Johns Hopkins University
Department of Applied Mathematics & Statistics
Lecturer: Maxim Bichuch

Fall 2015

EN 443
Financial Computing in C++
Final Project

Due Fri, Dec 18 at 11:59pm.

Instructions

This test is open notes. You may use the course notes, HW solutions, the code published on Blackboard, and published books (paper or electronic). Additionally, you may use C++ references websites on the Internet, such as cplusplus.com. You may not use **any** other resources without an **explicit** permission form the instructional staff of this course. Write all the references that you have used.

The exam is individual work. You may not discuss it with anybody, except the instructional staff of this course. In case of doubt/question please email the instructional staff. We will be happy to answer your questions. You may **not utilize any code** available on the web, with the exception of the one that has been posted on the Blackboard of this course.

Please comment your code thoroughly so we can understand what you are trying to do. This is especially important if you are not getting the results we expect. Produce a write up describing your solution and (if applicable) the algorithm you used in your implementation. Comment your results (e.g. in a finite difference method comment on the convergence). Please submit your answers packaged according to the HW instructions on blackboard. You should use C++ classes when designing solutions to problems. An extra 10 points will be given for a good design of every problem below. A crude sketch of a couple of classes that can be used to design the solution for both of the problems is attached. These classes rely on the classes of the Monte Carlo example (in the Examples folder on Blackboard).

Please write your full name, date and sign below.

I attest that I have completed this exam without unauthorized assistance from any person, materials, or device:

Full Name : Minhan Li

Signature : Minhan Li

See the Johns Hopkins Handbook on Graduate Student Misconduct Policy:
`engineering.jhu.edu/wp-content/uploads/2013/07/Grad-Student-Misconduct-Policy.pdf`

1. (60 points). **Swing Option:** Fix maturity date T and consider the standard Black-Scholes model for the stock price and its dynamics under the risk-neutral probability measure $\mathbb{P}$ given by:
$$dS_t = rS_t dt + \sigma S_t dW_t.$$
Let $0 < T_1 < ... < T_n = T$ be given dates and let $K > 0$ be a strike price. Let $m \leq n$. A swing option with m possible exercises confers the right to receive payment $S_{T_j} - K_j$ on at most m of the dates $T_1, .., T_n$. In other words, the owner of the option chooses on each date T_j whether or not to receive the payment $S_{T_j} - K$ on that date. He can choose to receive payments on at most m of these dates.

   Let $v(T_{j-}, S, k)$ denote the price of the swing option at time T_j immediately **before** the exercise decision if the asset price is S and there are k possible exercises remaining.

   Let $v(T_j, S, k)$ denote the price of the swing option at time immediately **after** the exercise decision at time T_j if the asset price is S and there are k possible exercises remaining after the exercise time T_j. If exercise is not optimal at time T_j then $v(T_{j-}, S, k) = v(T_j, S, k)$.

   a) Specify the formula for $v(T_{n-}, S, k)$. Note that the formula has two cases, depending on whether $k \geq 1$ or $k = 0$.
   b) For $j = n-1, .., 1$ specify the formula for $v(T_{j-}, S, k)$ in terms of $v(T_j, S, k)$ and provided $k \geq 1$ in terms of $v(T_j, S, k-1)$.
   c) Note that $v(t, S, k)$ is governed by the Black-Scholes equation between each pair of exercise dates. In particular in the interval $[T_j, T_{j+1})$ taking $v(T_{(j+1)-}, S_{T_{(j+1)}}, k)$ as the terminal condition, using Black-Scholes it is possible to solve for $v(t, S_t, k)$, for $t \in [T_j, T_{j+1})$. This would give $v(T_{j-}, S_{T_j}, k)$, by using Step 2. This can be done for every k.

      Use Crank-Nicolson finite difference scheme to solve for $v(0, 50, 2)$, i.e. with $t = 0, S_0 = 50, K = 50, k = 2, T = 1, T_1 = 1/3, T_2 = 2/3, T_3 = T, r = 0.07, \sigma = 0.2$.
   d) Can there be a closed form solution to this problem? Explain why or why not.

      Hint: Recall that it is never optimal to exercise an American option on a non dividend paying stock before maturity.

2. (40 points). **Interest rate caplet in Hull-White model:** In this question we are interested in computing the price of an interest rate caplet. In homework 11, we have already build and calibrated an interest rate tree for the Hull-White model:
$$dr_t = \alpha(m_t - r_t)\,dt + \sigma\,dW_t,$$
where $\alpha, \sigma > 0$ are positive constants, m_t is a deterministic process, and W is a standard Brownian Motion. Fix $T > 0$ and an integer N, and let $\Delta T = \frac{T}{N}$. Moreover recall that we have discretized the interest rate process as
$$r_{i+1} - r_i - = \alpha\left(m_{(i+1)\Delta T} - r_i\right)\Delta T + \sigma \Delta W_i.$$
Recall the definition of the **continuously compounded** forward rate at time s for investing between time t and $t + \delta$, $\delta > 0$
$$F(s, t, t+\delta) = \frac{1}{\delta} \log \frac{B(s,t)}{B(s, t+\delta)}.$$

The **simply compounded** forward rate is

$$L(s,t,t+\delta) = \frac{B(s,t) - B(s,t+\delta)}{\delta B(s,t+\delta)}.$$

Finally, if we assume that $s = t$, this rate becomes

$$L(t,t+\delta) = \frac{1 - B(t,t+\delta)}{\delta B(t,t+\delta)}.$$

This is known as the **LIBOR rate**, the simply compounded rate for a loan between times t and $t+\delta$ as set at time t. An interest rate caplet provides insurance against the LIBOR rate rising too high. Given a set period between times t and $(t+\delta)$, the caplet pays the difference between the LIBOR rate $L(t,t+\delta)$ and a set rate K if the LIBOR rate $L(t,t+\delta)$ is greater than the (cap) rate K. This payment happens, **at the end of the period** at time $(t+\delta)$. In other words the cash flow at time $(t+\delta)$ is $\bar{N}\delta(L(t,t+\delta)-K)^+$, where $\bar{N}$ is the notional. Thus its price at time zero is

$$\mathbb{E}\left[D_{t+\delta}\bar{N}\delta(L(t,t+\delta)-K)^+\right],$$

where $D_{t+\delta}$ is the discount factor at time $(t+\delta)$. Recall that in this setting we had

$$\begin{aligned} D_{i\Delta T} &= \mathrm{e}^{-\Delta T(r_0+r_1+\ldots+r_{i-1})}, \\ B(0,i\Delta T) &= \mathbb{E}\left[D_{i\Delta T}\right]. \end{aligned}$$

With these definitions compute the price of a caplet with maturity $t = 5, K = 1\%, \bar{N} = 1M, r(0) = 0, \delta = 0.25$, assume the model parameters are as before $\alpha = 1, \sigma = 0.2$.

Good Luck!