An introduction to the Cobordism Hypothesis

VII Congreso de Jóvenes Investigadores de la RSME

Santiago Pareja Pérez

Unizar–UCM

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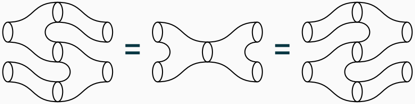

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We want to try to explain the following statement:

The Cobordism Hypothesis (Baez–Dolan, 1995)

The n-category of framed cobordisms $\mathbf{Bord}_n^{\mathrm{fr}}$ is...

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Unextended cobordisms and TQFTs

Cobordisms

We assume everything smooth and compact.

Let M and N be two closed $(n-1)$-manifolds.

A ***cobordism*** $B : M \to N$ is an n-manifold with boundary endowed with two embeddings $M \hookrightarrow \partial B \hookleftarrow N$ such that $\partial B \cong M \sqcup N$.

M is the ***in-boundary*** and N is the ***out-boundary***.

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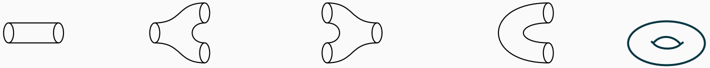

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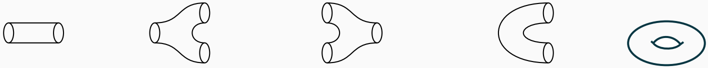

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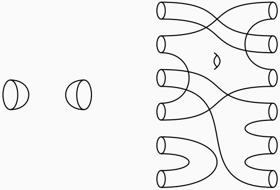

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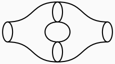
Gluing and adding cobordisms

We can ***compose*** cobordisms by gluing.



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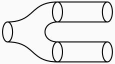

$\mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1$

The identities are the cylinders $M \times [0,1]\colon M \to M$.

We can add cobordisms by taking their disjoint union.

$\mathbb{S}^1 \sqcup (\mathbb{S}^1 \sqcup \mathbb{S}^1) \to (\mathbb{S}^1 \sqcup \mathbb{S}^1) \sqcup \mathbb{S}^1$

This is a ***monoidal structure*** with unit the empty manifold: $M \sqcup \varnothing \cong M$.

We can freely interchange connected components.
These *twist cobordisms* give a ***symmetric structure***.

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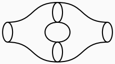

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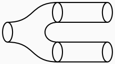

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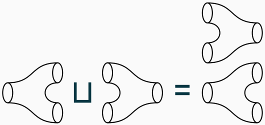

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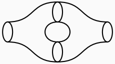

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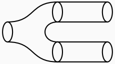

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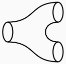
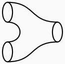
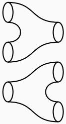
 $\sqcup$  $=$ 

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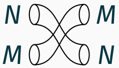

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$\mathbf{Cob}_n$, the category of n-cobordisms

n-cobordisms assemble into a symmetric monoidal category, $\mathbf{Cob}_n$:

Objects Closed $(n-1)$-manifolds M, N.

Morphisms n-cobordisms $B\colon M \to N$, up to diffeomorphism.

Identities "Cylinders" $M \times [0,1]\colon M \to M$.

Composition Gluing of cobordisms.

Monoid Disjoint union $B \sqcup B' : M \sqcup M' \to N \sqcup N'$.

Unit Empty manifold $\varnothing$.

Twists Twist cobordisms $M \sqcup N \to N \sqcup M$.

One well-understood symmetric monoidal category is $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$, the category of vector spaces equipped with tensor product and the usual interchange of factors.
(i.e., the twist maps $\sigma_{V,W}\colon V \otimes W \to W \otimes V$ are given by $v \otimes w \mapsto w \otimes v$).

We can use $\mathbf{Vect}_{\Bbbk}$ to study $\mathbf{Cob}_n$, via maps $\mathbf{Cob}_n \to \mathbf{Vect}_{\Bbbk}$.

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Topological Quantum Field Theories

Let $\mathcal{C}$ be a symmetric monoidal category.
A ***TQFT*** is a symmetric monoidal functor $Z\colon \mathbf{Cob}_n \to \mathcal{C}$.

Consider the case $\mathcal{C} = \mathbf{Vect}_{\Bbbk}$. A TQFT $Z\colon \mathbf{Cob}_n \to \mathbf{Vect}_{\Bbbk}$ assigns:

- closed $(n-1)$-manifold $M \rightsquigarrow$ $\Bbbk$-vector space $Z(M)$.
- n-cobordism $B\colon M \to N \rightsquigarrow$ $\Bbbk$-linear map $Z(B)\colon Z(M) \to Z(N)$.

And these must satisfy the axioms of a symmetric monoidal functor: for example, $Z(B \sqcup B') = Z(B) \otimes Z(B')$.

Usually our manifolds carry extra structure, such as an orientation ($\mathbf{Cob}_n^{\mathrm{or}}$) or a framing ($\mathbf{Cob}_n^{\mathrm{fr}}$).
All is the same: for example, an ***oriented TQFT*** is a map $Z\colon \mathbf{Cob}_n^{\mathrm{or}} \to \mathcal{C}$.

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We can give explicit generators and relations for $\mathbf{Cob}_2^{\mathrm{or}}$.

Generators:

- Objects: the circle $\mathbb{S}^1$. (Both orientations are isomorphic).
- Morphisms:

(Proof: Morse theory).

Relations:

And the mirrored ones.

TQFTs: the oriented 2D case (1)

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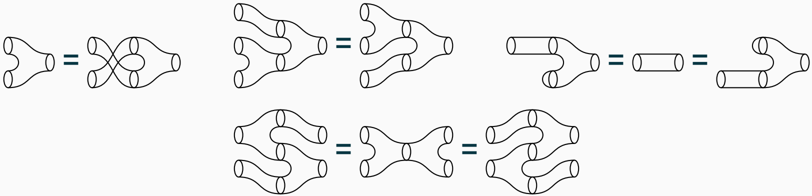

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Consider TQFTs $Z\colon \mathbf{Cob}_2^{\mathrm{or}} \to \mathbf{Vect}_{\Bbbk}$.

- Generators become algebraic structure.
- Relations become algebraic properties.

First consider the generators.

The circle becomes a vector space $A = Z(\mathbb{S}^1)$. It is equipped with:

- A ***unit*** $Z(\ )\colon \Bbbk \to A$.
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2D oriented TQFTs are commutative Frobenius algebras

Now consider the relations of $\mathbf{Cob}_2^{\mathrm{or}}$.

The vector space $A = Z(\mathbb{S}^1)$ becomes:

- A commutative, associative and unital algebra

- and also a cocommutative, coassociative and counital coalgebra

- in a compatible way:

Theorem (Folklore)

2D oriented TQFTs $Z\colon \mathbf{Cob}_2^{\mathrm{or}} \to \mathbf{Vect}_{\Bbbk}$ are the same as commutative Frobenius algebras.

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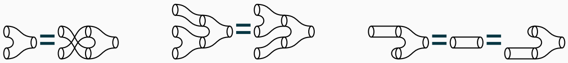

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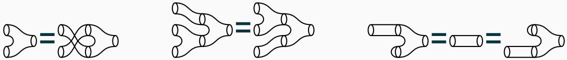

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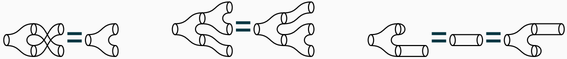

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Theorem (Folklore)

2D oriented TQFTs $Z\colon \mathbf{Cob}_2^{\mathrm{or}} \to \mathbf{Vect}_{\Bbbk}$ *are the same as commutative Frobenius algebras.*

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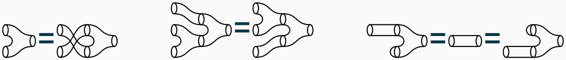

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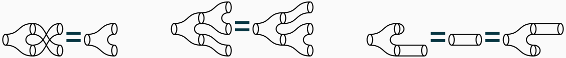

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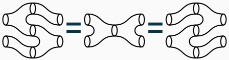

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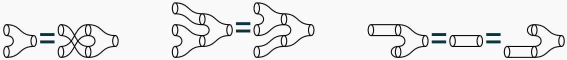

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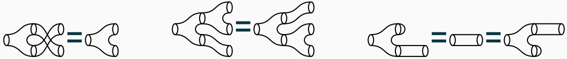

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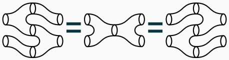

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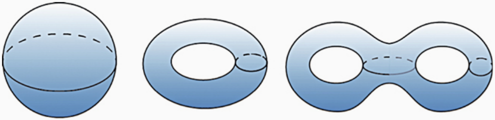
Extended cobordisms and TQFTs: The Cobordism Hypothesis

What about higher dimensions?

We would like to generalize this result for dimensions $n > 2$.
But notice that even the case $n = 3$ is a lot more complex: the category $\mathbf{Cob}_3^{\mathrm{or}}$ has infinitely many generating objects (the g-tori).



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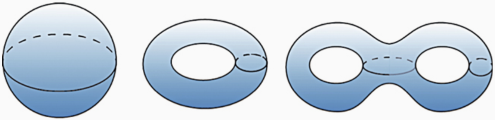

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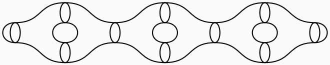

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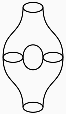

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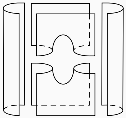

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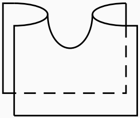

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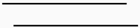
Target: 

We read 1-cobs from left to right, and 2-cobs from top to bottom.

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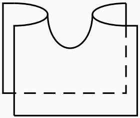

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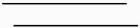
Target: 

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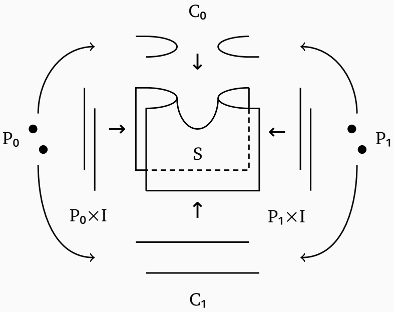
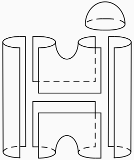


Gluing and adding extended cobordisms

We can compose extended cobordisms in multiple directions:



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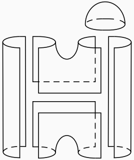

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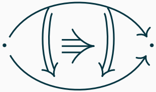
$\mathbf{Bord}_n$, the n-category of extended n-cobordisms

An *n-category* has k-morfisms between $(k-1)$-morfisms:



We can define an n-category of cobordisms, $\mathbf{Bord}_n$:

Objects Closed 0-manifolds (finite unions of points).

1-morphisms 1-cobordisms between 0-manifolds.

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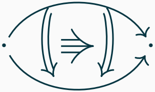

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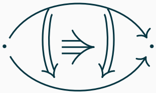

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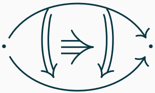

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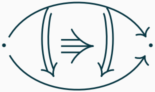

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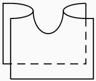
2-morphisms 2-cobordisms with corners. 

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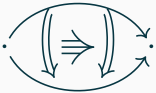

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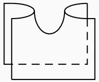
2-morphisms 2-cobordisms with corners. 

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The absurdity of specifying extended TQFTs

An ***extended TQFT*** is a symmetric monoidal functor of n-categories

$$Z\colon \mathbf{Bord}_n \to \mathcal{C}.$$

By definition, we need to assign a value to each k-morphism of $\mathbf{Bord}_n$. But we must ensure that everything commutes:

No matter how we cut up our manifolds, the result must be the same.

Extended TQFTs for $n \gg 0$ are absurdly hard to construct:

We have a lot of data to assign, but also a lot of constraints to satisfy.

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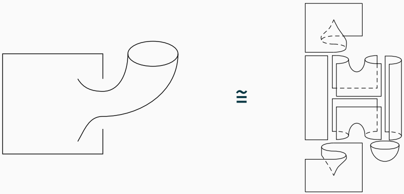

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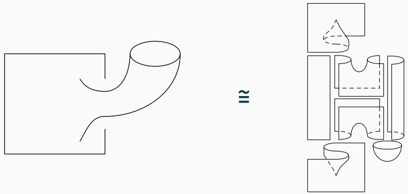

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Consider the 1-category $\mathbf{Bord}_1^{\mathrm{fr}} = \mathbf{Cob}_1^{\mathrm{fr}} = \mathbf{Cob}_1^{\mathrm{or}} = \mathbf{Bord}_1^{\mathrm{or}}$.

- Its objects are 1-framed 0-manifolds: finite unions of points.
- Its morphisms are 1-framed 1-cobordisms: circles and lines.

Every object can be written as a disjoint union $\coprod_p \mathrm{pt}^+ \sqcup \coprod_m \mathrm{pt}^-$.

And every morphism is generated from the following two:

There are two relations, known as "Zorro's Lemma":

Now consider TQFTs $Z\colon \mathbf{Bord}_1^{\mathrm{fr}} \to \mathcal{C}$.

These two diagrams impose conditions on the image of the point: the target object $Z(\mathrm{pt}^+) \in \mathcal{C}$ must be ***dualizable***.

So giving a TQFT $Z\colon \mathbf{Bord}_1^{\mathrm{fr}} \to \mathcal{C}$ is the same as selecting an object $Z(\mathrm{pt}^+) \in \mathcal{C}$, and that object must be dualizable.

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Say we want to construct an extended TQFT.
Assume that our manifolds are ***framed***, so that they are very rigid.

We can cut things up as much as we want, so maybe the $n = 1$ case generalizes to higher dimensions:

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That is ***The Cobordism Hypothesis***.

But our initial statement was actually stronger:

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Fully dualizable objects

An object $X \in \mathcal{C}$ is ***dualizable*** if there exists:

- A dual object $X^\vee \in \mathcal{C}$;
- Morphisms $\mathrm{ev}\colon X \square X^\vee \to \mathrm{I}$ and $\mathrm{coev}\colon \mathrm{I} \to X^\vee \square X$;
- such that the following compositions are the identities:

$$\begin{aligned} \mathrm{id}_X : X &\xrightarrow{\mathrm{id}_X \square \mathrm{coev}} X \square X^\vee \square X \xrightarrow{\mathrm{ev} \square \mathrm{id}_X} X, \\ \mathrm{id}_{X^\vee} : X^\vee &\xrightarrow{\mathrm{coev} \square \mathrm{id}_{X^\vee}} X^\vee \square X \square X^\vee \xrightarrow{\mathrm{id}_{X^\vee} \square \mathrm{ev}} X^\vee. \end{aligned}$$

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- X is dualizable;
- the morphisms ev and coev admit adjoints;
- the 2-morphisms witnessing those adjunctions admit adjoints;

$\vdots$

- the $(n-1)$-morphisms witnessing those adjunctions admit adjoints.

An n-category $\mathcal{C}$ is ***fully dualizable*** if every one of its objects is fully dualizable, and every one of its k-morphisms admits adjoints.

Any category has a maximal fully dualizable subcategory $\mathcal{C}^{\mathrm{fd}}$.

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- the $(n-1)$-morphisms witnessing those adjunctions admit adjoints.

An n-category $\mathcal{C}$ is ***fully dualizable*** if every one of its objects is fully dualizable, and every one of its k-morphisms admits adjoints.

Any category has a maximal fully dualizable subcategory $\mathcal{C}^{\mathrm{fd}}$.

Fully dualizable objects

An object $X \in \mathcal{C}$ is ***dualizable*** if there exists:

- A dual object $X^\vee \in \mathcal{C}$;
- Morphisms $\mathrm{ev}\colon X \square X^\vee \to \mathrm{I}$ and $\mathrm{coev}\colon \mathrm{I} \to X^\vee \square X$;
- such that the following compositions are the identities:

$$\mathrm{id}_X : X \xrightarrow{\mathrm{id}_X \square \mathrm{coev}} X \square X^\vee \square X \xrightarrow{\mathrm{ev} \square \mathrm{id}_X} X,$$
$$\mathrm{id}_{X^\vee} : X^\vee \xrightarrow{\mathrm{coev} \square \mathrm{id}_{X^\vee}} X^\vee \square X \square X^\vee \xrightarrow{\mathrm{id}_{X^\vee} \square \mathrm{ev}} X^\vee.$$

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The canonical $\mathrm{O}(n)$-action on $\mathbf{Bord}_n^{\mathrm{fr}}$

Consider the n-category $\mathbf{Bord}_n^{\mathrm{fr}}$.

Its k-morphisms are k-manifolds with corners M^k equipped with an *n-framing*: a trivialization $\mathrm{T}M \otimes \underline{\mathbb{R}}^{n-k} \to \underline{\mathbb{R}}^n$.

We can twist this framing by elements of $\mathrm{O}(n)$ in a natural way, by lifting the action $\mathrm{O}(n) \curvearrowright \mathbb{R}^n$. So $\mathrm{O}(n)$ acts on $\mathbf{Bord}_n^{\mathrm{fr}}$.

Now let $\mathcal{C}$ be any symmetric monoidal n-category.

The action $\mathrm{O}(n) \curvearrowright \mathbf{Bord}_n^{\mathrm{fr}}$ induces an action on $\mathrm{Hom}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C})$.

But remember:

Giving a map $Z\colon \mathbf{Bord}_n^{\mathrm{fr}} \to \mathcal{C}$ is the same as
selecting a fully dualizable object $Z(\mathrm{pt}^+) \in \mathcal{C}^{\mathrm{fd}}$.

In other words, $\mathrm{Hom}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \simeq (\mathcal{C}^{\mathrm{fd}})^{\sim}$, where $(\mathcal{C}^{\mathrm{fd}})^{\sim}$ is the space of fully dualizable objects of $\mathcal{C}$.

In conclusion:

Given any n-category $\mathcal{C}$, there is a canonical action $\mathrm{O}(n) \curvearrowright (\mathcal{C}^{\mathrm{fd}})^{\sim}$.

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Generalization to G-structured cobordisms

Given a subgroup $G \hookrightarrow \mathrm{O}(n)$ and a manifold M^n, a ***G-structure*** on M is a reduction of structure group of the frame bundle $\mathrm{Fr}\, M$ from $\mathrm{O}(n)$ to G.

(This makes sense up to homotopy).

- A $\{1\}$-structure is a framing.
- An $\mathrm{SO}(n)$-structure fixes an orientation.
- An $\mathrm{O}(n)$-structure gives no structure at all.

The Cobordism Hypothesis also has a G-structured version.

Thesis (Lurie, 2009)

*G-structured extended TQFTs $Z : \mathbf{Bord}_n^G \to \mathcal{C}$ correspond to **homotopy fixed points** of the canonical G-action $G \hookrightarrow \mathrm{O}(n) \curvearrowright (\mathcal{C}^{\mathrm{fd}})^{\sim}$:*

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Final example: 2D oriented extended TQFTs

Consider $\mathbf{Bord}_2^{\mathrm{or}}$.

- Notice that 1-morphisms $M, N \colon \varnothing \to \varnothing$ are closed 1-manifolds.
- And a 2-morphism $B \colon M \Rightarrow N$ is an unextended 2-cobordism.

So we can identify $\mathrm{Hom}_{\mathbf{Bord}_2^{\mathrm{or}}}(\varnothing, \varnothing) \simeq \mathbf{Cob}_2^{\mathrm{or}}$.

In other words, every extended TQFT yields an unextended TQFT. But the converse is not true.

When taking $\mathcal{C}$ to be the Morita bicategory $\mathbf{Alg}_2$, one can prove:

Theorem (Schommer-Pries, 2009)

2D oriented extended TQFTs $Z \colon \mathbf{Bord}_2^{\mathrm{or}} \to \mathbf{Alg}_2$ are the same as separable symmetric Frobenius algebras.

So in this context, a TQFT $Z \colon \mathbf{Cob}_2^{\mathrm{or}} \to \mathbf{Vect}_{\Bbbk}$ will extend down to points if and only if its corresponding Frobenius algebra is separable.

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Thank you!

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