Masters Project

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Introduction

In the study of partial differential equations, the advection-diffusion reaction equation is used to describe how physical quantities, such as liquids, change over time due to drift (or advection), random spreading (or diffusion), and a reaction or reactions such as generation or consumption. In this paper, we begin with the steady state advection-diffusion reaction equation:

$$-D\Delta u + \boldsymbol{V} \cdot \nabla u + \alpha u = f \quad \text{in } \Omega \tag{1}$$

with Dirichlet boundary conditions, $u = 0$ on $\partial\Omega$. Here,

- $D > 0$ is the constant of diffusivity
- $\boldsymbol{V} \in R^d, d \in \{1, 2, 3\}$ is a constant vector field
- $\alpha \geq 0$ generates a linear reaction
- We assume $f \in L^2(\Omega)$

Theory

For what assumptions on $f, \boldsymbol{V}$, and D does a weak solution exist?

Def. Coercive Functional

A bilinear functional ϕ on a normed space E is called coercive if there exists a positive constant K such that $\phi(x,x) \geq K||x||^2$ for all $x \in E$.

Theorem Lax-Milgram

Let ϕ be a bounded coercive bilinear form on a Hilbert space H. Then, for every bounded linear functional f on H, there exists a unique $x_f \in H$ such that $f(x) = \phi(x, x_f)$ for all $x \in H$.

Let $\Phi = H_0^1(\Omega)$, that is Φ is the space of functions with first-order weak derivatives that disappear in the boundary. on Ω. To use Lax-Milgram to guarantee existence of a weak solution, we must find $u \in \Phi$ such that

$$a(u, \phi) = l(\phi) \;\; \forall \phi \in \Phi,$$

where $|a(u,\phi) \leq C||u||_\Phi ||\phi||_\Phi$ and $a(\phi,\phi) \geq K||\phi||_\Phi^2$. Consider the case when $\boldsymbol{V} = 0$ so our equation becomes:

$$-D\Delta u + \alpha u = f.$$

Then

$$a(u,\phi) = \int_\Omega D\nabla u \cdot \nabla\phi + \int_\Omega \alpha u \phi.$$

Hence, for any $\phi \in \Phi$,

$$a(\phi, \phi) = D||\nabla \phi||^2 + \alpha||\phi||^2.$$

Recall that the norm on H^1 is given by

$$||\phi||_{H^1} = \left(||\phi||^2_{L^2(\Omega)} + ||\nabla \phi||^2_{L^2(\Omega)}\right)^{1/2}.$$

Then

$$a(\phi, \phi) \geq \min\{D, \alpha\}||\phi||^2_{H^1}.$$

This gives the coercivity condition for Lax-Milgram. For boundedness, recall the Poincare inequality:

$$||u - u_\Omega||_{L^2(\Omega)} \leq A||\nabla u||_{L^2(\Omega)} \text{ for some constant } A.$$

Define $C = D + \alpha A^2$ From above,

$$\begin{aligned}
a(u, \phi) &= \int_\Omega D\nabla u \cdot \nabla \phi + \int_\Omega \alpha u \phi \\
&\leq D||\nabla u||\ ||\nabla \phi|| + \alpha||u||\ ||\phi|| \\
&\leq D||\nabla u||\ ||\nabla \phi|| + \alpha A^2||\nabla u||\ ||\nabla \phi|| \\
&\leq C||\nabla u||\ ||\nabla \phi|| \\
&\leq C||u||_{H^1}||\phi||_{H^1}.
\end{aligned}$$

Thus, we have satisfied boundedness for Lax-Milgram. Hence, a weak solution exists.