COHOMOLOGY OF EILENBERG-MAC LANE SPACES: FIRST APPLICATIONS

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Abstract. The present document aims to compute the first stable homotopy groups for the sphere, following [1]. The main theoretical dependency is the computation of the mod 2 cohomology of Eilenberg-Mac Lane spaces. The computation will rely on a Postnikov tower-like argument and the Bockstein differentials to recover the 2^k-torsion, and to compute certain related fiber cohomologies.

1 Preliminaries

We first start with this application of Serre classes that resembles the usage of localization in (co)homology. The computation will only need the implicance (i) $\implies$ (vii).

Theorem 1 ($\mathcal{C}_p$ approximation theorem)**.** Let X and A be $1-$connected nice spaces (for example, CW complexes) such that $H_i(A;\mathbb{Z})$ and $H_i(X;\mathbb{Z})$ are finitely generated for each $i \in \mathbb{N}_0$. Let $f\colon A \to X$ be a map such that $f_*\colon \pi_2(A) \to \pi_2(X)$ is surjective. Then, conditions (i) $-$ (vi) are equivalent and imply condition (vii).

(i) $f^*\colon H^i(X;\mathbb{Z}/p\mathbb{Z}) \to H^i(A;\mathbb{Z}/p\mathbb{Z})$ is an isomorphism for $i < n$ injective for $i = n$.
(ii) $f_*\colon H_i(A;\mathbb{Z}/p\mathbb{Z}) \to H_i(X;\mathbb{Z}/p\mathbb{Z})$ is an isomorphism for $i < n$ and surjective for $i = n$.
(iii) $H_i(X, A;\mathbb{Z}/p\mathbb{Z}) = 0$ for $i \geq n$.
(iv) $H_i(X, A;\mathbb{Z}) \in \mathcal{C}_p$ for $i \geq n$.
(v) $\pi_i(X, A) \in \mathcal{C}_p$ for $i \geq n$.
(vi) $f_*\colon \pi_i(A) \to \pi_i(X)$ is a $\mathcal{C}_p-$isomorphism for $i < n$ and $\mathcal{C}_p-$surjective for $i = n$.
(vii) $\pi_i(A)_{(p)} \cong \pi_i(X)_{(p)}$ for $i < n$.

Proof. See [1]. $\square$

The preceding theorem reduces the problem of computing the $p-$component of $\pi_i(X)$ to that of finding a space A with the same $\mathbb{Z}/p\mathbb{Z}-$cohomology, together with a map $A \to X$ inducing isomorphisms in $\mathbb{Z}/p\mathbb{Z}-$cohomology.

We also require the following results related to the Bockstein morphism. These results allow us to detect mod 2 reduction of $\mathbb{Z}-$cohomology classes. The next result, in particular, is useful to compute $H^*(K(\mathbb{Z}/2^m\mathbb{Z},1);\mathbb{Z}/2\mathbb{Z})$.

Definition 1 (Bockstein exact couple)**.** The *Bockstein exact couple* is of the form

$$\begin{tikzcd}
D^1 = H^*(-;\mathbb{Z}) \arrow[rr, "i^1"] & & H^*(-;\mathbb{Z}) \arrow[dl, "j^1"] \\
& E^1 = H^*(-;\mathbb{Z}/2\mathbb{Z}) \arrow[ul, "k^1"] &
\end{tikzcd}$$

The morphism i^1 is induced by multiplication by 2 in $\mathbb{Z}$; j^1 is induced by the mod 2 reduction morphism $\rho\colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, and k^1 is the Bockstein morphism β. The differential $d = j^1k^1$ is the Bockstein morphism δ_2.

For convenience, we will denote the Bockstein differentials using subscripts, thus $d_1 = d, d_2 = d_2 = j^2k^2\colon E^2 \to E^2$, etc. The operation d_r acts as follows: take a cocycle in $\mathbb{Z}/2\mathbb{Z}-$coefficients; represent it by an integral cocycle; take its coboundary; divide by 2^r (this is possible because d_r is defined only on the kernel of d_{r-1}); and apply mod 2 reduction on the coefficients. Notice that $d_r\colon H^*(-;\mathbb{Z}/2\mathbb{Z}) \to H^{*+1}(-;\mathbb{Z}/2\mathbb{Z})$. By the universal coefficient theorem

$$H^p(X;G) = \mathrm{Hom}(H_p(X), G) \oplus \mathrm{Ext}(H_{p-1}(X), G).$$

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Therefore, a direct summand $\mathbb{Z}$ gives rise to a summand $\mathbb{Z}$ in $H^p(X;\mathbb{Z})$, and to a summand $\mathbb{Z}/2\mathbb{Z}$ in $H^p(X;\mathbb{Z}/2\mathbb{Z})$. A summand $\mathbb{Z}/2^n\mathbb{Z}$ in $H_p(X;\mathbb{Z})$ gives rises to a $\mathbb{Z}/2^n\mathbb{Z}$ in $H^{p+1}(X;\mathbb{Z})$ and to summands $\mathbb{Z}/2\mathbb{Z}$ in $H^p(X;)\mathbb{Z}/2\mathbb{Z}$ and $H^{p+1}(X;\mathbb{Z}/2\mathbb{Z})$.

Proposition 1 (Torsion detected by the Bockstein morphisms I)**.** Elements of $H^*(X;\mathbb{Z}/2\mathbb{Z})$ which come from *free* integral classes lie in $\mathrm{Ker}(d_r)$ for every $r \geq 0$ and not in $\mathrm{Im}(d_r)$ for all $r \geq 0$. If $z \in H^{n+1}(X;\mathbb{Z})$ generates a direct summand of order 2^r, then there exist cyclic direct summands of order 2 in $H^n(X;\mathbb{Z}/2\mathbb{Z})$, and $H^{n+1}(X;\mathbb{Z}/2\mathbb{Z})$ generated by z' and z'' respectively; $d_i(z'), d_i(z'') = 0$ for $i < r$, and $d_r(z') = z''$ (implicitly, z' and z'' are also not in $\mathrm{Im}(d_r)$ for $i < r$.)

Proof. Proven in [1]. □

Definition 2 (Persistent elements)**.** In the context of the previous result, we say that the image by ρ of the free subgroups of $H^*(X;\mathbb{Z})$ *persists* to E^∞ and that z' and z'' *persists* to E^r but not to E^{r+1}.

The following application is related to $\pi_*(S^n)$.

Corollary 1 (Torsion detected by the Bockstein morphisms II)**.** Let X be a space, such that $H^i(X;\mathbb{Z}/2\mathbb{Z}) = 0$ for $i < n$, and $H^n(X;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}\{z\}$. Then we can infer $H^n(X;\mathbb{Z})$, except for odd prime torsion. That is, $H^n(X;\mathbb{Z}) \cong \mathbb{Z}$ if $d_r z = 0$ for all $r \geq 1$, and $H^n(X;\mathbb{Z}) = \mathbb{Z}/2^n\mathbb{Z}$ if $d_i z = 0$ for $i < n$ and $d_n z \neq 0$.

Theorem 2 (Bockstein lemma)**.** Let $(E, p, B; F)$ be a fibre space. Let the class $u \in H^n(F;\mathbb{Z}/2\mathbb{Z})$ be transgressive, and suppose that, for some $i \in \mathbb{N}_{>0}$ and for some $v \in H^n(B;\mathbb{Z}/2\mathbb{Z})$, $d_i(v) = \tau(u)$. Then $d_{i+1}p^*v$ is defined, and moreover

$$j^* d_{i+1} p^*(v) = d_1(u)$$

where j is the inclusion $F \subseteq E$.

Proof. See [1]. □

Here the members of the formula $d_i(v) = \tau(u)$ and the formula of the conclusion lie in appropriate quotient groups of $H^{n+1}(F;\mathbb{Z}/2\mathbb{Z})$.

2 First stable homotopy groups

Now, we can state and prove the following result. By the previous results related to the minimal $p-$torsion in $\pi_*(S^n)$, it will be just needed to compute $\pi_i(S^n)_{(2)}$ for $i = 1, 2, 3$. The proof depends on the computation of $H^*(K(\mathbb{Z}/2\mathbb{Z}, n);\mathbb{Z}/2\mathbb{Z})$, $H^*(K(\mathbb{Z}, n);\mathbb{Z}/2\mathbb{Z})$ and $H^*(K(\mathbb{Z}/2^k\mathbb{Z}, n);\mathbb{Z}/2\mathbb{Z})$. The idea of the proof consists of constructing fibrations approximating the $n-$sphere by an inductive Postnikov tower-like argument. In each approximation, specific cohomology classes will be killed to make the fiber's canonical class a transgressive element. The role of the Bockstein morphisms is to detect the 2^k-torsion and to compute a basis for the cohomology of the fibers.

Theorem 3 (First stable homotopy groups)**.** We have the following isomorphisms.

(i) $\pi_1^s(S^0) \cong \mathbb{Z}/2\mathbb{Z}$.
(ii) $\pi_2^s(S^0) \cong \mathbb{Z}/2\mathbb{Z}$.
(iii) $\pi_3^s(S^0) \cong \mathbb{Z}/24\mathbb{Z}$.

Proof. Since the stated assertion is concerned with stable homotopy groups, assume that $n \gg 1$. This ensures that we will be working with the Steenrod algebra and that we will not need to consider cup products in the spectral sequence arguments. Consider the map $\mathrm{Sq}^2 \colon K(\mathbb{Z}, n) \to K(\mathbb{Z}/2\mathbb{Z}, n+2)$ which represents $\mathrm{Sq}^2(\iota_n) \in H^{n+2}(K(\mathbb{Z}, n);\mathbb{Z}/2\mathbb{Z})$. Essentially, we will construct a Postnikov tower inductively. The first step is the following. Take the path-loop fibration $K(\mathbb{Z}/2\mathbb{Z}, n+1) \to P \to K(\mathbb{Z}/2\mathbb{Z}, n+2)$. Now, take the pullback of this fibration under $\mathrm{Sq}^2 \colon K(\mathbb{Z}, n) \to K(\mathbb{Z}/2\mathbb{Z}, n+2)$, as the following commutative diagram shows.

$$\begin{tikzcd}
F := K(\mathbb{Z}/2\mathbb{Z}, n+1) \arrow[d, "i"] & K(\mathbb{Z}/2\mathbb{Z}, n+1) \arrow[d] \\
X_1 \arrow[r, "F"] \arrow[d, "p"] & P \arrow[d, "\mathrm{ev}_1"] \\
B := K(\mathbb{Z}, n) \arrow[r, "\mathrm{Sq}^2"] & K(\mathbb{Z}/2\mathbb{Z}, n+2)
\end{tikzcd}$$

By commutativity of the previous diagram, $\mathrm{Sq}^2 \circ p = \mathrm{ev}_1 \circ F$. Therefore, $p^* \circ \mathrm{Sq}^2 = F^* \circ (\mathrm{ev}_1)^*$. Because $P \cong \{*\}$, it follows that $p^* \circ \mathrm{Sq}^2 = 0$, hence $p^* \mathrm{Sq}^2 \iota_n = 0 \in H^{n+2}(X_1; \mathbb{Z}/2\mathbb{Z})$, i.e. we have killed the cohomology class $\mathrm{Sq}^2 \iota_n \in H^{n+2}(K(\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$. Since $K(\mathbb{Z}, n)$ is $1-$connected, by the cohomological Serre spectral sequence with $\mathbb{Z}/2\mathbb{Z}-$coefficients we have the following diagram.

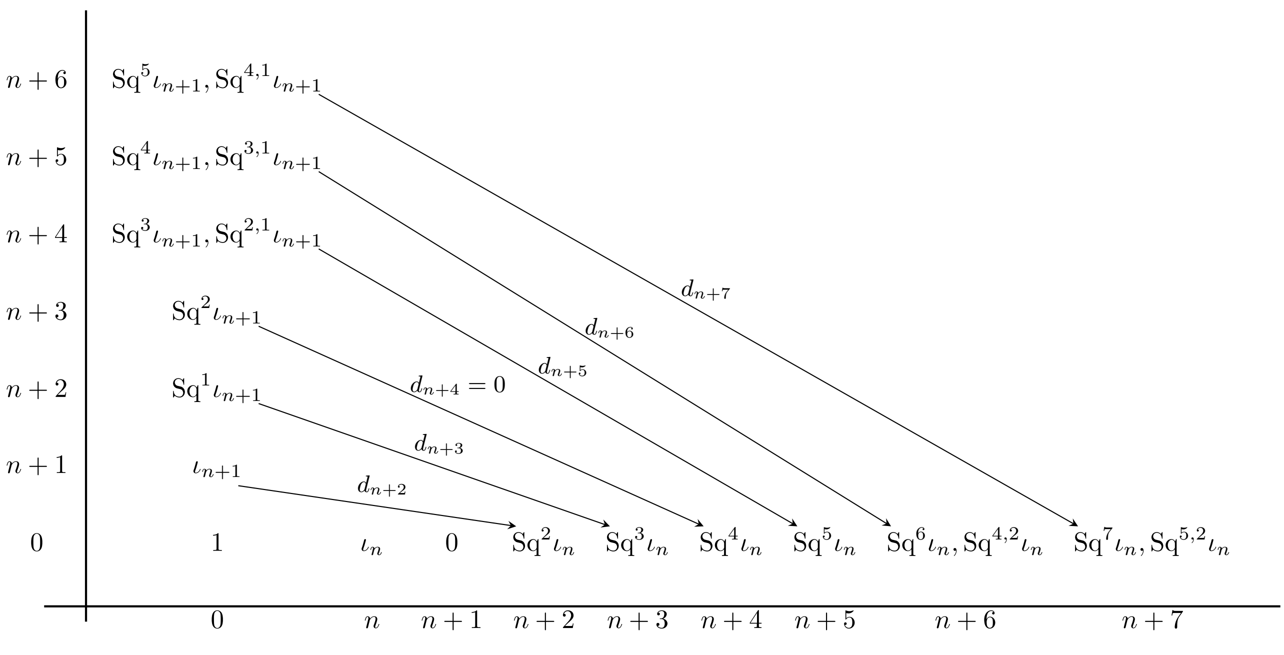

Figure 1. Serre spectral sequence associated with $K(\mathbb{Z}/2\mathbb{Z}, n+1) \to X_1 \to K(\mathbb{Z}, n)$.

Since we have killed $\mathrm{Sq}^2\iota_n \in H^{n+2}(X_1; \mathbb{Z}/2\mathbb{Z})$, we have that $\iota_{n+1} \in F = H^*(K(\mathbb{Z}/2\mathbb{Z}, n+1); \mathbb{Z}/2\mathbb{Z})$ is transgressive. Therefore, by the commutativity of transgressive differentials with the Steenrod operations, it follows that $d_{n+2}(\mathrm{Sq}^1\iota_{n+1}) = \mathrm{Sq}^1(d_{n+1}(\iota_{n+1})) = \mathrm{Sq}^1\mathrm{Sq}^2(\iota_n) = \mathrm{Sq}^3(\iota_n)$, applying the Adem relations. Moreover, $d_{n+4}(\mathrm{Sq}^2\iota_{n+1}) = \mathrm{Sq}^2\mathrm{Sq}^2\iota_n = \mathrm{Sq}^3\mathrm{Sq}^1\iota_n = 0$, and $d_{n+5}(\mathrm{Sq}^{2,1}\iota_{n+1}) = \mathrm{Sq}^{2,1,2}\iota_n = \mathrm{Sq}^5\iota_n + \mathrm{Sq}^{4,1}\iota_n = \mathrm{Sq}^5\iota_n$, $d_{n+5}(\mathrm{Sq}^3\iota_{n+1}) = \mathrm{Sq}^{3,2}\iota_n = 0$. Similarly, $d_{n+6}(\mathrm{Sq}^4\iota_{n+1}) = \mathrm{Sq}^{4,2}\iota_n$ and $d_{n+6}(\mathrm{Sq}^{3,1}\iota_{n+1}) = \mathrm{Sq}^{3,1,2}\iota_n = \mathrm{Sq}^{5,1}\iota_n = 0$. Analogously, $d_{n+7}(\mathrm{Sq}^5\iota_{n+1}) = \mathrm{Sq}^{5,2}\iota_n$ and $d_{n+7}(\mathrm{Sq}^{4,1}\iota_{n+1}) = \mathrm{Sq}^{5,2}\iota_n$.

Now, by the convergence of the Serre spectral sequence, and applying the same argument with d_{n+3}, we obtain that

$$\tilde{H}^i(X_1; \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z}, & i = n, \\ 0 & i < n \text{ or } n < i \leq n+2. \end{cases}$$

Additionally, the previous computation with the transgression gives us the following generators (as a $\mathbb{Z}/2\mathbb{Z}-$module) in the following degrees

$$H^{n+3}(X_1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{a\},$$
$$H^{n+4}(X_1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{b\} \oplus \mathbb{Z}/2\mathbb{Z}\{p^*(\mathrm{Sq}^4\iota_n)\},$$

$$H^{n+5}(X_1;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{c\},$$
$$H^{n+6}(X_1;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{d\} \oplus \mathbb{Z}/2\mathbb{Z}\{p^*(\mathrm{Sq}^6\iota_n)\},$$

where

$$\begin{aligned} i^*(a) &= \mathrm{Sq}^2\iota_{n+1}, \\ i^*(b) &= \mathrm{Sq}^3\iota_{n+1}, \\ i^*(c) &= \mathrm{Sq}^{3,1}\iota_{n+1}, \\ i^*(d) &= \mathrm{Sq}^5\iota_{n+1} + \mathrm{Sq}^{4,1}\iota_{n+1}. \end{aligned}$$

This will be required to compute a basis for $H^{n+i}(X_2;\mathbb{Z}/2\mathbb{Z})$ during the next step of this inductive argument.

Now let $f\colon S^n \to K(\mathbb{Z},n)$ represent the homotopy class of a generator of $\pi_n(K(\mathbb{Z},n)) = \mathbb{Z}$. The composition

$$S^n \xrightarrow{f} K(\mathbb{Z},n) \xrightarrow{\mathrm{Sq}^2} K(\mathbb{Z}/2\mathbb{Z},n+2)$$

is null-homotopic, since $\pi_n(K(\mathbb{Z}/2\mathbb{Z},n+2)) = 0$. Therefore, by the homotopy lifting property applied to the path-loop fibration, and the universal property of the pullback, this induces a map $f_1\colon S^n \to X_1$. Note that

$$(f_1)^*\colon H^i(X_1;\mathbb{Z}/2\mathbb{Z}) \to H^i(S^n;\mathbb{Z}/2\mathbb{Z})$$

is an isomorphism for $i \le n+1$ and surjective for $i = n+2$. By the homotopy exact sequence, $\pi_{n+1}(X_1)_{(2)} \cong \mathbb{Z}/2\mathbb{Z}$. Applying the $\mathcal{C}_p$ approximation theorem with $\mathcal{C}_2$, and noticing that $\pi_{n+2}(S^n)$ does not have odd $p-$torsion, it follows that

$$\pi_{n+1}(S^n) \cong \pi_{n+1}(S^n)_{(2)} \cong \pi_{n+1}(X_1)_{(2)} \cong \mathbb{Z}/2\mathbb{Z}.$$

Before we can continue, we will require the following lemma, which gives useful Bockstein relations in $H^*(X_1;\mathbb{Z}/2\mathbb{Z})$. These relations are needed because we do not know the exact behavior of the squaring operations in $H^*(X_1;\mathbb{Z}/2\mathbb{Z})$.

Lemma 1 (Bockstein relations in $H^*(X_1;\mathbb{Z}/2\mathbb{Z})$)**.** In $H^*(X_1;\mathbb{Z}/2\mathbb{Z})$, there are the following relations in terms of the Bockstein differentials $\beta_i\colon H^*(X_1;\mathbb{Z}/2\mathbb{Z}) \to H^{*+1}(X_1;\mathbb{Z}/2\mathbb{Z})$.

(a) $\beta_1(a) = b + k \cdot p^*(\mathrm{Sq}^4\iota_n)$ for some $k \in \mathbb{Z}$.
(b) $\beta_2(p^*(\mathrm{Sq}^4\iota_n)) = c$.
(c) $\beta_1(c) = 0$.

Proof.

(a) We have $i^*(a) = \mathrm{Sq}^2\iota_{n+1}$ by definition of $a \in H^{n+3}(X_1;\mathbb{Z}/2\mathbb{Z})$. Therefore $i^*(\beta_1 a) = \beta_1 i^*(a) = \beta_1\mathrm{Sq}^2\iota_{n+1} = \mathrm{Sq}^3\iota_{n+1}$. Moreover, $i^*(b) = \mathrm{Sq}^3\iota_{n+1}$. Since $p^*\mathrm{Sq}^4\iota_n \in \mathrm{Im}(p^*)$, $i^*p^*\mathrm{Sq}^4\iota_n = 0$, by the (cohomological) Serre exact sequence associated to the fibration $K(\mathbb{Z}/2\mathbb{Z},n+1) \to X_1 \to K(\mathbb{Z},n)$.

(b) Notice that $\beta_1\mathrm{Sq}^4\iota_n = \mathrm{Sq}^1\mathrm{Sq}^4\iota_n = \mathrm{Sq}^5\iota_n = d_{n+5}(\mathrm{Sq}^{2,1}\iota_{n+1})$. Thus $\beta_1 p^*\mathrm{Sq}^4\iota_n = 0$ since $p^*d_{n+5} = 0$ (remember that d_{n+5} is the transgression). By the Bockstein lemma (Lemma 2),

$$i^*\beta_2 p^*\mathrm{Sq}^4\iota_n = \beta_1(Sq^{2,1}\iota_{n+1}) = \mathrm{Sq}^{3,1}\iota_{n+1} = i^*(c).$$

Since $H^{n+5}(X_1;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{c\}$, the result follows.

(c) This follows from (b) since $\beta_1(u) = 0$ for all $u \in \mathrm{Im}(\beta_r)$.

■

The next step in this inductive computation is to kill the generator $H^{n+3}(X_1;\mathbb{Z}/2\mathbb{Z})$ in such a way that the canonical class in the fiber cohomology transgresses too. Consider the following commutative diagram, obtained analogously from the previous one.

$$\begin{tikzcd}
 & F_2 := K(\mathbb{Z}/2\mathbb{Z},n+2) \arrow[d, "i"] & K(\mathbb{Z}/2\mathbb{Z},n+2) \arrow[d] \\
 & X_2 \arrow[r] \arrow[d, "p_2"] & P \arrow[d] \\
S^n \arrow[ur, "\exists f_2"] \arrow[r, "f_1"] & X_1 \arrow[r, "a"] & K(\mathbb{Z}/2\mathbb{Z},n+3)
\end{tikzcd}$$

As in the previous step, consider the associated cohomological Serre spectral sequence, described in the next figure.

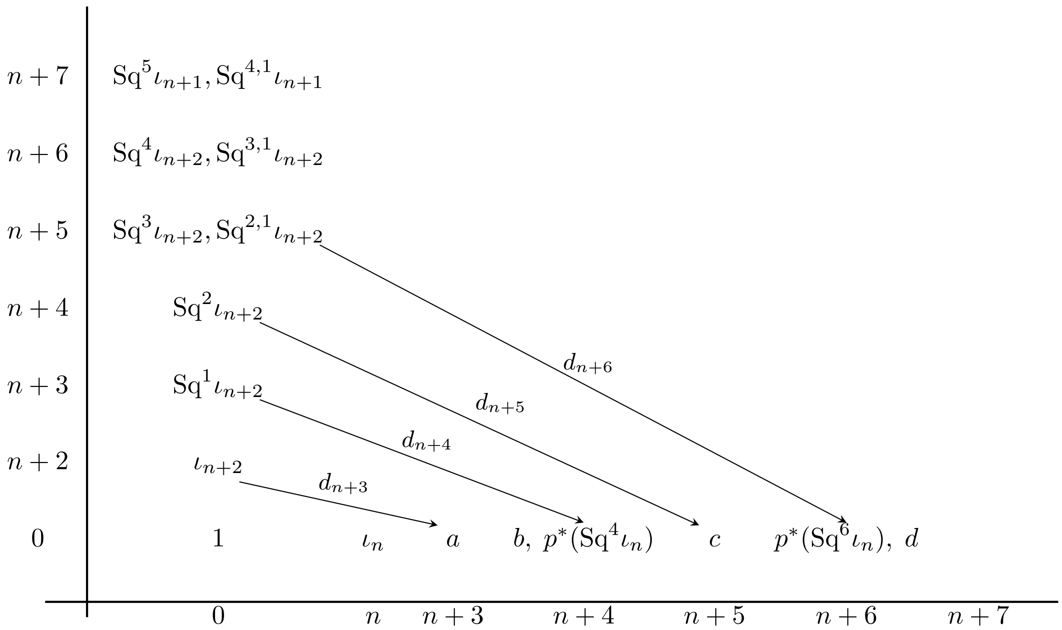

Figure 2. Serre spectral sequence associated to $K(\mathbb{Z}/2\mathbb{Z}, n+2) \to X_2 \to X_1$.

By construction $d_{n+3}(\iota_{n+2}) = a$. Then, $d_{n+4}(\mathrm{Sq}^1\iota_{n+2}) = \mathrm{Sq}^1(a) = \beta_1(a) = b + k \cdot p^*(\mathrm{Sq}^4\iota_n)$, and $d_{n+5}(\mathrm{Sq}^2\iota_{n+2}) = \mathrm{Sq}^2(a)$. We must verify if $\mathrm{Sq}^2(a) = c$ or 0. Note that $a = i^*(\mathrm{Sq}^2\iota_{n+1})$, so $\mathrm{Sq}^2(a) = i^*(\mathrm{Sq}^{2,2}\iota_{n+1}) = i^*(\mathrm{Sq}^{3,1}\iota_{n+1}) = i^*(c)$. Hence, $d_{n+5}(\mathrm{Sq}^2\iota_{n+2}) = \mathrm{Sq}^2 a = c$.

Similarly, $d_{n+6}(\mathrm{Sq}^{2,1}\iota_{n+2}) = d + m \cdot p^*\mathrm{Sq}^6\iota_n$, for some $m \in \mathbb{Z}$, since

$$i^*(\mathrm{Sq}^{2,1}a) = \mathrm{Sq}^{2,1}(i^*a) = \mathrm{Sq}^{2,1,2}(\iota_{n+1}) = \left(\mathrm{Sq}^5 + \mathrm{Sq}^{4,1}\right)(\iota_{n+1}) = i^*(d).$$

Hence, $\mathrm{Sq}^{2,1}a = d + m \cdot p^*\mathrm{Sq}^6\iota_n$ where $m \in \mathbb{Z}$. The same argument gives us $d_{n+6}(\mathrm{Sq}^3\iota_{n+2}) = 0 + n \cdot p^*\mathrm{Sq}^6\iota_n$, for some $n \in \mathbb{Z}$. Notice that $\mathrm{Sq}^3\iota_{n+2} = \mathrm{Sq}^{1,2}\iota_{n+2}$. Thus,

$$d_{n+6}(\mathrm{Sq}^3\iota_{n+2}) = \mathrm{Sq}^1\left(d_{n+5}(\mathrm{Sq}^2\iota_{n+2})\right) = \mathrm{Sq}^1(c).$$

But $\mathrm{Sq}^1(c) = 0$ by Lemma 1. Therefore, $d_{n+6}(\mathrm{Sq}^3\iota_{n+2}) = 0$. Now it follows that $\mathrm{Coker}(d_{n+6}) = \mathbb{Z}/2\mathbb{Z}\{(p_2)^*p^*\mathrm{Sq}^6(\iota_n)\}$.

Therefore, this computation gives us that $H^{n+3}(X_2; \mathbb{Z}/2\mathbb{Z}) = 0$ and $H^{n+4}(X_2; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}\{(p_2)^*p^*\mathrm{Sq}^4\iota_n\}$, assuming WLOG that $k = 0$ (since $\dim \mathrm{Coker}(\tau) = 1$, and thus we can identify the corresponding generator with $(p_2)^*p^*\mathrm{Sq}^4\iota_n$). Analogously, $H^{n+5}(X_2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{(p_2)^*p^*\mathrm{Sq}^6(\iota_n)\}$.

In summary, we have computed the following partial basis for $H^{n+i}(X_2; \mathbb{Z}/2\mathbb{Z})$.

$$\begin{aligned}
&H^{n+3}(X_2; \mathbb{Z}/2\mathbb{Z}) = 0, \\
&H^{n+4}(X_2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{(p_2)^*p^*\mathrm{Sq}^4\iota_n\}, \\
&H^{n+5}(X_2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{A\},
\end{aligned}$$

where

$$i^*(A) = \mathrm{Sq}^3(\iota_{n+2}).$$

Therefore, by the homotopy exact sequence, we obtain that $\pi_{n+2}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$.

We will need the following Bockstein relations.

Lemma 2 (Bockstein relations for $H^*(X_2; \mathbb{Z}/2\mathbb{Z})$)**.** In $H^*(X_2; \mathbb{Z}/2\mathbb{Z})$, $\beta_3\left((p_2)^*p^*\mathrm{Sq}^4\iota_n\right) = A$, where $A \in H^{n+5}(X_2; \mathbb{Z}/2\mathbb{Z})$ is defined by $i^*A = \mathrm{Sq}^3\iota_{n+2}$.

Proof. By the Bockstein lemma (Lemma 2) and Lemma 1, we have

$$\beta_2 p^* \mathrm{Sq}^4 \iota_n = c = d_{n+5}(\mathrm{Sq}^2 \iota_{n+2}) \in H^{n+5}(X_1; \mathbb{Z}/2\mathbb{Z}).$$

Thus,

$$i^* \left(\beta_3 (p_2)^* p^* \mathrm{Sq}^4 \iota_n\right) = \beta_1 (\mathrm{Sq}^2 \iota_{n+2}) = \mathrm{Sq}^3 \iota_{n+2} = i^* A.$$

■

Now, we would try to kill $H^{n+4}(X_2; \mathbb{Z}/2\mathbb{Z})$ representing $(p_2)^* p^* \mathrm{Sq}^4 \iota_n$ as a map $X_2 \to K(\mathbb{Z}/2\mathbb{Z}; n+4)$. However, it will follow that $d_{n+5}(\mathrm{Sq}^1 \iota_{n+3}) = 0$, since $\beta_1 (p_2)^* p^* \mathrm{Sq}^4 \iota_n = 0$. Since $\beta_1((p_2)^* p^* \mathrm{Sq}^4 \iota_n) = \beta_2((p_2)^* p^* \mathrm{Sq}^4 \iota_n) = 0$, by Proposition 1, $(p_2)^* p^* \mathrm{Sq}^4 \iota_n$ corresponds to the mod 2 reduction of some $\mathbb{Z}/8\mathbb{Z}-$cohomology class. Let $C \in H^{n+4}(X_2; \mathbb{Z}/8\mathbb{Z})$ be such class. Consider a map $X_2 \to K(\mathbb{Z}/8\mathbb{Z}, n+4)$ representing C.

Analogously as before, consider the following diagram that gives us X_3.

$$\begin{tikzcd}
 & K(\mathbb{Z}/8\mathbb{Z}, n+3) \arrow[d, "i"] & K(\mathbb{Z}/8\mathbb{Z}, n+3) \arrow[d] \\
 & X_3 \arrow[r] \arrow[d, "p_3"] & P \arrow[d] \\
S^n \arrow[ur, "\exists f_3"] \arrow[r, "f_2"] & X_2 \arrow[r, "\overline{(p_2)^* p^* \mathrm{Sq}^4}"] & K(\mathbb{Z}/8\mathbb{Z}, n+4)
\end{tikzcd}$$

Then $H^{n+4}(K(\mathbb{Z}/8\mathbb{Z}, n+3); \mathbb{Z}/2\mathbb{Z})$ is generated by $\beta_3 \iota_{n+3}$ which transgressses to $A = \beta_3 (p_2)^* p^* \mathrm{Sq}^4 \iota_n$. Applying the cohomological Serre spectral sequence, and our computation of $H^*(K(\mathbb{Z}/2^r\mathbb{Z}, m); \mathbb{Z}/2\mathbb{Z}) \cong H^*(K(\mathbb{Z}/2\mathbb{Z}, m); \mathbb{Z}/2\mathbb{Z})$ (as graded ring algebras) brings us:

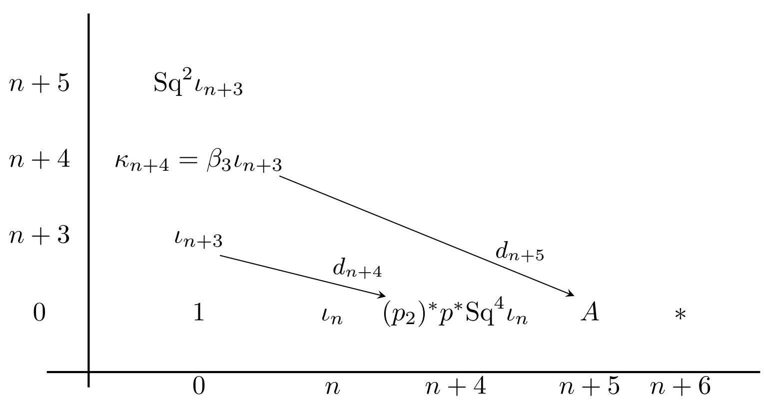

Figure 3. Serre spectral sequence associated to $K(\mathbb{Z}/8\mathbb{Z}, n+3) \to X_3 \to X_2$.

By the homotopy exact sequence, it follows that $\pi_{n+3}(X_3) = \mathbb{Z}/8\mathbb{Z}$. By the $\mathcal{C}_p-$approximation theorem, we conclude that $\pi_{n+3}(S^n)_{(2)} \cong \mathbb{Z}/8\mathbb{Z}$, for $n \gg 1$. □

References

[1] R. E. Mosher, M. C. Tangora, "Cohomology Operations and Applications in Homotopy Theory", Harper and Row, New York, 1968.

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