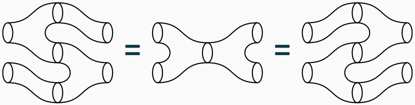
Classifying extended 2D TQFTs

XII Encuentro de Jóvenes Topólogos

Santiago Pareja Pérez

Unizar–UCM

24th October 2024



Outline: questions and keywords

Questions:

- What are TQFTs?
- How do we classify 2D TQFTs?
- Can we generalize this to higher dimensions?
- How does our generalization relate to the original 2D case?

Keywords:

- Symmetric monoidal functors.
- Frobenius algebras.
- *n*-categories and the Cobordism Hypothesis.
- Morita bicategory.

Outline: questions and keywords

Questions:

- What are TQFTs?
- How do we classify 2D TQFTs?
- Can we generalize this to higher dimensions?
- How does our generalization relate to the original 2D case?

Keywords:

- Symmetric monoidal functors.
- Frobenius algebras.
- n-categories and the Cobordism Hypothesis.
- Morita bicategory.

Outline: questions and keywords

Questions:

- What are TQFTs?
- How do we classify 2D TQFTs?
- Can we generalize this to higher dimensions?
- How does our generalization relate to the original 2D case?

Keywords:

- Symmetric monoidal functors.
- Frobenius algebras.
- *n*-categories and the Cobordism Hypothesis.
- Morita bicategory.

Outline: questions and keywords

Questions:

- What are TQFTs?
- How do we classify 2D TQFTs?
- Can we generalize this to higher dimensions?
- How does our generalization relate to the original 2D case?

Keywords:

- Symmetric monoidal functors.
- Frobenius algebras.
- n-categories and the Cobordism Hypothesis.
- Morita bicategory.

Outline: questions and keywords

Questions:

- What are TQFTs?
- How do we classify 2D TQFTs?
- Can we generalize this to higher dimensions?
- How does our generalization relate to the original 2D case?

Keywords:

- Symmetric monoidal functors.
- Frobenius algebras.
- n-categories and the Cobordism Hypothesis.
- Morita bicategory.

Outline: questions and keywords

Questions:

- What are TQFTs?
- How do we classify 2D TQFTs?
- Can we generalize this to higher dimensions?
- How does our generalization relate to the original 2D case?

Keywords:

- Symmetric monoidal functors.
- Frobenius algebras.
- *n*-categories and the Cobordism Hypothesis.
- Morita bicategory.

Outline: questions and keywords

Questions:

- What are TQFTs?
- How do we classify 2D TQFTs?
- Can we generalize this to higher dimensions?
- How does our generalization relate to the original 2D case?

Keywords:

- Symmetric monoidal functors.
- Frobenius algebras.
- n-categories and the Cobordism Hypothesis.
- Morita bicategory.

Outline: questions and keywords

Questions:

- What are TQFTs?
- How do we classify 2D TQFTs?
- Can we generalize this to higher dimensions?
- How does our generalization relate to the original 2D case?

Keywords:

- Symmetric monoidal functors.
- Frobenius algebras.
- n-categories and the Cobordism Hypothesis.
- Morita bicategory.

Cobordisms and TQFTs

We assume everything smooth and compact.

Let M and N be two closed oriented $(n-1)$-manifolds.

A ***cobordism*** $B : M \to N$ is an n-manifold with boundary endowed with a diffeomorphism $\partial B \cong \overline{M} \sqcup N$.

$\mathbb{S}^1 \to \mathbb{S}^1$ $\quad \mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1$ $\quad \mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathbb{S}^1$ $\quad \varnothing \to \mathbb{S}^1 \sqcup \mathbb{S}^1$ $\quad \varnothing \to \varnothing$

M is the ***in-boundary*** and N is the ***out-boundary***.

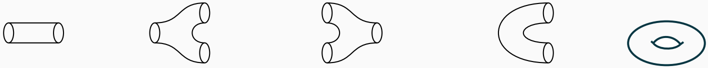
A cobordism needs not be connected:

$\mathbb{S}^1 \to \mathbb{S}^1$ $\quad \bigsqcup_6 \mathbb{S}^1 \to \bigsqcup_6 \mathbb{S}^1$

We assume everything smooth and compact.

Let M and N be two closed oriented $(n-1)$-manifolds.

A ***cobordism*** $B\colon M \to N$ is an n-manifold with boundary endowed with a diffeomorphism $\partial B \cong \overline{M} \sqcup N$.



$\mathbb{S}^1 \to \mathbb{S}^1$ $\quad$ $\mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1$ $\quad$ $\mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathbb{S}^1$ $\quad$ $\varnothing \to \mathbb{S}^1 \sqcup \mathbb{S}^1$ $\quad$ $\varnothing \to \varnothing$

M is the ***in-boundary*** and N is the ***out-boundary***.

A cobordism needs not be connected:

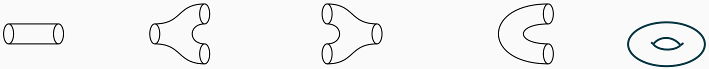
$\mathbb{S}^1 \to \mathbb{S}^1$ $\quad$ $\bigsqcup_6 \mathbb{S}^1 \to \bigsqcup_6 \mathbb{S}^1$

Cobordisms

We assume everything smooth and compact.

Let M and N be two closed oriented $(n-1)$-manifolds.

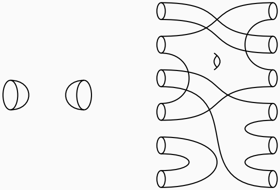
A ***cobordism*** $B\colon M \to N$ is an n-manifold with boundary endowed with a diffeomorphism $\partial B \cong \overline{M} \sqcup N$.



$\mathbb{S}^1 \to \mathbb{S}^1$ $\quad$ $\mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1$ $\quad$ $\mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathbb{S}^1$ $\quad$ $\varnothing \to \mathbb{S}^1 \sqcup \mathbb{S}^1$ $\quad$ $\varnothing \to \varnothing$

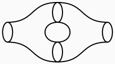
M is the ***in-boundary*** and N is the ***out-boundary***.

A cobordism needs not be connected:

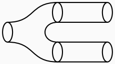


$\mathbb{S}^1 \to \mathbb{S}^1$ $\quad$ $\bigsqcup_6 \mathbb{S}^1 \to \bigsqcup_6 \mathbb{S}^1$

We can ***compose*** cobordisms by gluing.



$\mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathbb{S}^1$



$\mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1$

The identities are the cylinders $M \times [0,1] \colon M \to M$.

We can add cobordisms by taking their disjoint union.

$\mathbb{S}^1 \sqcup (\mathbb{S}^1 \sqcup \mathbb{S}^1) \to (\mathbb{S}^1 \sqcup \mathbb{S}^1) \sqcup \mathbb{S}^1$

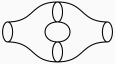
This is a ***monoidal structure*** with unit the empty manifold: $M \sqcup \varnothing \cong M$.

We can freely interchange connected components.
These *twist cobordisms* give a ***symmetric structure***.

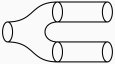
$M \sqcup N \to N \sqcup M$

Gluing and adding cobordisms

We can ***compose*** cobordisms by gluing.



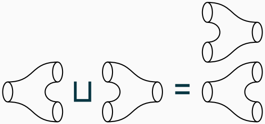
$\mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathbb{S}^1$



$\mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1$

The identities are the cylinders $M \times [0,1] \colon M \to M$.

We can add cobordisms by taking their disjoint union.



$\mathbb{S}^1 \sqcup (\mathbb{S}^1 \sqcup \mathbb{S}^1) \to (\mathbb{S}^1 \sqcup \mathbb{S}^1) \sqcup \mathbb{S}^1$

This is a ***monoidal structure*** with unit the empty manifold: $M \sqcup \varnothing \cong M$.

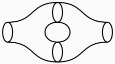
We can freely interchange connected components.

These *twist cobordisms* give a ***symmetric structure***.

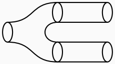
$M \sqcup N \to N \sqcup M$

Gluing and adding cobordisms

We can ***compose*** cobordisms by gluing.



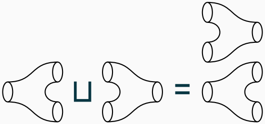
$\mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathbb{S}^1$



$\mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1$

The identities are the cylinders $M \times [0,1] \colon M \to M$.

We can add cobordisms by taking their disjoint union.



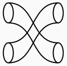
$\mathbb{S}^1 \sqcup (\mathbb{S}^1 \sqcup \mathbb{S}^1) \to (\mathbb{S}^1 \sqcup \mathbb{S}^1) \sqcup \mathbb{S}^1$

This is a ***monoidal structure*** with unit the empty manifold: $M \sqcup \varnothing \cong M$.

We can freely interchange connected components.

These *twist cobordisms* give a ***symmetric structure***.

N M



M N

$M \sqcup N \to N \sqcup M$

$\mathbf{Cob}_n$, the category of n-cobordisms

n-cobordisms assemble into a symmetric monoidal category, $\mathbf{Cob}_n$:

Objects Closed $(n-1)$-manifolds M, N.

Morphisms n-cobordisms $B\colon M \to N$, up to diffeomorphism.

Identities "Cylinders" $M \times [0,1]\colon M \to M$.

Composition Gluing of cobordisms.

Monoid Disjoint union $B \sqcup B' : M \sqcup M' \to N \sqcup N'$.

Unit Empty manifold $\varnothing$.

Twists Twist cobordisms $M \sqcup N \to N \sqcup M$.

One well-understood symmetric monoidal category is $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$, the category of vector spaces equipped with tensor product and the usual interchange of factors.
(i.e., the twist maps $\sigma_{V,W}\colon V \otimes W \to W \otimes V$ are given by $v \otimes w \mapsto w \otimes v$).

We can use $\mathbf{Vect}_{\Bbbk}$ to study $\mathbf{Cob}_n$, via maps $\mathbf{Cob}_n \to \mathbf{Vect}_{\Bbbk}$.

$\mathbf{Cob}_n$, the category of n-cobordisms

n-cobordisms assemble into a symmetric monoidal category, $\mathbf{Cob}_n$:

Objects Closed $(n-1)$-manifolds M, N.

Morphisms n-cobordisms $B\colon M \to N$, up to diffeomorphism.

Identities "Cylinders" $M \times [0,1]\colon M \to M$.

Composition Gluing of cobordisms.

Monoid Disjoint union $B \sqcup B' : M \sqcup M' \to N \sqcup N'$.

Unit Empty manifold $\varnothing$.

Twists Twist cobordisms $M \sqcup N \to N \sqcup M$.

One well-understood symmetric monoidal category is $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$, the category of vector spaces equipped with tensor product and the usual interchange of factors.
(i.e., the twist maps $\sigma_{V,W}\colon V \otimes W \to W \otimes V$ are given by $v \otimes w \mapsto w \otimes v$).

We can use $\mathbf{Vect}_{\Bbbk}$ to study $\mathbf{Cob}_n$, via maps $\mathbf{Cob}_n \to \mathbf{Vect}_{\Bbbk}$.

$\mathbf{Cob}_n$, the category of n-cobordisms

n-cobordisms assemble into a symmetric monoidal category, $\mathbf{Cob}_n$:

Objects Closed $(n-1)$-manifolds M, N.

Morphisms n-cobordisms $B\colon M \to N$, up to diffeomorphism.

Identities "Cylinders" $M \times [0,1]\colon M \to M$.

Composition Gluing of cobordisms.

Monoid Disjoint union $B \sqcup B' : M \sqcup M' \to N \sqcup N'$.

Unit Empty manifold $\varnothing$.

Twists Twist cobordisms $M \sqcup N \to N \sqcup M$.

One well-understood symmetric monoidal category is $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$, the category of vector spaces equipped with tensor product and the usual interchange of factors.
(i.e., the twist maps $\sigma_{V,W}\colon V \otimes W \to W \otimes V$ are given by $v \otimes w \mapsto w \otimes v$).

We can use $\mathbf{Vect}_{\Bbbk}$ to study $\mathbf{Cob}_n$, via maps $\mathbf{Cob}_n \to \mathbf{Vect}_{\Bbbk}$.

$\mathbf{Cob}_n$, the category of n-cobordisms

n-cobordisms assemble into a symmetric monoidal category, $\mathbf{Cob}_n$:

Objects Closed $(n-1)$-manifolds M, N.

Morphisms n-cobordisms $B\colon M \to N$, up to diffeomorphism.

Identities "Cylinders" $M \times [0,1]\colon M \to M$.

Composition Gluing of cobordisms.

Monoid Disjoint union $B \sqcup B' : M \sqcup M' \to N \sqcup N'$.

Unit Empty manifold $\varnothing$.

Twists Twist cobordisms $M \sqcup N \to N \sqcup M$.

One well-understood symmetric monoidal category is $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$, the category of vector spaces equipped with tensor product and the usual interchange of factors.
(i.e., the twist maps $\sigma_{V,W}\colon V \otimes W \to W \otimes V$ are given by $v \otimes w \mapsto w \otimes v$).

We can use $\mathbf{Vect}_{\Bbbk}$ to study $\mathbf{Cob}_n$, via maps $\mathbf{Cob}_n \to \mathbf{Vect}_{\Bbbk}$.

$\mathbf{Cob}_n$, the category of n-cobordisms

n-cobordisms assemble into a symmetric monoidal category, $\mathbf{Cob}_n$:

Objects Closed $(n-1)$-manifolds M, N.

Morphisms n-cobordisms $B\colon M \to N$, up to diffeomorphism.

Identities "Cylinders" $M \times [0,1]\colon M \to M$.

Composition Gluing of cobordisms.

Monoid Disjoint union $B \sqcup B'\colon M \sqcup M' \to N \sqcup N'$.

Unit Empty manifold $\varnothing$.

Twists Twist cobordisms $M \sqcup N \to N \sqcup M$.

One well-understood symmetric monoidal category is $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$, the category of vector spaces equipped with tensor product and the usual interchange of factors.
(i.e., the twist maps $\sigma_{V,W}\colon V \otimes W \to W \otimes V$ are given by $v \otimes w \mapsto w \otimes v$).

We can use $\mathbf{Vect}_{\Bbbk}$ to study $\mathbf{Cob}_n$, via maps $\mathbf{Cob}_n \to \mathbf{Vect}_{\Bbbk}$.

$\mathbf{Cob}_n$, the category of n-cobordisms

n-cobordisms assemble into a symmetric monoidal category, $\mathbf{Cob}_n$:

Objects	Closed $(n-1)$-manifolds M, N.
Morphisms	n-cobordisms $B\colon M \to N$, up to diffeomorphism.
Identities	"Cylinders" $M \times [0,1]\colon M \to M$.
Composition	Gluing of cobordisms.
Monoid	Disjoint union $B \sqcup B' \colon M \sqcup M' \to N \sqcup N'$.
Unit	Empty manifold $\varnothing$.
Twists	Twist cobordisms $M \sqcup N \to N \sqcup M$.

One well-understood symmetric monoidal category is $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$, the category of vector spaces equipped with tensor product and the usual interchange of factors.
(i.e., the twist maps $\sigma_{V,W}\colon V \otimes W \to W \otimes V$ are given by $v \otimes w \mapsto w \otimes v$).

We can use $\mathbf{Vect}_{\Bbbk}$ to study $\mathbf{Cob}_n$, via maps $\mathbf{Cob}_n \to \mathbf{Vect}_{\Bbbk}$.

Topological quantum field theories

A ***TQFT*** is a rule $Z\colon \mathbf{Cob}_n \to \mathbf{Vect}_{\Bbbk}$ which assigns

- closed $(n-1)$-manifold M $\rightsquigarrow$ $\Bbbk$-vector space $Z(M)$.
- n-cobordism $B\colon M \to N$ $\rightsquigarrow$ $\Bbbk$-linear map $Z(B)\colon Z(M) \to Z(N)$.

According to the following laws.

- Diffeomorphic cobordisms have equal image: $B \cong B' \rightsquigarrow Z(B) = Z(B')$.
- Cylinders go to identities: $Z(\;) = \mathrm{id}_{Z(M)}$.
- Gluing cobordism is composing functions: $Z(\;) = Z(\;) \circ Z(\;)$.

- Disjoint union is tensor product: $Z(\;) = Z(\;) \otimes Z(\;)$.
- The empty manifold goes to the ground field: $Z(\varnothing) = \Bbbk$.

- The interchange the order of factors: $Z(\;)\colon m \otimes n \mapsto n \otimes m$.

Symmetric monoidal functor.

Topological quantum field theories

A ***TQFT*** is a rule $Z\colon \mathbf{Cob}_n \to \mathbf{Vect}_{\Bbbk}$ which assigns

- closed $(n-1)$-manifold $M \rightsquigarrow$ $\Bbbk$-vector space $Z(M)$.
- n-cobordism $B\colon M \to N \rightsquigarrow$ $\Bbbk$-linear map $Z(B)\colon Z(M) \to Z(N)$.

According to the following laws.

- Diffeomorphic cobordisms have equal image: $B \cong B' \rightsquigarrow Z(B) = Z(B')$.
- Cylinders go to identities: $Z(\ ) = \mathrm{id}_{Z(M)}$.
- Gluing cobordism is composing functions: $Z(\ ) = Z(\ ) \circ Z(\ )$.

- Disjoint union is tensor product: $Z\left(\ \right) = Z(\ ) \otimes Z(\ )$.
- The empty manifold goes to the ground field: $Z(\varnothing) = \Bbbk$.

- The interchange the order of factors: $Z(\ )\colon m \otimes n \mapsto n \otimes m$.

Symmetric monoidal functor.

Topological quantum field theories

A ***TQFT*** is a rule $Z\colon \mathbf{Cob}_n \to \mathbf{Vect}_{\Bbbk}$ which assigns

- closed $(n-1)$-manifold M $\rightsquigarrow$ $\Bbbk$-vector space $Z(M)$.
- n-cobordism $B\colon M \to N$ $\rightsquigarrow$ $\Bbbk$-linear map $Z(B)\colon Z(M) \to Z(N)$.

According to the following laws.

- Diffeomorphic cobordisms have equal image: $B \cong B' \rightsquigarrow Z(B) = Z(B')$.
- Cylinders go to identities: $Z(\ ) = \mathrm{id}_{Z(M)}$.
- Gluing cobordism is composing functions: $Z(\ ) = Z(\ ) \circ Z(\ )$.

- Disjoint union is tensor product: $Z\left(\ \right) = Z(\ ) \otimes Z(\ )$.
- The empty manifold goes to the ground field: $Z(\varnothing) = \Bbbk$.

- The interchange the order of factors: $Z(\ )\colon m \otimes n \mapsto n \otimes m$.

Symmetric monoidal functor.

Topological quantum field theories

A ***TQFT*** is a rule $Z\colon \mathbf{Cob}_n \to \mathbf{Vect}_{\Bbbk}$ which assigns

- closed $(n-1)$-manifold $M \rightsquigarrow$ $\Bbbk$-vector space $Z(M)$.
- n-cobordism $B\colon M \to N \rightsquigarrow$ $\Bbbk$-linear map $Z(B)\colon Z(M) \to Z(N)$.

According to the following laws.

- Diffeomorphic cobordisms have equal image: $B \cong B' \rightsquigarrow Z(B) = Z(B')$.
- Cylinders go to identities: $Z(\quad) = \mathrm{id}_{Z(M)}$.
- Gluing cobordism is composing functions: $Z(\quad) = Z(\quad) \circ Z(\quad)$.

- Disjoint union is tensor product: $Z\left(\quad\right) = Z(\quad) \otimes Z(\quad)$.
- The empty manifold goes to the ground field: $Z(\varnothing) = \Bbbk$.

- The interchange the order of factors: $Z(\quad)\colon m \otimes n \mapsto n \otimes m$.

Symmetric monoidal functor.

Topological quantum field theories

A ***TQFT*** is a rule $Z\colon \mathbf{Cob}_n \to \mathbf{Vect}_{\Bbbk}$ which assigns

- closed $(n-1)$-manifold M $\rightsquigarrow$ $\Bbbk$-vector space $Z(M)$.
- n-cobordism $B\colon M \to N$ $\rightsquigarrow$ $\Bbbk$-linear map $Z(B)\colon Z(M) \to Z(N)$.

According to the following laws.

- Diffeomorphic cobordisms have equal image: $B \cong B' \rightsquigarrow Z(B) = Z(B')$.
- Cylinders go to identities: $Z(\;) = \mathrm{id}_{Z(M)}$.
- Gluing cobordism is composing functions: $Z(\;) = Z(\;) \circ Z(\;)$.

- Disjoint union is tensor product: $Z\left(\;\right) = Z(\;) \otimes Z(\;)$.
- The empty manifold goes to the ground field: $Z(\varnothing) = \Bbbk$.

- The interchange the order of factors: $Z(\;)\colon m \otimes n \mapsto n \otimes m$.

Symmetric monoidal functor.

Topological quantum field theories

A ***TQFT*** is a rule $Z\colon \mathbf{Cob}_n \to \mathbf{Vect}_\Bbbk$ which assigns

- closed $(n-1)$-manifold M $\rightsquigarrow$ $\Bbbk$-vector space $Z(M)$.
- n-cobordism $B\colon M \to N$ $\rightsquigarrow$ $\Bbbk$-linear map $Z(B)\colon Z(M) \to Z(N)$.

According to the following laws.

- Diffeomorphic cobordisms have equal image: $B \cong B' \rightsquigarrow Z(B) = Z(B')$.
- Cylinders go to identities: $Z($  $) = \mathrm{id}_{Z(M)}$.
- Gluing cobordism is composing functions: $Z($  $) = Z($  $) \circ Z($  $)$.

- Disjoint union is tensor product: $Z\Big($  $\Big) = Z($  $) \otimes Z($  $)$.
- The empty manifold goes to the ground field: $Z(\varnothing) = \Bbbk$.

- The  interchange the order of factors: $Z($  $)\colon m \otimes n \mapsto n \otimes m$.

Symmetric monoidal functor.

Topological quantum field theories

A ***TQFT*** is a rule $Z\colon \mathbf{Cob}_n \to \mathbf{Vect}_{\Bbbk}$ which assigns

- closed $(n-1)$-manifold M $\rightsquigarrow$ $\Bbbk$-vector space $Z(M)$.
- n-cobordism $B\colon M \to N$ $\rightsquigarrow$ $\Bbbk$-linear map $Z(B)\colon Z(M) \to Z(N)$.

According to the following laws.

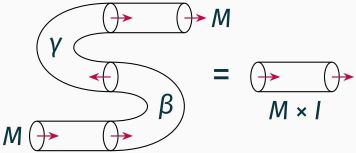
- ∘ Diffeomorphic cobordisms have equal image: $B \cong B' \rightsquigarrow Z(B) = Z(B')$.
- ∘ Cylinders go to identities: $Z(\ ) = \mathrm{id}_{Z(M)}$.
- ∘ Gluing cobordism is composing functions: $Z(\ ) = Z(\ ) \circ Z(\ )$.

- ▫ Disjoint union is tensor product: $Z\left(\ \right) = Z(\ ) \otimes Z(\ )$.
- ▫ The empty manifold goes to the ground field: $Z(\varnothing) = \Bbbk$.

- ◇ The interchange the order of factors: $Z(\ )\colon m \otimes n \mapsto n \otimes m$.

Symmetric monoidal functor.

Zorro's Lemma: decomposing cylinders



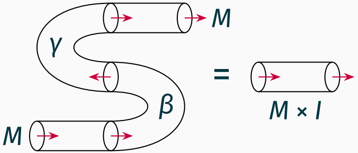
$$\begin{tikzcd} M & (M \sqcup \overline{M}) \sqcup M \arrow[r, "\beta\sqcup\mathrm{id}"] & \varnothing \sqcup M \arrow[d, equal] \\ M \sqcup \varnothing \arrow[u, equal] \arrow[r, "\mathrm{id}\sqcup\gamma"'] & M \sqcup (\overline{M} \sqcup M) \arrow[u, equal] & M \end{tikzcd} \quad = \quad M \times I.$$

Now we evaluate a TQFT Z on this diagram.
Let $V := Z(M)$ and $W := Z(\overline{M})$, and also $\mathrm{ev} := Z(\beta)$ and $\mathrm{coev} := Z(\gamma)$:

$$\begin{tikzcd} V & (V \otimes W) \otimes V \arrow[r, "\mathrm{ev}\otimes\mathrm{id}"] & \Bbbk \otimes V \arrow[d, equal] \\ V \otimes \Bbbk \arrow[u, equal] \arrow[r, "\mathrm{id}\otimes\mathrm{coev}"'] & V \otimes (W \otimes V) \arrow[u, equal] & V \end{tikzcd} \quad = \quad \mathrm{id}_V.$$

So $V \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} V \otimes W \otimes V \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} V$ is $\mathrm{id}_V : V \to V$.

Zorro's Lemma: decomposing cylinders



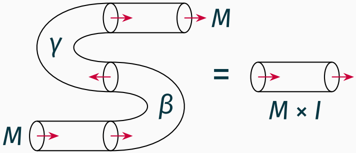
$$\begin{tikzcd}
M \arrow[d, equal] & (M \sqcup \overline{M}) \sqcup M \arrow[r, "\beta\sqcup\mathrm{id}"] \arrow[d, phantom, "\cong" rotate=90] & \varnothing \sqcup M \arrow[d, equal] \\
M \sqcup \varnothing \arrow[r, "\mathrm{id}\sqcup\gamma"'] & M \sqcup (\overline{M} \sqcup M) & M
\end{tikzcd} \quad = \quad M \times I.$$

Now we evaluate a TQFT Z on this diagram.
Let $V \coloneqq Z(M)$ and $W \coloneqq Z(\overline{M})$, and also $\mathrm{ev} \coloneqq Z(\beta)$ and $\mathrm{coev} \coloneqq Z(\gamma)$:

$$\begin{tikzcd}
V \arrow[d, equal] & (V \otimes W) \otimes V \arrow[r, "\mathrm{ev}\otimes\mathrm{id}"] \arrow[d, phantom, "\cong" rotate=90] & \Bbbk \otimes V \arrow[d, equal] \\
V \otimes \Bbbk \arrow[r, "\mathrm{id}\otimes\mathrm{coev}"'] & V \otimes (W \otimes V) & V
\end{tikzcd} \quad = \quad \mathrm{id}_V.$$

So $V \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} V \otimes W \otimes V \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} V$ is $\mathrm{id}_V : V \to V$.

Zorro's Lemma: decomposing cylinders



$$\begin{tikzcd}
M \arrow[d, equal] & (M \sqcup \overline{M}) \sqcup M \arrow[r, "\beta\sqcup\mathrm{id}"] \arrow[d, phantom, sloped, "\cong"] & \varnothing \sqcup M \arrow[d, equal] \\
M \sqcup \varnothing \arrow[r, "\mathrm{id}\sqcup\gamma"'] & M \sqcup (\overline{M} \sqcup M) & M
\end{tikzcd} \quad = \quad M \times I.$$

Now we evaluate a TQFT Z on this diagram.
Let $V \coloneqq Z(M)$ and $W \coloneqq Z(\overline{M})$, and also ev $\coloneqq Z(\beta)$ and coev $\coloneqq Z(\gamma)$:

$$\begin{tikzcd}
V \arrow[d, equal] & (V \otimes W) \otimes V \arrow[r, "\mathrm{ev}\otimes\mathrm{id}"] \arrow[d, phantom, sloped, "\cong"] & \Bbbk \otimes V \arrow[d, equal] \\
V \otimes \Bbbk \arrow[r, "\mathrm{id}\otimes\mathrm{coev}"'] & V \otimes (W \otimes V) & V
\end{tikzcd} \quad = \quad \mathrm{id}_V.$$

So $V \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} V \otimes W \otimes V \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} V$ is $\mathrm{id}_V \colon V \to V$.

Zorro's Lemma: finite dimensionality

The composition $V \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} V \otimes W \otimes V \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} V$ is id_V.
This forces V to have finite dimension, as follows.

- $\mathrm{coev}\colon \Bbbk \to W \otimes V$ is determined by its image at 1, say

$$\mathrm{coev}(1) =: \sum_{i=1}^{n} w_i \otimes v_i.$$

- We evaluate the composite $V \to V \otimes W \otimes V \to V$ at a generic $v \in V$:

$$v \longmapsto \sum_{i=1}^{n} v \otimes (w_i \otimes v_i) \longmapsto \sum_{i=1}^{n} \mathrm{ev}(v \otimes w_i) \cdot v_i = v.$$

Notice that $\mathrm{ev}(v \otimes w_i) \in \Bbbk$, so $\{v_1, \dots, v_n\}$ is a spanning set for V. $\square$

Zorro's Lemma: finite dimensionality

The composition $V \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} V \otimes W \otimes V \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} V$ is id_V.
This forces V to have finite dimension, as follows.

- $\mathrm{coev}\colon \Bbbk \to W \otimes V$ is determined by its image at 1, say
$$\mathrm{coev}(1) =: \sum_{i=1}^{n} w_i \otimes v_i.$$
- We evaluate the composite $V \to V \otimes W \otimes V \to V$ at a generic $v \in V$:
$$v \longmapsto \sum_{i=1}^{n} v \otimes (w_i \otimes v_i) \longmapsto \sum_{i=1}^{n} \mathrm{ev}(v \otimes w_i) \cdot v_i = v.$$

Notice that $\mathrm{ev}(v \otimes w_i) \in \Bbbk$, so $\{v_1, \dots, v_n\}$ is a spanning set for V. $\square$

Zorro's Lemma: finite dimensionality

The composition $V \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} V \otimes W \otimes V \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} V$ is id_V.
This forces V to have finite dimension, as follows.

- $\mathrm{coev}\colon \Bbbk \to W \otimes V$ is determined by its image at 1, say

$$\mathrm{coev}(1) =: \sum_{i=1}^{n} w_i \otimes v_i.$$

- We evaluate the composite $V \to V \otimes W \otimes V \to V$ at a generic $v \in V$:

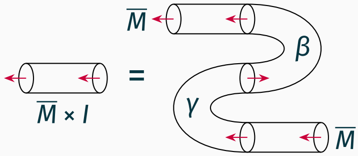
$$v \longmapsto \sum_{i=1}^{n} v \otimes (w_i \otimes v_i) \longmapsto \sum_{i=1}^{n} \mathrm{ev}(v \otimes w_i) \cdot v_i = v.$$

Notice that $\mathrm{ev}(v \otimes w_i) \in \Bbbk$, so $\{v_1, \dots, v_n\}$ is a spanning set for V. $\square$

Zorro's Lemma: dual objects

By similar arguments we can identify $W \equiv V^*$.

Here we need the dual diagram (the "Z" to our prior "S"):



An object X in a monoidal category $(\mathcal{C}, \Box, \mathrm{I})$ is called ***dualizable*** if it satisfies the conditions in both Zorro's diagrams.

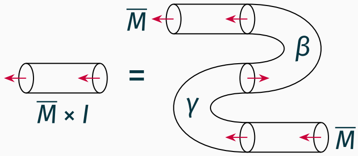
- There exists a ***dual object*** $X^\vee$,
- along with maps $\mathrm{ev} : X \Box X^\vee \to \mathrm{I}$ and $\mathrm{coev} : \mathrm{I} \to X^\vee \Box X$,
- such that the following compositions are the identities:

$$\mathrm{id}_X : X \xrightarrow{\mathrm{id}_X \Box \mathrm{coev}} X \Box X^\vee \Box X \xrightarrow{\mathrm{ev} \Box \mathrm{id}_X} X,$$
$$\mathrm{id}_{X^\vee} : X^\vee \xrightarrow{\mathrm{coev} \Box \mathrm{id}_{X^\vee}} X^\vee \Box X \Box X^\vee \xrightarrow{\mathrm{id}_{X^\vee} \Box \mathrm{ev}} X^\vee.$$

Zorro's Lemma: dual objects

By similar arguments we can identify $W \equiv V^*$.

Here we need the dual diagram (the "Z" to our prior "S"):



An object X in a monoidal category $(\mathcal{C}, \square, \mathrm{I})$ is called ***dualizable*** if it satisfies the conditions in both Zorro's diagrams.

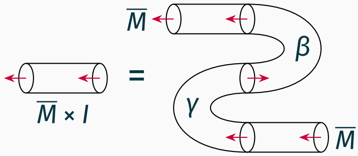
- There exists a ***dual object*** $X^\vee$,
- along with maps $\mathrm{ev} : X \square X^\vee \to \mathrm{I}$ and $\mathrm{coev} : \mathrm{I} \to X^\vee \square X$,
- such that the following compositions are the identities:

$$\mathrm{id}_X : X \xrightarrow{\mathrm{id}_X \square \mathrm{coev}} X \square X^\vee \square X \xrightarrow{\mathrm{ev} \square \mathrm{id}_X} X,$$
$$\mathrm{id}_{X^\vee} : X^\vee \xrightarrow{\mathrm{coev} \square \mathrm{id}_{X^\vee}} X^\vee \square X \square X^\vee \xrightarrow{\mathrm{id}_{X^\vee} \square \mathrm{ev}} X^\vee.$$

Zorro's Lemma: dual objects

By similar arguments we can identify $W \equiv V^*$.

Here we need the dual diagram (the "Z" to our prior "S"):

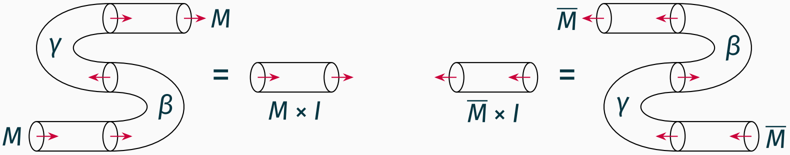


An object X in a monoidal category $(\mathcal{C}, \square, \mathrm{I})$ is called ***dualizable*** if it satisfies the conditions in both Zorro's diagrams.

- There exists a ***dual object*** $X^\vee$,
- along with maps $\mathrm{ev}\colon X \square X^\vee \to \mathrm{I}$ and $\mathrm{coev}\colon \mathrm{I} \to X^\vee \square X$,
- such that the following compositions are the identities:

$$\mathrm{id}_X : X \xrightarrow{\mathrm{id}_X \square \mathrm{coev}} X \square X^\vee \square X \xrightarrow{\mathrm{ev} \square \mathrm{id}_X} X,$$
$$\mathrm{id}_{X^\vee} : X^\vee \xrightarrow{\mathrm{coev} \square \mathrm{id}_{X^\vee}} X^\vee \square X \square X^\vee \xrightarrow{\mathrm{id}_{X^\vee} \square \mathrm{ev}} X^\vee.$$

Zorro's Lemma: recap



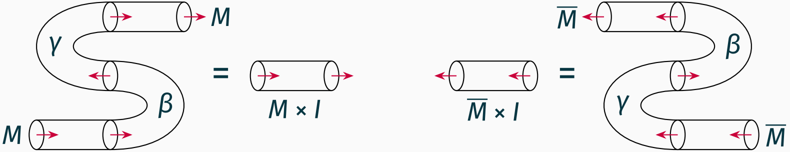
Zorro's Lemma

Every object $M \in \mathbf{Cob}_n$ is dualizable ($\mathbf{Cob}_n$ is ***rigid***).

TQFTs preserve duals (they are ***rigid*** functors).

In particular, each $Z(M) \in \mathbf{Vect}_{\Bbbk}$ has ***finite dimension***.

Zorro's Lemma: recap



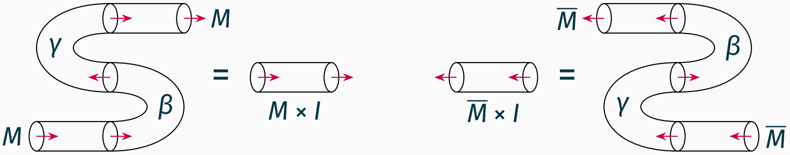
Zorro's Lemma

Every object $M \in \mathbf{Cob}_n$ is dualizable ($\mathbf{Cob}_n$ is ***rigid***).

TQFTs preserve duals (they are ***rigid*** functors).

In particular, each $Z(M) \in \mathbf{Vect}_{\Bbbk}$ has ***finite dimension***.

Zorro's Lemma: recap



Zorro's Lemma

Every object $M \in \mathbf{Cob}_n$ is dualizable ($\mathbf{Cob}_n$ is ***rigid***).

TQFTs preserve duals (they are ***rigid*** functors).

In particular, each $Z(M) \in \mathbf{Vect}_{\Bbbk}$ has ***finite dimension***.

1D TQFTs are determined by their image at the point

Here are all of the connected 1-cobordisms:

As a symmetric monoidal category, **Cob**$_1$ is generated by the two objects pt_+, pt_- and two of the arcs:

These satisfy two aditional relations, which are just Zorro's Lemma.

Theorem

Cob$_1$ *is the free symmetric monoidal category on a dualizable object.*

So oriented 1D TQFTs $Z\colon \mathbf{Cob}_1 \to \mathbf{Vect}_{\Bbbk}$ are determined by the image of the point $Z(\mathrm{pt}_+)$, which must be a finite-dimensional vector space.

1D TQFTs are determined by their image at the point

Here are all of the connected 1-cobordisms:

As a symmetric monoidal category, $\mathbf{Cob}_1$ is generated by the two objects pt_+, pt_- and two of the arcs:

These satisfy two aditional relations, which are just Zorro's Lemma.

Theorem

$\mathbf{Cob}_1$ *is the free symmetric monoidal category on a dualizable object.*

So oriented 1D TQFTs $Z\colon \mathbf{Cob}_1 \to \mathbf{Vect}_{\Bbbk}$ are determined by the image of the point $Z(\mathrm{pt}_+)$, which must be a finite-dimensional vector space.

1D TQFTs are determined by their image at the point

Here are all of the connected 1-cobordisms:

As a symmetric monoidal category, $\mathbf{Cob}_1$ is generated by the two objects pt_+, pt_- and two of the arcs:

These satisfy two aditional relations, which are just Zorro's Lemma.

Theorem

$\mathbf{Cob}_1$ *is the free symmetric monoidal category on a dualizable object.*

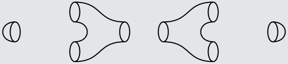
So oriented 1D TQFTs $Z\colon \mathbf{Cob}_1 \to \mathbf{Vect}_{\Bbbk}$ are determined by the image of the point $Z(\mathrm{pt}_+)$, which must be a finite-dimensional vector space.

2D oriented TQFTs are the same as commutative Frobenius Algebras

Generators of $\mathbf{Cob}_2$

Theorem

$\mathbf{Cob}_2$ *is generated by the object* $\mathbb{S}^1$ *and the four morphisms below.*



Proof:
Morse Theory.

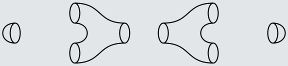
Morse function: smooth $f\colon B \to [0,1]$ without degenerate critical points nor repeated critical values.

Morse functions always exist.

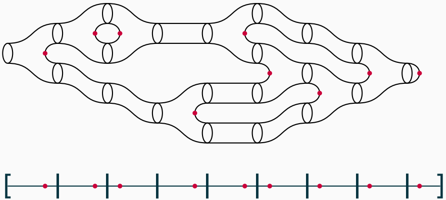
Generators of $\mathbf{Cob}_2$

Theorem

$\mathbf{Cob}_2$ *is generated by the object* $\mathbb{S}^1$ *and the four morphisms below.*



Proof:
Morse Theory.



Morse function: smooth $f\colon B \to [0,1]$ without degenerate critical points nor repeated critical values.

Morse functions always exist.

Relations for $\mathbf{Cob}_2$

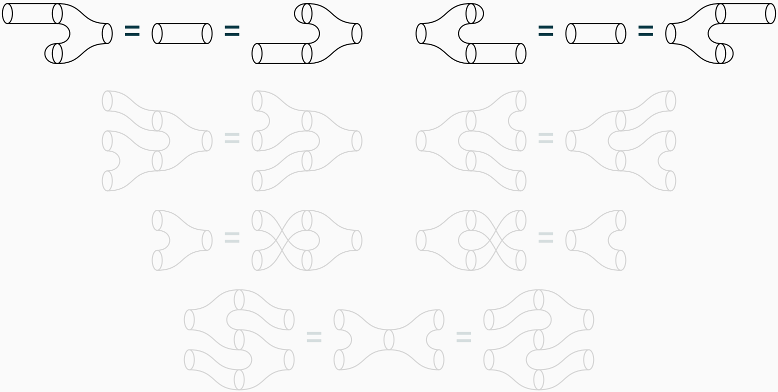
The following relations are sufficient.

Theorem

$\mathbf{Cob}_2$ is equivalent to the symmetric monoidal category generated by the morphisms , , , *and subject to the previous relations.*

Relations for $\mathbf{Cob}_2$

The following relations are sufficient.

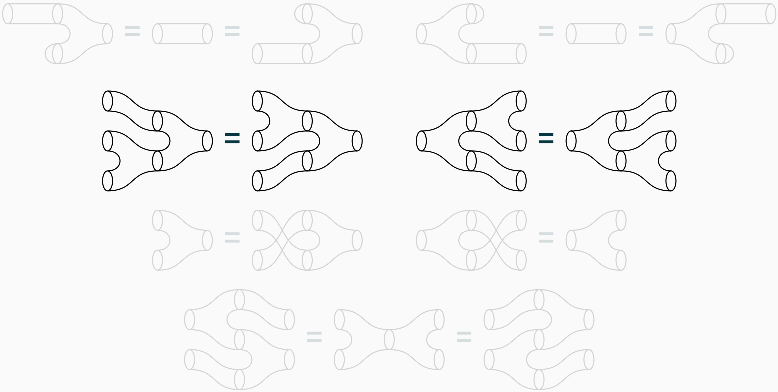


Theorem

$\mathbf{Cob}_2$ *is equivalent to the symmetric monoidal category generated by the morphisms* , , , *and subject to the previous relations.*

Relations for $\mathbf{Cob}_2$

The following relations are sufficient.

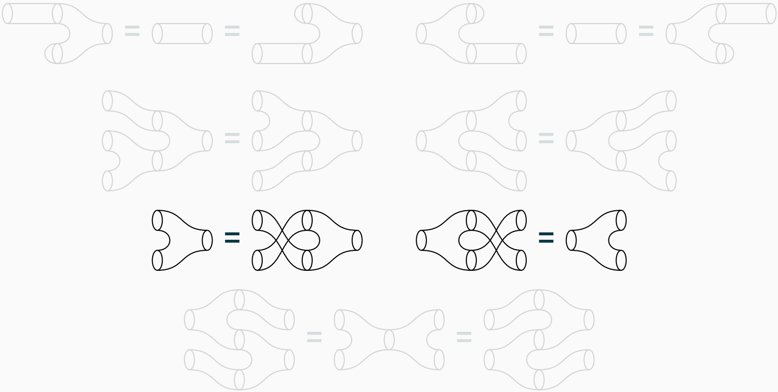


Theorem

$\mathbf{Cob}_2$ is equivalent to the symmetric monoidal category generated by the morphisms , , , and subject to the previous relations.

Relations for $\mathbf{Cob}_2$

The following relations are sufficient.

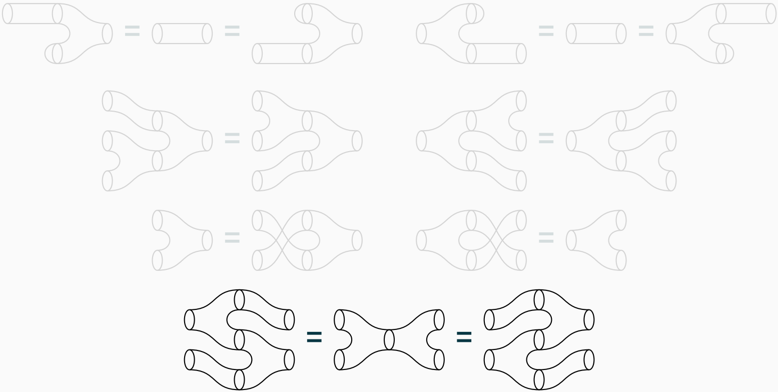


Theorem

$\mathbf{Cob}_2$ is equivalent to the symmetric monoidal category generated by the morphisms , , , *and subject to the previous relations.*

Relations for $\mathbf{Cob}_2$

The following relations are sufficient.

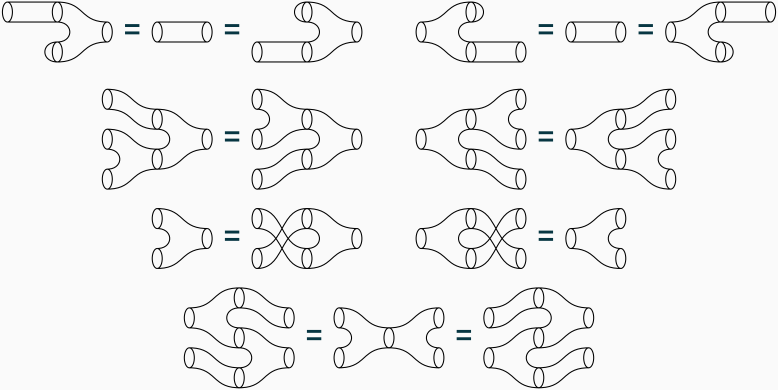


Theorem

$\mathbf{Cob}_2$ is equivalent to the symmetric monoidal category generated by the morphisms , , , *and subject to the previous relations.*

Relations for $\mathbf{Cob}_2$

The following relations are sufficient.



Theorem

$\mathbf{Cob}_2$ *is equivalent to the symmetric monoidal category generated by the morphisms* , , ,  *and subject to the previous relations.*

Algebras, graphically

We can give algebraic sense to the generators and relations of $\mathbf{Cob}_2$. Generators become algebraic structure. Relations become axioms.

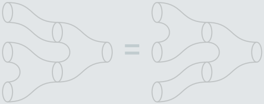
Algebras are unital and associative, but not necessarily commutative.

Definition

An *algebra* over a field $\Bbbk$ is a $\Bbbk$-vector space A equipped with linear maps

- multiplication $\mu : A \otimes A \to A$ (drawn),
- unit map $\eta : \Bbbk \to A$ (drawn),

satisfying



$(a \cdot b) \cdot c = a \cdot (b \cdot c)$,

$1 \cdot a = a = a \cdot 1$.

Algebras, graphically

We can give algebraic sense to the generators and relations of $\mathbf{Cob}_2$. Generators become algebraic structure. Relations become axioms.

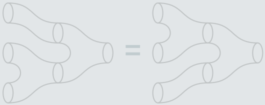
Algebras are unital and associative, but not necessarily commutative.

Definition

An ***algebra*** over a field $\Bbbk$ is a $\Bbbk$-vector space A equipped with linear maps

- multiplication $\mu : A \otimes A \to A$ (drawn ),
- unit map $\eta : \Bbbk \to A$ (drawn ),

satisfying



$(a \cdot b) \cdot c = a \cdot (b \cdot c),$

$1 \cdot a = a = a \cdot 1.$

Algebras, graphically

We can give algebraic sense to the generators and relations of $\mathbf{Cob}_2$. Generators become algebraic structure. Relations become axioms.

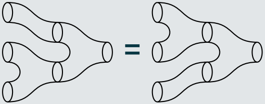
Algebras are unital and associative, but not necessarily commutative.

Definition

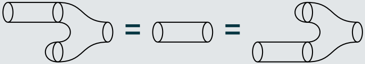
An ***algebra*** over a field $\Bbbk$ is a $\Bbbk$-vector space A equipped with linear maps

- multiplication $\mu\colon A \otimes A \to A$ (drawn ),
- unit map $\eta\colon \Bbbk \to A$ (drawn ),

satisfying



$$(a \cdot b) \cdot c = a \cdot (b \cdot c),$$



$$1 \cdot a = a = a \cdot 1.$$

Frobenius algebras

Definition

A ***Frobenius algebra*** (A, ε) is a $\Bbbk$-algebra A equipped with a linear "trace" map $\varepsilon : A \to \Bbbk$ whose kernel contains no non-trivial ideals.

Examples:
- Matrices $n \times n$ with the trace $\mathrm{tr} : \mathrm{M}_{\Bbbk}(n) \to \Bbbk$.
- Complex numbers with the real part $\mathfrak{Re} : \mathbb{C} \to \mathbb{R}$.

Let (A, ε) a Frobenius algebra. We draw ε as .

Define the ***pairing*** $\beta : A \otimes A \to \Bbbk$ (drawn) as

$$\beta(x \otimes y) \coloneqq \varepsilon(x \cdot y).$$

The pairing is ***associative***:

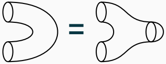
$$\beta((x \cdot a) \otimes y) = \beta(x \otimes (a \cdot y)).$$

Definition

A ***Frobenius algebra*** (A, ε) is a $\Bbbk$-algebra A equipped with a linear "trace" map $\varepsilon \colon A \to \Bbbk$ whose kernel contains no non-trivial ideals.

Examples:
- Matrices $n \times n$ with the trace $\mathrm{tr} \colon \mathrm{M}_{\Bbbk}(n) \to \Bbbk$.
- Complex numbers with the real part $\mathfrak{Re} \colon \mathbb{C} \to \mathbb{R}$.

Let (A, ε) a Frobenius algebra. We draw ε as .
Define the ***pairing*** $\beta \colon A \otimes A \to \Bbbk$ (drawn ) as



$$\beta(x \otimes y) \coloneqq \varepsilon(x \cdot y).$$

The pairing is ***associative***:

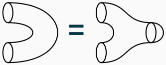
$$\beta((x \cdot a) \otimes y) = \beta(x \otimes (a \cdot y)).$$

Definition

A ***Frobenius algebra*** (A, ε) is a $\Bbbk$-algebra A equipped with a linear "trace" map $\varepsilon : A \to \Bbbk$ whose kernel contains no non-trivial ideals.

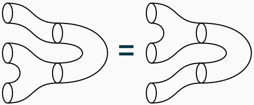
Examples:
- Matrices $n \times n$ with the trace $\mathrm{tr} : \mathrm{M}_{\Bbbk}(n) \to \Bbbk$.
- Complex numbers with the real part $\mathfrak{Re} : \mathbb{C} \to \mathbb{R}$.

Let (A, ε) a Frobenius algebra. We draw ε as .
Define the ***pairing*** $\beta : A \otimes A \to \Bbbk$ (drawn ) as



$$\beta(x \otimes y) \coloneqq \varepsilon(x \cdot y).$$

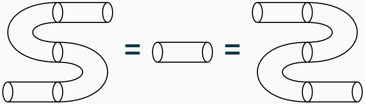
The pairing is ***associative***:



$$\beta((x \cdot a) \otimes y) = \beta(x \otimes (a \cdot y)).$$

Frobenius algebras in terms of pairings

The map β =  is a ***non-degenerate*** pairing:
there exists a ***copairing*** $\gamma\colon \Bbbk \to A \otimes A$ (drawn ) such that



(Since $\operatorname{Ker}\varepsilon$ contains no non-trivial ideals).

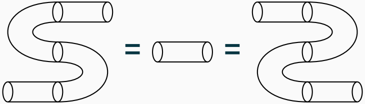
Characterization

A Frobenius algebra (A, β) is a $\Bbbk$-algebra A equipped with an associative non-degenerate pairing $\beta\colon A \otimes A \to \Bbbk$.

(Given β, we recover the previous definition by setting $\varepsilon = \beta(- \otimes 1_A)$).

Frobenius algebras in terms of pairings

The map $\beta =$  is a ***non-degenerate*** pairing:
there exists a ***copairing*** $\gamma\colon \Bbbk \to A \otimes A$ (drawn ) such that



(Since $\operatorname{Ker} \varepsilon$ contains no non-trivial ideals).

Characterization

A Frobenius algebra (A, β) is a $\Bbbk$-algebra A equipped with an associative non-degenerate pairing $\beta\colon A \otimes A \to \Bbbk$.

(Given β, we recover the previous definition by setting $\varepsilon = \beta(- \otimes 1_A)$).

Coalgebras

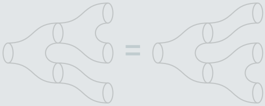
The dual concept to “algebra”.

Definition

An ***coalgebra*** over a field $\Bbbk$ is a $\Bbbk$-vector space A equipped with linear maps

- comultiplication $\delta\colon A \to A \otimes A$ (drawn ),
- counit map $\varepsilon\colon A \to \Bbbk$ (drawn ),

satisfying



(coassociativity)

(counit).

Coalgebras

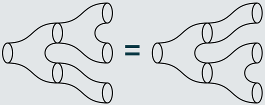
The dual concept to “algebra”.

Definition

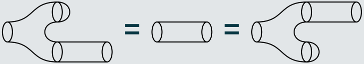
An ***coalgebra*** over a field $\Bbbk$ is a $\Bbbk$-vector space A equipped with linear maps

- comultiplication $\delta \colon A \to A \otimes A$ (drawn ),
- counit map $\varepsilon \colon A \to \Bbbk$ (drawn ),

satisfying



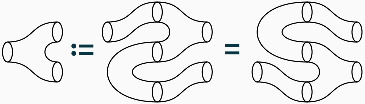
(coassociativity)



(counit).

Frobenius algebras in terms of coalgebras

Given a Frobenius algebra (A, ε), we define the comultiplication $\delta =$ 



with counit $\varepsilon =$ .

Characterization

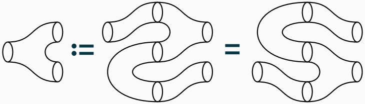
*A Frobenius algebra $(A, \mu, \eta, \delta, \varepsilon)$ is a $\Bbbk$-algebra (A, μ, η) which is also a $\Bbbk$-coalgebra (A, δ, ε), and such that the two structures satisfy the **Frobenius relation**:*

Example: $\varepsilon : \mathbb{C} \to \mathbb{R}$ $\quad \delta : \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$

$$z \mapsto \mathfrak{Re}(z), \qquad z \mapsto z \otimes 1 - iz \otimes i.$$

Frobenius algebras in terms of coalgebras

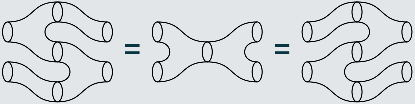
Given a Frobenius algebra (A, ε), we define the comultiplication $\delta =$ 



with counit $\varepsilon =$ .

Characterization

*A Frobenius algebra $(A, \mu, \eta, \delta, \varepsilon)$ is a $\mathbb{k}$-algebra (A, μ, η) which is also a $\mathbb{k}$-coalgebra (A, δ, ε), and such that the two structures satisfy the **Frobenius relation**:*

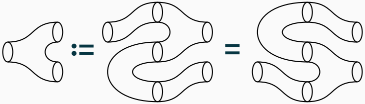


Example: $\varepsilon : \mathbb{C} \to \mathbb{R}$ $\quad$ $\delta : \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$

$z \mapsto \mathfrak{Re}(z),$ $\quad$ $z \mapsto z \otimes 1 - iz \otimes i.$

Frobenius algebras in terms of coalgebras

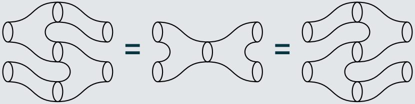
Given a Frobenius algebra (A, ε), we define the comultiplication $\delta =$ 



with counit $\varepsilon =$ .

Characterization

*A Frobenius algebra $(A, \mu, \eta, \delta, \varepsilon)$ is a $\Bbbk$-algebra (A, μ, η) which is also a $\Bbbk$-coalgebra (A, δ, ε), and such that the two structures satisfy the **Frobenius relation**:*



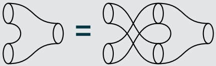
Example:

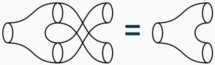
$$\varepsilon : \mathbb{C} \to \mathbb{R} \qquad \delta : \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$$
$$z \mapsto \mathfrak{Re}(z), \qquad z \mapsto z \otimes 1 - iz \otimes i.$$

Commutative and symmetric Frobenius algebras

Definition

A Frobenius algebra is ***commutative*** if it is a commutative algebra:



Equivalent to being a ***cocommutative*** coalgebra: 

Definition

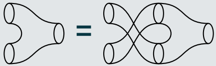
A Frobenius algebra is ***symmetric*** if the pairing β is symmetric:

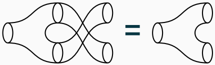
Equivalent to the copairing γ being ***symmetric***:

Commutative and symmetric Frobenius algebras

Definition

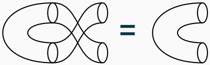
A Frobenius algebra is ***commutative*** if it is a commutative algebra:



Equivalent to being a ***cocommutative*** coalgebra: 

Definition

A Frobenius algebra is ***symmetric*** if the pairing β is symmetric:

Equivalent to the copairing γ being ***symmetric***: 

The correspondence

An oriented 2D TQFT Z determines a commutative Frobenius algebra.

Vector space	$\mathbb{S}^1 \xmapsto{Z} A,$
Multiplication	$\longmapsto \mu : A \otimes A \to A,$
Unit	$\longmapsto \eta : \Bbbk \to A,$
Comultiplication	$\longmapsto \delta : A \to A \otimes A,$
Counit	$\longmapsto \varepsilon : A \to \Bbbk.$

Likewise, a commutative Frobenius algebra determines a TQFT.

Theorem (folklore)

There is a natural equivalence between ***oriented 2D TQFTs*** *and* ***commutative Frobenius algebras****, given by the evaluation functor*

$$Z \longmapsto (Z(\mathbb{S}^1), Z(\;), Z(\;), Z(\;), Z(\;)).$$

The correspondence

An oriented 2D TQFT Z determines a commutative Frobenius algebra.

Vector space $\mathbb{S}^1 \xmapsto{Z} A,$

Multiplication  $\longmapsto \mu : A \otimes A \to A,$

Unit  $\longmapsto \eta : \Bbbk \to A,$

Comultiplication  $\longmapsto \delta : A \to A \otimes A,$

Counit  $\longmapsto \varepsilon : A \to \Bbbk.$

Likewise, a commutative Frobenius algebra determines a TQFT.

Theorem (folklore)

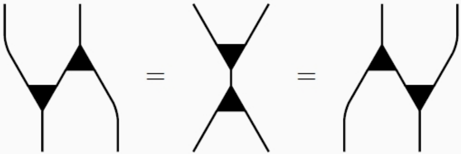
*There is a natural equivalence between **oriented 2D TQFTs** and **commutative Frobenius algebras**, given by the evaluation functor*

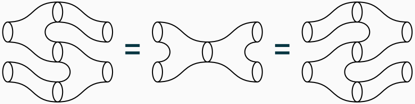
$Z \longmapsto (Z(\mathbb{S}^1), Z($  $), Z($  $), Z($  $), Z($  $)).$

A note on graphical calculus

We have obtained a system of graphical calculus for commutative Frobenius algebras: any “pants diagram” which holds true topologically will also hold algebraically.

This is usually done with string diagrams (read from top to bottom).

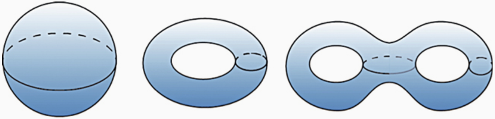


Compare with  .

Beyond 2D:
The Cobordism Hypothesis

The problem with increasing dimension

We would like to generalize this result for dimensions $n > 2$.
But notice that even the case $n = 3$ is a lot more complex: the category $\mathbf{Cob}_3$ has infinitely many generating objects (the g-tori).



We were able to reach our 2D classification by using Morse theory to cut our surfaces along closed curves.
But if we hope to generalize the results, cutting along codimension 1 submanifolds will not suffice.

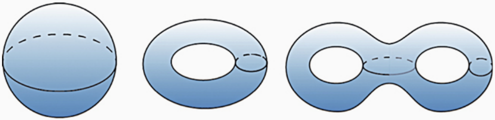
We need more degrees of freedom.

We need more directions in which to make our cuts.

We need n-categories.

The problem with increasing dimension

We would like to generalize this result for dimensions $n > 2$.
But notice that even the case $n = 3$ is a lot more complex: the category $\mathbf{Cob}_3$ has infinitely many generating objects (the g-tori).



We were able to reach our 2D classification by using Morse theory to cut our surfaces along closed curves.
But if we hope to generalize the results, cutting along codimension 1 submanifolds will not suffice.

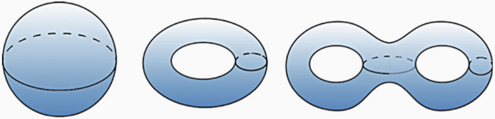
We need more degrees of freedom.

We need more directions in which to make our cuts.

We need n-categories.

The problem with increasing dimension

We would like to generalize this result for dimensions $n > 2$.
But notice that even the case $n = 3$ is a lot more complex: the category $\mathbf{Cob}_3$ has infinitely many generating objects (the g-tori).

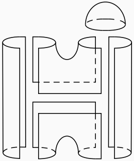


We were able to reach our 2D classification by using Morse theory to cut our surfaces along closed curves.
But if we hope to generalize the results, cutting along codimension 1 submanifolds will not suffice.

We need more degrees of freedom.

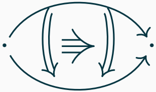
We need more directions in which to make our cuts.

We need n-categories.



n-categories of cobordisms

An n-category has k-morfisms between $(k-1)$-morfisms:



We can define an n-category of cobordisms, $\mathbf{Cob}_n(n)$:

Objects Closed 0-manifolds (finite unions of points).

1-morphisms 1-cobordisms between 0-manifolds.

2-morphisms 2-cobordisms with corners.

⋮

***n*-morphisms** n-cobordisms with corners, up to diffeomorphism.

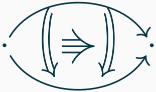
The compositions are given by gluing, in different directions.

We can cut along submanifold of arbitrary codimension.
Essentially, we can triangulate our cobordisms.

Multiple versions: oriented $\mathbf{Cob}_n^{\mathrm{or}}(n)$, framed $\mathbf{Cob}_n^{\mathrm{fr}}(n)$.

n-categories of cobordisms

An n-category has k-morfisms between $(k-1)$-morfisms:



We can define an n-category of cobordisms, $\mathbf{Cob}_n(n)$:

Objects Closed 0-manifolds (finite unions of points).

1-morphisms 1-cobordisms between 0-manifolds.

2-morphisms 2-cobordisms with corners.

⋮

n-morphisms n-cobordisms with corners, up to diffeomorphism.

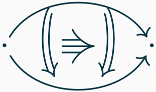
The compositions are given by gluing, in different directions.

We can cut along submanifold of arbitrary codimension.
Essentially, we can triangulate our cobordisms.

Multiple versions: oriented $\mathbf{Cob}_n^{\mathrm{or}}(n)$, framed $\mathbf{Cob}_n^{\mathrm{fr}}(n)$.

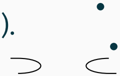
n-categories of cobordisms

An n-category has k-morfisms between $(k-1)$-morfisms:



We can define an n-category of cobordisms, $\mathbf{Cob}_n(n)$:

Objects Closed 0-manifolds (finite unions of points).



1-morphisms 1-cobordisms between 0-manifolds.

2-morphisms 2-cobordisms with corners.

⋮

n-morphisms n-cobordisms with corners, up to diffeomorphism.

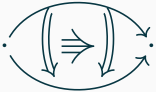
The compositions are given by gluing, in different directions.

We can cut along submanifold of arbitrary codimension.
Essentially, we can triangulate our cobordisms.

Multiple versions: oriented $\mathbf{Cob}_n^{\mathrm{or}}(n)$, framed $\mathbf{Cob}_n^{\mathrm{fr}}(n)$.

n-categories of cobordisms

An n-category has k-morfisms between $(k-1)$-morfisms:

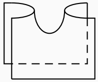


We can define an n-category of cobordisms, $\mathbf{Cob}_n(n)$:

Objects Closed 0-manifolds (finite unions of points).

1-morphisms 1-cobordisms between 0-manifolds.

2-morphisms 2-cobordisms with corners.



⋮

***n*-morphisms** n-cobordisms with corners, up to diffeomorphism.

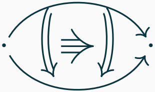
The compositions are given by gluing, in different directions.

We can cut along submanifold of arbitrary codimension.
Essentially, we can triangulate our cobordisms.

Multiple versions: oriented $\mathbf{Cob}_n^{\mathrm{or}}(n)$, framed $\mathbf{Cob}_n^{\mathrm{fr}}(n)$.

n-categories of cobordisms

An n-category has k-morfisms between $(k - 1)$-morfisms:

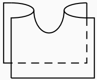


We can define an n-category of cobordisms, $\mathbf{Cob}_n(n)$:

Objects Closed 0-manifolds (finite unions of points).

1-morphisms 1-cobordisms between 0-manifolds.

2-morphisms 2-cobordisms with corners.



⋮

***n*-morphisms** n-cobordisms with corners, up to diffeomorphism.

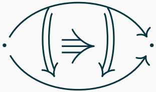
The compositions are given by gluing, in different directions.

We can cut along submanifold of arbitrary codimension.
Essentially, we can triangulate our cobordisms.

Multiple versions: oriented $\mathbf{Cob}_n^{\mathrm{or}}(n)$, framed $\mathbf{Cob}_n^{\mathrm{fr}}(n)$.

n-categories of cobordisms

An n-category has k-morfisms between $(k-1)$-morfisms:

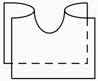


We can define an n-category of cobordisms, $\mathbf{Cob}_n(n)$:

Objects Closed 0-manifolds (finite unions of points).

1-morphisms 1-cobordisms between 0-manifolds.

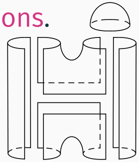
2-morphisms 2-cobordisms with corners.



⋮

***n*-morphisms** n-cobordisms with corners, up to diffeomorphism.

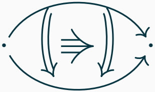
The compositions are given by gluing, in different directions.



We can cut along submanifold of arbitrary codimension.
Essentially, we can triangulate our cobordisms.

Multiple versions: oriented $\mathbf{Cob}_n^{\mathrm{or}}(n)$, framed $\mathbf{Cob}_n^{\mathrm{fr}}(n)$.

An n-category has k-morfisms between $(k-1)$-morfisms:

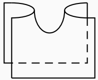


We can define an n-category of cobordisms, $\mathbf{Cob}_n(n)$:

Objects Closed 0-manifolds (finite unions of points).

1-morphisms 1-cobordisms between 0-manifolds.

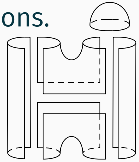
2-morphisms 2-cobordisms with corners.



⋮

***n*-morphisms** n-cobordisms with corners, up to diffeomorphism.

The compositions are given by gluing, in different directions.

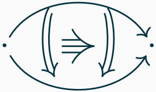


We can cut along submanifold of arbitrary codimension.
Essentially, we can triangulate our cobordisms.

Multiple versions: oriented $\mathbf{Cob}_n^{\mathrm{or}}(n)$, framed $\mathbf{Cob}_n^{\mathrm{fr}}(n)$.

n-categories of cobordisms

An n-category has k-morfisms between $(k - 1)$-morfisms:

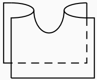


We can define an n-category of cobordisms, $\mathbf{Cob}_n(n)$:

Objects Closed 0-manifolds (finite unions of points).

1-morphisms 1-cobordisms between 0-manifolds.

2-morphisms 2-cobordisms with corners.

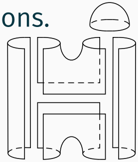


$\vdots$

***n*-morphisms** n-cobordisms with corners, up to diffeomorphism.

The compositions are given by gluing, in different directions.

We can cut along submanifold of arbitrary codimension.
Essentially, we can triangulate our cobordisms.



Multiple versions: oriented $\mathbf{Cob}_n^{\mathrm{or}}(n)$, framed $\mathbf{Cob}_n^{\mathrm{fr}}(n)$.

Extended TQFTs

An ***extended TQFT*** is a symmetric monoidal functor of n-categories

$$\mathbf{Cob}_n(n) \longrightarrow \mathcal{C}.$$

An extended TQFT requires specifying a colossal amount of data. However, this data must be subject to a colossal amount of conditions: no matter how we cut up a given cobordism, after taking images and composing back, the resulting n-morphism of $\mathcal{C}$ must be the same.

Framed manifolds are especially rigid. Given that we can decompose as much as we want, it is not unreasonable to think that framed extended TQFTs will be determined by their image at a single point.

Remember that this is what happened in the $n = 1$ case!

Extended TQFTs

An ***extended TQFT*** is a symmetric monoidal functor of n-categories

$$\mathbf{Cob}_n(n) \longrightarrow \mathcal{C}.$$

An extended TQFT requires specifying a colossal amount of data. However, this data must be subject to a colossal amount of conditions: no matter how we cut up a given cobordism, after taking images and composing back, the resulting n-morphism of $\mathcal{C}$ must be the same.

Framed manifolds are especially rigid. Given that we can decompose as much as we want, it is not unreasonable to think that framed extended TQFTs will be determined by their image at a single point.

Remember that this is what happened in the $n = 1$ case!

Extended TQFTs

An ***extended TQFT*** is a symmetric monoidal functor of n-categories

$$\mathbf{Cob}_n(n) \longrightarrow \mathcal{C}.$$

An extended TQFT requires specifying a colossal amount of data. However, this data must be subject to a colossal amount of conditions: no matter how we cut up a given cobordism, after taking images and composing back, the resulting n-morphism of $\mathcal{C}$ must be the same.

Framed manifolds are especially rigid. Given that we can decompose as much as we want, it is not unreasonable to think that framed extended TQFTs will be determined by their image at a single point.

Remember that this is what happened in the $n = 1$ case!

Extended TQFTs

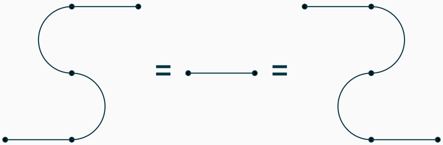
An ***extended TQFT*** is a symmetric monoidal functor of n-categories

$$\mathbf{Cob}_n(n) \longrightarrow \mathcal{C}.$$

An extended TQFT requires specifying a colossal amount of data. However, this data must be subject to a colossal amount of conditions: no matter how we cut up a given cobordism, after taking images and composing back, the resulting n-morphism of $\mathcal{C}$ must be the same.

Framed manifolds are especially rigid. Given that we can decompose as much as we want, it is not unreasonable to think that framed extended TQFTs will be determined by their image at a single point.

Remember that this is what happened in the $n = 1$ case!



The Cobordism Hypothesis

Thesis (Baez–Dolan Cobordism Hypothesis, 1995)

*A **framed extended TQFT** is determined by its image at a single point.*

$$\begin{aligned} \mathrm{Fun}^{\otimes}(\mathbf{Cob}_n^{\mathrm{fr}}(n), \mathcal{C}) &\longrightarrow \mathcal{C} \\ Z &\longmapsto Z(\mathrm{pt}_+). \end{aligned}$$

Restricting the image, we should have an equivalence of n-categories.

We believe it is true, with some caveats.
We want it to be true, so the definitions are constructed to make it so.
(Similar to the Homotopy Hypothesis in the theory of ∞-groupoids).

J. Lurie, 2009 Detailed proof sketch of the Hypothesis for general n. This includes multiple generalizations.

C. Schommer-Pries, 2009 Complete proof for the $n = 2$ case. This includes oriented and unoriented versions.

Ayala–Francis, 2017; Grady–Pavlov, 2021 Pre-prints claiming to fill the missing steps in Lurie's program.

The Cobordism Hypothesis

Thesis (Baez–Dolan Cobordism Hypothesis, 1995)

*A **framed extended TQFT** is determined by its image at a single point.*

$$\begin{aligned} \mathrm{Fun}^{\otimes}(\mathbf{Cob}_n^{\mathrm{fr}}(n), \mathcal{C}) &\longrightarrow \mathcal{C} \\ Z &\longmapsto Z(\mathrm{pt}_+). \end{aligned}$$

Restricting the image, we should have an equivalence of n-categories.

We believe it is true, with some caveats.
We want it to be true, so the definitions are constructed to make it so.
(Similar to the Homotopy Hypothesis in the theory of ∞-groupoids).

J. Lurie, 2009 Detailed proof sketch of the Hypothesis for general n. This includes multiple generalizations.

C. Schommer-Pries, 2009 Complete proof for the $n = 2$ case. This includes oriented and unoriented versions.

Ayala–Francis, 2017; Grady–Pavlov, 2021 Pre-prints claiming to fill the missing steps in Lurie's program.

Lurie's approach to the Cobordism Hypothesis

Since the n-morphisms of $\mathbf{Cob}^{\mathrm{fr}}_{n+1}(n+1)$ and $\mathbf{Cob}^{\mathrm{fr}}_{n}(n)$ are defined similarly, we hope to proceed by induction on the dimension n.

For this to work, we need to be careful with the data we track: taking diffeomorphism classes of n-cobordisms will discard important data. Working inside $\mathbf{Cob}^{\mathrm{fr}}_{n}(n)$ will not suffice.

So we work with (∞, n)-***categories***, which have infinite layers of k-morphisms — but all k-morphisms above layer n are invertible. (For example, $(\infty, 0)$-categories are ∞-groupoids).

This will let us avoid taking diffeomorphism classes.

Lurie's approach to the Cobordism Hypothesis

Since the n-morphisms of $\mathbf{Cob}_{n+1}^{\mathrm{fr}}(n+1)$ and $\mathbf{Cob}_{n}^{\mathrm{fr}}(n)$ are defined similarly, we hope to proceed by induction on the dimension n.

For this to work, we need to be careful with the data we track: taking diffeomorphism classes of n-cobordisms will discard important data. Working inside $\mathbf{Cob}_{n}^{\mathrm{fr}}(n)$ will not suffice.

So we work with ***(∞, n)-categories***, which have infinite layers of k-morphisms — but all k-morphisms above layer n are invertible. (For example, $(\infty, 0)$-categories are ∞-groupoids).

This will let us avoid taking diffeomorphism classes.

The (∞, n)-category of framed cobordisms

The (∞, n)-category of framed cobordisms, $\mathbf{Bord}_n^{\mathrm{fr}}$, consists of:

Objects	Closed framed 0-manifolds.
1-morphisms	Framed 1-cobordisms.
$\vdots$	
***n*-morphisms**	Framed n-cobordisms with corners.
$(n+1)$-morphisms	Diffeomorphisms between n-cobordisms.
$(n+2)$-morphisms	Diffeotopies between diffeomorphisms.
$(n+3)$-morphisms	Diffeotopies between diffeotopies.
$\vdots$	

An ***extended TQFT*** is a symmetric monoidal functor of (∞, n)-categories

$$\mathbf{Bord}_n^{\mathrm{fr}} \longrightarrow \mathcal{C}.$$

If $\mathcal{C}$ is an n-category, we recover our prior definition.

(An n-category "is" an (∞, n)-category where the only k-morphisms for $k > n$ are the identities).

The (∞, n)-category of framed cobordisms

The (∞, n)-category of framed cobordisms, $\mathbf{Bord}_n^{\mathrm{fr}}$, consists of:

Objects	Closed framed 0-manifolds.
1-morphisms	Framed 1-cobordisms.
$\vdots$	
n**-morphisms**	Framed n-cobordisms with corners.
$(n+1)$**-morphisms**	Diffeomorphisms between n-cobordisms.
$(n+2)$**-morphisms**	Diffeotopies between diffeomorphisms.
$(n+3)$**-morphisms**	Diffeotopies between diffeotopies.
$\vdots$	

An ***extended TQFT*** is a symmetric monoidal functor of (∞, n)-categories

$$\mathbf{Bord}_n^{\mathrm{fr}} \longrightarrow \mathcal{C}.$$

If $\mathcal{C}$ is an n-category, we recover our prior definition.

(An n-category "is" an (∞, n)-category where the only k-morphisms for $k > n$ are the identities).

The (∞, n)-category of framed cobordisms

The (∞, n)-category of framed cobordisms, $\mathbf{Bord}_n^{\mathrm{fr}}$, consists of:

Objects	Closed framed 0-manifolds.
1-morphisms	Framed 1-cobordisms.
$\vdots$	
n**-morphisms**	Framed n-cobordisms with corners.
$(n+1)$**-morphisms**	Diffeomorphisms between n-cobordisms.
$(n+2)$**-morphisms**	Diffeotopies between diffeomorphisms.
$(n+3)$**-morphisms**	Diffeotopies between diffeotopies.
$\vdots$	

An ***extended TQFT*** is a symmetric monoidal functor of (∞, n)-categories

$$\mathbf{Bord}_n^{\mathrm{fr}} \longrightarrow \mathcal{C}.$$

If $\mathcal{C}$ is an n-category, we recover our prior definition.

(An n-category "is" an (∞, n)-category where the only k-morphisms for $k > n$ are the identities).

Fully dualizable objects

The image objects of an extended TQFT must be ***fully dualizable***.

For $n = 1$, these are the dualizable objects:
there exist 1-morphisms ev = and coev = such that

For $n = 2$, we require the prior two 1-morphisms to admit adjoints:

there exist 2-morphisms u = and v = such that

$$\begin{array}{c} \mathrm{ev} \circ \mathrm{id}_{\varnothing} \\ \mathrm{id}_{\mathrm{ev}} \Downarrow \otimes u \\ \mathrm{ev} \circ \mathrm{ev}^{\vee} \circ \mathrm{ev} \\ v \otimes \Downarrow \mathrm{id}_{\mathrm{ev}} \\ \mathrm{id}_{+\sqcup -} \circ \mathrm{ev} \end{array}$$

and

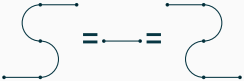
$$\begin{array}{c} \mathrm{id}_{\varnothing} \circ \mathrm{ev}^{\vee} \\ u \otimes \Downarrow \mathrm{id}_{\mathrm{ev}^{\vee}} \\ \mathrm{ev}^{\vee} \circ \mathrm{ev} \circ \mathrm{ev}^{\vee} \\ \mathrm{id}_{\mathrm{ev}^{\vee}} \Downarrow \otimes v \\ \mathrm{ev}^{\vee} \circ \mathrm{id}_{+\sqcup -} \end{array}$$

This pattern continues for general n.

Fully dualizable objects

The image objects of an extended TQFT must be ***fully dualizable***.

For $n = 1$, these are the dualizable objects:
there exist 1-morphisms ev =  and coev =  such that



For $n = 2$, we require the prior two 1-morphisms to admit adjoints:
there exist 2-morphisms u = and v = such that

$$\begin{array}{c} \mathrm{ev} \circ \mathrm{id}_{\varnothing} \\ \Downarrow {\scriptstyle \mathrm{id}_{\mathrm{ev}} \otimes u} \\ \mathrm{ev} \circ \mathrm{ev}^{\vee} \circ \mathrm{ev} \\ \Downarrow {\scriptstyle v \otimes \mathrm{id}_{\mathrm{ev}}} \\ \mathrm{id}_{+\sqcup-} \circ \mathrm{ev} \end{array}$$

and

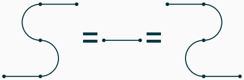
$$\begin{array}{c} \mathrm{id}_{\varnothing} \circ \mathrm{ev}^{\vee} \\ \Downarrow {\scriptstyle u \otimes \mathrm{id}_{\mathrm{ev}^{\vee}}} \\ \mathrm{ev}^{\vee} \circ \mathrm{ev} \circ \mathrm{ev}^{\vee} \\ \Downarrow {\scriptstyle \mathrm{id}_{\mathrm{ev}^{\vee}} \otimes v} \\ \mathrm{ev}^{\vee} \circ \mathrm{id}_{+\sqcup-} \end{array}$$

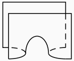
This pattern continues for general n.

Fully dualizable objects

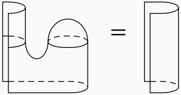
The image objects of an extended TQFT must be ***fully dualizable***.

For $n = 1$, these are the dualizable objects:
there exist 1-morphisms ev =  and coev =  such that



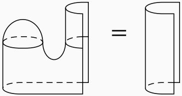
For $n = 2$, we require the prior two 1-morphisms to admit adjoints:
there exist 2-morphisms u =  and v =  such that

$$\begin{array}{c} \mathrm{ev}\circ \mathrm{id}_{\varnothing} \\ \Downarrow\ \mathrm{id}_{\mathrm{ev}}\otimes u \\ \mathrm{ev}\circ\mathrm{ev}^{\vee}\circ\mathrm{ev} \\ \Downarrow\ v\otimes \mathrm{id}_{\mathrm{ev}} \\ \mathrm{id}_{+\sqcup-}\circ\mathrm{ev}\end{array}$$



and

$$\begin{array}{c} \mathrm{id}_{\varnothing}\circ\mathrm{ev}^{\vee} \\ \Downarrow\ u\otimes \mathrm{id}_{\mathrm{ev}^{\vee}} \\ \mathrm{ev}^{\vee}\circ\mathrm{ev}\circ\mathrm{ev}^{\vee} \\ \Downarrow\ \mathrm{id}_{\mathrm{ev}^{\vee}}\otimes v \\ \mathrm{ev}^{\vee}\circ \mathrm{id}_{+\sqcup-}\end{array}$$

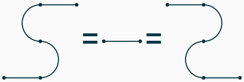


This pattern continues for general n.

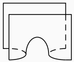
Fully dualizable objects

The image objects of an extended TQFT must be ***fully dualizable***.

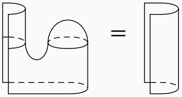
For $n = 1$, these are the dualizable objects:
there exist 1-morphisms ev =  and coev =  such that



For $n = 2$, we require the prior two 1-morphisms to admit adjoints:

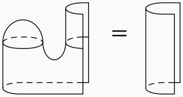
there exist 2-morphisms u =  and v =  such that

$$\begin{gathered}\mathrm{ev}\circ\mathrm{id}_{\varnothing} \\ \mathrm{id}_{\mathrm{ev}} \Downarrow \otimes\, u \\ \mathrm{ev}\circ\mathrm{ev}^{\vee}\circ\mathrm{ev} \\ v\otimes \Downarrow \mathrm{id}_{\mathrm{ev}} \\ \mathrm{id}_{+\sqcup -}\circ\mathrm{ev}\end{gathered}$$



and

$$\begin{gathered}\mathrm{id}_{\varnothing}\circ\mathrm{ev}^{\vee} \\ u\otimes \Downarrow \mathrm{id}_{\mathrm{ev}^{\vee}} \\ \mathrm{ev}^{\vee}\circ\mathrm{ev}\circ\mathrm{ev}^{\vee} \\ \mathrm{id}_{\mathrm{ev}^{\vee}} \Downarrow \otimes\, v \\ \mathrm{ev}^{\vee}\circ\mathrm{id}_{+\sqcup -}\end{gathered}$$



This pattern continues for general n.

More statements of the Cobordism Hypothesis

Thesis (Lurie, 2009)

*There is a bijection between **framed extended TQFTs** $Z\colon \mathbf{Bord}_n^{\mathrm{fr}} \to \mathcal{C}$ and **fully dualizable objects** of $\mathcal{C}$, induced by evaluation at the point:*

$$Z \longmapsto Z(\mathrm{pt}_+).$$

Thesis (Lurie, 2009)

$\mathbf{Bord}_n^{\mathrm{fr}}$ is the free symmetric monoidal (∞, n)-category on a single fully dualizable object.

More statements of the Cobordism Hypothesis

Thesis (Lurie, 2009)

There is a bijection between ***framed extended TQFTs*** $Z\colon \mathbf{Bord}_n^{\mathrm{fr}} \to \mathcal{C}$ *and* ***fully dualizable objects*** *of* $\mathcal{C}$*, induced by evaluation at the point:*

$$Z \longmapsto Z(\mathrm{pt}_+).$$

Thesis (Lurie, 2009)

$\mathbf{Bord}_n^{\mathrm{fr}}$ *is the free symmetric monoidal* (∞, n)*-category on a single fully dualizable object.*

The oriented Cobordism Hypothesis

Lurie also proves an oriented version, but it is a lot more technical.

Thesis (Lurie, 2009)

Oriented extended TQFTs *$Z\colon \mathbf{Bord}_n^{\mathrm{or}} \to \mathcal{C}$ correspond to* ***homotopy fixed points*** *of a certain canonical action $\mathrm{SO}(n) \curvearrowright \mathrm{Core}(\mathcal{C}^{\mathrm{fd}})$ on the core ∞-groupoid of the subcategory of fully dualizable objects of $\mathcal{C}$:*

$$\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^G, \mathcal{C}) \simeq (\mathrm{Core}(\mathcal{C}^{\mathrm{fd}}))^{\mathrm{h}\,G}.$$

Oriented TQFTs are still determined by their values on the point...

...but regarded as living inside $(\mathrm{Core}(\mathcal{C}^{\mathrm{fd}}))^{\mathrm{h}\,G}$. It carries extra data.

This is all very abstract and deep, so we will now explore the case of oriented extended 2D TQFTs.

The oriented Cobordism Hypothesis

Lurie also proves an oriented version, but it is a lot more technical.

Thesis (Lurie, 2009)

Oriented extended TQFTs *$Z\colon \mathbf{Bord}_n^{\mathrm{or}} \to \mathcal{C}$ correspond to* ***homotopy fixed points*** *of a certain canonical action $\mathrm{SO}(n) \curvearrowright \mathrm{Core}(\mathcal{C}^{\mathrm{fd}})$ on the core ∞-groupoid of the subcategory of fully dualizable objects of $\mathcal{C}$:*

$$\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{G}, \mathcal{C}) \simeq (\mathrm{Core}(\mathcal{C}^{\mathrm{fd}}))^{\mathrm{h}\,G}.$$

Oriented TQFTs are still determined by their values on the point…

…but regarded as living inside $(\mathrm{Core}(\mathcal{C}^{\mathrm{fd}}))^{\mathrm{h}\,G}$. It carries extra data.

This is all very abstract and deep, so we will now explore the case of oriented extended 2D TQFTs.

The oriented Cobordism Hypothesis

Lurie also proves an oriented version, but it is a lot more technical.

Thesis (Lurie, 2009)

Oriented extended TQFTs *$Z\colon \mathbf{Bord}_n^{\mathrm{or}} \to \mathcal{C}$ correspond to* ***homotopy fixed points*** *of a certain canonical action $\mathrm{SO}(n) \curvearrowright \mathrm{Core}(\mathcal{C}^{\mathrm{fd}})$ on the core ∞-groupoid of the subcategory of fully dualizable objects of $\mathcal{C}$:*

$$\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^G, \mathcal{C}) \simeq (\mathrm{Core}(\mathcal{C}^{\mathrm{fd}}))^{hG}.$$

Oriented TQFTs are still determined by their values on the point...

...but regarded as living inside $(\mathrm{Core}(\mathcal{C}^{\mathrm{fd}}))^{hG}$. It carries extra data.

This is all very abstract and deep, so we will now explore the case of oriented extended 2D TQFTs.

Returning to 2D:
(0, 1, 2)-TQFTs in the Morita bicategory

Schommer-Pries's approach to the Cobordism Hypothesis

(Schommer-Pries, 2009) finds generators and relations for $\mathbf{Cob}_2^{\mathrm{or}}(2)$.

Uses Cerf Theory: a sort of "parametrized Morse Theory".

Generators (up to orientation and permutation):

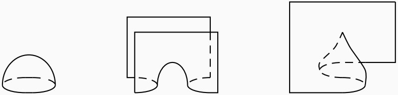
0D: •   1D: 2D:

Relations (up to orientation and permutation):

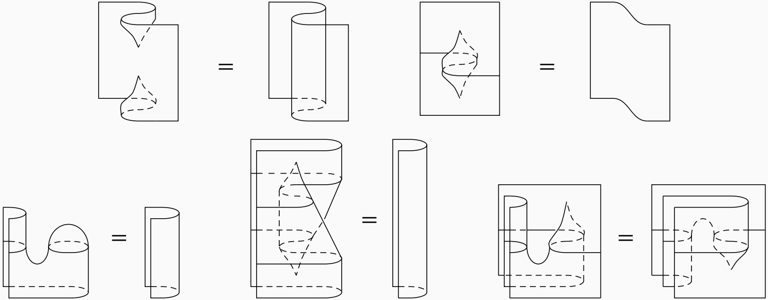
Schommer-Pries's approach to the Cobordism Hypothesis

(Schommer-Pries, 2009) finds generators and relations for $\mathbf{Cob}_2^{\mathrm{or}}(2)$.
Uses Cerf Theory: a sort of "parametrized Morse Theory".

Generators (up to orientation and permutation):

0D:  1D:  2D: 

Relations (up to orientation and permutation):



The Morita bicategory

This is one possible higher-categorical analogue of $\mathbf{Vect}_{\Bbbk}$.

Definition

Let A and B be algebras.
A ***B-A-bimodule*** ${}_BM_A$ is a left B-module which is also a right A-module in a compatible manner: $\quad b \cdot (m \cdot a) = (b \cdot m) \cdot a.$

The ***Morita bicategory*** $\mathbf{Alg}_2$ consists of:

- **Objects** $\Bbbk$-algebras A.
- **1-morphisms** An arrow $A \to B$ is an B-A-bimodule ${}_BM_A$.
- **2-morphisms** Bimodule homomorphisms.
- **Composition** of 1-morphisms: tensor product over the algebra.

$$ {}_CN_B \circ {}_BM_A := {}_C(N \otimes_B M)_A. $$

- **Identities** For each algebra A, the identity 1-morphism is ${}_AA_A$.

It is **symmetric monoidal** with monoid tensor product over $\Bbbk$:

$$ {}_AM_B \otimes {}_CN_D = {}_{A\otimes C}(M \otimes N)_{B\otimes D}. $$

The Morita bicategory

This is one possible higher-categorical analogue of $\mathbf{Vect}_{\Bbbk}$.

Definition

Let A and B be algebras.
A ***B-A-bimodule*** $_BM_A$ is a left B-module which is also a right A-module in a compatible manner: $\quad b \cdot (m \cdot a) = (b \cdot m) \cdot a.$

The ***Morita bicategory*** $\mathbf{Alg}_2$ consists of:

Objects $\Bbbk$-algebras A.

1-morphisms An arrow $A \to B$ is an B-A-bimodule $_BM_A$.

2-morphisms Bimodule homomorphisms.

Composition of 1-morphisms: tensor product over the algebra.

$$_CN_B \circ {_BM_A} := {_C(N \otimes_B M)_A}.$$

Identities For each algebra A, the identity 1-morphism is $_AA_A$.

It is **symmetric monoidal** with monoid tensor product over $\Bbbk$:

$$_AM_B \otimes {_CN_D} = {_{A\otimes C}(M \otimes N)_{B\otimes D}}.$$

The Morita bicategory

This is one possible higher-categorical analogue of $\mathbf{Vect}_{\Bbbk}$.

Definition

Let A and B be algebras.
A ***B-A-bimodule*** ${}_BM_A$ is a left B-module which is also a right A-module in a compatible manner: $b \cdot (m \cdot a) = (b \cdot m) \cdot a$.

The ***Morita bicategory*** $\mathbf{Alg}_2$ consists of:

Objects $\Bbbk$-algebras A.

1-morphisms An arrow $A \to B$ is an B-A-bimodule ${}_BM_A$.

2-morphisms Bimodule homomorphisms.

Composition of 1-morphisms: tensor product over the algebra.

$$ {}_CN_B \circ {}_BM_A := {}_C(N \otimes_B M)_A. $$

Identities For each algebra A, the identity 1-morphism is ${}_AA_A$.

It is **symmetric monoidal** with monoid tensor product over $\Bbbk$:

$$ {}_AM_B \otimes {}_CN_D = {}_{A\otimes C}(M \otimes N)_{B\otimes D}. $$

The Morita bicategory

This is one possible higher-categorical analogue of $\mathbf{Vect}_{\Bbbk}$.

Definition

Let A and B be algebras.
A ***B-A-bimodule*** ${}_BM_A$ is a left B-module which is also a right A-module in a compatible manner: $b \cdot (m \cdot a) = (b \cdot m) \cdot a$.

The ***Morita bicategory*** $\mathbf{Alg}_2$ consists of:

Objects $\Bbbk$-algebras A.

1-morphisms An arrow $A \to B$ is an B-A-bimodule ${}_BM_A$.

2-morphisms Bimodule homomorphisms.

Composition of 1-morphisms: tensor product over the algebra.

$$ {}_CN_B \circ {}_BM_A \coloneqq {}_C(N \otimes_B M)_A. $$

Identities For each algebra A, the identity 1-morphism is ${}_AA_A$.

It is **symmetric monoidal** with monoid tensor product over $\Bbbk$:

$$ {}_AM_B \otimes {}_CN_D = {}_{A\otimes C}(M \otimes N)_{B\otimes D}. $$

The Morita bicategory

This is one possible higher-categorical analogue of $\mathbf{Vect}_{\Bbbk}$.

Definition

Let A and B be algebras.
A ***B-A-bimodule*** ${}_BM_A$ is a left B-module which is also a right A-module in a compatible manner: $b \cdot (m \cdot a) = (b \cdot m) \cdot a$.

The ***Morita bicategory*** $\mathbf{Alg}_2$ consists of:

Objects $\Bbbk$-algebras A.
1-morphisms An arrow $A \to B$ is an B-A-bimodule ${}_BM_A$.
2-morphisms Bimodule homomorphisms.
Composition of 1-morphisms: tensor product over the algebra.

$$ {}_CN_B \circ {}_BM_A \coloneqq {}_C(N \otimes_B M)_A. $$

Identities For each algebra A, the identity 1-morphism is ${}_AA_A$.

It is **symmetric monoidal** with monoid tensor product over $\Bbbk$:

$$ {}_AM_B \otimes {}_CN_D = {}_{A\otimes C}(M \otimes N)_{B\otimes D}. $$

The Morita bicategory

This is one possible higher-categorical analogue of $\mathbf{Vect}_{\Bbbk}$.

Definition

Let A and B be algebras.
A ***B-A-bimodule*** ${}_BM_A$ is a left B-module which is also a right A-module in a compatible manner: $b \cdot (m \cdot a) = (b \cdot m) \cdot a$.

The ***Morita bicategory*** $\mathbf{Alg}_2$ consists of:

Objects $\Bbbk$-algebras A.
1-morphisms An arrow $A \to B$ is an B-A-bimodule ${}_BM_A$.
2-morphisms Bimodule homomorphisms.
Composition of 1-morphisms: tensor product over the algebra.

$${}_CN_B \circ {}_BM_A \coloneqq {}_C(N \otimes_B M)_A.$$

Identities For each algebra A, the identity 1-morphism is ${}_AA_A$.

It is **symmetric monoidal** with monoid tensor product over $\Bbbk$:

$${}_AM_B \otimes {}_CN_D = {}_{A\otimes C}(M \otimes N)_{B\otimes D}.$$

The Morita bicategory

This is one possible higher-categorical analogue of $\mathbf{Vect}_{\Bbbk}$.

Definition

Let A and B be algebras.
A ***B-A-bimodule*** $_BM_A$ is a left B-module which is also a right A-module in a compatible manner: $b \cdot (m \cdot a) = (b \cdot m) \cdot a$.

The ***Morita bicategory*** $\mathbf{Alg}_2$ consists of:

Objects $\Bbbk$-algebras A.
1-morphisms An arrow $A \to B$ is an B-A-bimodule $_BM_A$.
2-morphisms Bimodule homomorphisms.
Composition of 1-morphisms: tensor product over the algebra.

$$_CN_B \circ {_BM_A} \coloneqq {_C(N \otimes_B M)_A}.$$

Identities For each algebra A, the identity 1-morphism is $_AA_A$.

It is **symmetric monoidal** with monoid tensor product over $\Bbbk$:

$$_AM_B \otimes {_CN_D} = {_{A\otimes C}(M \otimes N)_{B\otimes D}}.$$

The Morita bicategory

This is one possible higher-categorical analogue of $\mathbf{Vect}_{\Bbbk}$.

Definition

Let A and B be algebras.
A ***B-A-bimodule*** ${}_BM_A$ is a left B-module which is also a right A-module in a compatible manner: $b \cdot (m \cdot a) = (b \cdot m) \cdot a$.

The ***Morita bicategory*** $\mathbf{Alg}_2$ consists of:

Objects $\Bbbk$-algebras A.

1-morphisms An arrow $A \to B$ is an B-A-bimodule ${}_BM_A$.

2-morphisms Bimodule homomorphisms.

Composition of 1-morphisms: tensor product over the algebra.

$$ {}_CN_B \circ {}_BM_A \coloneqq {}_C(N \otimes_B M)_A. $$

Identities For each algebra A, the identity 1-morphism is ${}_AA_A$.

It is **symmetric monoidal** with monoid tensor product over $\Bbbk$:

$$ {}_AM_B \otimes {}_CN_D = {}_{A\otimes C}(M \otimes N)_{B\otimes D}. $$

The 2D oriented Cobordism Hypothesis

Definition

A $\Bbbk$-algebra is (strongly) ***separable*** if:

- A has finite dimension.
- $A \otimes \mathbb{K}$ is semisimple for each field extension $\mathbb{K} \hookleftarrow \Bbbk$.

Separable algebras are the fully dualizable objects of $\mathbf{Alg}_2$.

Theorem (Schommer-Pries, 2009)

*There is a natural equivalence between **2D oriented extended TQFTs** $Z\colon \mathbf{Cob}_2^{\mathrm{or}}(2) \to \mathbf{Alg}_2$ and **separable symmetric Frobenius algebras**.*

- The algebra is given by $Z(\mathrm{pt}_+)$.
- The Frobenius form $\varepsilon\colon A \to \Bbbk$ is induced by $Z($ $)\colon Z(\mathbb{S}^1) \to \Bbbk$.
- Separability is a consequence of full dualizability. (Not trivial).
- Symmetry is a consequence of the identifications made.

The 2D oriented Cobordism Hypothesis

Definition

A $\Bbbk$-algebra is (strongly) ***separable*** if:

- A has finite dimension.
- $A \otimes \mathbb{K}$ is semisimple for each field extension $\mathbb{K} \hookleftarrow \Bbbk$.

Separable algebras are the fully dualizable objects of $\mathbf{Alg}_2$.

Theorem (Schommer-Pries, 2009)

There is a natural equivalence between ***2D oriented extended TQFTs*** $Z\colon \mathbf{Cob}_2^{\mathrm{or}}(2) \to \mathbf{Alg}_2$ *and* ***separable symmetric Frobenius algebras.***

- The algebra is given by $Z(\mathrm{pt}_+)$.
- The Frobenius form $\varepsilon\colon A \to \Bbbk$ is induced by $Z($ $)\colon Z(\mathbb{S}^1) \to \Bbbk$.
- Separability is a consequence of full dualizability. (Not trivial).
- Symmetry is a consequence of the identifications made.

The 2D oriented Cobordism Hypothesis

Definition

A $\Bbbk$-algebra is (strongly) ***separable*** if:

- A has finite dimension.
- $A \otimes \mathbb{K}$ is semisimple for each field extension $\mathbb{K} \hookleftarrow \Bbbk$.

Separable algebras are the fully dualizable objects of $\mathbf{Alg}_2$.

Theorem (Schommer-Pries, 2009)

There is a natural equivalence between ***2D oriented extended TQFTs*** $Z\colon \mathbf{Cob}_2^{\mathrm{or}}(2) \to \mathbf{Alg}_2$ *and* ***separable symmetric Frobenius algebras.***

- The algebra is given by $Z(\mathrm{pt}_+)$.
- The Frobenius form $\varepsilon\colon A \to \Bbbk$ is induced by $Z(\quad)\colon Z(\mathbb{S}^1) \to \Bbbk$.
- Separability is a consequence of full dualizability. (Not trivial).
- Symmetry is a consequence of the identifications made.

The 2D oriented Cobordism Hypothesis

Definition

A $\Bbbk$-algebra is (strongly) ***separable*** if:

- A has finite dimension.
- $A \otimes \mathbb{K}$ is semisimple for each field extension $\mathbb{K} \hookleftarrow \Bbbk$.

Separable algebras are the fully dualizable objects of $\mathbf{Alg}_2$.

Theorem (Schommer-Pries, 2009)

There is a natural equivalence between ***2D oriented extended TQFTs*** $Z\colon \mathbf{Cob}_2^{\mathrm{or}}(2) \to \mathbf{Alg}_2$ *and* ***separable symmetric Frobenius algebras***.

- The algebra is given by $Z(\mathrm{pt}_+)$.
- The Frobenius form $\varepsilon\colon A \to \Bbbk$ is induced by $Z(\quad)\colon Z(\mathbb{S}^1) \to \Bbbk$.
- Separability is a consequence of full dualizability. (Not trivial).
- Symmetry is a consequence of the identifications made.

The 2D oriented Cobordism Hypothesis

Definition

A $\Bbbk$-algebra is (strongly) ***separable*** if:

- A has finite dimension.
- $A \otimes \mathbb{K}$ is semisimple for each field extension $\mathbb{K} \hookleftarrow \Bbbk$.

Separable algebras are the fully dualizable objects of $\mathbf{Alg}_2$.

Theorem (Schommer-Pries, 2009)

There is a natural equivalence between ***2D oriented extended TQFTs*** $Z\colon \mathbf{Cob}_2^{\mathrm{or}}(2) \to \mathbf{Alg}_2$ *and* ***separable symmetric Frobenius algebras****.*

- The algebra is given by $Z(\mathrm{pt}_+)$.
- The Frobenius form $\varepsilon\colon A \to \Bbbk$ is induced by $Z(\quad)\colon Z(\mathbb{S}^1) \to \Bbbk$.
- Separability is a consequence of full dualizability. (Not trivial).
- Symmetry is a consequence of the identifications made.

The 2D oriented Cobordism Hypothesis

Definition

A $\Bbbk$-algebra is (strongly) ***separable*** if:

- A has finite dimension.
- $A \otimes \mathbb{K}$ is semisimple for each field extension $\mathbb{K} \hookleftarrow \Bbbk$.

Separable algebras are the fully dualizable objects of $\mathbf{Alg}_2$.

Theorem (Schommer-Pries, 2009)

There is a natural equivalence between ***2D oriented extended TQFTs*** $Z\colon \mathbf{Cob}_2^{\mathrm{or}}(2) \to \mathbf{Alg}_2$ *and* ***separable symmetric Frobenius algebras****.*

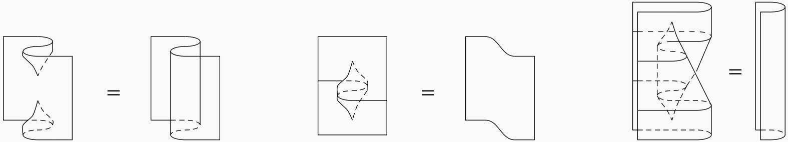
- The algebra is given by $Z(\mathrm{pt}_+)$.
- The Frobenius form $\varepsilon\colon A \to \Bbbk$ is induced by $Z($  $)\colon Z(\mathbb{S}^1) \to \Bbbk$.
- Separability is a consequence of full dualizability. (Not trivial).
- Symmetry is a consequence of the identifications made.

Identifying the value of the circle

We made multiple identifications which we should detail.

First, assign $Z(\mathrm{pt}_+) =: A$.

The cusp relations allow us to identify $Z(\mathrm{pt}_-) \simeq A^{\mathrm{op}}$:



(This is a ***Morita context***, an adjoint equivalence in $\mathbf{Alg}_2$).

We will abuse notation and treat this as an equality: $Z(\mathrm{pt}_-) =: A^{\mathrm{op}}$.

The two generating 1-morphisms must become bimodules:

$Z($ $) =: {}_{A \otimes A^{\mathrm{op}}}M \qquad Z($ $) =: N_{A \otimes A^{\mathrm{op}}}$

Consider the circle $\mathbb{S}^1 =$.

Set $V := Z(\mathbb{S}^1) = N \otimes_{A \otimes A^{\mathrm{op}}} M$; this is a $\Bbbk$-vector space.

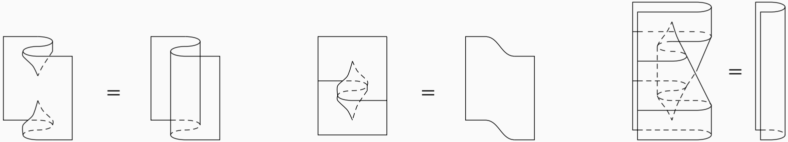
There exist identifications $V \cong A \otimes_{A \otimes A^{\mathrm{op}}} A \cong A/[A, A]$.

Identifying the value of the circle

We made multiple identifications which we should detail.

First, assign $Z(\mathrm{pt}_+) =: A$.

The cusp relations allow us to identify $Z(\mathrm{pt}_-) \simeq A^{\mathrm{op}}$:



(This is a ***Morita context***, an adjoint equivalence in $\mathbf{Alg}_2$).

We will abuse notation and treat this as an equality: $Z(\mathrm{pt}_-) =: A^{\mathrm{op}}$.

The two generating 1-morphisms must become bimodules:

$Z($$) =: {}_{A\otimes A^{\mathrm{op}}}M \qquad Z($$) =: N_{A\otimes A^{\mathrm{op}}}$

Consider the circle $\mathbb{S}^1 =$  .

Set $V := Z(\mathbb{S}^1) = N \otimes_{A\otimes A^{\mathrm{op}}} M$; this is a $\Bbbk$-vector space.

There exist identifications $V \cong A \otimes_{A\otimes A^{\mathrm{op}}} A \cong A/[A, A]$.

Defining the Frobenius form

We have identified $V \coloneqq Z(\mathbb{S}^1) \cong A/[A,A]$.

The cap  evaluates to a Frobenius form $V \to \Bbbk$.

Define $\varepsilon : A \to \Bbbk$ as the composition

$$\varepsilon : A \twoheadrightarrow A/[A,A] \cong V \to \Bbbk.$$

This forces (A, ε) to be symmetric: $\varepsilon(xy) = \varepsilon(yx)$.

We will now prove that $V \cong \mathrm{Center}(A)$, which will let us compare our results with the unextended case.

Defining the Frobenius form

We have identified $V \coloneqq Z(\mathbb{S}^1) \cong A/[A,A]$.

The cap  evaluates to a Frobenius form $V \to \Bbbk$.

Define $\varepsilon : A \to \Bbbk$ as the composition

$$\varepsilon : A \twoheadrightarrow A/[A,A] \cong V \to \Bbbk.$$

This forces (A, ε) to be symmetric: $\varepsilon(xy) = \varepsilon(yx)$.

We will now prove that $V \cong \mathrm{Center}(A)$, which will let us compare our results with the unextended case.

The value of the circle is the center

We can identify Center(A) with the A-A-bimodule maps $f\colon {}_AA_A \to {}_AA_A$.
(These f are of the form $f(a) = c \cdot a = a \cdot c$ for some $c \in$ Center(A)).

We will prove that $V \cong \mathrm{Hom}_{{}_A\mathbf{Bimod}_A}$. This is done graphically.

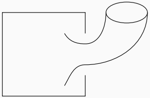
${}_AA_A \otimes V \to {}_AA_A$

$\mathrm{Hom}_{{}_A\mathbf{Bimod}_A} \to V$

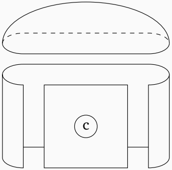
The value of the circle is the center

We can identify Center(A) with the A-A-bimodule maps $f\colon {}_AA_A \to {}_AA_A$. (These f are of the form $f(a) = c \cdot a = a \cdot c$ for some $c \in \mathrm{Center}(A)$).

We will prove that $V \cong \mathrm{Hom}_{{}_A\mathbf{Bimod}_A}$. This is done graphically.



${}_AA_A \otimes V \to {}_AA_A$

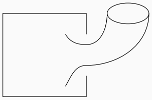


$\mathrm{Hom}_{{}_A\mathbf{Bimod}_A} \to V$

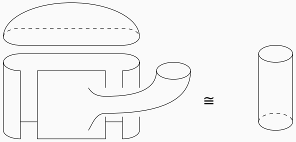
The value of the circle is the center

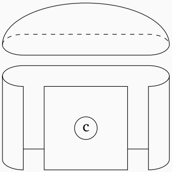
We can identify Center(A) with the A-A-bimodule maps $f\colon {}_AA_A \to {}_AA_A$.
(These f are of the form $f(a) = c \cdot a = a \cdot c$ for some $c \in$ Center(A)).

We will prove that $V \cong \mathrm{Hom}_{{}_A\mathbf{Bimod}_A}$. This is done graphically.

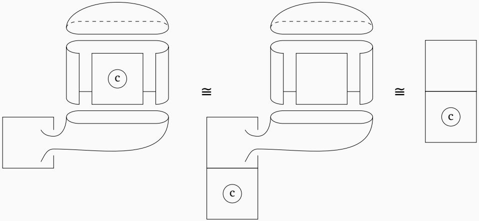


${}_AA_A \otimes V \to {}_AA_A$





$\mathrm{Hom}_{{}_A\mathbf{Bimod}_A} \to V$



Relating unextended and extended TQFTs

Given an extended TQFT $Z\colon \mathbf{Cob}_2^{\mathrm{or}}(2) \to \mathbf{Alg}_2$, one can obtain an unextended TQFT $\Omega Z\colon \mathbf{Cob}_2^{\mathrm{or}} \to \mathbf{Vect}_{\mathbb{k}}$ by taking loops:

$$\Omega\,\mathbf{Cob}_2^{\mathrm{or}}(2) = \mathrm{Map}_{\mathbf{Cob}_2^{\mathrm{or}}(2)}(\varnothing, \varnothing) \simeq \mathbf{Cob}_2, \quad \Omega\,\mathbf{Alg}_2 = \mathrm{Map}_{\mathbf{Alg}_2}(\mathbb{k}, \mathbb{k}) \simeq \mathbf{Vect}_{\mathbb{k}}.$$

This has the effect of taking centers in the Frobenius algebra analogy:

$$(\Omega Z)(\mathbb{S}^1) = \mathrm{Center}\, Z(\mathrm{pt}_Z) = Z(\mathbb{S}^1).$$

An unextended TQFT $W\colon \mathbf{Cob}_2^{\mathrm{or}} \to \mathbf{Vect}_{\mathbb{k}}$ extends down to points if and only if the Frobenius algebra $W(\mathbb{S}^1)$ is separable.

This extension is in general not unique.

Final recap

We compare our two kinds of 2D oriented TQFTs.

Unextended	**Extended**
1-categorical	2-categorical
Morse Theory	Cerf Theory
$\mathbf{Vect}_{\Bbbk}$	$\mathbf{Alg}_2$
$\mathbb{S}^1$,  and 	pt_+,  ,  ,  and 
Commutative Frobenius algebras	Separable symmetric Frobenius algebras

Thank you!

References

[1] J. C. Baez and J. Dolan. (1995). “Higher-dimensional Algebra and Topological Quantum Field Theory”. DOI: 10.1063/1.531236. arXiv: q-alg/9503002.

[2] J. Kock. (2003). *Frobenius Algebras and 2D Topological Quantum Field Theories*. DOI: 10.1017/CBO9780511615443.

[3] J. Lurie. (2009). “On the Classification of Topological Field Theories”. DOI: 10.4310/CDM.2008.v2008.n1.a3. arXiv: 0905.0465 [math.CT].

[4] C. J. Schommer-Pries. (2009). “The Classification of Two-Dimensional Extended Topological Field Theories”. arXiv: 1112.1000v1 [math.AT].

[5] S. Pareja Pérez. (2024). “2D Topological Quantum Field Theories, Frobenius Structures, and Higher Algebra”. DOI: 20.500.14352/105943.