Special Random Variables

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Plan

- The binomial distribution
- The Poisson distribution
- Hypergeometric distribution

The Binomial Random Variable

The Bernoulli Random Variable

- Suppose that a trial has only two outcomes, which are classified as either a "success" or as a "failure", such trial is known as **Bernoulli trial**
  - Define a random variable X as

$$X = \begin{cases} 1 & if\ the\ outcome\ is\ a\ success \\ 0 & if\ the\ outcome\ is\ a\ failure \end{cases}$$

The Bernoulli Random Variable

- X follows a Bernoulli distribution with parameter p and the probability mass function

$$P(X=1)=p$$
$$P(X=0)=1-p$$

- The probability mass function

$$P(X=x)=p^x(1-p)^{1-x}, \qquad x=0,1;\;\; 0\le p\le 1$$

  - $E(X)=p$ and $Var(X)=p(1-p)$

The binomial random variable

- Let p be the probability of success of a coin
- Suppose a coin is rolled two times, what is the probability distribution of X, the number of successes in two trials
- Suppose a coin is rolled three times, what is the probability distribution of X, the number of successes in two trials

The binomial random variable

- Binomial random variable deals with the distribution of the number of successes in n independent Bernoulli trials

- The probability success p remain constant from trial to trial

- Let $X_1, \ldots, X_n$ are independent and each follow a Bernoulli distribution with parameter p, and define

$$X = X_1 + \cdots + X_n$$

- X represents the number of success in n Bernoulli trials and X follows a binomial distribution with parameters n and p, i.e. $X \sim B(n, p)$

The binomial random variable

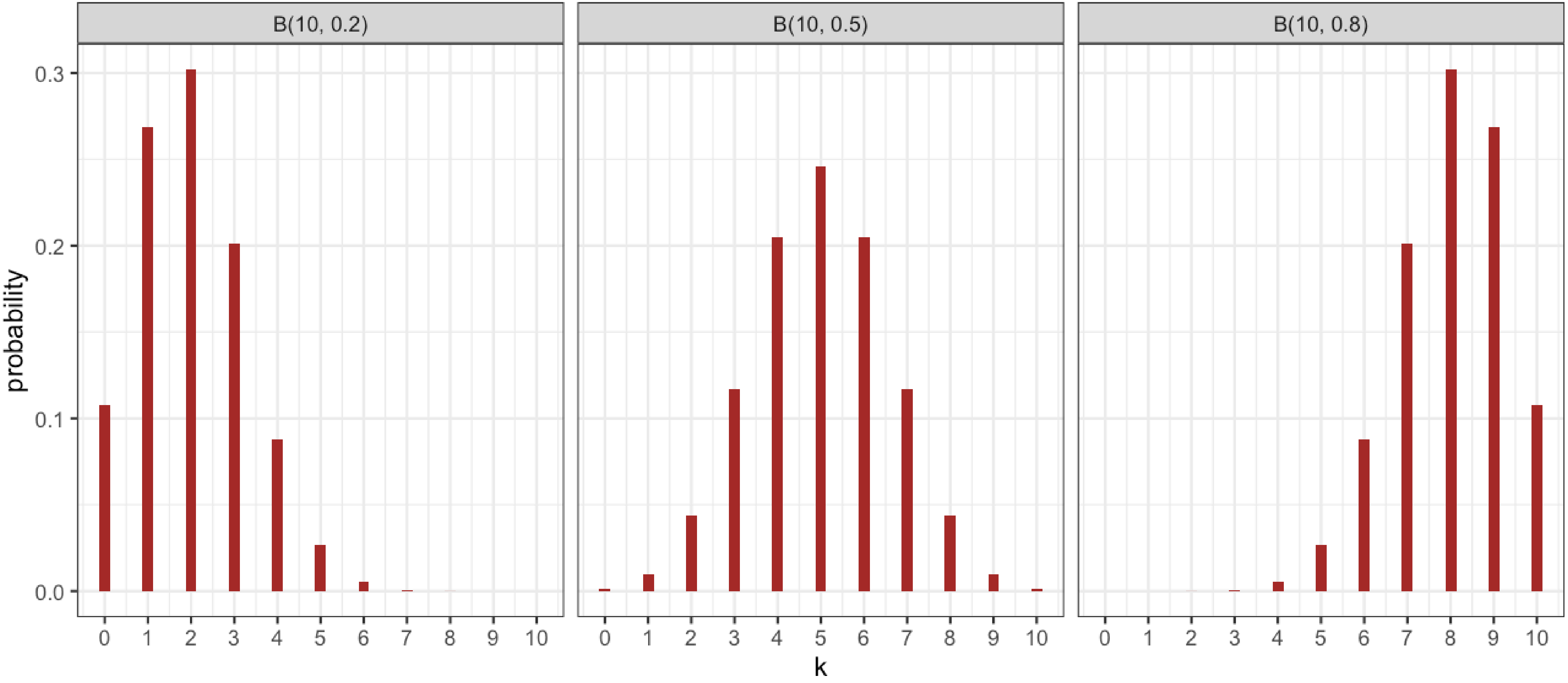
- The probability mass function of X

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n$$

  - $\binom{n}{x} \rightarrow$ number of ways x success can be obtained in n Bernoulli trials

Example 5.1a

- It is known that disks produced by a certain company will be defective with probability .01 independently of each other.
- The company sells the disks in packages of 10 and offers a money-back guarantee that at most 1 of the 10 disks is defective.
- What proportion of packages is returned?
- If someone buys three packages, what is the probability that exactly one of them will be returned?


B(10, 0.2)
B(10, 0.5)
B(10, 0.8)
probability
0.0
0.1
0.2
0.3
0 1 2 3 4 5 6 7 8 9 10
k

Expectation of binomial distribution

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^k (1-p)^{n-x} \\
&= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x} = np
\end{aligned}
$$

Variance of binomial distribution

$$
\begin{aligned}
E\big[X(X-1)\big] &= \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^k (1-p)^{n-x} \\
&= \sum_{x=2}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x} = n(n-1)p^2
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= E(X^2) - [E(X)]^2 \\
&= E\big[X(X-1)\big] + E(X) - [E(X)]^2 \\
&= n(n-1)p^2 + np - n^2p^2 \\
&= np(1-p)
\end{aligned}
$$

Binomial cumulative distribution function

- The cumulative distribution function of $X \sim B(n, p)$

$$P(X \leq a) = \sum_{x=a}^{n} \binom{n}{x} p^x (1-p)^{n-x}$$

- Following relationship is helpful to calculate cumulative distribution of binomial distribution

$$P(X = x + 1) = \frac{p}{1-p} \frac{n-x}{x+1} P(X = x)$$

Problems

- 1, 3, 5, 6, 7, 9

Geometric distribution

- Let p be the probability of success of a coin
- What is the probability distribution of X, the number of trials needed to get the success for the first time?

Geometric distribution

- Let X be the number of trials needed to get the success for the first time and p be the probability of success of the Bernoulli trial

- X follows a geometric distribution with parameter p, $X \sim G(p)$ and the corresponding probability mass function

$$p(x) = (1-p)^{x-1}p,\ x = 1, 2, \ldots$$

  - $E(X) = 1/p$ and $V(X) = (1-p)/p^2$
  - $P(X \leq a) = 1 - (1-p)^a$

Negative binomial distribution

- Let p be the probability of success of a coin
- What is the probability distribution of X, the number of trials needed to get the success for the rth $(r \geq 1)$ time?

Negative binomial distribution

- Let X be the number of trials needed to get the rth $(r \geq 1)$ success for the first time and p be the probability of success of the Bernoulli trial

- X follows a negative binomial distribution with parameters r and p, $X \sim NB(r,p)$ and the corresponding probability mass function

$$p(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \; x = r, r+1, \ldots$$

  - $E(X) = r/p$ and $V(X) = r(1-p)/p^2$

Exercise

- An archer hits a bull's-eye with a probability of 0.09, and the results of different attempts are assumed to be independent
- If a archer shoot a series of arrows, what is the probability that the first bull's-eye is scored with the fourth arrow?
- What is the probability that the third bull's-eye is scored with the tenth arrow?
- What is the expected number of arrows shot before the first bull's-eye is scored?
- What is the expected number of arrows shot before the third bull's-eye is scored?

The Poisson Random Variable

The Poisson Random Variable

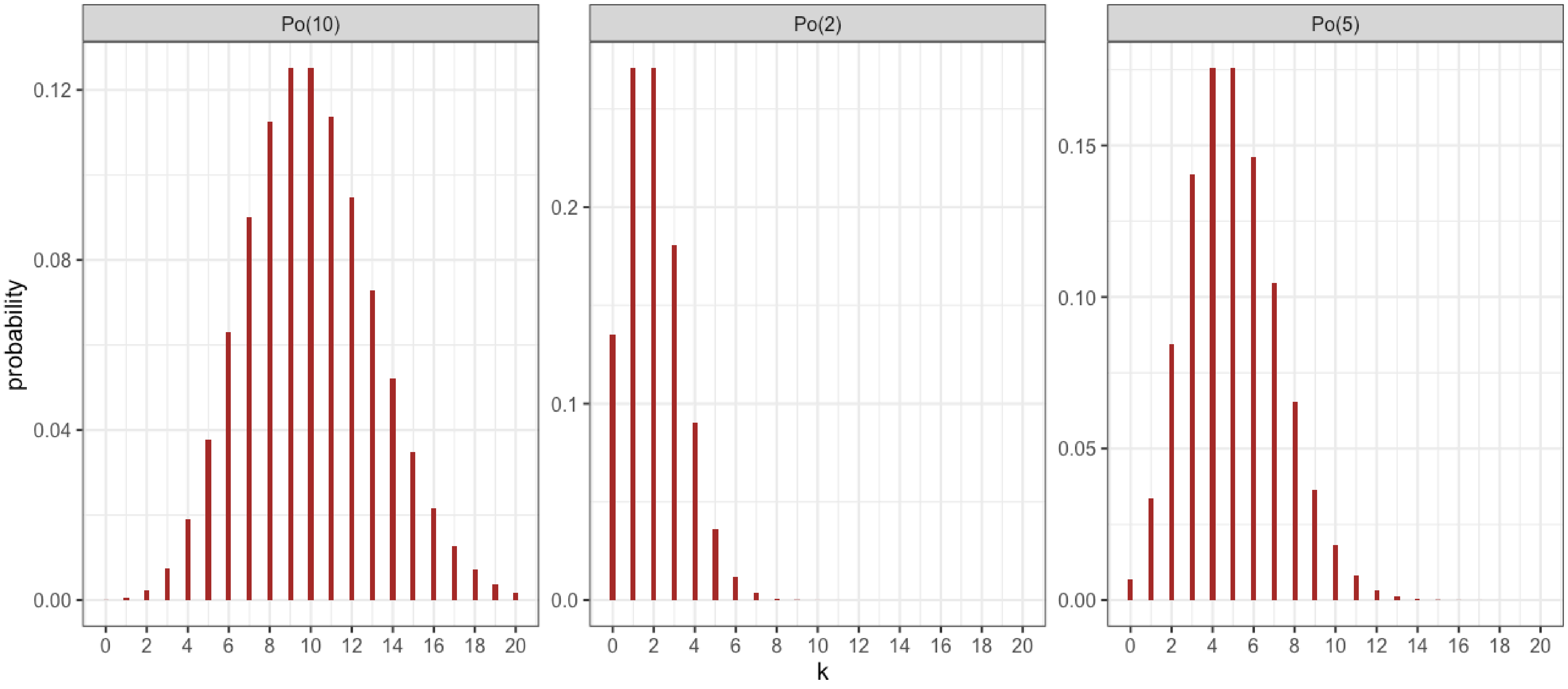
- A random variable X is said to be a Poisson random variable with parameter $\lambda > 0$, if its probability mass function is given by

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

- Using the relationship $e^\lambda = \sum_{x=0}^{\infty} \lambda^x / x!$, it can be shown that $\sum_{x=0}^{\infty} P(X = x) = 1$ for a Poisson distribution

- For $X \sim Po(\lambda)$

$$E(X) = V(X) = \lambda$$


Po(10)
Po(2)
Po(5)
probability
k
0.00
0.04
0.08
0.12
0.0
0.1
0.2
0.00
0.05
0.10
0.15
0 2 4 6 8 10 12 14 16 18 20

Examples of Poisson random variables

- The number of misprints on a page of a book
- The number of people in a community living to 100 year of age
- The number of wrong telephone numbers that are dialed in a day
- The number of transistors that fail on their first day of use
- The number of customers entering a post office on a given day

Expectation and variance of a Poisson random variable

$$E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!} = \lambda$$

- Similarly

$$E\big[X(X-1)\big] = \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!} = \lambda^2$$

$$Var(X) = E\big[X(X-1)\big] + E(X) - [E(X)]^2 = \lambda$$

The Poisson distribution

- The moment generating function of $X \sim Po(\lambda)$

$$M_X(t) = E\left[e^{tX}\right] = \sum_{x=0}^{\infty} e^{tx} e^{-\lambda} \lambda^x / (x!)$$

$$= e^{-\lambda} \sum_{x=0}^{n} (\lambda e^t)^x / (x!) = e^{-\lambda} e^{\lambda e^t} = e^{-\lambda(1-e^t)}$$

- Obtain $E(X)$ and $V(X)$

Poisson approximation of binomial distribution

- Suppose $X \sim B(n, p)$, for a large n and small p

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \simeq \frac{e^{-\lambda}\lambda^x}{x!}$$

  - $\lambda = np$

- **Examples:** $X \sim B(70, .1)$

$$P(X = 5) = \begin{cases} 0.1284 & for\ binomial \\ 0.1277 & for\ Poisson \end{cases}$$

Example 5.2a

- Suppose that the average number of accidents occurring weekly on a particular stretch of a highway equals 3.
- Calculate the probability that there is at least one accident this week.

Example 5.2b

- Suppose the probability that an item produced by a certain machine will be defective is .1.
- Find the probability that a sample of 10 items will contain at most one defective item.
- Assume that the quality of successive items is independent.

Example 5.2d

- If the average number of claims handled daily by an insurance company is 5, what proportion of days have less than 3 claims?
- What is the probability that there will be 4 claims in exactly 3 of the next 5 days?
- Assume that the number of claims on different days is independent.

Distribution of sum of two independent Poisson random variables

- If $X_1 \sim Po(\lambda_1)$ and $X_2 \sim Po(\lambda_2)$, then

$$Y = X_1 + X_2 \sim Po(\lambda_1 + \lambda_2)$$

- Moment generating function of X_1

$$M_{X_1}(t) = E\left[e^{tX}\right] = \exp\left[\lambda_1(e^t - 1)\right]$$

- Moment generating function of $Y = X_1 + X_2$

$$M_Y(t) = M_{X_1}(t)M_{X_2}(t) = \exp\left[(\lambda_1 + \lambda_2)(e^t - 1)\right]$$

Computation of Poisson distribution function

- For $X \sim Po(\lambda)$

$$\frac{P(X = x+1)}{P(X = x)} = \frac{\lambda}{x+1}, x = 1, 2, \ldots$$

Example 5.2f

- It has been established that the number of defective stereos produced daily at a certain plant is Poisson distributed with mean 4.
- Over a 2-day span, what is the probability that the number of defective stereos does not exceed 3?

Problems

- 13, 14, 18

Hypergeometric Distribution

Hypergeometric Distribution

- A bin contains $N + M$ batteries, of which N are of acceptable quality and M are defective

- A sample of size n is randomly chosen (without replacement) and all possible sampled subsets are equally likely

- The X denotes the number of acceptable batteries in the sample of size n and its probability mass function

$$P(X = x) = \frac{\binom{N}{x}\binom{M}{n-x}}{\binom{N+M}{n}}, \; k = 0, 1, \ldots, \min(n, N)$$

  - X follows a hypergeometric distribution with parameters N, M, and n

Example 5.3a

- The components of a 6-component system are to be randomly chosen from a bin of 20 used components.
- The resulting system will be functional if at least 4 of its 6 components are in working condition.
- If 15 of the 20 components in the bin are in working condition, what is the probability that the resulting system will be functional?
  - X, the number of components with working condition out of six selected components, follows a hypergeometric distribution with parameters $N = 20$, $M = 15$, and $n = 6$

Expectation and variance of hypergeometric distribution

- Let $X = \sum_{i=1}^{n} X_i$ be the number of acceptable batteries in n selection, where

$$X_i = \begin{cases} 1 & \text{if the ith selection is acceptable} \\ 0 & \text{otherwise} \end{cases}$$

- $E(X) = E\left(\sum_i X_i\right) = \sum_i E(X_i) = \sum_i P(X_i = 1) = \frac{nN}{N+M}$
- $Var(X) = Var\left(\sum_i X_i\right) = \sum_i Var(X_i) + 2\sum_{i>j} Cov(X_i, X_j)$

Expectation and variance of Hypergeometric distribution

$$Var(X) = Var\Big(\sum_i X_i\Big) = \sum_i Var(X_i) + 2\sum_{i>j} Cov(X_i, X_j)$$

$$Var(X_i) = P(X_i = 1)[1 - P(X_i = 1)] = \frac{N}{N+M}\frac{M}{N+M}$$

$$\begin{aligned}
Cov(X_i, X_j) &= E(X_i X_j) - E(X_i)E(X_j) \\
&= P(X_i X_j = 1) - \Big(\frac{N}{N+M}\Big)^2 \\
&= P(X_i = 1, X_j = 1) - \Big(\frac{N}{N+M}\Big)^2
\end{aligned}$$

Expectation and variance of Hypergeometric distribution

$$P(X_i = 1, X_j = 1) = P(X_i = 1)P(X_j = 1 \mid X_i = 1) = \frac{N}{N+M}\frac{N-1}{N+M-1}$$

$$\begin{aligned} Cov(X_i, X_j) &= \frac{N(N-1)}{(N+M)(N+M-1)} - \left(\frac{N}{N+M}\right)^2 \\ &= \frac{-NM}{(N+M)^2(N+M-1)} \end{aligned}$$

$$\begin{aligned} Var(X) &= \frac{nNM}{(N+M)^2} - \frac{n(n-1)NM}{(N+M)^2(N+M-1)} \\ &= \frac{nNM}{(N+M)^2}\left(1 - \frac{n-1}{N+M-1}\right) \end{aligned}$$

Hypergeometric Distribution

- Let $X \sim B(n, p)$ and $Y \sim B(m, p)$ then

$$P\left(X = i \,|\, X + Y = k\right) = \frac{\binom{n}{i}\binom{m}{k-i}}{\binom{n+m}{k}}$$

Multinomial distribution

- Consider a sequence of n independent trials where each individual trial can have k outcomes that occur with probability $p_1, \ldots, p_k$, respectively, where $\sum_{i=1}^{k} p_i = 1$
- Let X_i be the number of ith-type of occurrences, $X_i \sim B(n, p_i)$
- The counts $(X_1, \ldots, X_k)$ follows a multinomial distribution with parameters n and $p_1, \ldots, p_k$ and their joint probability mass function

$$p(X_1 = x_1, \ldots, X_k = x_k) = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$$

  - $\sum_{i=1}^{k} x_i = n$

Multinomial distribution

- Let X_i be the number of ith-type of occurrences, $X_i \sim B(n, p_i)$
- $E(X_i) = np_i$ and $V(X_i) = np_i(1 - p_i)$
- X_i's are not independent, $Cov(X_i, X_j) \neq 0$

The Uniform Random Variable

The Uniform Random Variable

- A random variable X is said to be uniformly distributed over the interval $[\alpha, \beta]$ if its pdf is given by

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

- Show that

$$E(X) = \frac{\alpha + \beta}{2} \;\; and \;\; Var(X) = \frac{(\beta - \alpha)^2}{12}$$

The Normal Random Variable

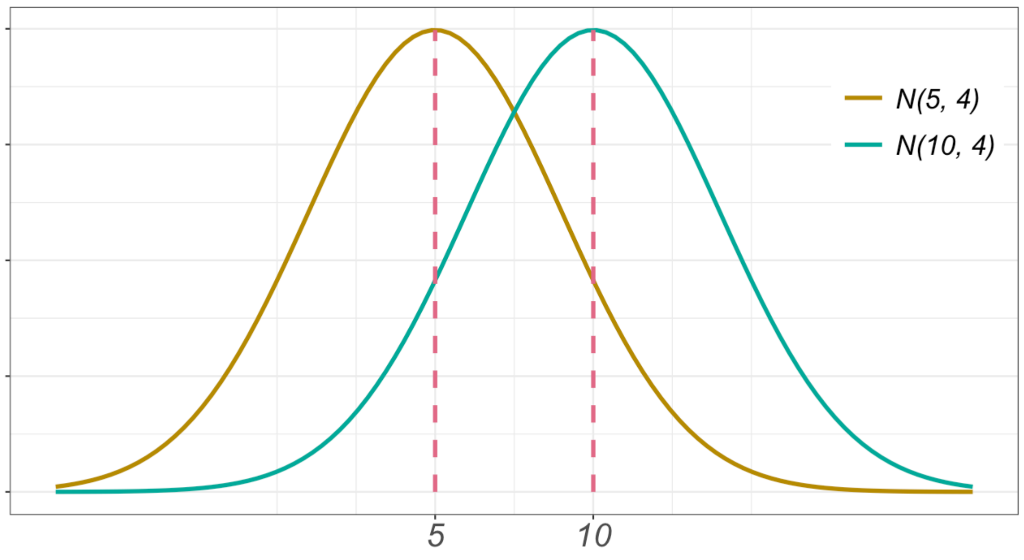
The Normal Random Variable

- A random variable X is said to be normally distributed random variable with parameters μ and σ^2,
  - i.e. $X \sim N(\mu, \sigma^2)$, if its density is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, \quad -\infty < x < \infty$$

- The normal density function is bell-shaped and symmetric about its mean μ
- The maximum value of the density function is $(\sigma\sqrt{2\pi})^{-1} \simeq 0.399/\sigma$ attains at $x = \mu$

Parameters of normal distribution and its density function



The normal random variable

- The expected value

$$E(X-\mu)=\int_{-\infty}^{\infty}(x-\mu)\frac{1}{\sqrt{2\pi\sigma^2}}\;e^{-\frac{1}{2\sigma^2}(x-\mu)^2}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}y\,e^{-y^2/2}\,dy=0$$

$$E(X)=\mu$$

The normal random variable

- The variance

$$Var(X) = E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}}\ e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

$$= \sigma^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2/2}\, dy = \sigma^2$$

- Integration by parts

$$\int uv\, dx = u \int v\, dx - \int \frac{du}{dx}\left(\int v\, dx\right) dx$$

Distribution of a linear combination of a normal distribution

- If $X \sim N(\mu, \sigma^2)$, what is the distribution of $Y = a + bX$, where a and b are constants?

- If $M(t)$ is the moment generating function (mgf) of X, the mgf of $Y = a + bX$

$$M_Y(t) = \int e^{ty} f(x) dx = \int e^{t(a+bx)} f(x)\, dx = e^{ta} M_X(t)$$

Distribution of a linear combination of a normal distribution

- The mgf of $X \sim N(\mu, \sigma^2)$

$$M_X(t) = e^{t\mu + (t^2\sigma^2/2)}$$

- The mgf of $Y = a + bX$

$$\begin{aligned} M_Y(t) &= e^{ta} M_X(tb) \\ &= e^{ta} e^{t\mu b + (t^2 b^2 \sigma^2/2)} \\ &= e^{t(b\mu + a) + (t^2 b^2 \sigma^2/2)} \end{aligned}$$

  - So, $Y \sim N\left(a + b\mu, \sigma^2 b^2\right)$

Standard normal distribution

- A normal distribution with mean 0 and variance 1 is known as the standard normal distribution, which is denoted by Z

$$Z \sim N(0, 1)$$

- The mgf of standard normal distribution: $M_Z(t) = e^{t^2/2}$

- The probability density function of a standard normal distribution

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty$$

Standard normal distribution

- If $X \sim N(\mu, \sigma^2)$ then

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

  - Proof this relationship!

- It can be shown that

$$E(Z) = E\left[\frac{X - \mu}{\sigma}\right] = 0 \;\; and \;\; Var(Z) = Var\left[\frac{X - \mu}{\sigma}\right] = 1$$

Cumulative distribution of the standard normal distribution

- The cdf of the standard normal distribution

$$\Phi(x) = P(Z < x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

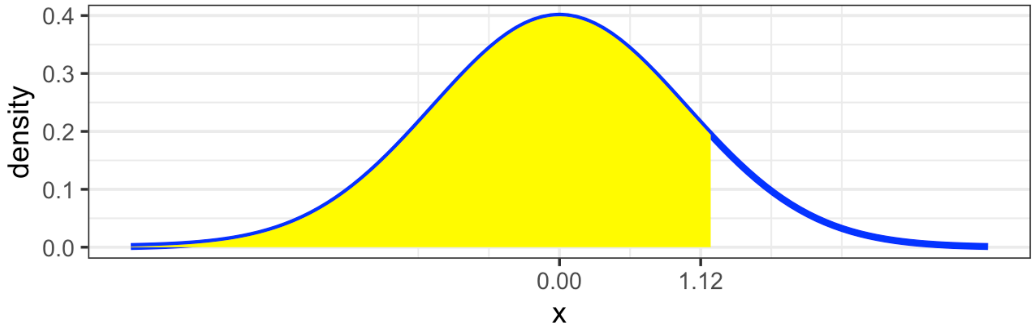
  - This integration cannot be evaluated agebraically
  - Most of the matematical statistics book has Z table, which provides $\Phi(x)$ values for different x (see page 642 of the textbook for the Z table)

- An important relationship

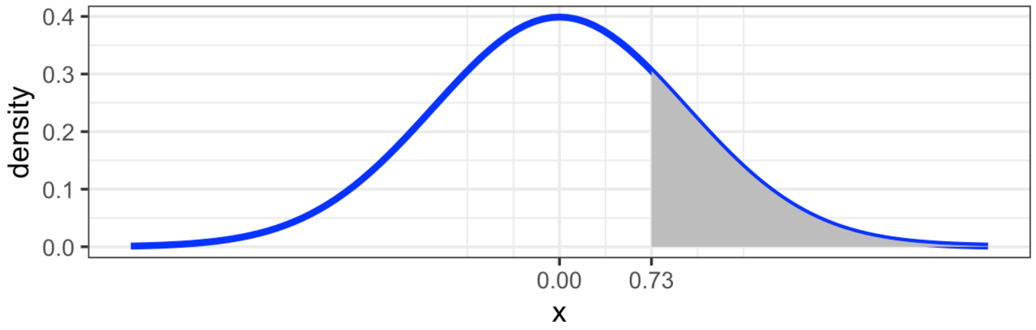
$$\Phi(x) + \Phi(-x) = 1$$

Probability calculation related to $Z \sim N(0,1)$

$$\begin{aligned} P(Z < 1.12) &= \Phi(1.12) \\ &= 0.8686 \end{aligned}$$

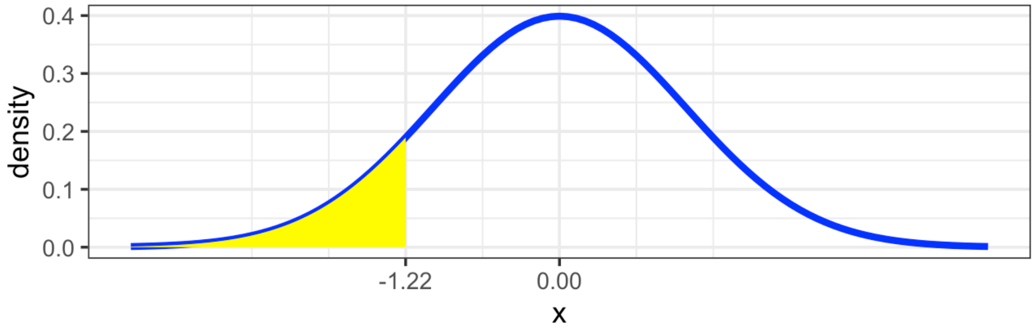


$$\begin{aligned} P(Z > .73) &= 1 - P(Z \leq .73) \\ &= 1 - \Phi(.73) \\ &= 1 - 0.7673 \\ &= 0.2327 \end{aligned}$$

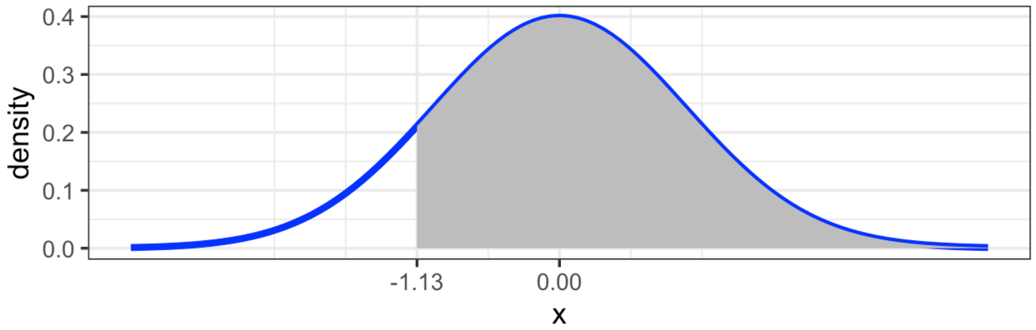


Probability calculation related to $Z \sim N(0,1)$

$$\begin{aligned} P(Z < -1.22) &= \Phi(-1.22) \\ &= 1 - \Phi(1.22) \\ &= 1 - 0.8686 \\ &= 0.1112 \end{aligned}$$

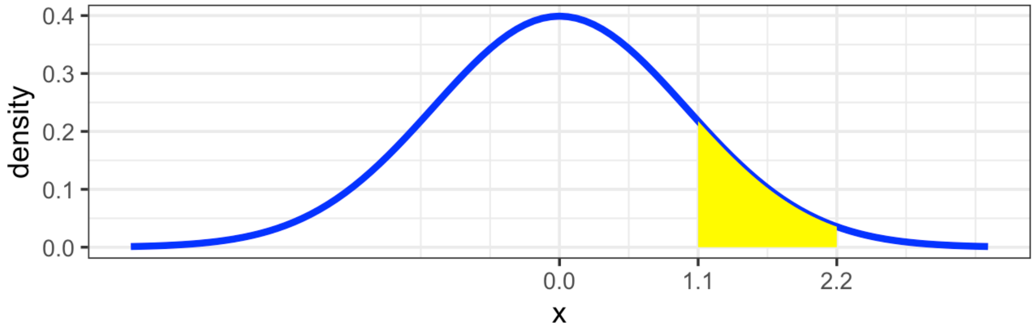


$$\begin{aligned} P(Z > -1.13) &= 1 - P(Z \leq -1.13) \\ &= 1 - \Phi(-1.13) \\ &= \Phi(1.13) \\ &= 0.8708 \end{aligned}$$

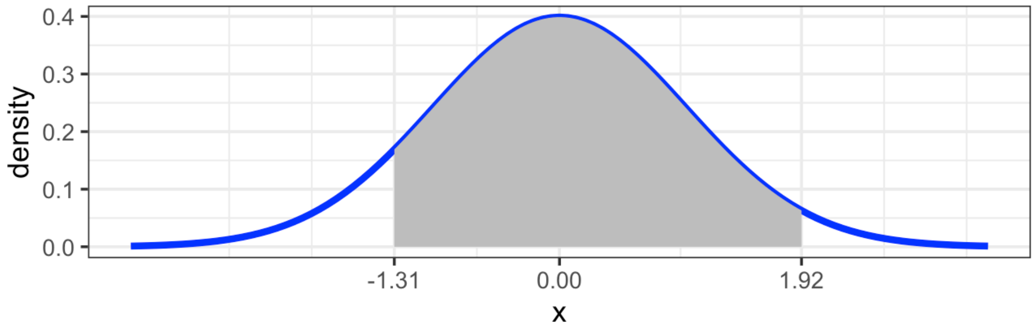


Probability calculation related to $Z \sim N(0,1)$

$$
\begin{aligned}
P(1.1 < Z < 2.2) &= \Phi(2.2) - \Phi(1.1) \\
&= 0.9861 - 0.8643 \\
&= 0.1218
\end{aligned}
$$



$$
\begin{aligned}
&P(-1.31 < Z < 1.92) \\
&= \Phi(1.92) - \Phi(-1.31) \\
&= \Phi(1.92) - 1 + \Phi(1.31) \\
&= 0.9726 - 1 + 0.9049 \\
&= 0.8775
\end{aligned}
$$



Probability calculation related to $X \sim N(\mu, \sigma^2)$

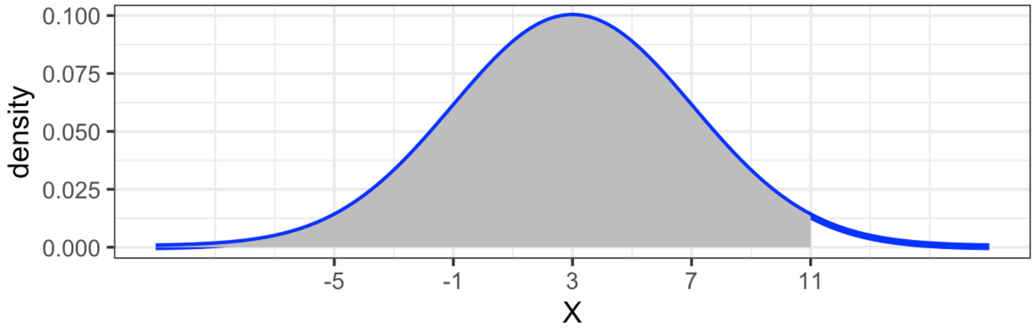
- For any $a < b$

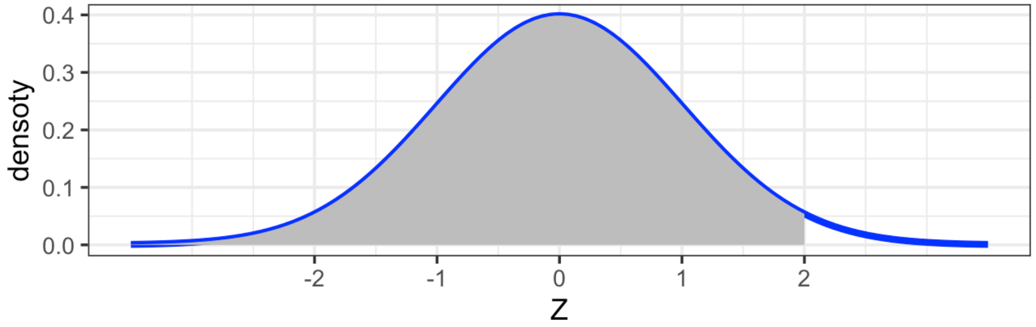
$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right)$$
$$= P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$
$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Probability calculation related to $X \sim N(\mu, \sigma^2)$

- For $X \sim N(3, 16)$

$$P(X < 11) = \Phi\left(\frac{11-3}{4}\right)$$
$$= \Phi(2)$$
$$= 0.9772$$





- Calculate

$$P(X > -1)$$

$$P(2 < X < 7)$$

Sum of several independent normal variables

- Let $X_1, \ldots, X_n$ be n independent normal random variables, where

$$X_i \sim N(\mu_i, \sigma_i^2)$$

- The distribution of $Y = \sum_{i=1}^{n} X_i$ follows a normal distribution with

$$mean = \sum_{i=1}^{n} \mu_i \quad and \quad variance = \sum_{i=1}^{n} \sigma_i^2$$

Exammple 5.5d

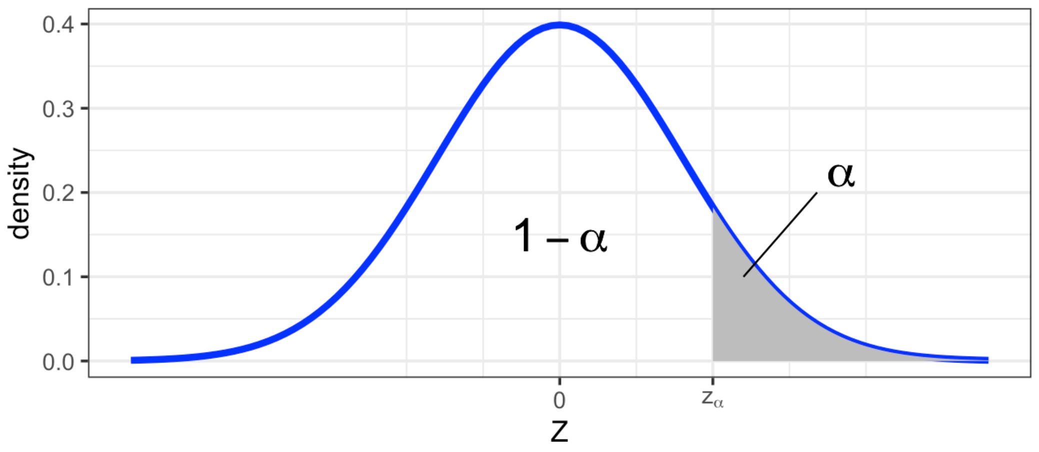
- Data from National Oceanic and Atmospheric Administration indicate that the yearly precipitation in Los Angles is a normal random variable with a mean of 12.08 inches and standard deviation of 3.1 inches.

- Find the probability that total precipitation during the next two years will exceed 25 inches.

Quantiles/percentiles of standard normal distribution

- $z_\alpha \rightarrow$ the $100(1-\alpha)$ percentile of standard normal distribution, i.e.,

$$P(Z < z_\alpha) = 1 - \alpha$$



- From the table, we can find
  - $z_{.05} = 1.645$
  - $z_{.025} = 1.96$
  - $z_{.01} = 2.33$

Exponential random variables

Exponential random variables

- An exponential random variable with parameter $\lambda > 0$ has the following probability density function

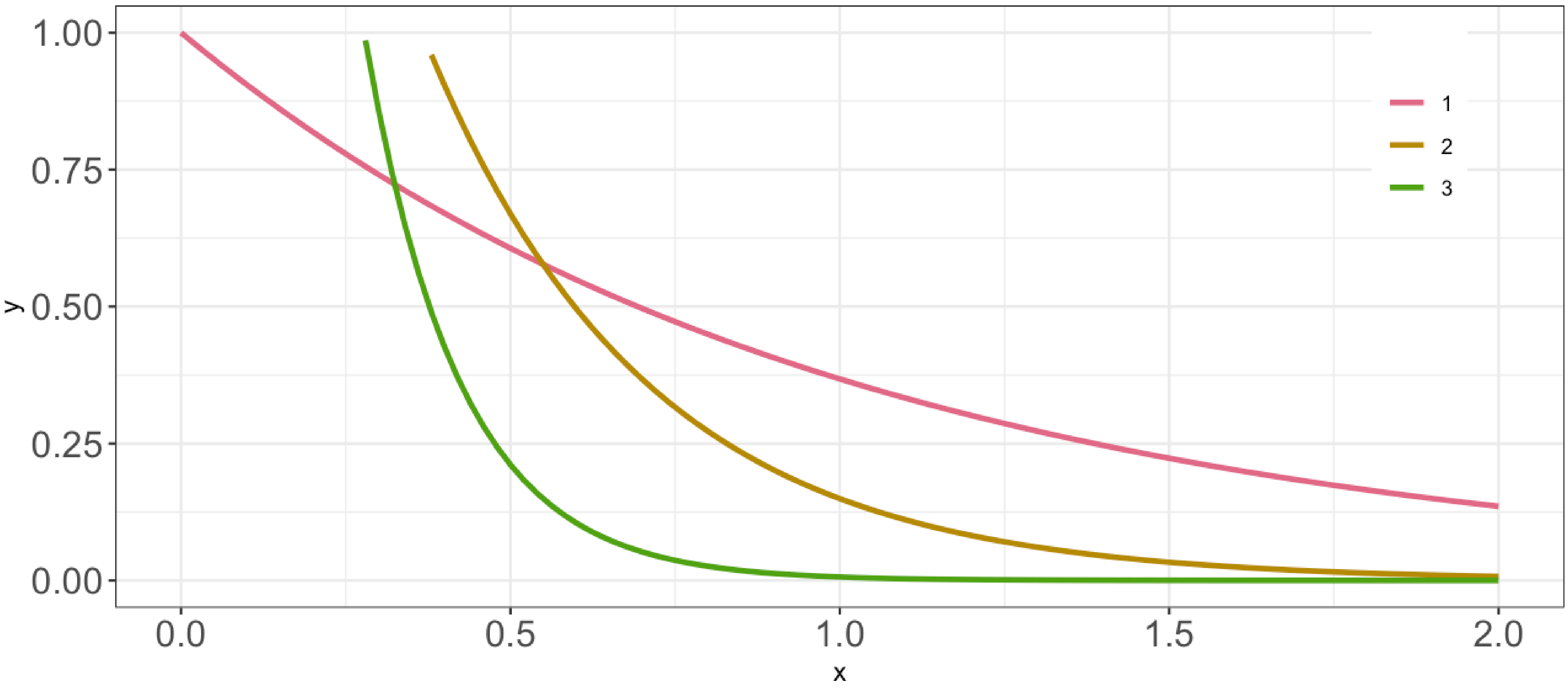
$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

- The moment generating function of a exponential random variable

$$M_X(t) = \lambda/(\lambda - t)$$

- Show that

$$E(X) = 1/\lambda \;\; and \;\; Var(X) = 1/\lambda^2$$


1.00
0.75
0.50
0.25
0.00
y
0.0
0.5
1.0
1.5
2.0
x
1
2
3

The cumulative distribution function

$$F(x) = \int_0^x \lambda e^{-\lambda y} dy = 1 - e^{-\lambda x}$$

- $P(a < X < b) = F(b) - F(a) = e^{-\lambda a} - e^{-\lambda b}$

Memoryless property of exponential random variable

- If X follows an exponential distribution with parameter λ

$$P(X > s + t \,|\, X > t) = P(X > s)$$

Exponential distribution

- If $X_1, \ldots, X_n$ are independent exponential random variables with parameters $\lambda_1, \ldots, \lambda_n$, respectively
- Then $Y = \min\{X_1, \ldots, X_n\}$ follows an exponential distribution with parameter $\sum_i \lambda_i$

The Gamma distribution

- A random variable is said to have a gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$ if its density function is given by

$$f(x) = \frac{1}{\Gamma(\alpha)} \lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}, \; x > 0$$

  - $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx$

The Gamma distribution

- The moment generating function of $X \sim G(\alpha, \lambda)$

$$M_X(t) = \Big(\frac{\lambda}{\lambda - t}\Big)^{\alpha}$$

- $E(X) = \alpha/\lambda$ and $V(X) = \alpha/\lambda^2$
- Let $X_1, \ldots, X_n$ are independent exponentially distributed random variables with a common parameter λ, then

$$\sum_{i=1}^{n} X_i \sim G(n, \lambda)$$

Beta distribution

- A random variable X follows a beta distribution with parameters a and b if the probability density function of X is

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\,x^{a-1}(1-x)^{b-1},\ 0 \leq x \leq 1$$

  - $\int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\ \Gamma(b)}{\Gamma(a+b)} = \mathrm{Beta}(a,b)$

- Expectation and variance
  - $E(X) = \frac{a}{(a+b)}$ and $V(X) = \frac{ab}{(a+b)^2(a+b+1)}$

Problems

- 1, 3, 5, 6, 7, 8, 10, 11, 13, 14, 23, 24, 25, 26, 33, 34, 35, 26, 37, 38