HOMOTOPY TYPE OF SPACES OF LONG KNOTS: EXTENDED COMPUTATIONS

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Abstract. The present document is intended to extend the computations given by Samuel Muñoz-Echániz in [Sam24], Proposition 6.7, using the classical Adams spectral sequence. There are still some uncertainties, but it is still possible to extend the computations even more.

Introduction

We will extend the computations given by Samuel Muñoz-Echániz in [Sam24], Proposition 6.7, for the case $\ell = m = 3$. We can proceed analogously in the other cases. The key elements involved in this argument are $(\pi_*^s)^\vee_{(3)}$ as specified in [Hat18] (for more details, see [Rav86]), and my program `minrv1` [Mor24] to compute minimal resolutions and Yoneda products. It is possible to reproduce these computations using [CCBFY22] software, although its interface does not allow filtering of the multiplicative structure, which is key to finding the correct Yoneda products. The diagrams were obtained using an AMD Ryzen 9 9950X 16-core processor and 192 GB of RAM, but the products can be verified in an average computer.

Let p be a prime. Let X, Y, Z be spectra, with Y and Z bounded below and $H_*(Y;\mathbb{F}_p)$ and $H_*(Z;\mathbb{F}_p)$ of finite type. By Moss' convergence theorem, the (opposite) Yoneda pairing

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_p}(H^*(Z;\mathbb{F}_p), H^*(Y;\mathbb{F}_p)) \otimes \mathrm{Ext}^{*,*}_{\mathcal{A}_p}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \to \mathrm{Ext}^{*,*}_{\mathcal{A}_p}(H^*(Z;\mathbb{F}_p), H^*(X;\mathbb{F}_p))$$

converges to the composition product in $\pi_*^X(Z)$. The pairings involved satisfy the Leibniz rule with respect to the differentials of the three spectral sequences. In particular, by taking $X = Y = \mathbb{S}$, i.e.

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_p}(H^*(Z;\mathbb{F}_p), H^*(\mathbb{S};\mathbb{F}_p)) \otimes \mathrm{Ext}^{*,*}_{\mathcal{A}_p}(H^*(\mathbb{S};\mathbb{F}_p), H^*(\mathbb{S};\mathbb{F}_p)) \to \mathrm{Ext}^{*,*}_{\mathcal{A}_p}(H^*(Z;\mathbb{F}_p), H^*(\mathbb{S};\mathbb{F}_p))$$

we can exploit the knowledge of $(\pi_*^s)_{(p)}$ and $\mathrm{Ext}^{*,*}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$ to obtain non-trivial differentials.

Therefore, in order to determine which differentials are non-trivial, we will need the following description of $\mathrm{Ext}^{*,*}_{\mathcal{A}_3}(\mathbb{F}_3, \mathbb{F}_3)$. It was obtained by constructing a minimal (free) $\mathcal{A}_3$−resolution of $\mathbb{F}_3$, computing the corresponding Yoneda products, and the knowledge of $(\pi_*^s)^\vee_{(3)}$ [Hat18].

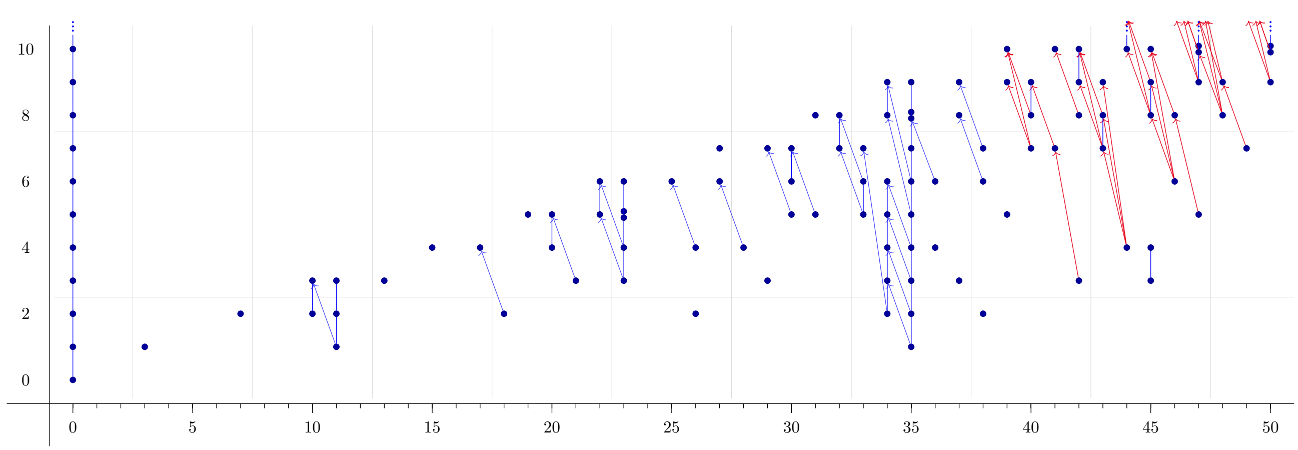

Figure 1. First non-trivial d_2 differentials in $\mathrm{Ext}^{s,t-s}_{\mathcal{A}_3}(\mathbb{F}_3, \mathbb{F}_3)$.

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The following chart will be used later for illustrative purposes regarding possible candidates for E_3-Massey products.

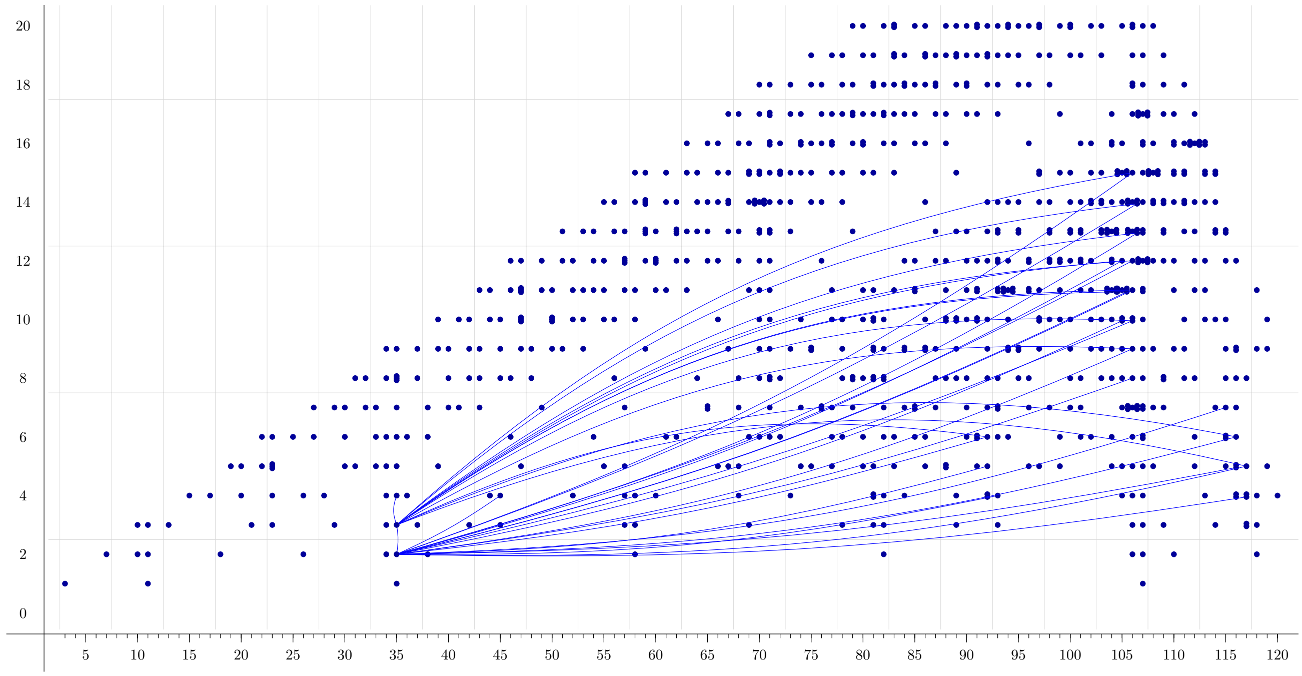

Figure 2. E_3-Massey product candidates in $\mathrm{Ext}^{s,t-s}_{\mathcal{A}_3}(\mathbb{F}_3, \mathbb{F}_3)$.

Computation

Before we can apply the classical Adams spectral sequence, remember Samuel's lemma [Sam24].

Lemma 1 (Cohomology of $\mathbb{S}^{\rho_3}_{hC_3}$)**.** The spectrum cohomology of $\mathbb{S}^{\rho_3}_{hC_3}$ is given by

$$H^*(\mathbb{S}^{\rho_3}_{hC_3}; \mathbb{F}_3) \cong \mathbb{F}_3\langle u\rangle \otimes_{\mathbb{F}_3} \mathbb{F}_3[\alpha, s]/(\alpha^2), \qquad |\alpha| = 1, \quad |s| = 2, \quad |u| = 3(d-p-2)$$

with

$$P^k(u\alpha^i s^j) = \left(\sum_{r=0}^{k} \binom{d-p-2}{r}\binom{j}{k-r}\right) u\alpha^i s^{j+2k}, \qquad \beta(u\alpha^i s^j) = \begin{cases} 0, & i = 0, \\ -us^{j+1}, & i = 1. \end{cases}$$

Moreover, $C_2 = D_3/C_3$ acts on $H^*(\mathbb{S}^{\rho_3}_{hC_3}; \mathbb{F}_3)$ by $u\alpha^i s^j \mapsto (-1)^{p+i+j} u\alpha^i s^j$.

Now, we can state and prove the following result. This is Samuel's proof [Sam24], except for the Adams spectral sequence part. There are modifications in redaction to fit my notes. Samuel computed $\pi^s_*(\mathbb{S}^{\rho_3}_{hD_3}) \otimes \mathbb{Z}_{(3)}$ for $* \leq 24$, with three uncertainties related to $* \in \{17, 18, 20, 21\}$ when p is even.

Proposition 1 (3-primary stable homotopy groups of $\mathbb{S}^{\rho_3}_{hD_3}$)**.** The first few homotopy groups $\pi^s_*(\mathbb{S}^{\rho_3}_{hD_3}) \otimes \mathbb{Z}_{(3)}$ when $d - p = 3$ are given in Table 1 for odd p and Table 2 for even p. Equally colored groups in this table correspond to the same case depending on the vanishing of certain differentials.

Proof. Consider the $\mathcal{A}_3-$submodules of $H^*(\mathbb{S}^{\rho_3}_{hC_3}; \mathbb{F}_3)$ given by

$$J_0 := \langle u\alpha^i s^j : i + j \equiv 0 \bmod 2\rangle, \qquad J_1 := \langle u\alpha^i s^j : i + j \equiv 1 \bmod 2\rangle.$$

Then $H^*(\mathbb{S}^{\rho_3}_{hC_3}) = J_0 \oplus J_1$ as $\mathcal{A}_3-$modules, and J_p, where p is taken mod 2, is the C_2-invariant vector space of the residual $C_2 = D_3/C_3-$action, because $\infty > |C_2| = 2 \in \mathbb{F}_3^\times$. Therefore, $H^*(\mathbb{S}^{\rho_3}_{hD_3}; \mathbb{F}_3) = J_p$ as an $\mathcal{A}_3-$module,

$*$	3	4	5	6	7	8	9	10	11	12	13
	0	$\mathbb{Z}/3\mathbb{Z}$	0	0	0	$\mathbb{Z}/3\mathbb{Z}$	0	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/9\mathbb{Z}$	0
$*$	14	15	16	17	18	19	20	21	22	23	24
	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	0	0	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	0	0	$\mathbb{Z}/9\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$
$*$	25	26	27	28	29	30	31	32	33	34	35
	0	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	0	$\mathbb{Z}/\mathbf{9}\mathbb{Z}$ or $\mathbb{Z}/9\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$	**?**
$*$	36	37	38	39							
	?	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$							

Table 1. $\pi^s_*\left(\mathbb{S}^{\rho_3}_{hD_3}\right)^\vee_{(3)}$, for $*\leq 38$ (p odd).

$*$	3	4	5	6	7	8	9	10	11	12	13
	$\mathbb{Z}_{(3)}$	0	0	$\mathbb{Z}/9\mathbb{Z}$	0	0	0	$\mathbb{Z}/9\mathbb{Z}$	0	0	$\mathbb{Z}/3\mathbb{Z}$
$*$	14	15	16	17	18	19	20	21	22	23	24
	$\mathbb{Z}/27\mathbb{Z}$	0	0	$\mathbb{Z}/\mathbf{3}\mathbb{Z}$ or $\mathbb{Z}/9\mathbb{Z}$	$\mathbb{Z}/\mathbf{81}\mathbb{Z}$ or $\mathbb{Z}/243\mathbb{Z}$	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/9\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	0
$*$	25	26	27	28	29	30	31	32	33	34	35
	0	$\mathbb{Z}/27\mathbb{Z}$	0	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/27\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$	0	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/9\mathbb{Z}$	0
$*$	36	37	38								
	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/24\mathbb{Z}$								

Table 2. $\pi^s_*\left(\mathbb{S}^{\rho_3}_{hD_3}\right)^\vee_{(3)}$, for $*\leq 38$ (p even).

and the classical Adams spectral sequence of J_p converges to the 3-primary part of the stable homotopy of $\mathbb{S}^{\rho_3}_{hD_3} = \left(\mathbb{S}^{\rho_3}_{hC_3}\right)_{hC_2}$.

Before diving into computer generated data and exploiting the multiplicative structure of the Adams spectral sequence, we will consider the Kahn-Priddy theorem when p is odd. Denote by $\hat{\theta}$ the D_3-representation pulled back from the standard $O(2)-$representation on $\mathbb{R}^2$. Observe that $\hat{\theta}|_{C_3}\equiv\theta$ in the notation of Lemma 1. Write $\underline{\mathrm{S}}(\hat{\theta}\otimes\sigma|_{C_3})$ for the unit sphere bundle of the associated vector bundle $S^1\to ED_3\times_{C_3}(\hat{\theta}\otimes\sigma)\to BC_3$. By the homotopy long exact sequence, it follows that $\underline{\mathrm{S}}(\hat{\theta}\otimes\sigma|_{C_3})\cong K(\pi,1)$ for some group π, and in fact, it must be homotopy equivalent to S^1 because $q\colon \underline{\mathrm{S}}(\hat{\theta})\otimes\sigma|_{C_3})\to BC_3$ does not admit a section, hence, the obtained short exact sequence cannot be split (its Euler class is $s\in H^2(BC_3;\mathbb{F}_3)=\mathbb{F}[\alpha,s]/(\alpha^2)$). Moreover, the homology class represented by q is the dual of α, and hence the residual $C_2=D_3/C_3-$action on $\underline{\mathrm{S}}(\hat{\theta}\otimes\sigma|_{C_3})\cong S^1$ must have degree -1. So, there is an equivalence of unbased spaced $\underline{\mathrm{S}}(\hat{\theta}\otimes\sigma|_{C_3})\cong S^\sigma$ which is C_2-equivariant up to homotopy. We thus get a cofibration

$$S^\sigma_+\cong\underline{\mathrm{S}}(\hat{\theta}\otimes\sigma|C_3)_+\xrightarrow{q}(BC_3)_+\twoheadrightarrow\mathrm{Th}(\hat{\theta}\otimes\sigma|_{C_3})\cong S^{-\sigma}\wedge\mathrm{Th}(\psi_3\otimes\sigma|_{C_3})$$

which is C_2-equivariant up to homotopy. By equipping both S^σ and BC_3 with distinguished basepoints that are fixed under the respective involutions and which match under q, we can get rid of the added basepoints and yield a homotopy cofibre sequence of C_2-spectra

$$\mathbb{S}^{(d+1)(\sigma-1)+2\sigma}\xrightarrow{q}S^{(d+1)(\sigma-1)+\sigma}\wedge\Sigma^\infty BC_3\twoheadrightarrow\mathbb{S}^{\rho_3}_{hC_3}.$$

Then, upon inverting 2 and taking homotopy C_2-orbits in the sequence above, we obtain equivalences of spectra

$$\mathbb{S}^{\rho_3}_{hD_3}\cong_{[\frac{1}{2}]}\begin{cases}(\Sigma^{\infty+1}BC_3)_{hC_2}\cong\Sigma^{\infty+1}BD_3, & d\text{ even (so }p\text{ is odd)},\\ \mathrm{hocofib}(q_{hC_2}\colon\mathbb{S}^1\to(S^\sigma\wedge\Sigma^\infty BC_3)_{hC_2}), & d\text{ odd (so }p\text{ is even)}.\end{cases}\tag{1}$$

Now, by the Kahn-Priddy theorem at the prime 3, the transfer-like map $\Sigma^{\infty+1}BD_3\to\tau_{>1}\mathbb{S}^1$ is split surjective on homotopy groups localised at 3, and hence by (1), $\pi^s(\mathbb{S}^{\rho_3}_{hD_3})$ split surjects onto $\pi^s_{*-1}\otimes\mathbb{Z}_{(3)}$ for $*>1$ when p is odd.

Let $p\in\mathbb{N}_{\geq 0}$ be even. It is not clear in this case if we can apply the Kahn-Priddy theorem or any of its versions. Except for $d_3\colon E_3^{18,0}\to E_3^{17,3}$, this will not be a problem in a considerable range, by considering the next figures.

We will denote the classes on $\mathrm{Ext}^{*,*}_{\mathcal{A}_3}(\mathbb{F}_3,\mathbb{F}_3)$ by $y^{(k)}_{s,t-s}$ where $(t-s,s)\in(\mathbb{N}_{\geq 0})^2$ are its coordinates and $k\in\mathbb{N}_{>0}$ is an index related to multiple occurrences of non-trivial cohomology classes in the same bidegree. We will use similar notation for the next pages. Analogously, consider the elements $x^{(k)}_{s,t-s}\in\mathrm{Ext}^{*,*}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3},\mathbb{F}_3)$, for $s,t-s\geq 0$ and $k>0$. By constructing a minimal free $\mathcal{A}_3-$resolution of $H^*(\mathbb{S}^{\rho_3}_{hD_3};\mathbb{F}_3$ and computing the Yoneda products, we obtain

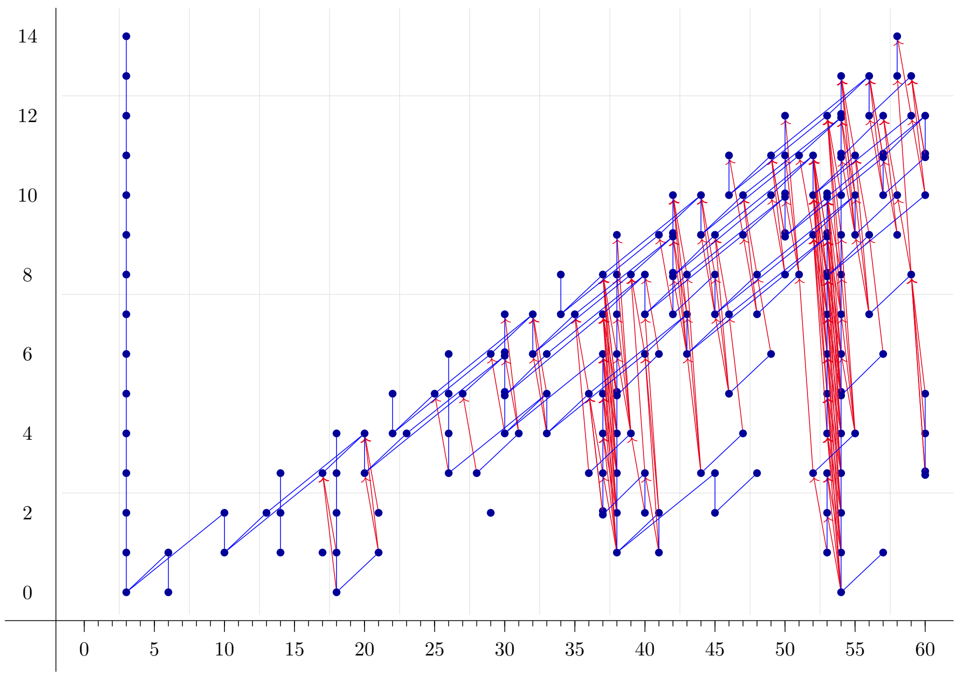

Figure 3. Multiplicative structure of $E_2^{*,*}$.

The red arrows represent unknown differentials. We are just displaying Yoneda products by lower degree classes. Consider $y^{(1)}_{1,0}\in\mathrm{Ext}^{0,1}_{\mathcal{A}_3}(\mathbb{F}_3,\mathbb{F}_3)$ and $x^{(1)}_{1,18}\in\mathrm{Ext}^{0,18}_{\mathcal{A}_3}(H^*(\mathbb{S}^{\rho_3}_{hD_3};\mathbb{F}_3),\mathbb{F}_3)$. By the previous diagram, $y^{(1)}_{1,0}\cdot x^{(1)}_{0,18}=kx^{(1)}_{1,18}$ for some $k\in\mathbb{F}_3^{\times}$. Since we are working with $\mathbb{F}_3-$vector spaces, we will not take care of constants of this sort in the rest of the computation, in particular, WLOG $k=1$. Then, since $d_2\colon\mathrm{Ext}^{*,*}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3},\mathbb{F}_3)\to\mathrm{Ext}^{*-1,*+2}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3},\mathbb{F}_3)$ is a derivation, it can be expressed in terms of itself and $d_2\colon\mathrm{Ext}^{*,*}_{\mathcal{A}_3}(\mathbb{F}_3,\mathbb{F}_3)\to\mathrm{Ext}^{*-1,*+2}_{\mathcal{A}_3}(\mathbb{F}_3,\mathbb{F}_3)$, i.e.

$$d_2\left(x^{(1)}_{1,18}\right)=d_2\left(y^{(1)}_{1,0}\cdot x^{(1)}_{0,18}\right)=d_2\left(y^{(1)}_{1,0}\right)\cdot x^{(1)}_{0,18}+(-1)^{18}y^{(1)}_{1,0}\cdot d_2\left(x^{(1)}_{0,18}\right)=0$$

since both differentials are trivial. By the same argument, $d_2\colon\mathrm{Ext}^{1,21}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3},\mathbb{F}_3)\to\mathrm{Ext}^{3,23}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3},\mathbb{F}_3)$ is trivial. The differential $d_3\colon\mathrm{Ext}^{0,18}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3},\mathbb{F}_3)\to\mathrm{Ext}^{3,17}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3},\mathbb{F}_3)$ turns out to be quite challenging. We will take a look at it at the end of this case. For bidegree reasons, it is not well placed and its corresponding Yoneda products have high degrees. Now, consider $d_2\colon\mathrm{Ext}^{2,21}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3},\mathbb{F}_3)\to\mathrm{Ext}^{1,23}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3},\mathbb{F}_3)$, and the following figure

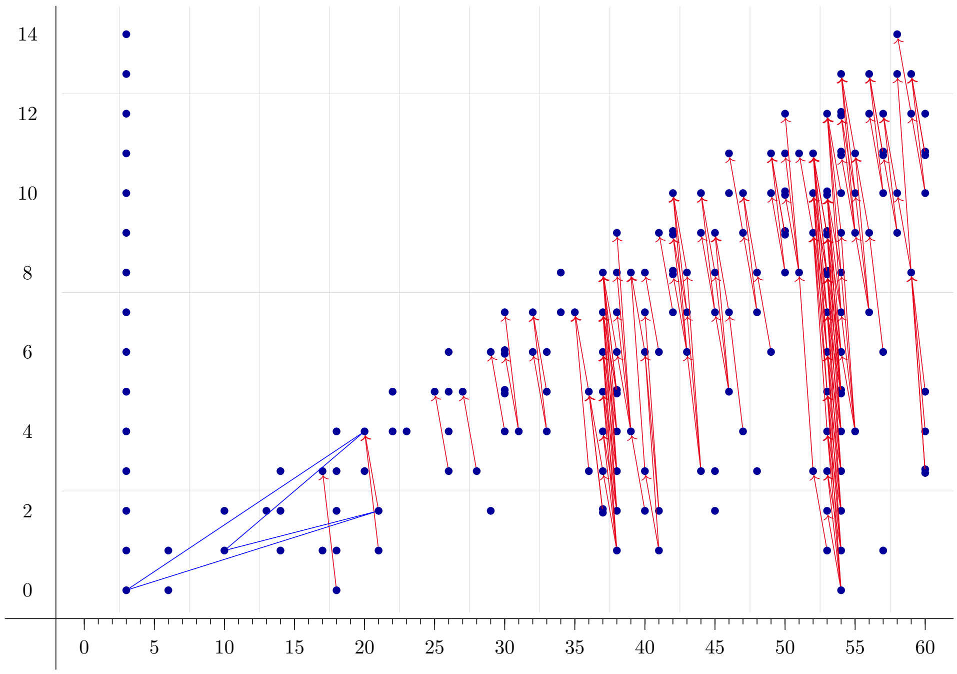

Figure 4. Multiplicative structure of $E_2^{*,*}$.

Then,

$$d_2\left(x^{(1)}_{2,21}\right) = d_2\left(y^{(1)}_{1,11}\cdot x^{(1)}_{1,10}\right) = d_2\left(y^{(1)}_{1,11}\right)\cdot x^{(1)}_{1,10} + (-1)^{11} y^{(1)}_{1,11}\cdot d_2\left(x^{(1)}_{1,10}\right) \neq 0$$

because $d_2(x^{(1)}_{1,10}) = 0$ and $d_2(y^{(1)}_{1,11}) \neq 0$ (Figure 1). Moreover, $d_3\colon \mathrm{Ext}^{1,21}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3}, \mathbb{F}_3) \to \mathrm{Ext}^{4,20}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3}, \mathbb{F}_3) = 0$.

Now, consider the next figure

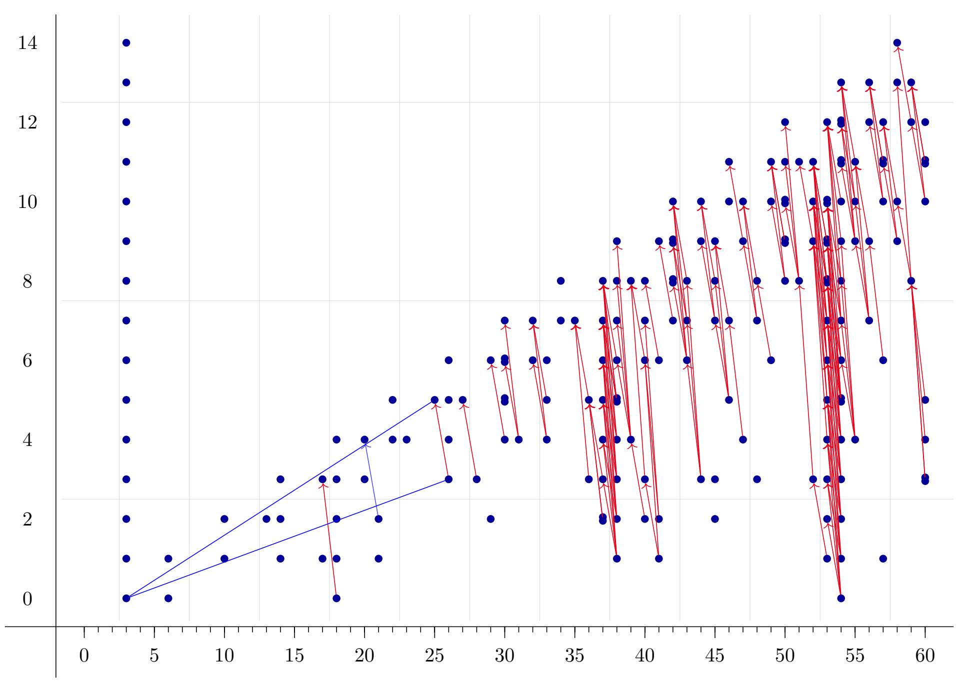

Figure 5. Multiplicative structure of $E_2^{*,*}$.

Since $x^{(1)}_{3,26} = y^{(1)}_{3,23} \cdot x^{(1)}_{0,3}$, and $d_2(y^{(1)}_{3,23}) \neq 0$, it follows that $d_2(x^{(1)}_{3,26}) \neq 0$. Then, we have

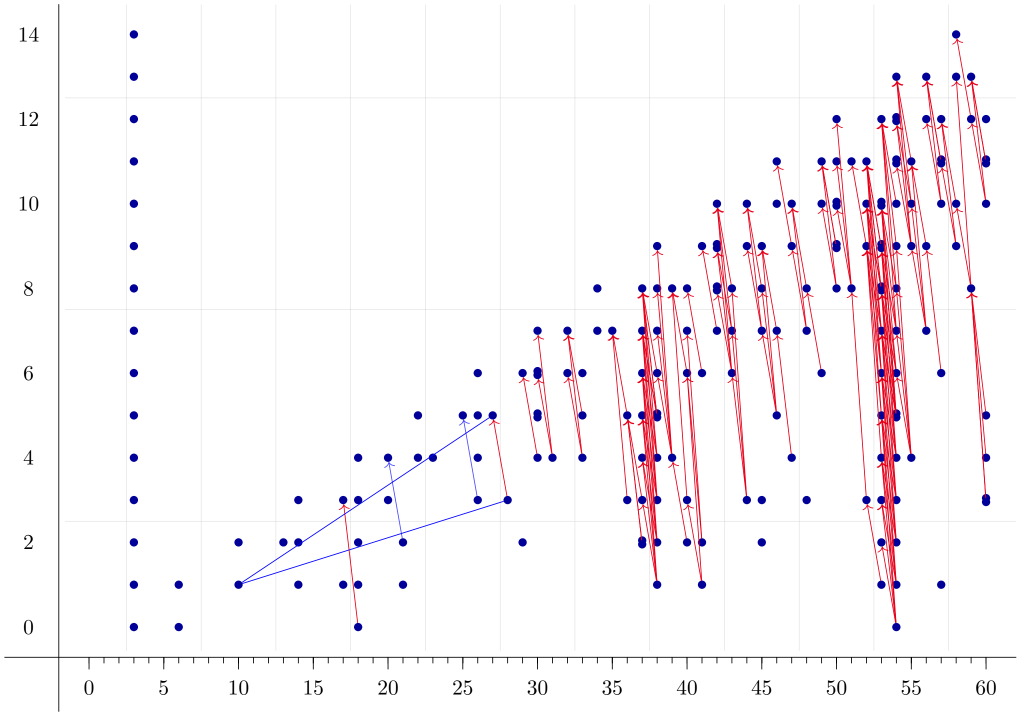

Figure 6. Multiplicative structure of $E_2^{*,*}$.

By a similar reasoning, $d_2(x^{(1)}_{3,28}) \neq 0$.

It will be preferable to skip some d_2 differentials for now and consider $d_2(x^{(1)}_{5,33})$. The relevant Yoneda products are displayed in the next diagram.

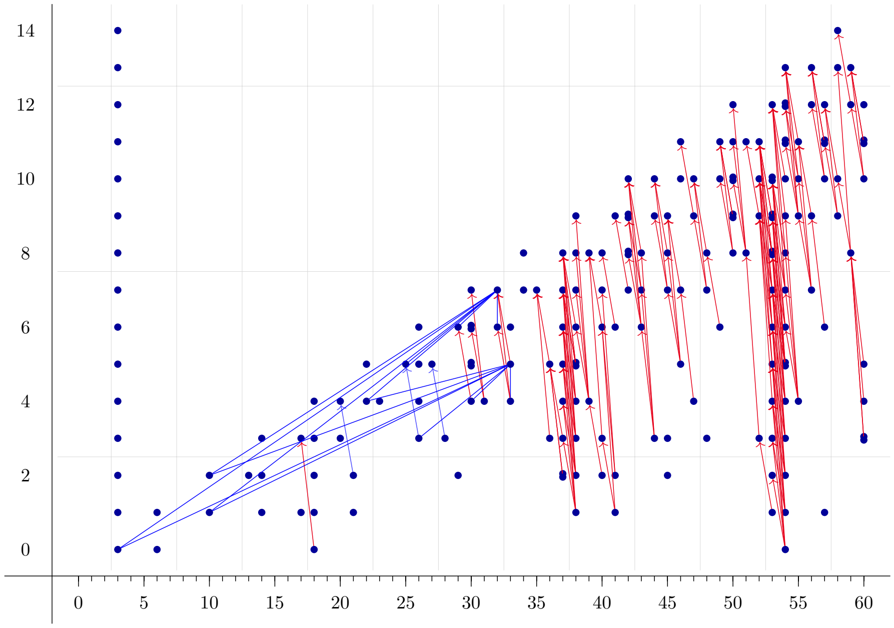

Figure 7. Multiplicative structure of $E_2^{*,*}$.

We have that $d_2(x^{(1)}_{5,33}) \neq 0$ because $d_2(y^{(1)}_{1,11}) \neq 0$. Assume by contradiction that $d_2(x^{(1)}_{4,33}) = 0$. By the derivation property, this would imply that $d_2(x^{1}_{5,33}) = 0$. Thus, $d_2(x^{(1)}_{4,33}) \neq 0$ and, similarly, $d_2(x^{(1)}_{4,30}) \neq 0$. Therefore, the corresponding higher differentials vanishes.

Similarly, $d_2(x^{(1)}_{4,31}), d_2(x^{(1)}_{5,36}) \neq 0$ as shown.

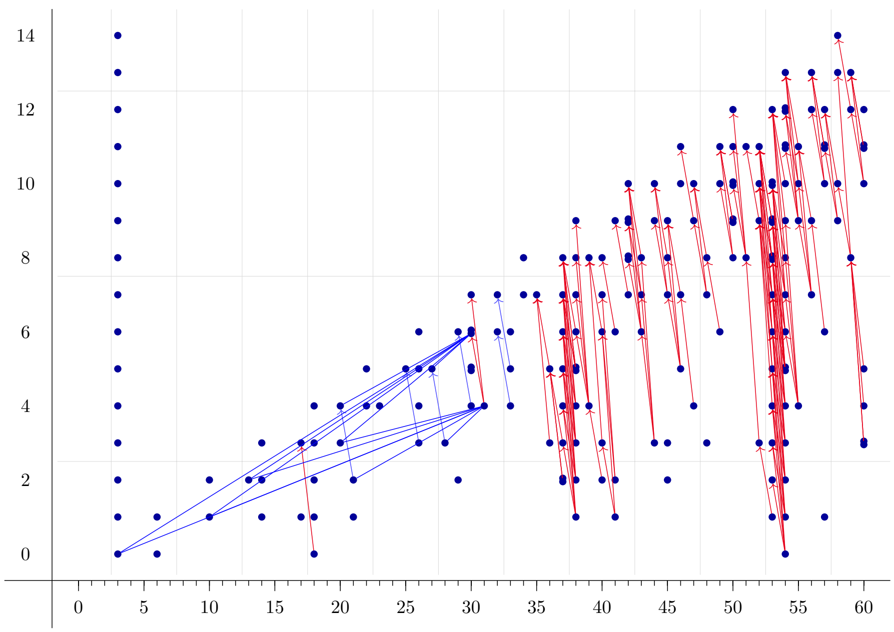

Figure 8. Multiplicative structure of $E_2^{*,*}$.

Since $d_2(y^{(1)}_{3,23}) \neq 0$, we have

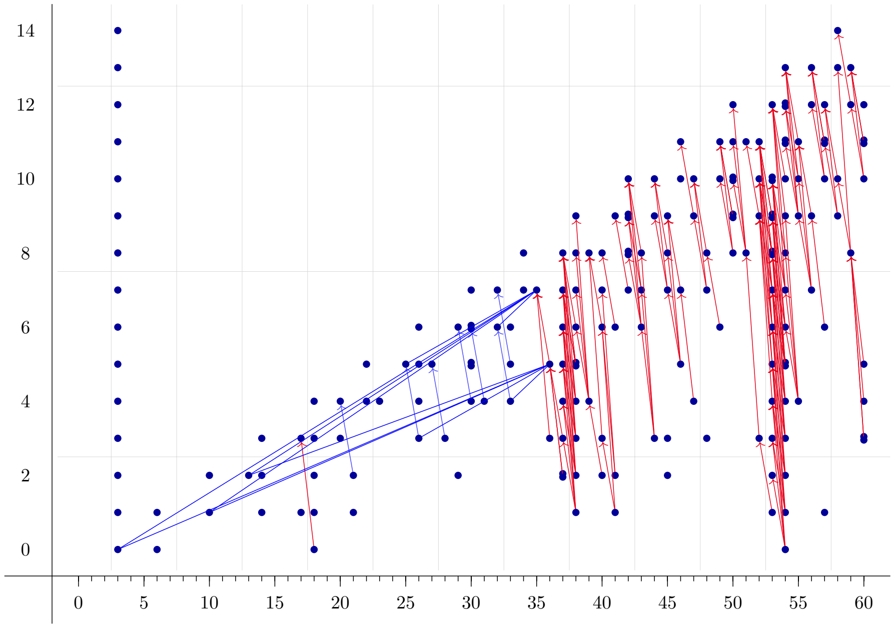

Figure 9. Multiplicative structure of $E_2^{*,*}$.

Note that this implies that $d_2(x^1_{3,37}) = 0$ since

$$\mathrm{Im}\left(d_2\colon \mathrm{Ext}^{3,37}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3}, \mathbb{F}_3) \to \mathrm{Ext}^{5,36}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3}, \mathbb{F}_3)\right) \subseteq \mathrm{Ker}\left(d_2\colon \mathrm{Ext}^{5,36}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3}, \mathbb{F}_3) \to \mathrm{Ext}^{7,35}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3}, \mathbb{F}_3)\right) = 0.$$

Finally, we will take advantage of the location of the column of differentials starting at $(1, 38)$. Consider the next sequence of diagrams.

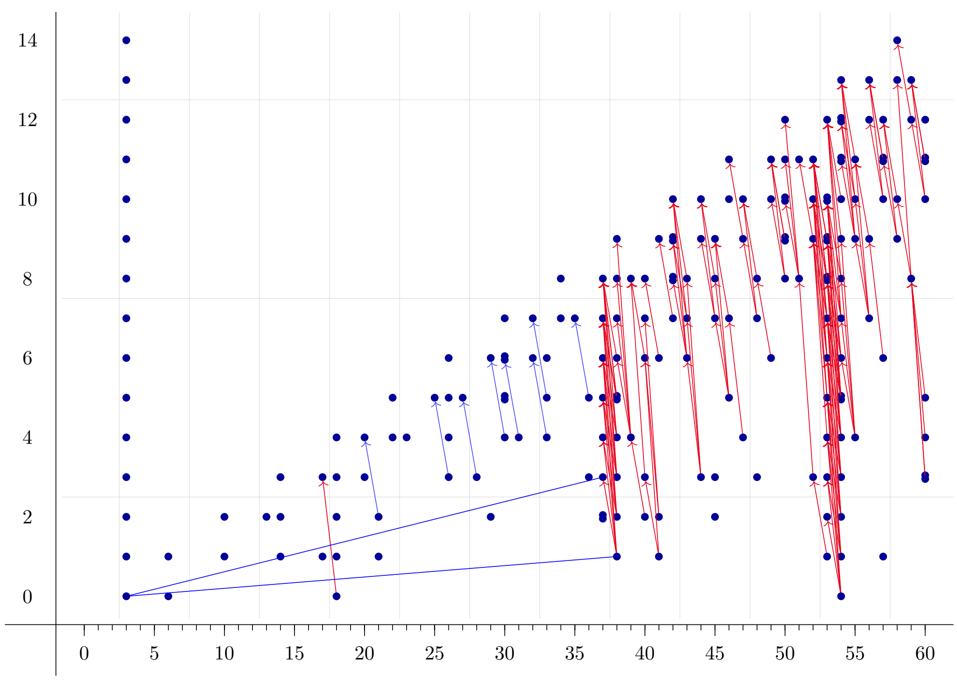

Figure 10. Multiplicative structure of $E_2^{*,*}$.

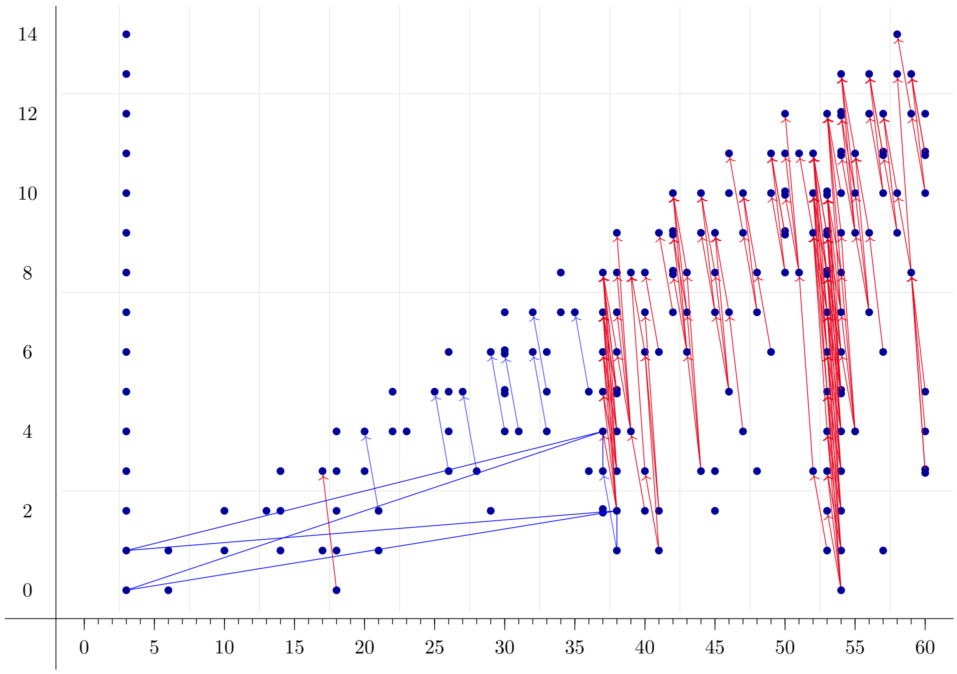

Figure 11. Multiplicative structure of $E_2^{*,*}$.

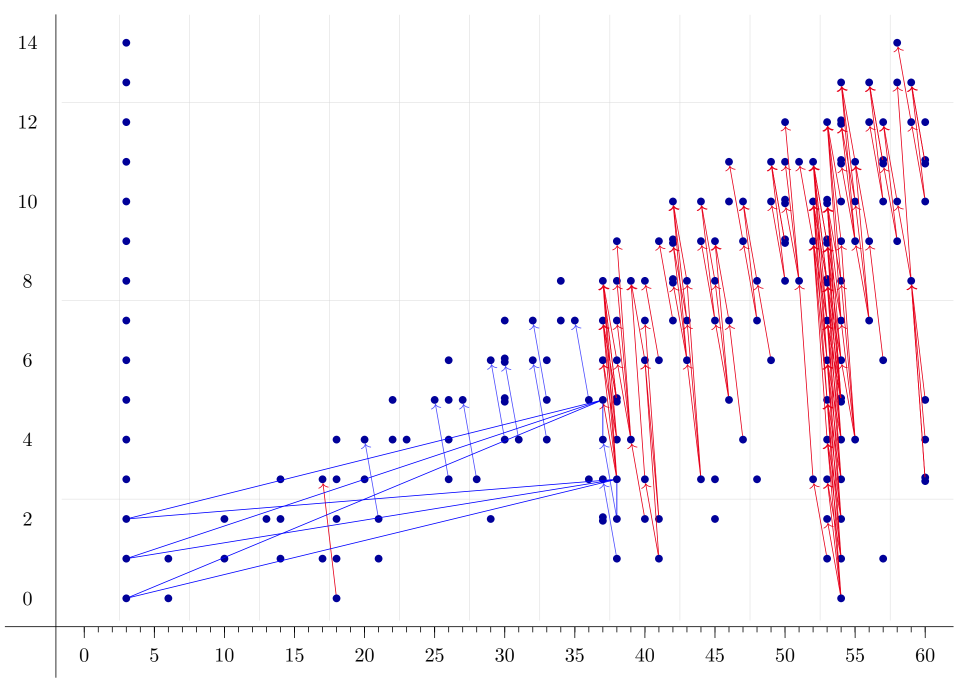

Figure 12. Multiplicative structure of $E_2^{*,*}$.

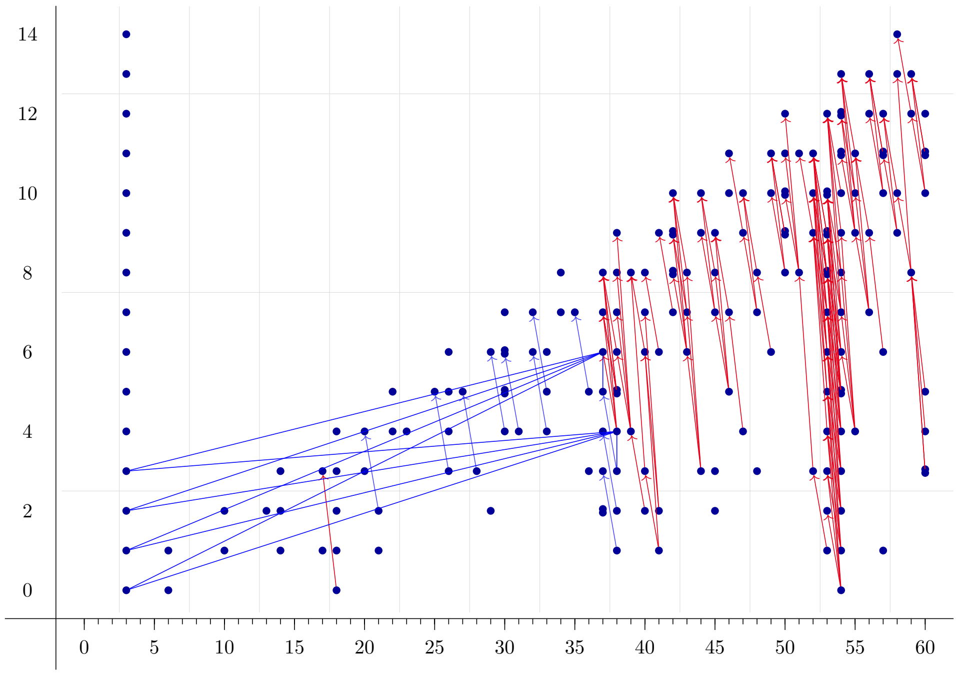

Figure 13. Multiplicative structure of $E_2^{*,*}$.

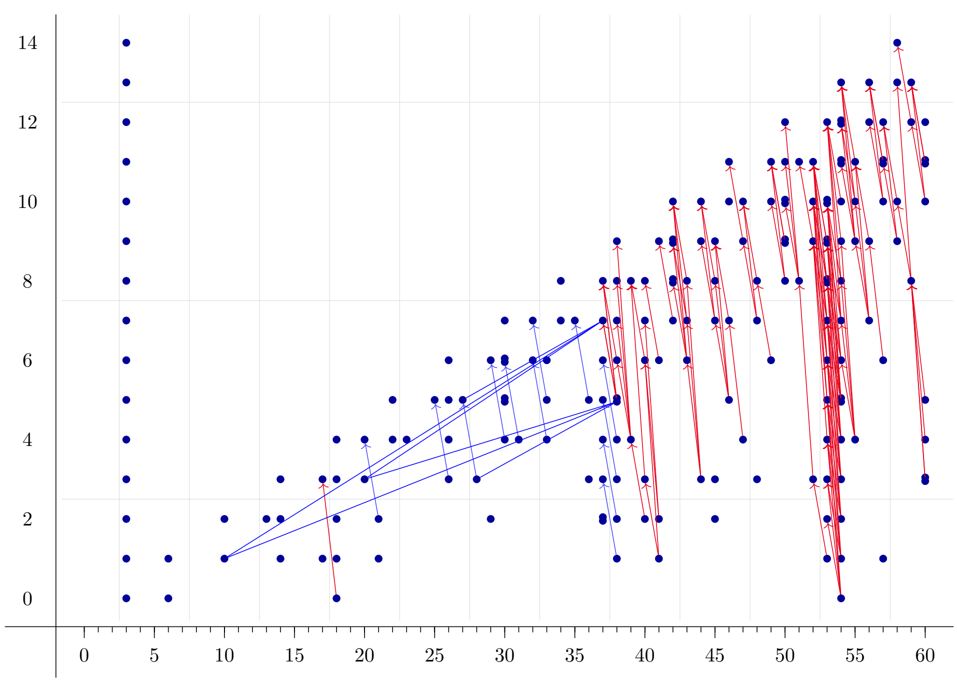

Figure 14. Multiplicative structure of $E_2^{*,*}$.

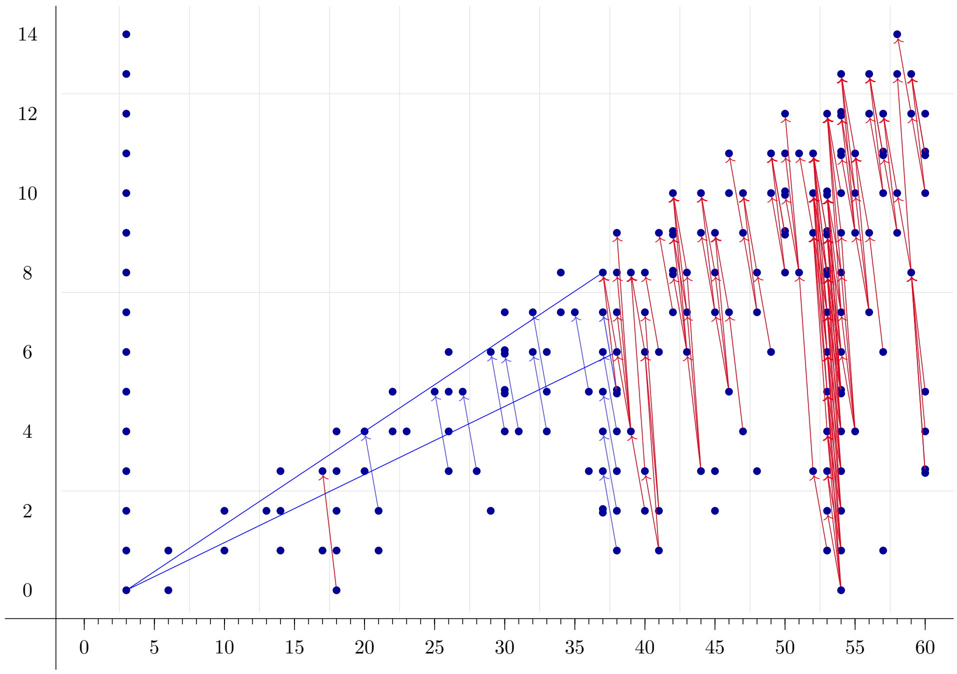

Figure 15. Multiplicative structure of $E_2^{*,*}$.

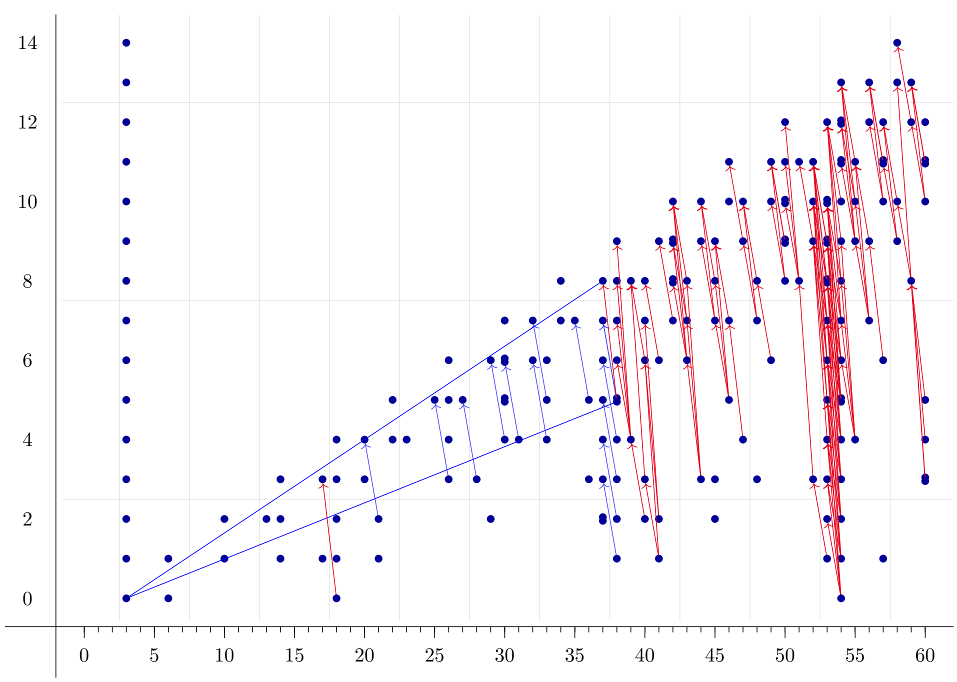

Figure 16. Multiplicative structure of $E_2^{*,*}$.

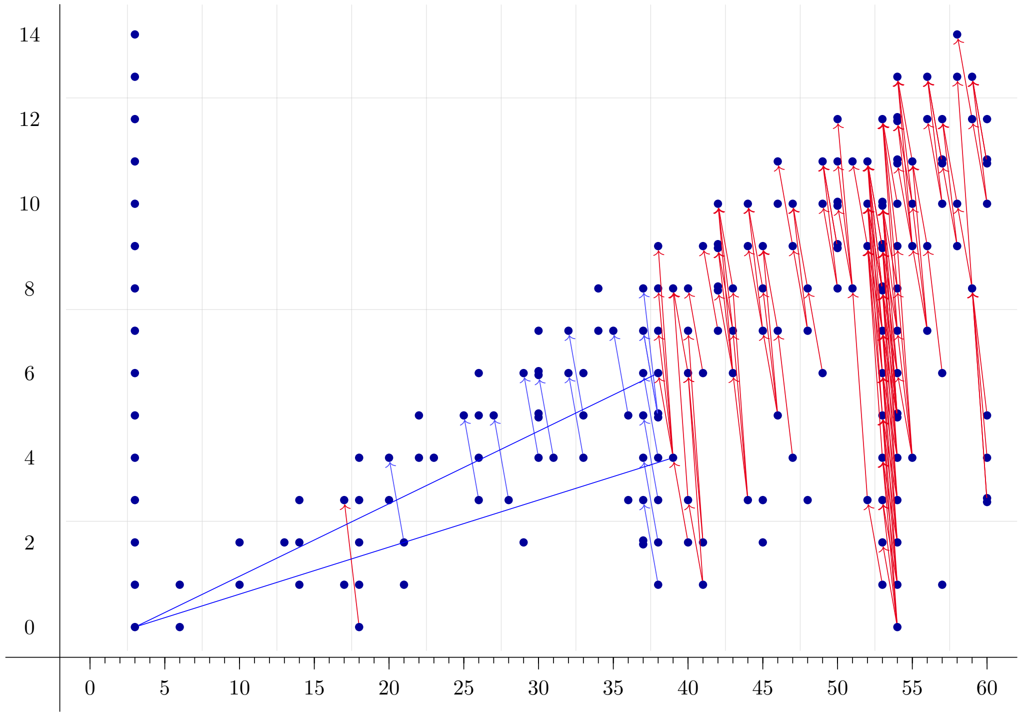

Figure 17. Multiplicative structure of $E_2^{*,*}$.

We have used that $d_2(y^{(1)}_{1,35}) \neq 0$, $d_2(y^{(1)}_{2,18}) \neq 0$ (and that $y^{(1)}_{2,18} \cdot x^{(1)}_{3,20} = x^{(2)}_{5,38}$, where the last element corresponds to the dot on the right). Hence, $d_2(x^{(2)}_{5,38}) \neq 0$. Notice that $d_2(x^{(1)}_{5,38}) = 0$ (see Figure 18, and take into account that the vertical products are trivial for $x^{(2)}_{5,38}$). It is also clear that $d_2(x^{(1)}_{6,38}) = 0$ and $d_3(x^{(1)}_{38,5}) \neq 0$. Moreover, $d_r(x^{(1)}_{4,36}) = 0$ for all $r \geq 2$, by Leibniz. In conclusion

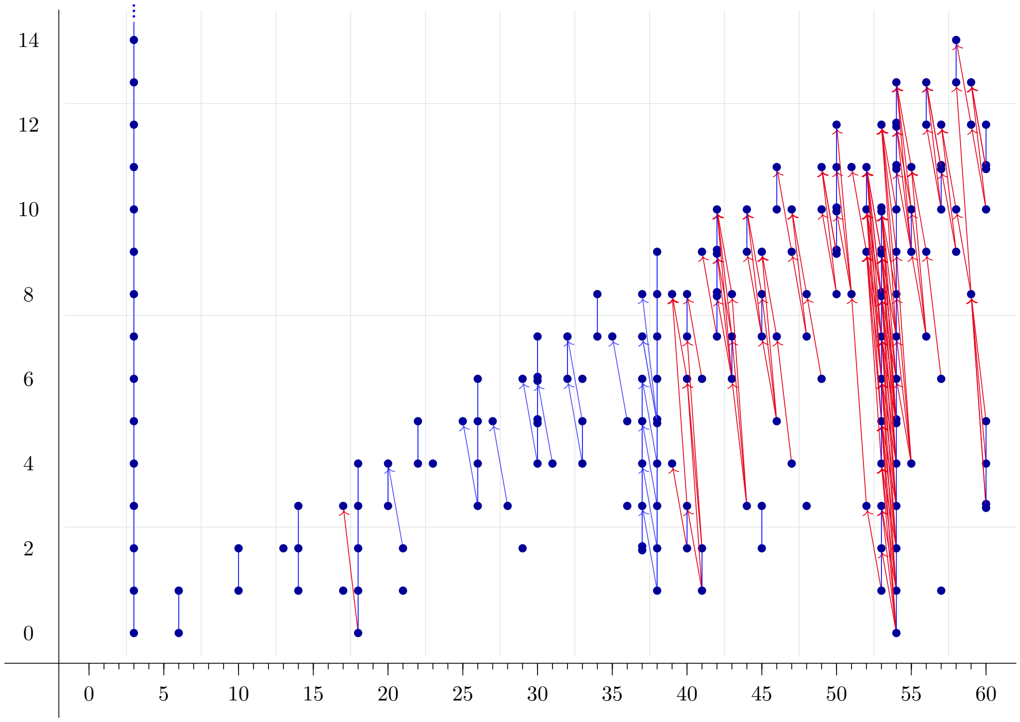

Figure 18. Multiplicative structure of $E_2^{*,*}$.

i.e. $\pi_*^s(\mathbb{S}^{\rho_3}_{hD_3})^\vee_{(3)}$, for $* \leq 38$, is given by

$*$	3	4	5	6	7	8	9	10	11	12	13
	$\mathbb{Z}_{(3)}$	0	0	$\mathbb{Z}/9\mathbb{Z}$	0	0	0	$\mathbb{Z}/9\mathbb{Z}$	0	0	$\mathbb{Z}/3\mathbb{Z}$
$*$	14	15	16	17	18	19	20	21	22	23	24
	$\mathbb{Z}/27\mathbb{Z}$	0	0	$\mathbb{Z}/\mathbf{3}\mathbb{Z}$ or $\mathbb{Z}/9\mathbb{Z}$	$\mathbb{Z}/\mathbf{81}\mathbb{Z}$ or $\mathbb{Z}/243\mathbb{Z}$	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/9\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	0
$*$	25	26	27	28	29	30	31	32	33	34	35
	0	$\mathbb{Z}/27\mathbb{Z}$	0	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/27\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	0	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/9\mathbb{Z}$	0
$*$	36	37	38								
	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/24\mathbb{Z}$								

Table 3. $\pi_*^s\left(\mathbb{S}^{\rho_3}_{hD_3}\right)^\vee_{(3)}$, for $* \leq 38$ (p even).

The following three charts give some insight into d_3 at $(0,18)$. The first chart concerns the high degree of the products related to $x^{(1)}_{0,18}$ and $x^{(1)}_{3,17}$. The second focuses on $x^{(1)}_{0,18}$. There are possible candidates to form E_3-Massey products in $(2,53)$ and $(3,53)$, assuming that the corresponding elements are killed by a d_2 differential. Hence, we need that $d_2(x^{(1)}_{0,54}) \neq 0$ (which is equivalent to $d_2(x^{(1)}_{1,54}) \neq 0$). It is unclear how to determine the (non)triviality of these differentials just by using Yoneda products. But it could still be possible to use E_2-Massey products. To form an E_3-Massey product using known data from the 3−sphere, it is also required a similar condition on $\mathrm{Ext}^{*,*}_{\mathcal{A}_3}(\mathbb{F}_3,\mathbb{F}_3)$ (see Figure 2). The third chart could be useful if d_3 at $(1,125)$ is non-trivial and $d_2(y^{(1)}_{1,107}) = 0$.

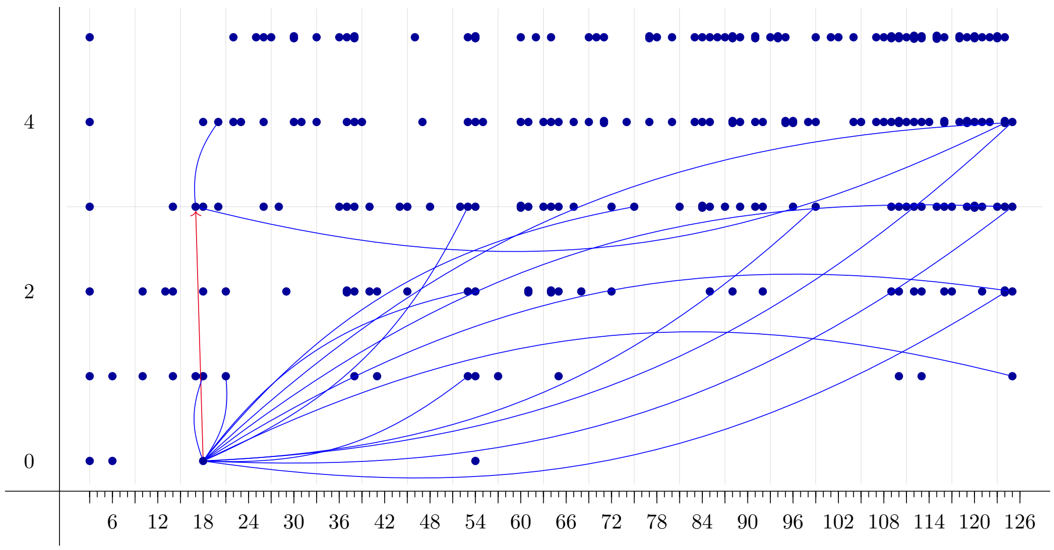

Figure 19. Products associated with $d_3\colon E_2^{0,18} \to E_2^{3,17}$.

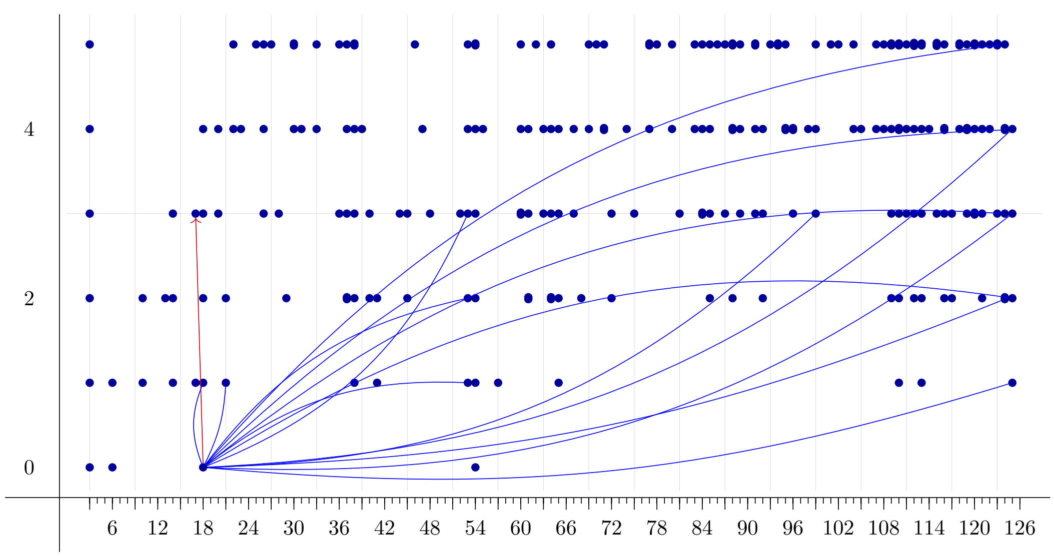

Figure 20. Products associated with $d_3\colon E_2^{0,18} \to E_2^{3,17}$.

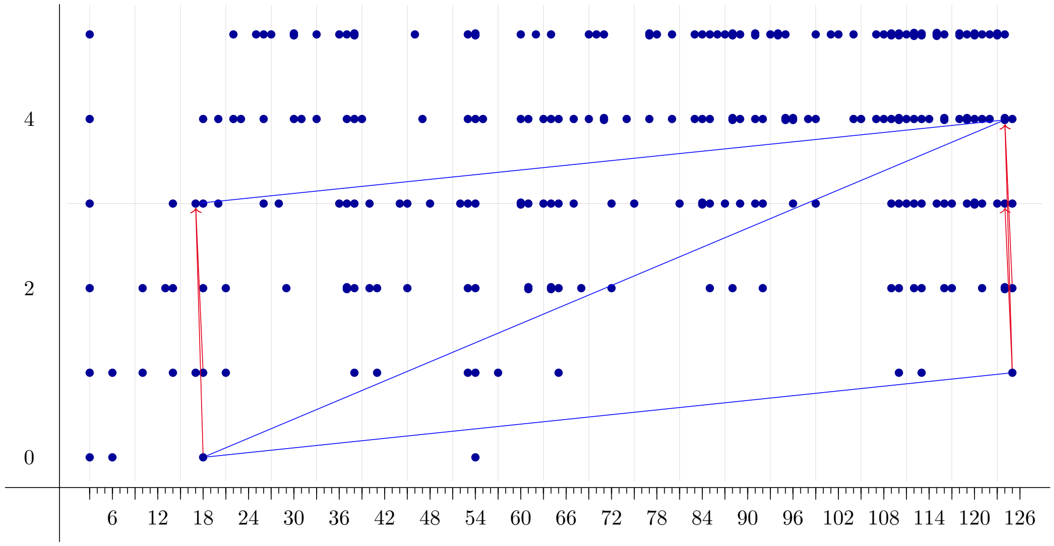

Figure 21. Products associated with $d_3\colon E_2^{0,18} \to E_2^{3,17}$.

We will now prove the stated assertion for odd p. Remember that now we have the Kahn-Priddy theorem at our disposal. However, it will not be required until later stages of the computation. The next figure corresponds to an overview of the multiplicative structure of the associated Adams spectral sequence.

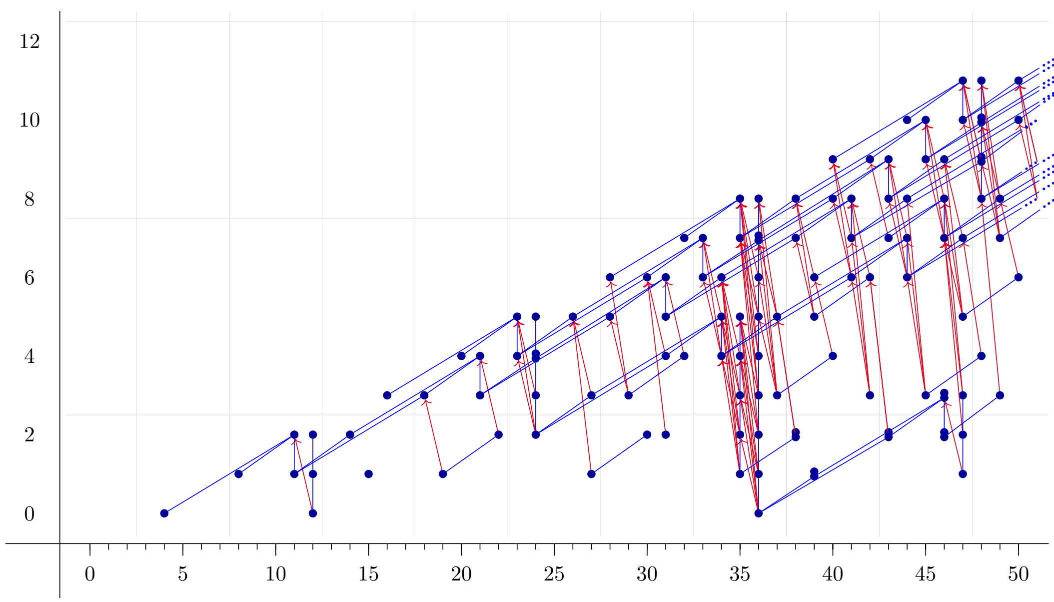

Figure 22. Multiplicative structure of $E_2^{*,*}$.

By degree reasons, the element $x_{0,12}^{(1)}$ is indecomposable. However, $x_{2,22}^{(1)}$ and $x_{5,21}^{(1)}$ can be written in three convenient ways as shown in the next figure.

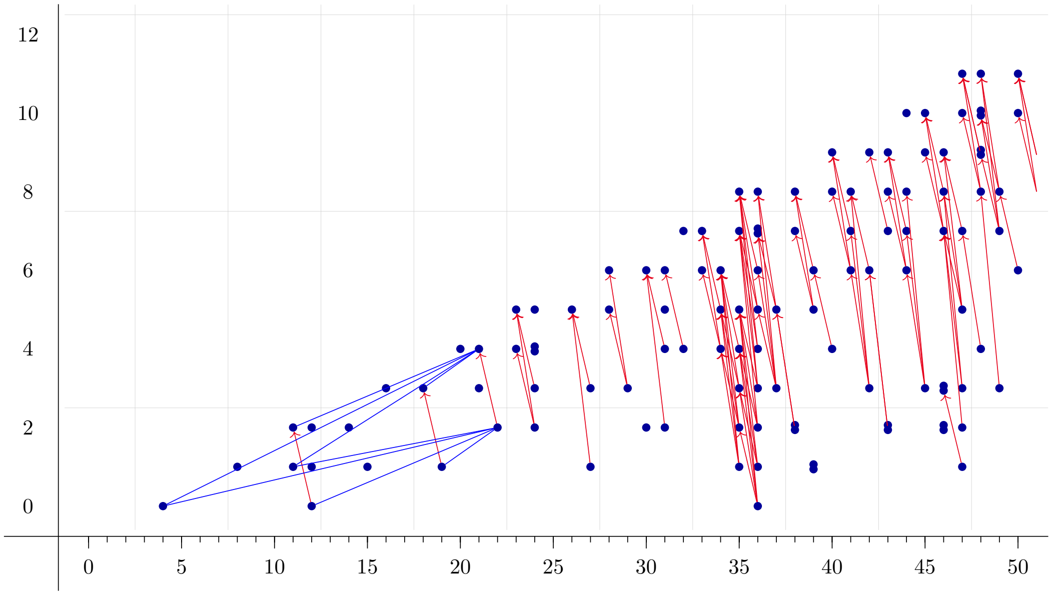

Figure 23. Multiplicative structure of $E_2^{*,*}$.

Observe that $x_{2,22}^{(1)} = y_{1,11}^{(1)} \cdot x_{1,11}^{(1)}$ and $0 \neq x_{4,21}^{(1)} = y_{3,10}^{(1)} \cdot x_{1,10}^{(1)} = d_2(y_{1,11}^{(1)}) \cdot x_{1,11}^{(1)}$. Hence, $d_2(x_{2,22}^{(1)}) \neq 0$ by Leibniz. Similarly, $d_2(x_{1,19}^{(1)}) \neq 0$. Moreover, these products imply that $d_2(x_{0,12}^{(1)}) \neq 0$.

Now, since $d_2(y_{3,23}^{(1)}) \neq 0$, we have $d_2(x_{3,27}^{(1)}) \neq 0$. Thus, $d_2(x_{2,24}^{(1)}) \neq 0 \neq d_2(x_{3,24}^{(1)})$ and $d_3(x_{2,24}^{(1)}) = 0 = d_4(x_{1,27}^{(1)})$. Repeating the same argument, consider the following diagrams.

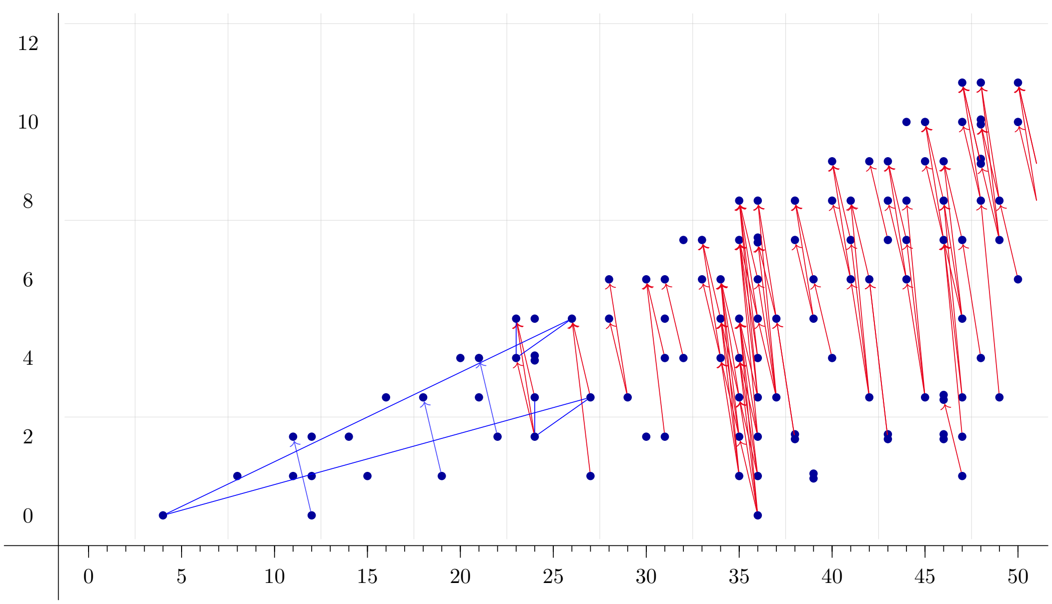

Figure 24. Multiplicative structure of $E_2^{*,*}$.

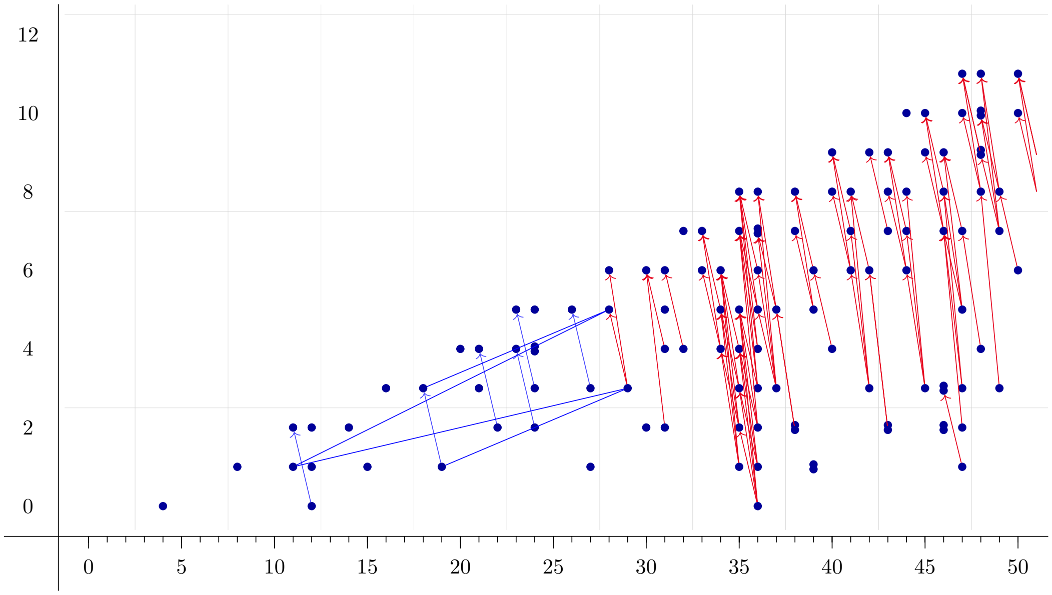

Figure 25. Multiplicative structure of $E_2^{*,*}$.

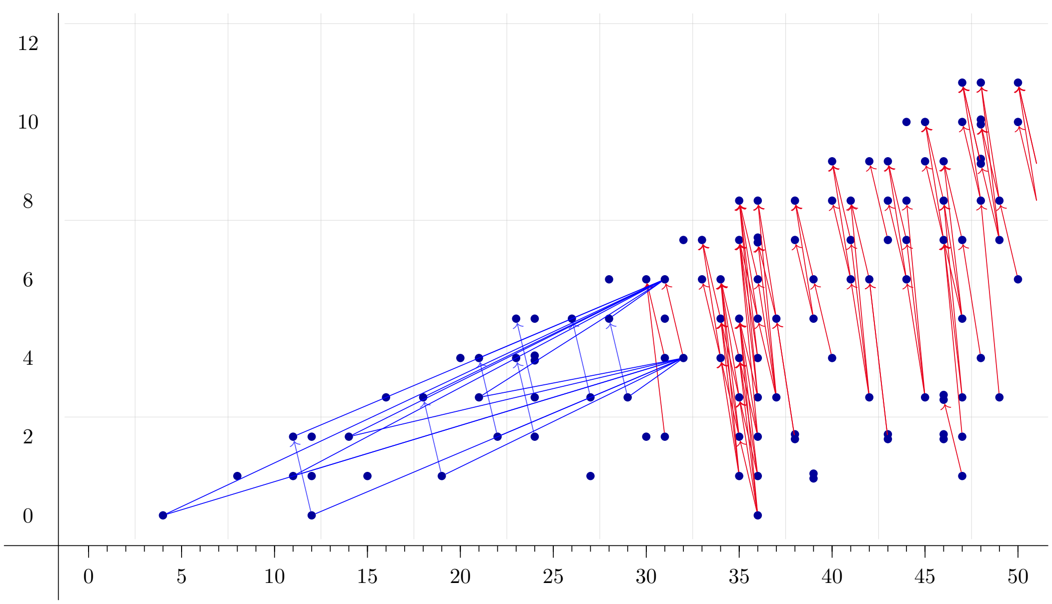

Figure 26. Multiplicative structure of $E_2^{*,*}$.

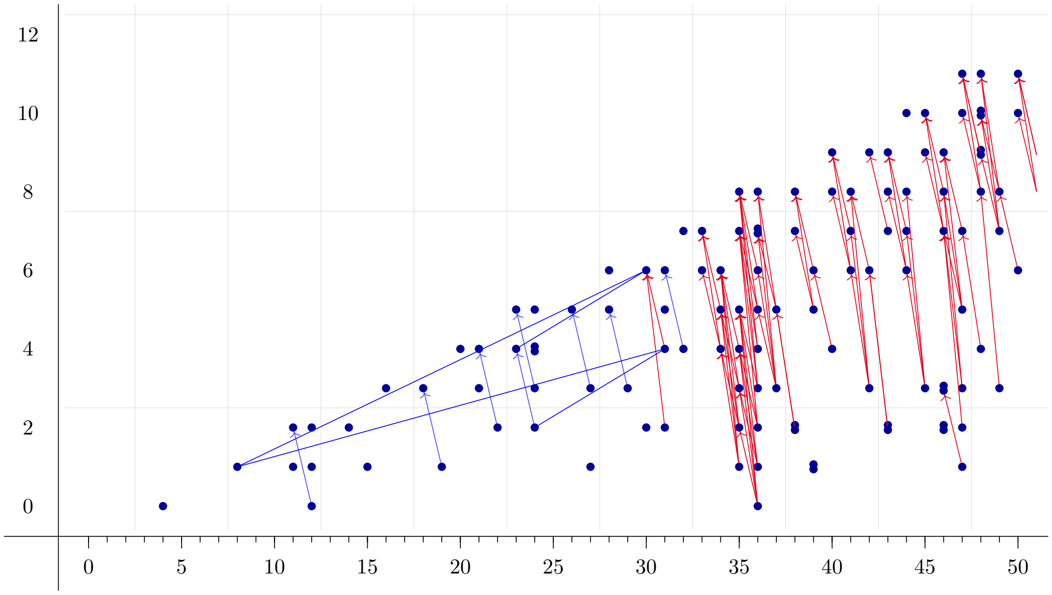

Figure 27. Multiplicative structure of $E_2^{*,*}$.

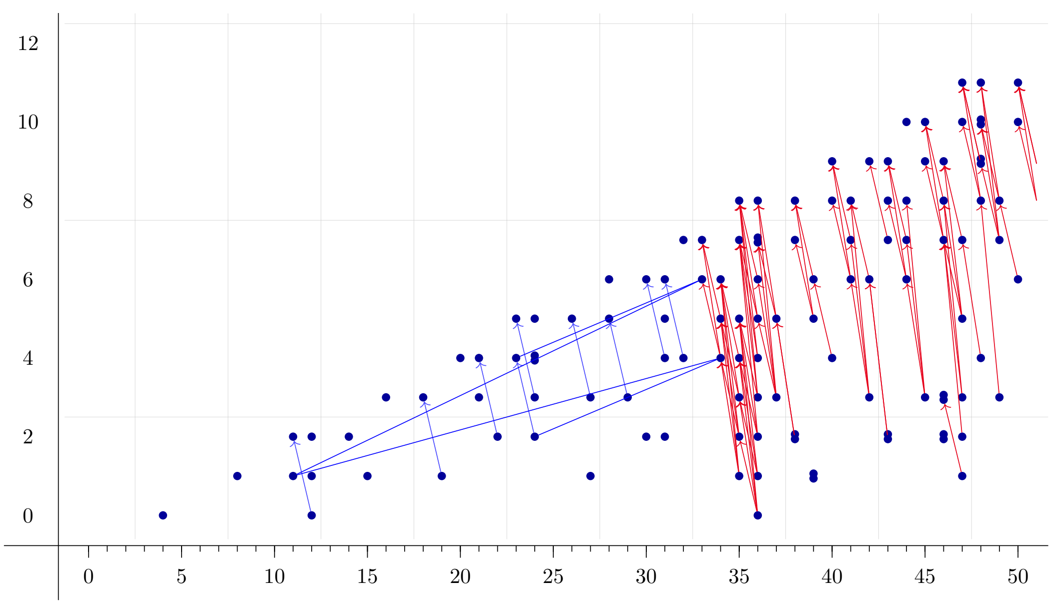

Figure 28. Multiplicative structure of $E_2^{*,*}$.

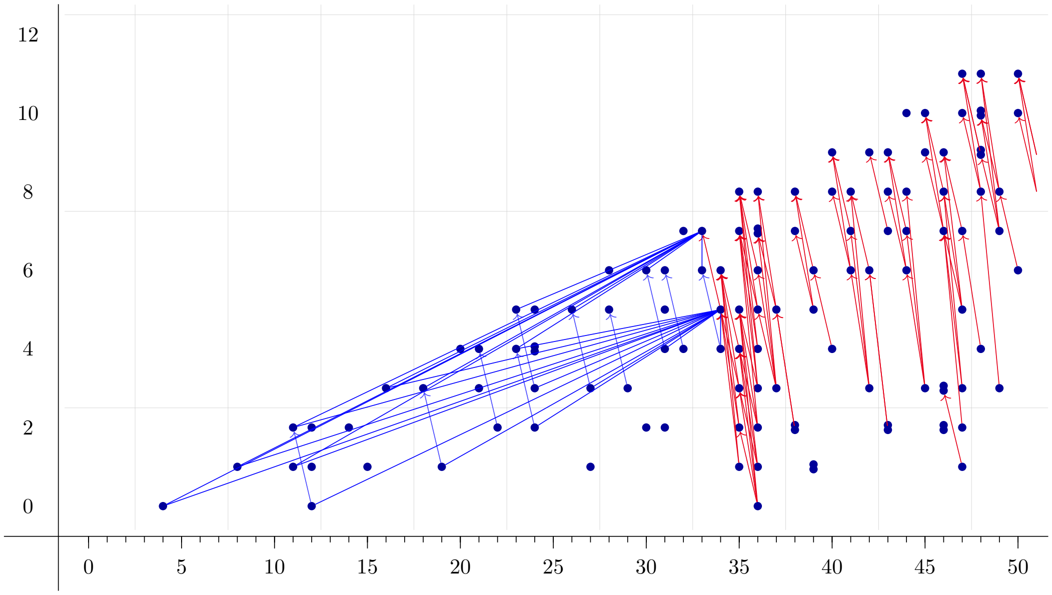

Figure 29. Multiplicative structure of $E_2^{*,*}$.

Note that $d_2(x^{(1)}_{2,35}) = 0 = d_2(x^{(1)}_{3,35})$ since $\mathrm{Im}(d_2) \subseteq \mathrm{Ker}(d_2)$. This could be obtained by using the multiplicative structure as the next figure suggests. However, we must trace the coefficients associated with each product explicitly.

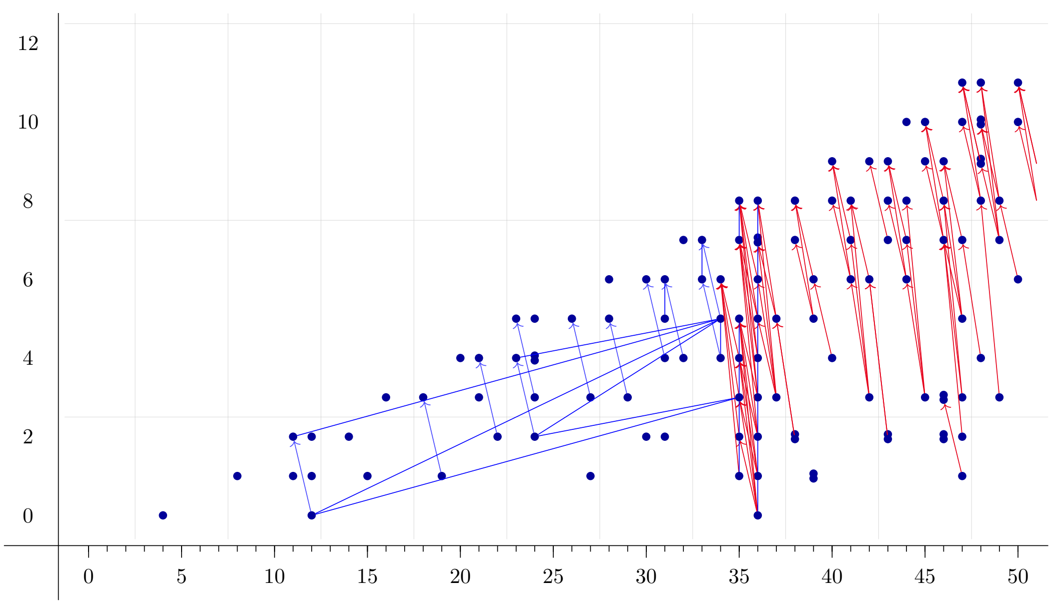

Figure 30. Multiplicative structure of $E_2^{*,*}$.

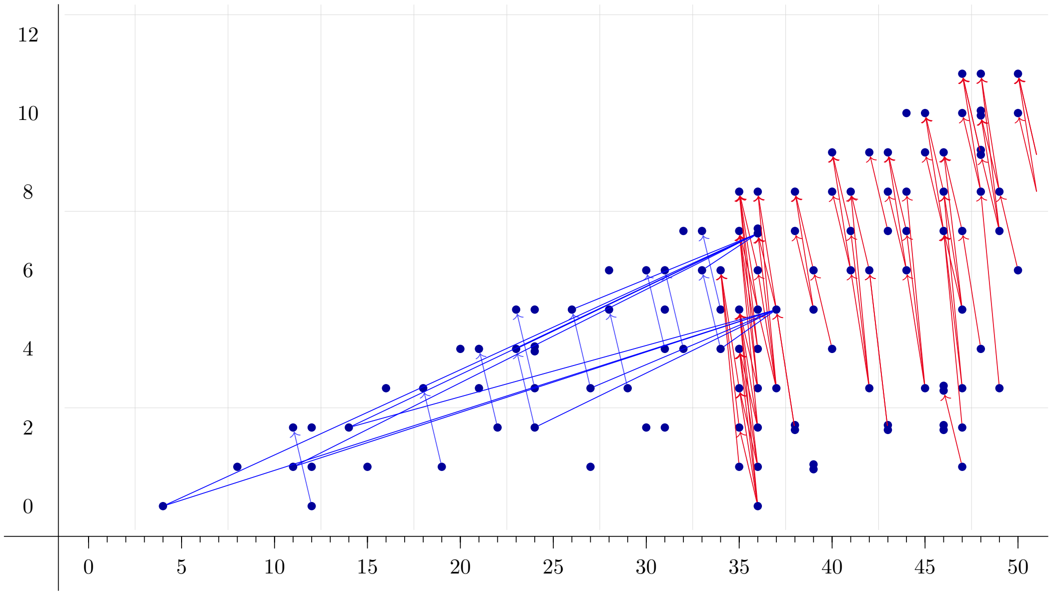

Figure 31. Multiplicative structure of $E_2^{*,*}$.

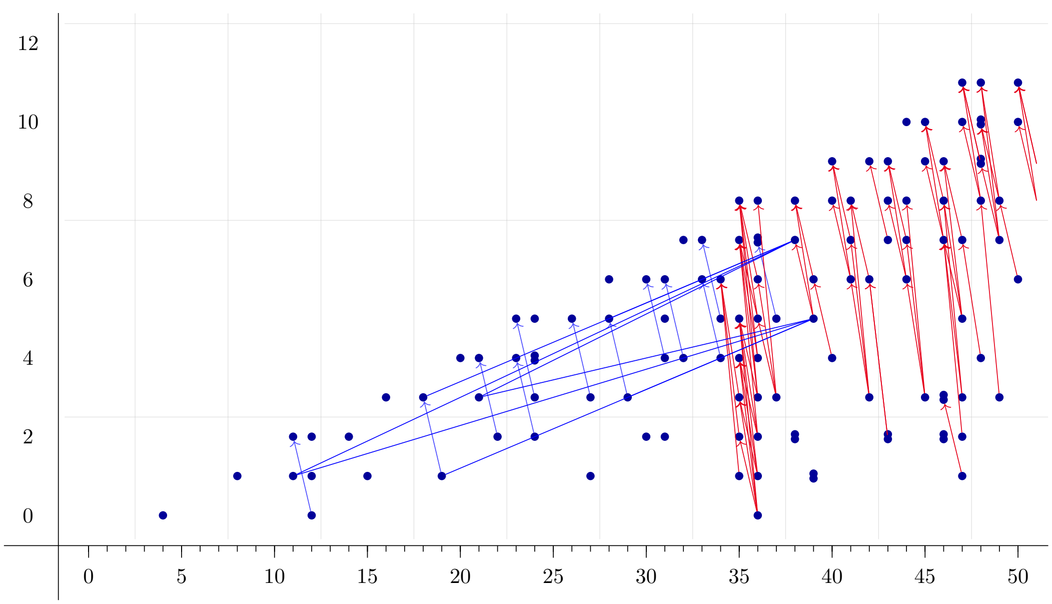

Figure 32. Multiplicative structure of $E_2^{*,*}$.

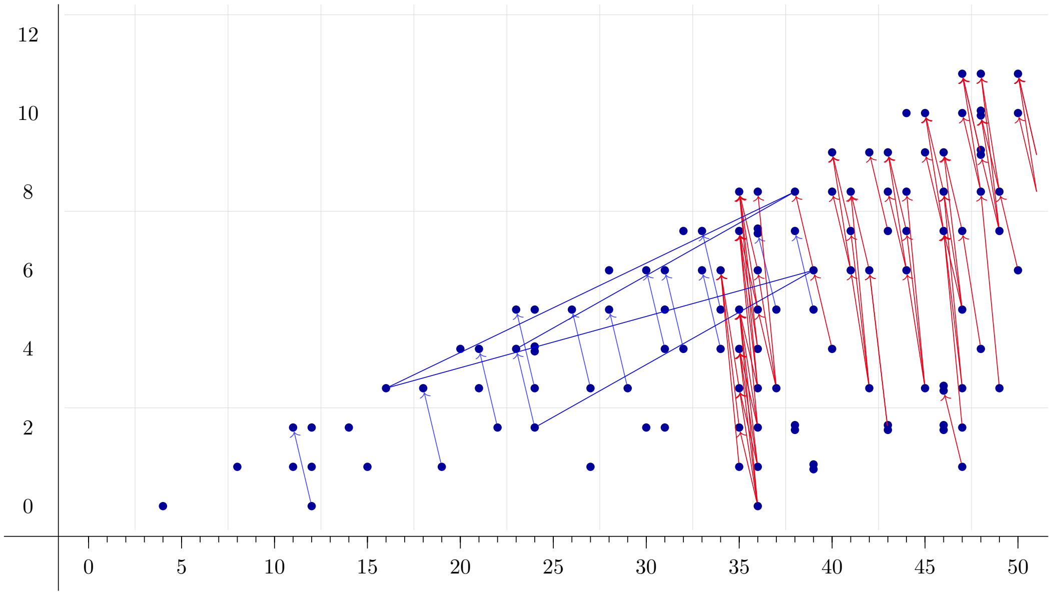

Figure 33. Multiplicative structure of $E_2^{*,*}$.

In the next figure, we have that $d_r(x^{(1)}_{3,37}) = 0$ for all $r \geq 2$, because $d_r(x^{(1)}_{1,27}) = 0 = d_r(y^{(1)}_{2,10})$ for all $r \geq 2$ (alternatively, by the Kahn-Priddy theorem we have that $\pi^s_{37}(\mathbb{S}^{\rho_3}_{hD_3})^\vee_{(3)} \twoheadrightarrow (\pi^s_{36})_{(3)} \cong \mathbb{Z}/3\mathbb{Z}$, thus $x^{(1)}_{3,37}$ survives in E_∞).

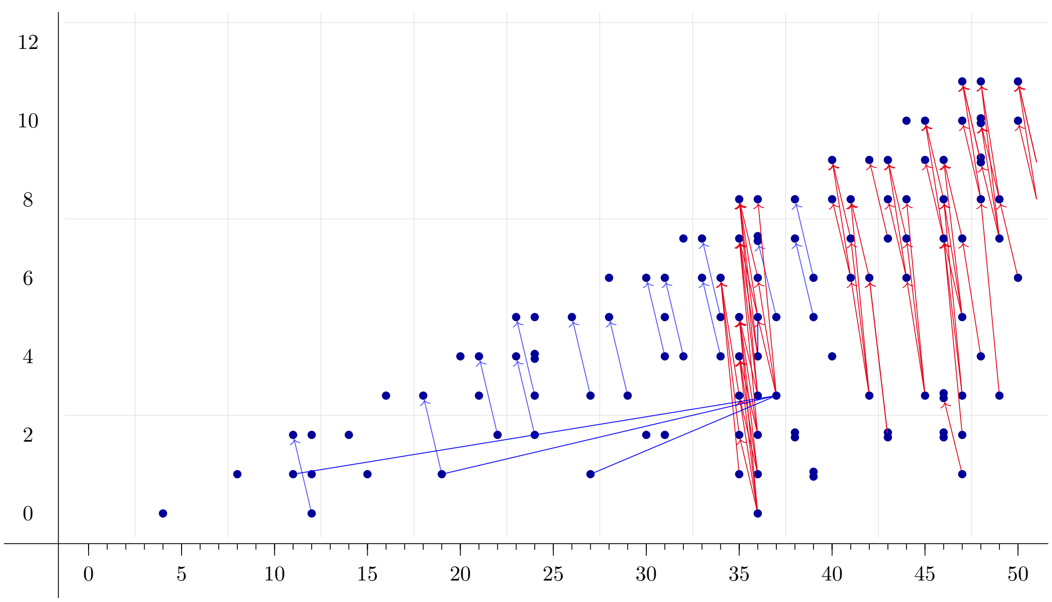

Figure 34. Multiplicative structure of $E_2^{*,*}$.

The columns $(*, 35)$ and $(*, 36)$ are harder to determine. Because there are no useful products hitting the $(*, 36)-$column. Moreover, the products involving the $(*, 36)$ have high degrees. The Kahn-Priddy theorem tells us that $\pi^s_{35}(\mathbb{S}^{\rho_3}_{hD_3})^\vee_{(3)} \twoheadrightarrow (\pi^s_{34})_{(3)} = 0$, but it is more useful in $(*, 36)$, because $\pi^s_{36}(\mathbb{S}^{\rho_3}_{hD_3})^\vee_{(3)} \twoheadrightarrow (\pi^s_{35})_{(3)} \cong \mathbb{Z}/27\mathbb{Z}$ is split surjective.

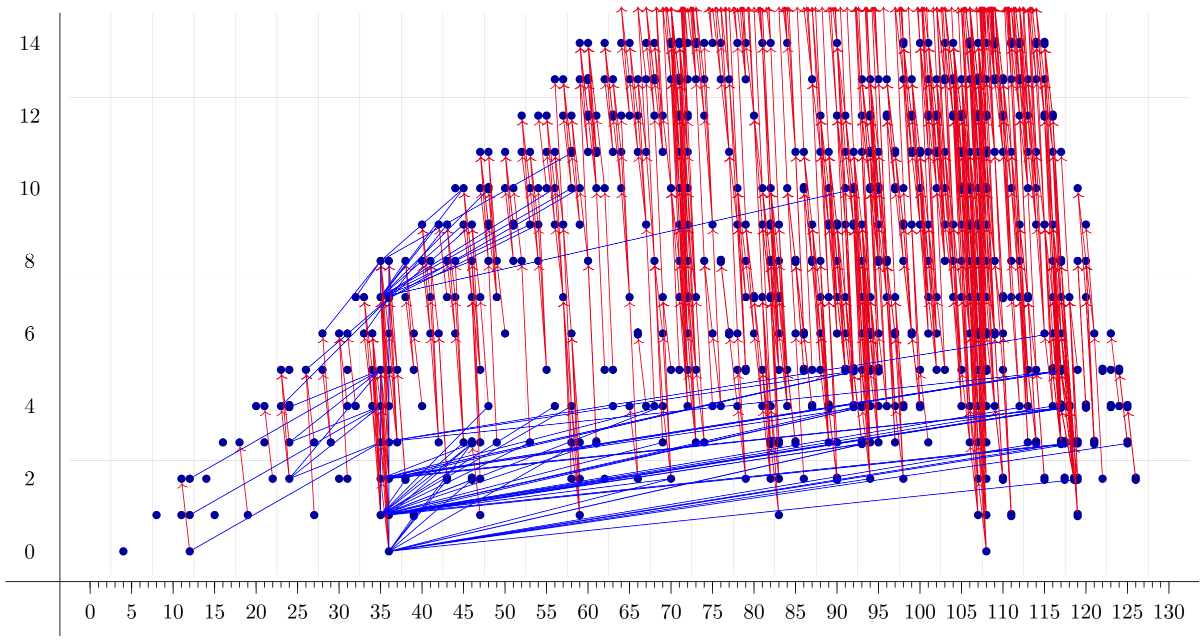

Figure 35. Multiplicative structure of $E_2^{*,*}$.

In conclusion

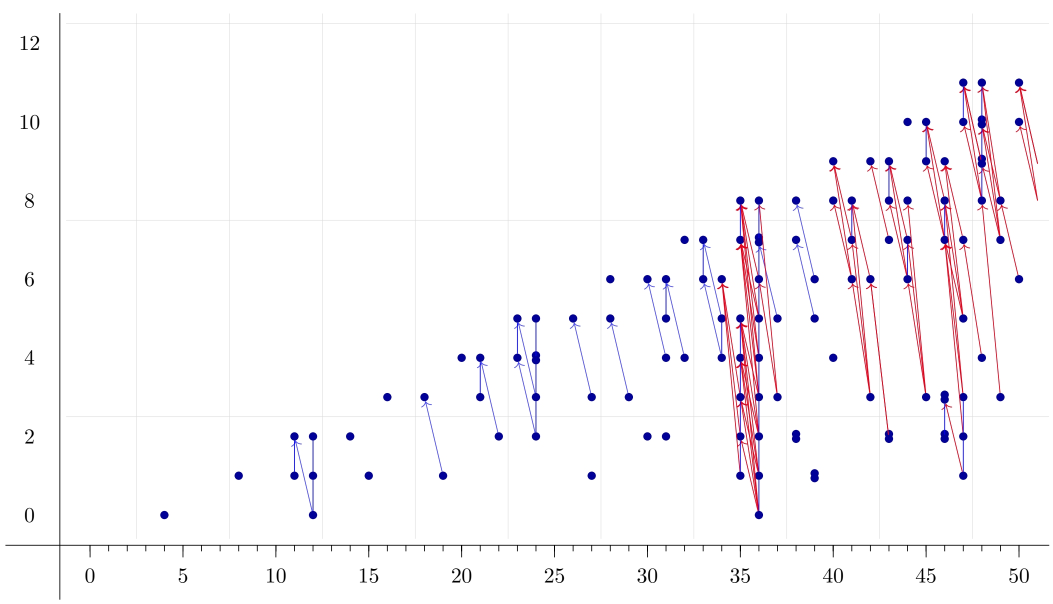

Figure 36. Multiplicative structure of $E_2^{*,*}$.

i.e.

$*$	3	4	5	6	7	8	9	10	11	12	13
	0	$\mathbb{Z}/3\mathbb{Z}$	0	0	0	$\mathbb{Z}/3\mathbb{Z}$	0	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/9\mathbb{Z}$	0
$*$	14	15	16	17	18	19	20	21	22	23	24
	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	0	0	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	0	0	$\mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$
$*$	25	26	27	28	29	30	31	32	33	34	35
	0	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	0	$\mathbb{Z}/\mathbf{9}\mathbb{Z}$ or $\mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	**?**
$*$	36	37	38	39							
	?	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$							

Table 4. $\pi_*^s\left(\mathbb{S}^{\rho_3}_{hD_3}\right)^\vee_{(3)}$, for $* \leq 38$ (p odd).

$\square$

Observe that in the previous argument, it is not clear how to compute all the non-trivial differentials with just the Leibniz rule and the Kahn-Priddy theorem, because both techniques rely on the knowledge of $(\pi_*^s)^\vee_{(3)}$, but these groups are currently determined up to degree 103. Moreover, due to the increasing complexity of the classical Adams spectral sequence, and the presence of higher differentials, it is still unclear if Kahn-Priddy would be enough to determine all the non-trivial differentials when p is odd.

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