problem is not too interesting at least from a conceptual viewpoint. For example, consider the discrete version of (4) and (5), we can always write

$$\begin{aligned} z_t &= h(x_t) = h(f(x_{t-1})) = h_1(x_{t-1}) \\ &= \cdots = h_t(x_0) \end{aligned} \tag{6}$$

for $t = 1, 2, \cdots$. If enough measurements are taken, (6) will represent a set of equations from which x_0 can be solved. Questions concerning the solvability and uniqueness of (6), of course, need to be investigated. However, they are of secondary importance when one considers the fact that in real life identification must invariably be carried out in a stochastic environment, i.e., Eqs. (4) and (5) should be replaced by

$$\dot{x} = f(x, w, t) \tag{7}$$

and

$$z(t) = h(x, v, t) \tag{8}$$

where w, and v are random processes whose description may only be partially known. This is essentially a nonlinear filtering or smoothing problem.

To the best of the writer's knowledge, there as yet exists no general solution to the nonlinear estimation problem posed by (7) and (8). However, two general approaches are worth mentioning.

A. Least Square Smoothing

Consider the criterion function

$$J = \int_{t_0}^{t_1} \| z - \hat{z} \|_{R^{-1}}^2 + \| w - \hat{w} \|_{Q^{-1}}^2 \, dt + \| x_0 - \hat{x}_0 \|_{P_0^{-1}}^2 \tag{9}$$

and the constraint equations

$$\dot{\hat{x}} = f(\hat{x}, \hat{w}, t); \quad x(t_0) = \hat{x}_0 \tag{10}$$

$$\hat{z} = h(\hat{x}, t) \tag{11}$$

where R, Q, and P_0 are the covariance matrices of v, w, x_0 respectively. We posed the problem of determining a particular $\hat{w}(t)$, $\hat{v}(t)$ and a $\hat{x}_0$ such that (9) is minimized subject to (10) and (11). This represents a typical deterministic optimization which can be solved by known techniques in the calculus of variations. Under suitable assumptions on the statistics of w and v, it is actually possible to obtain a probabilistic interpretation for the procedure. This is the approach of Bryson and Frazier (1962), and is mainly intended for nonreal time applications. The basic computational problem here is the time consuming task of solving the two point boundary value problem resulting from (9)–(11). Each time a new z is received, the t.p.b.v.p. must be re-solved.

B. Successive Linear Filtering

The idea of this procedure is very simple. Starting with an initial estimate of x, Eqs. (7) and (8) are linearized about this estimate. Based on these linearized equations of motion, we apply the Wiener-Kalman estimation procedure to obtain an improved estimate of x at the next instant. The procedure then repeats by linearizing (7) and (8) about the improved estimate. The method is ideally suited for real time computation and has actually been used with success by various authors (Schmidt, 1962; Kopp, 1963). The only difficulty is that no proof of and criterion for convergence has been obtained for this procedure. In other words, it is not possible to assert that the estimation error decreases or remains bounded with increasing measurement data. Experience, however, seems to indicate that this procedure can be expected to converge over wide ranges of the estimation error.

The purpose of this paper is to establish a *real time convergent* identification scheme for a restricted class of, namely linear, dynamic systems. The scheme was first mentioned in (Ho and Whalen, 1963), where the proof of convergence was not supplied. In this paper we shall study the particular scheme in much more detail with accompanying proof of convergence. Furthermore, experimental results of the application of the method to 2nd and 4th order systems will be presented. Thus, the practicality of the method will also be established.

II. THE IDENTIFICATION ALGORITHM

We shall restrict ourselves to the consideration of the linear, discrete version of (7) and (8). They are

$$x(t+1) = \Phi x(t) + dw_1(t) \tag{11a}$$

$$z_1(t) = h^T x(t) + v_1(t) \tag{12}$$

where Φ, d, h are constant but unknown matrix and vectors, w_1 and v_1 are independent, scalar, white random sequences with zero mean and finite variances. The problem is the identification of Φ based on the scalar measurements $z_1(1), z_1(2), \cdots, z_1(t), \cdots$. The identification procedure is required to satisfy the following two criteria:

(i) Real time realizability—The computation effort involved per measurement should not be increasing with the total number of measurements.

(ii) Stability—Regardless of the initial knowledge of Φ, the estimate of Φ should approach the true value of Φ as the number of measurements increases to infinity. One need not elaborate on the desirability of these criteria.

We shall further assume that

$$\text{Det}\,[d, \Phi\, d, \cdots, \Phi^{n-1}\, d] \neq 0 \tag{13}$$

i.e., (Φ, d) is a controllable pair

$$\text{Det}\,[h, \Phi^T h, \cdots, \Phi^{n-1^T} h] \neq 0 \tag{14}$$

i.e., (Φ, h) is an observable pair, and that Φ is a stable system matrix.

Remarks

(i) The assumptions of observability and controllability are not really restrictive. Since from the canonical decomposition theorem of Kalman (1963), the only part of a dynamic system that can be determined from input-output measurements is the part which is both controllable and observable.[1] Hence there is no loss of generality by assuming (13) and (14). The restriction on stable Φ matrix is necessary only for the purpose of the convergence proof. For finite time operation the procedure is applicable to even unstable systems (see numerical example later).

[1] Actually, the part which is observable but not controllable will transmit information to the output z_1. However, if the part is stable, then its influence on the output will die away. If it is not, then identification is meaningless in this case.

(ii) If (13) is true, then it is easily verified that

$$\sum_{i=0}^{n-1} \Phi^{n-1-i}\, d\, d^T \Phi^{n-1-i^T} > 0 \quad \text{(i.e. positive definite)} \tag{15}$$

(iii) If (14) is true, then (11) and (12) can be rewritten as

$$s(t+1) = \Phi^* s(t) + d^* w_1(t) \tag{16}$$

$$z_1(t) = h^{*T} s(t) + v_1(t) \tag{17}$$

where

$$\Phi^* = \begin{bmatrix} 0 & 1 & & \\ & \cdot \quad \cdot & \cdot \quad \cdot & \\ & \cdot & & \cdot \\ & 0 & & 1 \\ \varphi_1 & \cdots & & \varphi_n \end{bmatrix} \quad h^* = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad d^* \stackrel{\Delta}{=} \begin{bmatrix} d_1^* \\ \cdot \\ \cdot \\ \cdot \\ d_n^* \end{bmatrix} = T\, d \tag{18}$$

and $s(t) \stackrel{\Delta}{=} Tx(t)$

$$T = \begin{bmatrix} h^T \\ \hline h^T\Phi \\ \hline \vdots \\ \hline h^T\Phi^{n-1} \end{bmatrix} \tag{19}$$

In other words, it is possible to choose a different basis in the state space such that only n unknown instead of n^2 unknown are involved in Φ and that (16) and (17) are identical to (11) and (12) as far as $z_1(t)$ is concerned. Rewriting (17), we get

$$s_n(t+1) = \sum_{i=1}^{n} \varphi_i s_i(t) + d_n^* w_1(t) \tag{20}$$

and

$$s_i(t+1) = s_{i+1}(t) + d_i^* w_1(t); \quad i \leqq n-1 \tag{21}$$

Combining (20), (21) and (17), one gets after straightforward but messy manipulations

$$\begin{aligned} z_1(t+1) &= \sum_{i=1}^{n} \varphi_i z_1(t-n+i) + \omega(t) \\ &\stackrel{\Delta}{=} \varphi^T z(t) + \omega(t) \end{aligned} \tag{22}$$

where

$$z(t) \overset{\Delta}{=} \begin{bmatrix} z_1(t-n+1) \\ \vdots \\ z_1(t) \end{bmatrix}$$

and

$$\begin{aligned} \omega(t) = \sum_{i=1}^{n} \varphi_i[v_1(t-n+i) - \sum_{j=1}^{i-1} d_j^* w_1(t-n+i-j)] \\ + \sum_{i=1}^{n} d_i^* w_1(t-i+1) + v_1(t+1) \end{aligned} \tag{23}$$

Consequently, the identification problem can be restated as:

Given the system of (22) determine φ from measurements $z_1(1)$, $z_1(2), \cdots$, where the statistics of $\omega(t)$ are

$$E(\omega(t)) = 0 \ \forall t \tag{24}$$

$$E(\omega(t)\omega(t+i)) = \begin{cases} \text{finite} & i \leqq n+1 \\ 0 & i > n+1 \end{cases} \tag{25}$$

From a theoretical viewpoint, the fact that $\omega(t)$ represents a correlated random sequence is not very crucial. In view of (25), if we choose to look at (22) only for $t = n + 1, 2(n + 1), 3(n + 1), \cdots$ then the random sequence $\omega(n + 1), \omega(2(n + 1)), \cdots$ is uncorrelated. Hence, for the sake of notational simplicity and without loss of generality, we shall assume in the ensuing discussion that $\omega(t)$ is a white random sequence with zero mean and finite variance. This assumption is identically satisfied if $v_1(t) = 0$ for all t (i.e., no measurement noise) and $d_i^* = 0$ for $i = 1, \cdots, n - 1$. Otherwise, it is understood that (22) will only be used at t instants sufficiently far apart in time to ensure $\omega(t)$ as an uncorrelated sequence.

Now consider the specific estimation algorithm

$$\begin{aligned} \hat{\varphi}(t+1) &= \hat{\varphi}(t) + \rho(t)z(t)(z_1(t+1) - \hat{\varphi}(t)^T z(t)) \\ &\overset{\Delta}{=} \hat{\varphi}(t) + \rho(t)z(t)(z_1(t+1) - \hat{z}_1(t+1)) \\ \hat{\varphi}(1) &= \text{arbitrary} \end{aligned} \tag{26}$$

where $\rho(t)$ is an as yet unspecified scalar time sequence. The intuitive reasoning behind (26) is clear. It says that the estimate of the vector φ at time $t + 1$ based upon the receipt of the new information $z_1(t + 1)$ consists of two parts:

(i) the old estimate at time t

(ii) a correction term which is proportional to the difference between the actual measurement $z_1(t+1)$ and the predicted measurement $\hat{z}_1(t+1)$ calculated based on the old estimate.

Since the residue $\tilde{z}_1(t+1) \triangleq z_1(t+1) - \hat{z}_1(t+1)$ is caused partially by errors of the estimate $\hat{\varphi}(t)$ and partially by the random term $\omega(t)$, the term $\rho(t)z(t)$ can essentially be regarded as a weighting factor which appropriately distributes the residue according to the confidence one attaches to it. Another way to look at (26) is to note that

$$\operatorname{grad}_{\hat{\varphi}} \| z_1(t+1) - \hat{\varphi}(t)^T z(t) \|^2 = -z(t)(z_1(t+1) - \hat{\varphi}(t)^T z(t)) \tag{27}$$

Equation (26) can then be regarded as a descent scheme which proceeds in the direction to reduce the instantaneous estimation error of the measurement. The following assertion can now be stated:

Proposition: If one chooses

$$\rho(t) = 1/t \tag{28}$$

then (26) with *arbitrary* $\hat{\varphi}(1)$ implies

$$\lim_{t\to\infty} E\{\| \hat{\varphi}(t) - \varphi \|^2\} = 0 \tag{29}$$

i.e., the estimate converges to the true value in mean square. The proof of the proposition is somewhat lengthy and technical and will be presented in the Appendix. However, (28) is intuitively reasonable. Initially when the estimate $\hat{\varphi}(t)$ is expected to be poor, considerable weight is given to the residue term. As the estimate improves, less and less importance is attached to the correction term since more and more probably it simply represents the noise $\omega(t)$. Equation (28) quantitatively expresses the above consideration. In fact, it will be shown in the Appendix that

$$\begin{aligned} E \| \hat{\varphi}(t+1) - \varphi \|^2 \triangleq E \| \tilde{\varphi}(t+1) \|^2 &\leqq (1 - \rho(t)) E \| \tilde{\varphi}(t) \|^2 \\ &+ \text{terms proportional to } (\rho^2(t), \rho(t)\rho(t-i), \rho(t)\rho^2(t-i)) \end{aligned} \tag{30}$$

where $i \leqq n$. Now let $\alpha(\tau) \triangleq E \| \tilde{\varphi}(t) \|^2$ and using (28), it is easy to see that (30) will be dominated by the solution of the nonnegative scalar differential equation

$$\begin{gathered} \dot{\alpha} = -\frac{1}{t}\alpha + \frac{b}{t^2} + \sum_{i=1}^{n} \frac{c_i}{t(t+i)} + \sum_{i=1}^{n} \frac{c_i^*}{t(t+i)^2} \\ 0 < \alpha(1) = \text{arbitrary} < \infty \end{gathered} \tag{31}$$

with appropriate constants b, c_i, and c_i^*. Equation (31) can be directly solved to yield

$$\alpha(t) = \alpha(1)\frac{1}{t} + \int_1^t \frac{\tau}{t}\left[\frac{b}{\tau^2} + \sum_{i=1}^{n} \frac{c_i}{\tau(\tau + i)} + \sum_{i=1}^{n} \frac{c_i^*}{\tau(\tau + i)^2}\right] d\tau \quad (32)$$

which converges to zero as $t \to \infty$.

III. RELATIONSHIP TO OPTIMAL LINEAR FILTERING

Readers who are familiar with Wiener-Kalman filtering method undoubtedly have noticed the strong resemblance of (26) with the optimal linear filtering formula. In fact, (22) for successive values of t can be written in the vector matrix form

$$\begin{bmatrix} z_1(t+1) \\ z_1(t+2) \\ \vdots \\ z_1(t+k) \end{bmatrix} = \begin{bmatrix} z^T(t) \\ z^T(t+1) \\ z^T(t+k-1) \end{bmatrix} [\varphi] + \begin{bmatrix} \omega(t) \\ \omega(t+1) \\ \omega(t+k-1) \end{bmatrix} \quad (32)$$

which is in the form of

$$b = Ax + v \quad (33)$$

the regular form of the single stage estimation problem (Ho, 1963). Now if we deliberately ignore the correlation among elements of A, v, and b (i.e., among $z_1(t+1)$, $z(t)$, and $w(t)$) and apply the filtering formula formally, then we have,

$$\hat{\varphi}(t+1) = \hat{\varphi}(t) + P(t+1)z(t)(z_1(t+1) - \hat{\varphi}(t)^T z(t)) \quad (34)$$

where

$$P(t+1) = P(t) - P(t)z(t)(z(t)^T P(t)z(t) + \sigma^2)^{-1} z(t)^T P(t) \quad (35)$$

But it has already been shown that the matrix $P(t)$ asymptotically behaves as $1/t$ (Ho, 1963). Hence by an extension of the proof of convergence for (26) in the Appendix, we can show (34) to be convergent also despite the fact that the Wiener-Kalman method is not strictly applicable here. Of course, we can no longer claim any optimal property for (34)[2] in a stochastic sense. Deterministically, the estimate given by (34) still represents the least square fit to the measurement data.

[2] Note also that the approach of (34) and (35) is conceptually different from the method in (Kopp, 1963), where the system equation is considered as a nonlinear equation where the unknown parameters play the role of the additional state variables and where explicit linearization is involved before application of the filtering formula. No such linearization is present in this case.

IV. COMPUTATIONAL RESULTS

The identification scheme shown in Eqs. (34) and (35) were programmed for digital computation. Preliminary results indicate considerable promise. The results of two computational experiments will be presented. The first experiment deals with the identification of stationary linear system parameters. The second experiment deals with the tracking of nonstationary linear system.

Experiment I. A fourth order linear system is used. The basic dynamical relationships are

$$\begin{aligned} x_{(t+1)} &= \Phi^* x_{(t)} + \gamma w_1(t) \\ z_1(t) &= h^{*T} x(t) \end{aligned} \tag{36}$$

where

$$\Phi^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 \end{bmatrix} \qquad \gamma = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad h^* = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \varphi = \begin{bmatrix} -0.656 \\ +0.784 \\ -0.18 \\ +1.0 \end{bmatrix}$$

and $w_1(t)$ is a white gaussian noise with zero mean and variance σ^2. In terms of sampled-data language, this is equivalent to a z transfer function

$$G(z) = \frac{z^3}{(z^2 - 1.8z + 0.8z)(z^2 + 0.8z + 0.8)} \tag{37}$$

with the pole-zero configuration shown in Fig. 1. The dynamics of (37)

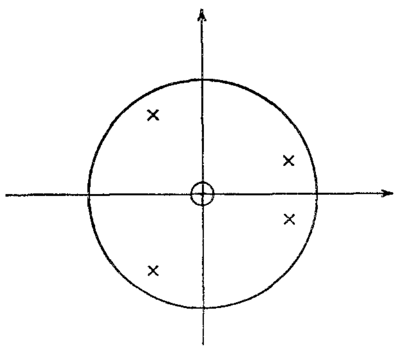

FIG. 1. Pole-zero pattern of 4th order system

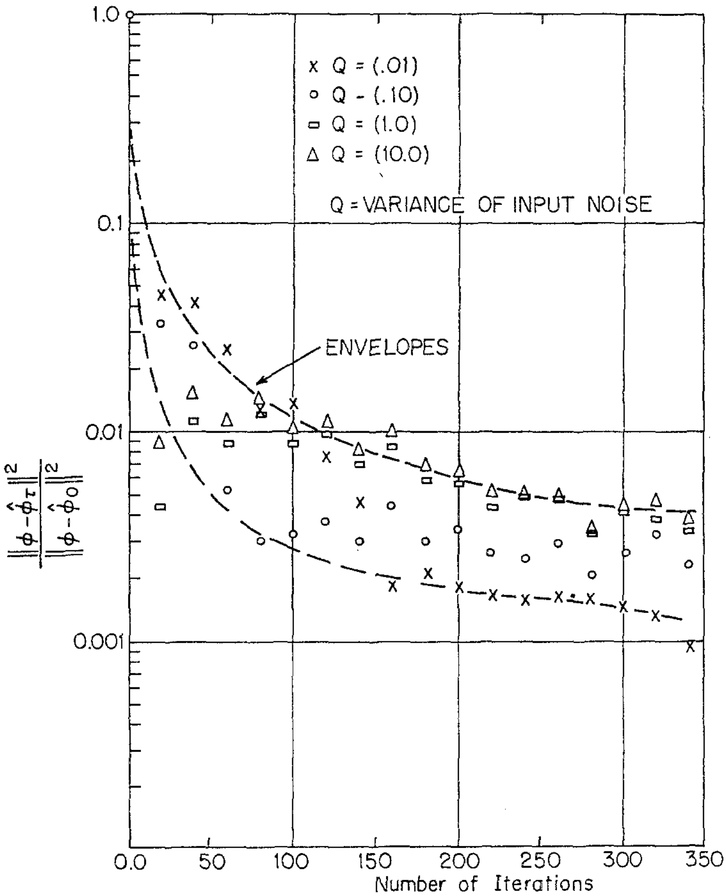

FIG. 2. Effect of noise amplitude upon the identification

can be considered as a hypothetical missile with the short period and the first bending mode included.

Four different noise levels ($\sigma^2 = 0.01, 0.1, 1.0, 10.0$) were used in the experiment. In all cases, the initial estimate $\hat{\varphi}_0$ is assumed to be exactly opposite to the true $\hat{\varphi}$ and the initial $P_0 = \text{diag}\,(100)$. The computational results are presented in Fig. 2. In all cases, the estimation error converges rapidly at first and then asymtotically approaches zero. The convergence properties for different noise levels are roughly the same. The "estimation error" is larger for higher noise level as is to be expected.

Computational results also showed (Lee, 1964) that the convergence properties of the identification are insensitive to the initial choice of P_0 and $\hat{\varphi}_0$. These properties are highly desirable in practical application where one has no apriori information P_0 and $\hat{\varphi}_0$.

Experiment II. Parameter tracking of nonstationary systems. For this experiment a 2nd order system is used. The dynamic relationships are as follows:

$$\begin{aligned} x_{(t+1)} &= \Phi^* x_{(t)} + \gamma w_{1(t)} \\ z_{1(t)} &= h^{*T} x_{(t)} \\ \varphi_{(t+1)} &= \psi \varphi_{(t)} \end{aligned} \tag{38}$$

where

$$\Phi^* = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \quad \gamma = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad h^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\psi = \begin{bmatrix} 1.02 & 0 \\ 0 & 1 \end{bmatrix} \quad \varphi_0 = \begin{bmatrix} -0.8 \\ +0.1 \end{bmatrix} \quad \hat{\varphi}_0 = \begin{bmatrix} +0.8 \\ -0.1 \end{bmatrix}$$

$$P_0 = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \quad \sigma^2 = 0.33$$

The trajectory of the system poles as a function of time is illustrated in Fig. 3. Note that as time progresses, the system becomes unstable. The

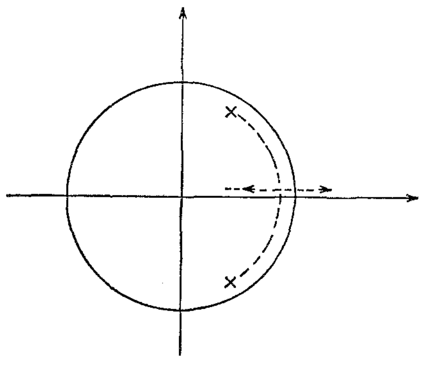

FIG. 3. Pole-zero pattern of 2nd order system

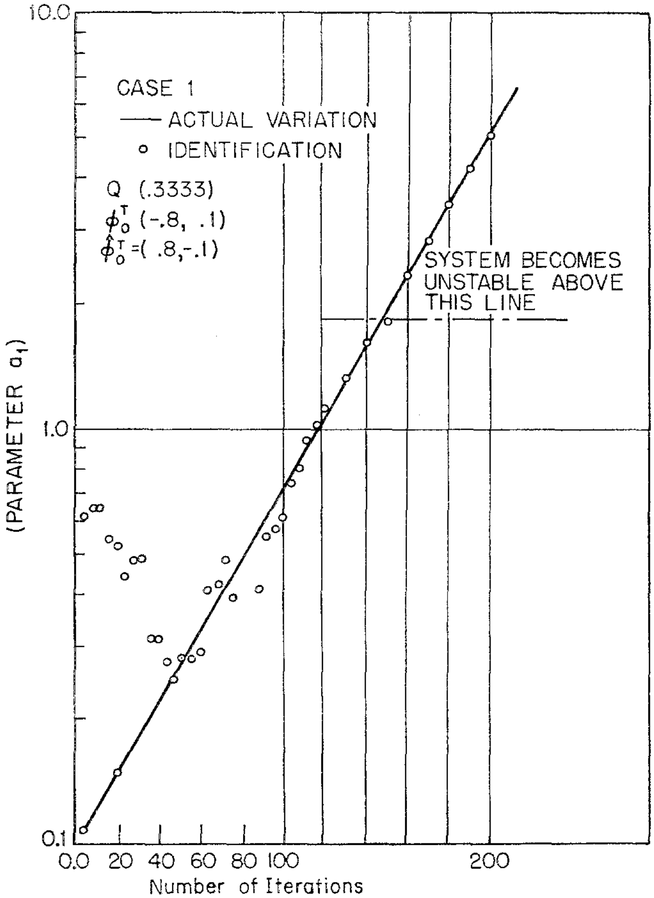

FIG. 4. Parameter tracking (nonstationary system 2nd order)

computational formula used are

$$\begin{aligned}\hat{\varphi}(t+1) &= \psi\hat{\varphi}(t) + M(t+1)z(t) \\ &\quad \cdot (z(t)^T M(t+1)z(t) + \sigma^2)^{-1}(z_1(t+1) - z^T(t)\cdot\psi\hat{\varphi}(t))\end{aligned} \tag{39}$$

$$M(t+1) = \psi P(t)\psi^T \tag{40}$$

$$\begin{aligned}P(t+1) &= M(t+1) - M(t+1)z(t) \\ &\quad \cdot (z(t)^T M(t+1)z(t) + \sigma^2)^{-1} z(t)M(t+1)\end{aligned} \tag{41}$$

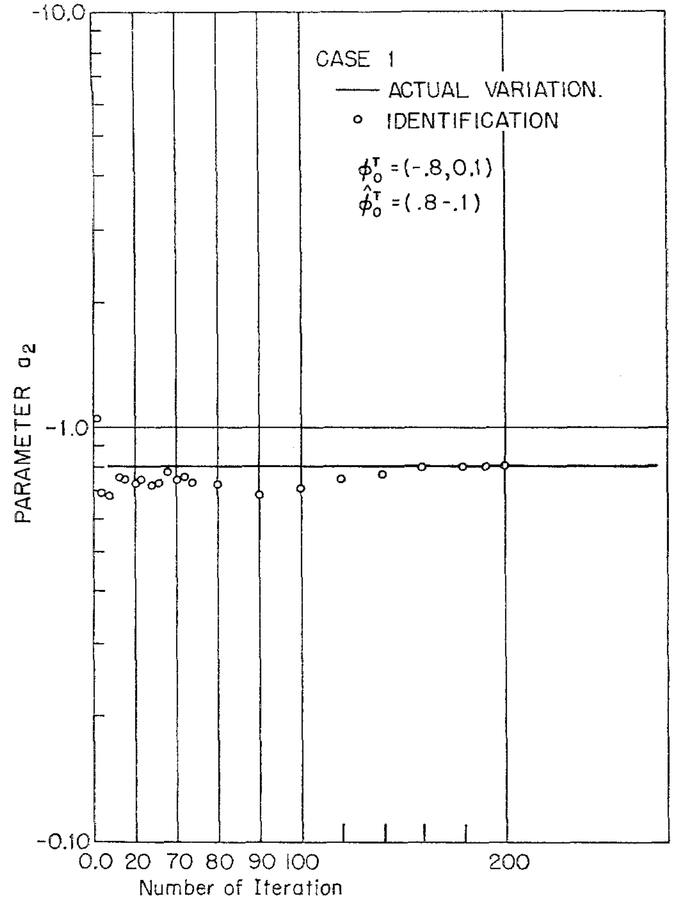

FIG. 5. Parameter tracking (nonstationary system)

These are simply the nonstationary versions of (34) and (35). The results are shown in Figs. 4 and 5. Excellent tracking behavior was demonstrated. Note that the identification is still optimal in the sense of least square fit over finite data.

Experiments were also performed for nonstationary systems whose parameter variations are further subjected to random perturbations. Excellent tracking behavior was also obtained in spite of the fact that the identification process is no longer optimal in any sense. The computational results are shown in Figs. 6 and 7.

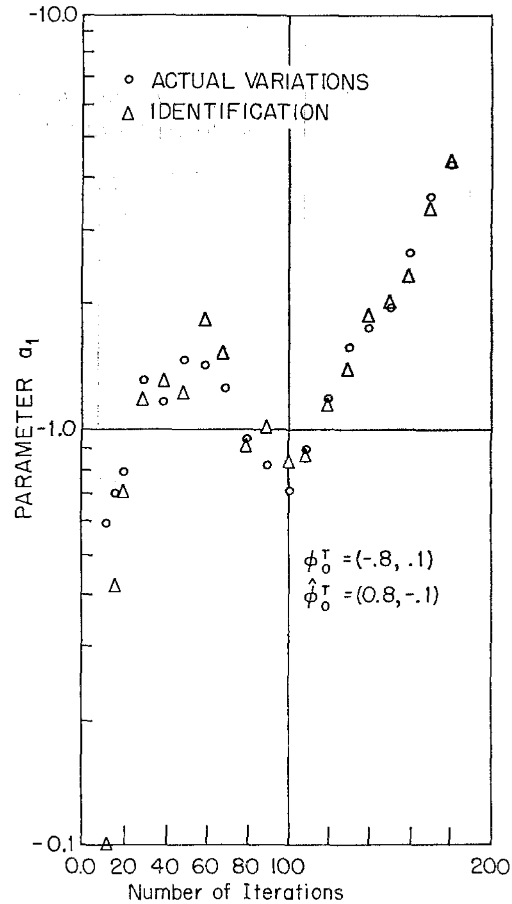

FIG. 6. Parameter tracking (nonstationary system) (parameter driven by random noise).

V. CONCLUSION

It has been shown that least square fit over finite data is not only a deterministically optimal procedure for identification purposes but also has the property of stochastic convergence. Experimental simulation further affirms the real time practicality of the method. Application of the method to adaptive filtering and control is obvious and immediate. For example, Eqs. (34) and (35) actually represent an adaptive predictor for $z_1(t + 1)$ which has the property that asymptotically it per-

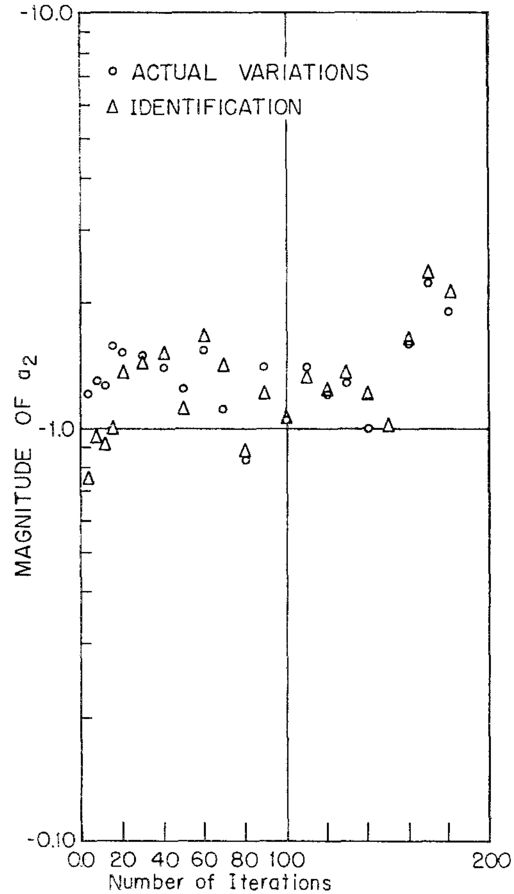

FIG. 7. Parameter tracking (system parameter driven by random noise)

forms as well as a Wiener-Kalman predictor which requires knowledge of φ. Thus, practically all of the knowledge for linear control and filtering theory can be extended via this approach to cases where certain parameters of the linear dynamic system in question are unknown.

APPENDIX. CONVERGENCE OF THE STOCHASTIC ESTIMATION METHOD

For simplicity of notation, we shall prove the convergence of the estimation scheme

$$\hat{\varphi}(t+1) = \hat{\varphi}(t) + \rho(t)(z_1(t+1) - \hat{\varphi}(t)^T s(t))s(t) \quad \text{(A. 1)}$$

only for the system

$$z_1(t+1) = \varphi^T s(t) + w_1(t) \quad \text{(A. 2)}$$

where

$$s(t) = \begin{bmatrix} z_1(t-n+1) \\ \vdots \\ z_1(t) \end{bmatrix} \qquad \Phi = \left[\begin{array}{c|c} 0 & \\ \vdots & I \\ 0 & \\ \hline \multicolumn{2}{c}{\varphi^T} \end{array}\right] \qquad d = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \text{(A. 3)}$$

$$s(t+1) = \Phi s(t) + d w_1(t) \quad \text{(A. 4)}$$

and

$$E(w_1(t)) = 0; \quad \mathrm{Var}(w_1(t)) < \infty \quad \text{(A. 5)}$$

As was pointed out in Section II this is no real loss of generality. Now combining (A. 1) and (A. 2), we get for the error of the estimate,

$$\begin{aligned} \tilde{\varphi}(t) &\overset{\Delta}{=} \hat{\varphi}(t) - \varphi \\ \tilde{\varphi}(t+1) &= (I - \rho(t)s(t)s(t)^T)\tilde{\varphi}(t) + \rho(t)s(t)w_1(t) \quad \text{(A. 6)} \\ &\overset{\Delta}{=} \Theta(t)\tilde{\varphi}(t) + \rho(t)s(t)w_1(t) \end{aligned}$$

Thus

$$\begin{aligned} \|\tilde{\varphi}(t+1)\|^2 &= \|\tilde{\varphi}(t)\|^2_{(I-2\rho(t)s(t)s(t)^T)} \\ &\quad + 2\rho(t)w_1(t)s(t)^T\Theta(t)\tilde{\varphi}(t) \quad \text{(A. 7)} \\ &\quad + \rho^2(t)[w_1^2(t)s(t)^T s(t) + \|\tilde{\varphi}(t)\|^2_{(s(t)s(t)^T)^2}] \end{aligned}$$

Since $w_1(t)$ is independent of $s(t)$ and $\tilde{\varphi}(t)$, we get after taking expectation of both sides of (A. 7)

$$E\|\tilde{\varphi}(t+1)\|^2 = E\{\|\tilde{\varphi}(t)\|^2_{(I-2\rho(t)s(t)s(t)^T)}\} + \mathcal{O}(\rho^2(t)) \quad \text{(A. 8)}$$

Now let us examine the second quadratic form in (A. 8) in detail.

$$\begin{aligned} &E\{\|\tilde{\varphi}(t)\|^2_{2\rho(t)s(t)s(t)^T}\} \\ &\quad = 2\rho(t)E\{\|\tilde{\varphi}(t-1)\|^2_{[\Theta^T(t-1)s(t)s(t)^T\Theta(t-1)+2\rho(t-1)s^T(t-1)\Theta^T(t-1)\cdot s(t)s(t)^T dw_1(t-1)+\rho^2(t-1)w_1^2(t-1)s(t-1)^T s(t)s(t)^T s(t-1)}\} \\ &\quad = 2\rho(t)E\{\|\tilde{\varphi}(t-1)\|^2_{\Theta(t-1)s(t)s(t)^T\Theta(t-1)}\} \end{aligned}$$

$$+ \Theta(\rho(t)\rho(t-1), \rho(t)\rho^2(t-1)) \quad \text{(A. 9)}$$

$$= 2\rho(t)E\{\|\tilde{\varphi}(t-1)\|^2_{[s(t)s(t)^T - 2\rho(t-1)s(t-1)s(t-1)^Ts(t)s(t)^T - \rho(t-1)s(t)s(t)^Ts(t-1)s(t-1) + \rho^2(t-1)s(t-1)s(t-1)^Ts(t)s(t)^Ts(t-1)s(t-1)^T]}\}$$
$$+ \Theta(\rho(t)\rho(t-1), \rho(t)\rho^2(t-1))$$
$$= 2\rho(t)E\{\|\tilde{\varphi}(t-1)\|^2_{s(t)s(t)^T}\} + \Theta(\rho(t)\rho(t-1), \rho(t)\rho^2(t-1))$$

Repeating the above reduction n times we get

$$E\{\|\tilde{\varphi}(t)\|^2_{2\rho s(t)s(t)^T}\} = 2\rho(t)E\{\|\tilde{\varphi}(t-n)\|^2_{s(t)s(t)^T}\} + \Theta(\rho(t)\rho(t-1), \rho(t)\rho^2(t-1), \cdots \rho(t)\rho^2(t-n)) \quad \text{(A. 10)}$$

But

$$s(t)s(t)^T = \Phi^n s(t-n)s(t-n)^T\Phi^{nT} + \sum_{i=0}^{n-1} \Phi^{n-1-i}\, dw_1^2(i)\, d^T\Phi^{n-1-i^T}$$
$$+ \text{cross product terms in } w_1(i)w_1(j) \quad \text{(A. 11)}$$

Since $w_1(i)w_1(j)$ are independent of $\tilde{\varphi}(t-n)$, $s(t-n)$ for all i, $j \geqq t-n$. Hence

$$E\{\|\tilde{\varphi}(t-n)\|^2_{s(t)s(t)^T}\} = E\{\|\tilde{\varphi}(t-n)\|^2_{\Phi^n s(t-n)s(t-n)^T\Phi^{nT}}\}$$
$$+ E\|\tilde{\varphi}(t-n)\|^2_{\left[\sum_{i=0}^{n-1}\Phi^{n-1-i}dE(w_1^2(i))d^T\Phi^{n-1-i^T}\right]} \quad \text{(A. 12)}$$
$$> 0 \quad \text{(by controllability Remark (ii))}$$

Let us define

$$R = \sum_{i=0}^{n-1} \Phi^{n-1-i}\, dE(w_1^2(i))\, d^T\Phi^{n-1-i}$$
$$\lambda_0 = \lambda_{\min}(R). \quad \text{(A. 13)}$$

then combining (A. 13), (A. 12), (A. 10), and (A. 8) we arrive at the result

$$E\|\tilde{\varphi}(t+1)\|^2 \leqq E\|\tilde{\varphi}(t)\|^2 - 2\rho(t)\lambda_0 E\|\tilde{\varphi}(t-n)\|^2 + \Theta(\rho(t)\rho(t-i), \rho(t)\rho^2(t-i)) \quad \text{(A. 14)}$$

which also implies

$$E\|\tilde{\varphi}(t)\|^2 < E\|\tilde{\varphi}(t-n)\|^2 + \text{second order terms in } \rho. \quad \text{(A. 15)}$$

Applying (A. 15) to (A. 14) we finally get

$$E \| \tilde{\varphi}(t+1) \|^2 \leqq E \| \tilde{\varphi}(t) \|^2 (1 - 2\rho(t)\lambda_0) + \mathcal{O}(\rho(t)\rho(t-i), \rho(t)\rho^2(t-i)) \quad i = 0, \cdots, n. \tag{A. 16}$$

which is the desired result claim in (30). It is easily verified (Bronwich, 1947) that if we choose

$$\rho(t) = 1/t \tag{A. 17}$$

then

$$\prod_{t=N}^{\infty} (1 - 2\rho(t)\lambda_0) = 0 \tag{A. 18}$$

$$\sum_{t=N}^{\infty} \rho(t)\rho^j(t-i) < \infty \qquad \begin{matrix} j = 1, 2 \\ i = 0, \cdots, n \end{matrix} \tag{A. 19}$$

Hence Dvoretzky's proof of the convergence of a stochastic approximation sequence applies (Dvoretzky (1956), p. 52). We have proved that

$$\lim_{t \to \infty} E \| \tilde{\varphi}(t) \|^2 = 0 \tag{A. 20}$$

RECEIVED: May 18, 1964

REFERENCES

BROMWICH, T. J. (1947), "Theory of Infinite Sines," 2nd ed., p. 105. Macmillan, New York.

BRYSON, A. E. AND FRAZIER, M. (1962), Smoothing in linear and nonlinear systems. *Proc. Optimal System Synthesis Conf., September 1962, Wright Field, Ohio.* ASD-TDR-63-119.

DVORETSKY, A. (1956), On stochastic approximation. *Third Berkeley Symp. Math. Statist. and Prob.*, p. 39–56. Univ. of California Press, 1956.

HO, Y. C. (1963), On optimal filtering and stochastic approximation. *J. Math. Anal. Appl.* **6**, 152–154.

HO, Y. C. AND WHALEN, B. (1963), An approach to the identification and control of linear dynamic systems with unknown parameters. *Trans. IEEE-PTGAC* **ACH-8**, 255–256.

KALMAN, R. E. (1963), Mathematical descriptions of linear dynamic system. *SIAM J. Control* **1**, No. 2.

KOPP, R. AND ORFORD, R., (1963), System identification for adaptive control systems. *AIAA J.* **1**, 2300–2306.

LEE, R. C. K. (1964), Optimal Estimation, Identification and Control. M.I.T. Sc.D. Thesis.

McLEAN, J. D., SCHMIDT, S. F., AND McGEE, L. A. (1962), Optimal filtering and linear prediction applied to a midcourse navigation system. NASA TND-1208.