Binary- and Multi- Class Logistic Regression

Outline

- Logistic Regression with Sparse Regularization
- Multi-Class Logistic

Logistic Regression

1. Load the binary dataset
   - $X \in \mathbb{R}^{n \times d}$, $y \in \{\pm 1\}^n$
2. Standardize the columns of X
   - Each column in X is standardized using z-score scaling
   - $X^j = \frac{(X^j - \mu_j)}{\sigma_j}$, $\mu_j = np.sqrt \frac{\sum_{i=1}^n X_i^j}{n}$, $\sigma_j = \frac{\sum_{i=1}^n (X_i^j - \mu_j)^2}{n}$
   - Each column has mean 0 and variance 1 but different ranges
3. Add bias (add a column of all ones to X)
4. Train logistic regression on X and y

Trained Logistic Regression

1. Standardize the columns of X_{validate} using μ_j and σ_j
2. Add bias (add a column of all ones to X)
3. Compute validation error on X_{valid} and y_{valid}
4. Compute the number of non-zeros for w

Fit logistic Regression

- Initialize $w \in \{0\}^d$
- Check gradient using the complex step method (it's given)
  - more accurate than finite differencing
- Set up the objective function and the gradient funObj (func, grad)
  - funObj[0] $= f(w) = \sum_{i=1}^{n} \log\left(1 + \exp(-y_i w^T x_i)\right)$
  - funObj[1] $= \nabla f(w) =$? (Derive)
- Run findMin
  - Update w^t for **maxEvals** iterations
  - $w^{t+1} = w^t - \alpha \nabla f(w^t)$
  - Uses line search to determine α that ensures
    - $f(w^{t+1}) < f(w^t)$
    - $w^{t+1} = w^t - \alpha \nabla f(w^t)$

Evaluate logistic Regression

- Compute the classification error on the validation set
- Compute the number of non-zeros on w

Logistic Regression L2

- Logistic regression

$$f(w) = \sum_{i=1}^{n} \log\left(1 + \exp(-y_i w^T x_i)\right)$$

- L2 Regularized Logistic Regression
- funObj[0] =

$$f(w) = \sum_{i=1}^{n} \log\left(1 + \exp(-y_i w^T x_i)\right) + \frac{\lambda}{2}||w||_2^2$$

- funObj[1] = $\nabla f(w) = ?$

Logistic Regression L1

- $f(w) = \sum_{i=1}^{n} \log\left(1 + \exp(-y_i w^T x_i)\right) + \lambda ||w||_1$
- $||w||_1$ is not differentiable at the origin

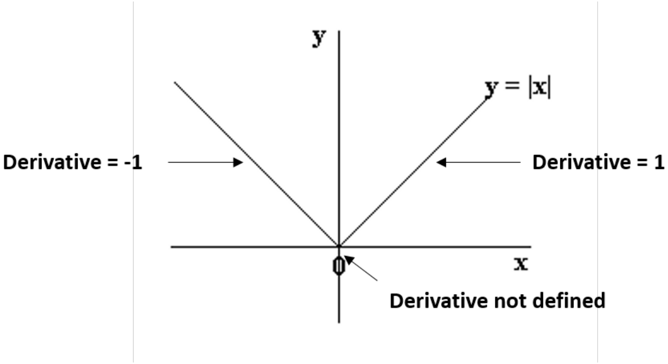


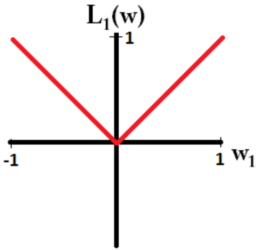
Figure 1:

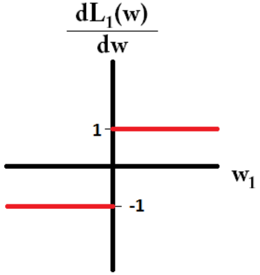
Logistic Regression L1

- $f(w) = \sum_{i=1}^{n} \log\left(1 + \exp(-y_i w^T x_i)\right) + \lambda ||w||_1$
- $||w||_1$ is not differentiable at the origin
- Define funObj the same way as Logistic regression
  - funObj[0] $= f(w) = \sum_{i=1}^{n} \log\left(1 + \exp(-y_i w^T x_i)\right)$
  - funObj[1] $= \nabla f(w) =?$
- Use findMinL1 instead of findMin to optimize the objective function
- findMinL1 uses proximal gradient to take care of the L1 norm

Logistic Regression L1

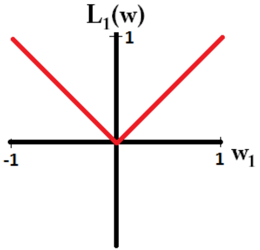
- You should expect some zero elements in w the parameter vector
- Distance to minimum is not negligible

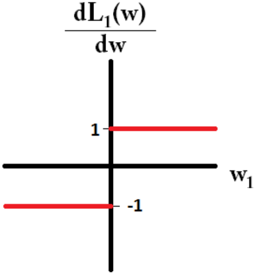




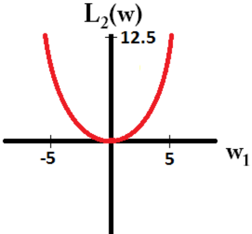
Logistic Regression L1

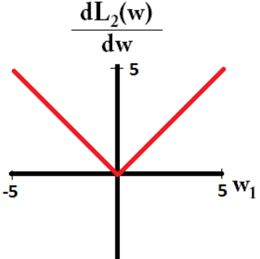
- Minimize $L_1(w) = \sum_i |w_i|_1$





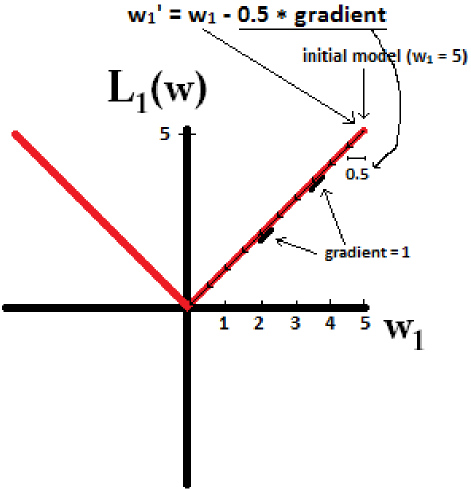
- Minimize $L_2(w) = \frac{1}{2} \sum_i w_i^2$





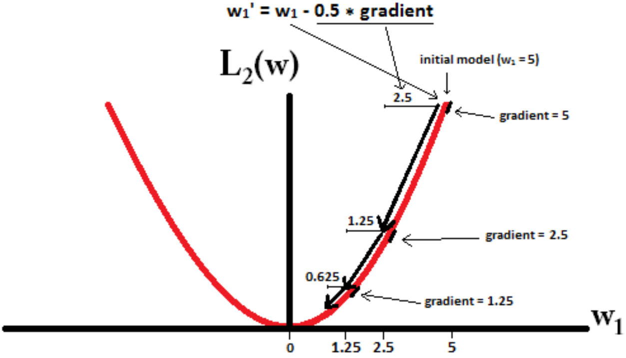
Logistic Regression L1

- Minimize $L_1(w) = \sum_i |w_i|_1$



Logistic Regression L1

- Minimize $L_2(w) = \frac{1}{2}\sum_i w_i^2$



Logistic Regression L0

- Objective:

$$f(w) = \sum_{i=1}^{n} \log\left(1 + \exp(-y_i w^T x_i)\right) + \lambda ||w||_0$$

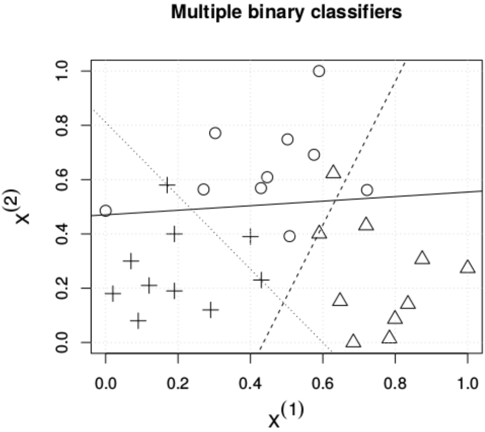
- Class inheritance (Demonstrate)
- Always include the bias
- Select best feature using forward selection
- Note that adding any feature adds the same penalty λ
- Stop when the decrease in the model error is less than the added penalty

leastSquaresClassifier

- Classify a multi-class dataset using one-vs-all method
- Consider a dataset of 3 samples
- How many linear models ?

leastSquaresClassifier

- Classify a multi-class dataset using one-vs-all method
- Consider a dataset of 3 samples



leastSquaresClassifier

- To make a prediction:
  - Compute $X_i W^k$ for each class k and take the largest.

```
def predict(self, X):
    yhat = np.dot(X, self.W)

    return np.argmax(yhat, axis=1)
```

logisticRegression

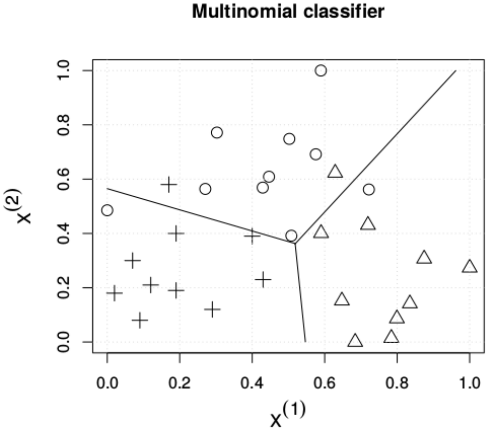
- Instead of leastSquares use logisticRegression
- $f(w) = \sum_{i=1}^{n} \log\left(1 + \exp(-y_i w^T x_i)\right)$

```
for i in range(self.n_classes):
    ytmp = y.copy().astype(float)
    ytmp[y==i] = 1
    ytmp[y!=i] = -1

    self.W[:, i] = np.linalg.lstsq(np.dot(X.T, X),
                                   np.dot(X.T, ytmp)
```

Multinomial logistic regression - SoftmaxClassifier

- Uses one classifier to classify points w.r.t all classes
- Learns a global distribution
  - Probability that a data point belongs to each class



Multinomial logistic regression - SoftmaxClassifier

- Same as with the other objective functions
  1. define funObj (func, grad)
  2. use findMin to minimize the function
- Set up the objective function and the gradient funObj (func, grad)
  - funObj[0] =

$$f(W) = \sum_{i=1}^{n} \left[-w_{y_i}^T x_i + \log \left(\sum_{c'=1}^{k} \exp(w_{c'}^T x_i) \right) \right].$$

  - funObj[1] $= \nabla f(w) = ?$
- Compute validation error