MF730 Final Project: Dynamic Asset Allocation

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Abstract

The goal of this project is to solve a dynamic asset allocation problem in the context of US stocks and treasury bills. After selecting a small investment set of securities to give us a complete market model, we try to find the optimal portfolio for a particular investor with CRRA utility, a predetermined amount of initial wealth, and a one year time horizon. We begin by specifying the dynamics of our model, in particular the short rate process r_t and the market price of risk process θ_t. The parameters of these two processes are calibrated using historical market data. Then, we quantify the efficient frontier, both with and without a risk-free asset in our investment set, which allows us to illustrate the benefits of a diversified portfolio. Next, we calculate the dynamic hedge portfolio and its decomposition into the mean-variance portfolio and the risk-free and market hedging portfolio. Finally, we analyze the performance of the portfolio in regards to its Sharpe ratio and information ratio by back testing over the historical time period.

1 Introduction: Asset Classes and Motivation

The problem of optimal asset allocation has been around for as long as securities have been publicly traded. Investors are always trying to maximize their wealth. To that end, this project attempts to apply modern portfolio theory to solve this dynamic asset allocation problem. Our procedure will closely follow Detemple, Garcia, and Rindisbacher (2003 - hereafter called DGR 2003).

We began by choosing to comprise our portfolio of 10 different stocks, from 5 different sectors, with the aim of representing a somewhat diversified portfolio. These 10 stocks were Unilever (UL), Goldman Sachs (GS), Pfizer (PFE), Google (GOOGL), Coca-Cola (KO), Bank of America (BAC), Johnson & Johnson (JNJ), Nike (NKE), Procter & Gamble (PG), and Apple (AAPL). We used the historical returns of these stocks along with historical data of 1-month maturity t-bills to calibrate our models.

The final goal of our investigation is the find the optimal allocation of assets for our chosen stocks. We plan to plot the efficient frontier, and from this establish the tangency portfolio which will give us the optimal portfolio. Plotting the efficient frontier will also help us to illustrate the benefits of a diversified portfolio. Our chosen time horizon for the portfolio is one year, however we will also look at longer time horizons in the hope that we again generate suitable weights.

2 Model settings

To begin our investigation we firstly needed to decide on several preferences and settings for our model, including our investor risk preferences and underlying dynamics.

For the risk preferences we decided to use the Constant Relative Risk Aversion (CRRA) model

$$u(c) = \frac{c^{1-R}}{1-R} \tag{1}$$

where R is constant and represents the relative risk aversion coefficient. We chose this as it is generally the default utility function, and varying R would allow us to further explore the possibilities for our optimal portfolio.

We then had to decide how we wanted to specify the underlying dynamics for our model, specifically for the short rate and market price of risk.

We used the following OU process for the short rate:

$$dr_t = \kappa_r(\bar{r} - r_t)dt + \sigma_r dW_t \tag{2}$$

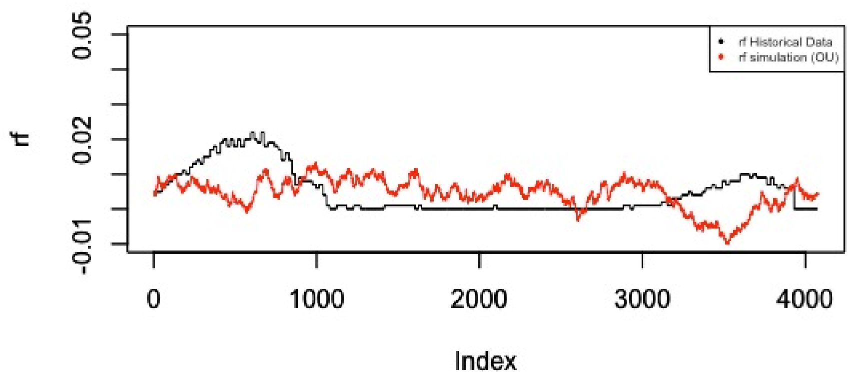
and calibrated the parameters $\kappa_r, \bar{r}, \sigma_r$ using the data from the 1-month maturity t bill.

We also modeled θ as an OU process

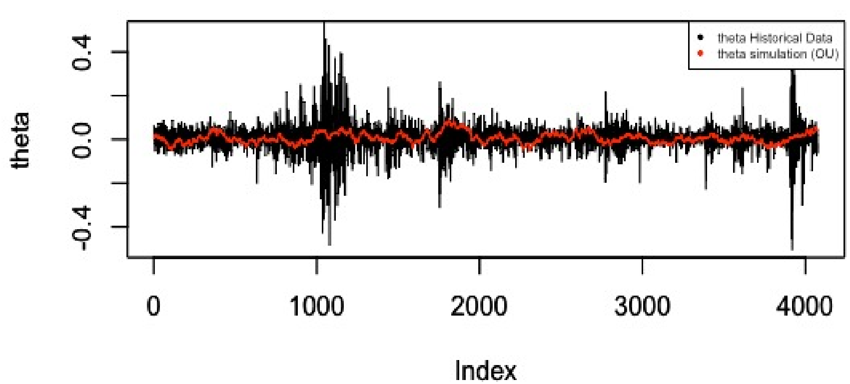
$$d\theta_t = \kappa_\theta(\bar{\theta} - \theta_t)dt + \sigma_\theta dW_t \tag{3}$$

calibrating the parameters from the data by using historical returns, realized volatility, and r from a simulation of the OU process in (2).

We chose these models by simulating r and θ, the plots of which are displayed below, and recognising patterns somewhat similar to OU processes.



Simulated risk-free rate vs historial risk-free rate



Simulated theta vs historical theta

3 Formulas for the Optimal Portfolio and Efficient Frontier

Here we outline the formulas, derived in DGR(2003), that we used in order to compute the optimal portfolios and efficient frontier.

- For the evolution of total wealth, we used the following dynamics:

$$dX_t = (X_t r_t - c_t)dt + X_t \pi_t' [(\mu_t - r_t 1)dt + \sigma_t dW_t] \tag{4}$$

  where c_t represents the consumption. In order to simplify our investigation, and focus on the final result rather than intermediate wealth, we chose to have zero consumption. The wealth dynamics therefore simplify to

$$dX_t = X_t r_t dt + X_t \pi_t' [(\mu_t - r_t 1)dt + \sigma_t dW_t] \tag{5}$$

- The general dynamic consumption-portfolio choice problem is given by

$$\max_{c,\pi,X_t} \mathbb{E}\left[\int_0^T u(c_v, v)dv + U(X_T, T)\right] \tag{6}$$

$$st \begin{cases} dX_t = (X_t r_t - c_t)dt + X_t \pi_t' [(\mu_t - r_t 1)dt + \sigma_t dW_t],\ X_0 = x \\ c_t \geq 0,\ t \in [0,T],\ X_T \geq 0 \\ X_t \geq 0,\ t \in [0,T] \end{cases} \tag{7}$$

  and from this we get the optimal consumption and bequest policies and optimal portfolio:

$$c_t^* = I^u(y^* \xi_t, t) \tag{8}$$

$$X_T^* = I^U(y^* \xi_T, T) \tag{9}$$

$$X_t^* \pi_t^* = X_t^* ({}_t')^{-1} + \xi_t^{-1} (\sigma_t')^{-1} \phi_t^* \tag{10}$$

$$\tag{11}$$

  where y* is the unique solution to:

$$x = \mathbb{E}\left[\int_0^T \xi_t I^u(y^* \xi_t, t) + \xi_T I^U(y^* \xi_T, T)\right] \tag{12}$$

  We identify ϕ_t^* using Clark-Ocone formula.

- The optimal portfolio has decomposition

$$X_t^* \pi_t^* = X_t^* [\pi_{1t}^* + \pi_{2t}^*] \tag{13}$$

where

$$X_t^* \pi_{1t}^* = \mathbb{E}_t \left[\int_t^T \xi_{t,v} \Gamma^u(c_v^*, v) dv + \xi_{t,T} \Gamma^U(X_T^*, T) \right] (\sigma_t')^{-1} \theta_t \tag{14}$$

$$X_t^* \pi_{2t}^* = -(\sigma_t')^{-1} \mathbb{E}_t \left[\int_t^T \xi_{t,v} (c_v^* - \Gamma^u(c_v^*, v)) H_{t,v} dv + \xi_{t,T} (X_T^* - \Gamma^U(X_T^*, T)) H_{t,T} \right] \tag{15}$$

The first part is the mean-variance portfolio, and the second part is the hedging portfolio.

- As we are assuming zero consumption, our formulas are simplified to

$$X_t^* \pi_{1t}^* = \mathbb{E}_t \left[\xi_{t,T} \Gamma^U(X_T^*, T) \right] (\sigma_t')^{-1} \theta_t \tag{16}$$

$$X_t^* \pi_{2t}^* = -(\sigma_t')^{-1} \mathbb{E}_t \left[\xi_{t,T} (X_T^* - \Gamma^U(X_T^*, T)) H_{t,T} \right] \tag{17}$$

To further decompose this, $H_{t,T}$ is given by

$$H_{t,T} = -\mathcal{D}_t log(\xi_t) \tag{18}$$

where

$$d\xi_t = -\xi_t (r_t dt + \sigma_t dW_t) \tag{19}$$

Therefore

$$H_{t,T} = \int_t^T \mathcal{D}_t r_s ds + \int_t^T (dw_s + \theta_s' ds) \mathcal{D}_t \theta_s \tag{20}$$

$$= \int_t^T \sigma_r e^{-\kappa_r t} ds + \int_t^T (dw_s + \theta_s' ds) \sigma_\theta e^{-\kappa_\theta t} \tag{21}$$

$$= H_{t,T}^r + H_{t,T}^\theta \tag{22}$$

We can also specify the definition

$$\Gamma^u = \frac{-u'(X)}{u''(X)} = \frac{X}{R} \tag{23}$$

since we are using $u(X)$ equal to the CRRA utility function.

- For the efficient frontier plot, we use the fact that frontier portfolios satisfy fund separation, and a portfolio $\bar{\pi}_t$ is a frontier portfolio if and only if it is given as the following combination of the mean and minimum variance portfolios:

$$\bar{\pi}_t(\bar{\mu}_t) = \pi_t^{mv} + \left(\frac{\bar{\mu}_t - \mu_t^{mv}}{\|\theta^{mv}\|^2} \right) \pi_t^m \tag{24}$$

where, with no riskfree asset, the mean and minimum variance portfolios are given by

$$\pi_t^{min.\ var} = \pi_t^{mv} = \frac{(\sigma_t \sigma_t')^{-1} 1_d}{\|\sigma_t^{-1} 1_d\|^2} \tag{25}$$

$$\pi_t^{mean.\ var} = \pi_t^m = (\sigma_t)^{-1} \theta_t^m \tag{26}$$

$$\theta_t^m = \sigma_t^{-1}(\mu_t - 1_d \mu_t^{mv} \tag{27}$$

and with a riskfree asset, the minimum variance portfolio consists solely of the riskfree asset so

$$\pi_t^{mv} = \pi_t^0 = 1 - \alpha_t 1_d (\sigma_t')^{-1} \theta_t^m \tag{28}$$

$$\pi_t^m = \alpha_t 1_d (\sigma_t')^{-1} \theta_t^m = \alpha_t \pi_t^m \tag{29}$$

where α_t is a scalar adapted process.

4 Methodology

- Our first step was to calibrate and fix the parameters for our short rate OU process. This was done by calibrating the parameters to the riskfree asset data as previously mentioned, estimating

$$\bar{r} = \mathbb{E}(r_t) \tag{30}$$

$$\kappa_r = \Delta \log \left(\frac{\mathrm{var}(r_t)}{\mathrm{cov}(r_t, r_{t-1})} \right) \tag{31}$$

$$\sigma_r = \Delta \mathrm{var}(r_t) \log \left(\frac{\mathrm{var}(r_t)}{\mathrm{cov}(r_t, r_{t-1})} \right) \tag{32}$$

  where n is the number of time intervals our data spanned. All code is included in an additional file.

- We then followed the same steps to calibrate the parameters for the θ OU process.

- Using our values for r and θ, we were able to simulate the state price density ξ_t

- Before plugging these values into our definitions for the portfolios π_1, π_2, we attempted to further simplify the definitions.

  Since we focused on the terminal wealth, the implicit solution for y* and definition of initial wealth x becomes

$$x = \mathbb{E}[\xi_T I^U (y^* \xi_T, T)] \tag{33}$$

and

$$X_T^* = I^U(y^*\xi_T, T) \tag{34}$$

so

$$x = \mathbb{E}[\xi_T X_T^*] \tag{35}$$

This simplifies the hedging portfolios to

$$X_t^* \pi_{1t}^* = \frac{1}{R}\xi_t(\sigma_t')^{-1}\theta_t \tag{36}$$

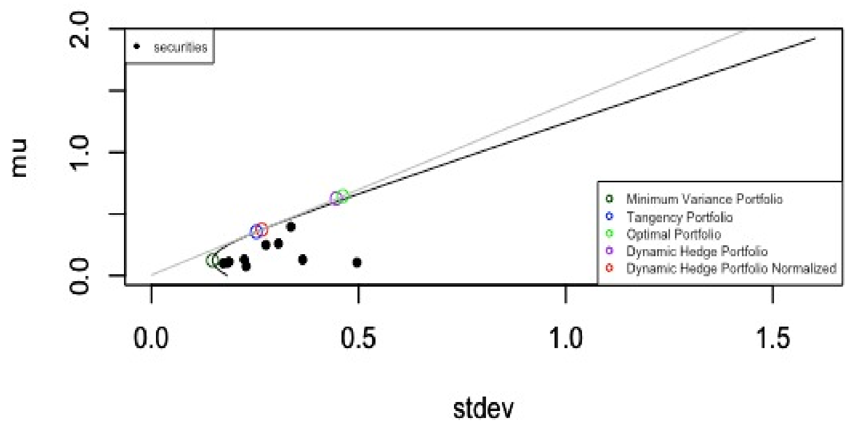
$$X_t^* \pi_{2t}^* = -(\sigma_t')^{-1}\xi_t\left(1 - \frac{1}{R}\right)\mathbb{E}[H_t, T] \tag{37}$$

- We then plugged the values for ξ_t, Γ^u, θ and σ into the expressions for the optimal portfolios to get explicit values for π_1, π_2

5 Results

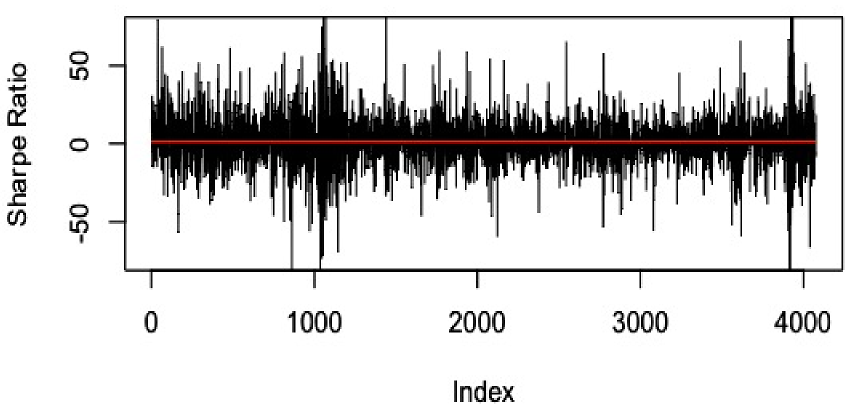
After deriving our optimal portfolio, we planned to investigate its performance in several ways. The first crucial check was whether or not the portfolio lies on the tangent to the efficient frontier, which becomes the efficient frontier when a risk free asset is involved. This is because a portfolio is optimal if and only if it lies on the efficient frontier. Secondly we planned to check the the location of the individual stocks. If these were not on the frontier but the portfolio was, we can conclude that our portfolio has strong diversification benefits. Finally, we will calculate the Sharpe Ratio and Information Ratio using the SP500 as our benchmark, to see how our portfolios performs relative to risk-free assets and the SP500.

The graph below shows the results of our first two performance checks.



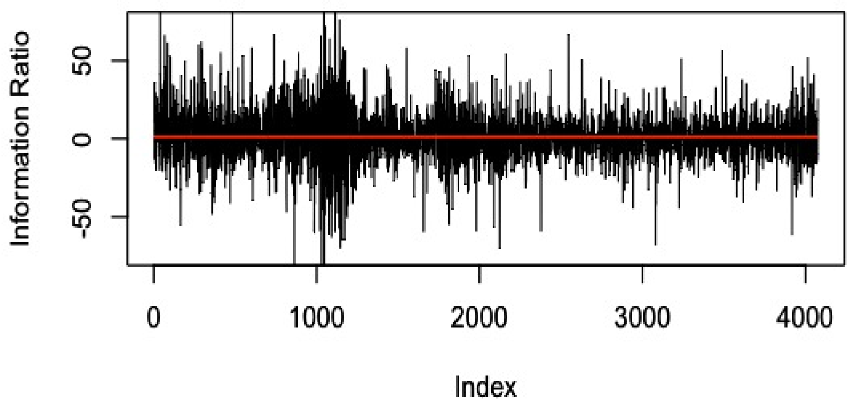
Efficient Frontier Optimal Portfolio for Risk Aversion = 3, Time horizon = 1 year

This graph displays the optimal portfolio generated for a risk aversion of 3 and a time horizon equal to 1 year. We can see from this that the optimal portfolio, dynamic hedge portfolio and normalized dynamic hedge portfolio (weights sum to 1) lie on the tangent to the efficient frontier, so from this we can conclude a positive result. Given that none of the stocks themselves (shown in black) lie on the frontier, we can see strong diversification benefits.



Sharpe Ratio over 1 year maturity

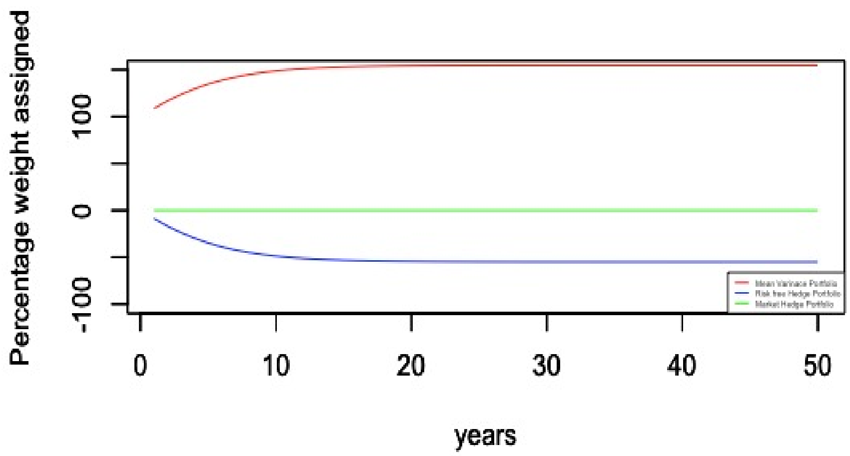
For our third performance criteria, we have the graph above showing the Sharpe ratio for the weights given by a 1-year maturity portfolio. As we can see there is a lot of fluctuation, however the mean Sharpe ratio was 1.383615 which is considered acceptable and sometimes good by investors. The mean excess return is 0.2655384 with respect to the risk-free rate, which is again good but with room for improvement.



Information Ratio w.r.t SP500 over 1 year maturity

We also have the Information Ratio over a 1 year time horizon with the SP500 as the benchmark. As displayed above, this follows a very similar pattern to the Sharpe ratio. The fluctuations are not ideal, however we have a mean Information Ratio of 1.240401 and a mean excess return of 0.3501087, therefore on average our optimal portfolio does beat the SP500.

As a final investigation we decided to see how the percentage of weight in the Mean Variance Portfolio vs the Risk Free Hedge (Minimum Variance) Portfolio changes with maturity. From the graph below we can clearly see that as maturity increases, almost all the weight is assigned to the Mean Variance Portfolio. This is to be expected as with portfolios in general, stocks lead to higher risk-adjusted returns than risk-free assets as maturity increases.



Percentage of weight assigned to Mean Variance and Risk Free Hedge Portfolios over time

6 Improvements

This paper made a number of assumptions in order to simplify the environment for achieving good results. If we were to extend this project, we would start by considering different processes to model r_t and θ_t. We can see from the graphs included in the settings section that they do not fully look like mean reverting processes.

Another possible area of improvement would be to consider adding constraints to our portfolio, for example, the no-short-sales constraint. This is a very common constraint in the real world that affects individual investors and mutual funds.

Given more time, we would also plan to further investigate the performance of our portfolio, for example, by using monthly data instead of daily data. This would make more sense for a longer investment horizon.

Another possible area for extension would be to include more assets in our investment set. We could add more stocks or other asset classes like commodities or corporate bonds. In reality, investors have many more than simply ten stocks to choose from.

Finally, we did not allow the investor to consume funds from the portfolio at any time, which may not be realistic. In fact, it is very common for investors to withdraw money from their investment accounts in order to pay for other expenses.

7 Conclusion

In conclusion, we were able to achieve our goal of creating an optimal portfolio that solved our dynamic asset allocation problem, as our portfolio lies on the efficient frontier. The Sharpe and Information Ratios also showed that our portfolio outperformed the risk-free rate and SP500 and therefore can be considered a strong strategy to follow. As previously mentioned, there are some improvements that can be made and that would likely lead to stronger performance, however overall we can conclude a positive result.

8 References

Detemple, J., Garcia, R. and Rindisbacher, M. (2003). A Monte Carlo Method for Optimal Portfolios. *The Journal of Finance*, 58, 401-446.

Detemple, J. (2014). Portfolio Selection: A Review. *Journal of Optimization Theory and Applications*, 161(1), 1-21.