# A guide for the Adams spectral sequence: a computational approach and applications

submitted by

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**Abstract**

The primary goal of this thesis is to provide a detailed guide to the Adams spectral sequence and exemplify its importance through applications to current research problems using `minrv1` [Mor24], a program that computes free $\mathcal{A}_p$−minimal resolutions and queries over the computed Yoneda products, developed by the author. We will see in Proposition 3.10.2.1 that the Adams spectral sequence is still useful even for equivariant problems, providing insight into the homotopy type of representation spheres, which are as ubiquitous as the sphere spectrum, extending some existing computations [Muñ24, Proposition 6.7].

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# Contents

# List of Figures

# Introduction

This work addresses the stable homotopy groups of the spheres that can be thought of as a particular instance of homotopy theory with the main advantage of being more computationally tractable. In fact, these groups form a generalized homology theory on a suitable category which we call the *(homotopy) category of spectra* or the *stable homotopy category*, allowing us to apply homological techniques such as multiplicative spectral sequences. The category of spectra appeared later than the stable homotopy functor, originally defined for spaces. This notion is a consequence of the Freudenthal Suspension Theorem, proven by H. Freudenthal in 1937 [Fre37], which describes the stable phenomena.

**Theorem 0.0.0.1** (Freudenthal Suspension Theorem (pp. 299-314, [Fre37]))**.** Let $X$ be a $(n-1)-$connected space, based on $x_0 \in X$. Then the suspension map

$$\begin{aligned} \Sigma\colon \pi_i(X) &\to \pi_{i+1}(\Sigma X) \\ [f\colon S^i \to X] &\mapsto [\Sigma f\colon \Sigma S^i \cong S^{i+1} \to \Sigma X] \end{aligned}$$

is a surjective morphism for all $0 \leq i \leq 2n-1$ and an isomorphism for all $0 \leq i \leq 2n-2$.

Nowadays it can be thought of as a foundational result in the subject. Its proof involves the Hurewicz Theorem, which constructs a bridge between homotopy and homology under an appropriate connectedness assumption. The range stated in the previous result is called the *stable range*, where homotopy satisfies the excision axiom, which is a property involved in the definition of a homology theory.

During that decade, H. Hopf also did some of the first computations regarding the homotopy groups of the spheres via Hopf fibrations. However, there was not a general procedure to compute homotopy groups, and to construct suitable fibrations was quite challenging. Additionally, via Serre classes, J. P. Serre had proven different results regarding the finiteness properties of the homotopy groups of spheres, together with its relation to the corresponding finiteness properties of its (co)homology. These finiteness properties offered us hope to compute such invariants.

Although these advancements were remarkable, the usual cup product structure on the cohomology ring was not enough to determine the mod $p$ cohomology of $K(\mathbb{Z}/p\mathbb{Z}, n)$, $n \geq 0$. This issue was solved by the notion of cohomology operations, with the Steenrod operations being the most important example. These operations were compatible with the suspension construction and other cases of interest, allowing us to study the stable case, being a reliable replacement for the cup product structure. Moreover, these operations were proven to be compatible with some special differentials, called the transgression, and with the suspension functor. This led to computing the mod $p$ cohomology of Eilenberg-Mac Lane spaces via the Serre spectral sequence as can be seen in J. P. Serre and H. Cartan's works [Car54], [Car55]. For these achievements and other works, J. P. Serre was awarded the Fields Medal in 1954. The role of the cohomology of Eilenberg-Mac Lane spaces was not restricted just to computing the homotopy groups of spheres. The mod $p$ cohomology of $K(\mathbb{Z}/p\mathbb{Z}, n)$ is also isomorphic to a portion of the Steenrod Algebra, which organizes the Steenrod operations as a Hopf Algebra. These computations were successful and even dealt with some unstable homotopy groups. However, as occurs with every spectral sequence, J. P. Serre struggled with extension problems that were unclear how to solve.

This led to a rearrangement of the theory expressed in the form of the classical Adams spectral sequence and the category of spectra. The category of spectra is a more suitable domain for the stable homotopy functor as it brings a better description of the stable homotopy phenomena. The Adams spectral sequence is a specialized spectral sequence to compute stable homotopy groups, resembling an algorithm that even led to computational approaches. However, it must not be confused with an algorithm since it is not clear whether it is possible to determine all the $E_r-$differentials. Further developments like the Adams-Novikov spectral sequence and recent works including [IWX23] bring us the following table describing the known range of $(\pi_k^S)^\wedge_{(p)}$.

| $p$ (prime) | known range | date |
|---|---|---|
| $p = 2$ | $0 \leq k \leq 90$ [IWX23] | 2023 |
| $p = 3$ | $0 \leq k \leq 103$ [Hat18], [Rav86] | 1975 |
| $p = 5$ | $0 \leq k \leq 999$ [Hat18], [Rav86] | |

**Table 1:** Known range of stable homotopy groups of $\mathbb{S}$.

By degree reasons in the Adams spectral sequence, these ranges are even better for larger primes $p$. Little is known about homotopy groups outside the stable range.

The primary goal of this thesis is to provide a detailed guide about how to use the Adams spectral sequence to compute some invariants called the stable homotopy groups related to a given spectrum. The Adams spectral sequence is the main tool to compute stable homotopy groups. One of the hardest open problems in stable homotopy theory is to determine the homotopy type $\pi_*^S$ of the sphere spectrum $\mathbb{S}$. The main result corresponds to Proposition 3.10.2.1. This work also illustrates how to perform several computations in lower degrees for $\pi_*^S$ (Theorem 3.9.0.1). Modern advancements [IWX23] in the knowledge of $\pi_*^S$ require further developments, such as the motivic Adams spectral sequence, the Adams-Novikov spectral sequence, and their corresponding comparison results.

In order to apply the Adams spectral sequence, the author developed a software called `minrv1`. It implements an algorithm [Mor24] to perform computer-assisted computations, including the construction of minimal resolutions describing the $E_2-$term of the spectral sequence, and computing its associated Yoneda products. These products are important for determining unknown differentials as they are derivations under this multiplicative structure. The information contained in the minimal resolution can also be used to compute $E_2-$Massey products, as can be seen in Proposition 3.7.0.3. These products are more technical, but maintain the same purpose as Yoneda products: to solve extension problems and determine unknown differentials.

The text distribution corresponds to the following.

- Chapter I introduces the preliminaries required to understand the construction of the Adams spectral sequence together with the motivation to define the stable homotopy functor, and more importantly, details why we are interested in computing the homotopy groups of the spheres, exemplified through the finiteness properties that these groups exhibit, and the later importance of the sphere spectrum $\mathbb{S}$. The topics discussed in this chapter involve applications of the Serre spectral sequence, following the lines of Serre's work about the homotopy of the spheres. First, we start with the Freudenthal Suspension Theorem, involving a more algebraic flavor to the homotopy functor in the *stable range*. Then, we discuss the notion of Serre classes, which allow us to understand how finiteness is preserved between homotopy and (co)homology of a given space. This journey includes important classical results such as the Hurewicz and the Whitehead theorems. With the aid of Serre classes, we are close to the construction of localization of spaces, and this can be seen through the results of rational homotopy and the minimal $p-$torsion in the (stable) homotopy of the spheres. Finally, we move on to the mod 2 cohomology of Eilenberg-Mac Lane spaces using the Serre spectral sequence and the mod 2 Steenrod operations since these spaces were relevant in previous advances and have a deep relation with cohomology operations. These results imply a considerable conceptual jump involving an infinite family of cohomology operations interacting with the differentials of the Serre spectral sequence rather than just the cup product.

- Chapter II consists of a small chapter describing Boardman's stable homotopy category of spectra, following the lines given in [Hat04] and [Swi75]. There are modern approaches, but this allows us to have a notion about the origins of the category of spectra, being another leap of insight that currently relates stable homotopy theory to Algebraic Geometry and Number theory by providing a general framework to treat the stable phenomena.

- Chapter III describes the classical Adams spectral sequence, including a linear exposure of both theoretical and computational results, including Moss' convergence theorems for Yoneda and Massey products (see [Mos68], [Mos70]), as well as a finiteness result consisting of the Adams vanishing line and an algorithm to construct minimal free $\mathcal{A}_p-$resolutions to compute the $E_2-$term and its products. The author suggests reading this chapter non-linearly, focusing on the examples, and returning to theoretical considerations when required. The first example is split into two parts as the theory develops, including all the required to perform computations. The second example, which involves representation spheres and $G-$spectra, is intended to illustrate how we can apply the Adams spectral sequence to current research even in the context of $G-$spectra. The source code written by the author and instructions to reproduce minimal resolutions are provided in [Mor24].

Finally, we specify further improvements highlighting the weaknesses of a classical approach for the Adams spectral sequence and future work: to write more efficient algorithms using GPU acceleration for the mod $p$ linear algebra routines, and explore how to implement a synthetic approach [Pst22] with a possible use of the secondary Steenrod algebra. Moreover, computing stable homotopy groups of representation spheres seems to be promising.

# Chapter 1

# Preliminaries

This chapter assumes familiarity with the Serre spectral sequence (see [RW21] for a comprehensive treatment) and the stable homotopy functor for spaces (pp. 384 [Hat02]). Nevertheless, this chapter will deal mostly with the unstable case, and we will not require prior knowledge of the stable homotopy functor except that it is a generalized homology theory on the category of pointed spaces. It is defined as the colimit of the directed system obtained by taking iterated suspensions and inclusions of a given space and it satisfies the wedge axiom.

## 1.1 The Freudenthal Suspension Theorem

### 1.1.1 Introduction

This section is intended to give a complete proof of the Freudenthal Suspension Theorem using the Serre spectral sequence, following the lines of [RW21]. The books [MT68], [Swi75] and [Rog10] are followed as a complement.

The main idea of the proof is to factorize the excision isomorphism in homology as the composite between the map induced by the unit of the suspension-loop adjunction, and the map induced by the path-loop fibration $\mathrm{ev}_1\colon P_*X \to X$. Such factorization, together with the Hurewicz Theorem, will allow us to construct a bridge between (stable) homotopy and homology. The technical tool that will let us prove the theorem corresponds to the Serre spectral sequence, together with its transgression, which measures the failure between homotopy and (co)homology.

### 1.1.2 Preliminaries

A useful preliminary is the *Serre exact sequence*. We could prove the Freudenthal Suspension Theorem directly, by just considering the Serre spectral sequence, however, the Serre exact sequence gives a cleaner proof. The following results are from [Swi75].

**Proposition 1.1.2.1** (Edge morphisms)**.** Let $F \to E \xrightarrow{p} B$ be a Serre fibration with path$-$connected basis $B$, such that $\pi_1(B, b_0)$ acts trivially on $H_*(F; G)$. Then

(a) The morphism $i_*\colon H_q(F) \to H_q(E)$ can be factored as

$$H_q(F) \cong E^2_{0,q} \to E^\infty_{0,q} \cong F_{0,q} \to H_q(E),$$

where the map on the left corresponds to the projection onto the infinity$-$page, and the right map is the inclusion.

(b) The morphism $p_*\colon H_p(E) \to H_p(B)$ can be factored as

$$H_p(E) \to E^\infty_{p,0} \to E^2_{p,0} \cong H_p(B),$$

where the left map is the projection $H_p(E) \cong F_{p,0} \to E^\infty_{p,0}$, and the right map is the inclusion.

*Proof.* Since the Serre spectral sequence is a first-quadrant spectral sequence, it follows that $E^r_{p,q} = 0$ for all $p < 0$. Therefore, $d^r \colon E^r_{0,q} \to E^r_{-r,q+r-1} = 0$ is trivial. Moreover, $E^{r+1}_{0,q}$ is a quotient of $E^r_{0,q}$. Take the surjective morphism $E^2_{0,q} \to E^\infty_{0,q}$ given by the composite between the projections $E^k_{0,q} \to E^{k+1}_{0,q}$. Note that $E^\infty_{0,q} = F_{0,q} \subseteq H_q(E)$ (since $E^\infty_{p,q} = 0$ for $p < 0$). We will prove the first assertion.

Let $p' \colon F \to \{b_0\}$ and the map between fibrations

$$\begin{tikzcd}
F \arrow[r, "i"] \arrow[d, "p'"] & E \arrow[d, "p"] \\
\{b_0\} \arrow[r, hook] & B
\end{tikzcd}$$

Note that $p' \colon F \to \{b_0\}$ corresponds to a Serre fibration with $1-$connected basis. Let ${E^r_{p,q}}'$ be the associated Serre spectral sequence. The following diagram commutes by naturality

$$\begin{tikzcd}
H_q(F) \cong E^2_{0,q} \arrow[r] & E^\infty_{0,q} \arrow[r] & H_q(E) \\
H_q(F) \cong {E^2_{0,q}}' \arrow[r] \arrow[u, "\mathrm{Id}"] & {E^\infty_{0,q}}' \arrow[r] \arrow[u, "i_*"] & H_q(E) \arrow[u, "i_*"]
\end{tikzcd}$$

Since ${E^2_{k,q}}' = 0$ for all $k \neq 0$, we have that ${E^\infty_{0,q}}' = H_q(F)$. By construction of the infinity-page, and by definition of the filtration, it follows that the morphism in the lower row is given by the identity, proving (a).

For the (b) statement, since the Serre spectral sequence is a first-quadrant spectral sequence, $E^r_{p,q} = 0$ for $q < 0$. Then, $E^{r+1}_{p,0} = \mathrm{Ker}(d^r) \subseteq E^r_{p,0}$. So $E^\infty_{p,0} \subseteq E^2_{p,0}$. Moreover, since $E^\infty_{p,q} = 0$ for $q < 0$, it follows that $F_{p,0} = H_p(E)$. Then, we have a surjective morphism $H_p(E) = F_{p,0} \to E^\infty_{p,0}$. The proof is similar to (a). Take the following map of fibrations

$$\begin{tikzcd}
E \arrow[r, "p"] \arrow[d, "p"] & B \arrow[d, "\mathrm{Id}"] \\
B \arrow[r, "\mathrm{Id}"] & B
\end{tikzcd}$$

By naturality of the Serre spectral sequence

$$\begin{tikzcd}
H_p(E) \arrow[r] \arrow[d, "p_*"] & E^\infty_{p,0} \arrow[r] \arrow[d, "p_*"] & E^2_{p,0} \cong H_p(B) \arrow[d, "="] \\
F'_{p,0} = H_p(B) \arrow[r] & {E^\infty_{p,0}}' \arrow[r] & {E^2_{p,0}}' \cong H_p(B)
\end{tikzcd}$$

Since the fiber of $\mathrm{Id} \colon B \to B$ is contractible, it follows that in the associated spectral sequence, all the rows are trivial except for the lower row. By convergence, ${E^\infty_{p,0}}' = H_p(B)$, implying that both morphisms in the below part of the diagram are the identity, proving the result. □

**Definition 1.1.2.1** (Edge morphism)**.** The morphisms of the previous result are called *edge morphisms.*

We have a homology analog of the homotopy exact sequence of a fibration. The following result was proven in [Rog10].

**Theorem 1.1.2.1** (Homology Serre exact sequence)**.** Let $F \to E \xrightarrow{p} B$ be a Hurewicz fibration, with $B$ simply−connected, and $F$ path−connected. Assume that $H_s(B) = 0$ for $1 \leq s < u$ and $H_t(F) = 0$ for $1 \leq t < v$. Then, there exists an exact sequence

$$H_{u+v-1}(F) \xrightarrow{i_*} H_{u+v-1}(E) \xrightarrow{p_*} H_{u+v-1}(B) \xrightarrow{d} H_{u+v-2}(F) \xrightarrow{i_*} H_{u+v-2}(E) \xrightarrow{p_*} H_{u+v-2}(B) \xrightarrow{d} \cdots \to H_1(E) \to 0,$$

where the boundary morphism $d\colon H_{u+v-k}(B) \to H_{u+v-k-1}(F)$ corresponds to the transgression.

*Proof.* By hypothesis, we can apply the Serre spectral sequence. By the description theorem of the $E^2$−term and the Universal Coefficients Theorem, we have that $E^2_{p,q}$ has the following description

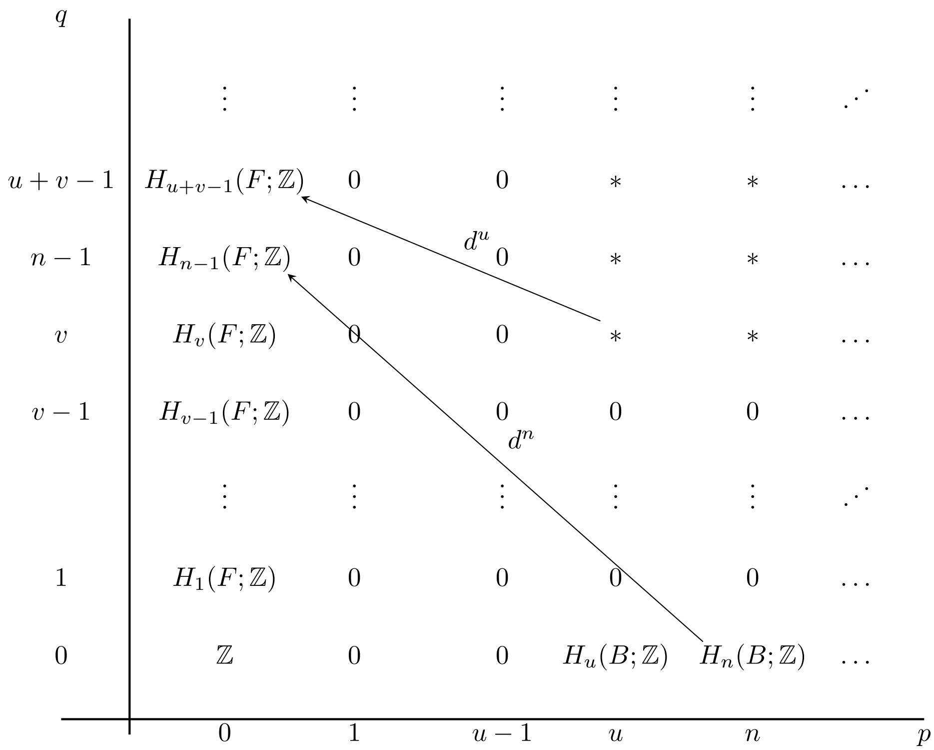


**Figure 1.1:** $E^2_{\bullet,\bullet}$ associated to the fibration $p$.

Denote by $*$ to the remaining possible non-trivial entries. Let $n < u+v$. Consider the transgressions $d^n\colon H_n(B) = E^n_{n,0} \to E^n_{0,n-1} = H_{n-1}(F)$. The first possibly non-trivial differential with total degree $u+v$ is $d^u\colon E^u_{u,v} \to H_{u+v-1}(F)$, as illustrated. This translates into the finite range stated in the theorem. Since the terms on the $E^2$−page with total degree $n$ are trivial except for the edges, by convergence we have the short exact sequence

$$0 \to E^\infty_{0,n} \to H_n(E) \to E^\infty_{n,0} \to 0,$$

where

$$E^\infty_{0,n} = \operatorname{Coker}\left(d^{n+1}\colon H_{n+1}(B) = E^{n+1}_{n+1,0} \to E^{n+1}_{0,n} = H_n(F)\right),$$
$$E^\infty_{n,0} = \operatorname{Ker}\left(d^n\colon H_n(B) = E^n_{n,0} \to E^n_{0,n-1} = H_{n-1}(F)\right).$$

By the previous proposition, $p_*\colon H_n(E) \to E^\infty_{n,0} \xrightarrow{i_*} H_n(B)$. Then $H_n(E) \xrightarrow{p_*} H_n(B) \xrightarrow{d^n} H_{n-1}(F)$ is exact. Similarly, $i_*\colon H_n(F) \to E^\infty_{0,n} \to H_n(E)$. Thus, $\operatorname{Im}(d^{n+1}) = \operatorname{Ker}(i_*)$, implying that

$$H_{n+1}(B) \xrightarrow{d^{n+1}} H_n(F) \xrightarrow{i_*} H_n(E) \xrightarrow{p_*} H_n(B) \xrightarrow{d^n} H_{n-1}(F)$$

is exact, for each $1 \leq n \leq u+v-2$. For $n = u+v-1$, note that $\operatorname{Cod}(d^{u+v})$ is quotient of $H_n(F)$, which translates into the desired truncated long exact sequence. $\square$

### 1.1.3 Freudenthal Suspension Theorem

At this point, we have the required tools to prove the Freudenthal Suspension Theorem at our disposal.

**Theorem 1.1.3.1** (Freudenthal Suspension Theorem)**.** Let $X$ be a $(n-1)-$connected space, based on $x_0 \in X$. Then the suspension map

$$\begin{aligned} \Sigma\colon \pi_i(X) &\to \pi_{i+1}(\Sigma X) \\ [f\colon S^i \to X] &\mapsto [\Sigma f\colon \Sigma S^i \cong S^{i+1} \to \Sigma X] \end{aligned}$$

is a surjective morphism for all $0 \leq i \leq 2n-1$ and an isomorphism for all $0 \leq i \leq 2n-2$.

*Proof.* Consider the path$-$loop fibration

$$\Omega\Sigma X \longrightarrow P_*\Sigma X \xrightarrow{\mathrm{ev}_1} \Sigma X,$$

and notice that $\Sigma X$ is $n-$connected (this can be proved with the Hurewicz Theorem, and the isomorphism $\tilde{H}_*(\Sigma X) \cong \tilde{H}_{*-1}(X)$), thus $\Omega\Sigma X$ is $(n-1)-$connected. This is enough to prove the case $i=0$. Since $\mathrm{ev}_1\colon P_*\Sigma X \to \Sigma X$ is a Serre fibration, and has $n-$connected basis $\Sigma X$ $(n \geq 1)$, by the Serre spectral sequence, we have that $E^2_{p,q} \cong H_p(\Sigma X; H_q(\Omega\Sigma X;\mathbb{Z}))$. Now, note that $\Omega\Sigma X$ is $(n-1)-$connected. By the Hurewicz Theorem, $\tilde{H}_j(\Omega\Sigma X;\mathbb{Z}) = 0$ for $0 \leq j \leq n-1$. On the other hand, by the Universal Coefficients Theorem, $H_p(\Sigma X; H_q(\Omega\Sigma X;\mathbb{Z})) \cong H_p(\Sigma X) \otimes_{\mathbb{Z}} H_q(\Omega\Sigma X) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(\Sigma X), H_q(\Omega\Sigma X))$. Similarly, by the Hurewicz Theorem, $\tilde{H}_j(\Sigma X;\mathbb{Z}) = 0$ for $0 \leq j \leq n$. Additionally, note that both the fiber and the basis are path$-$connected. Then, $H_0(\Sigma X) \cong \mathbb{Z}$, which is a free module. In summary

$$H_p(\Sigma X; H_q(\Omega\Sigma X;\mathbb{Z})) \cong (0), \qquad 1 \leq p \leq n, \text{ or } 1 \leq q \leq n-1.$$

In consequence, $E^2_{\bullet,\bullet}$ is given by

$$\begin{array}{c|cccccc}
q & & & & & & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
n+1 & H_{n+1}(\Omega\Sigma X;\mathbb{Z}) & 0 & 0 & H_{n+1}(\Sigma X; H_{n+1}(\Omega\Sigma X;\mathbb{Z})) & H_{n+2}(\Sigma X; H_{n+1}(\Omega\Sigma X;\mathbb{Z})) & \cdots \\
n & H_n(\Omega\Sigma X;\mathbb{Z}) & 0 & 0 & H_{n+1}(\Sigma X; H_n(\Omega\Sigma X;\mathbb{Z})) & H_{n+2}(\Sigma X; H_n(\Omega\Sigma X;\mathbb{Z})) & \cdots \\
n-1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \mathbb{Z} & 0 & 0 & H_{n+1}(\Sigma X; H_0(\Omega\Sigma X;\mathbb{Z})) & H_{n+2}(\Sigma X; H_0(\Omega\Sigma X;\mathbb{Z})) & \cdots \\
\hline
 & 0 & 1 & n & n+1 & n+2 & p
\end{array}$$

$d^{n+1}$, $d^{n+2}$

**Figure 1.2:** $E^2_{\bullet,\bullet}$ associated to $\mathrm{ev}_1$.

Notice that the diagram includes the transgressions $d^{n+1}$ and $d^{n+2}$ on their respective pages. Since the total space of $\mathrm{ev}_1\colon P_*\Sigma X \to \Sigma X$ is contractible, $E^\infty_{p,q} = 0$ except by $p=q=0$, where $E^\infty_{p,q} \cong \mathbb{Z}$. Observe that the differentials with codomain on the entry $(0,n)$ are trivial, except for the transgression $d^{n+1}$. Since $E^\infty_{0,n} = 0$ it follows that

$d^{n+1}\colon E^{n+1}_{n+1,0} \to E^{n+1}_{0,n} \cong H_n(\Omega\Sigma X;\mathbb{Z})$ is surjective. On the other hand, all the differentials exiting from the entry $(n+1,0)$ are trivial, except maybe for $d^{n+1}$. All the differentials entering $(n+1,0)$ are trivial. Therefore, that entry remains constant until $E^{n+1}_{\bullet,\bullet}$. Additionally, by convergence, $\mathrm{Ker}(d^{n+1}\colon E^{n+1}_{n+1,0} \cong H_{n+1}(\Sigma X;\mathbb{Z}) \to E^{n+1}_{0,n} \cong H_n(\Omega\Sigma X;\mathbb{Z})) = 0$, implying that $d^{n+1}$ is an isomorphism.

Analogously, the transgressions $d^j\colon H_j(\Sigma X;\mathbb{Z}) \to H_{j-1}(\Omega\Sigma X;\mathbb{Z})$ are isomorphisms for each $n+1 \le j \le 2n$. The argument is not valid for the entry $(0,2n)$, because it is hit by $d\colon E^{n+1}_{n+1,n} \to E^{n+1}_{0,2n}$ and $d\colon E^{2n+1}_{2n+1,0} \to E^{2n+1}_{0,2n}$, and it is not possible to ensure that $E^{n+1}_{n+1,n} = 0$. This restriction translates into the stipulated range of the theorem. Moreover, it is clear that $d^j = 0$ are isomorphisms, for each $2 \le j \le n$.

Now, since the transgressions $d^j$ are isomorphisms, we have the following situation

$$\begin{tikzcd}
 & H_j(P_*\Sigma X, \Omega\Sigma X) \arrow[r, "\cong"] \arrow[d, "\mathrm{ev}_{1*}"] & H_{j-1}(\Omega\Sigma X;\mathbb{Z}) \\
H_j(\Sigma X;\mathbb{Z}) \arrow[r, "\cong"] & H_j(\Sigma X, *) &
\end{tikzcd}$$

The left isomorphism is given by the characterization result of the transgression. The right isomorphism follows from the homology long exact sequence of the pair, together with the fact that $P_*\Sigma X \cong 0$. Since the transgressions are isomorphisms

$$\mathrm{ev}_{1*}\colon H_j(P_*\Sigma X, \Omega\Sigma X) \to H_j(\Sigma X, *)$$

is also an isomorphism, where $2 \le j \le 2n$.

Now, we require the following ingredient: the unit map of the suspension-loop adjunction. Define

$$\begin{aligned} \hat{f}\colon X &\to \Omega\Sigma X \\ x &\mapsto \hat{f}(x)(s) = [s,x], \quad s \in [0,1], \end{aligned}$$

and

$$\begin{aligned} f\colon CX &\to P_*\Sigma X \\ [t,s] &\mapsto (s \mapsto [st,x]), \quad s,t \in [0,1]. \end{aligned}$$

We are considering $CX, \Sigma X$ as quotients of $[0,1] \times X$. Also, we are considering $\Sigma X$ as a quotient of $CX$. This gives the following commutative square

$$\begin{tikzcd}
X \arrow[r, "\hat{f}"] \arrow[d, hook, "{[1,x]}"] & \Omega\Sigma X \arrow[d, hook] \\
CX \arrow[r, "f"] & P_*\Sigma X
\end{tikzcd}$$

By naturality of the homology long exact sequence, the following diagram commutes

$$\begin{tikzcd}
H_j(CX, X) \arrow[r, "\cong"] \arrow[d, "(f,\hat{f})_*"] \arrow[dd, bend right=60, "i_*"'] & H_{j-1}(X) \arrow[d, "\hat{f}_*"] \\
H_j(P_*\Sigma X, \Omega\Sigma X) \arrow[r, "\cong"] \arrow[d, "\mathrm{ev}_{1*}"] & H_{j-1}(\Omega\Sigma X) \\
H_j(\Sigma X, *) &
\end{tikzcd}$$

The isomorphisms given by the boundary differentials come from the fact that $CX, P_*\Sigma X \cong *$ Moreover, the left diagram commutes because $\mathrm{ev}_1 \circ f$ is homotopy equivalent to the inclusion, induced by the excision isomorphism. Taking $2 \leq j \leq 2n$ we deduce that $(f, \hat{f})_*$ is an isomorphism, i.e.

$$\hat{f}_* \colon H_{j-1}(X) \to H_{j-1}(\Omega\Sigma X)$$

is an isomorphism for each $1 \leq j-1 \leq 2n-1$. The case $n=0$ is trivial considering the argument for $i=0$. The $n=1$ case follows from the Hurewicz Theorem for $\pi_1$. Indeed, consider the following commutative square obtained by naturality of the Hurewicz morphism

$$\begin{tikzcd}
\pi_1(X) \arrow[r, "\hat{f}_*"] \arrow[d, two heads, "h"] & \pi_1(\Omega\Sigma X) \arrow[d, two heads, "h"] \\
H_1(X) \cong \pi_1(X)^{\mathrm{ab}} \arrow[r, "\hat{f}_*"] & H_1(\Omega\Sigma X) \cong \pi_1(\Omega\Sigma X)^{\mathrm{ab}} \cong \pi_1(\Omega\Sigma X) \cong \pi_2(\Sigma X)
\end{tikzcd}$$

This implies that the morphism in the lower row is an isomorphism. Moreover, the right morphism $h$ is an isomorphism because $\pi_2(\Sigma X)$ is abelian. By commutativity, it follows that $\hat{f}_* \colon \pi_1(X) \to \pi_1(\Omega\Sigma X)$ is surjective. Moreover,

$$\Sigma \colon \pi_1(X) \to \pi_2(\Sigma X)$$

is surjective, by the following commutative diagram

$$\begin{tikzcd}
\pi_1(X) \arrow[rr, "\Sigma"] \arrow[dr, "\hat{f}_*"'] & & \pi_2(\Sigma X) \\
& \pi_1(\Omega\Sigma X) \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

where

$$\begin{aligned} \sigma \colon \pi_1(\Omega\Sigma X) &\xrightarrow{\cong} \pi_2(\Sigma X) \\ [y \mapsto \gamma_y] &\mapsto [[t, y] \mapsto \gamma_y(t)]. \end{aligned}$$

Therefore, we have proven the statement for $n = 1$.

For the general case, assume that $n-1 \geq 1$, thus $\Omega\Sigma X$ is $1-$connected. Let $F$ be the homotopy fiber of $\hat{f}$. Up to homotopy, we can replace $\hat{f}$ with a Hurewicz fibration. Now, we can apply the Serre spectral sequence. W.L.O.G., suppose $\hat{f} \colon X \to \Omega\Sigma X$ is a fibration. Then

$$E^2_{p,q} \cong H_p(\Omega\Sigma X; H_q(F;\mathbb{Z})) \cong H_p(\Omega\Sigma X) \otimes_{\mathbb{Z}} H_q(F) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(\Omega\Sigma X), H_q(F)).$$

Since $\Omega\Sigma X$ is $(n-1)-$connected, by Hurewicz Theorem, $\tilde{H}_k(\Omega\Sigma X) = 0$ for all $0 \leq k \leq n-1$. Analogously, $H_k(X) = 0$, $0 \leq k \leq n-1$. Moreover, we have that $p = 0$, $E^2_{0,q} \cong H_q(F)$ because the basis is path$-$connected. On the other hand, $q = 0$, $E^2_{p,0} \cong H_p(\Omega\Sigma X)$ because $\mathbb{Z}$ is a free $\mathbb{Z}-$module. Notice that for each $1 \leq k \leq n-1$, $E^2_{k,q} = 0$. Additionally, $E^2_{0,0} \cong \mathbb{Z}$, because the fiber is path$-$connected (the total space and the basis are $1-$connected).

Therefore, the $E^2-$page of this spectral sequence has the following form

$$\begin{tikzcd}
q & & & & & & \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
n & H_n(F;\mathbb{Z}) & 0 & 0 & * & * & \cdots \\
n-1 & H_{n-1}(F;\mathbb{Z}) & 0 & 0 & * & * & \cdots \\
n-2 & H_{n-2}(F;\mathbb{Z}) & 0 & 0 & * & * & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
1 & H_1(F;\mathbb{Z}) & 0 & 0 & * & * & \cdots \\
0 & \mathbb{Z} & 0 & 0 & H_n(\Omega\Sigma X;\mathbb{Z}) \arrow[uuulll, "d^n"'] & H_{n+1}(\Omega\Sigma X;\mathbb{Z}) \arrow[uuuullll, "d^{n+1}"'] & \cdots \\
 & 0 & 1 & n-1 & n & n+1 & & p
\end{tikzcd}$$

**Figure 1.3:** $E^2_{\bullet,\bullet}$ associated to $\hat{f}$.

The entries $*$ correspond to those that could be non-trivial. The diagram also depicts the transgressions with illustrative purposes.

Remember that $X$ is $(n-1)-$connected. We will show that $H_k(F;\mathbb{Z}) = 0$ for all $1 \leq k \leq n-2$. For the case $n = 2$ there is nothing to prove. For $n \geq 3$, observe that all the differentials entering and exiting from the $(0,1)-$entry are trivial. By convergence of the Serre spectral sequence, there is a short exact sequence

$$0 \to E^\infty_{0,1} \to H_1(X) \to E^\infty_{1,0} \to 0,$$

leading to $E^\infty_{0,1} = H_1(F;\mathbb{Z}) = 0$. Consequently, the first row is trivial. Inductively we obtain the desired result. For $k = n-1$, we cannot apply the same argument, because $d^n$ could be non-trivial. Thus, $E^2_{p,k} = 0$ for $1 \leq k \leq n-2$. The next figure resembles the $E^2-$page.

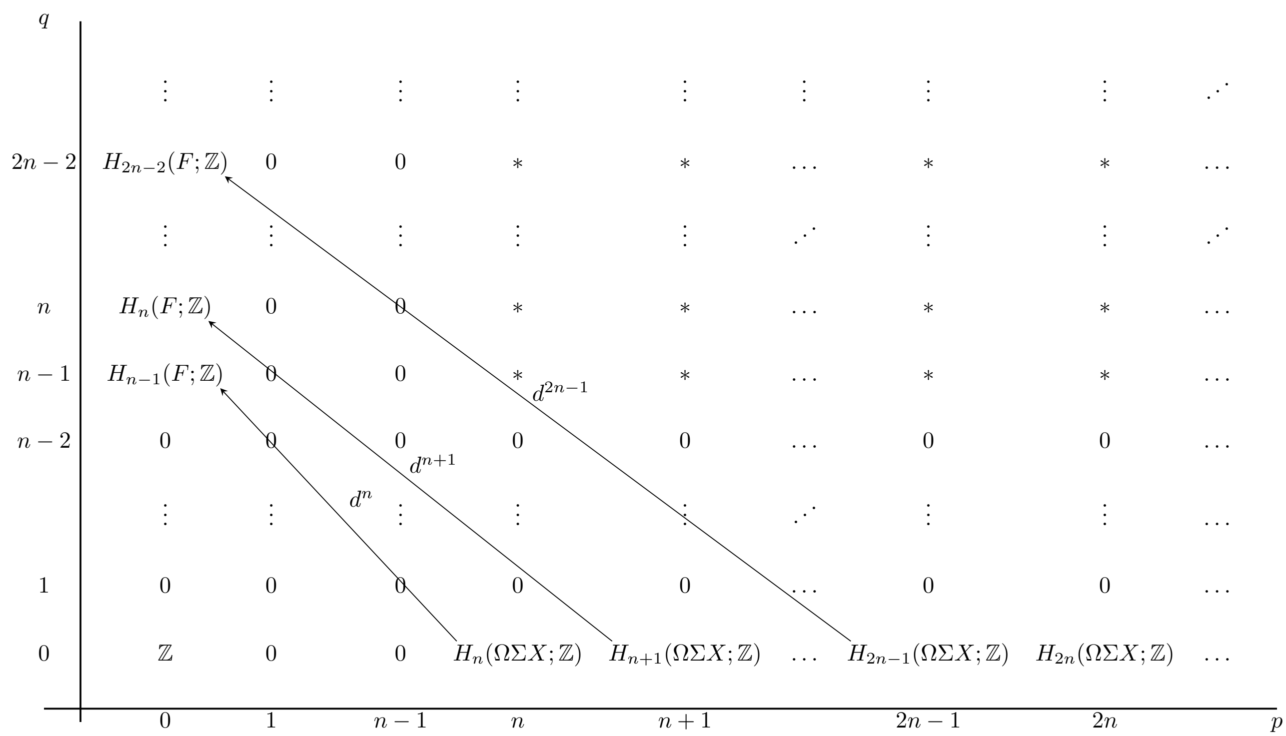


**Figure 1.4:** $E^2_{\bullet,\bullet}$ associated to $\hat{f}$.

Now, we will see that $H_k(F;\mathbb{Z}) = 0,\ 1 \leq k \leq n-2 < n-1 \leq k \leq 2n-2$. By the Serre exact sequence (which in this context can be deduced as a consequence of the Serre spectral sequence), since the fibration $\hat{f}\colon X \to \Omega\Sigma X$ is a Hurewicz fibration, with path$-$connected fiber and simply$-$connected basis, $H_k(\Omega\Sigma X;\mathbb{Z}) = 0$ for $1 \leq k \leq n-1$, and $H_l(F;\mathbb{Z}) = 0$ for $1 \leq l \leq n-2$, there exists an exact sequence

$$H_{2n-2}(F) \to H_{2n-2}(X) \xrightarrow{\hat{f}_*} H_{2n-2}(\Omega\Sigma X) \to H_{2n-3}(F) \to \cdots \to H_1(X) \to 0.$$

We know that $\hat{f}_*\colon H_l(X) \to H_l(\Omega\Sigma X)$ is an isomorphism for $1 \leq l \leq 2n-1$. Thus, $H_k(F;\mathbb{Z}) = 0$ for $1 \leq k \leq 2n-3$. This translates into having more trivial rows in the previous diagram. Applying the Serre exact sequence again implies that $H_{2n-2}(F;\mathbb{Z}) = 0$.

Finally, by the homotopy exact sequence associated to the fibration $\hat{f}$

$$\cdots \longrightarrow \pi_k(F) \longrightarrow \pi_k(X) \xrightarrow{\hat{f}} \pi_k(\Omega\Sigma X) \xrightarrow{\partial} \pi_{k-1}(F) \longrightarrow \ldots \tag{1.1}$$

Note that $\pi_1(F)$ is abelian since is quotient of $\pi_2(\Omega\Sigma X)$. By the Hurewicz Theorem, $\pi_1(F) \cong H_1(F;\mathbb{Z}) = 0$. Therefore, the fiber is 1$-$connected. By Hurewicz Theorem, $\pi_2(F) = 0$. Inductively, we have that $\pi_k(F) = 0$, for $1 \leq k \leq 2n-2$. By 1.1, it follows that $\hat{f}_*\colon \pi_k(X) \to \pi_k(\Omega\Sigma X)$ is an isomorphism for $1 \leq k \leq 2n-2$, and surjective for $1 \leq k \leq 2n-1$. Taking composite with $\sigma\colon \pi_t(\Omega\Sigma X) \to \pi_{t+1}(\Sigma X)$ proves the theorem. $\square$

## 1.2 Serre classes

### 1.2.1 Introduction

This section is intended to state and prove relevant results regarding Serre classes. They can be seen as a precursor to the localization of spaces. The main result of this section corresponds to the mod $\mathcal{C}$ Hurewicz theorem.

**Definition 1.2.1.1** (Serre classes)**.** A *Serre class* $\mathcal{C}$ is a class of abelian groups closed under taking subgroups, quotients, and forming extensions.

The following examples consist of the principal cases of interest.

**Example 1.2.1.1** (Important Serre classes)**.**

(a) The class of finitely generated abelian groups.

(b) The class of torsion abelian groups, where each element is killed by a product of prime numbers in some fixed set $\mathcal{P}$.

(c) The (sub)class of finite abelian groups in the previous class.

□

*Proof.*

(a) Since $\mathbb{Z}$ is noetherian, it follows that submodules of modules in $\mathcal{C}$ are finitely generated. As abelianness is preserved by taking subgroups, we have that $\mathcal{C}$ is closed under subgroups. The quotient case is immediate. The closedness under extensions follows from a standard result about finitely generated modules.

(b) This class is clearly closed under taking subgroups and quotients. Assume that $\mathcal{C}$ is not closed under extensions. Then the kernel contained in the central term of the associated short exact sequence must have an element of infinite order. By exactness, we obtain a contradiction.

(c) This proof is analogous to the previous case.

□

**Definition 1.2.1.2** (Mod $\mathcal{C}$ injectivity/surjectivity)**.** Let $\mathcal{C}$ be a Serre class. We say that a morphism $f\colon A \to B$ is

- *mod $\mathcal{C}$ injective* if $\mathrm{Ker}(f) \in \mathcal{C}$,
- *mod $\mathcal{C}$ surjective* if $\mathrm{Coker}(f) \in \mathcal{C}$,
- a *mod $\mathcal{C}$ isomorphism* if the previous conditions are met.

The next Serre subclass will be relevant in this section.

**Definition 1.2.1.3** (Serre ring)**.** We say that a Serre class is a *Serre ring* if it is closed under $\otimes_{\mathbb{Z}}$ $\mathrm{Tor}_1^{\mathbb{Z}}$.

The following lemma brings us a useful result for inductively spreading the property of belonging to a fixed Serre class through a fibration.

**Lemma 1.2.1.1.** Let $\mathcal{C}$ be a Serre ring (the previous examples satisfy this property). Let $p\colon E \to B$ be a Serre fibration with path-connected basis, and fiber $F = p^{-1}(b_0)$ such that $\pi_1(B, b_0)$ acts trivially on $H_*(F;\mathbb{Z})$. Then, if a pair between

$$\tilde{H}_*(F;\mathbb{Z}), \qquad \tilde{H}_*(E;\mathbb{Z}), \qquad \tilde{H}_*(B;\mathbb{Z})$$

belongs to $\mathcal{C}$, then the remaining homology also belongs to $\mathcal{C}$.

*Proof.* Firstly, suppose that $\tilde{H}_*(F;\mathbb{Z}), \tilde{H}_*(B;\mathbb{Z}) \in \mathcal{C}$. Since $B$ is $1-$connected, its fundamental group acts trivially on the homology of the fiber. Thus, by Serre spectral sequence, it follows that

$$E^2_{p,q} = H_p(B; H_q(F;\mathbb{Z})) \implies H_{p+q}(E;\mathbb{Z}).$$

Since $\mathcal{C}$ is closed under extensions, to prove that $H_{p+q}(E;\mathbb{Z}) \in \mathcal{C}$ for $p+q > 0$, it suffices to prove that $E^\infty_{p,q} \in \mathcal{C}$ for $p+q > 0$. Since $\mathcal{C}$ is closed under subgroups and subquotients, we just have to prove that $E^2_{p,q} \in \mathcal{C}$ where $p+q > 0$. By the Universal Coefficients Theorem,

$$0 \to H_p(B;\mathbb{Z}) \otimes H_q(F;\mathbb{Z}) \to H_p(B; H_q(F;\mathbb{Z})) \to \mathrm{Tor}(H_{p-1}(B;\mathbb{Z}), H_q(F;\mathbb{Z})) \to 0.$$

Since $\mathcal{C}$ is closed under $-\otimes_{\mathbb{Z}}-$ and $\mathrm{Tor}_1^{\mathbb{Z}}(-,-)$ (and $\mathbb{Z} \otimes_{\mathbb{Z}} A = A$, $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, A) = 0$), we obtain that $E^2_{p,q} \in \mathcal{C}$ for $p+q > 0$.

Now, assume that $\tilde{H}_*(F;\mathbb{Z}), \tilde{H}_*(E;\mathbb{Z}) \in \mathcal{C}$. The lower corner in the Serre spectral sequence gives us the following exact sequence

$$H_2(B;\mathbb{Z}) \xrightarrow{d^2} H_1(F;\mathbb{Z}) \to H_1(E;\mathbb{Z}) \to H_1(B;\mathbb{Z}) \to 0.$$

Therefore $H_1(B;\mathbb{Z})$ is a quotient of $H_1(E;\mathbb{Z})$, so that $H_1(B;\mathbb{Z}) \in \mathcal{C}$. To generalize this argument, we proceed inductively. Suppose that $H_p(B;\mathbb{Z}) \in \mathcal{C}$ for each $0 < p < k$ and fixed $k \geq 0$. Let $r \geq 0$. We want to exploit the convergence of the Serre spectral sequence by taking $r \gg 1$ large enough. By construction of the $(r+1)-$page, there is an exact sequence

$$0 \to E^{r+1}_{k,0} \to E^r_{k,0} \xrightarrow{d^r} E^r_{k-r,r-1}.$$

Moreover, $E^r_{k-r,r-1}$ is a subquotient of $E^2_{k-r,r-1} = H_{k-r}(B; H_{r-1}(F;\mathbb{Z}))$. By the inductive hypothesis, $H_{k-r}(B;\mathbb{Z})$, $H_{k-r-1}(B;\mathbb{Z}) \in \mathcal{C}$. Thus, by the Universal Coefficients Theorem, we have that $E^2_{k-r,r-1} \in \mathcal{C}$. Hence, $E^r_{k-r,r-1} \in \mathcal{C}$. Therefore, given $E^{r+1}_{k,0} \in \mathcal{C}$ we have $E^r_{k,0} \in \mathcal{C}$, by closedness under extensions (consider $\mathrm{Im}(d^r) \leq E^r_{k-r,r-1} \in \mathcal{C}$). Since $\tilde{H}_k(E;\mathbb{Z}) \in \mathcal{C}$, so are its filtration quotients, thus $E^{k+1}_{k,0} = E^\infty_{k,0} \in \mathcal{C}$. Inductively we conclude that $E^2_{k,0} = H_k(B;\mathbb{Z}) \in \mathcal{C}$.

Under the assumption that $\tilde{H}_*(B;\mathbb{Z}), \tilde{H}_*(E;\mathbb{Z}) \in \mathcal{C}$, an analogous argument lets us conclude that $\tilde{H}_*(F;\mathbb{Z}) \in \mathcal{C}$. ☐

**Lemma 1.2.1.2.** Let $\mathcal{C}$ be some of the Serre classes given in Example 1.2.1.1. If $\pi \in \mathcal{C}$ then $H_k(K(\pi,n);\mathbb{Z}) \in \mathcal{C}$ for all $n, k > 0$.

*Proof.* Using the path-loop fibration $K(\pi, n-1) \to P \to K(\pi, n)$ and the previous lemma, it suffices to show the case $n = 1$. For the general case, where $\mathcal{C}$ is either the class of finitely generated abelian groups or the class of finite abelian groups, then $\pi$ is a product of cyclic groups in $\mathcal{C}$. Since $K(G_1,1) \times K(G_2,1) \cong K(G_1 \times G_2, 1)$, by the Künneth Formula (or by the previous lemma), that reduces to proving the case when $\pi$ is cyclic. If $\pi = \mathbb{Z}$, $\mathcal{C}$ consists of finitely generated abelian groups, and $S^1 \cong K(\mathbb{Z},1)$, thus $H_k(S^1;\mathbb{Z}) \in \mathcal{C}$. If $\pi = \mathbb{Z}/m\mathbb{Z}$, by using an explicit model, we know that $H_k(K(\mathbb{Z}/m\mathbb{Z},1);\mathbb{Z})$ is $\mathbb{Z}/m\mathbb{Z}$ for each odd $k \geq 0$, and 0 for all even $k > 0$, since we can choose an infinitely-dimensional *lens* space for $K(\mathbb{Z}/m\mathbb{Z},1)$. Therefore, $H_k(K(\mathbb{Z}/m\mathbb{Z},1);\mathbb{Z}) \in \mathcal{C}$ for $k > 0$.

Let $\mathcal{P}$ be a fixed set of primes. For the class of torsion abelian groups, we can rely on the construction of the classifying space $K(\pi,1)$, i.e. the construction of $B\pi$, with the property that for each $G \subseteq \pi$, $BG$ is a subcomplex of $B\pi$. This is a consequence of the functoriality of such construction. Notice that $x \in H_k(B\pi;\mathbb{Z})$ where $k > 0$ is represented by a singular chain $\sum_i n_i \sigma_i$ with compact image (each chain has compact image by continuity) contained in some finite subcomplex of $B\pi$. This finite subcomplex involves only finitely many elements in $\pi$ and, thus, is contained in a subcomplex $BG$ for some finitely generated $G \leq \pi$. Because $G$ is a finite abelian group, by the first part of the proof, we know that the element in $H_k(BG;\mathbb{Z})$ represented by $\sum_i n_i \sigma_i$ has finite order, divisible only by primes in $\mathcal{P}$, implying the same for $x \in H_k(B\pi;\mathbb{Z})$. ☐

**Definition 1.2.1.4** (Acyclic Serre class)**.** We say that a Serre class is *acyclic* if it satisfies the consequence of the previous lemma.

### 1.2.2 The mod $\mathcal{C}$ Hurewicz Theorem

Now, we are ready to state and prove the main result of this section. Notice that we require a $1-$connected space $X$. This hypothesis is general enough for many cases of interest, like $X = S^n$. Nevertheless, this result remains valid for a path$-$connected abelian space $X$ (i.e. $\pi_1(X,x_0)$ acts trivially on $\pi_*(X,x_0)$).

**Theorem 1.2.2.1** (Mod $\mathcal{C}$ Hurewicz Theorem)**.** Let $\mathcal{C}$ be some of the following Serre classes

(a) The class of finitely generated abelian groups.

(b) The class of $\mathcal{P}-$torsion abelian groups.

(c) The class of finite $\mathcal{P}-$torsion abelian groups.

If $X$ is $1-$connected and $\pi_i(X,x_0) \in \mathcal{C}$ for $0 < i < n$, then $H_i(X;\mathbb{Z}) \in \mathcal{C}$ for $0 < i < n$, and

$$h\colon \pi_n(X,x_0) \to H_n(X;\mathbb{Z})$$

is a mod $\mathcal{C}$ isomorphism.

*Proof.* Since $X$ is $1-$connected, we will consider the Postnikov tower of $X$ (we could proceed similarly for the case when $X$ is a path$-$connected abelian space).

$$
\begin{tikzcd}
& \vdots \arrow[d] \\
& X_2 \arrow[d, "p_2"] \\
& X_1 \arrow[d, "p_1"] \\
X \arrow[r, "f_0"] \arrow[ur, "f_1"] \arrow[uur, "f_1"] & X_0
\end{tikzcd}
$$

Remember that the homotopy fiber of each $p_k\colon X_k \to X_{k-1}$ is a $K(\pi_k(X,x_0),k)$. Additionally, consider that each $f_k$ related to the Postnikov tower is an inclusion. Therefore, if $\pi_i(X,x_0) \in \mathcal{C}$ for each $0 < i < n$, then, by an inductive argument applied to the associated fibrations

$$K(\pi_i(X,x_0),i) \to X_i \overset{p_i}{\to} X_{i-1}$$

we obtain that $\tilde{H}_*(X_{n-1};\mathbb{Z}) \in \mathcal{C}$. Since $f_{n-1}\colon X \to X_{n-1}$ is $n-$connected, it follows that

$$\pi_i(X;\mathbb{Z}) \overset{\cong}{\to} \pi_i(X_{n-1};\mathbb{Z}), \qquad \text{for } i < n.$$

By the Whitehead Theorem (i.e. a consequence of the Hurewicz Theorem and the Serre Spectral Sequence),

$$H_i(X;\mathbb{Z}) \overset{\cong}{\to} H_i(X_{n-1};\mathbb{Z}), \qquad \text{for } i < n,$$

giving us that $H_i(X;\mathbb{Z}) \in \mathcal{C}$ for each $0 < i < n$. At this point, we have proven one of the implications of the Theorem 1.2.2.3.

Now, since $f_n\colon X \to X_n$ is $(n+1)-$connected, it is an isomorphism on degree $n$. Take the homotopy fiber sequence

$$K(\pi_n(X,x_0),n) \to X_n \overset{p_n}{\to} X_{n-1}.$$

Since $n \geq 2$, its basis must be $1-$connected. Observe that the fiber is clearly path$-$connected. By the homological Serre exact sequence, it follows that

$$
\begin{tikzcd}
H_{n+1}(X_{n-1};\mathbb{Z}) \arrow[r, "d^{n+1}"] & H_n(K(\pi_n,x_0),n);\mathbb{Z}) \arrow[r, "i_*"] \arrow[d, "h^{-1}(\cong)"] & H_n(X_n;\mathbb{Z}) \arrow[r, "(p_n)_*"] & H_n(X_{n-1};\mathbb{Z}) \\
& \pi_n(K(\pi_n(X,x_0),n),x_0) \arrow[d, "(j_*)^{-1}(\cong)"] & & \\
& \pi_n(X,x_0) \arrow[r, "h"] & H_n(X;\mathbb{Z}) \arrow[uu, "(f_n)_*(\cong)"'] &
\end{tikzcd}
$$

The groups in the extremes belong to $\mathcal{C}$, implying that $\mathrm{Ker}(i_*), \mathrm{Coker}(i_*) \in \mathcal{C}$. Moreover, by construction, $(f_n)_*$ is induced by the inclusion, and $K(\pi_n(X,x_0),n)$ consists of the same $n-$cells as $X$, thus the inclusion $j\colon X \to K(\pi_n(X,x_0),n)$ induces the isomorphism specified in the diagram, given by $[\tilde{g}\colon S^n \to K(\pi_n(X,x_0),n)] \mapsto [g\colon S^n \to X]$. Therefore, the center of the diagram commutes (which can be verified on elements), concluding that $h\colon \pi_n(X,x_0) \to H_n(X;\mathbb{Z})$ is an isomorphism mod $\mathcal{C}$. $\square$

Considering standard nomenclature we can rephrase the last result as follows.

**Theorem 1.2.2.2** (Mod $\mathcal{C}$ Hurewicz Theorem)**.** Let $\mathcal{C}$ be an acyclic Serre ring. If $X$ is 1−connected and $\pi_i(X, x_0) \in \mathcal{C}$ for $0 < i < n$, then $H_i(X; \mathbb{Z}) \in \mathcal{C}$ for all $0 < i < n$, and

$$h\colon \pi_n(X, x_0) \to H_n(X; \mathbb{Z})$$

is a mod $\mathcal{C}$ isomorphism.

A remarkable consequence corresponds to the following result, which says that a 1−connected space has finitely generated homotopy groups if and only if it has finitely generated homology groups. This is an interesting aspect where we can appreciate certain similitude between homotopy and (co)homology, where the first is topological and the second algebraic, and thus, more computationally tractable. For example, $\pi_i(S^n)$ is finitely generated for all $i, n \geq 0$, suggesting that we could be able to determine these invariants.

Previous to Serre's work, there was only known that these invariants are countable, by simplicial approximation techniques. In his proof, which follows similar lines to the results in this section, he used the Serre spectral sequence. However, Tammo tom Dieck proved that result without spectral sequences [tom08].

**Theorem 1.2.2.3** (Serre classes in homotopy and homology)**.** Let $X$ be a simply−connected space. Let $\mathcal{C}$ be an acyclic Serre ring. Then $\pi_n(X, x_0) \in \mathcal{C}$ for all $n > 0$ iff $H_n(X) \in \mathcal{C}$ for all $n > 0$.

*Proof.* Assume that $H_n(X; \mathbb{Z}) \in \mathcal{C}$ for all $n > 0$. Since $X$ is 1−connected, it follows that $\pi_1(X, x_0) = 0 \in \mathcal{C}$. By the mod $\mathcal{C}$ Hurewicz Theorem we have that $\pi_2(X) \cong H_2(X; \mathbb{Z})$ mod $\mathcal{C}$, i.e.

$$\mathrm{Ker}(h\colon \pi_2(X) \to H_2(X; \mathbb{Z})), \qquad \mathrm{Coker}(h\colon \pi_2(X) \to H_2(X; \mathbb{Z})) \in \mathcal{C}.$$

Hence, as $\mathcal{C}$ is closed under extensions, then $\pi_2(X) \in \mathcal{C}$. We proceed inductively.

The other implication was proven in the proof of the mod $\mathcal{C}$ Hurewicz Theorem. □

As in the case of the mod $\mathcal{C}$ Hurewicz Theorem, this result remains valid for a path−connected abelian space $X$. (see [Hat02]).

The next example shows that the previous theorem could fail if $X$ is not 1−connected.

**Example 1.2.2.1.** Let $X := S^1 \vee S^2$, and $x_0 \in X$. Let $\mathcal{C}$ be the acyclic Serre ring of finitely generated abelian groups. Since $X$ is a finite CW-complex, it has finitely generated (co)homology.

This space is not 1−connected. Indeed, by Seifert−van Kampen Theorem it follows that $\pi_1(X, x_0) \cong \pi_1(S^1, y_0) * \pi_1(S^2, z_0) \cong \mathbb{Z}$, where $y_0 \in S^1, z_0 \in S^2$.

We will verify that its homotopy is not finitely generated. Let $\tilde{X}$ be its universal covering. We can identify $\tilde{X}$ with $\mathbb{R}$, where we are adjoining a copy $S^2$ for each $x \in \tilde{X}$. Then, is immediate that $\pi_2(X, x_0) \cong \pi_2(\tilde{X}, x_0)$ has infinitely-many generators. □

The previous counterexample fails because $\pi_1(X, x_0)$ is not finite.

**Proposition 1.2.2.1** (Serre classes in homotopy and homology with relaxed conditions)**.** Let $X$ be a path−connected finite CW-complex, locally path−connected, and semi−locally simply−connected. Let $x_0 \in X$. Suppose that $\pi_1(X, x_0)$ is finite and that $H_i(X; \mathbb{Z})$ is finitely generated for each $i > 0$. Then $\pi_i(X, x_0)$ is finitely generated for each $i > 0$.

*Proof.* Since $X$ is path−connected, locally path−connected, and semi−locally simply−connected, there exists a unique (up to homeomorphism) universal covering of $X$. Let $\tilde{X}$ be the universal covering. Let $\tilde{x} \in \tilde{X}$. Then $\pi_i(X, x_0) \cong \pi_i(\tilde{X}, \tilde{x}_0)$, for all $i \geq 2$. Note that $\tilde{X}$ is 1−connected, and is a finite CW-complex since $\pi_1(X, x_0)$ is finite and $X$ is a finite CW-complex. In particular, $H_i(\tilde{X})$ is finitely generated for all $i > 0$. By the previous theorem applied to the acyclic Serre ring consisting of finitely generated abelian groups, it follows that $\pi_i(\tilde{X}, \tilde{x}_0)$ is finitely generated for all $i > 0$. Therefore, $\pi_i(X, x_0) \cong \pi_i(\tilde{X}, \tilde{x}_0)$ is finitely generated for all $i \geq 2$. We conclude the desired statement since $\pi_1(X, x_0)$ is finite. □

## 1.3 Rational homotopy

### 1.3.1 Rational cohomology of $K(\mathbb{Z}, n)$

This section consists of results about rational homotopy. This allows us to ignore the torsion part of the homotopy groups. A key ingredient corresponds to the rational homotopy of $K(\mathbb{Z}, m)$, $m \in \mathbb{N}_{>0}$.

**Theorem 1.3.1.1** (Rational cohomology of $K(\mathbb{Z}, n)$)**.** Let $n \in \mathbb{N}_{>0}$. We have the following isomorphism

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q}[x], & \text{even } n, \\ \Lambda_{\mathbb{Q}}[x], & \text{odd } n, \end{cases}$$

where $|x| = n$ and $\Lambda_{\mathbb{Q}}[x]$ denotes the exterior algebra over $\mathbb{Q}$ generated by $x$.

*Proof.* We proceed by induction applying the cohomological Serre spectral sequence. The case $n = 1$ is immediate, since $K(\mathbb{Z}, 1) \cong S^1$. The rational cohomology ring of $S^1$ satisfies the desired assertion. Similarly, we can consider an explicit model for $K(\mathbb{Z}, 2)$, i.e. $K(\mathbb{Z}, 2) \cong \mathbb{C}P^\infty$ (this comes from the fibration $S^1 \hookrightarrow S^\infty \to \mathbb{C}P^\infty$). By considering the CW-complex structure of the model space we have that

$$H^*(\mathbb{C}P^\infty); \mathbb{Q}) \cong \mathbb{Q} \oplus 0 \oplus \mathbb{Q} \oplus 0 \oplus \mathbb{Q} \oplus \cdots$$

as $\mathbb{Q}-$modules of cohomology. The corresponding multiplicative structure can be proven with the Serre spectral sequence. Therefore, $H^*(K(\mathbb{Z}, 2); \mathbb{Q}) \cong \mathbb{Q}[x]$, where $|x| = 2$.

Now, suppose that the stated assertion is valid for $n \geq 2$. We will prove that it is also valid for $n + 1$. Consider the path$-$loop fibration $K(\mathbb{Z}, n) \to PK(\mathbb{Z}, n + 1) \to K(\mathbb{Z}, n + 1)$. First, assume that $n$ is odd. Since the basis is $1-$connected, by the cohomological Serre spectral sequence with coefficients in $\mathbb{Q}$, we have the following $E_2$ description

$$\begin{array}{c|cccccc}
n+2 & 0 & 0 & 0 & 0 & & 0 \\
n+1 & 0 & 0 & 0 & 0 & & 0 \\
n & \mathbb{Q}\cdot a & * & * & * & & * \\
 & \searrow^{d_{n+1}} & & & & & \\
0 & \mathbb{Q}\cdot 1 & H^{n+1}(K(\mathbb{Z}, n+1); \mathbb{Q}) & H^{n+2}(K(\mathbb{Z}, n+1); \mathbb{Q}) & H^{n+3}(K(\mathbb{Z}, n+1); \mathbb{Q}) & & H^{2n+2}(K(\mathbb{Z}, n+1); \mathbb{Q}) \\
\hline
 & 0 & n+1 & n+2 & n+3 & \cdots & 2n+2
\end{array}$$

**Figure 1.5:** $E_2^{\bullet,\bullet}$ associated to the path$-$loop fibration (odd $n$).

By convergence of the spectral sequence, since the total space is contractible, $d_{n+1}$ is an isomorphism as shown. Suppose that $H^{n+1}(K(\mathbb{Z}, n+1); \mathbb{Q}) \cong \mathbb{Q} \cdot x$ where $|x| = n+1$. Consider a fixed entry $E_2^{p,0}$ where $n+2 \leq p \leq 2n+1$. Since the entering and exiting differentials are all trivial, by convergence to a contractible space it follows that $H^p(K(\mathbb{Z}, n); \mathbb{Q}) = 0$, $n + 2 \leq p \leq 2n + 1$. Moreover, the $n-$row corresponds to copies of $\mathbb{Q}$, according to the $E_2-$description theorem. Hence

$$
\begin{tikzcd}[row sep=small]
n+2 & 0 & 0 & 0 & 0 & & 0 \\
n+1 & 0 & 0 & 0 & 0 & & 0 \\
n & \mathbb{Q}\cdot a & \mathbb{Q}\cdot ax \arrow[drrrr, "d_{n+1}"] & 0 & 0 & & * \\
0 & \mathbb{Q}\cdot 1 & \mathbb{Q}\cdot x & 0 & 0 & & H^{2n+2}(K(\mathbb{Z},n+1);\mathbb{Q}) \\
 & 0 & n+1 & n+2 & n+3 & \cdots & 2n+2
\end{tikzcd}
$$

**Figure 1.6:** $E_2^{\bullet,\bullet}$ associated to the path−loop fibration (odd $n$).

The differential $d_{n+1}$ shown in the previous figure is an isomorphism applying the same argument of convergence. Then, $H^{2n+2}(K(\mathbb{Z},n+1);\mathbb{Q}) \cong \mathbb{Q}\cdot x^2$, since $d_{n+1}(ax) = d_{n+1}(a)x + ad_{n+1}(x) = x^2$. The entry denoted by $*$ is isomorphic to $\mathbb{Q}\cdot ax^2$. Proceeding inductively, we have shown that $H^*(K(\mathbb{Z},n+1);\mathbb{Q}) \cong \mathbb{Q}[x]$, where $|x| = n+1$.

Suppose that $n$ is even. The associated $E_2$−page is described as

$$
\begin{tikzcd}[row sep=small]
n+2 & \mathbb{Q}\cdot a^3 & & & & & \\
n+1 & \mathbb{Q}\cdot a^2 & & & & & \\
n & \mathbb{Q}\cdot a \arrow[dr, "d_{n+1}"] & & & & & \\
0 & \mathbb{Q}\cdot 1 & H^{n+1}(K(\mathbb{Z},n+1);\mathbb{Q}) & H^{n+2}(K(\mathbb{Z},n+1);\mathbb{Q}) & H^{n+3}(K(\mathbb{Z},n+1);\mathbb{Q}) & & H^{2n+2}(K(\mathbb{Z},n+1);\mathbb{Q}) \\
 & 0 & n+1 & n+2 & n+3 & \cdots & 2n+2
\end{tikzcd}
$$

**Figure 1.7:** $E_2^{\bullet,\bullet}$ associated to the path−loop fibration (even $n$).

For the same reason as in the previous case, we have that $d_{n+1}$ is an isomorphism as shown, thus $H^{n+1}(K(\mathbb{Z},n+1);\mathbb{Q}) \cong \mathbb{Q}\cdot x$, where $|x| = n+1$. Therefore

$$
\begin{tikzcd}[row sep=small]
n+2 & \mathbb{Q}\cdot a^3 & \mathbb{Q}\cdot a^3x & & & & \\
n+1 & \mathbb{Q}\cdot a^2 & \mathbb{Q}\cdot a^2x & & & & \\
n & \mathbb{Q}\cdot a \arrow[dr, "d_{n+1}"] & \mathbb{Q}\cdot ax & & & & \\
0 & \mathbb{Q}\cdot 1 & \mathbb{Q}\cdot x & H^{n+2}(K(\mathbb{Z},n+1);\mathbb{Q}) & H^{n+3}(K(\mathbb{Z},n+1);\mathbb{Q}) & & H^{2n+2}(K(\mathbb{Z},n+1);\mathbb{Q}) \\
 & 0 & n+1 & n+2 & n+3 & \cdots & 2n+2
\end{tikzcd}
$$

**Figure 1.8:** $E_2^{\bullet,\bullet}$ associated to the path−loop fibration (even $n$).

By the Leibniz rule, $d_{n+1}\colon \mathbb{Q}\cdot a^j \to \mathbb{Q}\cdot a^{j-1}x$ are are isomorphism for all $j \geq 1$, because $d_{n+1}(a^2) = 2ax$, $d_{n+1}(a^3) = 3a^2x, \ldots, d_{n+1}(a^j) = ja^{j-1}x$. On the other hand, consider the entries $E_2^{p,0}$ with $n+2 \leq p \leq 2n+1$. The differentials entering and exiting these entries are trivial, and this spectral sequence converges to a null−homotopic space. Therefore, these entries must be trivial. Their respective columns must be trivial too.

$$
\begin{tikzcd}
n+2 & \mathbb{Q}\cdot a^3 & \mathbb{Q}\cdot a^3x & 0 & 0 & & \\
n+1 & \mathbb{Q}\cdot a^2 & \mathbb{Q}\cdot a^2x & 0 & 0 & & \\
n & \mathbb{Q}\cdot a & \mathbb{Q}\cdot ax \arrow[drrrr, "d_{n+1}"] & 0 & 0 & & * \\
0 & \mathbb{Q}\cdot 1 & \mathbb{Q}\cdot x & 0 & 0 & & H^{2n+2}(K(\mathbb{Z},n+1);\mathbb{Q}) \\
 & 0 & n+1 & n+2 & n+3 & \cdots & 2n+2
\end{tikzcd}
$$

**Figure 1.9:** $E_2^{\bullet,\bullet}$ associated to the path−loop fibration (even $n$).

Additionally, the shown differential must be trivial, because

$$\operatorname{Ker}(d_{n+1}\colon \mathbb{Q}\cdot ax \to H^{2n+2}(K(\mathbb{Z},n+1);\mathbb{Q})) \supseteq \operatorname{Im}(d_{n+1}\colon \mathbb{Q}\cdot a^2 \to \mathbb{Q}\cdot ax) = \mathbb{Q}\cdot ax.$$

Therefore, all the differentials entering and exiting the entry associated to $H^{2n+2}(K(\mathbb{Z},n+1);\mathbb{Q})$ must be trivial. Since $PK(\mathbb{Z},n+1) \cong *$ it follows that $H^{2n+2}(K(\mathbb{Z},n+1);\mathbb{Q}) = 0$. Inductively, we conclude that $H^k(K(\mathbb{Z},n+1);\mathbb{Q}) = 0$ for all $k \geq n+2$, thus, $H^*(K(\mathbb{Z},n+1);\mathbb{Q}) = \Lambda_{\mathbb{Q}}[x]$ with $|x| = n+1$, finishing the proof. □

The next result deals with the case $X = S^n$. As a consequence, the free part is detected either by the degree or the Hopf invariant. In particular, the groups $\pi_i^s(S^0)$ are finite except by $i = 0$.

**Theorem 1.3.1.2.** The groups $\pi_i(S^n)$ are finite for $i > n$, except $\pi_{4k-1}(S^{2k})$, which is the direct sum between $\mathbb{Z}$ and a finite group.

*Proof.* Suppose that $n > 1$, since $n = 1$ is trivial. Note that the base spaces that follow in the proof are 1−connected so we can apply the Serre spectral sequence.

Take a map $S^n \to K(\mathbb{Z},n)$ inducing an isomorphism on $\pi_n$ (we could take the inclusion of CW-complexes), and consider W.L.O.G. the replacement fibration. From the homotopy exact sequence follows that $F$ is $n$−connected, and $\pi_i(F) \cong \pi_i(S^n)$ for all $i > n$. Apply the replacement fibration again to $F \to S^n$, obtaining a fibration

$$K(\mathbb{Z},n-1) \to X \to S^n,$$

where $X \cong F$. Consider the cohomological Serre spectral sequence with coefficients in $\mathbb{Q}$.

Assume that $n$ is odd. Since we know the rational cohomology of $K(\mathbb{Z},m)$, $m \geq 0$, $E_2^{\bullet,\bullet}$ has the following description

$$\begin{tikzcd}
3n-3 & \mathbb{Q}\cdot a^3 \arrow[dr, "d_{n+1}"] & \mathbb{Q}\cdot a^3x & 0 & \cdots \\
2n-2 & \mathbb{Q}\cdot a^2 \arrow[dr, "d_{n+1}"] & \mathbb{Q}\cdot a^2x & 0 & \cdots \\
n-1 & \mathbb{Q}\cdot a \arrow[dr, "d_{n+1}"] & \mathbb{Q}\cdot ax & 0 & \cdots \\
0 & \mathbb{Q}\cdot 1 & \mathbb{Q}\cdot x & 0 & \cdots \\
 & 0 & n & n+1 &
\end{tikzcd}$$

**Figure 1.10:** $E_2^{\bullet,\bullet}$ associated to the path−loop fibration (odd $n$).

The differential $d_{n+1}\colon \mathbb{Q}\cdot a \to \mathbb{Q}\cdot x$ is an isomorphism, since if not, it must be trivial, implying that $\mathbb{Q}\cdot a$ survives to $E_\infty^{\bullet,\bullet}$, contradicting the $(n-1)$−connectedness of $X$. The remaining differentials between the (possible) non-trivial columns are isomorphisms because of the previous observation, the Leibniz rule, and the commutativity of the cup product. Therefore, $\tilde{H}^*(X;\mathbb{Q}) = 0$. By the Universal Coefficients Theorem, the same is valid for homology, hence by Theorem 1.2.2.3 applied to the class of finite abelian groups, it follows that $\pi_i(X)$ is finite, concluding the same for $\pi_i(S^n)$, for each $i > n$, by the previous fibration.

Consider even $n$. Since the rational cohomology of $K(\mathbb{Z}, m)$, $m \geq 0$, is non-trivial in exactly two degrees, the associated spectral sequence only has two non-trivial rows. By a similar argument to the previous case, we have that $d_{n+1}\colon \mathbb{Q}\cdot a \to \mathbb{Q}\cdot x$ is an isomorphism. In particular, the term $\mathbb{Q}\cdot ax$ survives to $E_\infty^{\bullet,\bullet}$, implying that $H^{2n-1}(X;\mathbb{Q}) \cong \mathbb{Q}$, where we have used that we are working over a field to deal with the extension problem. Additionally, is immediate to verify that $H^*(X;\mathbb{Q}) \cong H^*(S^{2n-1};\mathbb{Q})$.

By the mod $\mathcal{C}$ Hurewicz theorem applied to the class of finite abelian groups, $\pi_i(S^n)$ is finite for $n < i < 2n-1$ and $\pi_{2n-1}(S^n)$ is isomorphic to $\mathbb{Z}$ except (possibly) by some finite abelian group. For the rest of the groups $\pi_i(S^n)$ with $i > 2n-1$, construct $Y$ from $X$ adjoining $2n$−cells or higher, to get rid of $\pi_i(X)$ for $i \geq 2n-1$. Consider the replacement fibration again. W.L.O.G., assume that $X \hookrightarrow Y$ is a fibration with fiber $Z$. Then $Z$ is $(2n-2)$−connected, and satisfies

$$\pi_i(Z) \cong \pi_i(X), \ \ i \geq 2n-1, \qquad \pi_i(Y) \cong \pi_i(X), \ \ i < 2n-1. \tag{1.2}$$

Consequently, the groups $\pi_*(Y)$ are finite. Applying 1.2.2.3, it must be that $\tilde{H}^*(Y;\mathbb{Q}) = 0$, and by the Serre spectral sequence, we obtain that $H^*(Z;\mathbb{Q}) \cong H^*(X;\mathbb{Q}) \cong H^*(S^{2n-1};\mathbb{Q})$. The following figure depicts the current situation.

$$\begin{array}{r|cc}
2n & \vdots & \vdots \\
2n-1 & H^{2n-1}(Z;\mathbb{Q}) & 0 \\
2n-2 & 0 & 0 \\
0 & \mathbb{Q} & 0 \\
\hline
 & 0 & 1
\end{array}$$

**Figure 1.11:** $E_2^{\bullet,\bullet}$ associated to the path−loop fibration (odd $n$).

Finally, we can apply the same argument given for odd $n$, with $Z$ replacing $S^n$, starting with a map $Z \to K(\mathbb{Z}, 2n-1)$ inducing an isomorphism on $\pi_{2n-1}$ modulo torsion, concluding that $\pi_i(Z)$ is finite for $i > 2n-1$. Therefore, (1.2), $\pi_i(Z) \cong \pi_i(X) \cong \pi_i(S^n)$ for $i > 2n-1$, proving the theorem. $\square$

**Corollary 1.3.1.1** (Finiteness of the stable homotopy of the sphere)**.** The stable homotopy groups of the sphere $\pi_i^s := \pi_i^s(S^0)$ are finite, except by $i = 0$.

### 1.3.2 Hurewicz morphism in rational stable homotopy

The last corollary is a general property, i.e. for all $i \geq 0$ and all $X$ we have that $\pi_i^s(X) \otimes \mathbb{Q} \cong H_i(X; \mathbb{Q})$. The text [Hat02] brings us two alternative proofs. We chose the second because it leads to a stronger result and illustrates an important technical manipulation related to the suspension, together with its relation with the Hurewicz morphism.

First, consider the following proposition which proves that the coefficients of the theory of homology $\pi_*^s(-)$ are $\pi_*^s(S^0)$.

**Proposition 1.3.2.1** (Coefficients of a reduced homology theory)**.** Let $h$ be a homology (in the sense of Eilenberg-Steenrod). Then $h_i(*) \cong \tilde{h}_i(S^0)$ for all $i \geq 0$.

*Proof.* Remember that $\tilde{h}_i(S^0) = h_i(S^0, *)$ for all $i \geq 0$. Since $h$ is a homology theory, by the exact sequence of a pair

$$\cdots \longrightarrow h_k(*) \xrightarrow{i_*} h_k(S^0) \longrightarrow h_k(S^0, *) \longrightarrow h_{k-1}(*) \xrightarrow{i_*} h_{k-1}(S^0) \longrightarrow h_{k-1}(S^0, *) \longrightarrow \dots$$

Since $h_k(S^0) \cong h_k(*) \oplus h_k(*)$ and the morphisms $h_j(*) \overset{i_*}{\hookrightarrow} h_j(S^0)$ are injective, by exactness $h_k(S^0, *) \cong h_k(*)$. The case $k = 0$ is completely analogous. $\square$

**Proposition 1.3.2.2** (Hurewicz morphism in rational stable homotopy)**.** The Hurewicz morphism $h\colon \pi_n(X) \to H_n(X)$ stabilizes on a rational isomorphism $h \otimes 1\colon \pi_n^s(X) \otimes \mathbb{Q} \to H_n(X) \otimes \mathbb{Q} \cong H_n(X; \mathbb{Q})$ for all $n > 0$.

*Proof.* Firstly, notice that the functor $\pi_*^s(-)$ corresponds to a reduced homology theory on the category of punctured CW-complexes. Since $\mathbb{Q}$ is a flat $\mathbb{Z}-$module, by the Universal Coefficients Theorem, the same is valid for $\pi_i^s \otimes \mathbb{Q}$. The coefficients of the homology theory $\pi_i^s(-) \otimes \mathbb{Q}$ are the same as in $H_i(-; \mathbb{Q})$, by the previous theorem.

We will verify that these homology theories coincide for every CW-complex. This is the point of the argument where the two proofs given in [Hat04] split. By convention, the suspension of a fundamental class is a fundamental class, making the following diagram commutative.

$$\begin{tikzcd}
\pi_i(X) \arrow[d, "h"] & \pi_{i+1}(CX, X) \arrow[l, "\cong"'] \arrow[r] \arrow[d, "h"] & \pi_{i+1}(SX, CX) \arrow[d, "h"] & \pi_{i+1}(SX) \arrow[l, "\cong"'] \arrow[d, "h"] \\
H_i(X) & H_{i+1}(CX, X) \arrow[l, "\cong"'] \arrow[r, "\cong"] & H_{i+1}(SX, CX) & H_{i+1}(SX) \arrow[l, "\cong"']
\end{tikzcd}$$

The isomorphisms on the right and the left are consequences of the homology/homotopy exact sequence of the pair. Note that $CX \cong *$. The middle isomorphism comes from the fact that the suspension is an isomorphism in homology.

Consequently, $h$ commutes with the suspension, inducing the stable Hurewicz map $h\colon \pi_n^s(X) \to H_n(X)$. Tensoring with $\mathbb{Q}$, the map $h \otimes 1\colon \pi_n^s(X) \otimes \mathbb{Q} \to H_n(X) \otimes \mathbb{Q} \cong H_n(X; \mathbb{Q}))$ is a natural transform between homology theories. When $X = S^k$, $k \geq 0$, this map is an isomorphism, by the Freudenthal Suspension Theorem.

Since the desired result is valid for the sphere, we will proceed to generalize it to arbitrary finite-dimensional CW-complexes by induction on the dimension. For $X^0$ the result is immediate because both homology theories coincide on the coefficients. Assume that the result is valid for $X^k$. Take the CW-pair $(X^{k+1}, X^k)$, where $X^{k+1}$ is an arbitrary CW-complex, obtained by adjoining $(k+1)-$cells to $X^k$. By the long exact sequence of the pair and the naturality of $h \otimes 1$, there is a commutative diagram

$$\begin{tikzcd}
H_{i+1}(X^{k+1},X^k;\mathbb{Q}) \arrow[r] \arrow[d, "h\otimes 1\ (\cong)"] & H_i(X^k;\mathbb{Q}) \arrow[r] \arrow[d, "h\otimes 1\ (\cong)"] & H_i(X^{k+1};\mathbb{Q}) \arrow[r] \arrow[d, "h\otimes 1"] & H_i(X^{k+1},X^k;\mathbb{Q}) \arrow[r] \arrow[d, "h\otimes 1\ (\cong)"] & H_{i-1}(X^k;\mathbb{Q}) \arrow[d, "h\otimes 1\ (\cong)"] \\
\pi^s_{i+1}(X^{k+1},X^k)\otimes\mathbb{Q} \arrow[r] & \pi^s_i(X^k)\otimes\mathbb{Q} \arrow[r] & \pi^s_i(X^{k+1})\otimes\mathbb{Q} \arrow[r] & \pi^s_i(X^{k+1},X^k)\otimes\mathbb{Q} \arrow[r] & \pi^s_{i-1}(X^k)\otimes\mathbb{Q}
\end{tikzcd}$$

where the isomorphisms on relative homotopy came from

$$H_r(X^{k+1},X^k;\mathbb{Q}) \cong H_r\left(\bigvee_\alpha S^{k+1};\mathbb{Q}\right) \cong \bigoplus_\alpha H_r(S^{k+1};\mathbb{Q}), \qquad \pi^s_r(X^{k+1},X^k)\otimes\mathbb{Q} \cong \bigoplus_\alpha \pi^s_r(S^{k+1})\otimes\mathbb{Q},$$

The last isomorphism follows from the fact that homotopy groups satisfy the excision axiom (and hence the wedge axiom) in the stable range, as seen in the proof of the Freudenthal Suspension Theorem, and the stable homotopy functor is the colimit of the corresponding directed system, which commutes with direct sums.

By the five lemma, the morphism in the middle is an isomorphism, concluding the inductive step. For an arbitrary CW-complex $X$, it suffices to consider an arbitrary finite-dimensional CW complex included in $X$. For an arbitrary space $X$, the result is obtained by CW approximation, concluding the proof. $\square$

## 1.4 First p-torsion in the stable homotopy of spheres

This section is about computing the first $p-$torsion in $\pi^S_*(S^0)$. It will follow [Hat04] and [Spa66]. The main result is that $\Sigma^2$ is an isomorphism on $p-$primary components. This result allows us to generalize the computation of the first $p-$torsion of $\pi_*(S^3)$.

### 1.4.1 First p-torsion in the homotopy of $S^3$

The next result only requires the cohomological Serre spectral sequence. Its importance relies on understanding the first stable $p-$torsion of $S^n$ reduces to this example, as we will see.

**Proposition 1.4.1.1** ($p-$torsion in $S^3$)**.** The first $p-$torsion in $\pi_*(S^3)$ corresponds to $\pi_{2p}(S^3) \cong \mathbb{Z}_p$.

*Proof.* We will prove that $\pi_*(S^3)$ are non-trivial for infinite degrees by considering its $p-$torsion subgroups. In particular, it will be verified that the $p-$torsion in $\pi_i(S^3)$ is trivial for $i < 2p$ and $\mathbb{Z}_p$ for $i = 2p$.

Let $S^3 \to K(\mathbb{Z},3)$ be the inclusion, which induces an isomorphism on $\pi_3$. Take the replacement fibration, denoting the fiber as $F$. Then, the fiber is $3-$connected since the inclusion induces isomorphisms on homotopy up to degree 3. Moreover, by the homotopy long exact sequence, $\pi_i(S^3) \cong \pi_i(F)$ for $i > 3$. Taking the replacement fibration again $F \to S^3$, we obtain a fibration $K(\mathbb{Z},2) \to X \to S^3$ with $X \cong F$. Since the cohomology ring of the fiber $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[x]$, with $|x| = 2$, $E_2^{\bullet;\bullet}$ is given by

$$\begin{tikzcd}
6 & \mathbb{Z}\cdot a^3 \arrow[dr, "d_3"] & \mathbb{Z}\cdot a^3x \\
4 & \mathbb{Z}\cdot a^2 \arrow[dr, "d_3"] & \mathbb{Z}\cdot a^2x \\
2 & \mathbb{Z}\cdot a \arrow[dr, "d_3"] & \mathbb{Z}\cdot ax \\
0 & \mathbb{Z}\cdot 1 & \mathbb{Z}\cdot x \\
 & 0 & 3
\end{tikzcd}$$

**Figure 1.12:** $E_2-$page associated to the replacement fibration $K(\mathbb{Z},2) \to X \to S^3$

Since $X$ is $3-$connected, $d_3\colon \mathbb{Z}\cdot a \to \mathbb{Z}\cdot x$ is an isomorphism. W.L.O.G., assume $d_3(a) = x$. By the Leibniz rule $d_3(a^n) = na^{n-1}x$, so

$$H^i(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}_n, & i = 2n+1, \\ 0, & i = 2n > 0, \end{cases} \qquad \text{i.e.} \qquad H_i(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}_n, & i = 2n > 0, \\ 0, & i = 2n-1, \end{cases} \tag{1.3}$$

by the Universal Coefficients Theorem. To see that, note that the basis and the fiber in $K(\mathbb{Z},2) \to X \to S^3$ have finitely generated homology. Thus, $H_*(X;\mathbb{Z})$ is finitely generated in each degree. By the Universal Coefficients Theorem

$$H^i(X;\mathbb{Z}) \cong \mathrm{Hom}(H_i(X),\mathbb{Z}) \oplus \mathrm{Ext}_{\mathbb{Z}}(H_{i+1}(X),\mathbb{Z}).$$

Then, for $i = 2n+1$

$$\mathbb{Z}_n \cong \mathrm{Hom}(H_{2n+1}(X),\mathbb{Z}) \oplus \mathrm{Ext}_{\mathbb{Z}}(H_{2n}(X),\mathbb{Z}).$$

Since these groups are finitely generated abelian groups, $H_{2n+1}(X)$ its free part is trivial. As $\mathrm{Ext}_{\mathbb{Z}}(-,-)$ commutes with finite direct sums, $H_{2n}(X) \cong \mathbb{Z}_n$. Analogously, $\mathrm{Ext}_{\mathbb{Z}}(H_{2n-1}(X),\mathbb{Z}) = 0$, the torsion part is trivial too, obtaining the previous description (1.3).

By the mod $\mathcal{C}$ Hurewicz Theorem, taking the class of torsion abelian groups without $p-$torsion, it follows that the $p-$torsion in $\pi_i(X) \cong \pi_i(S^3)$ for $i \geq 4$ corresponds to $\mathbb{Z}_p$ for the degree $i = 2p$. This concludes the proof, since we know $\pi_i(S^3)$ for $i \leq 3$. □

### 1.4.2 The double suspension

The book [Hat04] describes the minimal $p-$torsion in the homotopy of the spheres by an argument of localization of spaces. Although the construction of localization of spaces is important, since our main interest involves stable homotopy theory, we will follow the book [Spa66]. The following arguments illustrate how Serre classes are a precursor notion to localization.

**Definition 1.4.2.1.** Let $p$ be a prime. Let $\mathcal{C}_{\neg p}$ be the Serre class of $\mathcal{P}-$torsion, where we are excluding the $p-$torsion.

The next definition is a particular class of a Serre ring, leading to slightly stronger results.

**Definition 1.4.2.2** (Serre ideal)**.** We say that a Serre class $\mathcal{C}$ is a *Serre ideal* if for every $A \in \mathcal{C}$ and $B \in \mathrm{Obj}(\mathbf{Ab})$ then $A \otimes_{\mathbb{Z}} B \in \mathcal{C}$ and $\mathrm{Tor}(A,B) \in \mathcal{C}$.

In order to "localize" via Serre classes we will require

**Proposition 1.4.2.1** ($\mathcal{C}_{\neg p}$ is a Serre ideal)**.** The class $\mathcal{C}_{\neg p}$ is a Serre ideal.

*Proof.* By a straightforward argument, $\mathcal{C}_{\neg p}$ is a Serre class.

Let $A \in \mathcal{C}_{\neg p}, B \in \mathrm{Obj}(\mathbf{Ab})$. Since $A, B \in \mathrm{Obj}(\mathbf{Ab})$, then

$$A = \varinjlim_{i\in I} A_i, \qquad B = \varinjlim_{j\in J} B_j,$$

where the direct limits were taken over the corresponding finitely generated subgroups, ordered by the inclusion. Since $- \otimes_{\mathbb{Z}} -$ commutes with direct limits,

$$A \otimes_{\mathbb{Z}} B = \varinjlim_{i\in I} A_i \otimes_{\mathbb{Z}} \varinjlim_{j\in J} B_j \cong \varinjlim_{i\in I} \varinjlim_{j\in J} \left(A_i \otimes_{\mathbb{Z}} B_j\right).$$

Given that $A_i$ and $B_j$ are finitely generated abelian groups and $A_i$ is a torsion group

$$A_i \cong \bigoplus_{r=1}^{n_i} \mathbb{Z}/p_{i,r}^{\alpha^{i,r}}\mathbb{Z}, \qquad B_j \cong \mathbb{Z}^{d_j} \oplus \bigoplus_{s=1}^{m_j} \mathbb{Z}/q_{j,s}^{\beta^{j,s}}\mathbb{Z},$$

where $n_i, m_j, \alpha_{i,r}, \beta_{j,s} \in \mathbb{N}_{>0}$, and $p_{i,r}, q_{j,s} \in \mathbb{N}$ are primes. As $- \otimes_{\mathbb{Z}} -$ commutes with the direct sum

$$\begin{aligned}
A \otimes_{\mathbb{Z}} B = \varinjlim_{i\in I} \varinjlim_{j\in J} (A_i \otimes_{\mathbb{Z}} B_j) &\cong \varinjlim_{i\in I} \varinjlim_{j\in J} \left( \left[ \bigoplus_{r=1}^{n_i} \mathbb{Z}/p_{i,r}^{\alpha_{i,r}} \mathbb{Z} \right] \otimes_{\mathbb{Z}} \left[ \mathbb{Z}^{d_j} \oplus \bigoplus_{s=1}^{m_j} \mathbb{Z}/q_{j,s}^{\beta_{j,s}} \mathbb{Z} \right] \right) \\
&\cong \varinjlim_{i\in I} \varinjlim_{j\in J} \left( \bigoplus_{r=1}^{n_i} \bigoplus_{s=1}^{m_j} \left[ \mathbb{Z}/p_{i,r}^{\alpha_{i,r}} \mathbb{Z} \right] \otimes_{\mathbb{Z}} \left[ \mathbb{Z}^{d_j} \oplus \mathbb{Z}/q_{j,s}^{\beta_{j,s}} \mathbb{Z} \right] \right) \\
&\cong \varinjlim_{i\in I} \varinjlim_{j\in J} \left( \bigoplus_{r=1}^{n_i} \bigoplus_{s=1}^{m_j} \left[ \mathbb{Z}/p_{i,r}^{\alpha_{i,r}} \mathbb{Z} \right]^{d_j} \oplus \left[ \mathbb{Z}/p_{i,r}^{\alpha_{i,r}} \mathbb{Z} \right] \otimes_{\mathbb{Z}} \left[ \mathbb{Z}/q_{j,s}^{\beta_{j,s}} \mathbb{Z} \right] \right) \\
&\cong \varinjlim_{i\in I} \varinjlim_{j\in J} \left( \bigoplus_{r=1}^{n_i} \bigoplus_{s=1}^{m_j} \left[ \mathbb{Z}/p_{i,r}^{\alpha_{i,r}} \mathbb{Z} \right]^{d_j} \oplus \left[ \mathbb{Z}/\gcd\left(p_{i,r}^{\alpha_{i,r}}, q_{j,s}^{\beta_{j,s}}\right) \mathbb{Z} \right] \right).
\end{aligned}$$

Now, since $A_i$ has trivial $p-$torsion, the finitely generated abelian groups associated to the direct limit are torsion groups, with trivial $p-$torsion, concluding that $A \otimes_{\mathbb{Z}} B \in \mathcal{C}_{\neg p}$.

For the case of $\mathrm{Tor}_1^{\mathbb{Z}}(-,-)$ an analogous argument proves the desired result. ☐

Consider the following consequence of 1.4.1.1.

**Corollary 1.4.2.1** (Minimal $p-$torsion of $S^3$ (mod $\mathcal{C}_{\neg p}$))**.** The $3-$sphere $S^3$ is $(2p-1)-$connected (mod $\mathcal{C}_{\neg p}$) and $\left(\pi_{2p} S^3\right)_{(p)} \cong \mathbb{Z}/p\mathbb{Z}$.

*Proof.* This is an immediate consequence of Proposition 1.4.1.1. ☐

For completeness, we include the statement of the generalized Whitehead theorem. It can be proven using the theory of Serre classes.

**Theorem 1.4.2.1** (Generalized Whitehead Theorem (Theorem 22, pp. 512 [Spa66]))**.** Let $\mathcal{C}$ be an acyclic ideal of abelian groups and let $f\colon X \to Y$ be a map between simply$-$connected spaces. For $n \geq 1$ the following are equivalent.

(a) $f_*\colon \pi_i(X) \to \pi_i(Y)$ is a $\mathcal{C}-$isomorphism for $i \leq n$ and a $\mathcal{C}-$surjection for $i = n+1$.

(b) $f_*\colon H_i(X;\mathbb{Z}) \to H_i(Y;\mathbb{Z})$ is a $\mathcal{C}-$isomorphism for $i \leq n$ and a $\mathcal{C}-$surjection for $i = n+1$.

Another useful result corresponds to the generalized Wang homology sequence.

**Theorem 1.4.2.2** (Generalized homology Wang exact sequence (Theorem 2, pp. 482 [Spa66]))**.** Let $p\colon E \to B$ be a fibration with fiber $F$ and simply$-$connected base $B$ which is a homology $n-$sphere (over $R$) for some $n \geq 2$. Then there is an exact sequence

$$\cdots \to H_t(F;G) \xrightarrow{i_*} H_t(E;G) \to H_{t-n}(F;G) \to H_{t-1}(F;G) \xrightarrow{i_*} \cdots$$

However, we will use the cohomological version.

**Theorem 1.4.2.3** (Generalized cohomology Wang exact sequence (Theorem 1, pp. 498 [Spa66]))**.** Let $p\colon E \to B$ be a fibration with fiber $F$ and simply$-$connected base $B$ which is a cohomology $n-$sphere (over $R$) for some $n \geq 2$. Then there is an exact sequence

$$\cdots \to H^t(E;G) \xrightarrow{i^*} H^t(F;G) \xrightarrow{\theta} H^{t-n+1}(F;G) \to H^{t+1}(E;G) \xrightarrow{i^*} \cdots$$

in which $\theta(u \smile v) = \theta(u) \smile v + (-1)^{(n+1)\cdot|u|} u \smile \theta(v)$, the coefficients being suitably paired.

We will require the following technical result about the cohomological Serre spectral sequence. The goal of the following lemmas consists of understanding $H_*(\Omega^2 S^{n+2}; \mathbb{F}_p)$ in a specific range of degrees, allowing us to prove the main result of this section.

**Lemma 1.4.2.1** (Lemma 1, pp. 512 [Spa66])**.** Let $X$ be $1-$connected. Assume that there is an element $u \in H^n(X;R)$, with $n \geq 2$, such that $u^{m-1} \neq 0$ for some $m \geq 2$, and

$$\{1, u, u^2, \dots, u^{m-1}\} \subseteq H^*(X;R)$$

is a basis for degrees less than $mn$. Then, there exists an element $v \in H^{n-1}(\Omega X;R)$ such that $\{1, v\} \subseteq H^*(\Omega X;R)$ is a basis in degrees less than $mn-2$.

*Proof.* Since $X$ is $1-$connected, we will apply the Serre spectral sequence to $\Omega X \to PX \to X$. As $PX \cong \{*\}$, it follows that $E_\infty^{s,t} = 0$ for all $(s,t) \neq (0,0)$. Moreover, $X$ has trivial torsion in degrees $< mn$, so $E_2^{s,t} \cong H^s(X) \otimes H^t(\Omega X)$ for $s < mn$, where we are considering $R-$coefficients. Hence $E_2^{s,t} = 0$ if $s < mn$ and $s \neq 0, n, 2n, \dots, (m-1)n$. Since $d_r$ has bidegree $(r, 1-r)$, then for $s < mn$, $d_r\colon E_r^{s,t} \to E_r^{s+r,t-r+1}$ is trivial, except for $r = n, 2n, \dots, (m-1)n$. Therefore, $E_n^{s,t} \cong E_2^{s,t}$ for $s < mn$. If $t < n-1$, $E_n^{0,t} \cong E_\infty^{0,t}$, and if $0 < t < n-1$, we have

$$H^t(\Omega X) \cong E_2^{0,t} \cong E_\infty^{0,t} = 0,$$

so $H^t(\Omega X) = 0$ for $0 < t < n-1$. Moreover, there exists an exact sequence

$$0 \to E_\infty^{0,n-1} \to H^0(X) \otimes H^{n-1}(\Omega X) \xrightarrow{d_n} H^n(X) \otimes H^0(\Omega X) \to E_\infty^{n,0} \to 0.$$

Since $E_\infty^{0,n-1} = 0 = E_\infty^{n,0}$, there exists an element $v \in H^{n-1}(\Omega X)$ such that $d_n(1 \otimes v) = u \otimes 1$. Since $d_n$ is a derivation, $d_n(u^k \otimes v) = (-1)^{kn} u^{k+1} \otimes 1$. By the hypothesis relative to $H^*(X;R)$, for all $s < mn$, we have $d_n\colon E_n^{s-n,n-1} \xrightarrow{\cong} E_n^{s,0}$. As $d_n$ is a differential,

$$E_n^{s-2n,2n-2} \xrightarrow{d_n} E_n^{s-n,n-1} \xrightarrow{d_n} E_n^{s,0}$$

is trivial. Then, $d_n\colon E_n^{s-2t,2n-2} \to E_n^{2-n,n-1}$ is trivial for $s < mn$, and $E_{n+1}^{s-n,n-1} = 0 = E_{n+1}^{s,0}$ for $s < mn$. Therefore,

$$\begin{aligned} E_r^{s,t} &= 0 \qquad s < mn, t \leq n-1, r \geq n+1 \\ E_{n+1}^{0,2n-2} &= E_2^{0,2n-2}. \end{aligned}$$

Assume that the lemma is false, and let $q$ be the smallest integer such that $n-1 < q < mn-2$ and $H^q(\Omega X) \neq 0$. We will see that

$$E_\infty^{0,q} = E_2^{0,q} \cong H^q(\Omega X),$$

which is a contradiction.

We know that $E_n^{0,q} \cong E_2^{0,q}$. Moreover, $d_n\colon E_n^{0,q} \to E_n^{n,q-n+1}$ is trivial, because if $q-n+1 \neq n-1$, then $H^{q-n+1}(\Omega X) = 0$ and $E_r^{s,q-n+1} = 0$ for all $r$ and $s$, and if $q-n+1 = n-1$, then $E_{n+1}^{0,2n-2} = E_2^{0,2n-2}$. Then, $E_{n+1}^{0,q} \cong E_n^{0,q}$. By the assumption that $q$ is the minimal degree greater than $n-1$ such that $H^q(\Omega X) \neq 0$, it follows that $E_r^{s,t} = 0$ if $s < mn$, $t < q$, and $r \geq n+1$ (for $t \leq n-1$ this was verified previously). Hence, $d_r\colon E_r^{0,q} \to E_r^{r,q-r+1}$ is trivial for all $r \geq n+1$ and $E_\infty^{0,q} \cong E_{n+1}^{0,q}$ Therefore, we have the isomorphisms

$$E_\infty^{0,q} \cong E_{n+1}^{0,q} \cong E_n^{0,q} \cong E_2^{0,q},$$

proving the result. $\square$

The next result is a consequence of the generalized Wang exact sequence.

**Lemma 1.4.2.2** (Lemma 3, pp. 514 [Spa66])**.** Let $X$ be $1-$connected, with the cohomology of the $n-$sphere for odd $n > 1$. Then $H^*(\Omega X)$ has a basis $\{1, u_1, u_2, \dots\} \subseteq H^*(\Omega X)$, with $|u_k| = k(n-1)$, and $u_p \smile u_q = \frac{(p+q)!}{p!q!} \cdot u_{p+q}$.

*Proof.* We proceed by the Wang exact sequence (Theorem 1.4.2.3) for the fibration $PX \to X$. Since $PX \cong *$, we have that

$$\theta\colon H^t(\Omega X) \to H^{t-n+1}(\Omega X)$$

is an isomorphism for $t \neq 0$. Inductively, define $u_k \in H^{k(n-1)}(\Omega X)$ for $k \geq 0$, by the equations

$$u_0 = 1,$$

$$\theta(u_k) = u_{k-1}, \qquad k > 0.$$

Then, $\{1, u_1, u_2, \dots\} \subseteq H^*(\Omega X)$ is a basis, and we verify that it has the stated multiplicative property by double induction on $p$ and $q$. If $i = 0$ or $j = 0$, then $u_i \smile u_j = u_{i+j}$. Let $p > 0$ and $q > 0$. Assume that $u_i \smile u_j = \frac{(i+j)!}{i!j!} \cdot u_{i+j}$, if $i + j < p + q$, $i \geq 0$, and $j \geq 0$. Since $n$ is odd,

$$\begin{aligned}\theta(u_p \smile u_q) &= \theta(u_p) \smile u_q + u_p \smile \theta(u_q) = u_{p-1} \smile u_q + u_p \smile u_{q-1} \\ &= \left[\frac{(p+q-1)!}{(p-1)!q!} + \frac{(p+q-1)!}{p!(q-1)!}\right] \cdot u_{p+q-1} = \frac{(p+q)!}{p!q!} \cdot u_{p+q-1}.\end{aligned}$$

As $\theta$ is an injective morphism, we conclude that

$$u_p \smile u_q = \frac{(p+q)!}{p!q!} \cdot u_{p+q}.$$

□

The following lemma is important to understand the double suspension, which behaves better than the usual suspension for our purposes.

**Lemma 1.4.2.3** (Truncated $\mathbb{Z}_p$−homology of $\Omega^2 S^{n+2}$ (Lemma 10, pp. 516 [Spa66]))**.** Let $n \in \mathbb{N}_0$ be odd, and $p$ a prime. Then, $H_q(\Omega^2 S^{n+2}; \mathbb{Z}_p) = 0$ for $n < q < p(n+1) - 2$.

*Proof.* By the Lemma 1.4.2.2, we have the following basis

$$\{1, u_1, u_1^2, \dots, u_1^{p-1}\} \subseteq H^*(\Omega S^{n+2}; \mathbb{Z}/p\mathbb{Z})$$

in degrees $< p(n+1)$. By the Lemma 1.4.2.1, there exists an element $v \in H^n(\Omega^2 S^{n+2}; \mathbb{Z}/p\mathbb{Z})$ such that $\{1, v\} \subseteq H^*(\Omega^2 S^{n+2}; \mathbb{Z}/p\mathbb{Z})$ is a basis in degrees $< p(n+1) - 2$. The Universal Coefficients Theorem finishes the proof. □

The next result will allow us to compute the minimal stable $p$−torsion of the sphere.

**Theorem 1.4.2.1** (The double suspension preserves $p$−torsion)**.** The *double suspension* (i.e. taking two consecutive suspensions)

$$\Sigma^2 \colon \pi_i(S^n) \to \pi_{i+2}(S^{n+2})$$

is an isomorphism (mod $\mathcal{C}_{\neg p}$) for $i < p(n+1) - 3$, and $\mathcal{C}_{\neg p}$−surjective if $i = p(n+1) - 3$.

*Proof.* Let $p \colon S^n \to \Omega S^{n+1}$ be the adjoint of the suspension. By the Freudenthal Suspension Theorem, the composite

$$S^n \xrightarrow{p} \Omega S^{n+1} \xrightarrow{\Omega p} \Omega^2 S^{n+2}$$

induces an isomorphism $\pi_i(S^n) \cong \pi_i(\Omega^2 S^{n+2})$ for all $i \leq 2n - 2$. By the Whitehead Theorem, also induces an isomorphism on $H_i(S^n) \cong H_i(\Omega^2 S^{n+2})$, for all $i \leq n$. By the Universal Coefficients Theorem,

$$\begin{aligned}H_i(S^n; \mathbb{F}_p) &\cong (H_i(S^n) \otimes_{\mathbb{Z}} \mathbb{F}_p) \oplus \operatorname{Tor}(H_{i-1}(S^n), \mathbb{F}_p) \cong H_i(S^n) \otimes_{\mathbb{Z}} \mathbb{F}_p, \\ H_i(\Omega^2 S^{n+2}; \mathbb{F}_p) &\cong \left(H_i(\Omega^2 S^{n+2}) \otimes_{\mathbb{Z}} \mathbb{F}_p\right) \oplus \operatorname{Tor}(H_{i-1}(\Omega^2 S^{n+2}), \mathbb{F}_p).\end{aligned}$$

So, by the Lemma 1.4.2.3, and the previous isomorphism, we have $H_i(S^n; \mathbb{F}_p) \cong H_i(\Omega^2 S^{n+2}; \mathbb{F}_p)$, for all $i \leq p(n+1) - 3$, since $H_i(S^n) = 0$ for $i > n$. By the previous application of the Universal Coefficients Theorem, we also obtain that $H_i(S^n)$ and $H_i(\Omega^2 S^{n+2})$ have trivial $p$−torsion for $i \leq p(n+1) - 3$, so $H_i(S^n) \cong H_i(\Omega^2 S^{n+2})$ mod $\mathcal{C}_{\neg p}$, for $i \leq p(n+1) - 3$. By the generalized Whitehead Theorem, this induces a $\mathcal{C}_{\neg p}$−isomorphism $\pi_i(S^n) \cong \pi_i(\Omega^2 S^{n+2})$ for $i < p(n+1) - 3$, and a $\mathcal{C}_{\neg p}$−surjective map for $i = p(n+1) - 3$. Since the double suspension $\Sigma^2$ is in correspondence with the previously induced morphism via $\pi_i(\Omega^2 S^{n+2}) \cong \pi_{i+2}(S^{n+2})$, we conclude the proof. □

### 1.4.3 First p-torsion in the stable homotopy of the sphere

The following result allows us to move the computed $p-$torsion in $S^3$ to $S^n$.

**Proposition 1.4.3.1** (Torsion in the $n-$sphere coming from $S^3$)**.** Let $n \geq 3$ be odd and $p$ a prime. We have the following isomorphism

$$\pi_{i-n+3}(S^3)_{(p)} \cong \pi_i(S^n)_{(p)},$$

for $i < 4p + n - 6$.

*Proof.* By induction on $n$. For $n = 3$, there is nothing to prove. Suppose that $n \geq 5$. We must verify that $\Sigma^2 \colon \pi_{i-2}(S^{n-2}) \to \pi_i(S^n)$ is an isomorphism on the $p-$primary components. By Theorem 1.4.2.1, this result is valid for $i - 2 < p(n-1) - 3$, i.e. $i < p(n-1) - 1$. So, we just have to verify that

$$4p + n - 6 \leq p(n-1) - 1,$$

which is equivalent to $(p-1)(n-5) \geq 0$. □

The next result generalizes the previous statement about $S^3$.

**Proposition 1.4.3.2** ($p-$torsion in $S^n$)**.** Let $n \geq 3$ odd and $p$ a prime. We have the following isomorphism

$$(\pi_{n+m}(S^n))_{(p)} \cong \begin{cases} 0, & m < 2p-3 \\ \mathbb{Z}/p\mathbb{Z}, & m = 2p-3. \end{cases}$$

*Proof.* By Proposition 1.4.3.1, we have that

$$(\pi_{n+m}(S^n))_{(p)} \cong (\pi_{n+3}(S^3))_{(p)}.$$

If $n + m < 4p + n - 6$, i.e. $m < 4p - 6$, by Proposition 1.4.2.1,

$$(\pi_{n+m}(S^n))_{(p)} \cong (\pi_{n+3}(S^3))_{(p)} \cong \begin{cases} 0, & m+3 < 2p \\ \mathbb{Z}/p\mathbb{Z}, & m+3 = 2p \end{cases}.$$

□

Finally, we have the desired result.

**Proposition 1.4.3.3** (First $p-$torsion in stable homotopy of the sphere)**.** The following map is an isomorphism

$$(\pi_m^S(S^0))_{(p)} \cong \begin{cases} 0, & m < 2p-3 \\ \mathbb{Z}/p\mathbb{Z}, & m = 2p-3. \end{cases}$$

*Proof.* Let $n \gg 1$ be odd. By the Proposition 1.4.3.2, since $\pi_m^S \cong \pi_{n+m}(S^n)$, then

$$(\pi_m^S)_{(p)} \cong (\pi_{n+m}(S^n))_{(p)} \cong \begin{cases} 0, & m+3 < 2p \\ \mathbb{Z}/p\mathbb{Z} & m+3 = 2p. \end{cases}$$

□

## 1.5 Cohomology of Eilenberg-Mac Lane spaces

Our main goal will be to compute $H^*(K(\mathbb{Z}/2\mathbb{Z}, n), \mathbb{Z}/2\mathbb{Z})$. This is important due to its direct relation with $\mathcal{A}_2$, which will be key in the construction of the Adams spectral sequence. The Steenrod squares $\mathrm{Sq}^I$ play a fundamental role. The Borel Theorem has also a central role.

As seen before, computing the cohomology of Eilenberg-Mac Lane is far from being trivial. However, understanding their cohomology was required in almost all of the previous sections. The only Eilenberg-Mac Lane $K(\pi, n)$ spaces, with $n > 1$, whose (co)homology can be computed by elemental means corresponds to $K(\mathbb{Z}, 2) \cong \mathbb{C}P^\infty$, and its products (i.e. $K(\pi, 2)$, where $\pi$ is free abelian). The Serre spectral sequence allows us to go further by computing $H^*(K(\mathbb{Z}/2\mathbb{Z}, n), \mathbb{Z}/2\mathbb{Z})$, which turns out to be directly related to $\mathcal{A}_2$. This relation follows from the fact that computing $H^*(K(\mathbb{Z}/2\mathbb{Z}, n), \mathbb{Z}/2\mathbb{Z})$ is equivalent to determining all the cohomology operations with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

The book [Hat02] defines the Steenrod operations via the quadratic construction (cf. [Rog10, Theorem 7.2.1]). It has the advantage over the acyclic model approach of being induced by a construction at the level of spaces, providing more information than a purely algebraic construction. However, this advantage will not appear in this section. In that follows, we will assume this construction, the Representability Theorem for the cohomology functor of spaces, and the definition of the Steenrod algebra, together with its basic properties.

We will consider the case $p = 2$. The computation for odd $p$ primes is analogous, as can be seen in the work of Cartan [Car54], and in [McC00]. For odd primes, the Adem relations for reduced Steenrod powers are harder, so the proofs are longer and more technical.

### 1.5.1 Preliminaries

This section gives an overview of the basic properties of the mod 2 Steenrod algebra. The construction of the Steenrod squares brings us the following result.

**Theorem 1.5.1.1** (Existence of the mod 2 Steenrod operations)**.** There are natural morphisms between maps of pairs of spaces

$$\mathrm{Sq}^i \colon H^n(X, A; \mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(X, A; \mathbb{Z}/2\mathbb{Z}), \quad i \geq 0,$$

satisfying the axioms

(i) $\mathrm{Sq}^i(f^*(\alpha)) = f^*(\mathrm{Sq}^i(\alpha))$, for $f \colon X \to Y$.

(ii) $\mathrm{Sq}^i(\alpha + \beta) = \mathrm{Sq}^i(\alpha) + \mathrm{Sq}^i(\beta)$.

(iii) $\mathrm{Sq}^i(\alpha \smile \beta) = \sum_{j+k=i} \mathrm{Sq}^j(\alpha) \smile \mathrm{Sq}^k(\beta)$, i.e. the Cartan formula holds.

(iv) $\mathrm{Sq}^i(\sigma(\alpha)) = \sigma(\mathrm{Sq}^i(\alpha))$, where $\sigma \colon H^n(X; \mathbb{Z}/2\mathbb{Z}) \to H^{n+1}(\Sigma X; \mathbb{Z}/2\mathbb{Z})$ is the suspension isomorphism given by the reduced cross product $\times$ (i.e. the *smash product*) with a generator of $H^1(S^1; \mathbb{Z}/2\mathbb{Z})$.

(v) $Sq^i(\alpha) = \alpha^2$ if $i = |\alpha|$, and $\mathrm{Sq}^i(\alpha) = 0$ if $i > |\alpha|$.

(vi) $\mathrm{Sq}^0 = \mathrm{Id}$.

(vii) $\mathrm{Sq}^1$ is the Bockstein morphism $\beta$ in $\mathbb{Z}/2\mathbb{Z}$ associated to the coefficients sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

We will not require all those axioms. In particular, the Cartan formula will not be used. Note that the Steenrod operations commute with the suspension ($p \geq 2$). On the other hand, the cup product is trivial for products between positive degree classes in a suspension, as can be shown using the relative cup product. However, they are related: [MT68] defines the Steenrod squares via the cup$-$i construction, which generalizes the cup product. The last property is important to detect torsion in integral cohomology.

**Definition 1.5.1.1** (Notation)**.** Let $I := (i_1, \dots, i_k) \in \mathbb{N}^k_{>0}$, $k \in \mathbb{N}$. Define $\mathrm{Sq}^I := \mathrm{Sq}^{i_1} \dots \mathrm{Sq}^{i_k} := \mathrm{Sq}^{i_1} \circ \cdots \circ \mathrm{Sq}^{i_k}$.

**Definition 1.5.1.2** (Admissible Steenrod monomial)**.** Let $I := (i_1, \dots, i_k) \in \mathbb{N}^k_{>0}$, $k \in \mathbb{N}$. We say that $\mathrm{Sq}^I$ is *admissible* if $2i_j \leq i_{j-1}$ for all $j = 2, \dots, k$.

The following definition gives us a notion about how far a monomial is from being admissible.

**Definition 1.5.1.3** (Excess)**.** Consider $I := (i_1, \dots, i_k) \in \mathbb{N}^k_{>0}, k \geq 0$. We define the *excess* of an admissible $\mathrm{Sq}^I$ as

$$e(I) := \sum_j (i_j - 2i_{j+1}).$$

By convention, let $I := (i_1, \dots, i_k) = (i_1, \dots, i_k, 0, 0, \dots)$, so the last term in the sum is $i_k$.

**Definition 1.5.1.4** (Cohomology operation (pp. 2 [MT68]))**.** A *cohomology operation* of type $(\pi, n; G, m)$ is a family of functions $\theta_X \colon H^n(X;\pi) \to H^m(X;G)$ for a fixed space $X$, satisfying the naturality condition $f^*\theta_Y = \theta_X f^*$ for any map $f \colon X \to Y$.

**Example 1.5.1.1** (The cup-product square as a cohomology operation (pp. 2, [MT68]))**.** The cup-product square $u \mapsto u^2$ gives for each $n$ and each ring $\pi$ an operation $H^n(X;\pi) \to H^{2n}(X;\pi)$. This is not a morphism in general. ☐

**Definition 1.5.1.5** (Cohomology operations of type $(\pi, n; G, m)$ (pp. 2, [MT68]))**.** We will denote by $\mathcal{O}(\pi, n; G, m)$ the set of cohomology operations of type $(\pi, n; G, m)$.

The name given to the following result is nonstandard and it is used only in these notes, due to its relation with the Brown Representability Theorem.

**Theorem 1.5.1.1** (Representability Theorem (Theorem 2, pp. 4 [MT68]))**.** There is a one-to-one correspondence

$$\mathcal{O}(K(\pi, n); G, m) \cong H^m(K(\pi, n); G)$$

given by $\theta \leftrightarrow \theta(\iota_n)$, where $\iota_n$ is the fundamental class of $K(\pi, n)$.

**Definition 1.5.1.6** (Mod 2 Steenrod algebra (pp. 46, [MT68]))**.** Let $R := \mathbb{Z}/2\mathbb{Z}$, and let $M$ be the graded $\mathbb{Z}/2\mathbb{Z}-$module such that $M_i = \mathbb{Z}/2\mathbb{Z}\{\mathrm{Sq}^i\}$ for each $i \geq 0$. Consider its tensor algebra $\Gamma(M)$ which is bigraded. For each $(a, b) \in \mathbb{Z}^2$ such that $0 < a < 2b$, let

$$R(a, b) := \mathrm{Sq}^a \otimes \mathrm{Sq}^b + \sum_c \binom{b-c-1}{a-2c} \mathrm{Sq}^{a+b-c} \otimes \mathrm{Sq}^c.$$

Define the ideal $Q := \{1 + \mathrm{Sq}^0, R(a, b) \mid a, b \in \mathbb{Z}\} \leq \Gamma(M)$. The *mod 2 Steenrod algebra* is defined as the quotient algebra $\mathcal{A}_2 := \Gamma(M)/Q$. In other words,

$$\mathcal{A}_2 \cong \mathbb{Z}/2\mathbb{Z}\left[\mathrm{Sq}^i \mid i \geq 0\right] / \, I.$$

where $I$ is an ideal imposing the Adem relations.

Similarly, we can define the mod $p$ Steenrod algebra $\mathcal{A}_p$ considering the corresponding Adem relations.

The admissible monomials form a basis for the Steenrod algebra. The linear independence will be proven at the end of this section.

**Proposition 1.5.1.1** (Mod 2 Steenrod algebra generators)**.** The admissible monomials $\mathrm{Sq}^I$ generates $\mathcal{A}_2$ as a $\mathbb{Z}/2\mathbb{Z}-$module.

### 1.5.2 Computing the cohomology of Eilenberg-Mac Lane spaces

The next lemma describes the behavior of admissible monomials. This allows an inductive argument that brings us the generators of $H^*(K(\mathbb{Z}/2\mathbb{Z}, n), \mathbb{Z}/2\mathbb{Z})$, stated in the Serre Theorem.

**Lemma 1.5.2.1** (Admissibility lemma)**.**

(a) $\mathrm{Sq}^I(\iota_n) = 0$, si $\mathrm{Sq}^I$ is admissible, and $e(I) > n$.

(b) The admissible elements $\mathrm{Sq}^I$ such that $e(I) = n$ are in one-to-one correspondence with $\left(\mathrm{Sq}^J(\iota_n)\right)^{2^j}$ where $J$ is admissible and $e(J) < n$.

*Proof.* For a monomial $\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_k}$, by definition of $e(I)$, write $i_1 = e(I) + i_2 + \cdots + i_k$. Then, if $e(I) > n$, we have $i_1 > n + i_2 + \cdots + i_k = |\mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k}(\iota_n)|$, hence, $\mathrm{Sq}^I(\iota_n) = 0$, since $i_1 > |\iota_n|$.

If $e(I) = n$, then $i_1 = n + i_2 + \cdots + i_k$, thus

$$\mathrm{Sq}^I(\iota_n) = \mathrm{Sq}^{i_1}\left(\mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k}\right)(\iota_n) = \left(\mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k}(\iota_n)\right)^2.$$

Since $\mathrm{Sq}^I$ is admissible, by definition of excess, $e(i_2, \ldots, i_k) \leq e(I) = n$, thus $\mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k}$ has $e(i_2, \ldots, i_k) \leq n$. If $e(i_2, \ldots, i_k) = n$, we repeat the previous step until obtain an equation of the form

$$\mathrm{Sq}^I(\iota_n) = \left(\mathrm{Sq}^J(\iota_n)\right)^{2^j}$$

with $e(J) < n$. Note that truncating admissible monomials preserves admissibility.

For the converse, assume that $\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_k}$ is admissible, such that $e(i_2,\cdots,i_k)\leq n$, and let $i_1 := n+i_2+\cdots+i_k$, so

$$\mathrm{Sq}^{i_1}\cdots\mathrm{Sq}^{i_k}(\iota_n) = \left(\mathrm{Sq}^{i_2}\mathrm{Sq}^{i_3}\cdots(\iota_n)\right)^2.$$

Then, $(i_1,\ldots,i_k)$ is admissible, since $e(i_2,\ldots,i_k)\leq n$ implies $i_2\leq n+i_3+\cdots+i_k$, thus $i_1 = n+i_2+\cdots+i_k\geq 2i_2$. Moreover, $e(i_1,\ldots i_k) = n$ since $i_1 = n+i_2+\cdots+i_k$ because the sum telescopes. Inductively, we can write an admissible $2^j$−power of $\mathrm{Sq}^J(\iota_n)$ with $e(J)\leq n$ as an admissible monomial $\mathrm{Sq}^I(\iota_n)$ with $e(I)\leq n$. □

The next result will let us apply the Borel Theorem in the proof of the Serre Theorem, allowing us to construct a basis of transgressive elements.

**Lemma 1.5.2.2** ($\mathrm{Sq}^I$ commutes with transgression)**.** If $x\in H^*(F;\mathbb{Z}/2\mathbb{Z})$ is transgressive then also are $\mathrm{Sq}^i(x)$ and $\tau(\mathrm{Sq}^i(x)) = \mathrm{Sq}^i(\tau(x))$.

*Proof.* Consider the following diagram obtained by naturality of the cohomology long exact sequence

$$\begin{tikzcd} & H^r(B,b)\cong\tilde{H}^r(B) \arrow[r, "j^*"] \arrow[d, "p^*"] & H^r(B) \\ H^{r-1}(F) \arrow[r, "\delta"] & H^r(X,F) & \end{tikzcd}$$

Let $\tau := j^*(p^*)^{-1}\delta$ be the transgression. Let $x\in H^{r-1}(F)$ be a transgressive element. Then, $\delta x\in\mathrm{Im}(p^*)$, so $\mathrm{Sq}^i(x)\in\mathrm{Im}(p^*)$ by naturality and since $\mathrm{Sq}^i$ commutes with $\delta$ since the Steenrod squares commute with the suspension (and its inverse), and $\delta$ corresponds to the topological boundary map, which can be described in terms of the suspension (cf. [Hat02], pp. 200, and [May99], pp. 107-108). By naturality, it follows that $\tau(\mathrm{Sq}^i(x)) = \mathrm{Sq}^i(\tau(x))$. □

Before proving the Borel Theorem we require a spectral sequence comparison theorem.

**Theorem 1.5.2.1** (Comparison between cohomological Serre spectral sequences (Theorem 5.36, [Hat04]))**.** Let $\Phi$ be a morphism between first-quadrant cohomological spectral sequences, so $d_r\colon E_r^{p,q}\to E_r^{p+r,q-r+1}$. Assume that $E_2^{p,q} = E_2^{p,0}\otimes E_2^{0,q}$ for both spectral sequences with $d_2$−differentials coming from a tensor product of differentials located at $p=0$ or $q=0$. Then each pair between the following alternatives implies the third

(i) $\Phi$ is an isomorphism on $E_2^{p,0}$ terms.

(ii) $\Phi$ is an isomorphism on $E_2^{0,q}$ terms.

(iii) $\Phi$ is an isomorphism on $E_\infty$.

The following definition is required to state the Borel Theorem.

**Definition 1.5.2.1** (Simple system of generators)**.** Let $H^*$ be a graded commutative algebra over a ring $R$. We say that a set $\{x_1,x_2,\ldots\}$ is a *simple system of generators* if 1 and $x_{i_1}x_{i_2}\ldots x_{i_k}$ where $i_1<i_2<\cdots<i_k$ are an additive $R$−basis of $H^*$.

**Example 1.5.2.1** (Algebras having a simple system of generators ([Hat04]))**.**

(i) An exterior algebra has a simple system of generators.

(ii) A polynomial algebra $\mathbb{K}[x]$ has a simple system of generators, given by the terms $x^{2^i}$.

(iii) The truncated polynomial algebra $\mathbb{K}[x]/(x^{2^n})$ has a simple system of generators.

(iv) Tensor product between algebras that have a simple system of generators has a simple system of generators.

(v) As a consequence of the previous statement, multivariate polynomial algebras have a simple system of generators.

□

The next argument consists of constructing an algebraic model spectral sequence that resembles the Serre spectral sequence associated to the corresponding fibration, and proving that they are isomorphic using Theorem 1.5.2.1. This involves taking tensor products over spectral sequences over a field. In general, the tensor product between spectral sequences is not a spectral sequence. For a mod $p$ version of this result, see [McC00, Theorem 6.21].

**Theorem 1.5.2.2** (Borel Theorem (Theorem 5.34 [Hat04]))**.** Let $F \to X \to B$ be a fibration with $X \cong *$ and $B$ simply−connected. Assume that $H^*(F;\mathbb{K})$, where $\mathbb{K}$ is a field, has a basis (as a vector space) given by the products $x_{i_1} \cdots x_{i_k}$, $i_1 < \cdots < i_k$, of transgressive elements $x_1, x_2, \cdots \in H^*(F;\mathbb{K})$, of odd dimension if $\mathrm{Char}(\mathbb{K}) \neq 2$. Then

$$H^*(B;\mathbb{K}) \cong \mathbb{K}\left[y_1, y_2, \cdots \mid [y_i] := \tau(x_i)\right].$$

*Sketch proof.* We will construct an algebraic model of the Serre spectral sequence associated to $F \to X \to B$. The basic building block corresponds to a spectral sequence with 2−page given by $\Lambda_{\mathbb{K}}[\overline{x}_i] \otimes \mathbb{K}[\overline{y}_i]$, where $|\overline{x}_i| = |x_i|$ and $|\overline{y}_i| = |y_i|$. The non-trivial are given in the following diagram, i.e. $d_r(\overline{x}_i \otimes \overline{y}_i^m) := \overline{y}_i^{m+1}$ for $r = |\overline{y}_i|$.

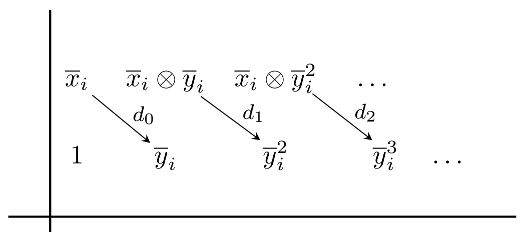


**Figure 1.13:** The basic building block for the model spectral sequence.

The cohomology of the fiber is $\Lambda_{\mathbb{K}}[\overline{x}_i]$, and that of the basis is $\mathbb{K}[\overline{y}_i]$. The $d_r$−differentials are (additive) isomorphisms because they map generators to generators. We impose the Leibniz rule arbitrarily. In consequence, $E_\infty$ is trivial, except by $E_\infty^{0,0} = \mathbb{K}$.

Since we are considering an $\mathbb{F}_2$−spectral sequence, by taking the tensor products of spectral sequences over all $i$, we obtain the desired (cohomological) model spectral sequence. The $E_2$−page is defined by $E_2^{p,q} := E_2^{p,0} \otimes E_2^{0,q}$, where the lower row is $\mathbb{K}[\overline{y}_1, \overline{y}_2, \dots]$, and the left column is $\Lambda_{\mathbb{K}}[\overline{x}_1, \overline{x}_2, \dots]$.

We are now at the critical point of the proof: define inductively the $d_r$−differentials, taking the tensors $d_{r,i} \otimes 1 + 1 \otimes d_{r,j}$, where we are taking care of the Koszul rule. Therefore, the $d_r$−differentials are derivations, and $\overline{x}_i$ are transgressive elements, with $d_r(\overline{x}_i) = \overline{y}_i$. In particular, we obtain $d_2(\overline{x}_i) = \overline{y}_i$, if $|\overline{x}_i| = 1$, and $d_2(\overline{x}_i) = 0$ in the other case.

It is also possible to describe the 3−page because we are working over a field, so the homology of a chain complex commutes with the tensor product by the algebraic Künneth Theorem. Consequently in the $E_3$−page, the generators $\overline{x_i}$ with degree $|\overline{x}_i| = 1$, and the generators $\overline{y}_i$ with degree $|\overline{y}_i| = 2$ are trivial, while the rest of them survives. The left column is the exterior algebra over the remaining $\overline{x}_i$ terms, and the bottom row is the polynomial algebra over the remaining $\overline{y}_i$ terms, and $E_3^{p,q} = E_3^{p,0} \otimes E_3^{0,q}$. The $d_3$−differentials are defined analogously, satisfying the derivation property, and such that $d_3(\overline{x}_i) = \overline{y}_i$, for $|\overline{x}_i| = 2$, and $d_3(\overline{x}_i) = 0$, in other case. We proceed inductively until construct $E_\infty$, with its entries all trivial, except $E_\infty^{0,0} = \mathbb{K}$.

Denote the terms in the Serre spectral sequence associated to the fibration by $E_r^{p,q}$, and the terms of the model spectral sequence by $\overline{E}_r^{p,q}$. Define the morphisms

$$\Phi\colon \overline{E}_2^{p,q} \to E_2^{p,q}$$

as follows. Let $\overline{x}_{i_1} \cdots \overline{x}_{i_r} \in \overline{E}_2^{0,q}$ a product of distinct generators. Define

$$\Phi(\overline{x}_{i_1} \cdots \overline{x}_{i_r}) := x_{i_1} \cdots x_{i_r}$$

by linear extension. On the terms $\overline{E}_2^{p,0}$ as the polynomial algebra is free, define $\Phi$ as the ring morphism such that $\Phi(\overline{y}_i) = y_i$, where $y_i$ is such that its image under the projection $E_2^{p,0} \to E_r^{p,0}$ is $\tau(x_i)$, for $r = |y_i|$. Then, define $\Phi$

on $\overline{E}_2^{p,q} = \overline{E}_2^{p,0} \otimes \overline{E}_2^{0,q}$ as the tensor product of the values defined on the factors. Note that $\Phi$ is just an additive morphism because $\mathrm{Char}(\mathbb{K}) = 2$, thus, for $\overline{x}_i \in \Lambda_{\mathbb{K}}[\overline{x}_i]$, $\overline{x}_i^2 = 0$, but we cannot determine whether $-x_i^2 = x_i^2 = 0$. However, in the rest of the argument suffices with the partial multiplicativity on the bottom row.

By construction, $\Phi$ commutes with $d_2$ inducing maps $\overline{E}_3^{p,q} \to E_3^{p,q}$. Inductively, $\Phi$ commutes with the rest of the differentials, inducing maps on the respective pages. Consequently, $\Phi\colon \overline{E}_r^{p,q} \to E_r^{p,q}$ is a map between spectral sequences.

Since $X \cong *$, $\Phi\colon \overline{E}_\infty^{p,q} \to E_\infty^{p,q}$ is an isomorphism. As the elements $x_i$ are a simple system of generators, $\overline{E}_2^{0,q} \cong E_2^{0,q}$, because $\Phi|_{\overline{E}_2^{0,q}}$ is surjective on each degree, it is also an isomorphism. By Theorem 1.5.2.1, it follows that $\Phi\colon \overline{E}_2^{p,0} \to E_2^{p,0}$ is an isomorphism. Moreover, $\Phi\colon \overline{E}_2^{p,0} \to E_2^{p,0}$ is a ring morphism, finishing the proof. $\square$

Consider the following remarks. We will only need (II).

(i) If $\mathrm{Char}(\mathbb{K}) \neq 2$, the elements $\{x_i\}_{i\in I}$, as being odd-dimensional, satisfies $x_i^2 = 0$, thus $H^*(F;\mathbb{K})$ is an exterior algebra. This follows from the commutativity of $\smile$, because we will have $2x_i^2 = 0$ and $\mathrm{Char}(\mathbb{K}) \neq 2$.

(ii) Since $X$ is contractible the fiber $F \cong \Omega B$ weakly. Then, if we assume that there are only finite $x_i \in H^j(F;\mathbb{K})$ for all $j \in \mathbb{N}_0$, then $H^*(F;\mathbb{K})$ is a commutative and associative Hopf algebra, hence [Hat02, Remark following Theorem 3C.4] a tensor product of exterior and polynomial algebras. When $\mathrm{Char}(\mathbb{K}) = p > 0$, also a product of truncated polynomial algebras. In particular, when $\mathbb{K} = \mathbb{Z}/2\mathbb{Z}$, $H^*(F;\mathbb{K})$ it has a simple system of generators. However, these generators are not transgressive in general.

(iii) Another result of Borel establishes that $H^*(B;\mathbb{K})$ is a polynomial algebra generated by even-dimensional elements if and only if $H^*(F;\mathbb{K})$ is an exterior algebra generated by odd-dimensional elements, without assumptions over the transgression. This can be proven either with the Serre spectral sequence, or with the Eilenberg-Moore spectral sequence, providing a more conceptual proof.

The following lemma will be required to compute $H^*(K(\mathbb{Z}/p\mathbb{Z},n),\mathbb{F}_p)$ for odd primes, although is not required for the mod 2 case. We provide a solution for [MT68, pp. 104, exercise].

**Lemma 1.5.2.3** (Transgression and fundamental class)**.** Let $\pi$ be an abelian group. Take the path$-$loop fibration $K(\pi,n) \to P \to K(\pi,n+1)$, where $n \in \mathbb{N}_{>0}$. Therefore, the transgression

$$\tau\colon H^n(K(\pi,n);\pi) \to H^{n+1}(K(\pi,n+1);\pi)$$

is an isomorphism and $\tau(\iota_n) = \iota_{n+1}$, where $\iota_j \in H^j(K(\pi,j);\pi)$, with $j \in \{n, n+1\}$, are the corresponding fundamental classes.

*Proof.* Since $K(\pi,n+1)$ is 1$-$connected, we can consider the Serre spectral sequence. Since $P \cong *$, from the $n-$connectedness of the fiber, and the $(n+1)-$connectedness of the basis, by convergence

$$\tau\colon H^n(K(\pi,n);\pi) \to H^{n+1}(K(\pi,n+1);\pi)$$

is an isomorphism.

Now, consider the following diagram given by the characterization of the transgression

$$\begin{tikzcd}
 & H^{n+1}(K(\pi,n+1),x_0;\pi) \cong \tilde{H}^{n+1}(K(\pi,n+1);\pi) \arrow[r, "j^*"] \arrow[d, "p^*"] & H^{n+1}(K(\pi,n+1);\pi) \\
H^n(K(\pi,n);\pi) \arrow[r, "\delta"] \arrow[d, "\alpha"] & H^{n+1}(P,K(\pi,n);\pi) \arrow[d, "\alpha"] & \\
\mathrm{Hom}(H_n(K(\pi,n)),\pi) \arrow[r, "\partial^*"] & \mathrm{Hom}(H_{n+1}(P,K(\pi,n)),\pi) &
\end{tikzcd}$$

The inferior commutative square follows from the relation between the boundary morphisms of homotopy and homology for a pair of spaces, respectively. Hence, the vertical morphisms are given by the Universal Coefficients

Theorem. This diagram commutes by a diagram chase argument, considering the diagram from the definition of the respective boundary morphisms. Since $P \cong *$, by the homology long exact sequence of the pair $(P, K(\pi, n))$

$$\cdots \to H_i(K(\pi,n)) \to H_i(P) \to H_i(P, K(\pi,n)) \to H_{i-1}(K(\pi,n)) \to \cdots$$

so $H_{i-1}(K(\pi,n)) \cong H_i(P, K(\pi,n))$ for all $i \geq 2$. This is also valid for $i = 1$, because $H_i(P, K(\pi,n)) = 0$, since $P$ is path$-$connected. Moreover, the terms related to the functor $\mathrm{Ext}^1_{\mathbb{Z}}(-,-)$ from the Universal Coefficients Theorem satisfy

$$\mathrm{Ext}(H_{n-1}(K(\pi,n)),\pi) = 0, \quad \mathrm{Ext}(H_n(P, K(\pi,n)),\pi) \cong \mathrm{Ext}(H_{n-1}(K(\pi,n)),\pi) = 0,$$

so the vertical morphisms $\alpha$ are isomorphisms. By the homotopy$-$homology ladder diagram, the following diagram commutes

$$\begin{tikzcd}
H_{n+1}(P, K(\pi,n)) \arrow[r, "\Phi_{n+1}"] \arrow[d, "\partial"] & \pi_{n+1}(P, K(\pi,n)) \arrow[d, "\hat{\partial}"] \\
H_n(K(\pi,n)) \arrow[r, "\Phi_n"] & \pi_n(K(\pi,n))
\end{tikzcd}$$

where $\Phi_n, \Phi_{n+1}$ are the inverse of the Hurewicz morphism, and $\hat{\partial}$ is the boundary morphism, which is an isomorphism, by a similar argument to the homological case. Since the short exact sequence from the Universal Coefficients Theorem is natural in the module of coefficients, we can extend the diagram of the transgression to

$$\begin{tikzcd}
 & H^{n+1}(K(\pi,n+1),x_0;\pi) \cong \tilde{H}^{n+1}(K(\pi,n+1);\pi) \arrow[r, "j^*"] \arrow[d, "p^*"] & H^{n+1}(K(\pi,n+1);\pi) \\
H^n(K(\pi,n);\pi) \arrow[r, "\delta"] \arrow[d, "\alpha"] & H^{n+1}(P, K(\pi,n);\pi) \arrow[r, "\beta"] \arrow[d, "\alpha"] & H^{n+1}(P, K(\pi,n);\pi_{n+1}(P, K(\pi,n))) \arrow[d, "\alpha"] \\
\mathrm{Hom}(H_n(K(\pi,n)),\pi) \arrow[r, "\partial^*"] & \mathrm{Hom}(H_{n+1}(P, K(\pi,n)),\pi) \arrow[r, "\beta"] & \mathrm{Hom}(H_{n+1}(P, K(\pi,n+1)),\pi)
\end{tikzcd}$$

The vertical morphisms in the lower right commutative square are isomorphisms, together with the horizontal morphisms $\beta$, which are induced by $\hat{\delta}^{-1}\colon \pi_n(K(\pi,n)) \to \pi_{n+1}(P, K(\pi,n))$. Now, note that $\beta \circ \partial^* \circ \alpha(\iota_n) = \beta(\Phi_n \circ \partial) = \hat{\delta}^{-1} \circ \Phi_n \circ \partial = \Phi_{n+1}$, by commutativity of the homotopy$-$homology ladder diagram. Therefore, $\alpha^{-1}(\beta \circ \partial^* \circ \alpha(\iota_n)) = \iota_{n+1} \in H^{n+1}(P, K(\pi,n);\pi_{n+1}(P, K(\pi,n)))$, by definition of the fundamental class. By commutativity, and because the isomorphism $\beta$ from the upper part comes from the Universal Coefficients Theorem, $\delta(\iota_n) = \iota_{n+1} \in H^{n+1}(P, K(\pi,n);\pi)$.

Finally, since the transgression is an isomorphism, $p^*$ is an isomorphism. Additionally, because $n+1 \geq 1$, W.L.O.G. we can assume that $j^*$ is an equality, i.e. we just have to prove that $p^*$ preserves the fundamental class. This will follow from the fact that $p^*$ is induced by a map between spaces, so we can proceed analogously to the previous argument, exploiting the naturality of the inverse of the Hurewicz morphism. Since $\mathrm{Ext}(H_n(K(\pi, n+1), x_0), \pi) = 0$, we have that $\alpha\colon H^{n+1}(K(\pi,n+1),x_0;\pi) \to \mathrm{Hom}(H_{n+1}(K(\pi,n+1),x_0),\pi)$ is an isomorphism. By naturality on spaces of the short exact sequence from the Universal Coefficients Theorem, we obtain the following commutative square

$$\begin{tikzcd}
H^{n+1}(K(\pi,n+1),x_0;\pi) \arrow[r, "\alpha"] \arrow[d, "p^*"] & \mathrm{Hom}(H_{n+1}(K(\pi,n+1),x_0),\pi) \arrow[d, "\mathrm{Hom}(p_*,\pi)"] \\
H^{n+1}(P, K(\pi,n);\pi) \arrow[r, "\alpha"] & \mathrm{Hom}(H_{n+1}(P, K(\pi,n)),\pi)
\end{tikzcd}$$

Let $\iota_{n+1} \in H^{n+1}(K(\pi, n+1), x_0; \pi), \iota_{P,n} \in H^{n+1}(P, K(\pi, n); \pi)$ be the corresponding fundamental classes. Taking the inverses of the Hurewicz morphisms

$$\alpha(\iota_{n+1}) =: \Phi_{n+1}\colon H_{n+1}(K(\pi, n+1), x_0) \to \pi_{n+1}(K(\pi, n+1), x_0)$$
$$\alpha(\iota_{P,n}) =: \Phi_{P,n}\colon H_{n+1}(P, K(\pi, n)) \to \pi_{n+1}(P, K(\pi, n)).$$

By naturality of the Hurewicz morphism, the following diagram commutes

$$\begin{tikzcd} H_{n+1}(K(\pi, n+1), x_0) \arrow[r, "\Phi_{n+1}"] & \pi_{n+1}(K(\pi, n+1), x_0) \\ H_{n+1}(P, K(\pi, n)) \arrow[u, "p_*"] \arrow[r, "\Phi_{P,n}"] & \pi_{n+1}(P, K(\pi, n)) \arrow[u, "\pi_{n+1}(p, p|_{K(\pi,n)})"'] \end{tikzcd}$$

Since the involved spaces are $n-$connected, it follows that the horizontal morphisms are isomorphisms. Moreover $K(\pi, n) \to P \xrightarrow{p} K(\pi, n+1)$ is a Serre fibration with path$-$connected basis, so $\pi_{n+1}\left(p, p|_{K(\pi,n)}\right)$ is an isomorphism. Consider the (iso)morphism of $\mathbb{Z}-$modules

$$\left(\pi_{n+1}\left(p, p|_{K(\pi,n)}\right)\right)^{-1} : \pi_{n+1}(K(\pi, n+1), x_0) \to \pi_{n+1}(P, K(\pi, n)).$$

By naturality on the coefficients,

$$\begin{tikzcd} H^{n+1}(K(\pi, n+1), x_0; \pi) \arrow[r, "\alpha"] \arrow[d, "p^*"] & \mathrm{Hom}(H_{n+1}(K(\pi, n+1), x_0), \pi) \arrow[d, "\mathrm{Hom}(p_*, \pi)"] \\ H^{n+1}(P, K(\pi, n); \pi) \arrow[r, "\alpha"] \arrow[d, "\gamma"] & \mathrm{Hom}(H_{n+1}(P, K(\pi, n)), \pi) \arrow[d, "\gamma"] \\ H^{n+1}(P, K(\pi, n); \pi_{n+1}(P, K(\pi, n))) \arrow[r, "\alpha"] & \mathrm{Hom}(H_{n+1}(P, K(\pi, n)), \pi_{n+1}(P, K(\pi, n))) \end{tikzcd}$$

where $\gamma$ is induced by $\left(\pi_{n+1}\left(p, p|_{K(\pi,n)}\right)\right)^{-1}$. Therefore, $\iota_{n+1} \in H^{n+1}(K(\pi, n+1), x_0; \pi)$, so

$$\begin{aligned} \gamma \circ \mathrm{Hom}(p_*, \pi) \circ \alpha(\iota_{n+1}) &= \gamma \circ \mathrm{Hom}(p_*, \pi)(\Phi_{n+1}) \\ &= \gamma(\Phi_{n+1} \circ p_*) \\ &= \gamma\left(\pi_{n+1}\left(p, p|_{K(\pi,n)}\right) \circ \Phi_{P,n}\right) \\ &= \left(\pi_{n+1}\left(p, p|_{K(\pi,n)}\right)\right)^{-1} \circ \pi_{n+1}\left(p, p|_{K(\pi,n)}\right) \circ \Phi_{P,n} \\ &= \Phi_{P,n}. \end{aligned}$$

Hence, $\gamma \circ p^*(\iota_{n+1}) = \iota_{P,n}$. Since $\gamma$ is an isomorphism and is obtained via the Universal Coefficients Theorem, $p^*(\iota_{n+1}) = \iota_{P,n} \in H^{n+1}(P, K(\pi, n); \pi)$. Now, $p^*$ is an isomorphism, because $\pi_{n+1}\left(p, p|_{K(\pi,n)}\right)$ is an isomorphism, and the inverse of the Hurewicz morphism is natural. In conclusion, $(p^*)^{-1}$ preserves the fundamental class finishing the proof. $\square$

The Borel Theorem is the main ingredient in the proof of the Serre Theorem, the main result of this section. The proof is an inductive argument taking the fibration $K(\mathbb{Z}/2\mathbb{Z}, n) \to P \to K(\mathbb{Z}/2\mathbb{Z}, n+1)$. The transgression is a key technical concept.

**Theorem 1.5.2.3** (Mod 2 cohomology $K(\mathbb{Z}/2\mathbb{Z}, n)$)**.** There is an isomorphism

$$H^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{F}_2\left[\mathrm{Sq}^I(\iota_n) \mid e(I) < n,\ \mathrm{Sq}^I \text{ an admissible monomial}\right],$$

where $\iota_n \in H^n(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$ is a generator.

*Proof.* By induction on $n \in \mathbb{N}$. For $n = 1$, note that

$$H^*(K(\mathbb{Z}/2\mathbb{Z}, 1); \mathbb{Z}/2\mathbb{Z}) \cong H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{F}_2[\mathrm{Sq}^0 \iota_1].$$

For the inductive step, take the path−loop fibration $K(\mathbb{Z}/2\mathbb{Z}, n) \to P \to K(\mathbb{Z}/2\mathbb{Z}, n+1)$. When $n = 1$, the fiber is $K(\mathbb{Z}/2\mathbb{Z}, 1)$, with a simple system of generators given by $i_1^{2^i} = \mathrm{Sq}^{2^{i-1}} \cdots \mathrm{Sq}^2 \mathrm{Sq}^1(\iota_1)$. Since we are considering a Serre fibration with 1−connected basis, we can apply the spectral sequence. Then $E_2^{p,q}$ has the following description

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots \\
2 & \mathbb{F}_2 \cdot \iota_1^2 & 0 & \mathbb{F}_2 \cdot \iota_1^2 \iota_2 \\
1 & \mathbb{F}_2 \cdot \iota_1 \arrow[drr, "d_2"] & 0 & \mathbb{F}_2 \cdot \iota_1 \iota_2 \\
0 & \mathbb{F}_2 \cdot 1 & 0 & \mathbb{F}_2 \cdot \iota_2 \\
 & 0 & 1 & 2
\end{tikzcd}$$

**Figure 1.14:** $E_2^{p,q}$ associated to $K(\mathbb{Z}/2\mathbb{Z}, n) \to P \to K(\mathbb{Z}/2\mathbb{Z}, n+1)$.

where $\iota_2 \in H^*(K(\mathbb{Z}/2\mathbb{Z}, 2); \mathbb{Z}/2\mathbb{Z})$ is the fundamental class. Since $P \cong *$, $d_2$ is an isomorphism.

These last elements are transgressive due to the previous lemma, since $\iota_1 \in H^1(K(\mathbb{Z}/2\mathbb{Z}, 1); \mathbb{Z}/2\mathbb{Z})$ is clearly transgressive: $d_2 \colon H^1(K(\mathbb{Z}/2\mathbb{Z}, 1); \mathbb{Z}/2\mathbb{Z}) \to H^2(K(\mathbb{Z}/2\mathbb{Z}, 2); \mathbb{Z}/2\mathbb{Z})$ is a morphism on its domain, and $d_2(\iota_1) = \iota_2$, by Lemma 1.5.2.3, since $d_2$ is an isomorphism. The Borel Theorem implies that

$$H^*(K(\mathbb{Z}/2\mathbb{Z}, 2); \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{F}_2 \left[ \mathrm{Sq}^{2^i} \cdots \mathrm{Sq}^2 \mathrm{Sq}^1(\iota_2) \mid d_2(\iota_1) = \iota_2 \right].$$

The general case is similar. If

$$H^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{F}_2 \left[ \mathrm{Sq}^I(\iota_n) \mid \mathrm{Sq}^I \text{ an admissible monomial, } \ e(I) < n \right]$$

then it has a simple system of generators given by the terms $\left(\mathrm{Sq}^I(\iota_n)\right)^{2^i}$, for $i = 0, 1, \ldots$. By the admissibility lemma, these terms correspond exactly to the admissible monomials $\mathrm{Sq}^I(\iota_n)$ such that $e(I) \leq n$. These elements are transgressive, because $\iota_n$ is transgressive, and the spectral sequence has trivial rows between the 0−th and the $(n+1)$−th rows (excluding the extremes). Since $d_n(\iota_n) = \iota_{n+1}$, $d_n(\mathrm{Sq}^I(\iota_n)) = \mathrm{Sq}^I(\iota_{n+1})$. By the Borel Theorem, we obtain the desired description for the cohomology of $K(\mathbb{Z}/2\mathbb{Z}, n+1)$, finishing the proof. □

### 1.5.3 Relation with the Steenrod algebra

The following result is key in the construction of the Adams spectral sequence. Observe that taking increasing values $n \in \mathbb{N}$ that the previous computation brings us $\tilde{H}^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$ a considerable part of the Steenrod algebra $\mathcal{A}_2$. Since the left side does not depend on $n$, taking an inverse limit, in the language of *spectra* this will mean that the mod 2 cohomology of the Eilenberg-Mac Lane spectrum associated to $\mathbb{Z}/2\mathbb{Z}$ will be isomorphic to $\mathcal{A}_2$. Nevertheless, in our context, inverse limits require special care.

**Corollary 1.5.3.1** (Steenrod algebra in degree $d$)**.** The morphism defined on admissible monomials

$$\begin{aligned} \Phi \colon \mathcal{A}_2 &\to \tilde{H}^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z}) \\ \mathrm{Sq}^I &\to \mathrm{Sq}^I(\iota_n) \end{aligned}$$

is an isomorphism when it is restricted to $(\mathcal{A}_2)_d$ over $H^{n+d}(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$ for $d \leq n$. In particular, the admissible monomials $\mathrm{Sq}^I$ form an additive base for $\mathcal{A}_2$.

*Proof.* The map is surjective, since the elements of $\tilde{H}^{n+d}(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$, for $d < n$, are linear polynomials on the variables $\mathrm{Sq}^I(\iota_n)$, and the unique non-linear term, for $d = n$, is $\iota_n^2 = \mathrm{Sq}^n(\iota_n)$.

For the injectivity, note that $d(I) \geq e(I)$. This can be proven inductively, observing that

$$e(i_1, \ldots, i_k) \leq d(i_1, \ldots, i_{k-1}) - i_k \leq d(i_1, \ldots, i_{k-1}) \leq d(i_1, \ldots, i_{k-1}, i_k).$$

Furthermore, $\mathrm{Sq}^n$ is the unique monomial such that $e(\mathrm{Sq}^n) = |\mathrm{Sq}^n| = n$, because

$$i_1 = e(I) + i_2 + \cdots + i_k, \qquad d(I) = i_1 \ldots i_k.$$

Replace $i_1$ on $d(I)$ to see that $k = 1$. Thus, $i_1 = n$, so $Sq^n$ is the only possibility.

Consequently, the admissible monomials $\mathrm{Sq}^I$ with $d(I) \leq n$ are mapped to linearly independent classes in $\tilde{H}^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$, since $\mathrm{Sq}^n(\iota_n) = (\mathrm{Sq}^0(\iota_n))^2$, and $H^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$ has a simple system of generators $(\mathrm{Sq}^{I'}(\iota_n))^{2^i}$, for $i = 0, 1, \ldots$, with $e(I') < n$.

As every monomial can be expressed as a linear combination of admissible monomials, by the Adem relation, we have proven the injectivity, together with the linear independence of the admissible monomials. $\square$

## 1.6 Cohomology of Eilenberg-Mac Lane spaces: first applications

This section aims to compute the first stable homotopy groups for the sphere, following [MT68]. The main theoretical dependency is the computation of the mod 2 cohomology of Eilenberg-Mac Lane spaces. The computation will rely on a Postnikov tower-like argument and the Bockstein homomorphism to recover the $2^k-$torsion, and to compute certain related fiber cohomologies.

### 1.6.1 Preliminaries

We first start with this application of Serre classes that resembles the usage of localization in (co)homology. The computation will only need the implicance (I) $\implies$ (VII).

**Theorem 1.6.1.1** ($\mathcal{C}_p$ approximation theorem (Theorem 4, pp. 100 [MT68])). Let $X$ and $A$ be $1-$connected nice spaces (for example, CW complexes) such that $H_i(A; \mathbb{Z})$ and $H_i(X; \mathbb{Z})$ are finitely generated for each $i \in \mathbb{N}_0$. Let $f\colon A \to X$ be a map such that $f_*\colon \pi_2(A) \to \pi_2(X)$ is surjective. Then, conditions (I) – (VI) are equivalent and imply condition (VII).

(i) $f^*\colon H^i(X; \mathbb{Z}/p\mathbb{Z}) \to H^i(A; \mathbb{Z}/p\mathbb{Z})$ is an isomorphism for $i < n$ injective for $i = n$.

(ii) $f_*\colon H_i(A; \mathbb{Z}/p\mathbb{Z}) \to H_i(X; \mathbb{Z}/p\mathbb{Z})$ is an isomorphism for $i < n$ and surjective for $i = n$.

(iii) $H_i(X, A; \mathbb{Z}/p\mathbb{Z}) = 0$ for $i \geq n$.

(iv) $H_i(X, A; \mathbb{Z}) \in \mathcal{C}_p$ for $i \geq n$.

(v) $\pi_i(X, A) \in \mathcal{C}_p$ for $i \geq n$.

(vi) $f_*\colon \pi_i(A) \to \pi_i(X)$ is a $\mathcal{C}_p-$isomorphism for $i < n$ and $\mathcal{C}_p-$surjective for $i = n$.

(vii) $\pi_i(A)_{(p)} \cong \pi_i(X)_{(p)}$ for $i < n$.

The preceding theorem reduces the problem of computing the $p-$component of $\pi_i(X)$ to that of finding a space $A$ with the same $\mathbb{Z}/p\mathbb{Z}-$cohomology, together with a map $A \to X$ inducing isomorphisms in $\mathbb{Z}/p\mathbb{Z}-$cohomology.

We also require the following results related to the Bockstein morphism. These results allow us to detect mod 2 reduction of $\mathbb{Z}-$cohomology classes. The next notion, in particular, is useful to compute $H^*(K(\mathbb{Z}/2^m\mathbb{Z}, 1); \mathbb{Z}/2\mathbb{Z})$.

**Definition 1.6.1.1** (Bockstein exact couple). The *Bockstein exact couple* is of the form

$$\begin{tikzcd} D^1 = H^*(-;\mathbb{Z}) \arrow[rr, "i^1"] & & H^*(-;\mathbb{Z}) \arrow[dl, "j^1"] \\ & E^1 = H^*(-;\mathbb{Z}/2\mathbb{Z}) \arrow[ul, "k^1"] & \end{tikzcd}$$

The morphism $i^1$ is induced by multiplication by 2 in $\mathbb{Z}$; $j^1$ is induced by the mod 2 reduction morphism $\rho\colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, and $k^1$ is the Bockstein morphism $\beta$. The differential $d = j^1k^1$ is the Bockstein morphism $\delta_2\colon H^p(K, L; \mathbb{F}_2) \to H^{p+1}(K, L; \mathbb{F}_2)$ associated to the exact sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$.

For convenience, we will denote the Bockstein differentials using subscripts, thus $d_1 = d, d_2 = d_2 = j^2k^2\colon E^2 \to E^2$, etc. The operation $d_r$ acts as follows: take a cocycle in $\mathbb{Z}/2\mathbb{Z}-$coefficients; represent it by an integral cocycle; take its coboundary; divide by $2^r$ (this is possible because $d_r$ is defined only on the kernel of $d_{r-1}$); and apply mod 2 reduction on the coefficients. Notice that $d_r\colon H^*(-;\mathbb{Z}/2\mathbb{Z}) \to H^{*+1}(-;\mathbb{Z}/2\mathbb{Z})$. By the universal coefficient theorem

$$H^p(X; G) = \mathrm{Hom}(H_p(X), G) \oplus \mathrm{Ext}(H_{p-1}(X), G).$$

Therefore, a direct summand $\mathbb{Z}$ gives rise to a summand $\mathbb{Z}$ in $H^p(X;\mathbb{Z})$, and to a summand $\mathbb{Z}/2\mathbb{Z}$ in $H^p(X;\mathbb{Z}/2\mathbb{Z})$. A summand $\mathbb{Z}/2^n\mathbb{Z}$ in $H_p(X;\mathbb{Z})$ gives rises to a $\mathbb{Z}/2^n\mathbb{Z}$ in $H^{p+1}(X;\mathbb{Z})$ and to summands $\mathbb{Z}/2\mathbb{Z}$ in $H^p(X;\mathbb{Z}/2\mathbb{Z})$ and $H^{p+1}(X;\mathbb{Z}/2\mathbb{Z})$.

**Proposition 1.6.1.1** (Torsion detected by the Bockstein morphisms I)**.** Elements of $H^*(X;\mathbb{Z}/2\mathbb{Z})$ which come from *free* integral classes lie in $\mathrm{Ker}(d_r)$ for every $r \geq 0$ and not in $\mathrm{Im}(d_r)$ for all $r \geq 0$. If $z \in H^{n+1}(X;\mathbb{Z})$ generates a direct summand of order $2^r$, then there exist cyclic direct summands of order 2 in $H^n(X;\mathbb{Z}/2\mathbb{Z})$, and $H^{n+1}(X;\mathbb{Z}/2\mathbb{Z})$ generated by $z'$ and $z''$ respectively; $d_i(z'), d_i(z'') = 0$ for $i < r$, and $d_r(z') = z''$ (implicitly, $z'$ and $z''$ are also not in $\mathrm{Im}(d_r)$ for $i < r$.)

**Definition 1.6.1.2** (Persistent elements)**.** In the context of the previous result, we say that the image by $\rho$ of the free subgroups of $H^*(X;\mathbb{Z})$ *persists* to $E^\infty$ and that $z'$ and $z''$ *persists* to $E^r$ but not to $E^{r+1}$.

The following application is useful to recover $2^k-$torsion in integral cohomology.

**Corollary 1.6.1.1** (Torsion detected by the Bockstein morphisms II)**.** Let $X$ be a space, such that $H^i(X;\mathbb{Z}/2\mathbb{Z}) = 0$ for $i < n$, and $H^n(X;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}\{z\}$. Then we can infer $H^n(X;\mathbb{Z})$, except for odd prime torsion. That is, $H^n(X;\mathbb{Z}) \cong \mathbb{Z}$ if $d_r z = 0$ for all $r \geq 1$, and $H^n(X;\mathbb{Z}) = \mathbb{Z}/2^n\mathbb{Z}$ if $d_i z = 0$ for $i < n$ and $d_n z \neq 0$.

The next result is quite technical but it will have an important role.

**Theorem 1.6.1.2** (Bockstein lemma (Theorem 1, pp. 106 [MT68]))**.** Let $(E, p, B; F)$ be a fibre space. Let the class $u \in H^n(F;\mathbb{Z}/2\mathbb{Z})$ be transgressive, and suppose that, for some $i \in \mathbb{N}_{>0}$ and for some $v \in H^n(B;\mathbb{Z}/2\mathbb{Z})$, $d_i(v) = \tau(u)$. Then $d_{i+1}p^*v$ is defined, and moreover

$$j^*d_{i+1}p^*(v) = d_1(u)$$

where $j$ is the inclusion $F \subseteq E$.

Here the members of the formula $d_i(v) = \tau(u)$ and the formula of the conclusion lie in appropriate quotient groups of $H^{n+1}(F;\mathbb{Z}/2\mathbb{Z})$.

The next result is a consequence of the results developed to compute $H^*(K(\mathbb{Z}/2\mathbb{Z}, n);\mathbb{Z}/2\mathbb{Z})$ and the Bockstein morphism.

**Theorem 1.6.1.3** (Mod 2 cohomology of $K(\mathbb{Z}, n)$ and $K(\mathbb{Z}/2^k\mathbb{Z}, n)$, (Theorem 5.37, [Hat04]))**.**

(a) For $n > 1$

$$H^*(K(\mathbb{Z}, n);\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{F}_2\left[\mathrm{Sq}^I(\iota_n) \mid \mathrm{Sq}^I \text{ admissible without } \mathrm{Sq}^1 \text{ term, } e(I) < n\right].$$

(b) For $n > 1$ and $k > 1$

$$H^*(K(\mathbb{Z}/2^k\mathbb{Z}, n);\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{F}_2\left[\mathrm{Sq}^I(\iota_n), \mathrm{Sq}^J(\kappa_{n+1}) \mid \mathrm{Sq}^I, \mathrm{Sq}^J \text{ admissible without } \mathrm{Sq}^1;\ e(I) < n,\ e(J) \leq n\right],$$

where $\kappa_{n+1} \in H^{n+1}(K(\mathbb{Z}/2^k\mathbb{Z}, n);\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{\kappa_{n+1}\}$.

We will prefer the following more explicit formulation.

**Theorem 1.6.1.1** (Mod 2 cohomology of $K(\mathbb{Z}/2^m\mathbb{Z}, q)$, (Theorem 4, pp. 90 [MT68]))**.** $H^*(K(\mathbb{Z}/2^m\mathbb{Z}, q);\mathbb{Z}/2\mathbb{Z})$ is the polynomial ring with generators $\{\mathrm{Sq}^{I_m}(\iota_q)\}$ where we define $\mathrm{Sq}^{I_m} := \mathrm{Sq}^I$ if $I$ terminates in $i_r > 1$ and $\mathrm{Sq}^{I_m} := \mathrm{Sq}^{i_1,\ldots,i_{r-1}}d_m$ (i.e. $\mathrm{Sq}^I$ with $d_m$ replacing $\mathrm{Sq}^{i_r}$) if $i_r = 1$; and where $I$ runs through admissible sequences of $e(I) < q$.

### 1.6.2 First stable homotopy groups

Now, we can state and prove the following result. By the previous results related to the minimal $p-$torsion in $\pi_*(S^n)$, it will be just needed to compute $\pi_i(S^n)_{(2)}$ for $i = 1, 2, 3$. The proof depends on the computation of $H^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$, $H^*(K(\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$ and $H^*(K(\mathbb{Z}/2^k\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$. The idea of the proof consists of constructing fibrations approximating the $n-$sphere by an inductive Postnikov tower-like argument. In each approximation, specific cohomology classes will be killed to make the fiber's canonical class a transgressive element. The role of the Bockstein morphisms is to detect the $2^k-$torsion and to compute a basis for the cohomology of the fibers.

**Theorem 1.6.2.1** (First stable homotopy groups)**.** We have the following isomorphisms.

(i) $\pi_1^s(S^0) \cong \mathbb{Z}/2\mathbb{Z}$.

(ii) $\pi_2^s(S^0) \cong \mathbb{Z}/2\mathbb{Z}$.

(iii) $\pi_3^s(S^0) \cong \mathbb{Z}/24\mathbb{Z}$.

*Proof.* Since the stated assertion is concerned with stable homotopy groups, assume that $n \gg 1$. This ensures that we will be working with the Steenrod algebra and that we will not need to consider cup products in the spectral sequence arguments. Consider the map $\mathrm{Sq}^2 \colon K(\mathbb{Z}, n) \to K(\mathbb{Z}/2\mathbb{Z}, n+2)$ which represents $\mathrm{Sq}^2(\iota_n) \in H^{n+2}(K(\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$. Essentially, we will construct a Postnikov tower inductively. The first step is the following. Take the path-loop fibration $K(\mathbb{Z}/2\mathbb{Z}, n+1) \to P \to K(\mathbb{Z}/2\mathbb{Z}, n+2)$. Now, take the pullback of this fibration under $\mathrm{Sq}^2 \colon K(\mathbb{Z}, n) \to K(\mathbb{Z}/2\mathbb{Z}, n+2)$, as the following commutative diagram shows.

$$\begin{tikzcd}
F := K(\mathbb{Z}/2\mathbb{Z}, n+1) \arrow[d, "i"] & K(\mathbb{Z}/2\mathbb{Z}, n+1) \arrow[d] \\
X_1 \arrow[r, "F"] \arrow[d, "p"] & P \arrow[d, "\mathrm{ev}_1"] \\
B := K(\mathbb{Z}, n) \arrow[r, "\mathrm{Sq}^2"] & K(\mathbb{Z}/2\mathbb{Z}, n+2)
\end{tikzcd}$$

By commutativity of the previous diagram, $\mathrm{Sq}^2 \circ p = \mathrm{ev}_1 \circ F$. Therefore, $p^* \circ \mathrm{Sq}^2 = F^* \circ (\mathrm{ev}_1)^*$. Because $P \cong \{*\}$, it follows that $p^* \circ \mathrm{Sq}^2 = 0$, hence $p^*\mathrm{Sq}^2\iota_n = 0 \in H^{n+2}(X_1; \mathbb{Z}/2\mathbb{Z})$, i.e. we have killed the cohomology class $\mathrm{Sq}^2\iota_n \in H^{n+2}(K(\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z})$. Since $K(\mathbb{Z}, n)$ is 1$-$connected, by the cohomological Serre spectral sequence with $\mathbb{Z}/2\mathbb{Z}-$coefficients we have the following diagram.

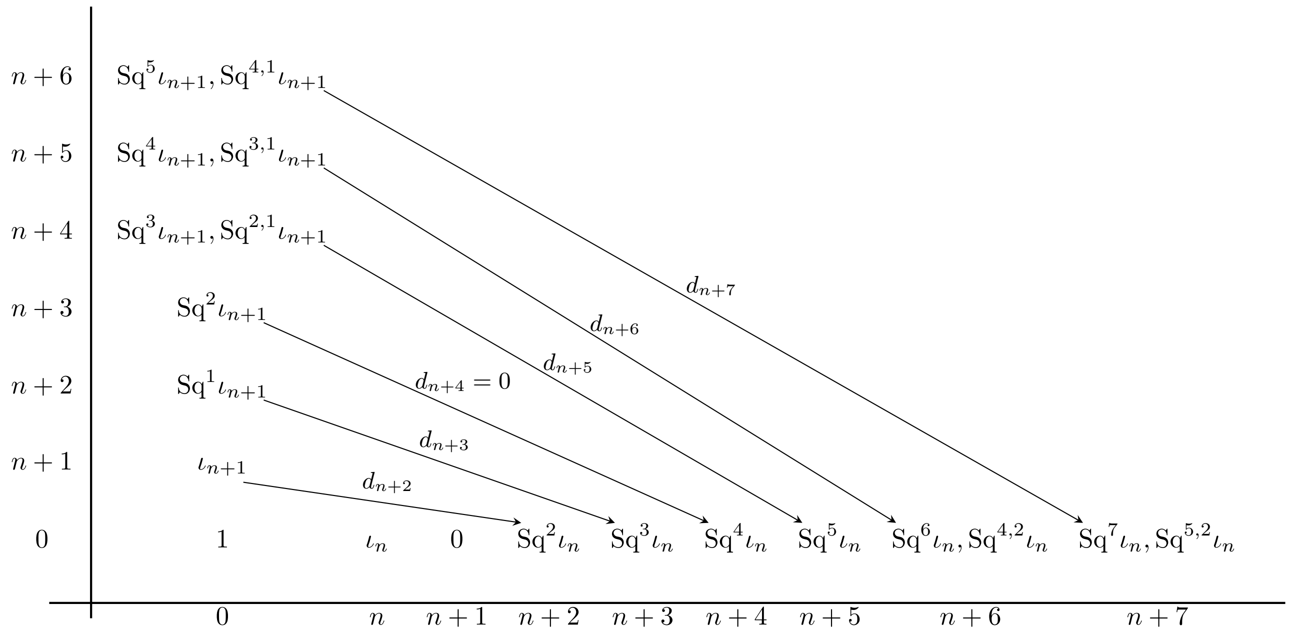


**Figure 1.15:** Serre spectral sequence associated to $K(\mathbb{Z}/2\mathbb{Z}, n+1) \to X_1 \to K(\mathbb{Z}, n)$.

Since we have killed $\mathrm{Sq}^2\iota_n \in H^{n+2}(X_1;\mathbb{Z}/2\mathbb{Z})$, we have that $\iota_{n+1} \in F = H^*(K(\mathbb{Z}/2\mathbb{Z}, n+1);\mathbb{Z}/2\mathbb{Z})$ is transgressive. Therefore, by the commutativity of transgressive differentials with the Steenrod operations, it follows that $d_{n+3}(\mathrm{Sq}^1\iota_{n+1}) = \mathrm{Sq}^1(d_{n+2}(\iota_{n+1})) = \mathrm{Sq}^1\mathrm{Sq}^2(\iota_n) = \mathrm{Sq}^3(\iota_n)$, applying the Adem relations. Moreover, $d_{n+4}(\mathrm{Sq}^2\iota_{n+1}) = \mathrm{Sq}^2\mathrm{Sq}^2\iota_n = \mathrm{Sq}^3\mathrm{Sq}^1\iota_n = 0$, and $d_{n+5}(\mathrm{Sq}^{2,1}\iota_{n+1}) = \mathrm{Sq}^{2,1,2}\iota_n = \mathrm{Sq}^5\iota_n + \mathrm{Sq}^{4,1}\iota_n = \mathrm{Sq}^5\iota_n$, $d_{n+5}(\mathrm{Sq}^3\iota_{n+1}) = \mathrm{Sq}^{3,2}\iota_n = 0$. Similarly, $d_{n+6}(\mathrm{Sq}^4\iota_{n+1}) = \mathrm{Sq}^{4,2}\iota_n$ and $d_{n+6}(\mathrm{Sq}^{3,1}\iota_{n+1}) = \mathrm{Sq}^{3,1,2}\iota_n = \mathrm{Sq}^{5,1}\iota_n = 0$. Analogously, $d_{n+7}(\mathrm{Sq}^5\iota_{n+1}) = \mathrm{Sq}^{5,2}\iota_n$ and $d_{n+7}(\mathrm{Sq}^{4,1}\iota_{n+1}) = \mathrm{Sq}^{5,2}\iota_n$.

Now, by the convergence of the Serre spectral sequence, and applying the same argument with $d_{n+3}$, we obtain

$$\tilde{H}^i(X_1;\mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z}, & i = n, \\ 0 & i < n \text{ or } n < i \leq n+2. \end{cases}$$

Additionally, the previous computation with the transgression gives us the following generators (as a $\mathbb{Z}/2\mathbb{Z}-$module) in the following degrees

$$\begin{aligned} &H^{n+3}(X_1;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{a\}, \\ &H^{n+4}(X_1;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{b\} \oplus \mathbb{Z}/2\mathbb{Z}\{p^*(\mathrm{Sq}^4\iota_n)\}, \\ &H^{n+5}(X_1;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{c\}, \\ &H^{n+6}(X_1;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{d\} \oplus \mathbb{Z}/2\mathbb{Z}\{p^*(\mathrm{Sq}^6\iota_n)\}, \end{aligned}$$

where

$$\begin{aligned} i^*(a) &= \mathrm{Sq}^2\iota_{n+1}, \\ i^*(b) &= \mathrm{Sq}^3\iota_{n+1}, \\ i^*(c) &= \mathrm{Sq}^{3,1}\iota_{n+1}, \\ i^*(d) &= \mathrm{Sq}^5\iota_{n+1} + \mathrm{Sq}^{4,1}\iota_{n+1}. \end{aligned}$$

This will be required to compute a basis for $H^{n+i}(X_2;\mathbb{Z}/2\mathbb{Z})$ during the next step of this inductive argument.

Now let $f\colon S^n \to K(\mathbb{Z}, n)$ represent the homotopy class of a generator of $\pi_n(K(\mathbb{Z}, n)) = \mathbb{Z}$. The composition

$$S^n \xrightarrow{f} K(\mathbb{Z}, n) \xrightarrow{\mathrm{Sq}^2} K(\mathbb{Z}/2\mathbb{Z}, n+2)$$

is null-homotopic, since $\pi_n(K(\mathbb{Z}/2\mathbb{Z}, n+2)) = 0$. Therefore, by the homotopy lifting property applied to the path-loop fibration, and the universal property of the pullback, this induces a map $f_1 \colon S^n \to X_1$. Note that

$$(f_1)^* \colon H^i(X_1; \mathbb{Z}/2\mathbb{Z}) \to H^i(S^n; \mathbb{Z}/2\mathbb{Z})$$

is an isomorphism for $i \leq n+1$ and surjective for $i = n+2$. By the homotopy exact sequence, $\pi_{n+1}(X_1)_{(2)} \cong \mathbb{Z}/2\mathbb{Z}$. Applying the $\mathcal{C}_p$ approximation theorem with $\mathcal{C}_2$, and noticing that $\pi_{n+2}(S^n)$ does not have odd $p-$torsion, it follows that

$$\pi_{n+1}(S^n) \cong \pi_{n+1}(S^n)_{(2)} \cong \pi_{n+1}(X_1)_{(2)} \cong \mathbb{Z}/2\mathbb{Z}.$$

Before we can continue, we will require the following lemma, which gives useful Bockstein relations in $H^*(X_1; \mathbb{Z}/2\mathbb{Z})$. These relations are needed because we do not know the exact behavior of the squaring operations in $H^*(X_1; \mathbb{Z}/2\mathbb{Z})$.

**Lemma 1.6.2.1** (Bockstein relations in $H^*(X_1; \mathbb{Z}/2\mathbb{Z})$)**.** In $H^*(X_1; \mathbb{Z}/2\mathbb{Z})$, there are the following relations in terms of the Bockstein differentials $\beta_i \colon H^*(X_1; \mathbb{Z}/2\mathbb{Z}) \to H^{*+1}(X_1; \mathbb{Z}/2\mathbb{Z})$.

(a) $\beta_1(a) = b + k \cdot p^*(\mathrm{Sq}^4 \iota_n)$ for some $k \in \mathbb{Z}$.

(b) $\beta_2(p^*(\mathrm{Sq}^4 \iota_n)) = c$.

(c) $\beta_1(c) = 0$.

*Proof.*

(a) We have $i^*(a) = \mathrm{Sq}^2 \iota_{n+1}$ by definition of $a \in H^{n+3}(X_1; \mathbb{Z}/2\mathbb{Z})$. Therefore $i^*(\beta_1 a) = \beta_1 i^*(a) = \beta_1 \mathrm{Sq}^2 \iota_{n+1} = \mathrm{Sq}^3 \iota_{n+1}$. Moreover, $i^*(b) = \mathrm{Sq}^3 \iota_{n+1}$. Since $p^* \mathrm{Sq}^4 \iota_n \in \mathrm{Im}(p^*)$, $i^* p^* \mathrm{Sq}^4 \iota_n = 0$, by the (cohomological) Serre exact sequence associated to the fibration $K(\mathbb{Z}/2\mathbb{Z}, n+1) \to X_1 \to K(\mathbb{Z}, n)$.

(b) Notice that $\beta_1 \mathrm{Sq}^4 \iota_n = \mathrm{Sq}^1 \mathrm{Sq}^4 \iota_n = \mathrm{Sq}^5 \iota_n = d_{n+5}(\mathrm{Sq}^{2,1} \iota_{n+1})$. Thus $\beta_1 p^* \mathrm{Sq}^4 \iota_n = 0$ since $p^* d_{n+5} = 0$ (remember that $d_{n+5}$ is the transgression). By the Bockstein lemma (Lemma 1.6.1.2),

$$i^* \beta_2 p^* \mathrm{Sq}^4 \iota_n = \beta_1 (Sq^{2,1} \iota_{n+1}) = \mathrm{Sq}^{3,1} \iota_{n+1} = i^*(c).$$

Since $H^{n+5}(X_1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{c\}$, the result follows.

(c) This follows from (b) since $\beta_1(u) = 0$ for all $u \in \mathrm{Im}(\beta_r)$.

■

The next step in this inductive computation is to kill the generator $H^{n+3}(X_1; \mathbb{Z}/2\mathbb{Z})$ in such a way that the canonical class in the fiber cohomology transgresses too. Consider the following commutative diagram, obtained analogously from the previous one.

$$\begin{tikzcd}
& F_2 := K(\mathbb{Z}/2\mathbb{Z}, n+2) \arrow[d, "i"] & K(\mathbb{Z}/2\mathbb{Z}, n+2) \arrow[d] \\
& X_2 \arrow[r] \arrow[d, "p_2"] & P \arrow[d] \\
S^n \arrow[ur, dashed, "f_2"] \arrow[r, "f_1"] & X_1 \arrow[r, "a"] & K(\mathbb{Z}/2\mathbb{Z}, n+3)
\end{tikzcd}$$

As in the previous step, consider the associated cohomological Serre spectral sequence, described in the next figure.

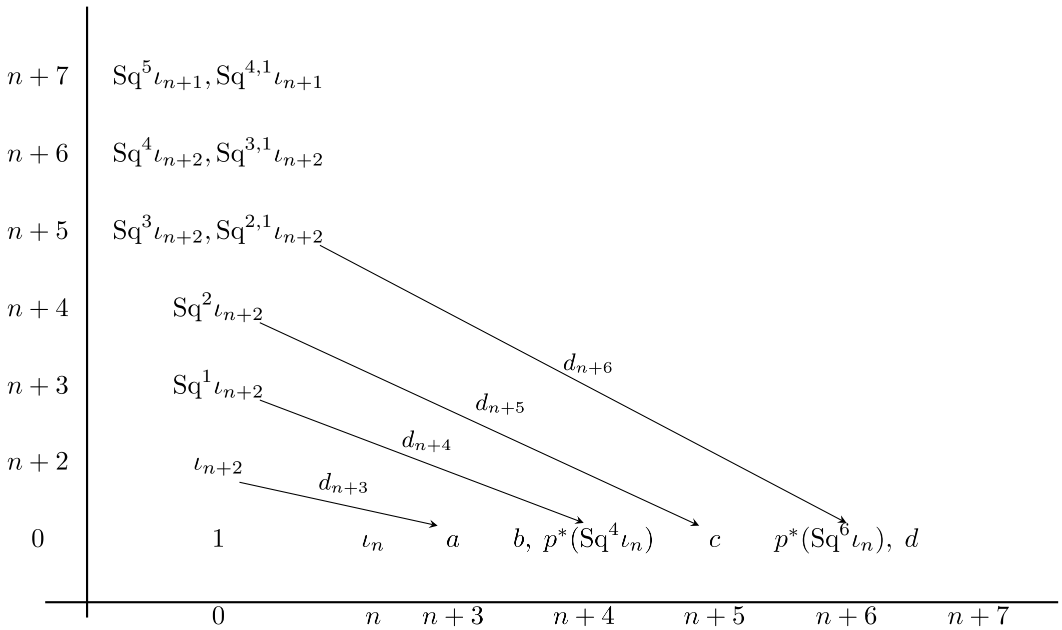


**Figure 1.16:** Serre spectral sequence associated to $K(\mathbb{Z}/2\mathbb{Z}, n+2) \to X_2 \to X_1$.

By construction $d_{n+3}(\iota_{n+2}) = a$. Then, $d_{n+4}(\mathrm{Sq}^1\iota_{n+2}) = \mathrm{Sq}^1(a) = \beta_1(a) = b + k \cdot p^*(\mathrm{Sq}^4\iota_n)$, and $d_{n+5}(\mathrm{Sq}^2\iota_{n+2}) = \mathrm{Sq}^2(a)$. We must verify if $\mathrm{Sq}^2(a) = c$ or 0. Note that $a = i^*(\mathrm{Sq}^2\iota_{n+1})$, so $\mathrm{Sq}^2(a) = i^*(\mathrm{Sq}^{2,2}\iota_{n+1}) = i^*(\mathrm{Sq}^{3,1}\iota_{n+1}) = i^*(c)$. Hence, $d_{n+5}(\mathrm{Sq}^2\iota_{n+2}) = \mathrm{Sq}^2 a = c$.

Similarly, $d_{n+6}(\mathrm{Sq}^{2,1}\iota_{n+2}) = d + m \cdot p^*\mathrm{Sq}^6\iota_n$, for some $m \in \mathbb{Z}$, since

$$i^*(\mathrm{Sq}^{2,1}a) = \mathrm{Sq}^{2,1}(i^*a) = \mathrm{Sq}^{2,1,2}(\iota_{n+1}) = \left(\mathrm{Sq}^5 + \mathrm{Sq}^{4,1}\right)(\iota_{n+1}) = i^*(d).$$

Hence, $\mathrm{Sq}^{2,1}a = d + m \cdot p^*\mathrm{Sq}^6\iota_n$ where $m \in \mathbb{Z}$. The same argument gives us $d_{n+6}(\mathrm{Sq}^3\iota_{n+2}) = 0 + n \cdot p^*\mathrm{Sq}^6\iota_n$, for some $n \in \mathbb{Z}$. Notice that $\mathrm{Sq}^3\iota_{n+2} = \mathrm{Sq}^{1,2}\iota_{n+2}$. Thus,

$$d_{n+6}(\mathrm{Sq}^3\iota_{n+2}) = \mathrm{Sq}^1\left(d_{n+5}(\mathrm{Sq}^2\iota_{n+2})\right) = \mathrm{Sq}^1(c).$$

But $\mathrm{Sq}^1(c) = 0$ by Lemma 1.6.2.1. Therefore, $d_{n+6}(\mathrm{Sq}^3\iota_{n+2}) = 0$. Now it follows that $\mathrm{Coker}(d_{n+6}) = \mathbb{Z}/2\mathbb{Z}\{(p_2)^*p^*\mathrm{Sq}^6(\iota_n)\}$.

Therefore, this computation gives us that $H^{n+3}(X_2; \mathbb{Z}/2\mathbb{Z}) = 0$ and $H^{n+4}(X_2; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}\{(p_2)^*p^*\mathrm{Sq}^4\iota_n\}$, assuming WLOG that $k = 0$ (since $\dim \mathrm{Coker}(\tau) = 1$, and thus we can identify the corresponding generator with $(p_2)^*p^*\mathrm{Sq}^4\iota_n$). Analogously, $H^{n+5}(X_2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{(p_2)^*p^*\mathrm{Sq}^6(\iota_n)\}$.

In summary, we have computed the following partial basis for $H^{n+i}(X_2; \mathbb{Z}/2\mathbb{Z})$.

$$\begin{aligned}
&H^{n+3}(X_2; \mathbb{Z}/2\mathbb{Z}) = 0, \\
&H^{n+4}(X_2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{(p_2)^*p^*\mathrm{Sq}^4\iota_n\}, \\
&H^{n+5}(X_2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\{A\},
\end{aligned}$$

where

$$i^*(A) = \mathrm{Sq}^3(\iota_{n+2}).$$

Therefore, by the homotopy exact sequence and the $\mathcal{C}_p-$approximation theorem, we obtain that $\pi_{n+2}(S^n) \cong \mathbb{Z}/2\mathbb{Z}$.

We will need the following Bockstein relations.

**Lemma 1.6.2.2** (Bockstein relations for $H^*(X_2; \mathbb{Z}/2\mathbb{Z})$)**.** In $H^*(X_2; \mathbb{Z}/2\mathbb{Z})$, $\beta_3\left((p_2)^*p^*\mathrm{Sq}^4\iota_n\right) = A$, where $A \in H^{n+5}(X_2; \mathbb{Z}/2\mathbb{Z})$ is defined by $i^*A = \mathrm{Sq}^3\iota_{n+2}$.

*Proof.* By the Bockstein lemma (Lemma 1.6.1.2) and Lemma 1.6.2.1, we have

$$\beta_2 p^*\mathrm{Sq}^4\iota_n = c = d_{n+5}(\mathrm{Sq}^2\iota_{n+2}) \in H^{n+5}(X_1;\mathbb{Z}/2\mathbb{Z}).$$

Thus,

$$i^*\left(\beta_3(p_2)^*p^*\mathrm{Sq}^4\iota_n\right) = \beta_1(\mathrm{Sq}^2\iota_{n+2}) = \mathrm{Sq}^3\iota_{n+2} = i^*A.$$

■

Now, we would try to kill $H^{n+4}(X_2;\mathbb{Z}/2\mathbb{Z})$ representing $(p_2)^*p^*\mathrm{Sq}^4\iota_n$ as a map $X_2 \to K(\mathbb{Z}/2\mathbb{Z};n+4)$. However, it will follow that $d_{n+5}(\mathrm{Sq}^1\iota_{n+3}) = 0$, since $\beta_1(p_2)^*p^*\mathrm{Sq}^4\iota_n = 0$. Since $\beta_1((p_2)^*p^*\mathrm{Sq}^4\iota_n) = \beta_2((p_2)^*p^*\mathrm{Sq}^4\iota_n) = 0$, by Proposition 1.6.1.1, $(p_2)^*p^*\mathrm{Sq}^4\iota_n$ corresponds to the mod 2 reduction of some $\mathbb{Z}/8\mathbb{Z}-$cohomology class. Let $C \in H^{n+4}(X_2;\mathbb{Z}/8\mathbb{Z})$ be such class. Consider a map $X_2 \to K(\mathbb{Z}/8\mathbb{Z},n+4)$ representing $C$.

Analogously as before, consider the following diagram that gives us $X_3$.

$$\begin{tikzcd}
 & K(\mathbb{Z}/8\mathbb{Z},n+3) \arrow[d, "i"] & K(\mathbb{Z}/8\mathbb{Z},n+3) \arrow[d] \\
 & X_3 \arrow[r] \arrow[d, "p_3"] & P \arrow[d] \\
S^n \arrow[ur, dashed, "f_3"] \arrow[r, "f_2"] & X_2 \arrow[r, "\overline{(p_2)^*p^*\mathrm{Sq}^4}"] & K(\mathbb{Z}/8\mathbb{Z},n+4)
\end{tikzcd}$$

Then, by Theorem 1.6.1.1 $H^{n+4}(K(\mathbb{Z}/8\mathbb{Z},n{+}3);\mathbb{Z}/2\mathbb{Z})$ is generated by $\beta_3\iota_{n+3}$ which transgresses to $A = \beta_3(p_2)^*p^*\mathrm{Sq}^4\iota_n$. Applying the cohomological Serre spectral sequence, and our computation of

$$H^*(K(\mathbb{Z}/2^r\mathbb{Z},m);\mathbb{Z}/2\mathbb{Z}) \cong H^*(K(\mathbb{Z}/2\mathbb{Z},m);\mathbb{Z}/2\mathbb{Z})$$

(as graded ring algebras) brings us:

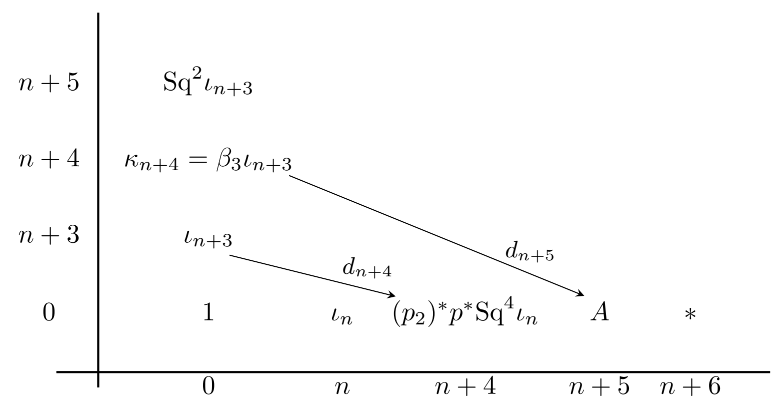


**Figure 1.17:** Serre spectral sequence associated to $K(\mathbb{Z}/8\mathbb{Z},n+3) \to X_3 \to X_2$.

By the homotopy exact sequence, it follows that $\pi_{n+3}(X_3) = \mathbb{Z}/8\mathbb{Z}$. By the $\mathcal{C}_p-$approximation theorem, we conclude that $\pi_{n+3}(S^n)_{(2)} \cong \mathbb{Z}/8\mathbb{Z}$, for $n \gg 1$. □

Observe that this method is not entirely inductive. Moreover, eventually, we must stop due to extension problems that are not trivial to solve. Nevertheless, it is also valid for computing unstable homotopy.

# Chapter 2

# The category of CW spectra

## 2.1 Introduction

This section gives a brief overview of the Boardman category of spectra. It will follow the same lines as [Hat04] and [Swi75]. For a comprehensive treatment, see [Ada74]. There are many reasons to consider spectra instead of spaces. The construction of the Adams spectral sequence is simplified using this notion. Furthermore, this is a rich subject on its own, and there are many modern approaches to spectra, providing interesting connections with Algebraic Geometry and Number Theory. Furthermore, the Brown Representability Theorem states that generalized cohomology theories are represented by spectra. However, this introduction will just define the basic notions required. In particular, the construction of the (handcrafted) smash product will be avoided, which is necessary to treat generalized cohomology theories defined from spectra other than the Eilenberg-Mac Lane spectrum.

The stable homotopy category, i.e. the (homotopy) category of spectra behaves more like abelian groups than spaces. This is more apparent when we consider the smash product, which corresponds to the tensor product in the category of spectra, endowing it with a symmetric closed monoidal structure. Its monoidal unit corresponds to the sphere spectrum $\mathbb{S}$, which is the homotopical analog of $\mathbb{Z}$. However, the construction of the smash product is non-trivial, following the lines in [Hat04] and [Swi75].

In the following, the subsequent sections will give an approach to define the category of spectra, together with their (co)homology and homotopy. The definition of the cohomology of spectra will be subtler than the rest due to the inverse limits involved in their definition. On the other hand, we will see that this category of spectra is more well-behaved than spaces: inclusions are always cofibrations, and the suspension functor has an inverse. Then, we will translate the Whitehead and the Hurewicz theorem to the language of spectra, construct the cofibre sequence, which is equivalent to the fibre sequence, concluding with the main result: the representability theorem for (cellular) cohomology of spectra. See [Lur17], [MMSS01], [HSS00] or [EKMM97] for more modern perspectives to spectra. Beware that each of these constructions has its advantages and disadvantages.

## 2.2 Spectra

We will replace topological spaces with more suitable objects for studying stable phenomena.

**Definition 2.2.0.1** (Spectrum)**.** A *spectrum* is a sequence of pointed spaces $\{X_n\}_{n\in\mathbb{N}_0}$ together with basepoint-preserving *structural* maps $\Sigma X_n \to X_{n+1}$.

There are two instances of spectra of our main interest. The next construction is functorial.

**Definition 2.2.0.2** (Suspension spectrum of a space)**.** Let $X$ be a space. Define $X_n := \Sigma^n X$ with $\mathrm{Id}\colon \Sigma X_n \to X_{n+1}$. This is the *suspension spectrum* $\Sigma^\infty X$.

**Definition 2.2.0.3** (Eilenberg-Mac Lane spectrum)**.** Let $G$ be an abelian group. Define $X_n := K(G,n)$ as a CW complex, and let $\Sigma K(G,n) \to K(G,n+1)$ be the adjoint of a map giving a CW-approximation $K(G,n) \xrightarrow{\cong} \Omega K(G,n+1)$. This is an *Eilenberg-Mac Lane spectrum*.

More generally, by shifting dimensions, we can also define the *shifted Eilenberg-Mac Lane spectrum* $X_n := K(G,m+n)$ for some fixed $m \in \mathbb{N}_0$, with maps $\Sigma K(G,m+n) \to K(G,m+n+1)$ as before.

Intuitively, spectra should be the natural domain of stable phenomena. Thus, the homotopy groups of a suspension spectrum associated with a space $X$ should correspond to $\pi_*^s(X)$.

**Definition 2.2.0.4** (Homotopy groups of a spectrum)**.** Let $X = \{X_n\}_{n\in\mathbb{N}_0}$ be a spectrum. Consider the following sequence

$$\cdots \longrightarrow \pi_{n+1}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \xrightarrow{(f_n)_*} \pi_{i+n+1}(X_{n+1}) \xrightarrow{\Sigma} \pi_{i+n+2}(\Sigma X_{n+1}) \longrightarrow \dots$$

where $f_n\colon \Sigma X_n \to X_{n+1}$ is part of the structure of the spectrum $X$. Define the *homotopy groups* of the spectrum $X$ as

$$\pi_i(X) := \underrightarrow{\operatorname{colim}}_{n\in\mathbb{N}_0} \pi_{n+i}(X_n).$$

For a suspension spectrum, the structural maps $f_n\colon \Sigma X_n \to X_{n+1}$ are just the identity. This proves the following proposition.

**Proposition 2.2.0.1** (Stable homotopy groups and the suspension spectrum)**.** Let $X$ be a space. Consider the associated suspension spectrum denoted by $\Sigma^\infty X$. Then,

$$\pi_*(\Sigma^\infty X) \cong \pi_*^s(X).$$

The Eilenberg-Mac Lane spectrum gives us an immediate analog of the Eilenberg-Mac Lane spaces.

**Proposition 2.2.0.2** (Eilenberg-Mac Lane spectrum)**.** Let $X := \{X_n := K(G, m+n)\}_{n\in\mathbb{N}_0}$ be a shifted Eilenberg-Mac Lane spectrum associated with the abelian group $G$. Then,

$$\pi_i(X) = \begin{cases} G & i = m, \\ 0 & i \neq m. \end{cases}$$

*Proof.* The Freudenthal suspension theorem implies that the structural map $\Sigma K(G, m+n) \to K(G, m+n+1)$ induces an isomorphism up to approximately $2(m+n)$, concluding the stated assertion. $\square$

At this point, it is possible to define homology and cohomology analogously. Nevertheless, these definitions will be postponed. This is because the definition of cohomology is more subtle, expressed as an inverse limit.

## 2.3 CW Spectra

It is preferable to restrict ourselves to a more structured class of spectra. It turns out, as in the case of CW complexes, that it will be enough for our purposes.

**Definition 2.3.0.1** (CW spectrum I)**.** Let $X$ be a spectrum, whose spaces $X_n$ are CW-complexes. $X$ is a *CW spectrum* if the structural maps $\Sigma X_n \to X_{n+1}$ are inclusions of subcomplexes.

By abuse of language, we will sometimes call such CW (sub)spectrum as a *(sub)spectrum*.

**Definition 2.3.0.2** (CW subspectrum)**.** A *CW subspectrum* $Y \subseteq X$ consists of subcomplexes $Y_n \subseteq X_n$ such that $\Sigma Y_n \subseteq Y_{n+1}$ for all $n \in \mathbb{N}_0$.

It is convenient to give a CW spectrum description to the Eilenberg-Mac Lane spectrum. It turns out that it is homotopy equivalent (in the category of spectra) to the previous definition.

**Proposition 2.3.0.1** (Eilenberg-Mac Lane CW spectrum)**.** Let $G$ be an abelian group. The Eilenberg-Mac Lane spectrum described before admits a cellular model.

*Proof.* Consider $X_n = K(G, m+n)$ as a CW complex. Define $X_{n+1}$ inductively from $\Sigma X_n$, attaching cells to kill $\pi_i(\Sigma X_{n+1})$ for $i > m+n+1$. The Freudenthal suspension theorem allows us to take attaching cells with dimension greater than approximately $2(m+n)$. $\square$

There is more analogy with CW complexes. The following definition makes use of the *reduced* suspension.

**Definition 2.3.0.3** (Stable $n$-cell)**.** Let $X$ be a CW spectrum. Consider all the nonbasepoint cells $e^i_\alpha$ of $X_n$, and take equivalence classes under the relation of being the $k$-suspension of another nonbasepoint cell, for $k \in \mathbb{Z}$. These equivalence classes are called the *stable cells* of $X$ or just *cells* if it's clear from the context. Thus, each equivalence class consists of cells $e^{k+n}_\alpha$ of $X_n$ for all $n \geq n_\alpha$, for some $n_\alpha$. In such a case, we say that the dimension of the stable cell is $k \in \mathbb{Z}$.

As suggested by the notation, a stable cell can have negative dimension. This means that we could index the structural spaces of a given spectrum by $\mathbb{Z}$.

**Example 2.3.0.1** (Stable cells of negative dimension)**.** Let $X$ be the CW spectrum with $X_n := S^1 \vee S^2 \vee \dots$, for all $n \in \mathbb{N}_0$, and the structural maps given by the inclusion $\Sigma X_n \hookrightarrow X_{n+1}$. In this case, we have one cell for each dimension in $\mathbb{Z}$. □

However, we are concerned with CW spectra with stable cells whose dimensions are bounded below.

**Definition 2.3.0.4** (Connective spectrum)**.** Let $X$ be a CW spectrum. We say that $X$ is *connective* if the dimensions of the stable cells are bounded below.

The following proposition justifies the previous name.

**Proposition 2.3.0.2** (Connectivity in connective spectra)**.** For a connective spectrum $X$, the connectivity of the spaces $X_n$ goes to infinity, as $n \to \infty$.

*Proof.* Assume that $X_n$ is $M-$connected but not $(M+1)-$connected for all $n \geq n_0$, where $n_0 \in \mathbb{Z}_{>0}$ is fixed, $M \geq 0$. Then we must have $(M+1)-$cells in each $X_n$, $n \geq n_0$. But this contradicts that $X$ is connective. □

We will define (co)homology for spectra.

**Definition 2.3.0.5** (Homology of a CW spectrum I)**.** Let $X$ be a CW spectrum. Let $G$ be an abelian group. Consider the following directed system of cellular chain complexes

$$\dots \hookrightarrow C_*(X_n; G) \hookrightarrow C_*(\Sigma X_n; G) \hookrightarrow C_*(X_{n+1}; G) \hookrightarrow \dots .$$

Let

$$C_*(X; G) := \underrightarrow{\operatorname{colim}}_{n \in \mathbb{N}_0} C_*(X_n; G).$$

Define the *(cellular) homology* groups of the spectrum $X$ as

$$H_i(X; G) = \operatorname{Ker}(\partial\colon C_i(X; G) \to C_{i-1}(X; G)) / \operatorname{Im}(\partial\colon C_{i+1}(X; G) \to C_i(X; G)) .$$

Notice that the chain complex associated with the spectrum $X$ is freely generated by the stable cells. Since homology commutes with colimits, we have the following alternative definition.

**Proposition 2.3.0.3** (Homology of a CW spectrum II)**.** Let $X$ be a CW spectrum. Then,

$$H_i(X; G) \cong \underrightarrow{\operatorname{colim}}\, H_{i+n}(X_n; G).$$

Observe that since we could have negative-dimensional (stable) cells, we could have non-trivial (co)homology in negative degrees.

**Definition 2.3.0.6** (Cohomology of a CW spectrum)**.** Let $X$ be a CW spectrum. Let $G$ be an abelian group. Define the following cochain complex as $C^*(X; G) := \operatorname{Hom}(C_*(X; \mathbb{Z}), G)$. We define the *(cellular) cohomology* groups of the spectrum $X$ as

$$H^i(X; G) = \operatorname{Ker}(\partial\colon C^i(X; G) \to C^{i+1}(X; G)) / \operatorname{Im}(\partial\colon C^{i-1}(X; G) \to C^i(X; G)) .$$

The next proposition is similar to the homotopy case.

**Proposition 2.3.0.4** ($\Sigma^\infty$ and (co)homology)**.** Let $X$ be a spectrum. Then $H_*(X; G) \cong H_*(\Sigma^\infty X; G)$ and $H^*(X; G) \cong H^*(\Sigma^\infty; G)$.

The definition of cohomology for a CW spectrum brings us the next results.

**Proposition 2.3.0.5** (Universal coefficient theorem for CW spectrum I (pp. 586. [Hat04])). Let $G$ be an abelian group and $X$ a CW spectrum. There exists a natural short exact sequence of abelian groups

$$0 \to H_k(X;\mathbb{Z}) \otimes_{\mathbb{Z}} G \to H_k(X;G) \to \operatorname{Tor}(H_{k-1}(X;\mathbb{Z}), G) \to 0$$

which splits (non-naturally).

*Proof.* The proof is the same as for the case of singular (co)homology of spaces. ☐

**Proposition 2.3.0.6** (Universal coefficient theorem for CW spectrum II (pp. 586. [Hat04])). Let $G$ be an abelian group and $X$ a CW spectrum. There exists a natural short exact sequence of abelian groups

$$0 \to \operatorname{Ext}(H_{k-1}(X;\mathbb{Z}), G) \to H^k(X;G) \to \operatorname{Hom}(H_k(X;\mathbb{Z}), G) \to 0$$

which splits (non-naturally).

*Proof.* The proof is the same as for the case of singular (co)homology of spaces. ☐

As expected, there are spectra of finite type.

**Definition 2.3.0.7** (Finite CW spectrum). A CW spectrum is said to be *finite* if it has just finitely many cells.

**Definition 2.3.0.8** (Spectrum of finite type). A CW spectrum is said to be of *finite type* if it has only finitely many cells in each dimension.

The next proposition is immediate.

**Proposition 2.3.0.7** ((Co)homology of finite type spectra). If $X$ is a finite type CW spectrum then for each $i \in \mathbb{Z}$ there is an $n \in \mathbb{N}_0$ such that $X_n$ contains all the $i-$cells of $X$. In particular, $H_i(X;G) = H_i(X_n;G)$ and $H^i(X;G) = H^i(X_n;G)$, for $n \gg 1$.

*Proof.* Since $X$ has finite type, it has finitely many stable cells in each dimension. Therefore, for each $i \in \mathbb{Z}$, the corresponding (co)limit stabilizes in finitely many steps. ☐

The corresponding statement for homotopy is not always true.

**Example 2.3.0.2** (pp. 586 [Hat04]). Let $X_n := \bigvee_k S^k$ for each $n \in \mathbb{N}_0$. Then, the groups $\pi_{i+n}(X_n)$ never stabilize since $\pi_{2p}(S^3) \cong \mathbb{Z}/p\mathbb{Z}$ for all primes $p$. ☐

## 2.4 Maps between CW spectra

This section will finish the construction of the category of spectra. The basic results in homotopy theory, such as the Whitehead theorem and the Hurewicz theorem, will be translated into the language of spectra.

**Definition 2.4.0.1** (Strict map between CW spectra). A *strict map* $f\colon X \to Y$ between CW spectra is a collection $\{f_n\colon X_n \to Y_n \mid n \in \mathbb{Z}\}$ of cellular maps such that the following diagram commutes

$$\begin{tikzcd}
\Sigma X_n \arrow[r] \arrow[d, "\Sigma f_n"] & X_{n+1} \arrow[d, "\Sigma f_{n+1}"] \\
\Sigma Y_n \arrow[r] & Y_{n+1}
\end{tikzcd}$$

The composition of strict maps is defined in an obvious way. The inclusion $i\colon Y \to X$ of a subspectrum $Y \subseteq X$ is a strict map and if $g\colon X \to Z$ is a strict map then $g|_Y = g \circ i$ is also a strict map.

The next definition is required to define maps between CW spectra.

**Definition 2.4.0.2** (Cofinal subspectrum). A CW subspectrum $Y \subseteq X$ is called *cofinal* if each cell in $X$ ultimately lies in $Y$, i.e. for any cell $e_n \subseteq X_n$ there is an $m \in \mathbb{N}_0$ such that $\Sigma^m e_n \subseteq Y_{n+m}$. If $Y$ is cofinal and $K_n \subseteq X_n$ is a finite subcomplex, then there is an $m \in \mathbb{N}_0$ such that $\Sigma^m K_n \subseteq Y_{n+m}$.

**Lemma 2.4.0.1** (Intersection between two cofinal subspectra)**.** The intersection of two cofinal subspectra is cofinal, and if $Z \subseteq Y \subseteq X$ are subspectra such that $Y$ is cofinal in $X$, and $Z$ is cofinal in $Y$, then $Z$ is cofinal in $X$. An arbitrary union of cofinal subspectra is cofinal.

**Definition 2.4.0.3** (Maps between CW spectra)**.** Let $X, Y$ be CW spectra. We consider the set $S$ of all pairs $(X', f')$ such that $X' \subseteq X$ is a cofinal subspectrum and $f' \colon X' \to Y$ is a strict map. On $S$ we introduce a relation as follows: $(X', f') \sim (X'', f'')$ iff there is a pair $(X''', f''')$ with $X''' \subseteq X' \cap X''$, $X'''$ is cofinal and $f'|_{X'''} = f''' = f''|_{X'''}$. From Lemma 2.4.0.1 it follows that $\sim$ is an equivalence relation. The equivalence classes are called *maps* from $X$ to $Y$, i.e. $\mathrm{Hom}(X, Y) = S/\sim$.

Intuitively, this tells us that we just need to define a map *eventually* on each cell. By abuse of language, we will call such maps as *maps of spectra*. The following lemma is needed to verify that the composition of maps between spectra is a map between spectra.

**Lemma 2.4.0.2** (Strict maps and cofinal spectra)**.** Let $X$ and $Y$ be CW spectra and $f \colon X \to Y$ a strict map. If $Y' \subseteq Y$ is a cofinal subspectrum, then there is a cofinal subspectrum $X' \subseteq X$ with $f(X') \subseteq Y'$.

*Proof.* Let $S$ be the set of all (CW) subspectra $Z \subseteq X$ such that $f(Z) \subseteq Y'$ and let $X' := \bigcup_{Z \in S} Z$. Then, $X' \subseteq X$ is a subspectrum, and $f(X') \subseteq Y'$. We must verify that $X'$ is cofinal. Let $e := \{e_n, \Sigma e_n, \dots\}$ be any cell of $X$ and $W \subseteq X_n$ a finite subcomplex containing $e_n$. Then, $f(W)$ is contained in a finite subcomplex $Q \subseteq Y_n$. Since $Y' \subseteq Y$ is cofinal, there is an $N \in \mathbb{N}_0$ such that $\Sigma^N Q \subseteq Y'_{n+N}$ and hence $f(\Sigma^N W) \subseteq Y'_{n+N}$. Therefore, $Z = \{\Sigma^N e_n, \dots\} \subseteq X'$, proving that $X'$ is cofinal. □

Now, we can compose maps between CW spectra.

**Proposition 2.4.0.1** (Composition of maps between CW spectra)**.** Let $f \colon X \to Y$ and $g \colon Y \to Z$ be maps of CW spectra. Then $g \circ f \colon X \to Z$ is a map between CW spectra. Moreover, the composition of maps is associative.

*Proof.* By Lemma 2.4.0.2, we obtain representative maps that can be composed, obtaining a well-defined map. This composition is clearly associative. □

**Proposition 2.4.0.2** (Inclusion of a subspectrum)**.** The strict map given by the inclusion $i \colon Y \to X$ of a subspectrum is a map. Then, for any map $g \colon X \to Z$, the restriction $g|_Y = g \circ i$ is also a map.

Now we can state the following.

**Proposition 2.4.0.3** (Homotopy functor of spectra)**.** There is a functor

$$\begin{aligned} \pi_* \colon \mathbf{CWSpec} &\to \mathbf{Ab} \\ X &\mapsto \pi_*(X) \end{aligned}$$

such that for each map of spectra $f \colon X \to Y$ we have an induced map

$$\pi_*(f) := \varinjlim_{n \in \mathbb{N}_0} (f_n \colon \pi_{*+n}(X_n) \to \pi_{*+n}(Y_n)) \colon \pi_*(X) \to \pi_*(Y).$$

*Proof.* Since $f \colon X \to Y$ is a map of (CW) spectra, it can be represented by $(X', f')$ where $f' \colon X' \to Y' \subseteq Y$ is a strict map and $X' \subseteq X$ and $Y' \subseteq Y$ are cofinal subspectra. WLOG, assume that $f' = f$. Note that $\varinjlim \colon [\mathbf{I}, \mathbf{Ab}] \to \mathbf{Ab}$ is a functor, for some associated indexing small category $\mathbf{I}$. Then the construction of $\pi_*(f) := \varinjlim(f_n \colon \pi_{*+n}(X_n) \to \pi_{*+n}(Y_n)) \colon \pi_*(X) \to \pi_*(Y)$ is clearly functorial. □

We have similar statements for (co)homology. The proofs are analogous. For cohomology, the only difference with the previous argument is that we must argue at the chain level before dualizing with the $\mathrm{Hom}_{\mathbb{Z}}(-; G)$ functor.

**Proposition 2.4.0.4** ((Co)homology functors of spectra)**.** There is a functor

$$\begin{aligned} H_*(-; G) \colon \mathbf{CWSpec} &\to \mathbf{Ab} \\ X &\mapsto H_*(X; G) \end{aligned}$$

such that for each map of spectra $f \colon X \to Y$ we have an induced map

$$H_*(f; G) := \varinjlim_{n \in \mathbb{N}_0} ((f_n)_* \colon H_{*+n}(X_n; G) \to H_{*+n}(Y_n; G)) \colon H_*(X; G) \to H_*(Y; G).$$

Moreover, there is a functor

$$H^*(-;G)\colon \mathbf{CWSpec} \to \mathbf{Ab}$$
$$X \mapsto H^*(X;G)$$

such that for each map of spectra $f\colon X \to Y$ we have an induced map

$$H^*(f;G)\colon H^*(Y;G) \to H^*(X;G)$$

given by passing the cochain maps

$$\mathrm{Hom}_{\mathbb{Z}}\left(\underset{n\in\mathbb{N}_0}{\underrightarrow{\mathrm{colim}}}((f_n)_*\colon C_{*+n}(X_n;G) \to C_{*+n}(Y_n;G)),\ G\right)\colon C^{*+n}(Y_n;G) \to C^{*+n}(X_n;G)$$

to their respective homology groups.

These functors form a (co)homology theory in the category of spectra.
We use the same notation as in the case of spaces.

**Definition 2.4.0.4** (Induced maps)**.** Let $f\colon X \to Y$ be a map of spectra. Define

$$f_* := \pi_*(f)\colon \pi_*(X) \to \pi_*(Y)$$
$$f_* := H_*(f;G)\colon H_*(X;G) \to H_*(Y;G)$$
$$f^* := H^*(f;G)\colon H^*(Y;G) \to H^*(X;G).$$

Also, remember that $\Sigma^\infty\colon \mathbf{Top}_* \to \mathbf{CWSpec}$ is a functorial construction.
We also can form the smash product between a spectrum and a space, leading to the following definition.

**Definition 2.4.0.5** (Homotopy of spectra)**.** A *homotopy* is a map $h\colon X \wedge I^+ \to Y$. There are two maps $i_0\colon X \to X \wedge I^+$, $i_1\colon X \to X \wedge I^+$ induced by the inclusions of $0, 1$ in $I^+$. Then we say that two maps of spectra $f_0, f_1\colon X \to Y$ are *homotopic* if there is a homotopy $h\colon X \wedge I^+ \to Y$ such that $h \circ i_0 = f_0$ and $h \circ i_1 = f_1$. We define $h_0 := h \circ i_0$ and $h_1 := h \circ i_1$.

The (handcrafted) smash product of spectra is considerably more expensive (cf. [Swi75]).
In terms of cofinal spectra we can say that two maps $f_0, f_1\colon X \to Y$ represented by $(X_0', f_0')$, $(X_1', f_1')$ respectively are homotopic if there is a cofinal subspectrum $X'' \subseteq X_0' \cap X_1'$ and a strict map $h''\colon X'' \wedge I^+ \to Y$ such that $h_0'' = f_0'|_{X''}$, $h_1'' = f_1'|_{X''}$.

**Proposition 2.4.0.5** (Homotopy equivalence is an equivalence relation)**.** Let $f, g\colon X \to Y$ be maps between CW spectra. Then the relation of being homotopically equivalent is an equivalence relation.

*Proof.* The proof is similar to the case of spaces. □

**Definition 2.4.0.6** (Homotopy equivalence classes)**.** Let $X$ and $Y$ be spectra. Define $[X, Y]$ to be the set of equivalence classes of maps $f\colon X \to Y$. Composition passes to homotopy classes.

**Remark 2.4.0.1** (Maps between spectra form an abelian group (Corollary 8.27 [Swi75]))**.** Let $X$ and $Y$ be spectra. Then, $[X, Y]$ is abelian.

**Remark 2.4.0.2** (Cofinal spectrum homotopy equivalence)**.** A spectrum $X$ is homotopy equivalent to a cofinal subspectrum via the inclusion.

We can replace the structural maps in the previous definition with inclusions.

**Proposition 2.4.0.6** (CW spectrum II)**.** Let $X$ be a spectrum, whose spaces $X_n$ are CW-complexes, where the base points are $0-$cells. Then, there exists an equivalent spectrum $X'$ such that the structural maps $\Sigma X_n \to X_{n+1}$ are inclusions of subcomplexes.

*Proof.* By the cellular approximation theorem, the structural maps are homotopy equivalent to cellular maps. Moreover, we can replace each $X_n$ by the union of the reduced mapping cylinders of the maps

$$\Sigma^n X_0 \to \Sigma^{n-1} X_1 \to \cdots \to \Sigma X_{n-1} \to X_n.$$

This shows the existence of the CW spectrum $X'$. □

We also can define the functor $\pi_*^Y(-) := [Y, -]$, for a fixed spectrum $Y$.

**Lemma 2.4.0.3** (Finiteness of $\pi_*^Y(Z)$ (pp. 595 [Hat04]))**.** The groups $\pi_t^Y(Z)$ are finitely generated for all connective spectrum of finite type $Z$.

**Example 2.4.0.1** (Suspension spectrum of a CW complex)**.** The suspension spectrum of a CW complex is a CW spectrum. □

**Proposition 2.4.0.7** (Stable homotopy functor as homotopy classes of maps)**.** Let $X := S^i$ the CW suspension spectrum of the sphere $S^i$. Then $[S^i, Y] = \pi_i(Y)$.

*Proof.* Notice that spectrum maps $S^i \to Y$ are space maps $S^{i+n} \to Y_n$ for some $n \in \mathbb{Z}$, and spectrum homotopies $S^i \wedge I^+ \to Y$ are space homotopies $S^{i+n} \wedge I^+ \to Y_n$ for some $n \in \mathbb{Z}$. □

However, we could have homotopies between spectra that are not induced by homotopies of spaces.
The next definition will be useful for defining cofibrations of spectra.

**Definition 2.4.0.7** (Mapping cone)**.** Given a map $f\colon X \to Y$ of spectra, we can construct the *mapping cone* $Y \cup_f CX$. We give $I$ the base point 0 and define $CX$ to be $X \wedge I$. We take $Y \cup_f CX$ to be the spectrum with $(Y \cup_f CX)_n = Y_n \cup_{f'_n} (X'_n \wedge I)$, where $(X', f')$ represents $f$.

**Proposition 2.4.0.8** (The mapping cone is well-defined)**.** Let $f\colon X \to Y$ be a map of spectra. If $(X', f')$ and $(X'', f'')$ are representatives of $f$, then $\{Y_n \cup_{f'_n} (X'_n \wedge I)\}_{n\in\mathbb{Z}}$ and $\{Y_n \cup_{f''_n} (X''_n \wedge I)\}_{n\in\mathbb{Z}}$ have a mutual cofinal subspectrum $\{Y_n \cup_{f'''_n} (X'''_n \wedge I)\}_{n\in\mathbb{Z}}$, hence they are equivalent.

The next definition is required to prove the Whitehead theorem.

**Definition 2.4.0.8** (Mapping cylinder)**.** Let $f\colon X \to Y$ be a map of spectra. Consider a $\{X', f'\}$ representative for $f$. Define the *mapping cylinder* $M_f$ as $M_f := \{Y_n \cup_{f'_n} X'_n \wedge I^+\}_{n\in\mathbb{Z}}$.

This is also well-defined.

**Proposition 2.4.0.9** (Basic properties of the mapping cylinder)**.** Let $f\colon X \to Y$ be a map of spectra. Then the mapping cylinder $M_f$ is a spectrum having the same homotopy type as $Y$ and containing $X'$ as a subspectrum.

**Definition 2.4.0.9** (Suspension of a CW spectrum)**.** Let $X = \{X_n\}_{n\in\mathbb{Z}}$ be a spectrum. We define the *suspension* $\Sigma X$ of $X$ as the spectrum with $\Sigma X_n := X_{n+1}$, $n \in \mathbb{Z}$. For any strict map $f\colon X \to Y$ we define the strict map $\Sigma f\colon \Sigma X \to \Sigma Y$ as $(\Sigma f)_n := f_{n+1}$. Then for any map $f\colon X \to Y$ represented by $(X', f')$ we can define $\Sigma f\colon \Sigma X \to \Sigma Y$ to be the map represented by $(\Sigma X', \Sigma f')$.

This construction is functorial, and it has an inverse called the desuspension $\Sigma^{-1}$.

**Definition 2.4.0.10** (Desuspension functor)**.** Let $X$ be a spectrum. Define the *desuspension* functor $\Sigma^{-1}X$ by $(\Sigma^{-1}X)_n := X_{n-1}$ and $(\Sigma^{-1}f)_n := f_{n-1}$.

**Remark 2.4.0.3** (Desuspension functor)**.** To prove that the suspension functor is invertible, it must be proven that $X \wedge S^1 \cong \Sigma X$, and this can be done with the Whitehead Theorem for spectra (Theorem 2.4.0.1).

**Definition 2.4.0.11** (Wedge sums of spectra)**.** Given a collection $\{X^\alpha\}_{\alpha\in A}$ of spectra, we define their *wedge sum* $\bigvee_\alpha X^\alpha$ by $(\bigvee_\alpha X^\alpha)_n := \bigvee_\alpha X_n^\alpha$. Since $\Sigma(\bigvee_\alpha X_n^\alpha) = \bigvee_\alpha \Sigma X_n^\alpha \subseteq \bigvee_\alpha X_{n+1}^\alpha$, this is a (CW) spectrum.

The next result states that the inclusion of spectra satisfies the homotopy extension property, i.e. inclusions are cofibrations.

**Lemma 2.4.0.4** (Homotopy extension property (Theorem 8.20 [Swi75]))**.** Let $E$ and $H$ be spectra, $F \subseteq E$ a subspectrum and $G$ a cofinal subspectrum of $E \wedge \{0\}^+ \cup F \wedge I^+$. Given a function $g\colon G \to H$, we can find a cofinal subspectrum $K$ of $E \wedge I^+$ containing $G$ and an extension of $g$ to a function $k\colon K \to H$. Moreover, if $G = E \wedge \{0\}^+ \cup F \wedge I^+$, we can choose $K = E \wedge I^+$.

The proof of the following result is similar to the proof for the case of spaces.

**Theorem 2.4.0.1** (Whitehead Theorem for spectrum (Theorem 8.25 [Swi75]))**.** A map of spectra is a weak homotopy equivalence if and only if it is a homotopy equivalence.

Provided with Theorem 2.4.0.1 it is now possible to prove that $\Sigma X \cong X \wedge S^1$ [Swi75, Theorem 8.26].
Finally, we have a Hurewicz Theorem for spectra.

**Theorem 2.4.0.2** (Hurewicz theorem for spectra (Proposition 5.44 [Hat04]))**.** Let $X$ be a $n-$connected CW spectrum, i.e. such that $\pi_i(X) = 0$ for $i \leq n$. Then $X$ is homotopy equivalent to a CW spectrum with no cells of dimension $\leq n$.

**Remark 2.4.0.4.** As a consequence of the Hurewicz theorem for spectra, it follows that if $X$ is a $n-$connected CW spectrum, then the Hurewicz morphism $\pi_{n+1}(X) \to H_{n+1}(X)$ is an isomorphism, since $X$ has no cells of dimension $\leq n$, and the Hurewicz morphism is the direct limit of the Hurewicz morphisms $\pi_{n+1+k}(X_k) \to H_{n+1+k}(X_k)$ for spaces, which are isomorphisms.

## 2.5 Cofibration sequences

This section addresses the construction of the cofibre sequences of spectra. This will lead to the wedge axiom in the context of generalized (co)homology theories associated with spectra. We will also obtain that, in the context of spectra, cofibre sequences are equivalent to fibre sequences. The next definition is required to construct Adams resolutions.

**Definition 2.5.0.1** (Quotient of spectra (pp. 154 [Ada74]))**.** Let $X$ be a CW spectrum, $A$ a subspectrum. We say that $A$ is a *closed* if for every finite subcomplex $K \subseteq X_n, \Sigma^m K \subseteq A_{m+n}$ implies $K \subseteq A_n$. That is, if a cell gets into $A$ later, we put it into $A$ to start with. It is equivalent to saying that $A \subseteq B \subseteq X$, $A$ cofinal in $B$ implies that $A = B$. Suppose that $i \colon X \to Y$ is the inclusion of a closed subspectrum. Then we can form $Y/X$, with the $n$th term $Y_n/X_n$. In this case, there is a map

$$r \colon Y \cup_i CX \to Y/X$$

with computes

$$Y_n \cup_{i_n} CX_n \to Y_n/X_n.$$

The map $r$ is an equivalence by the Whitehead Theorem for spectra (Theorem 2.4.0.1).

**Definition 2.5.0.2** (Special/general cofibre sequence)**.** For any map $f \colon X \to Y$ of spectra, we call the sequence

$$X \xrightarrow{f} Y \xrightarrow{j} X \cup_f CX$$

a *special cofibre sequence*. A *general cofibre sequence*, or simply a *cofibre sequence*, is any sequence

$$X \xrightarrow{g} Y \xrightarrow{j} Z$$

for which there is a homotopy commutative diagram

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "\alpha"] & Q \arrow[r, "h"] \arrow[d, "\beta"] & R \arrow[d, "\gamma"] \\ X \arrow[r, "f"] & Y \arrow[r, "j"] & Y \cup_f CX \end{tikzcd}$$

in which $\alpha, \beta, \gamma$ are homotopy equivalences.

In the previous diagram, one may even assume that $f$ is an inclusion of spectra: if $\{X', f'\}$ is a representative for $f$, then we can consider its mapping cylinder $X' \subseteq M_f \cong Y$.

**Proposition 2.5.0.1** (Elementary cofibre sequences (Proposition 8.30 [Swi75]))**.** In the sequence

$$X \xrightarrow{f} Y \xrightarrow{j} Y \cup_f CX \xrightarrow{k'} X \wedge S^1 \xrightarrow{f \wedge 1} Y \wedge S^1,$$

each pair of consecutive maps forms a cofibre sequence.

**Lemma 2.5.0.1** (Ladder of cofibre sequences)**.** Given a homotopy commutative diagram of spectra and maps as follows

$$\begin{tikzcd} G \arrow[r, "g"] \arrow[d, "\alpha"] & H \arrow[r, "h"] \arrow[d, "\beta"] & K \arrow[r, "k"] \arrow[d, dashed, "\gamma"] & G \wedge S^1 \arrow[d, "\alpha\wedge 1"] \\ G' \arrow[r, "g'"] & H' \arrow[r, "h'"] & K' \arrow[r, "k'"] & G' \wedge S^1 \end{tikzcd}$$

in which the rows are cofibre sequences, we can find a map $\gamma\colon K \to K'$ such that the resulting diagram is homotopy commutative.

*Proof.* We may assume the rows are special cofibre sequences (i.e. $K = H \cup_g CG$, $K' = H' \cup_{g'} CG'$, and $h, h'$ are inclusions) and even that $g$ is an inclusion (using the mapping cylinder $M_f$). We choose representatives $(B, \beta')$ for $\beta$, $(A', f')$ for $g'$ and $(A, \alpha')$ for $\alpha$ so that $g(A) \subseteq B$ and $\alpha'(A) \subseteq A'$. Then we have a homotopy commutative diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\alpha'"] & B \arrow[r] \arrow[d, "\beta'"] & B \cup_g CA \arrow[r] \arrow[d, dashed, "\gamma'"] & A \wedge S^1 \arrow[d, "\alpha'\wedge 1"] \\ A' \arrow[r, "f'"] & H' \arrow[r] & H' \cup_{f'} CA' \arrow[r] & A' \wedge S^1 \end{tikzcd}$$

whose rows are cofibre sequences and $g$ is an inclusion.

By the homotopy extension property of spectra (Lemma 2.4.0.4), we can find a $\beta''\colon B \to H'$ with $\beta'' \circ g = f \circ \alpha'$ and $\beta'' \cong \beta'$. We define $\gamma'|_B := \beta''$, $\gamma'|_{CA} := C\alpha'\colon CA \to CA'$. Then $\gamma'$ is a function that makes the diagram commute strictly if $\beta'$ is replaced by $\beta''$. Hence the diagram commutes up to homotopy even with $\beta'$. ☐

**Proposition 2.5.0.2** (Cofibre exact sequences)**.** If $G \xrightarrow{g} H \xrightarrow{h} K$ is a cofibre sequence, then for any spectrum $X$ the sequences

$$[X, G] \xrightarrow{g_*} [X, H] \xrightarrow{h_*} [X, K]$$

$$[G, X] \xleftarrow{g^*} [H, X] \xleftarrow{h^*} [K, X]$$

are exact.

*Proof.*

(i) Since $h \circ g \cong 0$, it follows $h_* \circ g_* \cong 0$. Suppose $f\colon X \to H$ satisfies $h_*[f] = 0$. We apply Lemma 2.5.0.1, i.e. the previous ladder of cofibre sequences, to the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & X \wedge I \arrow[r] \arrow[d, "\overline{h}"] & X \wedge S^1 \arrow[r, "1"] \arrow[d, "k"] & X \wedge S^1 \arrow[d, "f\wedge 1"] \\ H \arrow[r, "h"] & K \arrow[r] & G \wedge S^1 \arrow[r, "g\wedge 1"] & H \wedge S^1 \end{tikzcd}$$

where $\overline{h}\colon X \wedge I \to K$ is a null-homotopy of $h \circ f$. We obtain a map $k\colon X \wedge S^1 \to G \wedge S^1$ such that $(g \wedge 1) \circ k \cong f \wedge 1$. From the natural equivalence $\sigma\colon [X, G] \to [X \wedge S^1, G \wedge S^1]$ we get a map $k'\colon X \to G$ with $k \cong k' \wedge 1$. Then $(g \circ k') \wedge 1 = (g \wedge 1) \circ (k' \wedge 1) \cong (g \wedge 1) \circ k \cong f \wedge 1$. Since $\sigma\colon [X, G] \to [X \wedge S^1, H \wedge S^1]$ is injective, it follows $g \circ k' \cong f$, i.e. $g_*[k'] = [f]$.

(ii) Again $g^* \circ h^* = 0$ follows from $h \circ g \cong 0$. Suppose given $f\colon H \to X$ such that $g^*[f] = 0$. We apply Lemma 2.5.0.1 to the diagram

$$\begin{tikzcd} G \arrow[r, "g"] & H \arrow[r, "h"] \arrow[d, "f"] & K \arrow[r] \arrow[d, "f'"] & G \wedge S^1 \\ * \arrow[r] & X \arrow[r, "1"] & X \arrow[r] & * \end{tikzcd}$$

to obtain a map $f' \colon K \to X$ such that $f' \circ h \cong f$; that is, $h^*[f'] = [f]$.

□

Thus for cofibre of spectra, the notions of "cofibre sequence" and "fibre sequence" coincide.

## 2.6 Cohomology of Eilenberg-Mac Lane spectra

In this section we state the representability theorem for (cellular) cohomology of spectra by Eilenberg-Mac Lane spectra, following the lines given in [Hat02]. We are avoiding the smash product, although it would lead to a more general statement. The previously constructed exact sequences can be extended indefinitely in both directions, since spectra can always be desuspended. This allows to prove that $h^i(X) := [\Sigma^{-i}X, Y]$ defines a generalized cohomology theory for a fixed spectrum $Y$.

This weaker version of the Brown Representability Theorem will be a key element in the construction of the Adams spectral sequence.

**Proposition 2.6.0.1** (Representability theorem for (cellular) cohomology (Proposition 5.45 [Hat04]))**.** Let $X$ be a CW spectrum. There are natural isomorphisms $H^n(X;G) \cong [X, K(G,n)]$ given by $[f] \mapsto f^*(\alpha)$ for some $\alpha \in \tilde{H}^n(K(G,n);G)$; for all abelian group $G$ and $n \in \mathbb{Z}$.

**Remark 2.6.0.1** (Cohomology operations for spectra)**.** As a consequence of Proposition 2.6.0.1, we can define cohomology operations for spectra, and from the work of Serre and Cartan, we obtain $[K(\mathbb{F}_p,n), K(\mathbb{F}_p,m)] \cong H^m(K(\mathbb{F}_p,n);\mathbb{F}_p)$ for all primes $p$.

The next proposition will also be needed in the construction of the Adams spectral sequence to deal with Adams resolutions of generalized Eilenberg-Mac Lane spectra (i.e. wedges of Eilenberg-Mac Lane spectra). Observe that the involved product does not need to be finite. The proof of this statement involves the Milnor exact sequence.

**Proposition 2.6.0.2** (Wedges of spectra and products (Proposition 5.46 [Hat04]))**.** The natural map

$$\left[X, \bigvee_{i\in I} K(G,n_i)\right] \to \prod_{i\in I}[X, K(G,n_i)]$$

is an isomorphism if $X$ is a connective CW spectrum of finite type and $n_i \to \infty$ as $i \to \infty$.

Provided with the language of spectra and its related basic results, we can now construct the classical Adams spectral sequence.

# Chapter 3

# The classical Adams spectral sequence

## 3.1 Introduction

This chapter is intended to give an exposure to the construction and basic properties of the classical Adams spectral sequence, together with relevant examples. The Adams spectral sequence can be thought of as a refinement of the Serre spectral sequence, being the main tool to compute stable homotopy for a given spectrum. It can be constructed as a functor $\mathbf{hoTop}_* \to \mathbf{GrAb}$ from the homotopy category of pointed topological spaces to the category of graded abelian groups, without the need of having a category of spectra at our disposal. However, it is more convenient to construct the Adams spectral sequence using the language of spectra, which is more appropriate to deal with $\pi_*^Y(-)$.

This chapter will start with the construction of the spectral sequence and a convergence result that fits with practical applications. After that, it will be given the definition of a minimal resolution which allows us to perform computations on the $E_2-$term. Together with that result, we also need the Adams vanishing theorem, providing an important finiteness condition that makes computational approaches possible. Later, we will define Yoneda and Massey products and their topological counterparts, the composition product and Toda brackets. We will relate these multiplicative structures via Moss' convergence theorem. After that, we will introduce the notion of *hidden extensions* and we will deal with the problem of solving such extensions through examples.

We will not deal with or require the smash product explicitly. However, the smash product of spectra induces the smash product pairing in the Adams spectral sequence. Further, the composition product on $(\pi_*^Y)_*^\wedge$ can be described in terms of the smash product. The smash product of spectra also allows us to consider the Adams spectral sequence based on a generalized homology theory ([Rav86]) such as the $MU-$based Adams spectral sequence (i.e. the Adams$-$Novikov spectral sequence).

## 3.2 Constructing the Adams spectral sequence

The $E_2-$page of the Adams spectral sequence is described in terms of Ext functor and projective resolutions, as we will see soon. The next definition addresses the geometrical realization of such resolutions.

**Definition 3.2.0.1** (Adams resolution, (2.1.3. Definition [Rav86]))**.** Let $X$ be a spectrum and $E^\bullet$ a generalized cohomology theory represented by a spectrum $E$, then an $E-$Adams resolution of $X$ is a diagram of the form

$$\begin{tikzcd}
\vdots \arrow[d] & \\
F_2 \arrow[r, "f_2"] \arrow[d] & K_2 \\
F_1 \arrow[r, "f_1"] \arrow[d] & K_1 \\
F_0 := X \arrow[r, "f_0"] & K_0
\end{tikzcd}$$

where

(a) Each $K_i$ is a wedge of suspensions of $E$.

(b) Each $F_{n+1} \to F_n \to K_n$ is a homotopy fiber sequence.

(c) Each $f_n$ is a surjection on cohomology.

The Adams resolution is also known as an *Adams tower* (cf. [Hat04]), because it is a reminiscent of a Postnikov tower. Observe that we can use cofiber sequences in the previous definition as well.

The following result gives the construction of the Adams spectral sequence for connective CW spectra of finite type, such as the suspension spectrum of a finite CW complex. This result heavily relies on the construction of the Steenrod algebra and the Representability Theorem. The idea of the proof is to construct a resolution of a given spectrum by wedges of Eilenberg-Mac Lane spectra, also obtaining a free $\mathcal{A}_p-$resolution in cohomology via cofibrations. These cofibrations are arranged as a *staircase diagram* in stable homotopy which leads to an exact couple. We describe the $E_1-$term via the Representability Theorem. The bidegree in the Ext term comes from the Steenrod algebra.

**Theorem 3.2.0.1** (The Adams spectral sequence (pp. 594 [Hat04]))**.** Let $X$ be a connective CW spectrum of finite type, $Y$ be a finite CW spectrum, and $p \in \mathbb{N}$ a prime number. There is a spectral sequence with $E_2-$page given by

$$E_2^{s,t} := \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(X;\mathbb{F}_p), H^*(Y;\mathbb{F}_p)).$$

*Proof.* We will construct the following (Adams) resolution for the spectrum $X$ by wedges of Eilenberg-Mac Lane spectra.

$$\begin{tikzcd}
X \arrow[r] & K_0 \arrow[rr] \arrow[dr] & & K_1 \arrow[rr] \arrow[dr] & & K_2 \arrow[rr] \arrow[dr] & & K_3 \arrow[r] & \dots \\
& & X_1 := K_0/X \arrow[ur] & & X_2 := K_1/X_1 \arrow[ur] & & X_3 := K_2/X_2 \arrow[ur] & &
\end{tikzcd}$$

Since $X$ has finite type, choose generators $(\alpha_{n,i})_{n\in\mathbb{Z},i\in I_n} \subseteq H^*(X;\mathbb{F}_p)$ as an $\mathcal{A}_p-$module, satisfying $(\alpha_{n,i})_{n\in\mathbb{Z},i\in I_n} \subseteq H^n(X;\mathbb{F}_p)$ where $|I_n| < \infty$, for each $n \in \mathbb{Z}$. By the representability theorem, we obtain maps $f_{n,i} \colon X \to K(\mathbb{F}_p, n)$. As $X$ is a connective CW spectrum of finite type, via $\left[X, \bigvee_{n\in\mathbb{Z},i\in I_n} K(\mathbb{F}_p,n)\right] \cong \prod_{n\in\mathbb{Z},i\in I_n}[X, K(\mathbb{F}_p,n)]$, we obtain a map $X \to K_0 := \bigvee_{n\in\mathbb{Z},i\in I_n} K(\mathbb{F}_p,n)$. It follows that $K_0$ has finite type. Now, replace $X \to K_0$ with an inclusion and define $X_1 := K_0/X$. This is again a connective spectrum of finite type. Hence, inductively, we obtain the desired resolution of $X$.

Consider the cofiber sequences $X_s \to K_s \to X_{s+1}$ and their related cohomology long exact sequences. Splitting them brings us to the next exact diagram.

$$\begin{tikzcd}
0 & H^*(X;\mathbb{F}_p) \arrow[l] & H^*(K_0;\mathbb{F}_p) \arrow[l, "j^*"'] & & H^*(K_1;\mathbb{F}_p) \arrow[ll] \arrow[dl, "j^*"] & & H^*(K_2;\mathbb{F}_p) \arrow[ll] \arrow[dl, "j^*"] & & \dots \arrow[ll] \arrow[dl, "j^*"] \\
& & & H^*(X_1;\mathbb{F}_p) \arrow[ul, "k^*"] \arrow[dl] & & H^*(X_2;\mathbb{F}_p) \arrow[ul, "k^*"] \arrow[dl] & & H^*(X_3;\mathbb{F}_p) \arrow[ul, "k^*"] \arrow[dl] & \\
& & 0 & & 0 \arrow[ul] & & 0 \arrow[ul] & &
\end{tikzcd}$$

This is a free $\mathcal{A}_p-$resolution of $H^*(X;\mathbb{F}_p)$, since $[H\mathbb{F}_p, H\mathbb{F}_p] \cong H^*(H\mathbb{F}_p;\mathbb{F}_p) \cong \mathcal{A}_p$ and $H^*(-;\mathbb{F}_p)$ satisfies the wedge axiom. Observe that by defining $X^n := \Sigma^{-n}X_n$ and $K^n := \Sigma^{-n}K_n$ we obtain the Adams resolution from Definition 3.2.0.1, so we are actually working with the Adams resolution using a different indexing.

We will proceed to construct an exact couple. Fix a finite CW spectrum $Y$ and consider the functors

$$\begin{aligned} \pi_t^Y \colon \mathbf{CWSpec} &\to \mathbf{Ab} \\ Z &\mapsto [\Sigma^t Y, Z]. \end{aligned}$$

Applied to the cofibrations $X_s \to K_s \to X_{s+1}$, these maps give us long exact sequences forming a staircase diagram. This will bring us the required exact couple by considering the morphisms from each cofibration.

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\dots \arrow[r] & \pi^Y_{t+1}(X_s) \arrow[r] \arrow[d, "i_*"] & \pi^Y_{t+1}(K_s) \arrow[r] & \pi^Y_{t+1}(X_{s+1}) \arrow[r] \arrow[d] & \pi^Y_{t+1}(K_{s+1}) \arrow[r] & \pi^Y_{t+1}(X_{s+2}) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \pi^Y_{t}(X_{s-1}) \arrow[r, "j_*"] \arrow[d] & \pi^Y_{t}(K_{s-1}) \arrow[r, "k_*"] & \pi^Y_{t}(X_{s}) \arrow[r] \arrow[d] & \pi^Y_{t}(K_{s}) \arrow[r] & \pi^Y_{t}(X_{s+1}) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \pi^Y_{t-1}(X_{s-2}) \arrow[r] \arrow[d] & \pi^Y_{t-1}(K_{s-2}) \arrow[r] & \pi^Y_{t-1}(X_{s-1}) \arrow[r] \arrow[d] & \pi^Y_{t-1}(K_{s-1}) \arrow[r] & \pi^Y_{t-1}(X_{s}) \arrow[r] \arrow[d] & \dots \\
 & \vdots & & \vdots & & \vdots &
\end{tikzcd}$$

Analogously to the construction of the Serre spectral sequence, this staircase diagram leads to the Adams spectral sequence. The groups $\pi_t^Y(Z)$ are finitely generated for all connective spectrum of finite type $Z$.

Now, we will describe the $E_1$ and $E_2$ pages of this spectral sequence. Since $K_s = \bigvee_{s,i} K_{s,i}$, we have

$$\begin{aligned}
\mathrm{Hom}^0_{\mathcal{A}_p}(H^*(K_s;\mathbb{F}_p), H^*(Y;\mathbb{F}_p)) &\cong \mathrm{Hom}^0_{\mathcal{A}_p}\left(\bigoplus_i \Sigma^{n_i}\mathcal{A}_p, H^*(Y;\mathbb{F}_p)\right) \\
&\cong \prod_i \mathrm{Hom}^0_{\mathcal{A}_p}(\Sigma^{n_i}\mathcal{A}_p, H^*(Y;\mathbb{F}_p)) \\
&\cong \prod_i H^{n_i}(Y;\mathbb{F}_p) \\
&\cong \prod_i [Y, K_{s,i}] \\
&\cong [Y, K_s]
\end{aligned}$$

for appropriate $(n_i)_{i=1,\dots,m} \subseteq \mathbb{Z}$. Since $H^*(K_s;\mathbb{F}_p)$ is a free $\mathcal{A}_p-$module, it follows that the natural map

$$[Y, K_s] \xrightarrow{\cong} \mathrm{Hom}^0_{\mathcal{A}_p}(H^*(K_s;\mathbb{F}_p), H^*(Y;\mathbb{F}_p))$$

given by the module of induced maps is an isomorphism. This isomorphism allows us to translate stable homotopy groups into the Hom functor applied to a pair of cohomology $\mathcal{A}_p-$modules, leading to the Ext description of $E_2$. Here $\mathrm{Hom}^r$ denotes the morphisms that shift degree by $r$. Replacing $Y$ by $\Sigma^t Y$, we obtain

$$\left[\Sigma^t Y, K_s\right] \cong \mathrm{Hom}^0_{\mathcal{A}_p}\left(H^*(K_s;\mathbb{F}_p), H^*(\Sigma^t Y;\mathbb{F}_p)\right) \cong \mathrm{Hom}^t_{\mathcal{A}_p}\left(H^*(K_s;\mathbb{F}_p), H^*(Y;\mathbb{F}_p)\right).$$

Thus, if we set $E_1^{s,t} := \pi_t^Y(K_s)$, we have $E_1^{s,t} \cong \mathrm{Hom}^t_{\mathcal{A}_p}\left(H^*(K_s;\mathbb{F}_p), H^*(Y;\mathbb{F}_p)\right)$. The differential $d_1 \colon \pi_t^Y(K_s) \to \pi_t^Y(K_{s+1})$ is induced by the map $K_s \to K_{s+1}$ in the resolution of $X$ constructed earlier. This implies that the $E_1-$page of the spectral sequence consists of the complexes

$$0 \to \mathrm{Hom}^t_{\mathcal{A}_p}(H^*(K_0;\mathbb{F}_p), H^*(Y;\mathbb{F}_p)) \to \mathrm{Hom}^t_{\mathcal{A}_p}(H^*(K_1;\mathbb{F}_p), H^*(Y;\mathbb{F}_p)) \to \dots$$

The homology groups of this complex are by definition $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(X;\mathbb{F}_p), H^*(Y;\mathbb{F}_p))$, i.e.

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(X;\mathbb{F}_p), H^*(Y;\mathbb{F}_p)).$$

□

**Remark 3.2.0.1** (Change of indexing)**.** We usually prefer the indexing obtained by the usage of Definition 3.2.0.1 to construct the staircase diagram. Under this indexing, the entries of the $E_r-$term are described by $E_r^{s,t-s}$, for $s \in \mathbb{N}_0$, $t-s \in \mathbb{Z}$. Moreover, the staircase diagram from the previous proof corresponds to

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\dots \arrow[r] & \pi^Y_{t-s+1}(X^s) \arrow[r] \arrow[d, "i_*"] & \pi^Y_{t-s+1}(K^s) \arrow[r] & \pi^Y_{t-s}(X^{s+1}) \arrow[r] \arrow[d] & \pi^Y_{t-s}(K^{s+1}) \arrow[r] & \pi^Y_{t-s-1}(X^{s+2}) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \pi^Y_{t-s+1}(X^{s-1}) \arrow[r, "j_*"] \arrow[d] & \pi^Y_{t-s+1}(K^{s-1}) \arrow[r, "k_*"] & \pi^Y_{t-s}(X^s) \arrow[r] \arrow[d] & \pi^Y_{t-s}(K^s) \arrow[r] & \pi^Y_{t-s-1}(X^{s+1}) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \pi^Y_{t-s+1}(X^{s-2}) \arrow[r] \arrow[d] & \pi^Y_{t-s+1}(K^{s-2}) \arrow[r] & \pi^Y_{t-s}(X^{s-1}) \arrow[r] \arrow[d] & \pi^Y_{t-s}(K^{s-1}) \arrow[r] & \pi^Y_{t-s-1}(X^s) \arrow[r] \arrow[d] & \dots \\
 & \vdots & & \vdots & & \vdots &
\end{tikzcd}$$

Before stating and proving the convergence theorem for the connective case, we must note that resolutions by Eilenberg-Mac Lane spectra behave like projective resolutions. First, we introduce a few definitions.

**Definition 3.2.0.2** (Complex of spectra)**.** A sequence of maps of spectra

$$Z \to L_0 \to L_1 \to \cdots \to 0$$

will be called a *complex* on $Z$ if each composition of two successive maps is nullhomotopic.

**Definition 3.2.0.3** (Eilenberg-Mac Lane complex)**.** If the $L_i$'s from the previous definition are wedges of Eilenberg-Mac Lane spectra, we call it an *Eilenberg-Mac Lane complex.*

**Definition 3.2.0.4** (Resolution of spectra)**.** A complex for which the induced sequence $0 \leftarrow H^*(Z;\mathbb{F}_p) \leftarrow H^*(L_0;\mathbb{F}_p) \leftarrow \dots$ is exact is a *resolution* of $Z$.

**Lemma 3.2.0.1** (Lifting lemma (Lemma 5.48 [Hat04]))**.** Suppose we have the diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "f"] & L_0 \arrow[r] \arrow[d, dashed, "f_0"] & L_1 \arrow[r] \arrow[d, dashed, "f_1"] & L_2 \arrow[r] \arrow[d, dashed, "f_2"] & \dots \\
X \arrow[r] & K_0 \arrow[r] & K_1 \arrow[r] & K_2 \arrow[r] & \dots
\end{tikzcd}$$

where the first row is a resolution and the second row is an Eilenberg-Mac Lane complex. Then, there exist maps $f_i \colon L_i \to K_i$ forming homotopy-commutative squares.

*Proof.* Since the compositions in a complex are nullhomotopic we may start with an enlarged diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[dd, "f"] & L_0 \arrow[rr] \arrow[dr] \arrow[dd, dashed, "f_0"] & & L_1 \arrow[rr] \arrow[dr] \arrow[dd, dashed, "f_1"] & & L_2 \arrow[rr] \arrow[dr] \arrow[dd, dashed, "f_2"] & & \dots \\
 & & L_0/Z = Z_1 \arrow[ur] \arrow[dd, dashed] & & L_1/Z_1 = Z_2 \arrow[ur] \arrow[dd, dashed] & & \dots \arrow[ur] & \\
X \arrow[r] & K_0 \arrow[rr] \arrow[dr] & & K_1 \arrow[rr] \arrow[dr] & & K_2 \arrow[rr] \arrow[dr] & & \dots \\
 & & K_0/X = X_1 \arrow[ur] & & K_1/X_1 = X_2 \arrow[ur] & & \dots \arrow[ur] &
\end{tikzcd}$$

where the triangles are homotopy-commutative. The map $X \to K_0$ is equivalent to a collection of classes $\alpha_j \in H^*(X;\mathbb{F}_p)$. Since $H^*(L_0;\mathbb{F}_p) \to H^*(Z;\mathbb{F}_p)$ is surjective by assumption, there are classes $\beta_j \in H^*(L_0;\mathbb{F}_p)$ mapping to classes $f^*(\alpha_j) \in H^*(Z;\mathbb{F}_p)$. These $\beta_j$'s give a map $f_0\colon L_0 \to K_0$ making a homotopy-commutative square with $f$. This square induces a map $L_0/Z \to K_0/X$ making another homotopy-commutative square. The exactness property of the upper row implies that the map $H^*(L_1;\mathbb{F}_p) \to H^*(L_0;\mathbb{F}_p)$ is surjective, so we can repeat the argument with $Z$ and $X$ replaced by $Z_1 = L_0/Z$ and $X_1 = K_0/X$ to construct the map $f_1$, and so on inductively for all the $f_i$'s. □

The following result is concerned with the convergence of the Adams spectral sequence. Note that we are not invoking localization of spaces or spectra. In more generality, the spectral sequences converge to $[Y, X_p^\wedge]_s$ where $X_p^\wedge$ is the $p-$completion of $X$, but we will restrict ourselves to connective spectra of finite type. These are more computationally tractable as we will see soon.

**Theorem 3.2.0.2** (Convergence theorem (Theorem 5.47 [Hat04])). Let $X$ be a connective CW spectrum of finite type, $Y$ be a finite CW spectrum, and $p \in \mathbb{N}$ a prime number. Then

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(X;\mathbb{F}_p), H^*(Y;\mathbb{F}_p)) \implies (\pi^Y_{t-s}(X))_{(p)}$$

strongly, i.e.

(i) For fixed $s,t \geq 0$ the groups $E_r^{s,t}$ are independent of $r$ for $r \gg 1$, and the stable groups satisfy $E_\infty^{s,t} \cong F^{s,t}/F^{s+1,t+1}$ for the filtration of $\pi^Y_{t-s}(X)$ by the images

$$F^{s,t} := \mathrm{Im}\left(\pi^Y_t(X_s) \to \pi^Y_{t-s}(X)\right) = \mathrm{Im}\left(\pi^Y_{t-s}(X^s) \to \pi^Y_{t-s}(X)\right)$$

where the last inequality is a notation abuse that arises from Remark 3.2.0.1.

(ii) $\bigcap_{n\in\mathbb{N}_0} F^{s+n,t+n} \leq \pi^Y_{t-s}(X^s)$ consisting of torsion elements of order prime to $p$.

*Proof.* First, we will show (b). As noted earlier, $E_1^{s,t} = \mathrm{Hom}^t_{\mathcal{A}_p}(H^*(K_s;\mathbb{F}_p), H^*(Y;\mathbb{F}_p))$ are $\mathbb{F}_p-$vector spaces, so by exactness all the vertical maps in the diagram are isomorphisms on non-$p$ torsion. This implies that the non-$p$ torsion in $\pi^Y_{t-s}(X)$ is contained in $\bigcap_{n\in\mathbb{Z}} F^{s+n,t+n}$.

We will prove the opposite direction. First, consider the case where $\pi^*(X)$ is equal to its $p$-torsion. By hypothesis, $\pi^*(X)$ are finite. We construct a special Eilenberg-Mac Lane complex (which is not a resolution) of the form $X \to L_0 \to L_1 \to \dots$ in the following way. Let $\pi_n(X)$ be the first non-trivial group of $X$. Then, let $L_0$ be a wedge of $K(\mathbb{F}_p, n)$'s with one factor for each element of a basis for $H^n(X;\mathbb{F}_p)$, so there is a map $X \to L_0$ inducing $H^n(X;\mathbb{F}_p) \cong H^n(L_0;\mathbb{F}_p)$. We also have that $H_n(X;\mathbb{F}_p) \cong H_n(L_0;\mathbb{F}_p)$, obtaining a morphism $\pi_n(X) \to \pi_n(X)\otimes_{\mathbb{Z}}\mathbb{F}_p$ by the Hurewicz Theorem for connective spectra. Assume W.L.O.G. that $X \to L_0$ is the inclusion. Therefore, the cofiber $Z_1 := L_0/X$ has $\pi_i(Z_1) = 0$ for $i \leq n$ and

$$\pi_{n+1}(Z_1) = \mathrm{Ker}\left(\pi_n(X) \to \pi_n(L_0)\right)$$

which has smaller order than $\pi_n(X)$. Repeat the process with $Z_1$ instead of $X$ to construct a map $Z_1 \to L_1$ inducing the map $\pi_{n+1}(Z_1) \to \pi_{n+1}(Z_1)\otimes_{\mathbb{Z}}\mathbb{F}_p$, so the cofiber $Z_2 := L_1/Z_1$ has its first nontrivial homotopy group $\pi_{n+2}(Z_2)$ with $|\pi_{n+2}(Z_2)| < |\pi_{n+1}(Z_1)|$. After finitely many steps we obtain $Z_k$ with $\pi_{n+k}(Z_k) = 0$ as well as all the lower homotopy groups. At this point, we switch our attention to $\pi_{n+k+1}(Z_k)$ and repeat the steps again. This infinite process yields the complex $X \to L_0 \to L_1 \to \cdots$.

Now consider the associated tower

$$\cdots \to Z^2 \to Z^1 \to X$$

where $Z^k := \Sigma^{-k} Z_k$. Here the first map $Z^1 \to X$ induces an isomorphism on all homotopy groups except $\pi_n$, where it induces an inclusion of a proper subgroup. The same is true for $Z^2 \to Z^1$, and after finitely many steps, this descending chain $\pi_n(X) \geq \pi_n(Z^1) \geq \pi_n(Z^2) \geq \cdots \geq 0$ becomes zero and we move on to $\pi_{n+1}(X)$, eventually reducing this to zero, and so on up the tower, killing each $\pi_i(X)$ in turn. Thus, for each $i$ the groups $\pi_i(Z^k) \cong 0$ for all $k \gg 1$. The same is true for the groups $\pi_i^Y(Z^k)$ when $Y$ is a finite spectrum, since a map $\Sigma^i Y \to Z^k$ can be homotoped to a constant map one cell at a time if all the groups $\pi_j(Z^k)$ vanish for $j$ less than or equal to the largest dimension of the cells of $\Sigma^i Y$. By the Lemma 3.2.0.1, using the notation of Remark 3.2.0.1 we obtain a map of towers inducing the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_i^Y(X^2) \arrow[r] \arrow[d] & \pi_i^Y(X^1) \arrow[r] \arrow[d] & \pi_i^Y(X) \arrow[d, equal] \\
\cdots \arrow[r] & \pi_i^Y(Z^2) \arrow[r] & \pi_i^Y(Z^1) \arrow[r] & \pi_i^Y(X)
\end{tikzcd}$$

Assume that $\alpha \in \pi_i^Y(X)$ can be pulled back arbitrarily in the upper row. By commutativity, it can also be pulled back arbitrarily in the lower row. However, by our previous construction, it is impossible. Hencen $\bigcap_{n\in\mathbb{N}_0} F^{s+n,t+n} = \emptyset$, which proves (b) when $\pi_*(X) = \pi_*(X)_{(p)}$.

In the general case, let $\alpha \in \pi_n^Y(X)$ such that $|\alpha| = p^r$ or $\infty$, $r \in \mathbb{N}_0$. Then there exists $k \in \mathbb{Z}_{>0}$ such that $p^k \nmid \alpha$. Consider the map $X \xrightarrow{p^k} X = \mathrm{Id} + \cdots + \mathrm{Id} \in [X, X]$ ($p^k$ times). Thus, we obtain a cofibration $X \xrightarrow{p^k} X \to Z$ inducing a long exact sequence

$$\cdots \to \pi_i(X) \xrightarrow{p^k} \pi_i(X) \to \pi_i(Z) \to \cdots$$

where the map $p^k$ corresponds to multiplication by $p^k$. From exactness, it follows that $\pi^*(Z)$ is equal to its $p-$torsion. By the Lemma 3.2.0.1, $X \to Z$ induces a map between the corresponding Adam towers. The map $\pi_n^Y(X) \to \pi_n^Y(Z)$ maps $\alpha \mapsto \beta \in \pi_n^Y(Z)$ where $\beta \neq 0$, by our choice of $\alpha$ and $k$, using the exactness of $\pi_n^Y(X) \xrightarrow{p^k} \pi_n^Y(X) \to \pi_n^Y(Z)$. If $\alpha$ is pulled back arbitrarily far in the tower on $X$ then the same would happen to $\beta$. This is impossible as we have shown before. Hence (b) holds in general.

To prove (a) consider the portion of the $r^{th}-$derived exact couple as follows.

$$\begin{tikzcd}
& & & E_r^{\bullet,\bullet} \\
& E_r^{s,t} \arrow[r, "k_r"] & A_r^{\bullet,\bullet} \arrow[d, "i_r"] & \\
& & A_r^{\bullet,\bullet} \arrow[uur, "j_r"] & \\
A_r^{\bullet,\bullet} \arrow[d] & & & \\
A_r^{\bullet,\bullet} \arrow[uuur] & & &
\end{tikzcd}$$

We claim that $i_r$ is injective for $r \gg 1$. For the non $p-$torsion part, this follows from exactness since the $E$ columns are $\mathbb{F}_p-$vector spaces, so $\mathrm{Ker}(i_r)$ is contained in the $p-$torsion part of $A_r^{\bullet,\bullet}$. For the $p-$torsion part, it follows from (b) that $A_r^{s,t}$ contains no $p-$torsion for $r \gg 1$, since $A_r^{s,t}$ consists of the elements of $A_1^{s,t}$ that pull back $r-1$ units vertically, by construction of the derived exact couple.

Since $i_r$ is injective for $r \gg 1$, by exactness, $k_r = 0$ for $r \gg 1$. So $d_r \colon E_r^{s,t} \xrightarrow{k_r} A_r^{\bullet,\bullet} \xrightarrow{j_r} E_r^{\bullet,\bullet} = 0$. The differential $d_r$ mapping to $E_r^{s,t}$ is also zero for $r \gg 1$, by degree reasons since the bottom rows of the initial staircase diagram are all trivial. Thus, $E_r^{s,t} = E_{r+1}^{s,t}$ for $r \gg 1$.

Moreover, since $k_r \colon E_r^{s,t} \to A_r^{\bullet,\bullet} = 0$ for $r \gg 1$, by exactness, $E_r^{s,t} = \mathrm{Coker}(i_r \colon A_r^{\bullet,\bullet} \to A_r^{\bullet,\bullet})$. This vertical map is just the inclusion $F^{s+1,t+1} \hookrightarrow F^{s,t}$ for $r \gg 1$, since there is no $p-$torsion for $r \gg 1$, proving the desired result. □

## 3.3 Minimal free $\mathcal{A}_p-$resolutions

Constructing minimal resolutions over the Steenrod algebra $\mathcal{A}_p$ offers an algorithmic method of computing the $E_2-$page. It turns out that a minimal $\mathcal{A}_p-$resolution also allows us to define multiplicative structures such as the Yoneda products and has secondary information that leads to $E_2-$Massey products, i.e. Massey products on the $E_2-$term, as we will see later.

**Definition 3.3.0.1** (Minimal $\mathcal{A}_p-$resolution)**.** A *minimal resolution* of $H^*(X;\mathbb{F}_p)$ is a free resolution

$$\cdots \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow H^*(X;\mathbb{F}_p) \longrightarrow 0$$

where each $F_i$ has the minimum number of free generators in all degrees.

The next result justifies the importance of the previous definition.

**Lemma 3.3.0.1** (Dual complex of a minimal resolution)**.** For a minimal free resolution, all the boundary maps in the dual complex

$$\cdots \longleftarrow \mathrm{Hom}_{\mathcal{A}_p}(F_2, \mathbb{F}_p) \longleftarrow \mathrm{Hom}_{\mathcal{A}_p}(F_1, \mathbb{F}_p) \longleftarrow \mathrm{Hom}_{\mathcal{A}_p}(F_0, \mathbb{F}_p) \longleftarrow 0$$

are zero. Hence, $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(X;\mathbb{F}_p), \mathbb{F}_p) \cong \mathrm{Hom}^t_{\mathcal{A}_p}(F_s, \mathbb{F}_p)$.

*Proof.* Let $\mathcal{A}_p^+ := \mathrm{Ker}(\varepsilon\colon \mathcal{A}_p \to (\mathcal{A}_p)_0 \cong \mathbb{F}_p)$ where $\varepsilon$ corresponds to the augmentation. Observe that $\mathrm{Ker}(\varphi) \leq \mathcal{A}_p^+ F_i$ since if $x \in \mathrm{Ker}(\varphi_i)$, and consider a basis (as an $\mathcal{A}_p-$module) $\{x_{i,j}\} \subseteq F_i$, then $x = \sum_j a_j x_{i,j}$ where $a_j \in \mathcal{A}_p$. Then if $x \notin \mathcal{A}_p^+ F_i$, then some $0 \neq a_j \in (\mathcal{A}_p)_0 \cong \mathbb{F}_p$ and we can solve the equation $0 = \varphi_i(x) = \sum_j a_j \varphi_i(x_{i,j})$ for $\varphi_i(x_{i,j})$, which says that $\varphi_i(x_{i,j})$ is redundant, contradicting minimality. By exactness $\varphi_{i-1}\varphi_i = 0$, so $\varphi(x) \in \mathrm{Ker}(\varphi_{i-1})$ for each $x \in F_i$. By the preceding paragraph, we obtain $\varphi_i(x) = \sum_j a_j x_{i-1,j}$ with $a_j \in \mathcal{A}_p^+$. Hence, for each $f \in \mathrm{Hom}_{\mathcal{A}_p}(F_{i-1}, \mathbb{F}_p)$ we have $\varphi_i^*(f(x)) = f\varphi_i(x) = \sum_j a_j f(x_{i-1}, j) = 0$ by degree reasons, since $a_j \in \mathcal{A}_p^+$ and $f(x_{i-1,j}) \in \mathbb{F}_p \cong A_p/A_p^+$. $\square$

The following algorithm (cf. [Hat04], pp. 602) describes how we can construct minimal resolutions inductively. The algorithm is also valid for other $A-$modules as well.

**Input:** A truncated free resolution $F_{s_0} \overset{f_{s_0}}{\to} \cdots \overset{f_1}{\to} F_0 \overset{f_0}{\to} M$, $s_0 \in \mathbb{N}_0$, with each $F_j$ truncated for degrees $\geq t_0 \in \mathbb{Z}$.

**Output:** An extension $F_{s_0+1} \overset{f_{s_0+1}}{\to} \cdots \overset{f_1}{\to} F_0 \overset{f_0}{\to} M$, $s_0 \in \mathbb{N}_0$, with each $F_j$ truncated for degrees $\geq t_0 \in \mathbb{Z}$.

1: **while** $0 \leq r < t_0 - s_0$ **do** ▷ $r$: relative degree $t - s$
2: Choose a $\mathbb{F}_p-$basis for $(F_{s_0})_{r+1}$ and $(F_{s_0-1})_{r+2}$
3: Compute the corresponding matrix for $(f_{s_0})_{r+1}\colon (F_{s_0})_{r+1} \to (F_{s_0-1})_{r+2}$
4: $K \leftarrow \mathrm{Ker}((f_{s_0})_{r+1}\colon (F_{s_0})_{r+1} \to (F_{s_0-1})_{r+2})$
5: Choose a $\mathbb{F}_p-$basis for $(F_{s_0+1})_r$
6: Compute the corresponding matrix for $(f_{s_0+1})_r\colon (F_{s_0+1})_r \to (F_{s_0})_{r+1}$
7: $I \leftarrow \mathrm{Im}((f_{s_0+1})_r\colon (F_{s_0+1})_r \to (F_{s_0})_{r+1})$
8: $d \leftarrow \dim_{\mathbb{F}_p}(K/I)$
9: Lift the $d$ elements from $K/I$ and construct $(F_{s_0+1})_r$
10: Update the morphism $(f_{s_0+1})_r$ in order to kill $K$
11: **end while**

**Figure 3.1:** Construction of minimal free $\mathcal{A}_p-$resolutions

The first map $f_0\colon F_0 \to M$ is given by the action of the Steenrod algebra on the cohomology $M$ of the spectrum of interest. This is a general algorithm, so it neglects information about the Steenrod algebra (cf. [Nas19]), but it is quite simple to implement. Observe that in step 10 we are ensuring that we will obtain an exact sequence. The minimal property follows from the construction, but it is important to take care of $f_0\colon F_0 \to M$ by choosing a minimal presentation of $M$. For a live example of this algorithm, see [Mor24].

## 3.4 Adams vanishing theorem

Let $r \in \mathbb{N}_0$. Let $(\mathcal{A}_2)_r \subseteq \mathcal{A}_2$ be the subalgebra generated by $\mathrm{Sq}^1, \mathrm{Sq}^2, \ldots, \mathrm{Sq}^{2^r}$. Also, define $(\mathcal{A}_2)_\infty = \mathcal{A}_2$. The next result is relevant in order to perform computations because it ensures that our filtrations are finite and bounded by the *Adams vanishing line*.

**Theorem 3.4.0.1** (Adams vanishing theorem ($p = 2$), (Theorem 2.1 [Ada66]))**.** Suppose that $L$ is a left module over $\mathcal{A}_r$, a free left (or right) module over $(\mathcal{A}_2)_0$, and that $L_t = 0$ for all $t < l$. Define $T\colon \mathbb{Z} \to \mathbb{Z}$ as

$$T(4k) = 12k, \qquad T(4k+1) = 12k+2, \qquad T(4k+2) = 12k+4, \qquad T(4k+3) = 12k+7,$$

where $k \in \mathbb{Z}$. Then $\mathrm{Ext}^{s,t}_{\mathcal{A}_2}(L, \mathbb{F}_2) = 0$ if $t < l + T(s)$. This is also valid for $r = \infty$.

We also have a version for odd primes.

**Theorem 3.4.0.2** (Adams vanishing theorem ($p$ odd prime), (Theorem 1 [Liu63])). Let $p$ be an odd prime. Let $M$ be any $(\mathcal{A}_p)_0$−free $\mathcal{A}_p$−module (i.e. it is acted on freely by the Bockstein morphism) such that $M_t = 0$ for $t < m$. Then $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(M, \mathbb{F}_p) = 0$ for $t < m + (2p-1)s - 1$, $s \geq 1$.

We have an important consequence for $p = 2$. Although it is a straightforward result, it is not immediate.

**Corollary 3.4.0.1** (Adams vanishing line for the 2−local sphere, (Corollary 5.5 (proof) [Ada66])). $\mathrm{Ext}^{s,t}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2) = 0$ for $0 < t - s < f(s)$, where $f(s) = 2s - \varepsilon$ and $\varepsilon = 1$ for $s \equiv 0, 1 \bmod 4$, $\varepsilon = 2$ for $s \equiv 2$ and $\varepsilon \equiv 3$ for $s \equiv 3$.

Of course, there is an analogous statement for odd primes.

**Corollary 3.4.0.2** (Adams vanishing line for the $p$−local sphere ($p$ odd prime), (Corollary 1 [Liu63])). $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p) = 0$ for $t < (2p-1)s - 2$, $s \geq 1$.

Observe that in [Ada66] it is also stated and proven the *Adams periodicity theorem*. However, it is a consequence of Massey products so the corresponding results will be introduced later in this document.

## 3.5 Example: the $E_2-$page of the sphere spectrum ($p = 2$)

The following example illustrates the (lack of) computational capability of the previous results. Since we are using the relative degree convention (Remark 3.2.0.1) in the horizontal axis, the differentials are of the form $d_r\colon E_r^{s,t-s} \to E_r^{s+r,t-s-1}$.

**Proposition 3.5.0.1** ($E_2-$page converging to $\pi_*^S$ ($p = 2$))**.** Let $p := 2$ and $\mathbb{S}$ be the sphere spectrum. Then, the truncated $E_2-$page of the Adams spectral sequence converging to the $2-$primary stable homotopy of the spheres has the following description.

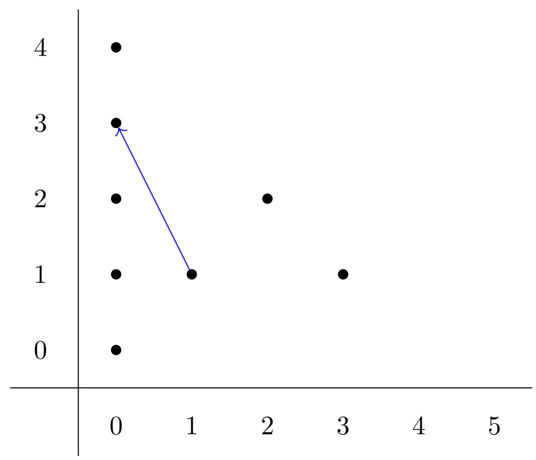


**Figure 3.2:** $E_2^{s,t-s} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t-s}(\mathbb{F}_2, \mathbb{F}_2)$, $t = (t-s) + s < 5$.

where each dot denotes a copy of $\mathbb{F}_2$.

*Proof.* The suspension spectrum $\mathbb{S} = \Sigma^\infty S^0$ is clearly connective and has finite type, since it has finite stable cells in each dimension. Therefore, Theorem 3.2.0.2 applies, so the associated Adams spectral sequence converges (strongly)

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(H^*(\mathbb{S};\mathbb{F}_2), H^*(\mathbb{S};\mathbb{F}_2)) \implies (\pi_{t-s}^{\mathbb{S}}(\mathbb{S}))_{(2)}.$$

In order to proceed via Lemma 3.3.0.1, we need to construct a minimal free $\mathcal{A}_2-$resolution of $H^*(\mathbb{S};\mathbb{F}_2) \cong \mathbb{F}_2\{x\}$, $|x| = 0$. This means that we must understand how $\mathcal{A}_2$ acts on $H^*(\mathbb{S};\mathbb{F}_2)$. Since $\mathrm{Sq}^i(x) = 0$ for $i > |x|$, we can identify $H^*(\mathbb{S};\mathbb{F}_2) \cong \mathbb{F}_2\{x\} \cong \mathcal{A}_2/(\mathcal{A}_2)^+$, i.e. $H^*(\mathbb{S};\mathbb{F}_2) \cong \mathrm{Ker}(f_0\colon \mathcal{A}_2 \to \mathbb{F}_2\{x\})$ where $f_0 := \varepsilon$ is the augmentation map, being a map of graded $\mathcal{A}_2-$modules. Define $F_0 := \mathcal{A}_2$. Note that $f_0\colon F_0 \to \mathbb{F}_2$ is a truncated minimal $\mathcal{A}_2-$free resolution. In order to prove the statement, we want to extend this truncated minimal resolution to

$$\begin{array}{c} F_4 = \bigoplus_{i_4 \in I_4} \Sigma^{t_{i_4}} \mathcal{A}_2 \\ \big\downarrow f_4 \\ \vdots \\ \big\downarrow f_1 \\ F_0 = \bigoplus_{i_0 \in I_0} \Sigma^{t_{i_0}} \mathcal{A}_2 \\ \big\downarrow f_0 \\ \mathbb{F}_2 \cong \mathcal{A}_2/\mathcal{A}_2^+ \end{array}$$

We will apply Algorithm 3.1. Denote the $\mathcal{A}_2-$module generator of $F_0$ by $\alpha_{0,1}^0 \in F_0$. In general, $\alpha_{t,k}^s \in \Sigma^t \mathcal{A}_2$, $k \geq 1$, denotes a generator for a copy of $\Sigma^t \mathcal{A}_2 \leq F_s$. Consider the following diagram resembling the first step of the algorithm, where the $s$th row denotes $F_s$ and the horizontal axis denotes the relative degree $t - s$ of the corresponding elements.

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | $\alpha^0_{0,1}$ | $\mathrm{Sq}^1\alpha^0_{0,1}$ | $\mathrm{Sq}^2\alpha^0_{0,1}$ | $\mathrm{Sq}^{2,1}\alpha^0_{0,1}$ $\mathrm{Sq}^3\alpha^0_{0,1}$ | $\mathrm{Sq}^4\alpha^0_{0,1}$ $\mathrm{Sq}^{3,1}\alpha^0_{0,1}$ | |

**Figure 3.3:** Truncated minimal $\mathcal{A}_2-$resolution of $\mathbb{F}_2$.

Since $\mathrm{Ker}(f_0\colon F_0 \to \mathbb{F}_2) = \mathcal{A}^+$, according to Algorithm 3.1, we need to construct a morphism $f_1\colon F_1 \to F_0$ of graded $\mathcal{A}_2-$modules such that $F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} \mathbb{F}_2$ is exact. Denote by $(f_j)_r\colon (F_j)_r \to (F_{j-1})_r$ to the $r$th (absolute) degree part of the map $f_j$. Assume that $F_1 := 0$ and consider a morphism $f_1\colon F_1 \to F_0$. Then $\mathrm{Ker}((f_0)_1) = \langle \mathrm{Sq}^1\alpha^0_{0,1}\rangle_{\mathbb{F}_2}$ and $\mathrm{Im}((f_1)_1)) = 0$. Thus, we need more generators to obtain the exactness property. Define $F_1 := \Sigma^1\mathcal{A}_2$, and denote its generator as $\alpha^1_{1,1}$. Now, define $f_1\colon F_1 \to F_0$ such that $f_1(\alpha^1_{1,1}) = \mathrm{Sq}^1\alpha^0_{0,1}$.

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 1 | $\alpha^1_{1,1}$ | $\mathrm{Sq}^1\alpha^1_{1,1}$ | | | | |
| 0 | $\alpha^0_{0,1}$ | $\mathrm{Sq}^1\alpha^0_{0,1}$ | $\mathrm{Sq}^2\alpha^0_{0,1}$ | $\mathrm{Sq}^{2,1}\alpha^0_{0,1}$ $\mathrm{Sq}^3\alpha^0_{0,1}$ | $\mathrm{Sq}^4\alpha^0_{0,1}$ $\mathrm{Sq}^{3,1}\alpha^0_{0,1}$ | |

**Figure 3.4:** Truncated minimal $\mathcal{A}_2-$resolution of $\mathbb{F}_2$.

By the Adem relations, observe that $f_1(\mathrm{Sq}^1\alpha^1_{1,1}) = \mathrm{Sq}^1\mathrm{Sq}^1\alpha^0_{0,1} = 0$. Therefore, we require another generator $\alpha^1_{2,1}$, and $f_1$ must satisfy $f_1(\alpha^1_{2,1}) = \mathrm{Sq}^2\alpha^0_{0,1}$.

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 1 | $\alpha^1_{1,1}$ | $\mathrm{Sq}^1\alpha^1_{1,1}$ $\alpha^1_{2,1}$ | $\mathrm{Sq}^1\alpha^1_{2,1}$ $\mathrm{Sq}^2\alpha^1_{1,1}$ | | | |
| 0 | $\alpha^0_{0,1}$ | $\mathrm{Sq}^1\alpha^0_{0,1}$ | $\mathrm{Sq}^2\alpha^0_{0,1}$ | $\mathrm{Sq}^{2,1}\alpha^0_{0,1}$ $\mathrm{Sq}^3\alpha^0_{0,1}$ | $\mathrm{Sq}^4\alpha^0_{0,1}$ $\mathrm{Sq}^{3,1}\alpha^0_{0,1}$ | |

**Figure 3.5:** Truncated minimal $\mathcal{A}_2-$resolution of $\mathbb{F}_2$.

Iterating this process, we obtain

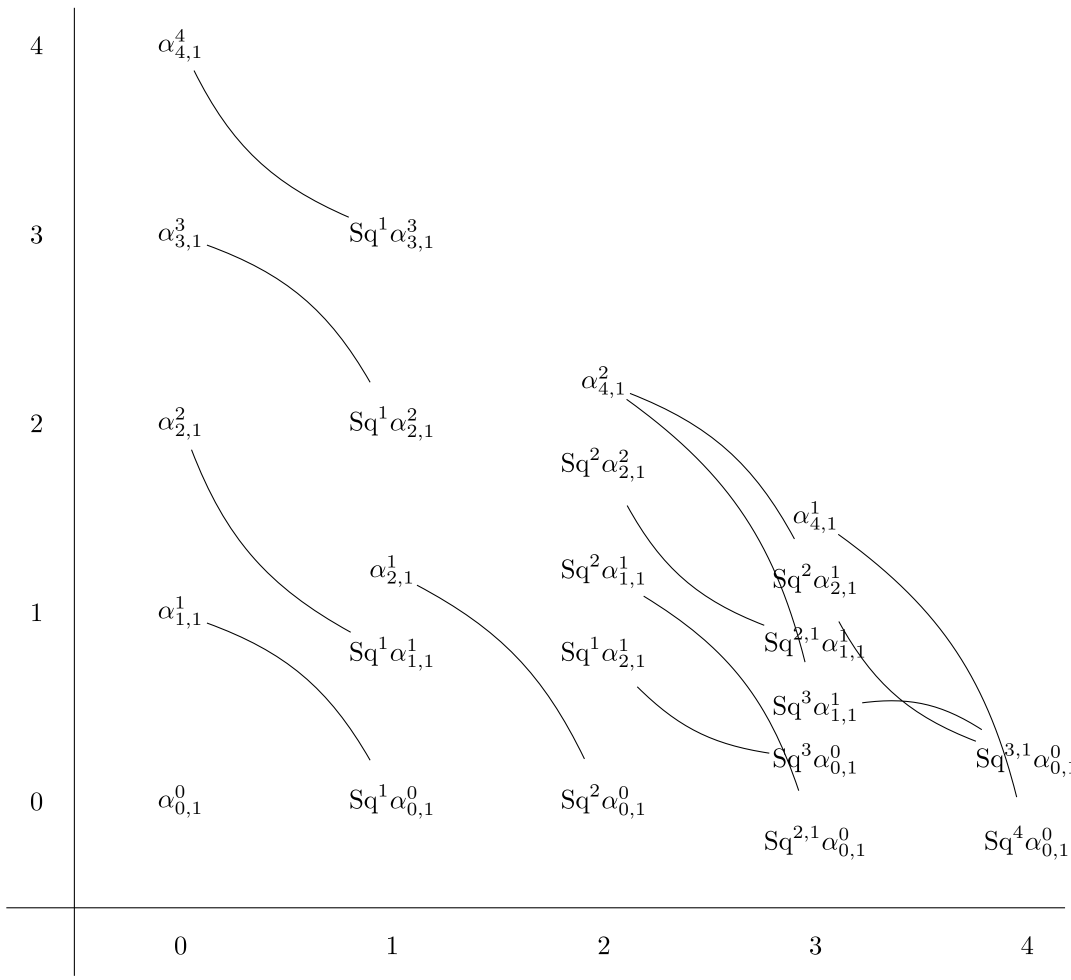


**Figure 3.6:** Truncated minimal $\mathcal{A}_2$−resolution of $\mathbb{F}_2$.

Since in each step, we are adding a minimal number of copies of $\mathcal{A}_2$, we have a truncated minimal free resolution of $\mathbb{F}_2$. By Lemma 3.3.0.1, applying $\mathrm{Hom}_{\mathcal{A}_2}(-, \mathbb{F}_2)$, we get the following description of the associated $E_2$−page. Remember that we are considering $\mathbb{F}_2 \cong \mathcal{A}_2/\mathcal{A}_2^+$ as a graded algebra.

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | $\gamma^4_{4,1}$ | | | | | |
| 3 | $\gamma^3_{3,1}$ | | | | | |
| 2 | $\gamma^2_{2,1}$ | | $\gamma^2_{4,1}$ | | | |
| 1 | $\gamma^1_{1,1}$ | $\gamma^1_{2,1}$ | | $\gamma^1_{4,1}$ | | |
| 0 | $\gamma^0_{0,1}$ | | | | | |
| | 0 | 1 | 2 | 3 | 4 | 5 |

**Figure 3.7:** $E_2^{s,t-s} = \mathrm{Ext}^{s,t-s}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$, $t < 5$.

where the $\gamma^i_{j,k}$ comes from dualizing $\alpha^i_{j,k}$. This concludes the proof. $\square$

By the Adams Vanishing Theorem (Theorem 3.4.0.1), we have determined $E_2^{*,1}$ completely. The 0th column corresponds to the non-trivial entries $E_2^{s,0} = \mathbb{F}_2\{\gamma^s_{s,1}\}$ for all $s \geq 0$, which comes from the fact that $\pi_0^S \cong \mathbb{Z}$, but we need to define a multiplicative structure on $E_r$ to prove this (Proposition 3.6.0.1). Furthermore, observe that we have possible non-trivial differentials between the 0th and the 1st column. For degree reasons, this does not happen for odd primes.

There is a more detailed minimal resolution (`https://issyl-m.github.io/andresmoranl.github.io/mix/Mod2FreeResolutionSphere/`) computed using the author's program [Mor24], and more detailed descriptions of the $E_2-$term for different primes (`https://issyl-m.github.io/andresmoranl.github.io/mix/Mod2FreeResolutionSpherePlot/`). Observe that the computations are easier for higher primes, with $p = 2$ as the hardest case. This is due to the grading on $\mathcal{A}_p$ leading to gaps of trivial groups, producing a sparseness effect, and reducing the amount of possibly non-trivial differentials. Further, the mod 2 linear algebra may be more computationally demanding. Therefore, even with a highly optimized computational approach, we require a deeper understanding of the Steenrod algebra to extend the computations significantly.

## 3.6 Yoneda products

There are two important pairings that describe the multiplicative structure on the $E_2-$term and $\pi_*^Y(X)$ for a given spectrum $X$, corresponding to the Yoneda pairing and the composition pairing respectively. The Yoneda pairing resembles the composition pairing algebraically. It is a pairing between spectral sequences, i.e. it is compatible with the differentials, satisfying the Leibniz rule. Moreover, the Yoneda pairing converges to the composition pairing via Moss' Convergence Theorem, descending to the $E_r-$pages, for $r > 2$. In other words, the Yoneda pairing allows us, via Moss' Convergence Theorem, to detect composition products in $\pi_*^Y(X)$.

In practice, it turns out that the (induced) Yoneda products are not enough to determine $\pi_*^Y(X)$ completely, i.e. there are situations where the induced products on the $E_\infty-$page do not detect all of the possible products on $\pi_*^Y(X)$. In such cases, we require further multiplicative structures of *higher order* such as the Massey pairing, which converges to the Toda brackets, by another convergence theorem from Moss. Roughly speaking, Toda brackets encode secondary information related to the composition product. Hence, these products could be used to detect hidden relations in $\pi_*^Y(X)$ that are not directly available for Yoneda products. Additionally, the $E_r-$Massey products satisfy a generalized Leibniz rule that could be used to determine some $d_r-$differentials.

First, we will introduce the Yoneda pairing.

**Theorem 3.6.0.1** (The Yoneda pairing (Theorem 9.5, [McC00])). Suppose that $\Gamma$ is an algebra over a field $k$. Let $L, M$ and $N$ be left $\Gamma-$modules. Then there is a bilinear, associative pairing, called the *(Yoneda) composition product*, defined for all $p, t, q, t' \geq 0$,

$$\circ\colon \mathrm{Ext}_\Gamma^{p,t}(L, M) \otimes \mathrm{Ext}_\Gamma^{q,t'}(M, N) \to \mathrm{Ext}_\Gamma^{p+q,t+t'}(L, N).$$

The next result is relevant in our context.

**Proposition 3.6.0.1** (The opposite Yoneda pairing (pp 82. [Rog15])). Let $X$, $Y$ and $Z$ be spectra. Then there is a bilinear associative pairing called the *(opposite) Yoneda pairing*

$$\mathrm{Ext}_{\mathcal{A}_p}^{*,*}(H^*(Z;\mathbb{F}_p), H^*(Y;\mathbb{F}_p)) \otimes \mathrm{Ext}_{\mathcal{A}_p}^{*,*}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \to \mathrm{Ext}_{\mathcal{A}_p}^{*,*}(H^*(Z;\mathbb{F}_p), H^*(X;\mathbb{F}_p)).$$

This pairing is described as follows. Let $[f] \in \mathrm{Ext}_{\mathcal{A}_p}^{*,*}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p))$ and $[g] \in \mathrm{Ext}_{\mathcal{A}_p}^{*,*}(H^*(Z;\mathbb{F}_p), H^*(Y;\mathbb{F}_p))$. Assume that $f\colon P_s \to H^*(X;\mathbb{F}_p)$ and $g\colon Q_u \to H^*(Y;\mathbb{F}_p)$ are morphisms of degree $t, v \in \mathbb{Z}$ respectively, where $P_*$ is a free resolution of $H^*(Y;\mathbb{F}_p)$ and the same for $Q_*$ and $H^*(Z;\mathbb{F}_p)$. Then, $g$ lifts to a chain map $g_* = \{g_n\colon Q_{u+n} \to P_n\}_n$, where $|g_n| = v$ as a morphism, making the diagram

$$\begin{tikzcd}
 & H^*(X;\mathbb{F}_p) & & & & & \\
\cdots \arrow[r] & P_s \arrow[u, "f"] \arrow[r, "\partial_s"] & \cdots \arrow[r] & P_1 \arrow[r, "\partial_1"] & P_0 \arrow[r, "\varepsilon"] & H^*(Y;\mathbb{F}_p) \\
\cdots \arrow[r] & Q_{u+s} \arrow[u, "g_s"] \arrow[r, "\partial_{u+s}"] & \cdots \arrow[r] & Q_{u+1} \arrow[u, "g_1"] \arrow[r, "\delta_{u+1}"] & Q_u \arrow[u, "g_0"] \arrow[ur, "g"] &
\end{tikzcd}$$

commute. The composite $fg_s\colon Q_{u+s} \to H^*(X;\mathbb{F}_p)$ is then an $\mathcal{A}_p-$module morphism of degree $(v+t)$, and satisfies $fg_s\partial_{u+s+1} = 0$. Its cohomology class is by definition the Yoneda product of $[g]$ and $[f]$. This product is well-defined.

We can consider Proposition 3.6.0.1 as our definition of the Yoneda pairing.

In practice, we choose $Y := X := \mathbb{S}$ as the sphere spectrum since its $p-$primary stable homotopy is relatively known for low degrees (this situation improves for $p \gg 1$). Intuitively, notice that we are taking products between a class from the sphere spectrum and a class for a given spectrum $Z$. This is not a coincidence, since $\mathbb{S}$ is a categorification of $\mathbb{Z}$ ([Sto, pp. 7]).

The next example illustrates how to compute Yoneda products in an important case.

**Example 3.6.0.1** (Yoneda products on the $E_2-$page of $\mathbb{S}$, (Example 13.2 [Rog15]))**.** Let $X = Y = Z = \mathbb{S}$ and let $P_* = Q_*$ be the minimal resolution of $\mathbb{F}_2 \cong \mathcal{A}_2/\mathcal{A}_2^+$. We can compute the Yoneda product

$$\mathrm{Ext}^{u,v}_{\mathcal{A}_2}(\mathbb{F}_2,\mathbb{F}_2) \otimes \mathrm{Ext}^{s,t}_{\mathcal{A}_2}(\mathbb{F}_2,\mathbb{F}_2) \to \mathrm{Ext}^{u+s,v+t}_{\mathcal{A}_2}(\mathbb{F}_2,\mathbb{F}_2)$$

that makes $\mathrm{Ext}^{*,*}_{\mathcal{A}_2}(\mathbb{F}_2,\mathbb{F}_2)$ into a bigraded algebra. Choose cocycle representatives $f\colon P_s \to \mathbb{F}_2 \cong \mathcal{A}_2/\mathcal{A}_2^+$ and $g\colon P_u \to \mathbb{F}_2 \cong \mathcal{A}_2/\mathcal{A}_2^+$. Consider a lifting $g_*\colon P_{u+*} \to P_*$ and compute the composite $fg_s$.

Let $f := h_0\colon P_1 \to \mathbb{F}_2$ be dual to $\alpha^1_{1,1} \in P_1$ and let $g := h_2\colon P_1 \to \mathbb{F}_2$ be dual to $\alpha^1_{4,1} \in P_1$ (the notation stands for the Hopf maps). A lift $g_0\colon P_1 \to P_0$ of $g$ is given by $\alpha^1_{4,1} \mapsto \alpha^0_{0,1}$ and $\alpha^1_{i,1} \mapsto 0$ for $i \neq 4$.

$$\begin{tikzcd}
\mathbb{F}_2 & & \\
P_1 \arrow[u, "f=h_0"] \arrow[r, "\partial_1"] & P_0 \arrow[r, "\varepsilon"] & \mathbb{F}_2 \\
P_2 \arrow[u, "g_1"] \arrow[r, "\partial_2"] & P_1 \arrow[u, "g_0"] \arrow[ur, "g=h_2"] &
\end{tikzcd}$$

The composite $g_0\partial_2\colon P_2 \to P_0$ is then given by $\alpha^2_{2,1} \mapsto 0$, $\alpha^2_{4,1} \mapsto 0$, $\alpha^2_{5,1} \mapsto \mathrm{Sq}^1\alpha^0_{0,1}$, $\alpha^2_{8,1} \mapsto \mathrm{Sq}^4\alpha^0_{0,1} + \mathrm{Sq}^{3,1}\alpha^0_{0,1}$ and so on. A lift $g_1\colon P_2 \to P_1$ is given by $\alpha^2_{2,1} \mapsto 0, \alpha^2_{4,1} \mapsto 0$, $\alpha^2_{5,1} \mapsto \alpha^1_{1,1}$, $\alpha^2_{8,1} \mapsto \alpha^1_{4,1} + \mathrm{Sq}^3\alpha^1_{1,1}$ and so on. Hence $fg_1\colon P_2 \to \mathbb{F}_2$ is given by $\alpha^2_{5,1} \mapsto 1$ and $\alpha^2_{i,1} \mapsto 0$ for $i \neq 5$, by degree reasons. Therefore, $[fg_1]$ is the dual of $\alpha^2_{5,1}$, denoted by $\gamma^2_{5,1}$. Thus $h_0h_2 = \gamma^2_{5,1}$. □

The following result summarizes the multiplicative structure of the Adams spectral sequence.

**Theorem 3.6.0.2** (Moss' convergence theorem (Theorem 2.1, [Mos68]))**.** Let $X$, $Y$ and $Z$ be spectra with $Y$ and $Z$ connective of finite type. Then, there is a natural pairing of spectral sequences

$$E_r^{s_1,t_1}(Y,Z) \otimes E_r^{s_2,t_2}(X,Y) \to E_r^{s,t}(X,Z)$$

where $s = s_1 + s_2$ and $t = t_1 + t_2$, such that

(i) It is the Yoneda pairing for $r = 2$.

(ii) $d_r$ is a derivation with respect to this product.

(iii) The pairing at $E_{r+1}$ is induced by the pairing at $E_r$.

(iv) The pairing at $E_\infty$ is induced by the composition of maps.

Furthermore, observe that each page is a DGA (i.e. a differential graded algebra) so we can define $E_r-$Massey products for $r \geq 2$ by taking a suitable differential (bi)graded category.

## 3.7 $E_r-$Massey products

This section is intended to define the $E_r-$Massey products and provide the corresponding Moss' Convergence Theorem to Toda brackets.

**Definition 3.7.0.1** (Differential bigraded category (dbg))**.** A *differential bigraded category (DBG)* is a category $\mathcal{C}$ such that if $X, X' \in \mathrm{Obj}(\mathcal{C})$, then the set of morphisms from $X'$ to $X$, $\mathcal{C}(X', X)$ is a bigraded group $(\mathcal{C}^s_t(X', X))_{s,t\in\mathbb{Z}}$, with a differential $d\colon \mathcal{C}^s_t \to \mathcal{C}^{s+r-1}_{t+r-2}$ where $r \geq 2$ is a fixed integer, and if $a \in \mathcal{C}^s_t(X', X)$ and $a' \in \mathcal{C}^{s'}_{t'}(X'', X')$ then

$$d(aa') = (da)a' + (-1)^{t-s}a(da').$$

The product is the composition of maps in the category $\mathcal{C}$.

**Definition 3.7.0.2** (Induced pairing in homology)**.** Define $H^s_t(X', X) = (\mathrm{Ker}(d)/\mathrm{Im}(d))^s_t$. There is a pairing

$$H^s_t(X', X) \otimes H^{s'}_{t'}(X'', X') \to H^{s+s'}_{t+t'}(X'', X).$$

The previous pairing is well-defined.

As in the case of a DGA, we can construct Massey products on a DBG.

**Definition 3.7.0.3** (Massey product pairing for a DBG)**.** Let $\mathcal{C}$ be a DBG. Let $X, X', X'' \in \mathrm{Obj}(\mathcal{C})$. Let $a \in H^s_t(X', X)$, $a' \in H^{s'}_{t'}(X'', X')$ and $a'' \in H^{s''}_{t''}(X''', X'')$ be such that $aa' = 0 = a'a''$. Let $p := s + s' + s'' - r + 1$, $p' := s + s' - r + 1$, $p'' := s' + s'' - r + 1$, $q := t + t' + t'' - r + 1$, $q' := t + t' - r + 1$, and $q'' := t' + t'' - r + 1$. We define the *Massey product*

$$\langle a, a', a''\rangle \in \frac{H^p_{q+1}(X''', X)}{H^{p'}_{q'+1}(X'', X)a'' + aH^{p''}_{q''+1}(X''', X')}$$

as follows. Choose representatives $x \in \mathcal{C}^s_t$, $x' \in \mathcal{C}^{s'}_{t'}$ and $x'' \in \mathcal{C}^{s''}_{t''}$ for $a$, $a'$ and $a''$ respectively. Define $\overline{y} := (-1)^{|y|}y = (-1)^{t-s}y$ for $y \in \mathcal{C}^s_t$. Then, define $\langle a, a', a''\rangle := \{ux'' - \overline{x}v \mid d(u) = xx',\ d(v) = x'x''\}$.

**Proposition 3.7.0.1** (The Massey product is well-defined)**.** The Massey product for a DBG is well-defined.

*Proof.* The same proof as for DGAs applies. □

**Definition 3.7.0.4** (Indeterminacy)**.** The *indeterminacy* of a Massey product is defined as

$$\mathrm{Ind}(\langle a, a', a''\rangle) := \{a - b \mid a, b \in \langle a, a', a'''\rangle\}.$$

Now, consider the definition of Massey products in the Adams spectral sequence.

**Definition 3.7.0.5** (Massey products on $E_r$ ($r \geq 3$) (pp. 295 [Mos70]))**.** Let $r \geq 3$. Let $X, X', X''$ and $X'''$ be connective spectra of finite type. Define $\mathcal{C}^s_t(X', X) := E^{s,t}_{r-1}(X', X)$, $\mathcal{C}^{s'}_{t'}(X'', X') := E^{s',t'}_{r-1}(X'', X')$, and $\mathcal{C}^{s''}_{t''}(X''', X'') := E^{s'',t''}_{r-1}(X''', X'')$. Using Theorem 3.6.0.2, we define the composition in the underlying category as the product induced on $E_{r-1}$ by the Yoneda pairing. This defines a DBG and the $E_r-$Massey product, for $r \geq 3$.

Notice that the previous definition requires a pairing on the $E_{r-1}-$page. Since there is no such pairing of $E_1-$terms, the $E_2-$Massey products require another definition.

**Definition 3.7.0.6** (Massey products on $E_2$ (pp. 295 [Mos70]))**.** Let $p_0$ be a prime. Let $\mathcal{C}$ be the category whose objects are chain complexes of projective $\mathcal{A}_{p_0}-$modules, i.e. objects of the form

$$0 \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_{s-1} \overset{d_s}{\leftarrow} C_s \leftarrow \cdots$$

with $C_s = 0$ for $s < 0$ and $d_s(C_{s,t}) \subseteq C_{s-1,t}$. Define

$$\mathcal{C}^s_{-t}(C', C) := \mathcal{C}^{s,t}(C', C) := \prod_n \mathrm{Hom}^t_{\mathcal{A}_{p_0}}(C'_n, C_{n-s})$$

so that, if $f\colon C' \to C \in \mathcal{C}^{s,t}$ then $f = (f_n)$ where $f_n(C'_{n,m}) \subseteq C_{n-s,m-t}$. Now, let $d\colon \mathcal{C}^{s,t} \to \mathcal{C}^{s+1,t}$ be defined by requiring

$$(df)_n = d_{n-s}f_n - (-1)^{t+s}f_{n-1}d_n$$

obtaining a DBG $\mathcal{C}$. Now if $M, M' \in \mathrm{Obj}(\mathcal{A}_{p_0}\text{-Mod})$ with projective resolutions $C$ and $C'$, we have a natural isomorphism

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_{p_0}}(M', M) \cong H^{s,t}(C', C),$$

independent of the choice of the projective resolutions. This defines the $E_2-$Massey product

$$\langle a, a', a''\rangle \in E_2^{p,q+1}(X''', X)$$

such that

$$\langle a, a', a''\rangle = (-1)^n \langle a'', a', a\rangle$$

where $n := i'i'' + i''i + ii' + 1$ and $i := t - s, i' := t' - s', i'' := t'' - s''$.

The $E_r-$Massey products satisfy the following properties, which are general facts. In particular, we have linearity. The other properties are important as well, since they are applied to detect hidden relations when solving extension problems.

**Proposition 3.7.0.2** (Properties of $E_r-$Massey products (pp. 295 [Mos70]))**.** The $E_r-$Massey products, $r \geq 2$, are linear in each variable. Furthermore,

$$\begin{aligned}
b\langle a, a', a''\rangle &\subseteq (-1)^{m-n}\langle ba, a', a''\rangle, \\
\langle ab, a', a''\rangle &\subseteq \langle a, ba', a''\rangle \\
\langle a, a', ba''\rangle &\subseteq \langle a, a'b, a''\rangle \\
\langle a, a', a''\rangle b &\subseteq \langle a, a', a''b\rangle
\end{aligned}$$

where $b \in H^m_n$.

Moss [Mos70] also defined *matric Massey products* which are more general than the triple Massey product discussed here.

In order to compute $E_r-$Massey products, $r \geq 3$, is enough with the definition. The $E_2-$case is slightly different. The next computational result exploits the relation between Massey products and Toda brackets in a different way.

**Proposition 3.7.0.3** ($E_2-$Massey products (Theorem 2.6.5 [Bru]))**.** Let $\mathbb{K}$ be a finite field. Let $A$ be a connected $\mathbb{K}-$algebra and let $M, N$ be $A-$modules. Let $a \in \mathrm{Ext}^1_A(\mathbb{K}, \mathbb{K})$, let $x \in \mathrm{Ext}^{s_2}_A(M, \mathbb{K})$ and $y \in \mathrm{Ext}^{s_3}_A(N, M)$ so that $\langle a, x, y\rangle \in \mathrm{Ext}^{s_2+s_3}_A(N, \mathbb{K})$. Let $C \to \mathbb{K}$, $D \to M$ and $E \to N$ be resolutions of $\mathbb{K}$, $M$ and $N$, respectively. Let $h\colon xy \cong 0$ be a chain homotopy with degree 1 component $h_1\colon E_{s_2+s_3} \to C_1$. If $g \in E_{s_2+s_3}$, write $\tilde{y}_{s_2}(g) := \sum_i a_i g_i \in D_{s_2}$. Then

$$\langle a, x, y\rangle(g) = \sum_i \overline{a}(a_i)x(g_i)$$

where $\overline{a}\colon IA \to \mathbb{K}$, $IA := \mathrm{Ker}(A \xrightarrow{\varepsilon} \mathbb{K})$, is given by the commutative diagram

$$\begin{tikzcd}
IA \arrow[d, "\overline{a}"'] & C_1 \arrow[l, "d"'] \arrow[d] \arrow[dl, "a"] \\
\mathbb{K} & C_0 \arrow[l]
\end{tikzcd}$$

The element $a \in \mathrm{Ext}^1_A(\mathbb{K}, \mathbb{K})$ seen as an element of the $E_2-$page of the Adams spectral sequence corresponds to $a \in E_2^{1,r}$ for $r \in \mathbb{Z}$. Although the chain homotopy $h$ is not used in the computations, it is required to prove the stated result. In order to perform computations, the most important step is to compute the lifting $\tilde{y}$. Moreover, a common case of use for this result corresponds to $M := \mathbb{K} := \mathbb{F}_p$. There are more methods to compute Massey products, such as the May spectral sequence as well as cobar resolutions for $E_2-$Massey products [McC00, pp. 426].

Toda brackets can be defined in triangulated categories, and it turns out that the definition for the category of chain complexes coincides with the definition of Massey products.

**Definition 3.7.0.7** (Toda bracket (Definition 9.41 [McC00]))**.** Let $\gamma \in [X, Y]$, $\beta \in [Y, Z]$ and $\alpha \in [Z, W]$. Suppose that $\beta \circ \gamma \cong 0$ and $\alpha \circ \beta \cong 0$. Let $c, b, a$ be mappings representing $\gamma, \beta$ and $\alpha$ respectively. There are extensions of $b \circ c$ and $a \circ b$ to $CX \to Z$ and $CY \to W$, which we denote by $B$ and $C$, respectively. Write $\Sigma X = C^+X \cup C^-X$ and consider the mapping $\Sigma X \to W$ given on $C^+X$ as a $a \circ B$ and on $C^-X$ as $C \circ c$. The set of all such mappings is denoted $\langle \alpha, \beta, \gamma\rangle \subseteq [\Sigma X, W]$, called the *Toda bracket* of $\alpha, \beta$ and $\gamma$. It has indeterminacy given by the subset $\mathrm{Im}(\alpha_*) + \mathrm{Im}((\Sigma\gamma)^*)$.

The Massey products satisfy a generalized Leibniz rule. The original statement explicitly involves matric Massey products.

**Theorem 3.7.0.1** (Generalized Leibniz rule (Theorem 1 (ii), [Mos70]))**.** Let $a \in E_r^{s,t}(X', X)$, $a' \in E_r^{s',t'}(X'', X')$, $a'' \in E_r^{s'',t''}(X''', X'')$ be such that $aa' = 0 = a'a''$. If $a, a', a''$ also satisfy $ad_r(a') = 0 = a'd_r(a'')$ then

$$d_r\langle a, a', a''\rangle \subseteq -\langle d_r(a), a', a''\rangle - (-1)^{t-s}\langle a, d_r(a'), a''\rangle - (-1)^{(t-s)+(t'-s')}\langle a, a', d_r(a'')\rangle.$$

**Definition 3.7.0.8** (Permanent cycle)**.** In the context of the $E_{r_0}-$page of the Adams spectral sequence, $r_0 \geq 2$, we say that a cycle is *permanent* if it belongs to $\mathrm{Ker}(d_r \colon E_r^{s,t-s} \to E_r^{s+r,t-s-1})$ for all $r \geq r_0$.

Observe that a permanent cycle could be a trivial class. Otherwise, the class is a *survivor*, just as in the case of the Serre spectral sequence.

There is also a Moss' Convergence Theorem for Massey products, which relates them to Toda brackets. This result is an important tool to detect permanent cycles and *hidden extensions*. This result is also valid when the involved Adams spectral sequences are weakly convergent, i.e. $E_\infty^{s,t} = \bigcap_{r>s} E_r^{s,t}$, for all $s, t \in \mathbb{Z}$. We assume a stronger condition, that is, $X, X', X''$ are connective spectra of finite type.

**Theorem 3.7.0.2** (Moss' convergence theorem for Massey products (Theorem 2, [Mos70]))**.** Let $X, X', X''$ be connective spectra of finite type. Let $a \in E_r^{s,t}(X', X)$, $a' \in E_r^{s',t'}(X'', X')$, $a'' \in E_r^{s'',t''}(X''', X'')$ be permanent cycles such that $aa' = 0 = a'a''$. Let $w, w', w''$ be homotopy classes realizing $a, a', a'' \in E_\infty^{*,*}$, and suppose that $ww' = 0 = w'w''$. Then $\langle a, a', a''\rangle$ contains a permanent cycle that is realized by an element of the Toda bracket $\langle w, w', w''\rangle$ provided all elements of the following groups are permanent cycles

$$E_{s+s'-n+1}^{n,i+i'-n+1}(X'', X), \qquad \text{where } 0 \leq n \leq s + s' - r,$$
$$E_{s'+s''-n+1}^{n,i'+i''-n+1}(X''', X'), \qquad \text{where } 0 \leq n \leq s' + s'' - r,$$

and $i := t - s, i' := t' - s', i'' := t'' - s'' \in \mathbb{Z}$.

The last technical condition means that we must not have *crossing differentials.* The next figure resembles a crossing differential $d_{r+k}$ for $k \geq 1$.

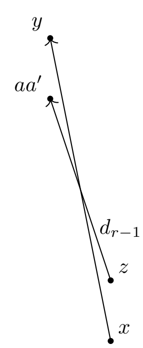


**Figure 3.8:** Crossing differential.

We usually choose $X'' := X' := X := \mathbb{S}$ because it is a *ring spectrum* (i.e. a spectrum with a ring-like multiplication). Rather than being just a reasonable choice, Toda brackets behave better for ring spectra. The importance of Theorem 3.7.0.2, is exemplified in Example 3.9.0.1.

An interesting consequence of the Massey products corresponds to the next result. See [Ada66] for a more general statement ($p = 2$).

**Theorem 3.7.0.3** (Periodicity theorem (Theorem 3.4.6 [Rav86]))**.** Let $p$ be a prime and $q := 2p - 2$.

(a) For $p = 2$ and $n \geq 1$, $\mathrm{Ext}_{\mathcal{A}_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}_{\mathcal{A}_2}^{s+2^{n+1},t+3\cdot 2^{n+1}}(\mathbb{F}_2, \mathbb{F}_2)$ for

$$0 < t - s < \min\left(g(s) + 2^{n+2}, h(s)\right),$$

where $g(s) = 2s - 4 - \tau$ with $\tau = 2$ if $s \equiv 0, 1 \bmod 4$, $\tau = 1$ if $s \equiv 3$, and $\tau = 0$ if $s \equiv 2$, where $h(s)$ is defined by the following table

| $s$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\geq 9$ |
|---|---|---|---|---|---|---|---|---|---|
| $h(s)$ | 1 | 1 | 7 | 10 | 17 | 22 | 25 | 32 | $5s-7$ |

(b) For $p > 2$ and $n \geq 0$, $\mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathrm{Ext}^{s+p^n, s+(q+1)/p^n}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$ for

$$0 < t - s < \min\left(g(s) + p^n q, h(s)\right),$$

where $g(s) = qs - 2p - 1$ and $h(s) = 0$ for $s = 1$ and $h(s) = (p^2 - p - 1)s - \tau$ with $\tau = 2p^2 - 2p + 1$ for even $s > 1$ and $\tau = p^2 + p - 2$ for odd $s > 1$.

## 3.8 Hidden extensions

We have two important tools to help us resolve extension problems: the Moss' Convergence Theorem 3.6.0.2 and the Moss' Convergence Theorem for Massey products 3.7.0.2. However, determining the multiplicative structure on $E_\infty$ is not enough to determine $\pi^Y_*(X)$ in general, since we could be missing further multiplicative relations required to solve the corresponding extension problems. This leads to the notion of *hidden extensions*.

**Definition 3.8.0.1** (Hidden extension (naive definition) (pp. 53 [Isa19]))**.** Let $p$ be a prime. Let $X$ be a connective spectrum of finite type and $Y$ be a spectrum. Then,

$$E_2^{s,t} \implies \left(\pi^Y_{t-s}(X)\right)^\wedge_p$$

and the Moss Convergence Theorem applies. Let $\alpha, \beta \in \pi^Y_*(X)$ such that $\alpha\beta \neq 0$ and are detected by $a, b \in E_\infty^{*,*}$ respectively, with $ab = 0$. We call to this situation a *hidden $\alpha-$extension.*

This definition is too general, introducing trivialities that should be avoided. D. C. Isaksen stated a more technical definition ([Isa19, Definition 4.2]) which says that the product $\alpha\beta$ associated to $ab = 0$, rather than being trivial, it could be realized in higher filtration (i.e. for a higher $s \geq 0$ coordinate).

There are several strategies to solve hidden extensions, although these could be considerably non-trivial. Our main goal is to solve $p-$extensions, i.e. the hidden extensions associated to multiplication (composition) with $p$ (the map induced by multiplication by $p\colon S^1 \to S^1$). Hidden relations can be propagated via the multiplicative structures described in the previous section. They can also be solved using comparison theorems such as the algebraic [WX18] or the geometric Kahn-Priddy Theorem (Proposition 3.10.2.1) or even by comparing the Adams spectral sequence with other spectral sequences. Moreover, higher multiplicative structures such as Massey products can detect hidden extensions via Moss's Convergence Theorem 3.7.0.2, being a common approach. More recently, there are also synthetic approaches [Burk21].

The following example illustrates a situation where the Yoneda products are not enough to decide whether certain products in stable homotopy are non-trivial or not.

**Example 3.8.0.1** ((Possible) hidden extension)**.** Let $p$ be a prime. Let $X$ be a connective spectrum of finite type and consider $Y := \mathbb{S}$. Let $\gamma := \gamma^{(1)}_{0,1} \in E^{1,0}_\infty(\mathbb{S}, \mathbb{S})$. Assume that the $E_\infty^{*,17}-$column associated to $(\pi^Y_*(X))_{(p)}$ has the following form

Take the corresponding Adams filtrations

$$\pi^S_0(\mathbb{S})_{(p)} = G^0 \geq G^1 \geq G^2 \geq \dots \qquad \text{and} \qquad \pi^S_{17}(X)_{(p)} = F^0 \geq F^1 \geq F^2 \geq \dots$$

For degree reasons, we know that $\gamma x = 0 \in E^{2,17}_\infty = F^2/F^3$. Moreover, by the Moss' Convergence Theorem, this product is realized in stable homotopy by $\tilde{\gamma}\tilde{x} \in F^{1+1} \leq (\pi^S_{17}(X))_{(p)}$, where $\tilde{\gamma} \in G^1$ and $\tilde{x} \in F^1$ induce $\gamma$ and $x$ respectively, as the composition product on stable homotopy preserves filtration. Since the product on $E_\infty^{*,*}$ is induced by the composition product, $\overline{\tilde{\gamma}\tilde{x}} = 0 \in F^2/F^3$, so $\tilde{\gamma}\tilde{x} \in F^3$.

Consider the following short exact sequences associated to the filtration $\{F^k\}_{k \in \mathbb{N}_0}$, that is

$$0 \to F^2/F^3 \cong 0 \to F^1/F^3 \to F^1/F^2 \cong \mathbb{Z}/p\mathbb{Z}\{x\} \to 0$$

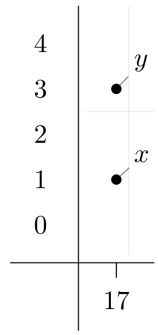


**Figure 3.9:** Possible hidden $\gamma-$extension.

$$0 \to F^3/F^4 \cong \mathbb{Z}/p\mathbb{Z}\{y\} \to F^1/F^4 \to F^1/F^3 \cong \mathbb{Z}/p\mathbb{Z}\{x\} \to 0$$

where the stated isomorphisms were obtained by the Adams Convergence Theorem. Observe that we know that $\tilde{\gamma}\tilde{x} \in F^3$. However, it is not clear whether this element is trivial in $F^1/F^4 \cong F^0/F^4 \cong (\pi^S_{17}(X))_{(p)}$ or not.

The reason why it is not possible to determine the multiplicative structure on stable homotopy just by considering the induced Yoneda products on $E_\infty^{*,17}$ is that there is a trivial entry $E_\infty^{2,17} = 0$. If this entry were non-trivial, then we would have similar short exact sequences, but we would be taking quotients by $F^3$ instead of $F^4$, ensuring that the product remains trivial in stable homotopy. Heuristically, these islands of sequences of points could indicate the possibility of hidden extensions. $\square$

For an example where the corresponding product turns out to be non-trivial, see [WX18].

## 3.9 Example: the $E_2-$page of the sphere spectrum ($p = 2$) revisited

Since we can construct minimal resolutions and compute Yoneda products, we are ready to start with basic computations such as solving extension problems and even compute our first differential for $\mathrm{Ext}^{s,t}_{\mathcal{A}_2}(\pi^S_{t-s})_{(2)}$, $t-s < 40$. Assume that we have constructed a minimal $\mathcal{A}_2-$free resolution of $\mathbb{F}_2$ large enough to obtain the following description and computed the chain map lifting of each cochain representing a class in $\mathrm{Ext}^{s,t-s}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$, $t-s < 40$, that is, that we have computed all the Yoneda products in that range. This was done computationally using `minrv1` [Mor24]. Then, we have the following result.

**Proposition 3.9.0.1** (Truncated $E_2-$page associated to $\mathbb{S}^{\wedge}_2$)**.** Let $p := 2$. Let $\mathbb{S}$ be the sphere spectrum. Then, the associated Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2) \implies (\pi^S_{t-s})_{(2)}$$

has its $E_2^{s,t-s}-$term, $t-s < 40, s \geq 0$, given by

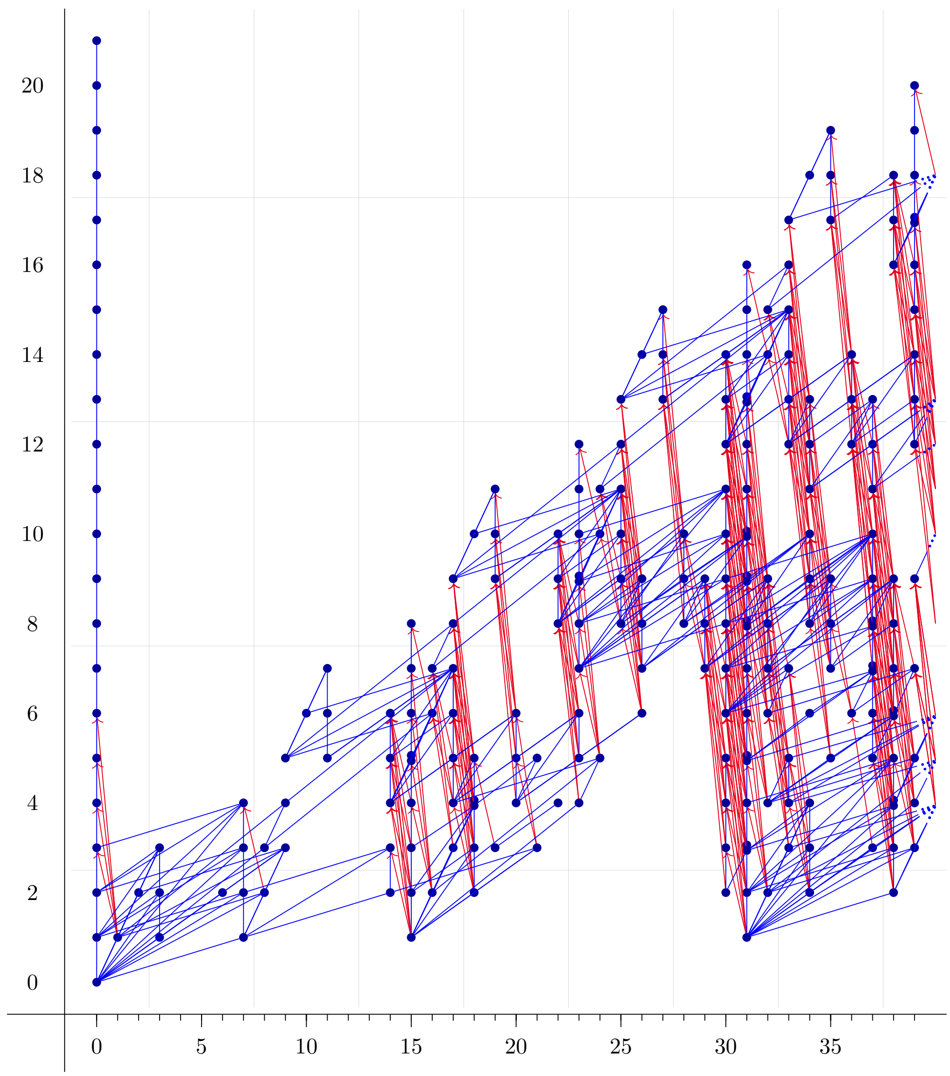


**Figure 3.10:** $\mathrm{Ext}^{s,t-s}_{\mathcal{A}_2}(\mathbb{F}_2, \mathbb{F}_2)$, $t-s < 40$, $s \geq 0$.

The red lines resemble differentials, and the blue lines represent Yoneda products by classes in the sphere $E_2-$term. Even in a small range, the diagram is quite complicated, due to the presence of higher differentials, and we can also appreciate the Adams Periodicity Theorem (Theorem 3.7.0.3) in action. The Adams Vanishing Line goes through the periodic classes at the top, with a slope of $\frac{1}{2}$. We will now focus on a smaller area, and prove that there are no differentials for $t - s \leq 13$.

**Theorem 3.9.0.1** (First 13 stable stems of $\mathbb{S}_2^\wedge$ (pp. 599 [Hat04])). The first $2-$primary stable stems for $\mathbb{S}$ are described in the following table.

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\mathbb{Z}_{(2)}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/16\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $*$ | 11 | 12 | 13 | | | | | | | | |
| | $\mathbb{Z}/8\mathbb{Z}$ | 0 | 0 | | | | | | | | |

**Table 3.1:** $(\pi_*^S)_2^\wedge$, $* \leq 13$.

Moreover, the first non-trivial differential corresponds to $d_2 \colon E_2^{1,15} \to E_2^{3,14}$.

*Proof.* First, consider the next diagram resembling the associated $E_2-$term. We will show that the differentials are trivial in lower degrees using the derivation rule.

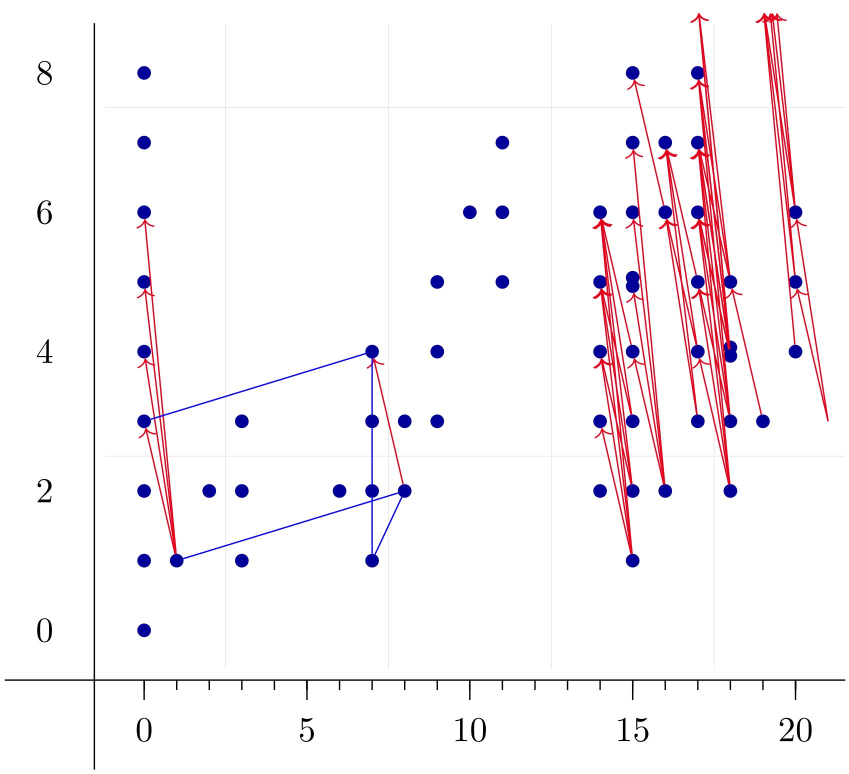


**Figure 3.11:** $\mathrm{Ext}_{\mathcal{A}_2}^{s,t-s}(\mathbb{F}_2, \mathbb{F}_2)$, $t - s < 40$, $s \geq 0$.

In order to use more standard notation, consider the classes $h_i \in E_2^{1,2^i-1}$, $i = 0, 1, \ldots$. This notation comes from the fact that these classes converge to the classical Hopf maps $2, \eta, \nu, \sigma, \ldots$ respectively. This notation is also handy for expressing decomposable elements.

Now, assume that $d_2(h_1) \neq 0$, i.e. $d_2(h_1) = h_0^3$. Since $d_2$ is a derivation with respect to the Yoneda product, and since $h_0 h_1 = 0$ then

$$0 = d_2(h_0 h_1) = d_2(h_0)h_1 + h_0 d_2(h_1) = 0 + h_0^4 \neq 0.$$

We deduce that $d_2(h_1) = 0$. The same argument applies to prove that $d_r(h_1) = 0$ for $r \geq 2$. Moreover,

$$d_2(h_1h_3) = d_2(h_1)h_3 + h_1d_2(h_3) = h_1d_2(h_3) = 0.$$

Now, we have that $(\pi_0^S)_{(2)} \cong \mathbb{Z}_{(2)}$. Indeed, consider the filtration (using a relative degree indexing)

$$(\pi_0^S)_{(2)} = F^{0,0} \supseteq F^{1,0} \supseteq \dots .$$

By convergence, we have the isomorphisms $E_\infty^{s,0} \cong F^{s,0}/F^{s+1,0}$, for all $s \geq 0$. Let $2 \in (\pi_0^S)_{(2)}$ be the class that induces $h_0 \in F^{1,0}/F^{0,0} \cong E_\infty^{1,0}$ (think about it as a symbol for a moment). By the Moss Convergence Theorem (Theorem 3.6.0.2), the pairing on $E_\infty^{*,*}$ is induced by the composition pairing on $F^{0,0} = (\pi_0^S)_{(2)}$. Therefore, $2^k \mapsto h_0^k \neq 0$, $k \geq 0$. Consider the exact sequences

$$0 \to F^{s,0}/F^{s+1,0} \to F^{s-1,0}/F^{s+1,0} \to F^{s-1,0}/F^{s,0} \to 0.$$

Taking $s = 1$, we obtain

$$0 \to \mathbb{Z}/2\mathbb{Z}\{h_0\} \to F^{0,0}/F^{2,0} \to \mathbb{Z}/2\mathbb{Z}\{1\} \to 0.$$

By our previous observation, $F^{0,0}/F^{2,0} \cong \mathbb{Z}/4\mathbb{Z}$. Inductively, by considering appropriate quotients, $F^{0,0}/F^{s,0} \cong \mathbb{Z}/2^s\mathbb{Z}$ for $s \geq 1$, and $F^{0,0} = (\pi_0^S)_{(2)} \to \cdots \to \mathbb{Z}/2^s\mathbb{Z} \to \cdots \to \mathbb{Z}/2\mathbb{Z} \to 0$ is a projective system. Therefore,

$$(\pi_0^S)_{(2)} = \varprojlim_s \mathbb{Z}/2^s\mathbb{Z} \cong \mathbb{Z}_2^\wedge.$$

Therefore, our element $2 \in (\pi_0^S)_{(2)}$ corresponds to multiplication by 2. As a map, $2\colon \mathbb{S} \to \mathbb{S}$ is induced by $2\colon S^1 \to S^1$ given by $z \mapsto z^2$. Therefore, we can think about each vertical line as multiplication by 2.

Before analyzing the extension problems, we will also prove that $0 \neq d_2\colon E_2^{1,15} \to E_2^{3,14}$. Consider the class $h_3 \in E_2^{1,7}$. We have that $h_3^2 \in E_2^{2,14} = E_\infty^{2,14}$ because there are no differentials to consider. Let $\sigma \in (\pi_7^S)_{(2)}$ be the class inducing $h_3$. Therefore, since the (induced) pairing on $E_\infty$ and the composition pairing agree, we have that $\sigma^2 = (-1)^{|\sigma|^2}\sigma^2 = (-1)^{49}\sigma^2$. Hence, $2\sigma^2 = 0 \in (\pi_{14}^S)_{(2)}$. On the other hand, $0 \neq h_0h_3^2 \in E_2^{3,14}$. Therefore it must be killed by a differential, being $0 \neq d_2\colon E_2^{1,15} \to E_2^{3,14}$ the only possibility.

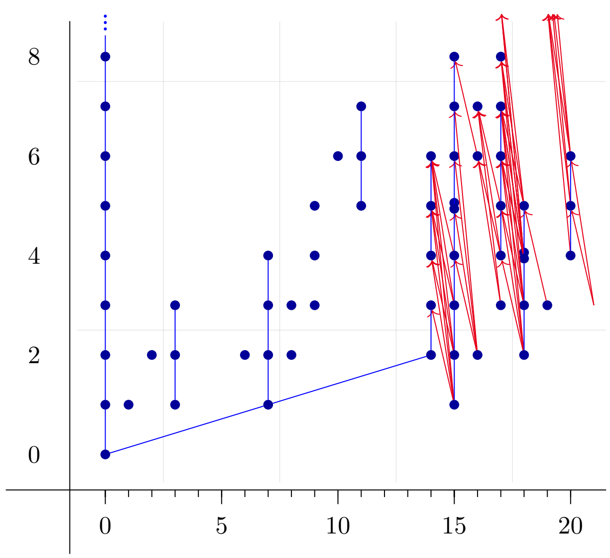


**Figure 3.12:** $\mathrm{Ext}_{\mathcal{A}_2}^{s,t-s}(\mathbb{F}_2, \mathbb{F}_2)$, $t - s < 40$, $s \geq 0$.

Consider the vertical lines induced by multiplication by 2 on $(\pi_*^S)_{(2)}$. This brings us to the desired result by a similar argument to the $* = 0$ case since there are no hidden 2−extensions in this range for degree reasons.

| $*$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\mathbb{Z}_{(2)}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/8\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/16\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ |
| $*$ | 11 | 12 | 13 | | | | | | | | |
| | $\mathbb{Z}/8\mathbb{Z}$ | 0 | 0 | | | | | | | | |

**Table 3.2:** $(\pi_*^S)_2^\wedge$, $* \leq 13$.

As a computational heuristic, each sequence of vertical lines denotes a cyclic group of the corresponding order, and if we have zero $h_0-$products between them or multiple points in the same bidegree, we take a direct sum between these cyclic groups provided that there are no hidden $2-$extensions. $\square$

Observe that the proof is mostly computational, instead of Theorem 1.6.2.1 which uses the Serre spectral sequence and the Steenrod operations.

Now, we will revisit Massey products to illustrate their importance. The $E_r-$Massey products ($r \geq 3$) are easier than the $E_2-$case. However, higher Massey products are stronger and require specific conditions to be non-trivial.

**Example 3.9.0.1** ($E_r-$Massey products (Example (i) [Mos70])). Consider the following diagram. Assume that we know that $0 \neq d_2\colon E_2^{1,31} \to E_2^{3,30}$, i.e. $d_2(h_5) = h_0h_4^2$, as depicted.

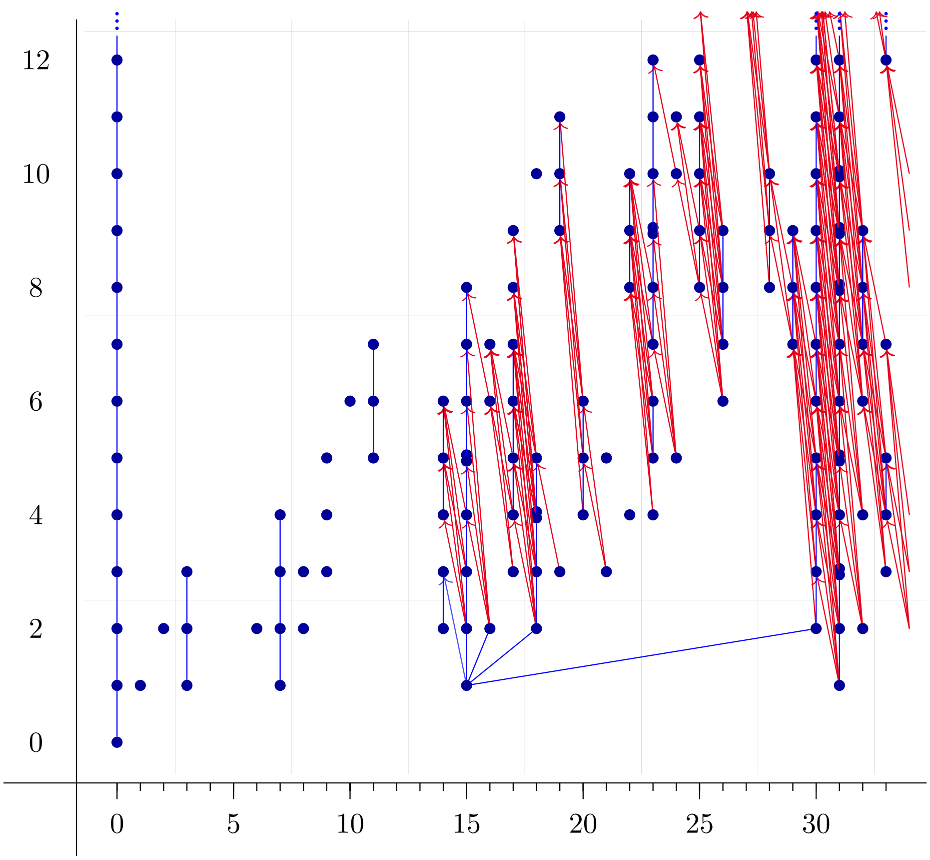


**Figure 3.13:** $\mathrm{Ext}_{\mathcal{A}_2}^{s,t-s}(\mathbb{F}_2,\mathbb{F}_2)$, $t-s<40$, $s \geq 0$.

Let $\eta \in (\pi_1^S)_{(2)}$ be the class inducing $h_1$. Take $\theta \in (\pi_{30}^S)_{(2)}$ to be the element realizing the permanent class $h_4^2$. Then, $\theta^2 = -\theta^2$, so $2\theta = 0$. Moreover, $\eta \cdot 2 = 0$, so we can consider the Toda bracket $\langle \eta, 2, \theta\rangle$. By our previous assumption, $d_2(h_5) = h_0h_4^2$ (in fact, $h_4 = 0 \in E_3^{1,15}$), and note that $h_1h_0 = 0$. Consider the $E_3-$Massey product

$\langle h_1, h_0, h_4^2\rangle$. By definition,

$$\langle h_1, h_0, h_4^2\rangle = \{0 \cdot h_4^2 - (-1)^1 h_1 \cdot x \mid d_2(0) = h_1h_0,\ d_2(x) = h_0h_4^2\} = \{h_1h_5\} \in \frac{E_3^{2,32}}{h_1 \cdot E_3^{1,31} + E_3^{0,2} \cdot h_4^2}.$$

The classes forming the Massey product are all permanent, considering $h_4 = 0$. Moreover, the crossing differential hypothesis is satisfied since $E_2^{0,31} = 0$. By the Moss' Convergence Theorem for $E_r-$Massey products (Theorem 3.7.0.2) we have that $h_1h_5 \in E_\infty^{2,32}$ is a permanent cycle realized by the Toda bracket $\langle \eta, 2, \theta\rangle$. Since there are no non-trivial differentials that could hit $h_1h_5$, this class survives to $E_\infty$. In this way, we have avoided computing higher differentials using Massey products. $\square$

## 3.10 Example: stable homotopy type of representation spheres

### 3.10.1 Introduction

We will extend the computations given by Samuel Muñoz-Echániz in [Muñ24], Proposition 6.7, for the case $\ell = m = 3$. For $m \geq 0$, let $\psi_m$ denote the real $m-$dimensional representation of the dihedral group $D_m \leq \Sigma_m$ (the symmetric group of $m$ elements) given by permuting the factors in $\mathbb{R}^m$, and let $\sigma\colon C_2 := \{\pm 1\} \hookrightarrow \mathbb{R}^\times$ be the sign representation (also regarded as a $D_m-$representation by restriction along the determinant $D_m \hookrightarrow O(2) \overset{\det}{\to} C_2$). For $p \leq d-3$ and $d \geq 5$, define the virtual $D_m-$representations

$$\rho_m := (d+1)(\sigma - 1) + \psi_m \otimes (d - p - 3 + \sigma).$$

In the following, we will consider the homotopy orbit spectrum $\mathbb{S}^{\rho_3}_{hC_3}$ (which can be thought of as a concept resembling what should be the quotient $\mathbb{S}^{\rho_3}/C_3$ where $C_3$ stands for the cyclic group of three elements) of the virtual representation sphere $\mathbb{S}^{\rho_3}$ associated to the virtual representation $\rho_m = \rho_3$. However, we will not require further knowledge of $G-$spectra (see [HHR21]) except for establishing the proper context before applying the Adams spectral sequence. In this scope, $\ell \in \mathbb{N}$ corresponds to the prime where we will localize the stable homotopy groups. Our main goal will be to compute the stable homotopy groups of $\mathbb{S}^{\rho_3}_{hD_3}$.

We can proceed analogously in the other cases. The key elements involved in this argument are $(\pi_*^s)_3^\wedge$ as specified in [Hat18] (for more details, see [Rav86]), and the author's program `minrv1` [Mor24] to compute minimal resolutions and Yoneda products. It is possible to reproduce these computations using [CCBFY22] software, although its interface does not allow filtering of the multiplicative structure, which is key to finding the correct Yoneda products. The diagrams were obtained using an AMD Ryzen 9 9950X 16-core processor and 192 GB of RAM, but the products can be verified in an average computer.

Let $p$ be a prime. Let $X, Y$, $Z$ be spectra, with $Y$ and $Z$ bounded below and $H_*(Y;\mathbb{F}_p)$ and $H_*(Z;\mathbb{F}_p)$ of finite type. By Moss' convergence theorem, the (opposite) Yoneda pairing

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_p}(H^*(Z;\mathbb{F}_p), H^*(Y;\mathbb{F}_p)) \otimes \mathrm{Ext}^{*,*}_{\mathcal{A}_p}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)) \to \mathrm{Ext}^{*,*}_{\mathcal{A}_p}(H^*(Z;\mathbb{F}_p), H^*(X;\mathbb{F}_p))$$

converges to the composition product in $\pi_*^X(Z)$. The pairings involved satisfy the Leibniz rule with respect to the differentials of the three spectral sequences. In particular, by taking $X = Y = \mathbb{S}$, i.e.

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_p}(H^*(Z;\mathbb{F}_p), H^*(\mathbb{S};\mathbb{F}_p)) \otimes \mathrm{Ext}^{*,*}_{\mathcal{A}_p}(H^*(\mathbb{S};\mathbb{F}_p), H^*(\mathbb{S};\mathbb{F}_p)) \to \mathrm{Ext}^{*,*}_{\mathcal{A}_p}(H^*(Z;\mathbb{F}_p), H^*(\mathbb{S};\mathbb{F}_p))$$

we can exploit the knowledge of $(\pi_*^s)_{(p)}$ and $\mathrm{Ext}^{*,*}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$ to obtain non-trivial differentials.

Therefore, in order to determine which differentials are non-trivial, we will need the following description of $\mathrm{Ext}^{*,*}_{\mathcal{A}_3}(\mathbb{F}_3, \mathbb{F}_3)$. It was obtained by constructing a minimal (free) $\mathcal{A}_3-$resolution of $\mathbb{F}_3$, computing the corresponding Yoneda products, and the knowledge of $(\pi_*^s)_3^\wedge$ [Hat18].

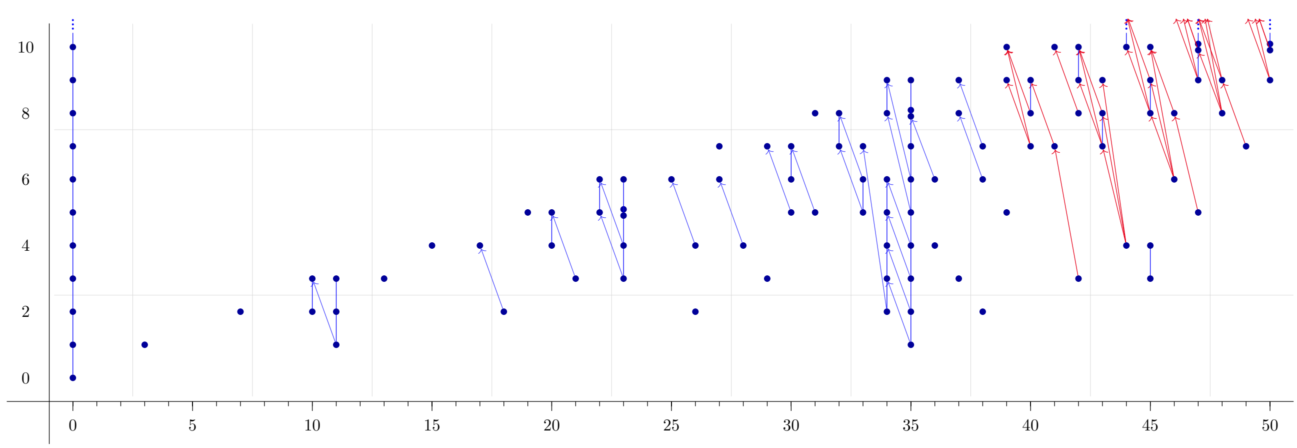


**Figure 3.14:** First non-trivial $d_2$ differentials in $\mathrm{Ext}^{s,t-s}_{\mathcal{A}_3}(\mathbb{F}_3, \mathbb{F}_3)$.

The following chart will be used later for illustrative purposes regarding possible candidates for $E_3-$Massey products.

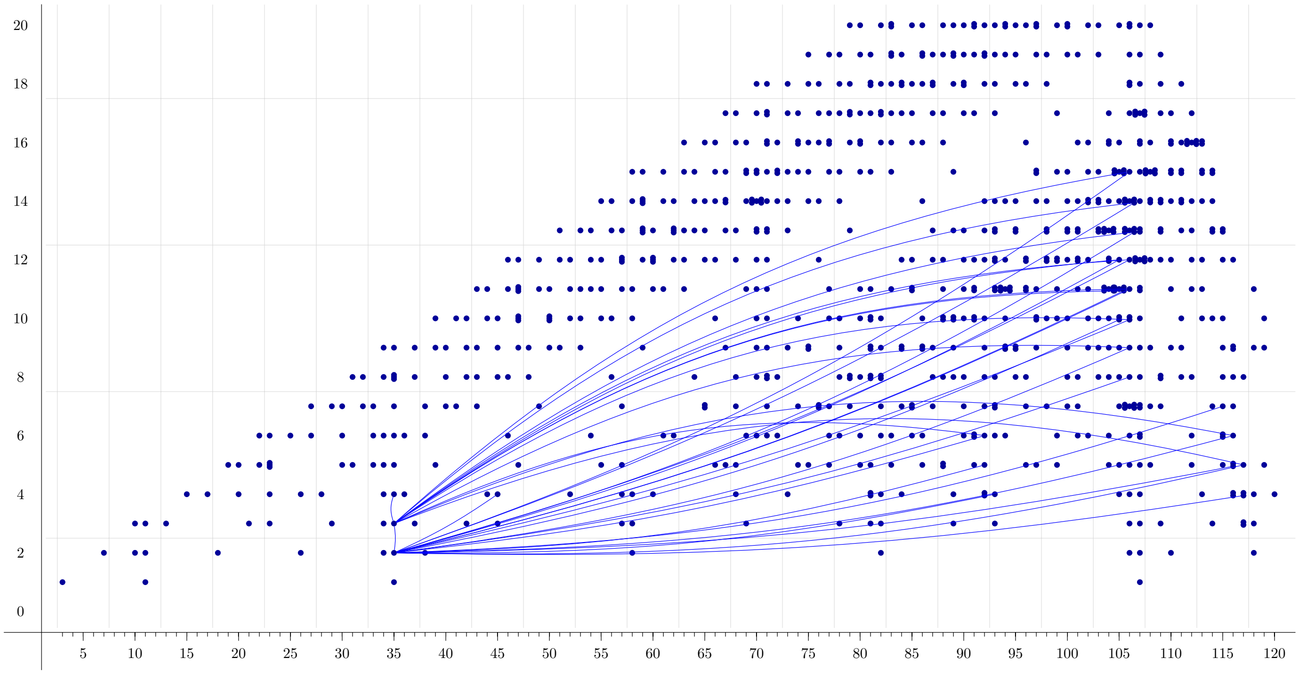


**Figure 3.15:** $E_3-$Massey product candidates in $\mathrm{Ext}^{s,t-s}_{\mathcal{A}_3}(\mathbb{F}_3, \mathbb{F}_3)$.

### 3.10.2 Computation

Before we can apply the classical Adams spectral sequence, remember Muñoz-Echániz's lemma [Muñ24].

**Lemma 3.10.2.1** (Cohomology of $\mathbb{S}^{\rho_3}_{hC_3}$)**.** The spectrum cohomology of $\mathbb{S}^{\rho_3}_{hC_3}$ is given by

$$H^*(\mathbb{S}^{\rho_3}_{hC_3}; \mathbb{F}_3) \cong \mathbb{F}_3\langle u\rangle \otimes_{\mathbb{F}_3} \mathbb{F}_3[\alpha, s]/(\alpha^2), \qquad |\alpha| = 1, \quad |s| = 2, \quad |u| = 3(d-p-2)$$

with

$$P^k(u\alpha^i s^j) = \left(\sum_{r=0}^{k} \binom{d-p-2}{r}\binom{j}{k-r}\right) u\alpha^i s^{j+2k}, \qquad \beta(u\alpha^i s^j) = \begin{cases} 0, & i = 0, \\ -us^{j+1}, & i = 1. \end{cases}$$

Moreover, $C_2 = D_3/C_3$ acts on $H^*(\mathbb{S}^{\rho_3}_{hC_3}; \mathbb{F}_3)$ by $u\alpha^i s^j \mapsto (-1)^{p+i+j} u\alpha^i s^j$.

In order to solve hidden extensions and compute some differentials, we will require the Kahn-Priddy theorem.

**Theorem 3.10.2.1** ([Ada73], Formulation 2.4)**.** Let $p$ be a prime. Let $L$ be 1-connected, $p$-primary, and $p$-equivalent to the classifying space of the $p$-symmetric group $B\Sigma_p$. Write $L^n$ for the skeleton of $L$ consisting of cells of dimension $\leq 2(p-1)n$. For $n \in \mathbb{N}_{>0} \cup \{\infty\}$, there is a morphism $\varphi_n \colon \Sigma^\infty L^n \to \Sigma^\infty S^0$ in the category of CW-spectra such that the induced morphism

$$(\varphi_n)_* \colon [\Sigma^\infty W, \Sigma^\infty L^n]_{\mathbf{CWSpec}} \to [\Sigma^\infty W, \Sigma^\infty S^0]_{\mathbf{CWSpec}}$$

is a split epimorphism whenever $W$ is a connected $p$-primary CW-complex of dimension $< 2(p-1)n$.

Now, we can state and prove the following result. This is Muñoz-Echániz's proof [Muñ24], except for the Adams spectral sequence part. There are modifications in redaction to fit the author's notes. Muñoz-Echániz computed $\pi^s_*(\mathbb{S}^{\rho_3}_{hD_3}) \otimes \mathbb{Z}_{(3)}$ for $* \leq 24$, with three uncertainties related to $* \in \{17, 18, 20, 21\}$ when $p$ is even. The author extended these computations and solved the uncertainties in degrees 20 and 21 for the even case.

| $*$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/9\mathbb{Z}$ | 0 |
| $*$ | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}/9\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$ |
| $*$ | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 |
| | 0 | 0 | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | $\mathbb{Z}/\mathbf{9}\mathbb{Z}$ or $\mathbb{Z}/9\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$ | **?** |
| $*$ | 36 | 37 | 38 | 39 | | | | | | | |
| | **?** | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$ | | | | | | | |

**Table 3.3:** $\pi_*^s\left(\mathbb{S}^{\rho_3}_{hD_3}\right)^\wedge_3$, for $*\leq 39$ ($p$ odd).

| $*$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\mathbb{Z}_{(3)}$ | 0 | 0 | $\mathbb{Z}/9\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}/9\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}/3\mathbb{Z}$ |
| $*$ | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| | $\mathbb{Z}/27\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}/\mathbf{3}\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/9\mathbb{Z}$ | $\mathbb{Z}/\mathbf{81}\mathbb{Z}$ or $\mathbb{Z}/243\mathbb{Z}$ | 0 | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/9\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 |
| $*$ | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 |
| | 0 | $\mathbb{Z}/27\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/27\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/9\mathbb{Z}$ | 0 |
| $*$ | 36 | 37 | 38 | | | | | | | | |
| | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/81\mathbb{Z}$ | | | | | | | | |

**Table 3.4:** $\pi_*^s\left(\mathbb{S}^{\rho_3}_{hD_3}\right)^\wedge_3$, for $*\leq 38$ ($p$ even).

**Proposition 3.10.2.1** (3-primary stable homotopy groups of $\mathbb{S}^{\rho_3}_{hD_3}$)**.** The first few homotopy groups $\pi_*^s(\mathbb{S}^{\rho_3}_{hD_3})\otimes\mathbb{Z}_{(3)}$ when $d-p=3$ are given in Table 3.3 for odd $p$ and Table 3.4 for even $p$. Equally colored groups in this table correspond to the same case depending on the vanishing of certain differentials.

*Proof.* Consider the $\mathcal{A}_3-$submodules of $H^*(\mathbb{S}^{\rho_3}_{hC_3};\mathbb{F}_3)$ given by

$$J_0 := \langle u\alpha^i s^j : i+j\equiv 0 \bmod 2\rangle, \qquad J_1 := \langle u\alpha^i s^j : i+j\equiv 1 \bmod 2\rangle.$$

Then $H^*(\mathbb{S}^{\rho_3}_{hC_3}) = J_0\oplus J_1$ as $\mathcal{A}_3-$modules, and $J_p$, where $p$ is taken mod 2, is the $C_2-$invariant vector space of the residual $C_2 = D_3/C_3-$action, because $\infty > |C_2| = 2\in\mathbb{F}_3^\times$. Therefore, $H^*(\mathbb{S}^{\rho_3}_{hD_3};\mathbb{F}_3) = J_p$ as an $\mathcal{A}_3-$module, and the classical Adams spectral sequence of $J_p$ converges to the 3-primary part of the stable homotopy of $\mathbb{S}^{\rho_3}_{hD_3} = \left(\mathbb{S}^{\rho_3}_{hC_3}\right)_{hC_2}$. Since $J_p$ is bounded below and is acted on freely by the Bockstein, by the Adams vanishing theorem (for odd primes) [Liu63, Theorem 1], the $E_2^{s,t}-$page of the associated Adams spectral sequence has trivial entries for $t < m + (2\cdot 3-1)s - 1$, $s\geq 1$ where $m=3$ for the even case and $m=4$ for the odd case. Observe that the Adams vanishing line goes just over the periodicity classes. This ensures that there are no remaining classes to compute for $s\gg 1$ (remember that we are taking *truncated* minimal resolutions).

Before diving into computer generated data and exploiting the multiplicative structure of the Adams spectral sequence, we will consider the Kahn-Priddy theorem when $p$ is odd. Denote by $\hat{\theta}$ the $D_3-$representation pulled back from the standard $O(2)-$representation on $\mathbb{R}^2$. Observe that $\hat{\theta}|_{C_3}\equiv\theta$ in the notation of Lemma 3.10.2.1. Write $\underline{\mathrm{S}}(\hat{\theta}\otimes\sigma|_{C_3})$ for the unit sphere bundle of the associated vector bundle $S^1\to ED_3\times_{C_3}(\hat{\theta}\otimes\sigma)\to BC_3$. By the homotopy long exact sequence, it follows that $\underline{\mathrm{S}}(\hat{\theta}\otimes\sigma|_{C_3})\cong K(\pi,1)$ for some group $\pi$, and in fact, it must be homotopy equivalent to $S^1$ because $q\colon \underline{\mathrm{S}}(\hat{\theta}\otimes\sigma|_{C_3})\to BC_3$ does not admit a section, hence, the obtained short exact sequence cannot be split (its Euler class is $s\in H^2(BC_3;\mathbb{F}_3) = \mathbb{F}[\alpha,s]/(\alpha^2)$). Moreover, the homology class represented by $q$ is the dual of $\alpha$, and hence the residual $C_2 = D_3/C_3-$action on $\underline{\mathrm{S}}(\hat{\theta}\otimes\sigma|_{C_3})\cong S^1$ must have degree $-1$. So, there is an equivalence of unbased spaces $\underline{\mathrm{S}}(\hat{\theta}\otimes\sigma|_{C_3})\cong S^\sigma$ which is $C_2-$equivariant up to homotopy. We thus get a cofibration

$$S^\sigma_+\cong\underline{\mathrm{S}}(\hat{\theta}\otimes\sigma|_{C_3})_+\xrightarrow{q}(BC_3)_+\twoheadrightarrow\mathrm{Th}(\hat{\theta}\otimes\sigma|_{C_3})\cong S^{-\sigma}\wedge\mathrm{Th}(\psi_3\otimes\sigma|_{C_3})$$

which is $C_2-$equivariant up to homotopy. By equipping both $S^\sigma$ and $BC_3$ with distinguished basepoints that are fixed under the respective involutions and which match under $q$, we can get rid of the added basepoints and yield a homotopy cofibre sequence of $C_2-$spectra

$$\mathbb{S}^{(d+1)(\sigma-1)+2\sigma} \xrightarrow{q} S^{(d+1)(\sigma-1)+\sigma} \wedge \Sigma^\infty BC_3 \twoheadrightarrow \mathbb{S}^{\rho_3}_{hC_3}.$$

Then, upon inverting 2 and taking homotopy $C_2-$orbits in the sequence above, we obtain equivalences of spectra

$$\mathbb{S}^{\rho_3}_{hD_3} \cong_{[\frac{1}{2}]} \begin{cases} (\Sigma^{\infty+1}BC_3)_{hC_2} \cong \Sigma^{\infty+1}BD_3, & d \text{ even (so } p \text{ is odd)},\\ \operatorname{hocofib}(q_{hC_2}\colon \mathbb{S}^1 \to (S^\sigma \wedge \Sigma^\infty BC_3)_{hC_2}), & d \text{ odd (so } p \text{ is even)}. \end{cases} \tag{3.1}$$

Now, by the Kahn-Priddy theorem (Theorem 3.10.2.1) at the prime 3, the transfer-like map $\Sigma^{\infty+1}BD_3 \to \tau_{>1}\mathbb{S}^1$ is split surjective on homotopy groups localised at 3, and hence by (3.1), $\pi^s(\mathbb{S}^{\rho_3}_{hD_3})$ split surjects onto $\pi^s_{*-1} \otimes \mathbb{Z}_{(3)}$ for $* > 1$ when $p$ is odd.

Let $p \in \mathbb{N}_{\geq 0}$ be even. It is not clear in this case if we can apply the Kahn-Priddy theorem or any of its versions. Except for $d_3\colon E_3^{18,0} \to E_3^{17,3}$, this will not be a problem in a considerable range, by considering the next figures. We will denote the classes in $\operatorname{Ext}^{*,*}_{\mathcal{A}_3}(\mathbb{F}_3, \mathbb{F}_3)$ by $y^{(k)}_{s,t-s}$ where $(t-s, s) \in (\mathbb{N}_{\geq 0})^2$ are its coordinates and $k \in \mathbb{N}_{>0}$ is an index related to multiple occurrences of non-trivial cohomology classes in the same bidegree. We will use similar notation for the next pages. Analogously, consider the elements $x^{(k)}_{s,t-s} \in \operatorname{Ext}^{*,*}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3}, \mathbb{F}_3)$, for $s, t-s \geq 0$ and $k > 0$. By constructing a minimal free $\mathcal{A}_3-$resolution of $H^*(\mathbb{S}^{\rho_3}_{hD_3}; \mathbb{F}_3)$ and computing the Yoneda products, we obtain

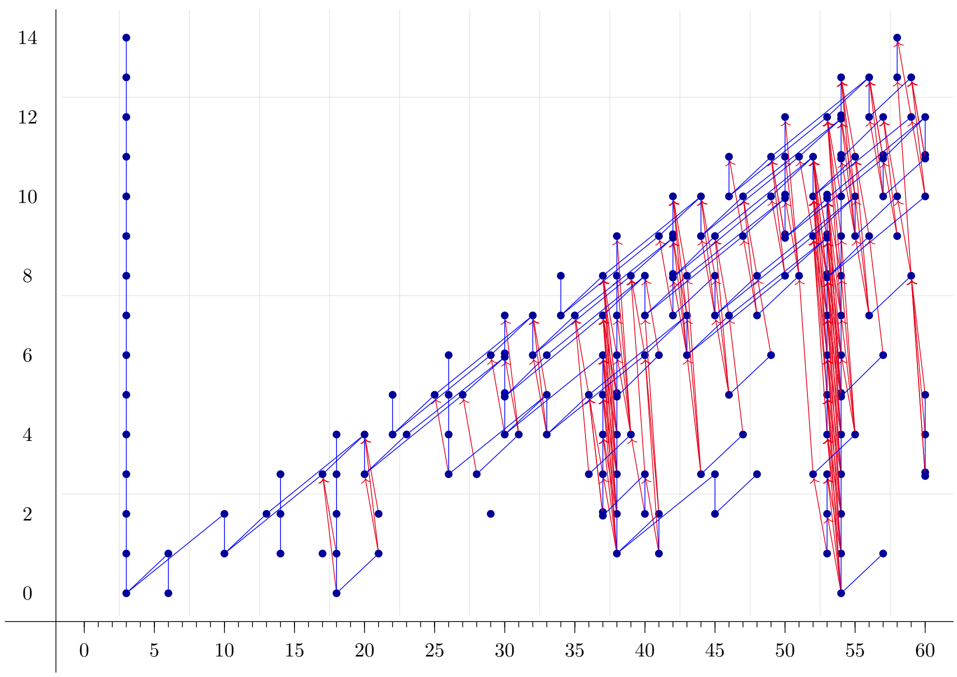


**Figure 3.16:** Multiplicative structure of $E_2^{*,*}$.

The red arrows represent unknown differentials. We are just displaying Yoneda products by lower degree classes. Consider $y^{(1)}_{1,0} \in \operatorname{Ext}^{0,1}_{\mathcal{A}_3}(\mathbb{F}_3, \mathbb{F}_3)$ and $x^{(1)}_{1,18} \in \operatorname{Ext}^{0,18}_{\mathcal{A}_3}(H^*(\mathbb{S}^{\rho_3}_{hD_3}; \mathbb{F}_3), \mathbb{F}_3)$. By the previous diagram, $y^{(1)}_{1,0} \cdot x^{(1)}_{0,18} = kx^{(1)}_{1,18}$ for some $k \in \mathbb{F}_3^\times$. Since we are working with $\mathbb{F}_3-$vector spaces, we will not take care of constants of this sort in the rest of the computation, in particular, WLOG $k = 1$. Then, since $d_2\colon \operatorname{Ext}^{*,*}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3}, \mathbb{F}_3) \to \operatorname{Ext}^{*-1,*+2}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3}, \mathbb{F}_3)$ is a derivation, it can be expressed in terms of itself and $d_2\colon \operatorname{Ext}^{*,*}_{\mathcal{A}_3}(\mathbb{F}_3, \mathbb{F}_3) \to \operatorname{Ext}^{*-1,*+2}_{\mathcal{A}_3}(\mathbb{F}_3, \mathbb{F}_3)$, i.e.

$$d_2\left(x^{(1)}_{1,18}\right) = d_2\left(y^{(1)}_{1,0} \cdot x^{(1)}_{0,18}\right) = d_2\left(y^{(1)}_{1,0}\right) \cdot x^{(1)}_{0,18} + (-1)^{18} y^{(1)}_{1,0} \cdot d_2\left(x^{(1)}_{0,18}\right) = 0$$

since both differentials are trivial. By the same argument, $d_2\colon \operatorname{Ext}^{1,21}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3}, \mathbb{F}_3) \to \operatorname{Ext}^{3,23}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3}, \mathbb{F}_3)$ is trivial. The differential $d_3\colon \operatorname{Ext}^{0,18}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3}, \mathbb{F}_3) \to \operatorname{Ext}^{3,17}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3}, \mathbb{F}_3)$ turns out to be quite challenging. We will take a look at it at the end of this case. For bidegree reasons, it is not well placed and its corresponding Yoneda products have high degrees. Now, consider $d_2\colon \mathrm{Ext}^{2,21}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3},\mathbb{F}_3) \to \mathrm{Ext}^{1,23}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3},\mathbb{F}_3)$, and the following figure

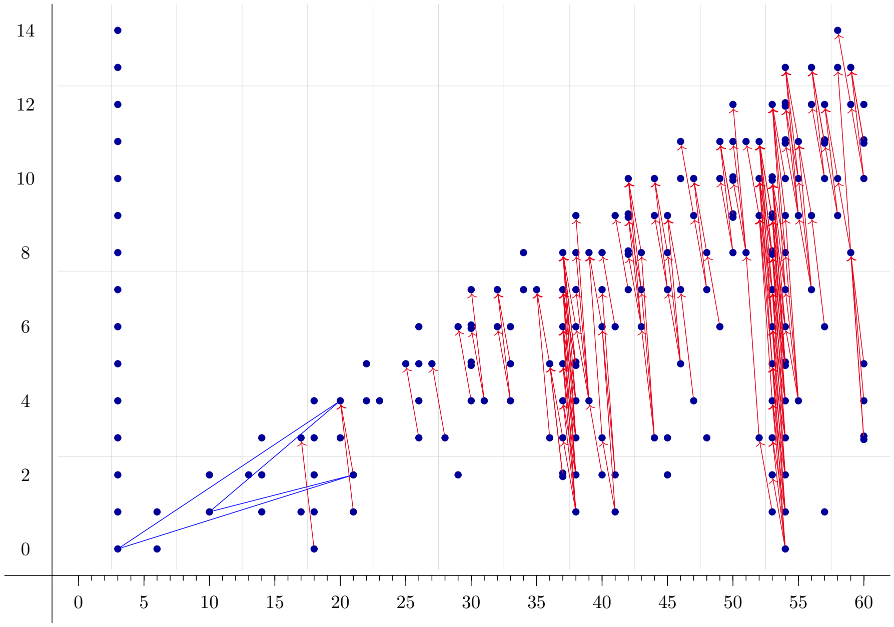


**Figure 3.17:** Multiplicative structure of $E_2^{*,*}$.

Then,

$$d_2\left(x^{(1)}_{2,21}\right) = d_2\left(y^{(1)}_{1,11}\cdot x^{(1)}_{1,10}\right) = d_2\left(y^{(1)}_{1,11}\right)\cdot x^{(1)}_{1,10} + (-1)^{11} y^{(1)}_{1,11}\cdot d_2\left(x^{(1)}_{1,10}\right) \neq 0$$

because $d_2(x^{(1)}_{1,10}) = 0$ and $d_2(y^{(1)}_{1,11}) \neq 0$ (Figure 3.14). Moreover, $d_3\colon \mathrm{Ext}^{1,21}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3},\mathbb{F}_3) \to \mathrm{Ext}^{4,20}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3},\mathbb{F}_3) = 0$. Now, consider the next figure

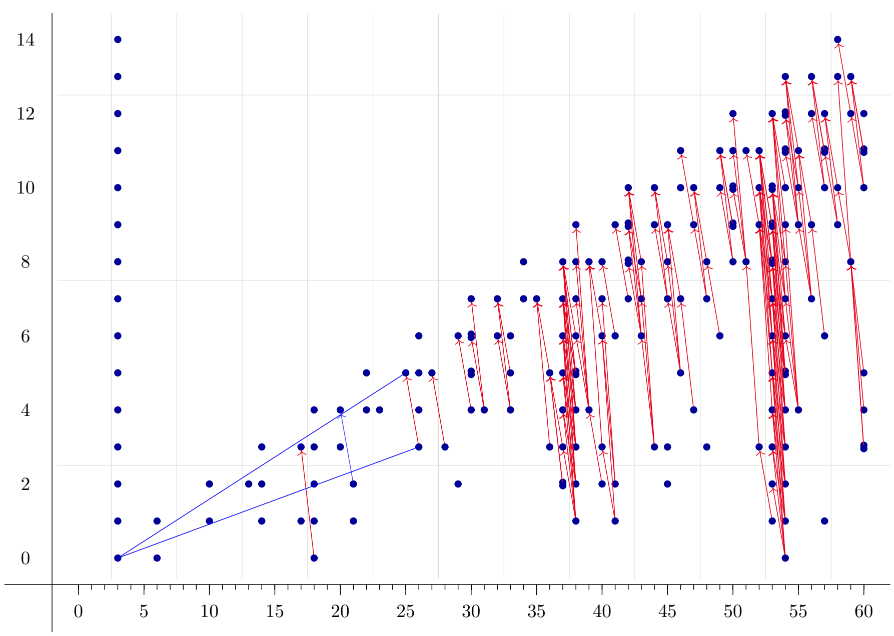


**Figure 3.18:** Multiplicative structure of $E_2^{*,*}$.

Since $x^{(1)}_{3,26} = y^{(1)}_{3,23} \cdot x^{(1)}_{0,3}$, and $d_2(y^{(1)}_{3,23}) \neq 0$, it follows that $d_2(x^{(1)}_{3,26}) \neq 0$. Then, we have

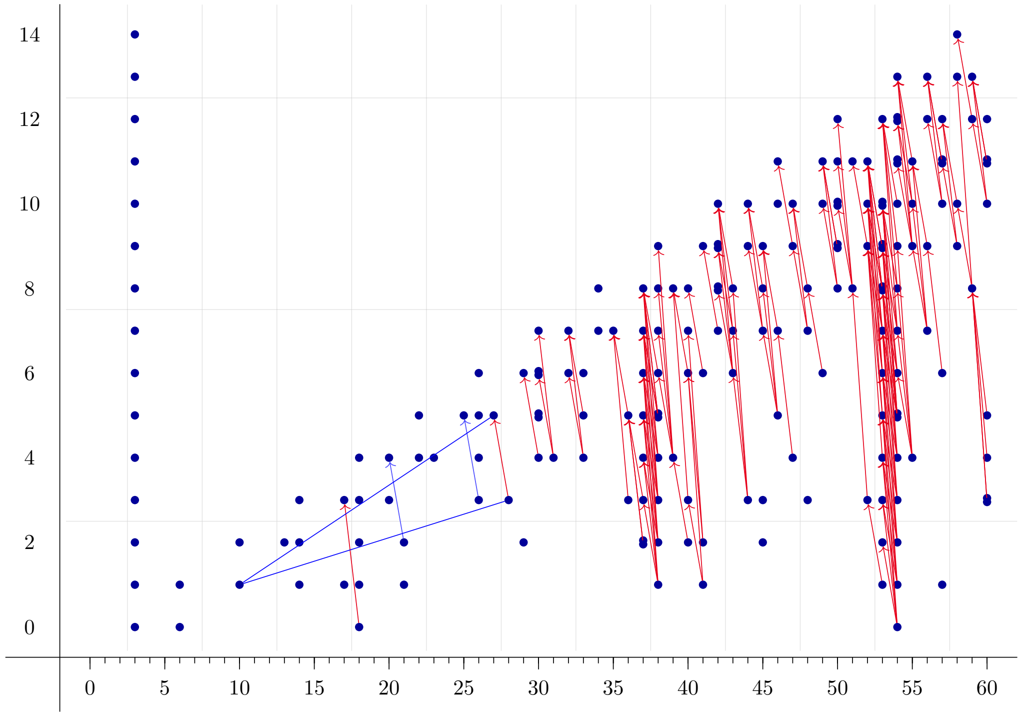


**Figure 3.19:** Multiplicative structure of $E_2^{*,*}$.

By a similar reasoning, $d_2(x^{(1)}_{3,28}) \neq 0$.

It will be preferable to skip some $d_2$ differentials for now and consider $d_2(x^{(1)}_{5,33})$. The relevant Yoneda products are displayed in the next diagram.

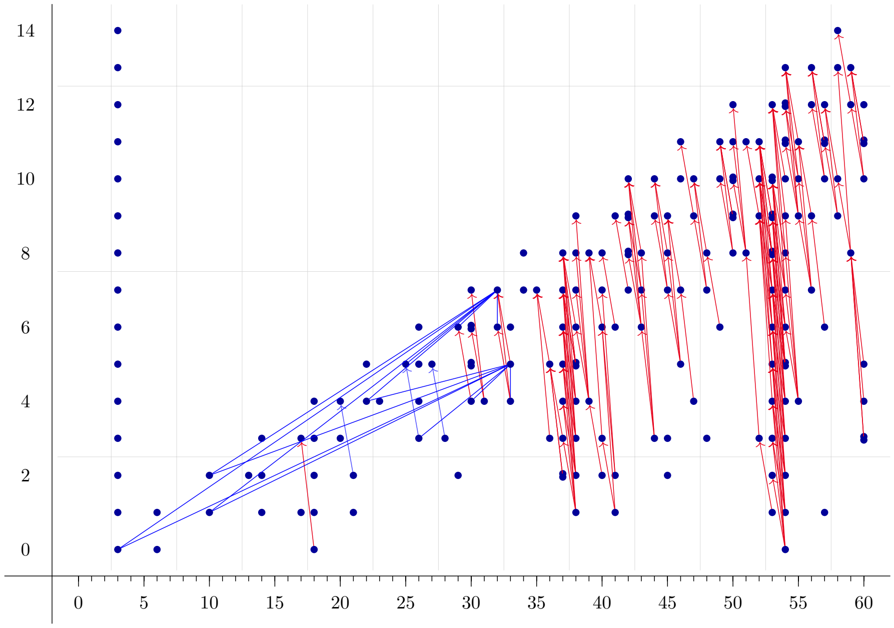


**Figure 3.20:** Multiplicative structure of $E_2^{*,*}$.

We have that $d_2(x^{(1)}_{5,33}) \neq 0$ because $d_2(y^{(1)}_{1,11}) \neq 0$. Assume by contradiction that $d_2(x^{(1)}_{4,33}) = 0$. By the derivation property, this would imply that $d_2(x^{1}_{5,33}) = 0$. Thus, $d_2(x^{(1)}_{4,33}) \neq 0$ and, similarly, $d_2(x^{(1)}_{4,30}) \neq 0$. Therefore, the corresponding higher differentials vanishes.

Similarly, $d_2(x^{(1)}_{4,31}), d_2(x^{(1)}_{5,36}) \neq 0$ as shown.

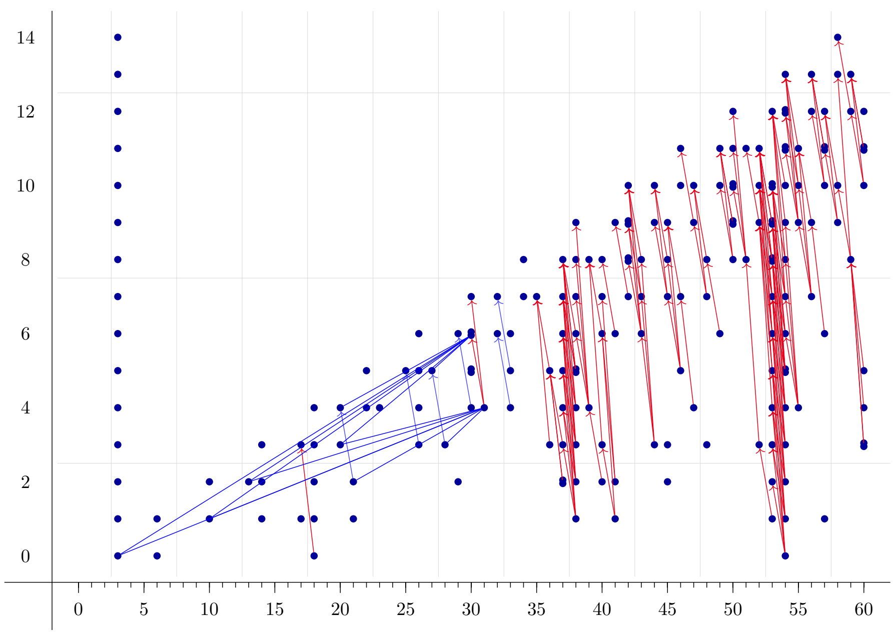


**Figure 3.21:** Multiplicative structure of $E_2^{*,*}$.

Since $d_2(y^{(1)}_{3,23}) \neq 0$, we have

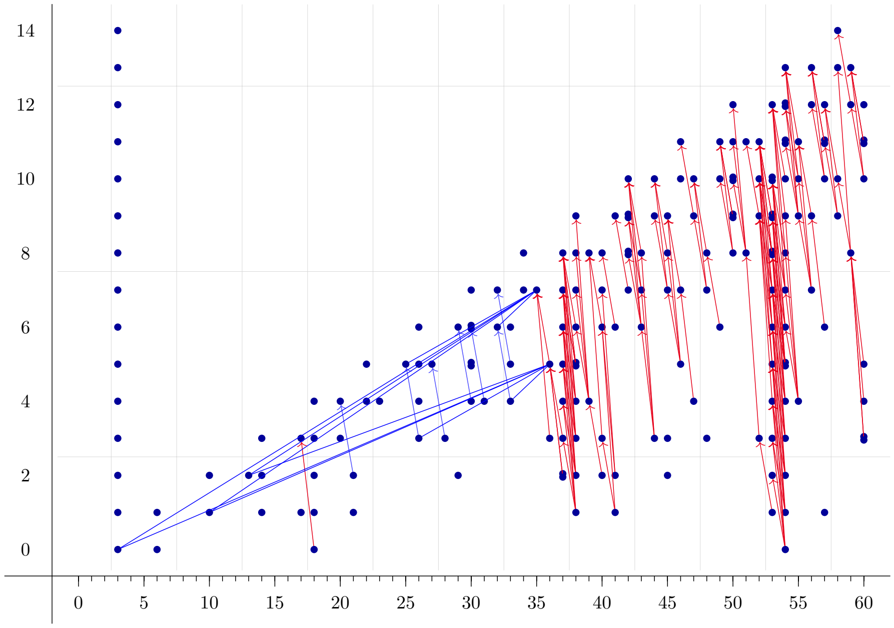


**Figure 3.22:** Multiplicative structure of $E_2^{*,*}$.

Note that this implies that $d_2(x^1_{3,37}) = 0$ since

$$\operatorname{Im}\left(d_2\colon \operatorname{Ext}^{3,37}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3}, \mathbb{F}_3) \to \operatorname{Ext}^{5,36}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3}, \mathbb{F}_3)\right) \subseteq \operatorname{Ker}\left(d_2\colon \operatorname{Ext}^{5,36}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3}, \mathbb{F}_3) \to \operatorname{Ext}^{7,35}_{\mathcal{A}_3}(\mathbb{S}^{\rho_3}_{hD_3}, \mathbb{F}_3)\right) = 0.$$

Finally, we will take advantage of the location of the column of differentials starting at $(1, 38)$. Consider the next sequence of diagrams.

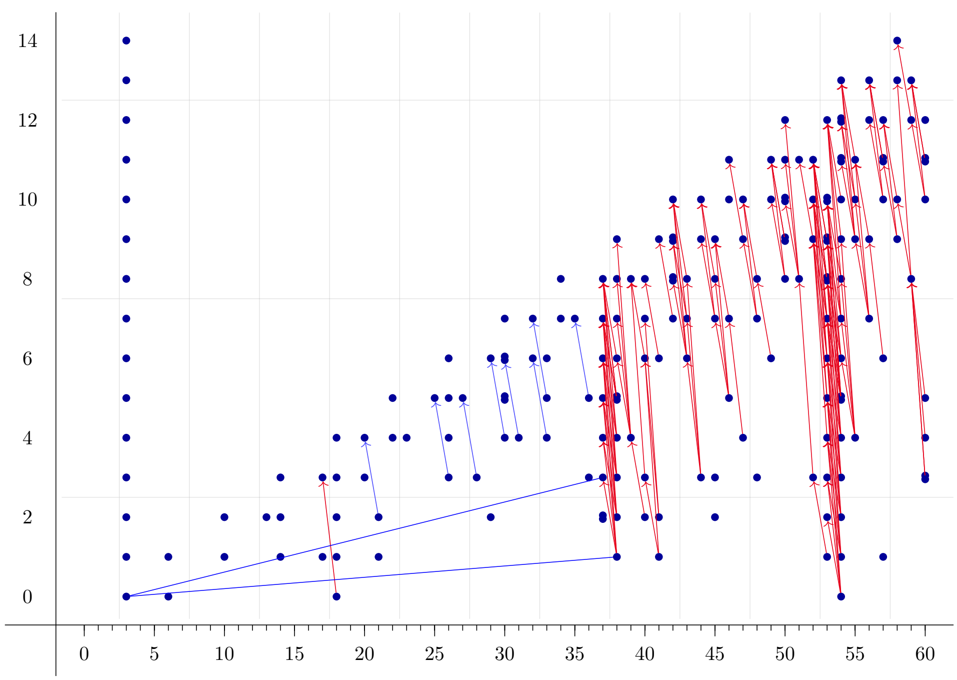


**Figure 3.23:** Multiplicative structure of $E_2^{*,*}$.

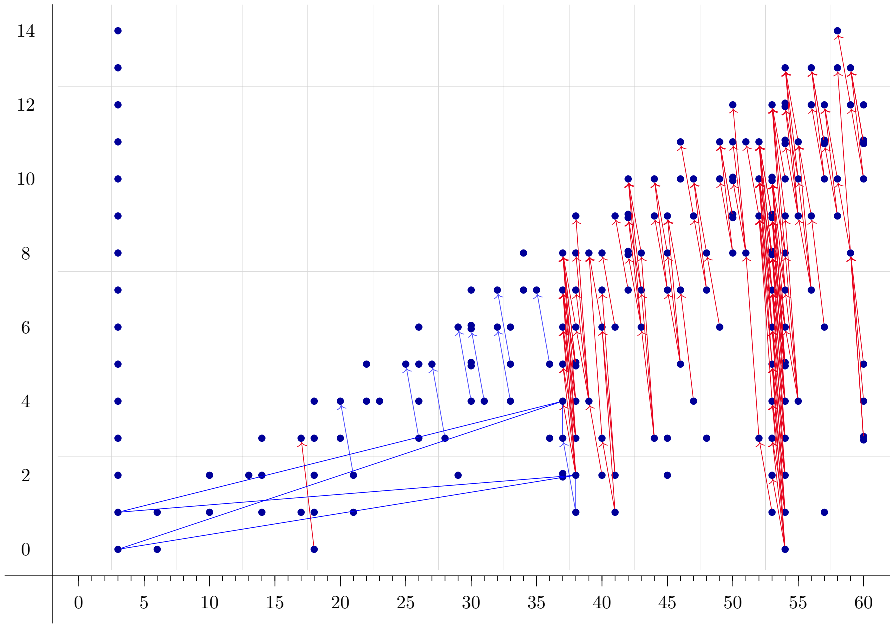


**Figure 3.24:** Multiplicative structure of $E_2^{*,*}$.

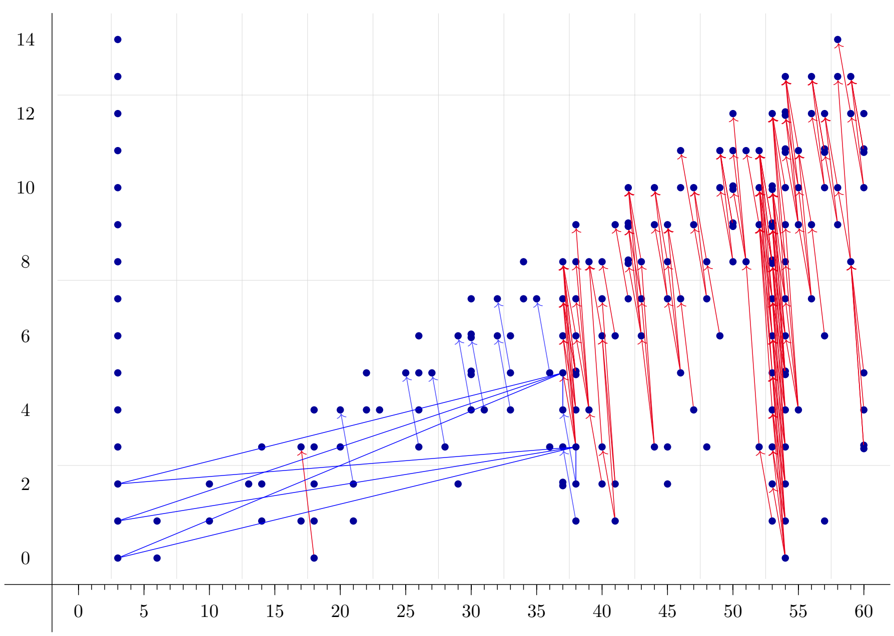


**Figure 3.25:** Multiplicative structure of $E_2^{*,*}$.

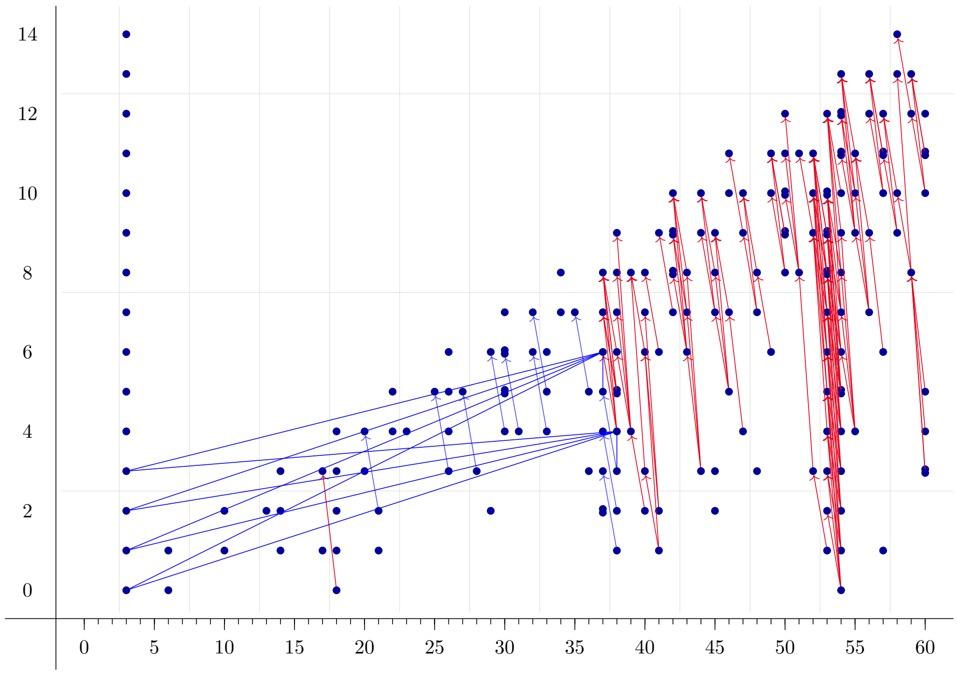


**Figure 3.26:** Multiplicative structure of $E_2^{*,*}$.

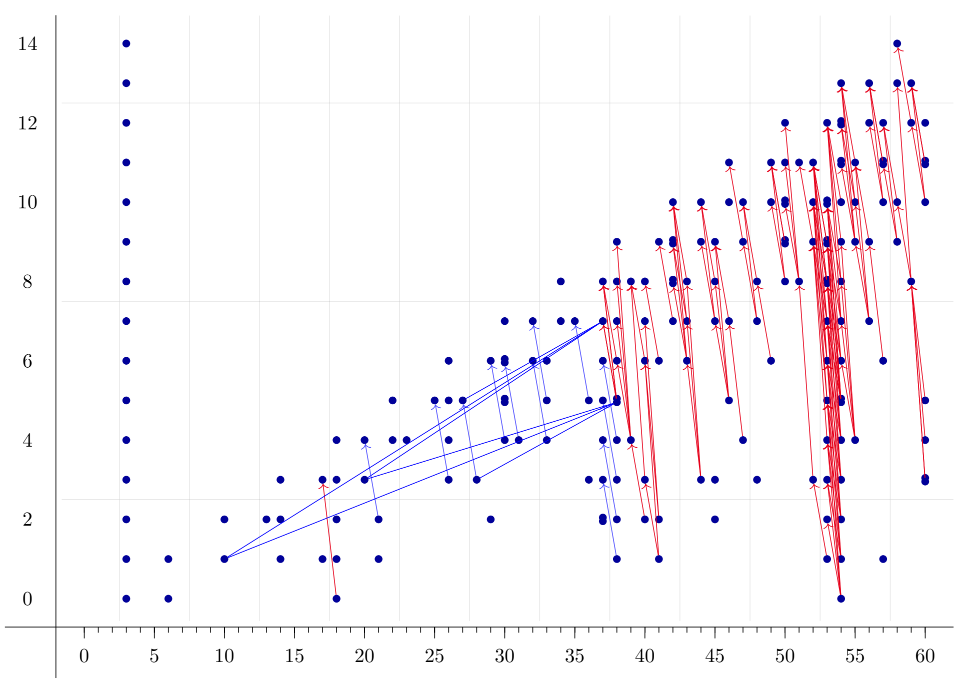


**Figure 3.27:** Multiplicative structure of $E_2^{*,*}$.

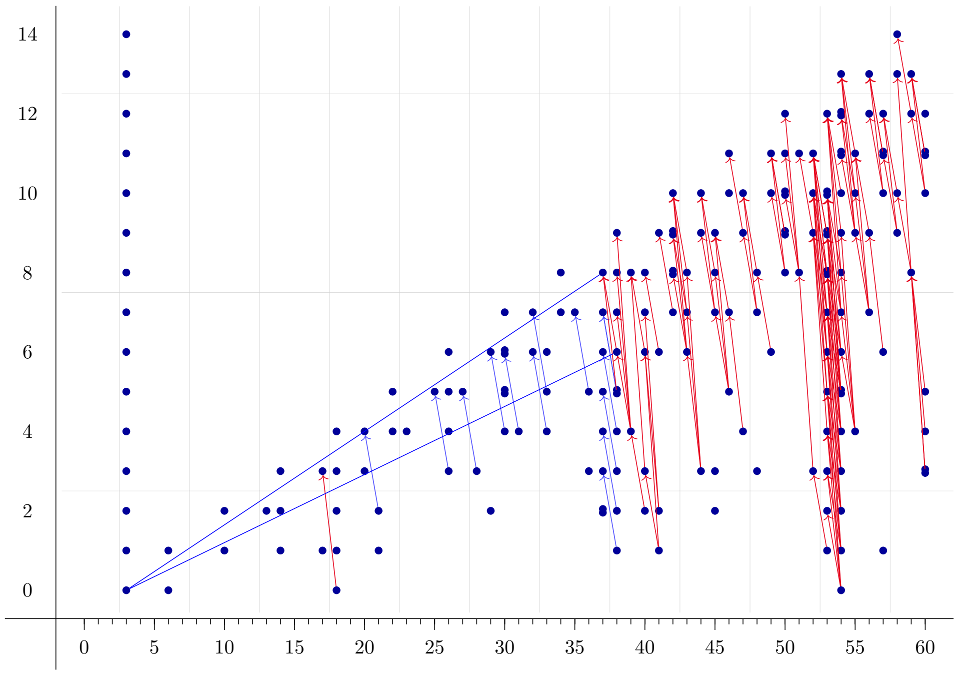


**Figure 3.28:** Multiplicative structure of $E_2^{*,*}$.

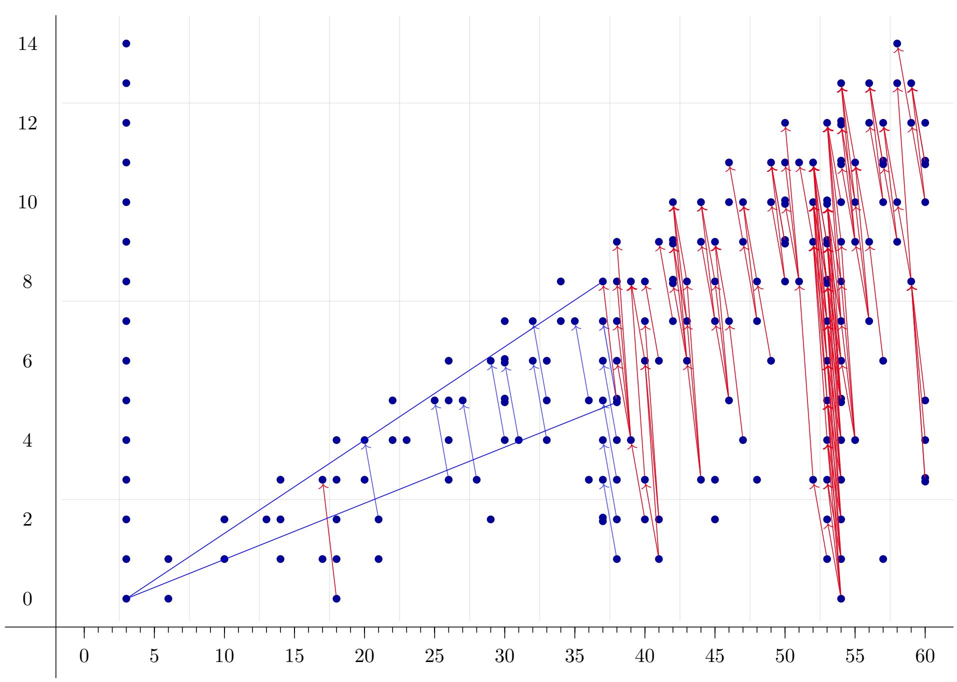


**Figure 3.29:** Multiplicative structure of $E_2^{*,*}$.

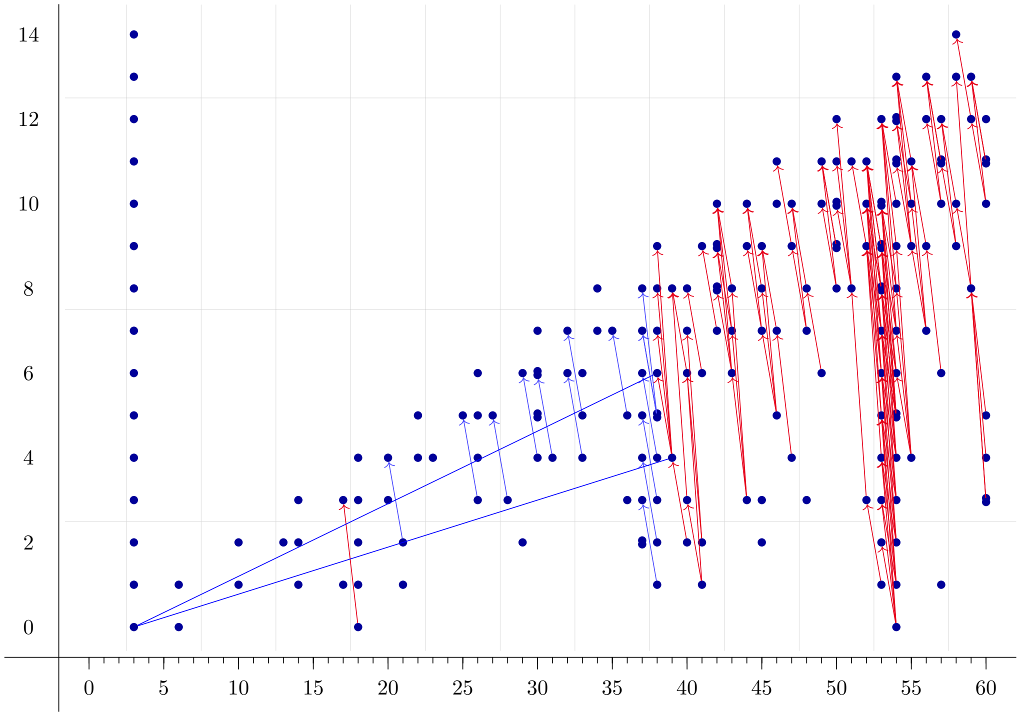


**Figure 3.30:** Multiplicative structure of $E_2^{*,*}$.

We have used that $d_2(y^{(1)}_{1,35}) \neq 0$, $d_2(y^{(1)}_{2,18}) \neq 0$ (and that $y^{(1)}_{2,18} \cdot x^{(1)}_{3,20} = x^{(2)}_{5,38}$, where the last element corresponds to the dot on the right). Hence, $d_2(x^{(2)}_{5,38}) \neq 0$. Notice that $d_2(x^{(1)}_{5,38}) = 0$ (see Figure 3.31, and take into account that the vertical products are trivial for $x^{(2)}_{5,38}$). It is also clear that $d_2(x^{(1)}_{6,38}) = 0$ and $d_3(x^{(1)}_{38,5}) \neq 0$. Moreover, $d_r(x^{(1)}_{4,36}) = 0$ for all $r \geq 2$, by Leibniz. In conclusion

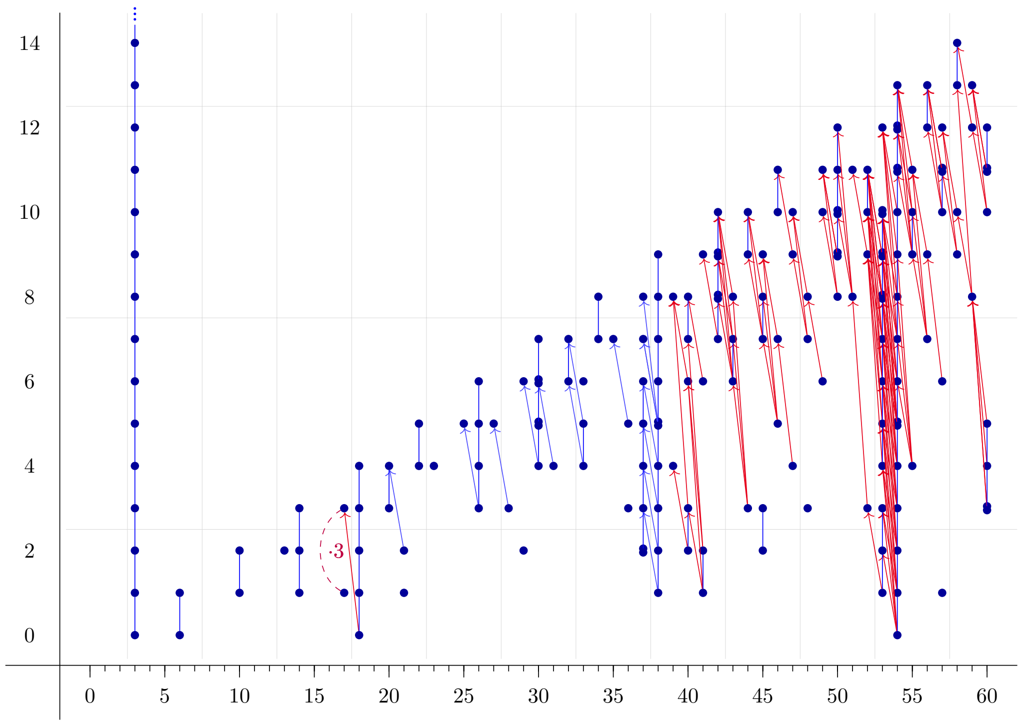


**Figure 3.31:** Multiplicative structure of $E_2^{*,*}$.

Since we have determined $E_\infty^{s,t-s}$, $t-s \leq 38$, $s \geq 0$, by degree reasons, there are no hidden 3−extensions except (possibly) for $t-s=17$. It follows that $\pi_*^s(\mathbb{S}^{\rho_3}_{hD_3})^\wedge_3$, for $* \leq 38$, is given by

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $*$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| | $\mathbb{Z}_{(3)}$ | 0 | 0 | $\mathbb{Z}/9\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}/9\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}/3\mathbb{Z}$ |
| $*$ | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| | $\mathbb{Z}/27\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}/\mathbf{3}\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/9\mathbb{Z}$ | $\mathbb{Z}/\mathbf{81}\mathbb{Z}$ or $\mathbb{Z}/243\mathbb{Z}$ | 0 | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/9\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 |
| $*$ | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 |
| | 0 | $\mathbb{Z}/27\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/27\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/9\mathbb{Z}$ | 0 |
| $*$ | 36 | 37 | 38 | | | | | | | | |
| | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/81\mathbb{Z}$ | | | | | | | | |

**Table 3.5:** $\pi_*^s\left(\mathbb{S}^{\rho_3}_{hD_3}\right)^\wedge_3$, for $* \leq 38$ ($p$ even).

The following three charts give some insight into $d_3$ at $(0, 18)$. The first chart concerns the high degree of the products related to $x^{(1)}_{0,18}$ and $x^{(1)}_{3,17}$. The second focuses on $x^{(1)}_{0,18}$. There are possible candidates to form $E_3-$Massey products in $(2, 53)$ and $(3, 53)$, assuming that the corresponding elements are killed by a $d_2$ differential. Hence, we need that $d_2(x^{(1)}_{0,54}) \neq 0$ (which is equivalent to $d_2(x^{(1)}_{1,54}) \neq 0$). It is unclear how to determine the (non)triviality of these differentials just by using Yoneda products. But it could still be possible to use $E_2-$Massey products. To form an $E_3-$Massey product using known data from the 3$-$sphere, it is also required a similar condition on $\mathrm{Ext}^{*,*}_{\mathcal{A}_3}(\mathbb{F}_3, \mathbb{F}_3)$ (see Figure 3.15). The third chart could be useful if $d_3$ at $(1, 125)$ is non-trivial and $d_2(y^{(1)}_{1,107}) = 0$.

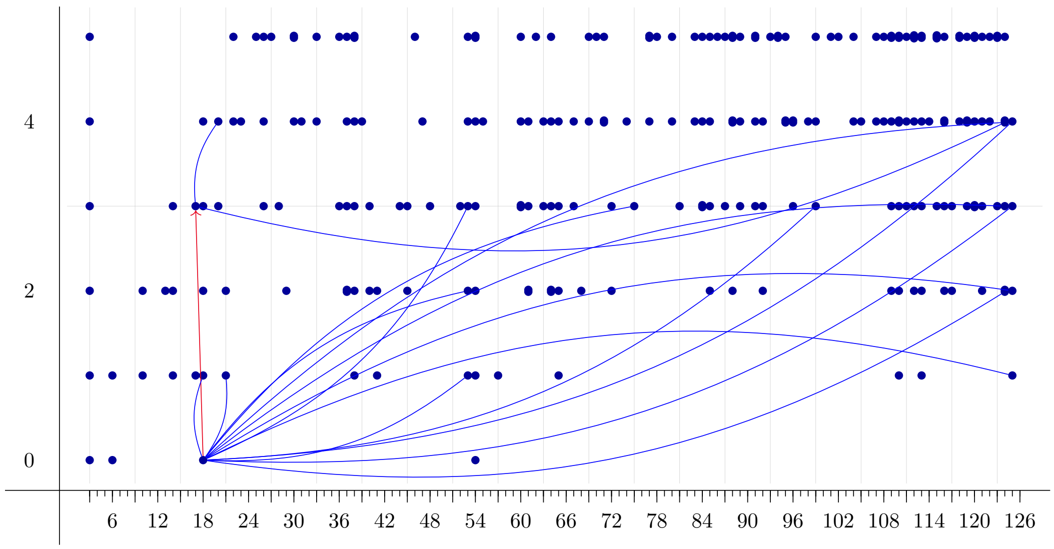


**Figure 3.32:** Products associated with $d_3\colon E_2^{0,18} \to E_2^{3,17}$.

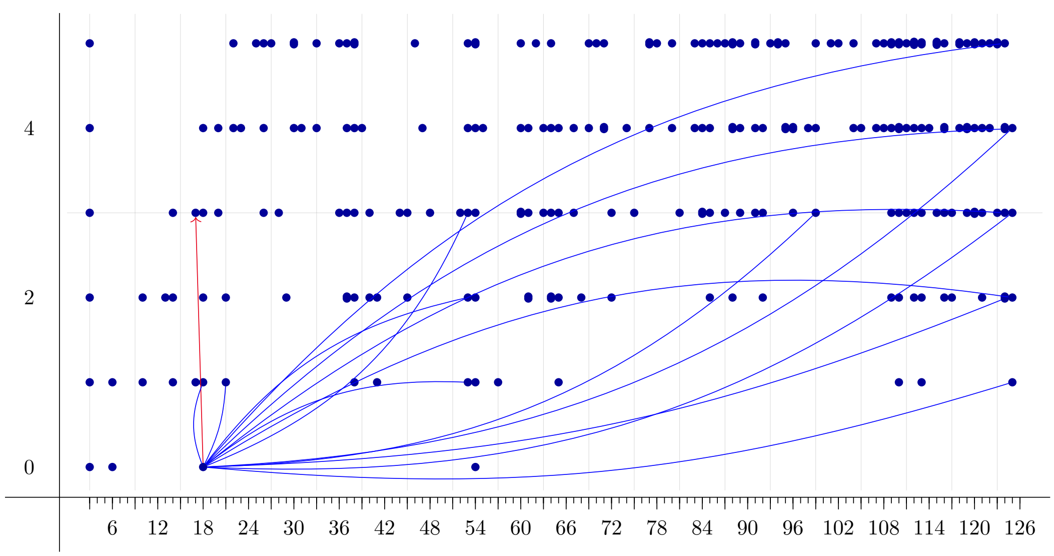


**Figure 3.33:** Products associated with $d_3\colon E_2^{0,18} \to E_2^{3,17}$.

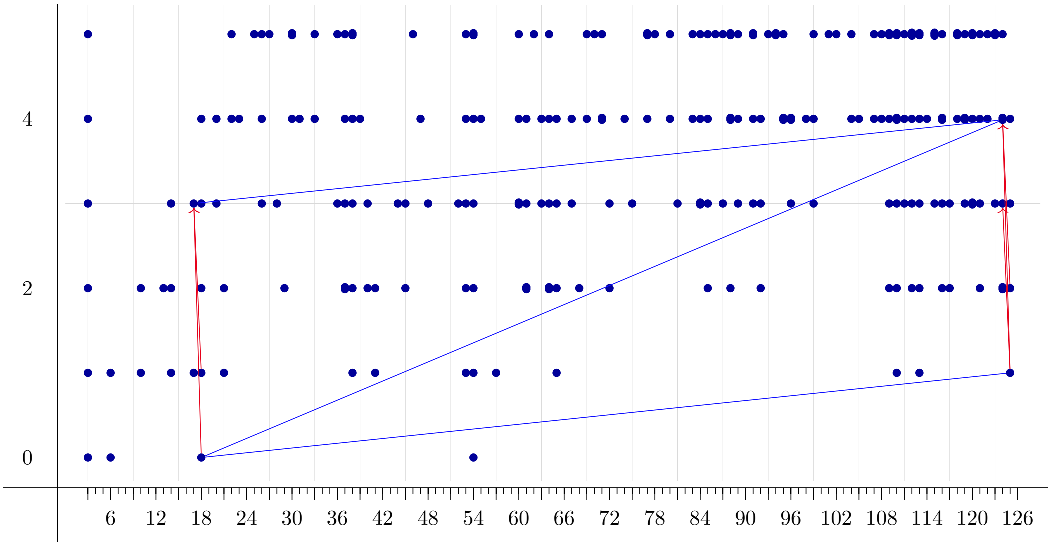


**Figure 3.34:** Products associated with $d_3 \colon E_2^{0,18} \to E_2^{3,17}$.

We will now prove the stated assertion for odd $p$. Remember that now we have the Kahn-Priddy theorem at our disposal. However, it will not be required until later stages of the computation. The next figure corresponds to an overview of the multiplicative structure of the associated Adams spectral sequence.

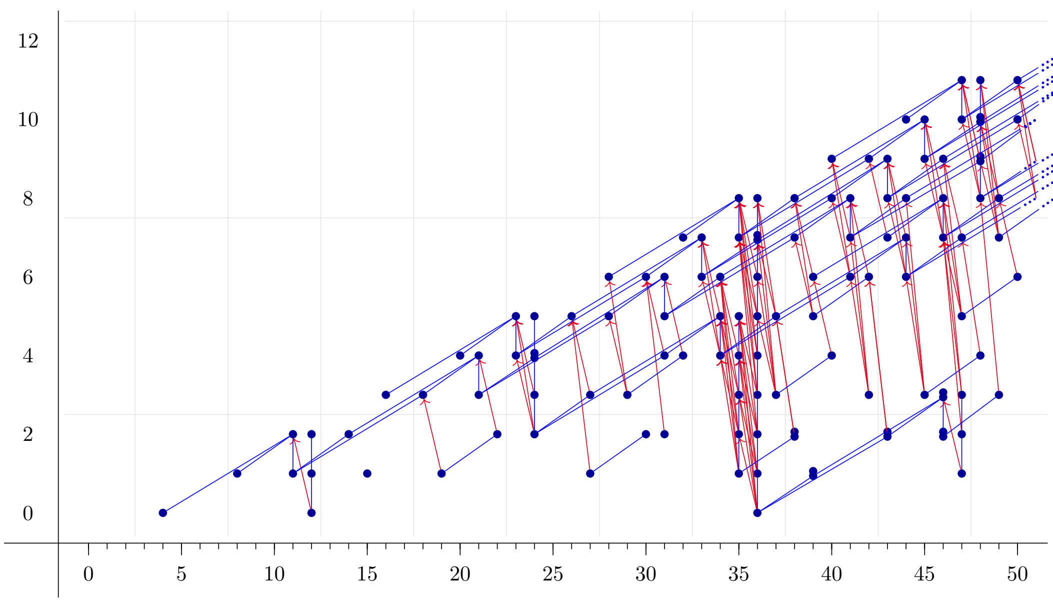


**Figure 3.35:** Multiplicative structure of $E_2^{*,*}$.

By degree reasons, the element $x_{0,12}^{(1)}$ is indecomposable. However, $x_{2,22}^{(1)}$ and $x_{5,21}^{(1)}$ can be written in three convenient ways as shown in the next figure.

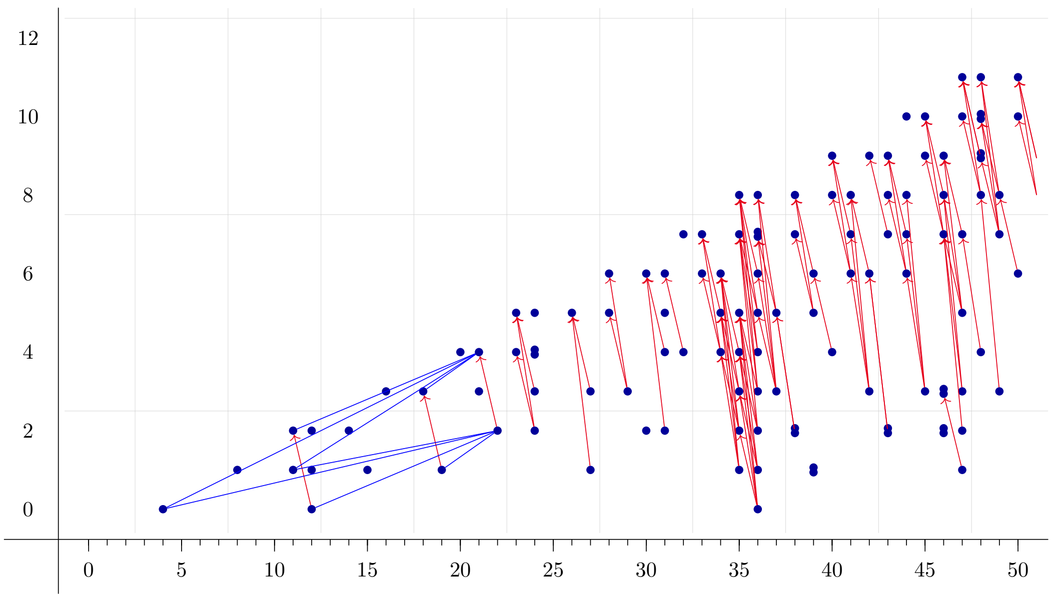


**Figure 3.36:** Multiplicative structure of $E_2^{*,*}$.

Observe that $x^{(1)}_{2,22} = y^{(1)}_{1,11} \cdot x^{(1)}_{1,11}$ and $0 \neq x^{(1)}_{4,21} = y^{(1)}_{3,10} \cdot x^{(1)}_{1,10} = d_2(y^{(1)}_{1,11}) \cdot x^{(1)}_{1,11}$. Hence, $d_2(x^{(1)}_{2,22}) \neq 0$ by Leibniz. Similarly, $d_2(x^{(1)}_{1,19}) \neq 0$. Moreover, these products imply that $d_2(x^{(1)}_{0,12}) \neq 0$.

Now, since $d_2(y^{(1)}_{3,23}) \neq 0$, we have $d_2(x^{(1)}_{3,27}) \neq 0$. Thus, $d_2(x^{(1)}_{2,24}) \neq 0 \neq d_2(x^{(1)}_{3,24})$ and $d_3(x^{(1)}_{2,24}) = 0 = d_4(x^{(1)}_{1,27})$. Repeating the same argument, consider the following diagrams.

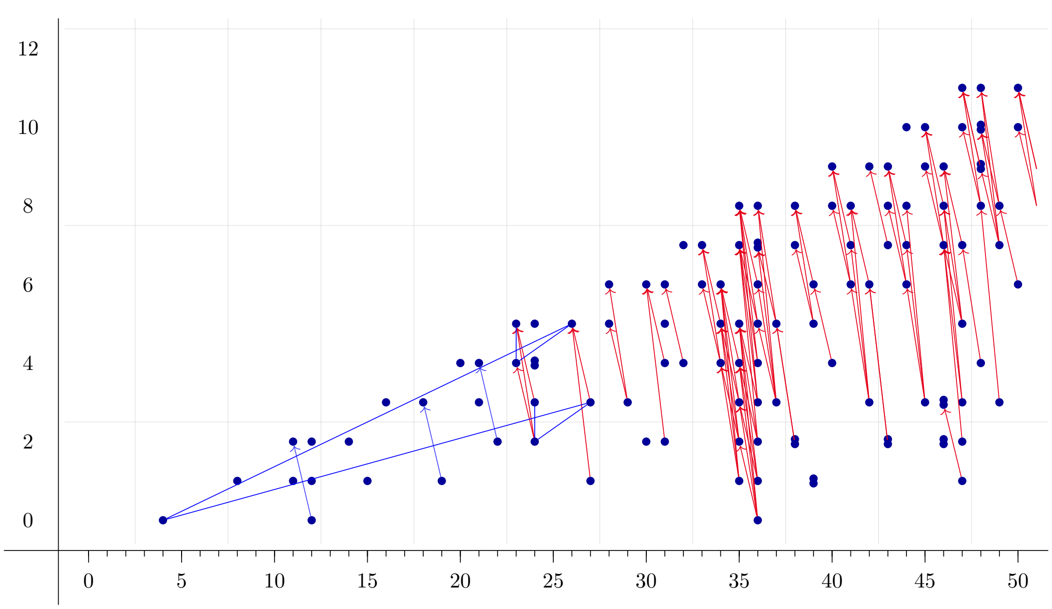


**Figure 3.37:** Multiplicative structure of $E_2^{*,*}$.

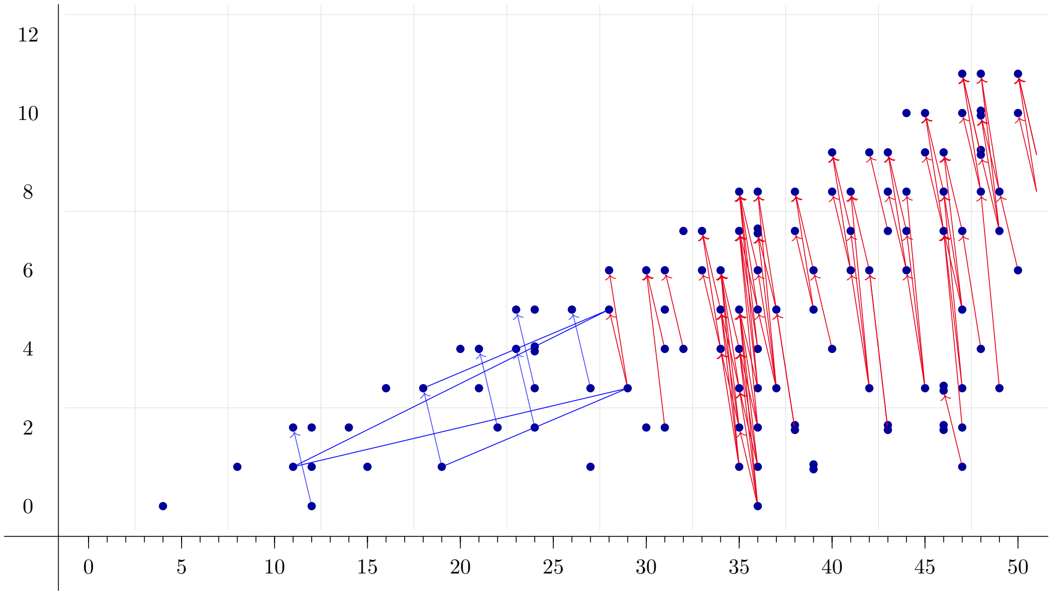


**Figure 3.38:** Multiplicative structure of $E_2^{*,*}$.

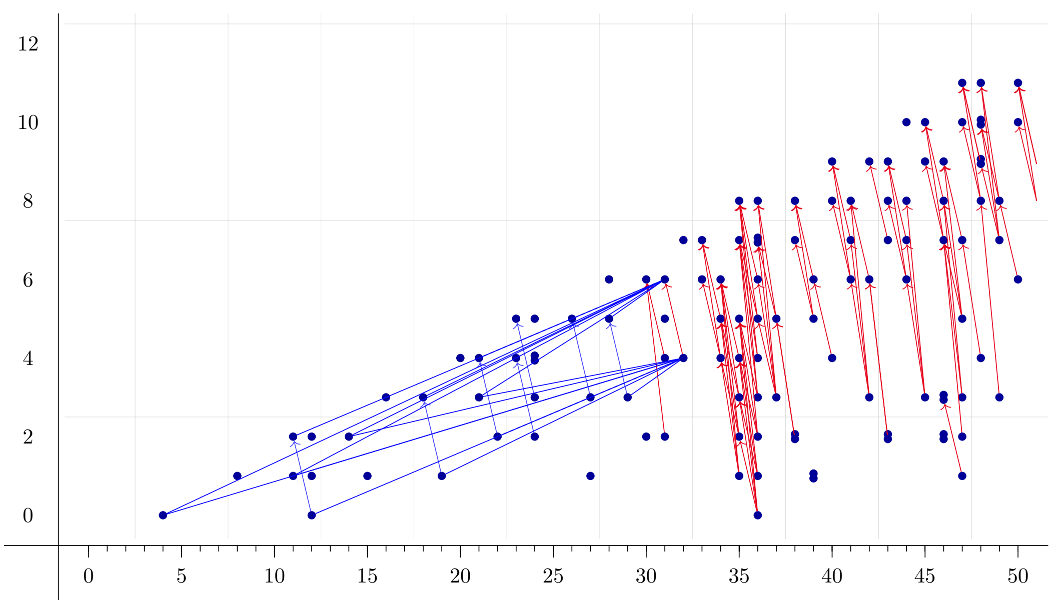


**Figure 3.39:** Multiplicative structure of $E_2^{*,*}$.

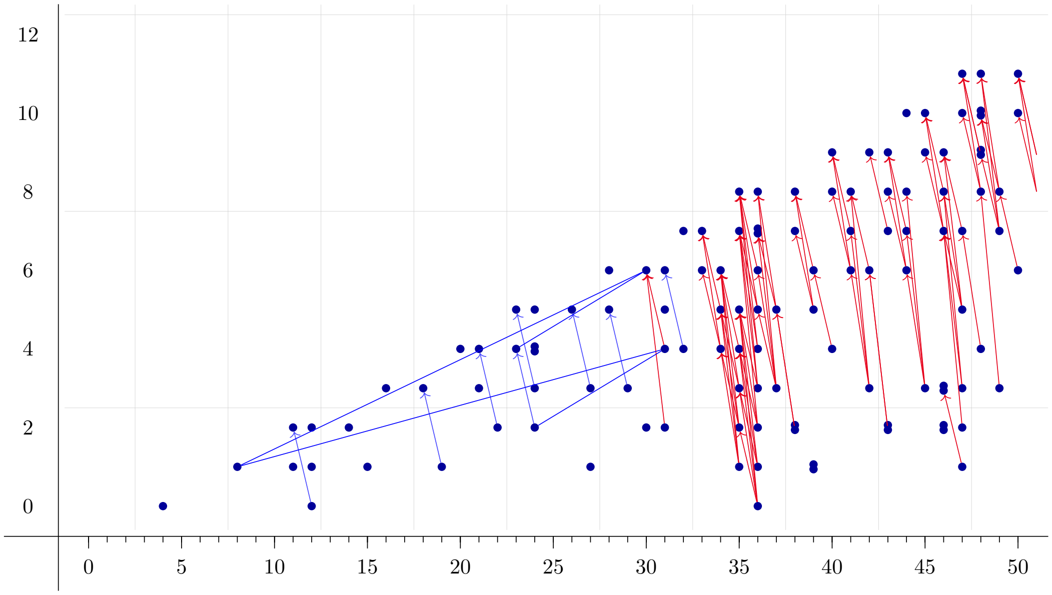


**Figure 3.40:** Multiplicative structure of $E_2^{*,*}$.

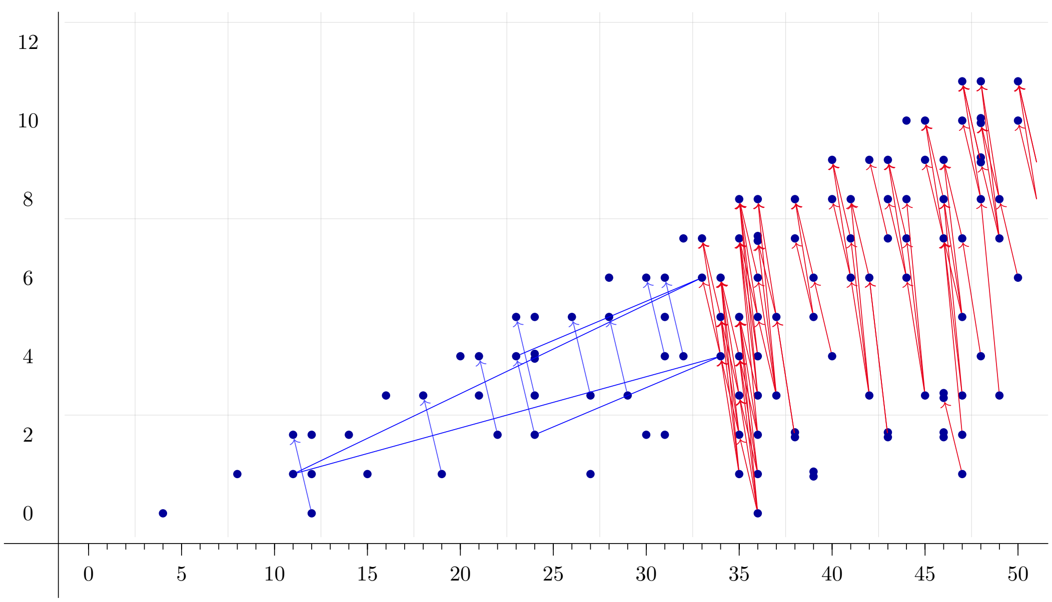


**Figure 3.41:** Multiplicative structure of $E_2^{*,*}$.

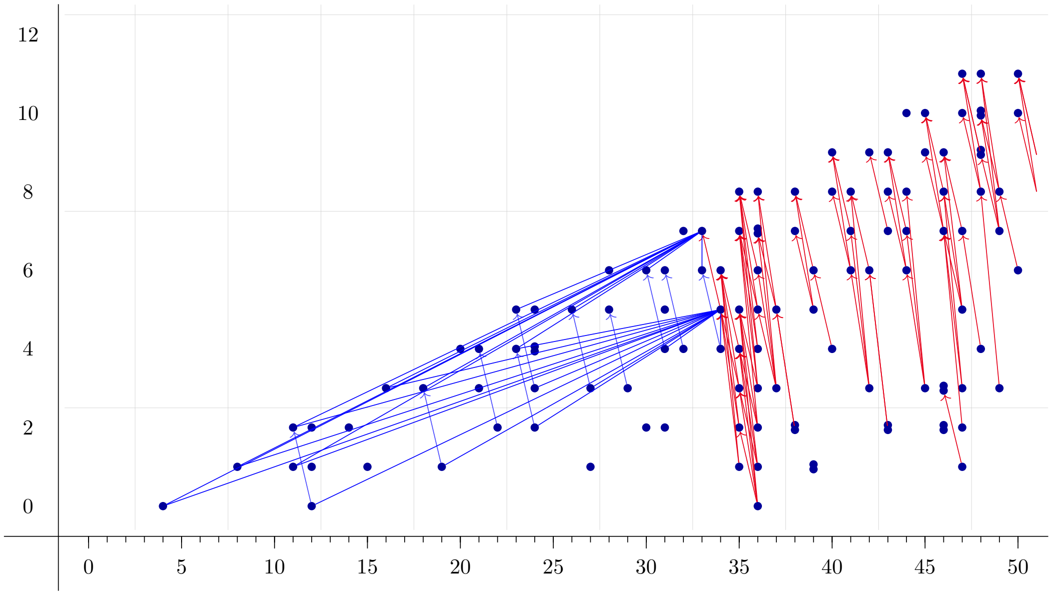

**Figure 3.42:** Multiplicative structure of $E_2^{*,*}$.

Note that $d_2(x_{2,35}^{(1)}) = 0 = d_2(x_{3,35}^{(1)})$ since $\mathrm{Im}(d_2) \subseteq \mathrm{Ker}(d_2)$. This could be obtained by using the multiplicative structure as the next figure suggests. However, we must trace the coefficients associated with each product explicitly.

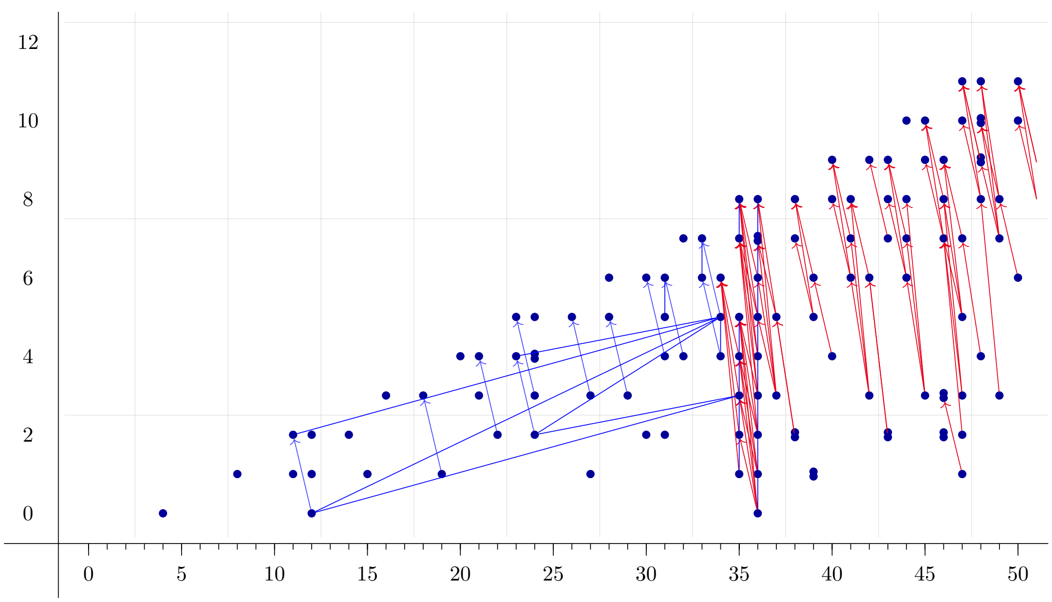

**Figure 3.43:** Multiplicative structure of $E_2^{*,*}$.

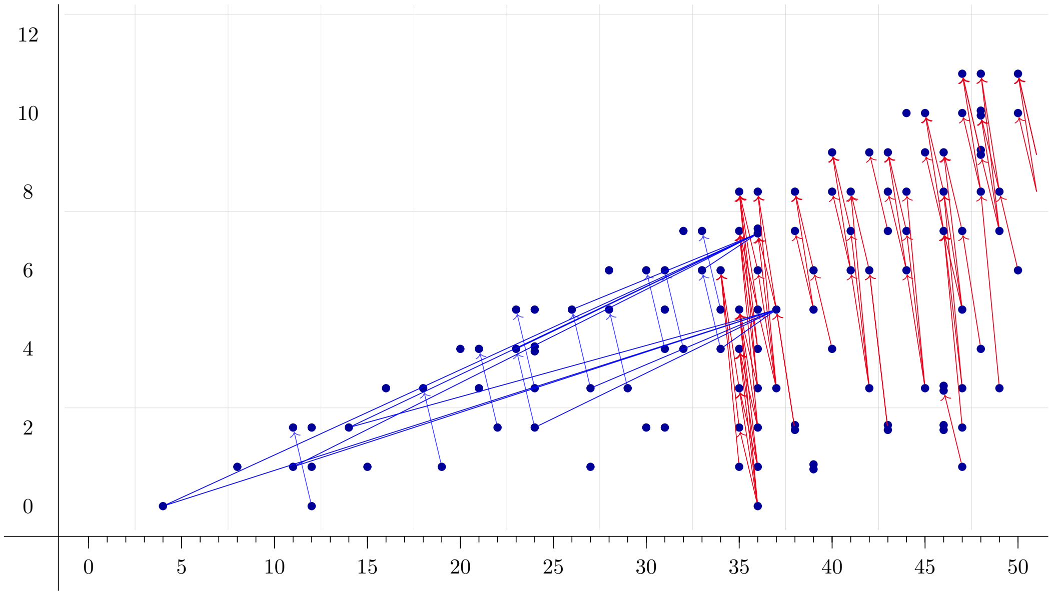


**Figure 3.44:** Multiplicative structure of $E_2^{*,*}$.

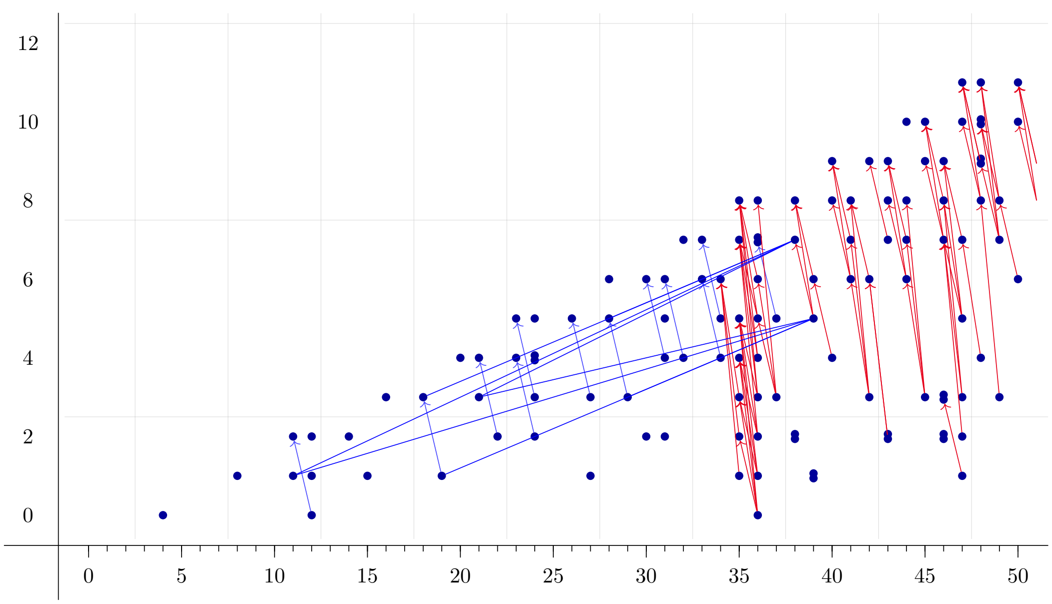


**Figure 3.45:** Multiplicative structure of $E_2^{*,*}$.

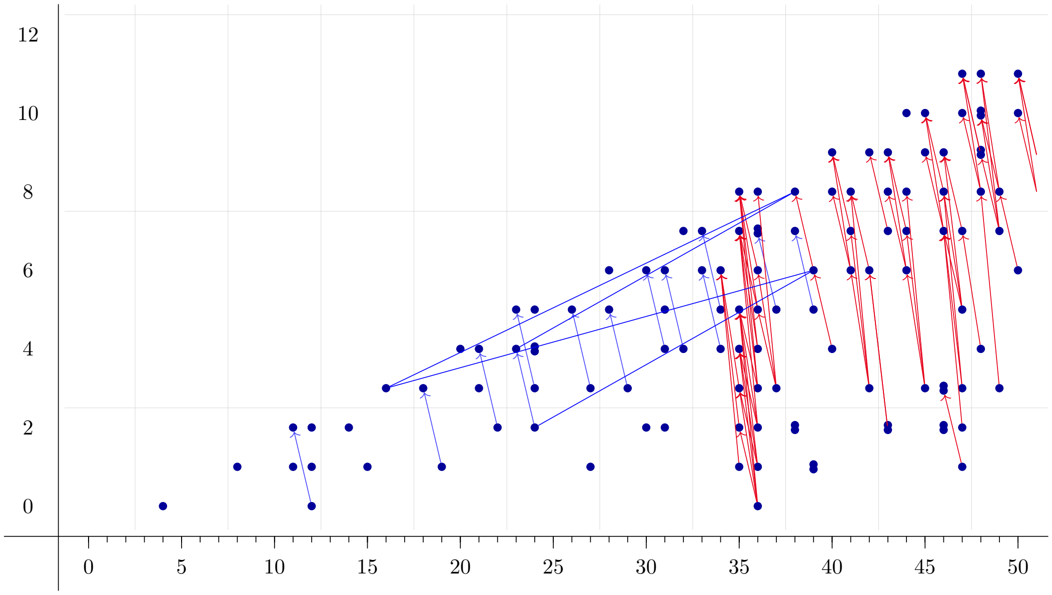

**Figure 3.46:** Multiplicative structure of $E_2^{*,*}$.

In the next figure, we have that $d_r(x^{(1)}_{3,37}) = 0$ for all $r \geq 2$, because $d_r(x^{(1)}_{1,27}) = 0 = d_r(y^{(1)}_{2,10})$ for all $r \geq 2$ (alternatively, by the Kahn-Priddy theorem we have that $\pi^s_{37}(\mathbb{S}^{\rho_3}_{hD_3})^\wedge_3 \twoheadrightarrow (\pi^s_{36})_{(3)} \cong \mathbb{Z}/3\mathbb{Z}$, thus $x^{(1)}_{3,37}$ survives in $E_\infty$).

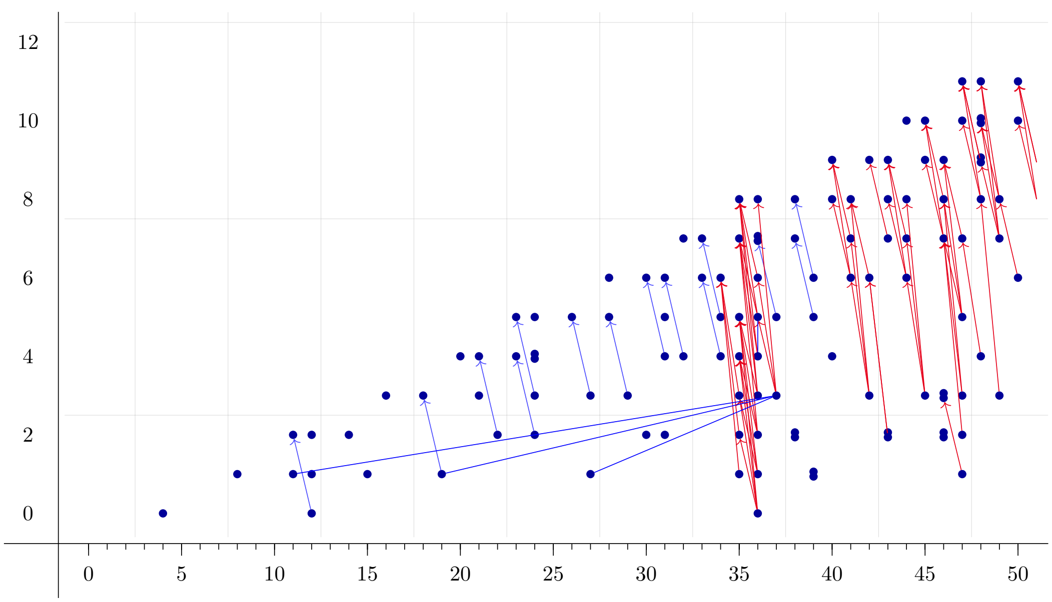

**Figure 3.47:** Multiplicative structure of $E_2^{*,*}$.

The columns $(*, 35)$ and $(*, 36)$ are harder to determine. Because there are no useful products hitting the $(*, 36)-$column. Moreover, the products involving the $(*, 36)$ have high degrees. The Kahn-Priddy theorem tells us that $\pi^s_{35}(\mathbb{S}^{\rho_3}_{hD_3})^\wedge_3 \twoheadrightarrow (\pi^s_{34})_{(3)} = 0$, but it is more useful in $(*, 36)$, because $\pi^s_{36}(\mathbb{S}^{\rho_3}_{hD_3})^\wedge_3 \twoheadrightarrow (\pi^s_{35})_{(3)} \cong \mathbb{Z}/27\mathbb{Z}$ is split surjective.

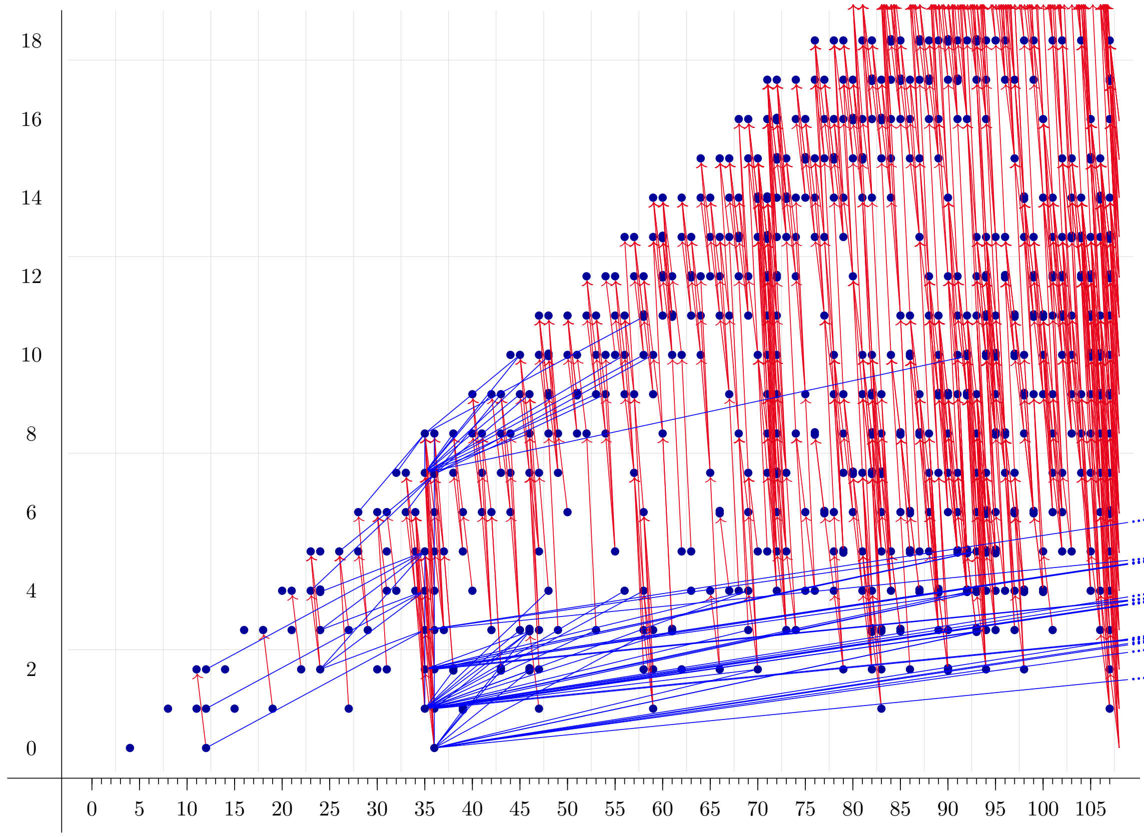


**Figure 3.48:** Multiplicative structure of $E_2^{*,*}$.

In conclusion

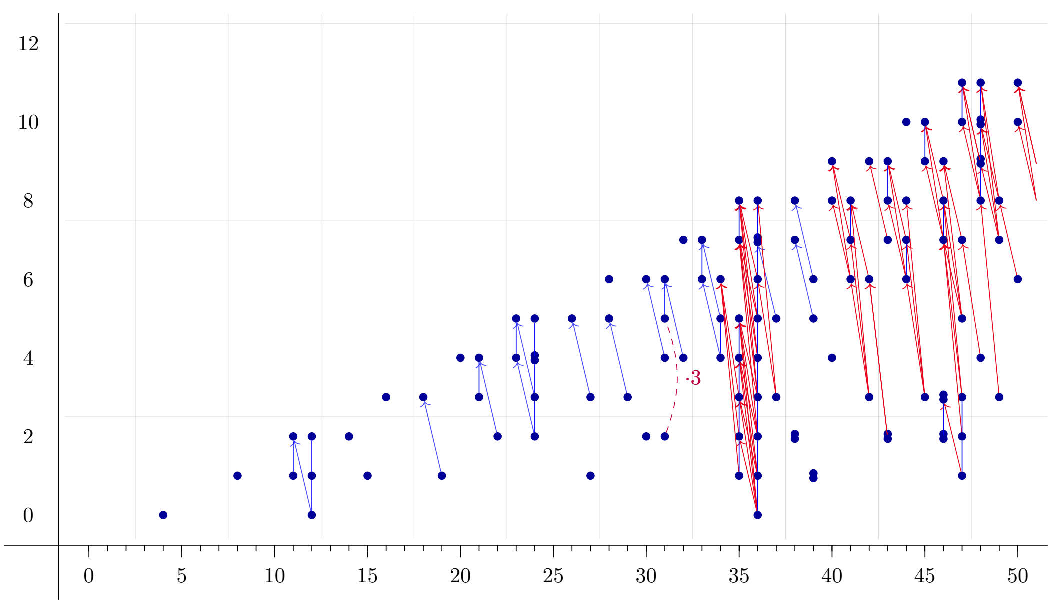


**Figure 3.49:** Multiplicative structure of $E_2^{*,*}$.

For degree reasons, there is a possible hidden 3−extension in the 31st column, as depicted by the dashed line. By the Kahn-Priddy theorem, $\pi^s_{31}(\mathbb{S}^{\rho_3}_{hD_3})^\wedge_3 \twoheadrightarrow (\pi^s_{30})_{(3)} \cong \mathbb{Z}/3\mathbb{Z}$ is split-surjective. Therefore $\pi^s_{31}(\mathbb{S}^{\rho_3}_{hD_3})^\wedge_3$ must have a $\mathbb{Z}/3\mathbb{Z}$ direct summand. Thus, there are no hidden 3−extensions, obtaining

| $*$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/9\mathbb{Z}$ | 0 |
| $*$ | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $*$ | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 |
| | 0 | 0 | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}$ | 0 | $\mathbb{Z}/\mathbf{9}\mathbb{Z}$ or $\mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ | **?** |
| $*$ | 36 | 37 | 38 | 39 | | | | | | | |
| | **?** | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ | | | | | | | |

**Table 3.6:** $\pi^s_*\left(\mathbb{S}^{\rho_3}_{hD_3}\right)^\wedge_3$, for $* \leq 39$ ($p$ odd).

□

Observe that in the previous argument, it is not clear how to compute all the non-trivial differentials with just the Leibniz rule and the Kahn-Priddy theorem, because both techniques rely on the knowledge of $(\pi^s_*)^\wedge_3$, but these groups are currently determined up to degree 103. Moreover, due to the increasing complexity of the classical Adams spectral sequence, and the presence of higher differentials, it is still unclear if Kahn-Priddy would be enough to determine all the non-trivial differentials when $p$ is odd. However, as it can be seen, the Kahn-Priddy theorem is quite useful to solve hidden 3−extensions.

# Further work

From a computational perspective, the program `minrv1` [Mor24] can be improved in several ways. Indeed, we can rewrite the relevant routines in a low-level language like `C`, removing the `SageMath` dependencies, which are a considerable bottleneck since its objects are bloated for our purposes (although the author chose `SageMath` initially to achieve a working sample dealing with mod $p$ linear algebra as soon as possible). Moreover, we could implement GPU acceleration, which has shown an increasing computing capability due to the rise in AI investment nowadays. Other improvements could include a better interface and distributed computations.

Of course, these improvements are not a replacement for a better theoretical framework but a complement. The classical Adams spectral sequence has several variants, such as the Adams-Novikov, motivic, and synthetic Adams spectral sequences, being more efficient in computing stable homotopy groups. The existence and success of these spectral sequences are related to a deeper understanding of the category of spectra and the Steenrod algebra, being the sources of its computability capability. Additionally, knowledge of vanishing lines of subalgebras of the Steenrod algebra leads to significant optimizations without the need for considerable optimizations at the software-level [Nas19]. There is also a direct interplay between secondary cohomology operations and synthetic spectra conducing to an algorithm [Chua22] capable of computing the $d_2$-differentials in the Adams spectral sequence for the sphere spectrum up to arbitrary degree having a reasonable computational time in contrast with a classical approach. A more recent significant advance corresponds to [IWX23], determining the first 90 stable stems of $(\pi_*^S)_{(2)}$, involving the motivic setting.

A possible future use of these tools could involve analyzing representation spheres in the equivariant setting, as well as studying the corresponding equivariant Adams spectral sequence.

# Bibliography


[Ada66] J. F. Adams, "A periodicity theorem in homological algebra", *Proc. Cambridge Phil. Soc. 62*, 1966.

[Ada73] J. F. Adams, "The Kahn-Priddy theorem", *Math. Proc. Cambridge Philos. Soc.* **73**, 1973, pp. 45-55.

[Ada74] J. F. Adams, "Stable Homotopy and Generalised Homology", *Univ. of Chicago Press*, 1974.

[Bru] R. R. Bruner, "An Adams Spectral Sequence Primer", URL: `http://www.rrb.wayne.edu/papers/adams.pdf`.

[Burk21] R. Burklund, "An extension in the Adams spectral sequence in dimension 54", Bull. Lond. Math. Soc. **53**, 2021, no. 2, 404–407, DOI 10.1112/blms.12428. MR4239183.

[Car54] H. Cartan, "Sur les groupes d'Eilenberg-MacLane. II", *Proc. Nat. Acad. Sci.*, U.S.A., 1954, 704–707, DOI 10.1073/pnas.40.8.704 (French).

[Car55] H. Cartan, "Détermination des algèbres $H_*(\pi, n; Z_p)$ et $H^*(\pi, n; Z_p)$, $p$ premier impair", *Séminaire Henri Cartan*, Volume 7 (1954-1955) no. 1, Talk no. 9, 10 p, URL: `http://www.numdam.org/item/SHC_1954-1955__7_1_A9_0/` .

[CCBFY22] D. Chua, H. Chatham, J. Beauvais-Feisthauer, and M. Young, "The spectral sequences project (v0.0.4)", 2022, URL: `https://spectralsequences.github.io/sseq/`.

[Chua22] D. Chua, "The E3 Page of the Adams Spectral Sequence", *ProQuest LLC*, Ann Arbor, MI, 2022, arXiv:2105.07628 URL: `https://arxiv.org/pdf/2105.07628`.

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, "Rings, modules, and algebras in stable homotopy theory", Mathematical Surveys and Monographs **47**, American Mathematical Society, Providence, RI, 1997, MR1417719, with an appendix by M Cole.

[Fre37] H. Freudenthal, "Über die Klassen der Sphärenabbildungen I", *Compositio Math.* **5**, 1937, pp. 299-314.

[Hat02] A. Hatcher, "Algebraic Topology", *Cambridge University Press*, Cambridge, 2002.

[Hat04] A. Hatcher, "Spectral Sequences in Algebraic Topology", Unpublished, 2004, URL: `https://pi.math.cornell.edu/%7Ehatcher/AT/ATch5.pdf` .

[Hat18] A. Hatcher, "Pictures of stable homotopy groups of spheres", Unpublished, URL: `https://pi.math.cornell.edu/%7Ehatcher/stemfigs/stems.pdf` .

[HHR21] M. A. Hill, M. J. Hopkins, and D. C. Ravenel, "Equivariant stable homotopy theory and the Kervaire invariant problem", New Mathematical Monographs, Cambridge University Press, 2021.

[HSS00] M. Hovey, B. Shipley, and J. Smith, "Symmetric spectra", J. Amer. Math. Soc. 13.1, 2000, pp. 149–208.

[Isa19] D. C. Isaksen, "Stable stems", Mem. Amer. Math. Soc. **262**, 2019, no. 1269, viii+159, DOI. 10.1090/memo/1269. MR4046815.

[IWX23] D. C. Isaksen, G. Wang and Z. Xu, "Stable homotopy groups of spheres: From dimension 0 to 90", Publ. Math. Inst. Hautes Études Sci. **137**, 2023.

[Liu63] A. Liulevicius, "Zeroes of the cohomology of the Steenrod algebra", *Proc. Amer. Math. Soc.* **14**, 1963, pp. 972-976.

[Lur17] J. Lurie, “Higher algebra”, September 18, 2017, URL: `https://people.math.harvard.edu/~lurie/papers/HA.pdf`.

[May99] J. P. May, “A concise course in algebraic topology”, *The University of Chicago Press*, Chicago, 1999.

[McC00] J. McCleary. “A User's Guide to Spectral Sequences”, second edition, *Cambridge University Press*, Cambridge, 2000.

[MMSS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley, “Model categories of diagram spectra”, Proc. London Math. Soc. (3) 82.2, 2001, pp. 441–512.

[Mor24] A. Morán, “MinimalResolution”, *GitHub*, GitHub repository, 2024, URL: `https://github.com/Issyl-m/MinimalResolution/blob/main/minrv1.py` , 6ba9a25.

[Mos68] R. M. F. Moss, “On the composition pairing of Adams spectral sequences”, *Proc. London Math. Soc.* (3) **18**, 1968, pp. 179-192, MR0220294 (36 #3360).

[Mos70] R. M. F. Moss, “Secondary compositions and the Adams spectral sequence”, *Mathematische Zeitschrift* **115**, 1970, no. 4, pp. 283–310.

[MT68] R. E. Mosher, M. C. Tangora, “Cohomology Operations and Applications in Homotopy Theory”, Harper and Row, New York, 1968.

[Muñ24] S. Muñoz-Echániz, “A Weiss-Williams theorem for spaces of embeddings and the homotopy type of spaces of long knots”, Unpublished, 2024, URL: `https://arxiv.org/pdf/2311.05541v2` .

[Nas19] C. Nassau, “Computing a minimal resolution over the Steenrod algebra”, Unpublished, 2019, URL: `https://arxiv.org/pdf/1910.04063` .

[Pst22] P. Pstrągowski, “Synthetic spectra and the cellular motivic category”, *Inventiones mathematicae*, 2022.

[Rav86] D. C. Ravenel, “Complex cobordism and stable homotopy groups of spheres”, Pure Applied Math, *Academic Press*, Orlando, 1986.

[Rog10] J. Rognes, “Spectral sequences”, 2010, URL: `https://www.uio.no/studier/emner/matnat/math/MAT9580/v21/dokumenter/spseq.pdf`.

[Rog15] J. Rognes, “Introduction to the Adams Spectral sequence”, 2015, URL: `https://www.uio.no/studier/emner/matnat/math/MAT9580/v15/undervisningsmateriale/adams-sp-seq.010615.pdf`.

[RW21] O. R. Williams, “Homotopy Theory”, Lecture notes, University of Cambridge, 2021, URL: `https://www.dpmms.cam.ac.uk/%7Eor257/teaching/notes/HomotopyTheory.pdf` .

[Spa66] E. H. Spanier, “Algebraic topology”, *McGraw-Hill Book Co.*, New York-Toronto, Ont.-London, 1966.

[Sto] B. Stonek, “Introduction to stable homotopy theory”, URL: `http://bruno.stonek.com/stable-homotopy-2022/stable-online.pdf`.

[Swi75] R. M. Switzer, “Algebraic topology: homotopy and homology”, *Die Grundlehren der mathematischen Wissenschaften*, Band 212, Springer-Verlag, New York-Heidelberg, 1975.

[tom08] T. Dieck, “Algebraic topology”, EMS Textbooks in Mathematics, *European Mathematical Society (EMS)*, Zürich, 2008.

[WX18] G. Wang and Z. Xu, “Some extensions in the Adams spectral sequence and the 51-stem”, Algebr. Geom. Topol. **18** 2018, no. 7, 3887–3906.