# Topic to discuss

→ LU Decomposition Method. or
LU Factorization Method or
Dolittle LU decomposition Method

→ Numerical problem

→ Homework problem.

# Triangular Factorisation Methods OR LU Decomposition Method

The co-efficient matrix A of a system of Linear equations can be factorised (or decomposed) into two triangular matrices L and U such that

$$A = LU$$

where,

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix} \text{ and, } U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

$L$ is known as lower triangular matrix and
$U$ is known as upper triangular matrix.

Once $A$ is factorised into $L$ and $U$, the system of equations,
$$Ax = b$$
can be expressed as follows
$$(LU)x = b$$
$$\text{or,} \quad L(Ux) = b$$

Let us assume that
$$Ux = y$$

Where $y$ is an unknown vector.

Substituting $Ux = y$ in equation —

we get, $Ly = b$

Now we can solve the system

$$Ax = b$$

in two stages:

1. Solve the equation, $Ly = b$ for $y$ by forward substitution

2. Solve the equation $Ux = y$ for $x$ using $z$ (found in stage 1) by back substitution.

## Summary

$$AX = B \quad \text{— (1)}$$

let $A = LU$

Eq (1) becomes,

$$LUX = B$$

$$L(UX) = B \quad \text{— (2)}$$

Let $UX = Y$ — (3)

Eqn (2) becomes

$$LY = B \quad \text{— (4)}$$

Solve eqn — (4) & (3)

Q: Solve the following system of equations by LU factorization method.

$$2x - 6y + 8z = 24$$
$$5x + 4y - 3z = 2$$
$$3x + y + 2z = 16$$

Solution: Lets convert the given system of equation into matrix form, $AX = B$.

$$\begin{bmatrix} 2 & -6 & 8 \\ 5 & 4 & -3 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix}$$

Let, $A = LU$

where, $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

i.e. $A = LU$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

$$A = LU$$

$$\begin{bmatrix} 2 & -6 & 8 \\ 5 & 4 & -3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

Now, comparing the matrix, by equating both sides, (Row wise).

clearly, $\boxed{u_{11} = 2}$ , $\boxed{u_{12} = -6}$ , $\boxed{u_{13} = 8}$

$\rightarrow l_{21}u_{11} = 5$

$$\boxed{l_{21} = \frac{5}{2}}$$

$l_{21}u_{12} + u_{22} = 4$

$$u_{22} = 4 + \frac{5}{2} \times 6$$

$$\boxed{u_{22} = 19}$$

$\rightarrow l_{21}u_{13} + u_{23} = -3$

$$u_{23} = -3 - \frac{5}{2} \times 8$$

$$\boxed{u_{23} = -23}$$

$\rightarrow l_{31}u_{11} = 3$

$$\boxed{l_{31} = \frac{3}{2}}$$

$\rightarrow l_{31}u_{12} + l_{32}u_{22} = 1$

$$l_{32} = \frac{1 + \frac{3}{2} \times 6}{19}$$

$$\boxed{l_{32} = \frac{10}{19}}$$

$\rightarrow l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2$

$$u_{33} = 2 - \frac{3}{2} \times 8 + \frac{10}{19} \times 23$$

$$u_{33} = \frac{38 - 228 + 230}{19}$$

$$\boxed{u_{33} = \frac{40}{19}}$$

It follows,

$$A = LU$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{5}{2} & 1 & 0 \\ \frac{3}{2} & \frac{10}{19} & 1 \end{bmatrix} \begin{bmatrix} 2 & -6 & 8 \\ 0 & 19 & -23 \\ 0 & 0 & \frac{40}{19} \end{bmatrix}$$

And we have the system, which can be written as,

$$AX = B$$

$$LUX = B$$

$$L(UX) = B$$

$$LY = B \quad (1) \quad \text{where, } UX = Y \quad (2)$$

As, $LY = B$ —— 1

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{5}{2} & 1 & 0 \\ \frac{3}{2} & \frac{10}{19} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix}$$

$$= \begin{bmatrix} y_1 \\ \frac{5}{2} y_1 + y_2 \\ \frac{3}{2} y_1 + \frac{10}{19} y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix}$$

Hence, $\boxed{y_1 = 24}$

$$\frac{5}{2} y_1 + y_2 = 2$$

$$y_2 = 2 - \frac{5}{2} \times 24$$

$$\boxed{y_2 = -58}$$

$$\frac{3}{2} y_1 + \frac{10}{19} y_2 + y_3 = 16$$

$$y_3 = 16 - \frac{3}{2} \times 24 + \frac{10}{19} \times 58$$

$$\boxed{y_3 = \frac{200}{19}}$$

and putting the values of $y_1, y_2$ & $y_3$ in equation — 2

$$UX = Y$$

$$\begin{bmatrix} 2 & -6 & 8 \\ 0 & 19 & -23 \\ 0 & 0 & \frac{40}{19} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 \\ -58 \\ \frac{200}{19} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x - 6y + 8z \\ 19y - 23z \\ \frac{40}{19} z \end{bmatrix} = \begin{bmatrix} 24 \\ -58 \\ \frac{200}{19} \end{bmatrix}$$

Now equating both sides,

$$\frac{40}{19} z = \frac{200}{19}$$

$$\therefore \boxed{z = 5}$$

$$19y - 23z = -58$$

$$y = \frac{-58 + 23 \times 5}{19}$$

$$\therefore \boxed{y = 3}$$

$$2x - 6y + 8z = 24$$

$$x = \frac{24 + 6 \times 3 - 8 \times 5}{2}$$

$$\therefore \boxed{x = 1}$$

# Homework Problem

Q: Solve the linear system of equations by matrix factorisation (LU method)

$$3x + 2y - 4z = 12$$
$$-x + 5y + 2z = 1$$
$$2x - 3y + 4z = -3$$

Solution :- Lets write the system of equation in matrix form $AX = B$.

where,

$$A = \begin{bmatrix} 3 & 2 & -4 \\ -1 & 5 & 2 \\ 2 & -3 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 12 \\ 1 \\ -3 \end{bmatrix}$$

Let, $A = LU$

where,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

i.e, $A = LU$

$$\Rightarrow \begin{bmatrix} 3 & 2 & -4 \\ -1 & 5 & 2 \\ 2 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Apply matrix multiplication,

$$\begin{bmatrix} 3 & 2 & -4 \\ -1 & 5 & 2 \\ 2 & -3 & 4 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Now comparing the matrix, by equating both sides, (Row wise)

so, clearly, $\boxed{u_{11} = 3}$

$\boxed{u_{12} = 2}$

$\boxed{u_{13} = -4}$

$\rightarrow \quad l_{21} u_{11} = -1$

$$\boxed{l_{21} = \frac{-1}{3}}$$

$\rightarrow l_{21}u_{12} + u_{22} = 5$

$$u_{22} = 5 - \left(-\frac{1}{3}\right) \times 2$$

$$= 5 + \frac{2}{3}$$

$$\boxed{u_{22} = \frac{17}{3}}$$

$\rightarrow l_{21}u_{13} + u_{23} = 2$

$$u_{23} = 2 - \left(-\frac{1}{3}\right) \times (-4)$$

$$= 2 - \frac{4}{3}$$

$$\boxed{u_{23} = \frac{2}{3}}$$

$\rightarrow l_{31} u_{11} = 2$

$$\boxed{l_{31} = \frac{2}{3}}$$

$\rightarrow l_{31} u_{12} + l_{32} u_{22} = -3$

$$l_{32} = \frac{-3 - \left(\frac{2}{3}\right) \times 2}{\frac{17}{3}}$$

$$= \frac{-3 - \frac{4}{3}}{\frac{17}{3}} = \frac{\frac{-13}{3}}{\frac{17}{3}}$$

$$\boxed{l_{32} = -\frac{13}{17}}$$

$$\rightarrow \quad l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4$$

$$u_{33} = 4 - \left(\frac{2}{3}\right)\times(-4) - \left(-\frac{13}{17}\right)\times\left(\frac{2}{3}\right)$$

$$= 4 + \frac{8}{3} + \frac{26}{51}$$

$$= \frac{204+136+26}{51}$$

$$= \frac{366}{51}$$

$$\boxed{u_{33} = \frac{122}{17}}$$

So, we have,

$u_{11} = 3$, $u_{12} = 2$, $u_{13} = -4$

$u_{22} = \frac{17}{3}$, $u_{23} = \frac{2}{3}$, $u_{33} = \frac{122}{17}$

$l_{21} = -\frac{1}{3}$, $l_{31} = \frac{2}{3}$, $l_{32} = -\frac{13}{17}$

As it follows,

$$A = LU$$

Now, we have, $L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 2/3 & \frac{-13}{17} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 3 & 2 & -4 \\ 0 & 17/3 & 2/3 \\ 0 & 0 & \frac{122}{17} \end{bmatrix}$

We have the system, which can be written as –

$$AX = B$$

$$\Rightarrow LUX = B \qquad [\text{As, } A = LU]$$

$$\Rightarrow L(UX) = B$$

$$\Rightarrow LY = B \quad \text{—— (1)}$$

Where, $UX = Y$ —— (2)

Equation (1) can be written as,

$$LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ 2/3 & -\frac{13}{17} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 1 \\ -3 \end{bmatrix}$$

Let solve this,

$$\begin{bmatrix} y_1 \\ -\frac{1}{3} y_1 + y_2 \\ \frac{2}{3} y_1 - \frac{13}{17} y_2 + y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 1 \\ -3 \end{bmatrix}$$

Clearly, $\boxed{y_1 = 12}$

$$\rightarrow \quad -\frac{1}{3}y_1 + y_2 = 1$$

$$y_2 = 1 + \frac{\cancel{12}^{\,4}}{\cancel{3}}$$

$$\boxed{y_2 = 5}$$

$$\rightarrow \quad \frac{2}{3}y_1 - \frac{13}{17}y_2 + y_3 = -3$$

$$y_3 = -3 - \frac{2}{\cancel{3}} \times \cancel{12}^{\,4} + \frac{13}{17} \times 5$$

$$= -3 - 8 + \frac{65}{17}$$

$$\boxed{y_3 = \frac{-122}{17}}$$

Now, Equation — 2 $Ux = y$ can be written as,

$$\begin{bmatrix} 3 & 2 & -4 \\ 0 & 17/3 & 2/3 \\ 0 & 0 & \frac{122}{17} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Or,

$$\begin{bmatrix} 3x + 2y - 4z \\ \frac{17}{3}y + \frac{2}{3}z \\ \frac{122}{17}z \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \\ \frac{-122}{17} \end{bmatrix}$$

So, after solving, we get

$x = -24$, $y = 16$, $z = -1$

Now compare,

$$\frac{122}{17}z = -\frac{122}{17} \quad \therefore z = -1$$

or, $$\frac{17}{3}y + \frac{2}{3}z = 5$$

$$\Rightarrow y = \frac{5 + \frac{2}{3}}{\frac{17}{3}} \quad \therefore y = 1$$

and

$$3x + 2y - 4z = 12$$

$$x = \frac{12 - 2 + 4}{3} \quad \therefore x = 2$$

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