MF772 Final Project: Vasicek Models for Credit Derivative Pricing

Issy Anand, Allen Chacko, Henry Gagne, Bhavin Shah, Divyam Singh

December 2020

Abstract

In this paper we aim to implement the stochastic based intensity model for the default risk of a single obligor. This paper focuses on pricing and calibrating default-able and non defaultable zero coupon bonds under the two factor Vasicek model framework. We use the zero coupon bonds issued by J.P.Morgan to calibrate the defaultable bond and simulate the survival probability.

Keywords: Vasicek Model, Stochastic intensity model, Survival Probability, Calibration, Monte Carlo

1 Introduction

Stochastic intensity based models allow us to capture the risk in the change of the credit quality of the obligor. Unlike deterministic models we can now account for the correlation between default intensities and default free interest rates. Empirical studies estimate this correlation to be around -0.2, this gives us a ballpark figure for our correlation estimate.

To price the defaultable zero coupon bond we need the dynamics of short rate r(t) and the intensity process $\lambda(t)$. The specification of dynamics of r(t) and $\lambda(t)$ are an extension of Vasicek (1977) model in the interest rate modelling literature. This model dynamics allows for credit spread and interest rate to be negative with a positive probability and should therefore be viewed as an approximation of reality.

The Vasicek model, otherwise known as the two factor Gaussian model , defines the dynamics of the short rate to be:

$$dr(t) = (\kappa(t) - \alpha r(t))dt + \sigma(t)dW(t) \tag{1}$$

and similarly the dynamics of the default intensity λ to be

$$d\lambda(t) = (\bar{\kappa}(t) - \bar{\alpha}\lambda(t))dt + \bar{\sigma}(t)d\bar{W}(t) \tag{2}$$

where W and $\bar{W}$ are two Brownian motions with correlation

$$dWd\bar{W} = \rho dt$$

We have used the US Treasury yield data to calibrate the short rate parameters, fixed these and assumed intensity parameters for our chosen company. Using the dynamics, we were then able to calculate the survival probabilities and compare these to market data to check the validity of our parameters.

2 Formulas and Proofs

We would firstly like to prove the explicit solution for the Vasicek equation short rate dynamics. We have that

$$dr(t) = (\kappa(t) - \alpha r(t))dt + \sigma(t)dW(t) \tag{3}$$

So if we multiply both sides by $e^{\alpha t}$ we get

$$e^{\alpha t}dr(t) + \alpha e^{\alpha t}dt = e^{\alpha t}\kappa(t)dt + e^{\alpha t}\sigma(t)dW(t)$$

$$d(e^{\alpha t}r(t)) = e^{\alpha t}\kappa(t)dt + e^{\alpha t}\sigma(t)dW(t)$$

$$e^{\alpha t}r(t) = r(0) + \int_0^t e^{\alpha s}\kappa(s)ds + \int_0^t e^{\alpha s}\sigma(s)dW(s)$$

$$r(t) = r(0)e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)}\kappa(s)ds + \int_0^t e^{-\alpha(t-s)}\sigma(s)dW(s)$$

We now aim to prove the formulas we will use to calculate the default free bond price and survival probabilities from our calibrated parameters. From the Schonbucher textbook, we have that the default free bond price under the Vasicek model is given by

$$\begin{aligned} B(0,T) &= \mathbb{E}[e^{-\int_0^T r(t)dt}] \\ B(0,T) &= e^{\mathcal{A}(T)-\mathcal{B}(T)r(0)} \end{aligned}$$

where,

$$\begin{aligned} \mathcal{B}(T) &= \frac{1}{\alpha}(1-e^{-\alpha T}) \\ \mathcal{A}(T) &= \frac{1}{2}\int_0^T \sigma^2(s)\mathcal{B}(s)^2 ds - \int_0^T \mathcal{B}(s)\kappa(s)ds \end{aligned}$$

Before proving this formula, we will firstly simplify the expression for $\mathcal{A}(T)$. We do this by taking α, κ, and σ constant in our model and calculating the integrals as follows:

$$\begin{aligned} \frac{1}{2}\int_0^T \sigma^2\mathcal{B}(s)^2 ds &= \frac{1}{2}\int_0^T \sigma^2\frac{1}{\alpha^2}(1-e^{-\alpha s})^2 ds \\ &= \frac{\sigma^2}{2\alpha^2}\int_0^T (1-2e^{-\alpha s}+e^{-2\alpha s})ds \\ &= \frac{\sigma^2}{2\alpha^2}\left[s+\frac{2}{\alpha}e^{-\alpha s}-\frac{1}{2\alpha}e^{-2\alpha s}\right]_0^T \\ &= \frac{\sigma^2}{2\alpha^2}\left(T+\frac{2}{\alpha}(e^{-\alpha T}-1)-\frac{1}{2\alpha}(e^{-2\alpha T}-1)\right) \end{aligned}$$

$$\begin{aligned} \int_0^T \frac{1}{\alpha}\left(1-e^{-\alpha s}\right)\kappa ds &= \frac{\kappa}{\alpha}\int_0^T (1-e^{-\alpha s})ds \\ &= \frac{\kappa}{\alpha}\left[s+\frac{1}{\alpha}e^{-\alpha s}\right]_0^T \\ &= \frac{\kappa}{\alpha}\left(T+\frac{1}{\alpha}(e^{-\alpha T}-1)\right) \end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathcal{A}(T) &= \frac{\sigma^2}{2\alpha^2}\left(T + \frac{2}{\alpha}(e^{-\alpha T}-1) - \frac{1}{2\alpha}(e^{-2\alpha T}-1)\right) - \frac{\kappa}{\alpha}\left(T + \frac{1}{\alpha}(e^{-\alpha T}-1)\right) \\
&= T\left(\frac{\sigma^2}{2\alpha^2} - \frac{\kappa}{\alpha}\right) + \frac{1}{\alpha}\left(1-e^{-\alpha T}\right)\left(\frac{\kappa}{\alpha} - \frac{\sigma^2}{\alpha^2}\right) - \frac{\sigma^2}{4\alpha^3}\left(e^{-2\alpha T}-1\right) \\
&= T\left(\frac{\sigma^2}{2\alpha^2} - \frac{\kappa}{\alpha}\right) + \frac{1}{\alpha}\left(1-e^{-\alpha T}\right)\left(\frac{\kappa}{\alpha} - \frac{\sigma^2}{\alpha^2}\right) - \frac{\sigma^2}{4\alpha}\frac{1}{\alpha^2}\left(1-2e^{-\alpha T}+e^{2\alpha T}\right) + \frac{1}{\alpha}\frac{\sigma^2}{2\alpha^2}\left(1-e^{-\alpha T}\right) \\
&= T\left(\frac{\sigma^2}{2\alpha^2} - \frac{\kappa}{\alpha}\right) + \frac{1}{\alpha}\left(1-e^{-\alpha T}\right)\left(\frac{\kappa}{\alpha} - \frac{\sigma^2}{2\alpha^2}\right) - \frac{\sigma^2}{4\alpha}\frac{1}{\alpha^2}\left(1-e^{-\alpha T}\right)^2 \\
&= \left(\frac{\kappa}{\alpha} - \frac{\sigma^2}{2\alpha^2}\right)\left(\mathcal{B}(T)-T\right) - \frac{\sigma^2}{4\alpha}\mathcal{B}(T)^2
\end{aligned}
$$

For the proof of the bond price formula, let us consider a $u \geq t$. Then we have

$$
\begin{aligned}
r_u &= e^{-\alpha(u-t)}r_t + \int_t^u e^{-\alpha(u-s)}\kappa(s)ds + \int_t^u e^{-\alpha(u-s)}\sigma(s)dW(s) \\
&= e^{-\alpha(u-t)}r_t + \kappa(1-e^{-\alpha(u-t)}) + \int_t^u e^{-\alpha(u-s)}\sigma(s)dW(s)
\end{aligned}
$$

and we can express the default free bond price as

$$
B(t,T) = \mathbb{E}\left[e^{\int_t^T r_u(r_t)du}\right] \tag{4}
$$

Therefore, taking the differential w.r.t r_t we have that

$$
\begin{aligned}
\frac{\partial r_u(r_t)}{\partial r_t} &= e^{-\alpha(u-t)} \\
\implies \int_t^T \frac{\partial r_u(r_t)}{\partial r_t} &= \int_t^T e^{-\alpha(u-t)} \\
&= \frac{1}{\alpha}(1-e^{-\alpha(T-t)})
\end{aligned}
$$

Also,

$$
\begin{aligned}
\frac{\partial B(t,T,r_t)}{\partial r_t} &= \mathbb{E}\left[-\left(\frac{\partial r_u(r_t)}{\partial r_t}\right)e^{\int_t^T r_u(r_t)du}\right] \\
&= \frac{1}{\alpha}(1-e^{-\alpha(T-t)})\mathbb{E}\left[e^{\int_t^T r_u(r_t)du}\right] \\
&= -\mathcal{B}(t,T)B(t,T,r_t)
\end{aligned}
$$

Therefore,

$$B(t,T,r_t) = \mathcal{A}(t,T)e^{-\mathcal{B}(t,T)r_t} \tag{5}$$

for some function $\mathcal{A}(t,T)$

We know that the PDE for the vasicek model is

$$-r_tB(t,T,r_t) + \frac{\partial}{\partial t}B(t,T,r_t) + \frac{\partial}{\partial r_t}B(t,T,r_t)(\alpha(\kappa - r_t)) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial r_t^2}B(t,T,r_t) = 0$$

So comparing this to the derivatives of

$$B(T,T,r_t) = \mathcal{A}(t,T)e^{-\mathcal{B}(t,T)r_t} \tag{6}$$

we get the following:

$$\begin{aligned}
\frac{\partial B}{\partial t} &= \frac{\partial \mathcal{A}}{\partial t}e^{-\mathcal{B}(t,T)r_t} - \mathcal{A}\frac{\partial \mathcal{B}}{\partial t}e^{-\mathcal{B}(t,T)r_t} \\
\frac{\partial B}{\partial r_t} &= -\mathcal{A}\mathcal{B}e^{-\mathcal{B}(t,T)r_t} \\
\frac{\partial^2 B}{\partial r_t^2} &= \mathcal{A}\mathcal{B}^2e^{-\mathcal{B}(t,T)r_t}
\end{aligned}$$

Substituting these into the Vasicek PDE, we get

$$-r_t\mathcal{A} + \frac{\partial \mathcal{A}}{\partial t} - \mathcal{A}\frac{\partial \mathcal{B}}{\partial t}r_t - \mathcal{A}\mathcal{B}(\alpha(\kappa - r_t)) + \frac{\sigma^2}{2}\mathcal{A}\mathcal{B}^2 = 0 \tag{7}$$

and with $B(t,T,0) = \mathcal{A}(t,T)$ and $r_t = 0$, we get

$$\frac{\partial \mathcal{A}}{\partial t} - \alpha\kappa\mathcal{A}\mathcal{B} + \frac{\sigma^2}{2}\mathcal{A}\mathcal{B}^2 = 0 \tag{8}$$

Solving this backwards ODE with $\mathcal{A}(T,T) = 1$ leads us to

$$\begin{aligned}
\mathcal{A}(t,T) &= exp\left[\frac{\alpha\kappa}{\alpha}\int_t^T(1 - e^{-\alpha(T-u)})du + \frac{\sigma^2}{2\alpha^2}\int_t^T(1 - e^{-\alpha(T-u)})^2du\right] \\
&= exp\left[-\kappa(T-t) + \frac{\kappa}{\alpha}(1 - e^{-\alpha(T-t)}) + \frac{\sigma^2}{2\alpha^2}(T-t) + \frac{\sigma^2}{4\alpha^3}(1 - e^{-2\alpha(T-t)}) - \frac{\sigma^2}{\alpha^3}(1 - e^{-\alpha(T-t)})\right] \\
&= exp\left[\left(\frac{\kappa}{\alpha} - \frac{\sigma^2}{2\alpha}\right)((1 - e^{-\alpha(T-t)}) - (T-t)) + \frac{\sigma^2}{4\alpha} * \frac{1}{\alpha^2}(1 - e^{-\alpha(T-t)})^2\right] \\
&= exp\left[\left(\frac{\kappa}{\alpha} - \frac{\sigma^2}{2\alpha}\right)(\mathcal{B}(t,T) - (T-t)) + \frac{\sigma^2}{4\alpha}\mathcal{B}(T,t)^2\right]
\end{aligned}$$

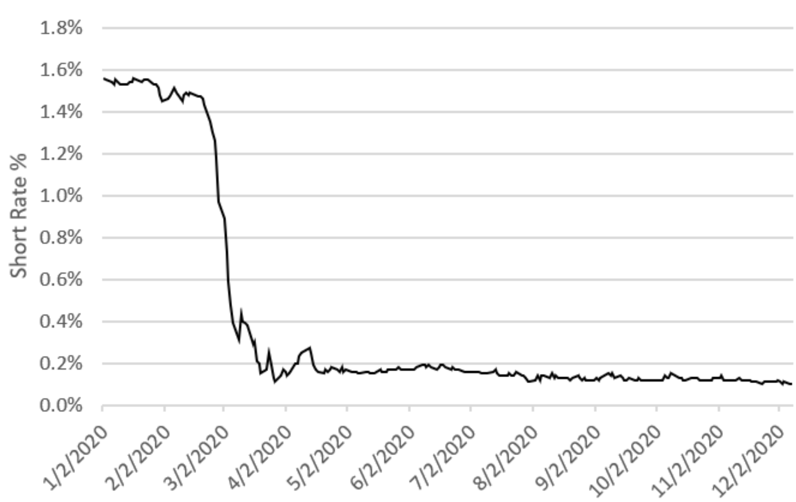
Since the default intensity $\lambda(t)$ follows analagous dynamics to $r(t)$, we can find a very similar express for the survival probability

$$\mathbb{P} = \mathbb{E}\left[e^{\int_t^T \lambda_u(r_t)du}\right] \tag{9}$$

by simply changing α, κ, λ to $\bar{\alpha}, \bar{\kappa}, \bar{\lambda}$

3 Model Framework

We collect the interest rate data for the year 2020 from the treasury website. We can see from the graph below the sudden decrease in the interest rate during the months of March and April. This jump is caused primarily due to the changes in Fed policies accompanying the COVID-19 crisis. A simple Vasicek Model cannot capture this change in the interest rate regime. In-order to capture this drop we need sophisticated models like the Jump Diffusion Model. Any attempt to model this data using Vasicek Model will result in sub-optimal model parameters. Therefore we exclude the data prior to April 2020 for our calibration of interest rate to the Vasicek Model.



Graph 1 : Interest rate Jan'20 - Dec'20

We use the structure of yields as of 12/8/2020 to calculate the zero coupon bond prices at different maturities. We rely on the treasury website to obtain the structure of yields.

Inorder to numerically compute the prices of default-able and non defaultable zero coupon we discritize the equations (1) and (2) and run Monte Carlo simulation for a fixed set a of model parameters. We leverage this Monte Carlo simulation later on during the calibration of non defaultable zero coupon bond to calculate the survival probability.

4 Model Calibration

We start by calibrating the short rate to get the Vasicek model parameters. For our calibration we use treasury yield data for ZCB with maturities of 1mo, 2mo, 3mo, 6mo, 1yr, 2yr, 3yr, 5yr, 7yr, 10yr, 20yr, and 30yr. Starting with the short rate model:

$$dr(t) = (\kappa(t) - \alpha r(t))dt + \sigma(t)dW(t) \tag{10}$$

This process discretized corresponds to an AR(1) process. To see this, we look at

$$r_{t+1} = ar_t + b + c\epsilon$$

Which, when you let

$$a = -\kappa\Delta t, \quad b = \kappa\alpha\Delta, \quad c = \sigma\sqrt{\Delta t}$$

then

$$r_{t+1} = \kappa(\alpha - r_t)\Delta t + \sigma\epsilon\sqrt{\Delta t} \tag{11}$$

which corresponds exactly to a Euler discretization of the Vasicek short rate process at

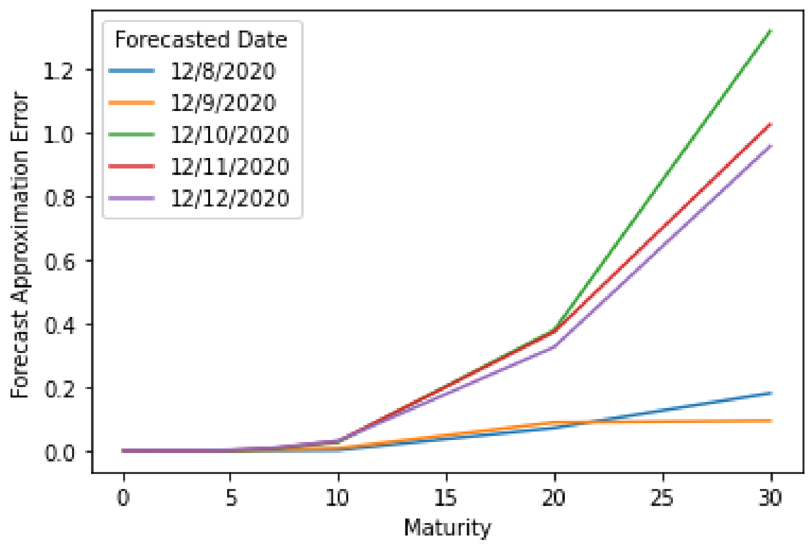
$$t = i\Delta * t$$

This shows that we are able to perform an OLS regression of the market short rates against the market short rates lagged by one index. The regression will output a slope, intercept, and a residual error, which we can use to recover the calibrated Vasicek parameters.

$$\kappa = -\frac{ln(slope)}{\Delta t}, \;\; \alpha = \frac{intercept}{1 - slope}, \;\; \sigma = sd(\epsilon)\sqrt{\frac{-2ln(slope)}{\Delta t(1 - slope^2)}} \tag{12}$$

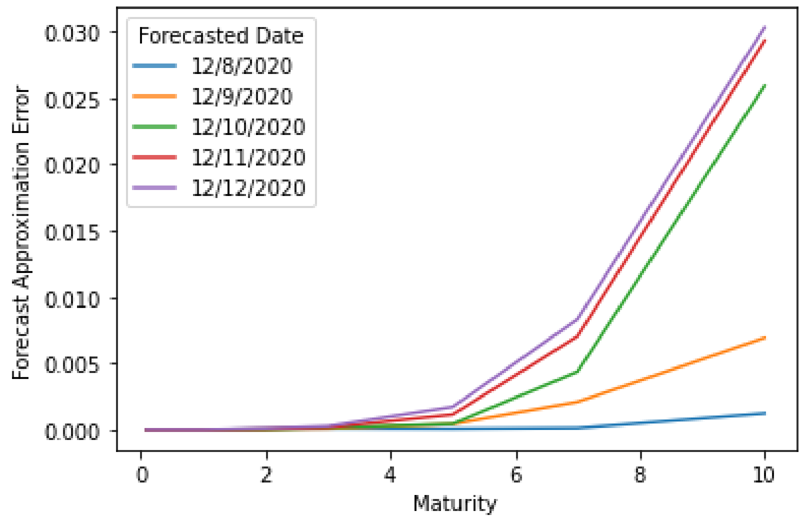
4.1 Calibration Results for short rate

We calibrated the short rate process for all maturities available from 1mo to 30yrs. Using the calibrate parameters, we can calculate an exact *model* ZCB price. Then, to judge the accuracy of the calibration for different maturities, we calculate the error between the Vasicek model price and the market price implied by the treasury yields.



Graph 2 : Short rate Vasicek model error (All Maturities)

We infer from the graph above that error increases with time to maturities, so our model better performs for derivatives of shorter maturity. The errors jump substantially for the 20 and 30 year bonds, so we decide to discard those from our analysis. Zooming in on just the errors from 1mo to 10yrs, we get the following plots

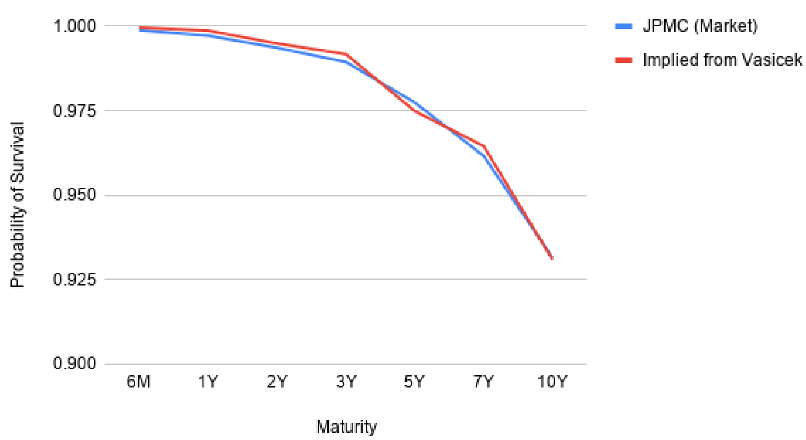


Graph 3 : Short rate Vasicek model error (Up to 10 Yrs)

For the 10yr bonds, the maximum calibrated price error was only 0.03% of the market price, so we can feel confident that our calibration is working as expected.

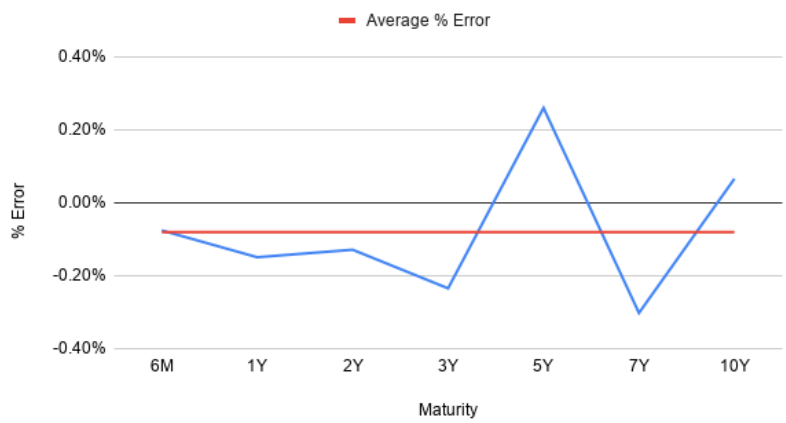
4.2 Calibration Results for defaultable ZCB

To calibrate our intensity process parameters, we first obtained CDS spreads and survival probability for our chosen company JPMC from Bloomberg. (see appendix 1) The implied survival probability almost entirely matches the market survival probability, both decreasing for longer maturities.



Graph 4 : Calibration Survival Probability of J.P.Morgan

We see that the error is relatively small with an average error of 0.08%



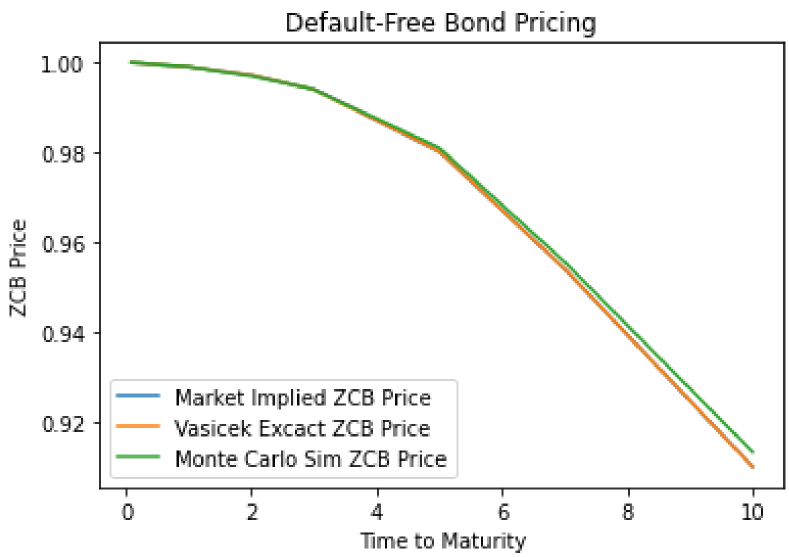
Graph 5 : Error in Calibration of Survival Probability

5 Model Testing

Bond prices are expected to decrease monotonically as the maturity increases. We can see this monotone behavior in *Graph 6*, this gives us the initial sense check that our Vasicek model parameters are calibrated correctly. Further testing and detailed analysis of the calibrated model is presented in this section.

5.1 Back testing

To test the model, we first compare the default free Vasicek bond prices to the market ZCB prices. Additionally, we can simulate the Vasicek short rate process, and use an integral approximation to estimate a Monte Carlo ZCB price. This results in the plot below, where we can see that all three methods give similar pricing for all maturities from 1mo to 10yrs



Graph 6: Comparison of Default Free Bond Prices

Next, we are interested in comparing the defaultable prices from the Vasicek model to the defaultable bond prices from JPMorgan. We assume the parameters for the intensity process

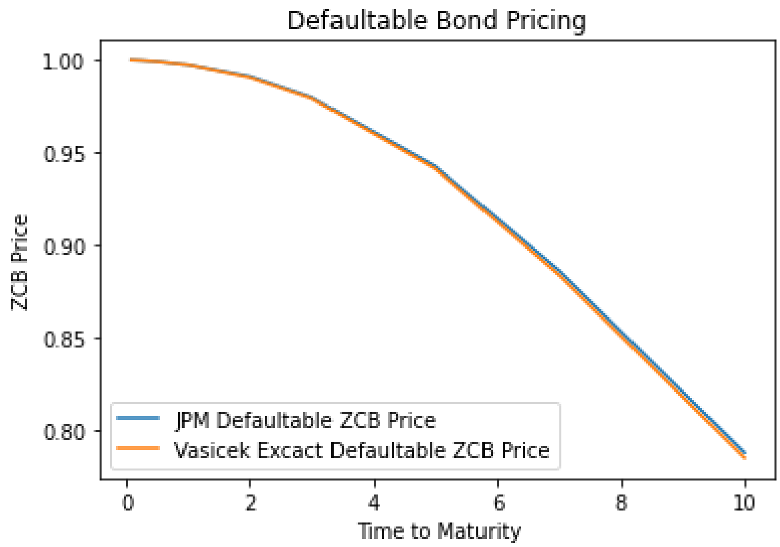
$$\bar{\lambda} = .06$$

$$\bar{\sigma} = .07$$

$$\bar{\kappa} = 3$$

$$\bar{\lambda_0} = .06$$

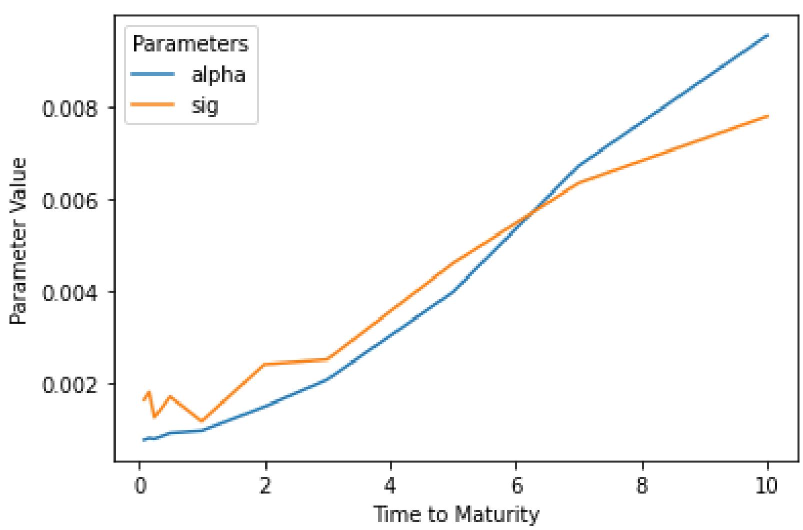
$$\rho = .5$$



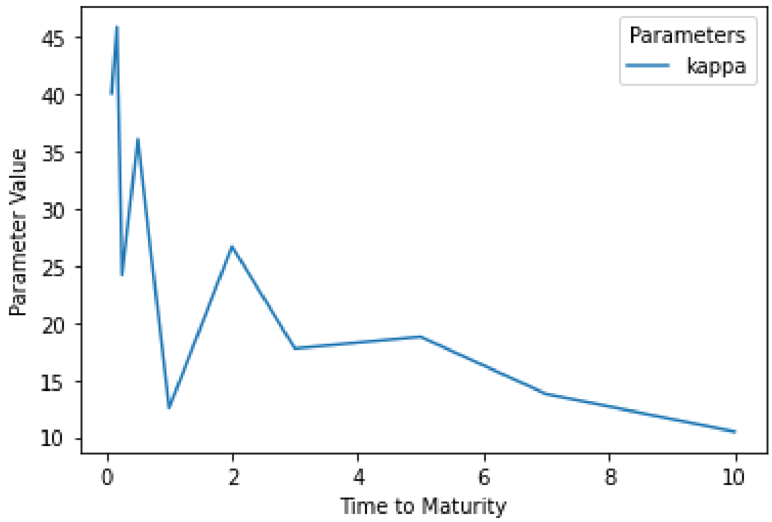
Graph 7: Comparison of Defaultable Bond Prices

5.2 Parameter Sensitivity

We are interested in how the calibrated short rate parameters change for different maturities, so we select a single forecast date, of 12/12/2020, and plot the fitted parameters for each maturity.



Graph 8 : Calibrated Alpha and Sigma by maturity



Graph 9 : Calibrated Kappa by maturity

The above plots reveal an increasing structure for alpha and sigma, and a choppy, but decreasing structure for kappa. We can understand the upward sloping alpha and sigma as a result of longer dated treasuries giving a higher yield to maturity. As a direct result of this, the long term mean will increase. Interestingly, we also see sigma increasing at the same rate as alpha, indicating that the volatility of the short rate process itself increases for longer dated treasuries as well. This is consistent with an understanding that longer term bonds have more sensitivity to changes in yield. For kappa, we see the mean reversion speed bounce around for short maturities, and then smooth out to a downward sloping line. This shows us that the mean reversion rate is the most sensitive to changes in the short rate process, so we have less reliability in the calibrated kappa. However, for short maturities, the mean reversion rate does not have as large of an impact on the exact ZCB price. This is reinforced by *Graph 2*, where we only see very small calibration errors for short maturities.

6 Conclusion

In conclusion, we have successfully managed to calibrate the two-factor Vasicek model to Treasury Yield ZCB data and JPMorgan survival probabilities. Whilst we did start to see larger errors for long-term maturities in our short rate calibration, the magnitude of the errors for less than 10 years is sufficiently small, as are all errors for our intensity process calibration up to 10 years. We therefore achieved our aim and were able to confidently estimate defaultable bond prices for JPMorgan. Going forwards, we would feel confident to re-calibrate this model to other firms and use it as a general template for estimating survival probabilities and bond prices.

References

1. Philipp.J.Schonbucher, *Credit Derivative Pricing Models - Models, Pricing and Implementation*. John Wiley and Sons Ltd.

2. Rogemar S. Mamon, *Three Ways to Solve for Bond Prices in the Vasicek Model*. Journal of Applied Mathematics and Decision Sciences, 8(1), 1-14

3. Thijs van den berg, *Calibrating the Ornstein-Uhlenbeck (Vasicek) model*

Appendix

1.

Maturity	**CDS Spreads (bps)**	**Cumulative Survival Probability**
6M	22.904	0.9988
1Y	26.76	0.9972
2Y	31.188	0.9936
3Y	34.647	0.9894
5Y	44.914	0.9774
7Y	55.007	0.9616
10Y	69.904	0.9315

Table 1: JPMC data from Bloomberg