Type Inference & Unification

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Notation

Notation

$$thing ::=$$
$$stuff$$
$$more\ stuff$$
$$\dots$$

Notation

$$\frac{assumption_1 \qquad assumption_2 \qquad \ldots \qquad assumption_n}{conclusion} \; (\text{RULE-NAME})$$

Notation

$$Context ::= $$
$$\bullet$$
$$Context, var : Type$$

Notation

$$Context \vdash term : Type$$

STLC

STLC

$$Type ::=$$
$$\qquad Type \rightarrow Type$$
$$\qquad \texttt{Bool}$$

$$term ::=$$
$$\qquad var$$
$$\qquad \lambda var\!:\!Type.\ term$$
$$\qquad term\ term$$
$$\qquad \texttt{true}$$
$$\qquad \texttt{false}$$
$$\qquad \texttt{if}\ term\ \texttt{then}\ term\ \texttt{else}\ term$$

STLC

$$\overline{\Gamma, x : T \vdash x : T} \tag{T-var}$$

$$\frac{\Gamma, x : S \vdash e : T}{\Gamma \vdash \lambda x{:}S.\, e \; : \; S \to T} \tag{T-lam-ann}$$

$$\frac{\Gamma \vdash f : S \to T \qquad \Gamma \vdash x : S}{\Gamma \vdash f\, x \; : \; T} \tag{T-app}$$

$$\overline{\Gamma \vdash \texttt{true} : \texttt{Bool}} \tag{T-true}$$

$$\overline{\Gamma \vdash \texttt{false} : \texttt{Bool}} \tag{T-false}$$

$$\frac{\Gamma \vdash b : \texttt{Bool} \qquad \Gamma \vdash x : T \qquad \Gamma \vdash y : T}{\Gamma \vdash \texttt{if } b \texttt{ then } x \texttt{ else } y \; : \; T} \tag{T-if}$$

STLC

λb:Bool. if b then false else true

STLC

$$\text{Bool} \to \text{Bool}$$

$$\lambda b.\ \text{if } b \text{ then false else true}$$

STLC

$$term ::=$$
$$\cdots$$
$$\lambda var . \; term$$

STLC

$$\overline{\rule{4cm}{0pt}} \qquad\qquad\qquad (\text{T-lam})$$

STLC

$$\frac{\qquad\qquad\qquad\qquad}{\Gamma \vdash \lambda x.\ e\ :} \qquad\qquad (\text{T-lam})$$

STLC

$$\frac{\Gamma, x : \quad \vdash\ e\ :}{\Gamma\ \vdash\ \lambda x.\ e\quad :} \qquad \text{(T-lam)}$$

STLC

$$\frac{\Gamma, x : \text{??} \vdash e :}{\Gamma \vdash \lambda x.\, e \ :} \qquad \text{(T-LAM)}$$

STLC

$$\frac{\Gamma, x : \text{??} \vdash e : \boxed{T}}{\Gamma \vdash \lambda x .\, e \ :} \qquad \text{(T-\textsc{lam})}$$

STLC

$$\frac{\Gamma, x : \text{??} \vdash e : T}{\Gamma \vdash \lambda x . e \ : \ \text{??} \to T} \qquad (\text{T-lam})$$

Type Inference

Type Inference

$$\Gamma \vdash x \Rightarrow T$$

Type Inference

$$\Gamma \vdash x \Rightarrow T$$

Type Inference

$$\Gamma \;\vdash\; x \;\Rightarrow\; \boxed{T}$$

Type Inference

$$\overline{\Gamma, \, x : T \vdash x \Rightarrow T} \qquad\qquad (\text{T-\textsc{var}})$$

$$\frac{\Gamma, \, x : S \vdash e \Rightarrow T}{\Gamma \vdash \lambda x{:}S.\, e \;\Rightarrow\; S \rightarrow T} \qquad\qquad (\text{T-\textsc{lam-ann}})$$

$$\overline{\Gamma \vdash \texttt{true} \Rightarrow \texttt{Bool}} \qquad\qquad (\text{T-\textsc{true}})$$

$$\overline{\Gamma \vdash \texttt{false} \Rightarrow \texttt{Bool}} \qquad\qquad (\text{T-\textsc{false}})$$

Type Inference

$$\frac{\Gamma, x : S \vdash e \Rightarrow T}{\Gamma \vdash \lambda x{:}S.\, e \;\Rightarrow\; S \to T} \qquad \text{(T-LAM-ANN)}$$

Type Inference

$$\frac{\Gamma, x : S \vdash e \Rightarrow T}{\Gamma \vdash \lambda x : S . \ e \ \Rightarrow \ S \to T} \qquad \text{(T-lam-ann)}$$

Type Inference

$$\frac{\Gamma, x : S \vdash e \Rightarrow T}{\Gamma \vdash \lambda x{:}S.\ e \;\;\Rightarrow\;\; S \rightarrow T} \qquad (\text{T-lam-ann})$$

Type Inference

$$\frac{\Gamma, x : S \vdash e \Rightarrow T}{\Gamma \vdash \lambda x{:}S.\, e \;\Rightarrow\; S \to T} \qquad (\text{T-lam-ann})$$

Type Inference

$$\frac{\Gamma, x : S \vdash e \Rightarrow T}{\Gamma \vdash \lambda x{:}S.\ e \ \Rightarrow \ \boxed{S \to T}} \qquad \text{(T-lam-ann)}$$

Type Inference

$$meta ::=$$
$$?^0$$
$$?^1$$
$$\ldots$$
$$?^n$$

$$Type ::=$$
$$\ldots$$
$$meta \text{ (only during type-checking)}$$

Type Inference

$$\frac{new(?^n) \qquad \Gamma, \, x : ?^n \, \vdash \, e \, \Rightarrow \, T}{\Gamma \, \vdash \, \lambda x . \, e \quad \Rightarrow \quad ?^n \to T} \qquad \text{(T-LAM)}$$

Type Inference

$$\frac{new(?^n) \qquad \Gamma, \, x : ?^n \, \vdash \, e \, \Rightarrow \, T}{\Gamma \, \vdash \, \lambda x. \, e \, \Rightarrow \, ?^n \to T} \qquad \text{(T-LAM)}$$

Type Inference

$$\frac{new(?^n) \qquad \Gamma, x : ?^n \vdash e \Rightarrow T}{\Gamma \vdash \lambda x.\, e \quad \Rightarrow \quad ?^n \to T} \qquad \text{(T-LAM)}$$

Type Inference

$$\frac{new(?^n) \qquad \Gamma, x : ?^n \vdash e \Rightarrow T}{\Gamma \vdash \lambda x . e \quad \Rightarrow \quad ?^n \to T} \qquad \text{(T-LAM)}$$

Type Inference

$$\frac{new(?^n) \qquad \Gamma, \, x : ?^n \; \vdash \; e \; \Rightarrow \; \boxed{T}}{\Gamma \; \vdash \; \lambda x . \, e \quad \Rightarrow \quad ?^n \to T} \qquad \text{(T-\textsc{lam})}$$

Type Inference

$$\frac{new(?^n) \qquad \Gamma, \, x : ?^n \, \vdash \, e \, \Rightarrow \, T}{\Gamma \, \vdash \, \lambda x . \, e \, \Rightarrow \, ?^n \rightarrow T} \qquad (\text{T-lam})$$

Type Inference

$$\rule{6cm}{0.4pt} \qquad (\text{T-APP})$$

Type Inference

$$\frac{}{\Gamma \vdash f\ x\ \Rightarrow} \qquad (\text{T-App})$$

Type Inference

$$\frac{\Gamma \vdash f \Rightarrow}{\Gamma \vdash f\ x \quad \Rightarrow} \qquad (\text{T-App})$$

Type Inference

$$\frac{\Gamma \;\vdash\; f \;\Rightarrow\; S}{\Gamma \;\vdash\; f\,x \;\;\Rightarrow\;} \qquad\qquad (\text{T-App})$$

Type Inference

$$\frac{\Gamma \vdash x \Rightarrow \quad \Gamma \vdash f \Rightarrow S}{\Gamma \vdash f \, x \quad \Rightarrow} \qquad \text{(T-App)}$$

Type Inference

$$\frac{\begin{array}{l} \Gamma \;\vdash\; x \;\Rightarrow\; \boxed{T} \\ \Gamma \;\vdash\; f \;\Rightarrow\; S \end{array}}{\Gamma \;\vdash\; f\,x \;\;\Rightarrow} \qquad (\text{T-APP})$$

Type Inference

$$\frac{\begin{array}{c} \Gamma \ \vdash \ x \ \Rightarrow \ T \qquad \boxed{new(?^n)} \\ \Gamma \ \vdash \ f \ \Rightarrow \ S \end{array}}{\Gamma \ \vdash \ f \ x \quad \Rightarrow} \qquad (\text{T-APP})$$

Type Inference

$$\frac{\Gamma \vdash x \Rightarrow T \qquad new(?^n) \\ \Gamma \vdash f \Rightarrow S \qquad \boxed{[S = T \rightarrow ?^n]}}{\Gamma \vdash f\,x \ \Rightarrow} \qquad (\text{T-APP})$$

Type Inference

$$\frac{\Gamma \vdash x \Rightarrow T \qquad new(?^n) \qquad \Gamma \vdash f \Rightarrow S \qquad [S = T \to ?^n]}{\Gamma \vdash f\,x \Rightarrow ?^n} \quad (\text{T-App})$$

Type Inference

$$\rule{8cm}{0.4pt} \quad \text{(T-IF)}$$

Type Inference

$$\frac{\rule{0pt}{0pt}\hspace{6cm}}{\Gamma \vdash \texttt{if } b \texttt{ then } x \texttt{ else } y \quad \Rightarrow} \tag{T-IF}$$

Type Inference

$$\frac{\Gamma \vdash b \Rightarrow}{\Gamma \vdash \text{if } b \text{ then } x \text{ else } y \Rightarrow} \quad \text{(T-IF)}$$

Type Inference

$$\frac{\Gamma \;\vdash\; b \;\Rightarrow\; B}{\Gamma \;\vdash\; \texttt{if } b \texttt{ then } x \texttt{ else } y \;\Rightarrow} \qquad (\text{T-IF})$$

Type Inference

$$\frac{\Gamma \vdash x \Rightarrow \quad \Gamma \vdash b \Rightarrow B}{\Gamma \vdash \text{if } b \text{ then } x \text{ else } y \ \Rightarrow} \tag{T-IF}$$

Type Inference

$$\Gamma \vdash x \Rightarrow \boxed{T_1}$$
$$\Gamma \vdash b \Rightarrow B$$
$$\overline{\qquad\qquad \Gamma \vdash \text{if } b \text{ then } x \text{ else } y \quad \Rightarrow \qquad\qquad} \qquad (\text{T-IF})$$

Type Inference

$$\frac{\Gamma \vdash x \Rightarrow T_1 \qquad \Gamma \vdash y \Rightarrow}{\Gamma \vdash b \Rightarrow B} \tag{T-IF}$$
$$\Gamma \vdash \text{if } b \text{ then } x \text{ else } y \quad \Rightarrow$$

Type Inference

$$\frac{\Gamma \vdash x \Rightarrow T_1 \qquad \Gamma \vdash y \Rightarrow \boxed{T_2}}{\Gamma \vdash b \Rightarrow B} \qquad \qquad (\text{T-IF})$$
$$\Gamma \vdash \text{if } b \text{ then } x \text{ else } y \quad \Rightarrow$$

Type Inference

$$\frac{\Gamma \vdash x \Rightarrow T_1 \qquad \Gamma \vdash y \Rightarrow T_2 \qquad [B = \texttt{Bool}]}{\Gamma \vdash \texttt{if } b \texttt{ then } x \texttt{ else } y \quad \Rightarrow} \qquad \text{(T-IF)}$$

$$\Gamma \vdash b \Rightarrow B$$

Type Inference

$$\frac{\Gamma \vdash x \Rightarrow T_1 \qquad \Gamma \vdash y \Rightarrow T_2 \qquad [B = \texttt{Bool}]}{\Gamma \vdash b \Rightarrow B \qquad\qquad [T_1 = T_2]}{\Gamma \vdash \texttt{if } b \texttt{ then } x \texttt{ else } y \quad \Rightarrow} \quad (\text{T-\textsc{if}})$$

Type Inference

$$\frac{\Gamma \vdash x \Rightarrow T_1 \qquad \Gamma \vdash y \Rightarrow T_2 \qquad [B = \texttt{Bool}]}{\Gamma \vdash b \Rightarrow B \qquad\qquad\qquad [T_1 = T_2]}{\Gamma \vdash \texttt{if } b \texttt{ then } x \texttt{ else } y \quad \Rightarrow \quad T_1} \tag{T-IF}$$

Type Inference

$$\dfrac{new(?^0) \quad \dfrac{\overline{\Gamma,\ b:?^0 \vdash b \Rightarrow ?^0}\ \text{(T-VAR)} \quad \overline{\Gamma,\ b:?^0 \vdash \texttt{false} \Rightarrow \text{Bool}}\ \text{(T-FALSE)} \quad \overline{\Gamma,\ b:?^0 \vdash \texttt{true} \Rightarrow \text{Bool}}\ \text{(T-TRUE)} \quad [?^0 = \text{Bool}]\ [\text{Bool} = \text{Bool}]}{\Gamma,\ b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \ \Rightarrow\ \text{Bool}}\ \text{(T-IF)}}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \ \Rightarrow\ ?^0 \to \text{Bool}}\ \text{(T-LAM)}$$

$$\lambda b.\ \texttt{if } b \texttt{ then false else true}$$

Type Inference

$$\frac{new(?^0) \qquad \dfrac{\overline{\Gamma, b : ?^0 \vdash b \Rightarrow ?^0} \; (\text{T-VAR}) \qquad \overline{\Gamma, b : ?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}} \; (\text{T-FALSE}) \qquad \overline{\Gamma, b : ?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}} \; (\text{T-TRUE}) \qquad [?^0 = \texttt{Bool}] \; [\texttt{Bool} = \texttt{Bool}]}{\Gamma, b : ?^0 \vdash \texttt{if } b \texttt{ then false else true} \; \Rightarrow \; \texttt{Bool}} \; (\text{T-IF})}{\Gamma \vdash \lambda b. \texttt{ if } b \texttt{ then false else true} \; \Rightarrow \; ?^0 \to \texttt{Bool}} \; (\text{T-LAM})$$

$$\frac{new(\; ?^n \;) \qquad \Gamma, \; x \; : \; ?^n \; \vdash \; e \; \Rightarrow \; T}{\Gamma \vdash \lambda \, x \, . \; e \; \Rightarrow \; ?^n \; \to \; T} \qquad (\text{T-LAM})$$

Type Inference

$$\frac{\begin{array}{c} \dfrac{}{\Gamma, \, b : ?^0 \vdash b \Rightarrow ?^0} \; \text{(T-VAR)} \quad \dfrac{}{\Gamma, \, b : ?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}} \; \text{(T-FALSE)} \quad \dfrac{}{\Gamma, \, b : ?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}} \; \text{(T-TRUE)} \quad [?^0 = \texttt{Bool}] \; [\texttt{Bool} = \texttt{Bool}] \\ new(?^0) \qquad\qquad\qquad \Gamma, \, b : ?^0 \vdash \texttt{if } b \texttt{ then false else true} \; \Rightarrow \; \texttt{Bool} \end{array}}{\Gamma \vdash \lambda b. \; \texttt{if } b \texttt{ then false else true} \; \Rightarrow \; ?^0 \rightarrow \texttt{Bool}} \; \text{(T-LAM)} \quad \text{(T-IF)}$$

$$\frac{new(\; ?^n \;) \qquad \Gamma, \; \boxed{x} : \; ?^n \; \vdash \; e \; \Rightarrow \; T}{\Gamma \vdash \lambda \, \boxed{x} . \; e \; \Rightarrow \; ?^n \; \rightarrow \; T} \qquad \text{(T-LAM)}$$

Type Inference

$$\cfrac{new(?^0) \qquad \cfrac{\overline{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\ \text{(T-VAR)} \qquad \overline{\Gamma,\, b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\ \text{(T-FALSE)} \qquad \overline{\Gamma,\, b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\ \text{(T-TRUE)} \qquad [?^0 = \texttt{Bool}]\ [\texttt{Bool} = \texttt{Bool}]}{\Gamma,\, b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}}\ \text{(T-IF)}}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \Rightarrow ?^0 \to \texttt{Bool}}\ \text{(T-LAM)}$$

$$\cfrac{new(\ ?^n\ ) \qquad \Gamma,\, b:\ ?^n \vdash e \Rightarrow T}{\Gamma \vdash \lambda\, b.\ e \ \Rightarrow\ ?^n \to T} \qquad \text{(T-LAM)}$$

Type Inference

$$
\frac{new(?^0) \qquad \dfrac{\overline{\Gamma, b : ?^0 \vdash b \Rightarrow ?^0}\ (\text{T-VAR}) \quad \overline{\Gamma, b : ?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\ (\text{T-FALSE}) \quad \overline{\Gamma, b : ?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\ (\text{T-TRUE}) \quad [?^0 = \texttt{Bool}]\ [\texttt{Bool} = \texttt{Bool}]}{\Gamma, b : ?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}}\ (\text{T-IF})}{\Gamma \vdash \lambda b. \texttt{ if } b \texttt{ then false else true} \Rightarrow ?^0 \to \texttt{Bool}}\ (\text{T-LAM})
$$

$$
\frac{new(\ ?^n\ ) \qquad \Gamma,\ b\ :\ ?^n \vdash e \Rightarrow T}{\Gamma \vdash \lambda\ b\ .\ e \quad \Rightarrow \quad ?^n \to T} \qquad (\text{T-LAM})
$$

Type Inference

$$\dfrac{new(?^0) \qquad \dfrac{\overline{\Gamma,\ b:?^0 \vdash b \Rightarrow ?^0}\ (\text{T-var}) \quad \overline{\Gamma,\ b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\ (\text{T-false}) \quad \overline{\Gamma,\ b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\ (\text{T-true}) \quad [?^0 = \texttt{Bool}]\ [\texttt{Bool} = \texttt{Bool}]}{\Gamma,\ b:?^0 \vdash \texttt{if } b \texttt{ then false else true } \Rightarrow \texttt{Bool}}\ (\text{T-if})}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true } \Rightarrow ?^0 \to \texttt{Bool}}\ (\text{T-lam})$$

$$\dfrac{new(\ ?^n\ ) \qquad \Gamma,\ b\ :\ ?^n\ \vdash\ \boxed{e} \Rightarrow\ T}{\Gamma \vdash \lambda\ b\ .\ \boxed{e}\ \Rightarrow\ ?^n\ \to\ T} \qquad (\text{T-lam})$$

Type Inference

$$
\dfrac{new(?^0) \qquad \dfrac{\overline{\Gamma,\, b : ?^0 \vdash b \Rightarrow ?^0}\ (\text{T-VAR}) \quad \overline{\Gamma,\, b : ?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\ (\text{T-FALSE}) \quad \overline{\Gamma,\, b : ?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\ (\text{T-TRUE}) \qquad [?^0 = \texttt{Bool}]\ [\texttt{Bool} = \texttt{Bool}]}{\Gamma,\, b : ?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}}\ (\text{T-IF})}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \Rightarrow ?^0 \to \texttt{Bool}}\ (\text{T-LAM})
$$

$$
\dfrac{new(\ ?^n\ ) \qquad \Gamma,\ b\ :\ ?^n \vdash\ \boxed{\texttt{if } b \texttt{ then false else true}}\ \Rightarrow\ T}{\Gamma \vdash \lambda\, b\, .\ \boxed{\texttt{if } b \texttt{ then false else true}}\ \Rightarrow\ ?^n\ \to\ T}\ (\text{T-LAM})
$$

Type Inference

$$\cfrac{new(?^0) \quad \cfrac{\overline{\Gamma,\, b : ?^0 \vdash b \Rightarrow ?^0} \text{ (T-VAR)} \quad \overline{\Gamma,\, b : ?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}} \text{ (T-FALSE)} \quad \overline{\Gamma,\, b : ?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}} \text{ (T-TRUE)} \quad [?^0 = \texttt{Bool}] \; [\texttt{Bool} = \texttt{Bool}]}{\Gamma,\, b : ?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}} \text{ (T-IF)}}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \Rightarrow ?^0 \to \texttt{Bool}} \text{ (T-LAM)}$$

$$\cfrac{new(\ ?^n\ ) \quad \Gamma,\ b\ :\ ?^n \vdash\ \texttt{if } b \texttt{ then false else true} \Rightarrow\ T}{\Gamma \vdash \lambda\ b\ .\ \texttt{if } b \texttt{ then false else true} \Rightarrow\ ?^n \to\ T} \text{ (T-LAM)}$$

Type Inference

$$\frac{new(?^0) \qquad \dfrac{\dfrac{}{\Gamma, \, b : ?^0 \vdash b \Rightarrow ?^0}\,\text{(T-VAR)} \quad \dfrac{}{\Gamma, \, b : ?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\,\text{(T-FALSE)} \quad \dfrac{}{\Gamma, \, b : ?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\,\text{(T-TRUE)} \quad [?^0 = \texttt{Bool}] \;\; [\texttt{Bool} = \texttt{Bool}]}{\Gamma, \, b : ?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}}\,\text{(T-IF)}}{\Gamma \vdash \lambda b. \texttt{ if } b \texttt{ then false else true} \Rightarrow ?^0 \rightarrow \texttt{Bool}}\,\text{(T-LAM)}$$

$$\frac{new(\,?^n\,) \qquad \Gamma, \; b \; : \; ?^n \vdash \texttt{ if } b \texttt{ then false else true} \; \Rightarrow \; T}{\Gamma \vdash \lambda \, b \, . \;\; \texttt{if } b \texttt{ then false else true} \;\; \Rightarrow \; ?^n \rightarrow T}\,\text{(T-LAM)}$$

Type Inference

$$\frac{new(?^0) \qquad \dfrac{\overline{\Gamma,\ b:?^0 \vdash b \Rightarrow ?^0}\ (\text{T-VAR}) \quad \overline{\Gamma,\ b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\ (\text{T-FALSE}) \quad \overline{\Gamma,\ b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\ (\text{T-TRUE}) \quad [?^0 = \texttt{Bool}]\ [\texttt{Bool} = \texttt{Bool}]}{\Gamma,\ b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}}\ (\text{T-IF})}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \Rightarrow\ ?^0 \to \texttt{Bool}}\ (\text{T-LAM})$$

$$\frac{new(\ ?^0\ ) \qquad \Gamma,\ b\ :\ ?^0 \vdash\ \texttt{if } b \texttt{ then false else true}\ \Rightarrow\ T}{\Gamma \vdash \lambda\ b\ .\ \texttt{if } b \texttt{ then false else true}\ \Rightarrow\ ?^0 \to\ T}\ (\text{T-LAM})$$

Type Inference

$$\frac{new(?^0) \qquad \cfrac{\cfrac{}{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\ (\text{T-VAR}) \quad \cfrac{}{\Gamma,\, b:?^0 \vdash \text{false} \Rightarrow \text{Bool}}\ (\text{T-FALSE}) \quad \cfrac{}{\Gamma,\, b:?^0 \vdash \text{true} \Rightarrow \text{Bool}}\ (\text{T-TRUE}) \quad [?^0 = \text{Bool}]\ [\text{Bool} = \text{Bool}]}{\Gamma,\, b:?^0 \vdash \text{if } b \text{ then false else true} \Rightarrow \text{Bool}}\ (\text{T-IF})}{\Gamma \vdash \lambda b.\ \text{if } b \text{ then false else true} \Rightarrow ?^0 \rightarrow \text{Bool}}\ (\text{T-LAM})$$

$$\frac{new(\ ?^0\ ) \qquad \Gamma,\ b\ :\ ?^0\ \vdash\ \text{if } b \text{ then false else true} \Rightarrow T}{\Gamma \vdash \lambda\, b\,.\ \text{if } b \text{ then false else true} \quad \Rightarrow \quad ?^0\ \rightarrow\ T}\ (\text{T-LAM})$$

Type Inference

$$\frac{\begin{array}{c} new(?^0) \end{array} \quad \dfrac{\dfrac{}{\Gamma, b:?^0 \vdash b \Rightarrow ?^0} \text{(T-VAR)} \quad \dfrac{}{\Gamma, b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}} \text{(T-FALSE)} \quad \dfrac{}{\Gamma, b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}} \text{(T-TRUE)} \quad [?^0 = \texttt{Bool}] \quad [\texttt{Bool} = \texttt{Bool}]}{\Gamma, b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}} \text{(T-IF)}}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \Rightarrow ?^0 \to \texttt{Bool}} \text{(T-LAM)}$$

$$\frac{\begin{array}{ll} \Gamma \vdash x \Rightarrow T_1 \qquad \Gamma \vdash y \Rightarrow T_2 & [\,B = \texttt{Bool}\,] \\ \Gamma \vdash b \Rightarrow B & [\,T_1 = T_2\,] \end{array}}{\Gamma \vdash \texttt{if } b \texttt{ then } x \texttt{ else } y \Rightarrow T_1} \quad \text{(T-IF)}$$

Type Inference

$$\cfrac{new(?^0) \quad \cfrac{\overline{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\ (\text{T-VAR}) \quad \overline{\Gamma,\, b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\ (\text{T-FALSE}) \quad \overline{\Gamma,\, b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\ (\text{T-TRUE}) \quad [?^0 = \texttt{Bool}]\ [\texttt{Bool} = \texttt{Bool}]}{\Gamma,\, b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}}\ (\text{T-IF})}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \Rightarrow ?^0 \to \texttt{Bool}}\ (\text{T-LAM})$$

$$\cfrac{\Gamma \vdash x \Rightarrow T_1 \qquad \Gamma \vdash y \Rightarrow T_2 \qquad [\, B = \texttt{Bool}\,] \\ \Gamma \vdash b \Rightarrow B \qquad\qquad\qquad\qquad [\, T_1 = T_2\,]}{\Gamma \vdash \texttt{if } b \texttt{ then } x \texttt{ else } y \Rightarrow T_1} \quad (\text{T-IF})$$

Type Inference

$$\frac{new(?^0) \quad \frac{\overline{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\ (\text{T-VAR}) \quad \overline{\Gamma,\, b:?^0 \vdash \mathtt{false} \Rightarrow \mathtt{Bool}}\ (\text{T-FALSE}) \quad \overline{\Gamma,\, b:?^0 \vdash \mathtt{true} \Rightarrow \mathtt{Bool}}\ (\text{T-TRUE}) \quad [?^0 = \mathtt{Bool}]\ [\mathtt{Bool} = \mathtt{Bool}]}{\Gamma,\, b:?^0 \vdash \mathtt{if}\ b\ \mathtt{then}\ \mathtt{false}\ \mathtt{else}\ \mathtt{true}\ \Rightarrow\ \mathtt{Bool}}\ (\text{T-IF})}{\Gamma \vdash \lambda b.\ \mathtt{if}\ b\ \mathtt{then}\ \mathtt{false}\ \mathtt{else}\ \mathtt{true}\ \Rightarrow\ ?^0 \to \mathtt{Bool}}\ (\text{T-LAM})$$

$$\frac{\Gamma,\, b:?^0 \vdash x \Rightarrow T_1 \qquad \Gamma,\, b:?^0 \vdash y \Rightarrow T_2 \qquad [\,B = \mathtt{Bool}\,] \\ \Gamma,\, b:?^0 \vdash b \Rightarrow B \qquad\qquad\qquad\qquad\qquad\quad [\,T_1 = T_2\,]}{\Gamma,\, b:?^0 \vdash \mathtt{if}\ b\ \mathtt{then}\ x\ \mathtt{else}\ y \ \Rightarrow\ T_1}\ (\text{T-IF})$$

Type Inference

$$\dfrac{new(?^0) \quad \dfrac{\overline{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\;\text{(T-VAR)} \quad \overline{\Gamma,\, b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\;\text{(T-FALSE)} \quad \overline{\Gamma,\, b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\;\text{(T-TRUE)} \quad [?^0 = \texttt{Bool}] \;\; [\texttt{Bool} = \texttt{Bool}]}{\Gamma,\, b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \;\Rightarrow\; \texttt{Bool}}\;\text{(T-IF)}}{\Gamma \vdash \lambda b.\; \texttt{if } b \texttt{ then false else true} \;\Rightarrow\; ?^0 \to \texttt{Bool}}\;\text{(T-LAM)}$$

$$\dfrac{\Gamma,\, b:?^0 \vdash x \Rightarrow T_1 \qquad \Gamma,\, b:?^0 \vdash y \Rightarrow T_2 \qquad [\, B = \texttt{Bool}\,] \\[4pt] \Gamma,\, b:?^0 \vdash b \Rightarrow B \qquad\qquad\qquad\qquad\qquad\quad [\, T_1 = T_2 \,]}{\Gamma,\, b:?^0 \vdash \texttt{if } b \texttt{ then } x \texttt{ else } y \;\Rightarrow\; T_1}\;\text{(T-IF)}$$

Type Inference

$$
\dfrac{new(?^0) \quad \dfrac{\overline{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\ \text{(T-VAR)} \quad \overline{\Gamma,\, b:?^0 \vdash \mathtt{false} \Rightarrow \mathtt{Bool}}\ \text{(T-FALSE)} \quad \overline{\Gamma,\, b:?^0 \vdash \mathtt{true} \Rightarrow \mathtt{Bool}}\ \text{(T-TRUE)} \quad [?^0 = \mathtt{Bool}]\ [\mathtt{Bool} = \mathtt{Bool}]}{\Gamma,\, b:?^0 \vdash \mathtt{if}\ b\ \mathtt{then\ false\ else\ true} \;\Rightarrow\; \mathtt{Bool}}\ \text{(T-IF)}}{\Gamma \vdash \lambda b.\ \mathtt{if}\ b\ \mathtt{then\ false\ else\ true} \;\Rightarrow\; ?^0 \to \mathtt{Bool}}\ \text{(T-LAM)}
$$

$$
\dfrac{\begin{array}{lll} \Gamma,\, b:?^0 \vdash x \Rightarrow T_1 & \Gamma,\, b:?^0 \vdash y \Rightarrow T_2 & [\,B = \mathtt{Bool}\,] \\[4pt] \Gamma,\, b:?^0 \vdash b \Rightarrow B & & [\,T_1 = T_2\,] \end{array}}{\Gamma,\, b:?^0 \vdash \mathtt{if}\ b\ \mathtt{then}\ x\ \mathtt{else}\ y \;\Rightarrow\; T_1}\ (\text{T-IF})
$$

Type Inference

$$\dfrac{new(?^0) \quad \dfrac{\dfrac{}{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\;(\text{T-VAR}) \quad \dfrac{}{\Gamma,\, b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\;(\text{T-FALSE}) \quad \dfrac{}{\Gamma,\, b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\;(\text{T-TRUE}) \quad [?^0 = \texttt{Bool}]\;\;[\texttt{Bool} = \texttt{Bool}]}{\Gamma,\, b:?^0 \vdash \text{if } b \text{ then false else true} \;\Rightarrow\; \texttt{Bool}}\;(\text{T-IF})}{\Gamma \vdash \lambda b.\ \text{if } b \text{ then false else true} \;\Rightarrow\; ?^0 \to \texttt{Bool}}\;(\text{T-LAM})$$

$$\dfrac{\begin{array}{ll} \Gamma,\, b:?^0 \vdash x \Rightarrow T_1 \qquad \Gamma,\, b:?^0 \vdash y \Rightarrow T_2 \qquad [\,B = \texttt{Bool}\,] \\[2mm] \Gamma,\, b:?^0 \vdash b \Rightarrow B \qquad\qquad\qquad\qquad\qquad\quad [\,T_1 = T_2\,] \end{array}}{\Gamma,\, b:?^0 \vdash \texttt{if } b \texttt{ then } x \texttt{ else } y \;\Rightarrow\; T_1}\;(\text{T-IF})$$

Type Inference

$$
\cfrac{
  \cfrac{
    \overline{\Gamma,\, b : ?^0 \vdash b \Rightarrow ?^0}\ \text{(T-VAR)}
    \quad
    \overline{\Gamma,\, b : ?^0 \vdash \mathtt{false} \Rightarrow \mathtt{Bool}}\ \text{(T-FALSE)}
    \quad
    \overline{\Gamma,\, b : ?^0 \vdash \mathtt{true} \Rightarrow \mathtt{Bool}}\ \text{(T-TRUE)}
    \quad
    [?^0 = \mathtt{Bool}]\ \ [\mathtt{Bool} = \mathtt{Bool}]
  }{
    \Gamma,\, b : ?^0 \vdash \mathtt{if}\ b\ \mathtt{then\ false\ else\ true} \Rightarrow \mathtt{Bool}
  }\ \text{(T-IF)}
}{
  \Gamma \vdash \lambda b.\ \mathtt{if}\ b\ \mathtt{then\ false\ else\ true} \Rightarrow ?^0 \to \mathtt{Bool}
}\ \ new(?^0)\ \ \text{(T-LAM)}
$$

$$
\cfrac{
  \begin{array}{lll}
    \Gamma,\, b : ?^0 \vdash x \Rightarrow T_1 \quad & \Gamma,\, b : ?^0 \vdash y \Rightarrow T_2 \quad & [\, B = \mathtt{Bool}\,] \\[4pt]
    \Gamma,\, b : ?^0 \vdash b \Rightarrow B & & [\, T_1 = T_2 \,]
  \end{array}
}{
  \Gamma,\, b : ?^0 \vdash \mathtt{if}\ b\ \mathtt{then}\ x\ \mathtt{else}\ y \Rightarrow T_1
}\ \ (\text{T-IF})
$$

Type Inference

$$\cfrac{new(?^0) \quad \cfrac{\overline{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\,(\text{T-VAR}) \quad \overline{\Gamma,\, b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\,(\text{T-FALSE}) \quad \overline{\Gamma,\, b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\,(\text{T-TRUE}) \quad [?^0 = \texttt{Bool}] \;\; [\texttt{Bool} = \texttt{Bool}]}{\Gamma,\, b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}}\,(\text{T-IF})}{\Gamma \vdash \lambda b.\, \texttt{if } b \texttt{ then false else true} \Rightarrow ?^0 \to \texttt{Bool}}\,(\text{T-LAM})$$

$$\cfrac{\begin{array}{c} \Gamma,\, b:?^0 \vdash \boxed{\texttt{false}} \Rightarrow T_1 \qquad \Gamma,\, b:?^0 \vdash y \Rightarrow T_2 \qquad [\,B = \texttt{Bool}\,] \\[4pt] \Gamma,\, b:?^0 \vdash b \Rightarrow B \qquad\qquad\qquad\qquad\qquad [\,T_1 = T_2\,] \end{array}}{\Gamma,\, b:?^0 \vdash \texttt{if } b \texttt{ then } \boxed{\texttt{false}} \texttt{ else } y \Rightarrow T_1}\,(\text{T-IF})$$

Type Inference

$$\cfrac{new(?^0) \qquad \cfrac{\cfrac{}{\Gamma, \, b: ?^0 \vdash b \Rightarrow ?^0}\text{(T-VAR)} \quad \cfrac{}{\Gamma, \, b: ?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\text{(T-FALSE)} \quad \cfrac{}{\Gamma, \, b: ?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\text{(T-TRUE)} \quad [?^0 = \texttt{Bool}] \;\; [\texttt{Bool} = \texttt{Bool}]}{\Gamma, \, b: ?^0 \vdash \texttt{if } b \texttt{ then false else true} \;\Rightarrow\; \texttt{Bool}}\text{(T-IF)}}{\Gamma \vdash \lambda b. \texttt{ if } b \texttt{ then false else true} \;\Rightarrow\; ?^0 \to \texttt{Bool}}\text{(T-LAM)}$$

$$\cfrac{\begin{array}{ccc} \Gamma, \, b: ?^0 \vdash \texttt{false} \;\Rightarrow\; T_1 & \Gamma, \, b: ?^0 \vdash y \;\Rightarrow\; T_2 & [\, B = \texttt{Bool}] \\ \Gamma, \, b: ?^0 \vdash b \;\Rightarrow\; B & & [\, T_1 \;=\; T_2 \,] \end{array}}{\Gamma, \, b: ?^0 \vdash \texttt{if } b \texttt{ then false else } y \;\Rightarrow\; T_1}\,\text{(T-IF)}$$

Type Inference

$$\frac{\overline{\Gamma,\ b:?^0 \vdash b \Rightarrow ?^0}\ (\text{T-VAR}) \quad \overline{\Gamma,\ b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\ (\text{T-FALSE}) \quad \overline{\Gamma,\ b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\ (\text{T-TRUE}) \quad [?^0 = \texttt{Bool}]\ [\texttt{Bool} = \texttt{Bool}]}{\dfrac{new(?^0) \qquad \Gamma,\ b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \ \Rightarrow \ \texttt{Bool}}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \ \Rightarrow\ ?^0 \to \texttt{Bool}}\ (\text{T-LAM})}\ (\text{T-IF})$$

$$\frac{\Gamma,\ b:?^0 \vdash \texttt{false} \Rightarrow T_1 \qquad \Gamma,\ b:?^0 \vdash \boxed{y} \Rightarrow T_2 \qquad [\,B = \texttt{Bool}\,] \qquad \Gamma,\ b:?^0 \vdash b \Rightarrow B \qquad\qquad\qquad [\,T_1 = T_2\,]}{\Gamma,\ b:?^0 \vdash \texttt{if } b \texttt{ then false else } \boxed{y} \ \Rightarrow\ T_1}\ (\text{T-IF})$$

Type Inference

$$\dfrac{new(?^0) \quad \dfrac{\overline{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\ \text{(T-VAR)} \quad \overline{\Gamma,\, b:?^0 \vdash \text{false} \Rightarrow \text{Bool}}\ \text{(T-FALSE)} \quad \overline{\Gamma,\, b:?^0 \vdash \text{true} \Rightarrow \text{Bool}}\ \text{(T-TRUE)} \quad [?^0 = \text{Bool}] \quad [\text{Bool} = \text{Bool}]}{\Gamma,\, b:?^0 \vdash \text{if } b \text{ then false else true} \ \Rightarrow \ \text{Bool}}\ \text{(T-IF)}}{\Gamma \vdash \lambda b.\ \text{if } b \text{ then false else true} \ \Rightarrow \ ?^0 \to \text{Bool}}\ \text{(T-LAM)}$$

$$\dfrac{\Gamma,\, b:?^0 \ \vdash \ \text{false} \ \Rightarrow \ T_1 \qquad \Gamma,\, b:?^0 \ \vdash \ \text{true} \ \Rightarrow \ T_2 \qquad [\, B = \text{Bool}] \qquad \Gamma,\, b:?^0 \ \vdash \ b \ \Rightarrow \ B \qquad\qquad\qquad\qquad\qquad [\, T_1 \ = \ T_2 \,]}{\Gamma,\, b:?^0 \ \vdash \text{if } b \text{ then false else true} \ \Rightarrow \ T_1}\ \text{(T-IF)}$$

Type Inference

$$\dfrac{\dfrac{new(?^0)}{\Gamma \vdash \lambda b.\ \text{if } b \text{ then false else true} \ \Rightarrow \ ?^0 \to \text{Bool}} \ \dfrac{\overline{\Gamma, b : ?^0 \vdash b \Rightarrow ?^0} \ (\text{T-VAR}) \quad \overline{\Gamma, b : ?^0 \vdash \text{false} \Rightarrow \text{Bool}} \ (\text{T-FALSE}) \quad \overline{\Gamma, b : ?^0 \vdash \text{true} \Rightarrow \text{Bool}} \ (\text{T-TRUE}) \quad [?^0 = \text{Bool}] \ [\text{Bool} = \text{Bool}]}{\Gamma, b : ?^0 \vdash \text{if } b \text{ then false else true} \ \Rightarrow \ \text{Bool}} \ (\text{T-IF})}{} (\text{T-LAM})$$

$$\dfrac{\begin{array}{ccc} \Gamma, b : ?^0 \vdash \text{false} \Rightarrow T_1 & \Gamma, b : ?^0 \vdash \text{true} \Rightarrow T_2 & [\,B = \text{Bool}\,] \\ \Gamma, b : ?^0 \vdash b \Rightarrow B & & [\,T_1 = T_2\,] \end{array}}{\Gamma, b : ?^0 \vdash \text{if } b \text{ then false else true} \ \Rightarrow \ T_1} \ (\text{T-IF})$$

Type Inference

$$\dfrac{\dfrac{new(?^0) \quad \dfrac{\overline{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\ (\text{T-VAR}) \quad \overline{\Gamma,\, b:?^0 \vdash \mathtt{false} \Rightarrow \mathtt{Bool}}\ (\text{T-FALSE}) \quad \overline{\Gamma,\, b:?^0 \vdash \mathtt{true} \Rightarrow \mathtt{Bool}}\ (\text{T-TRUE}) \quad [?^0 = \mathtt{Bool}]\ [\mathtt{Bool} = \mathtt{Bool}]}{\Gamma,\, b:?^0 \vdash \mathtt{if}\ b\ \mathtt{then}\ \mathtt{false}\ \mathtt{else}\ \mathtt{true}\ \Rightarrow\ \mathtt{Bool}}\ (\text{T-IF})}{\Gamma \vdash \lambda b.\ \mathtt{if}\ b\ \mathtt{then}\ \mathtt{false}\ \mathtt{else}\ \mathtt{true}\ \Rightarrow\ ?^0 \to \mathtt{Bool}}\ (\text{T-LAM})$$

$$\overline{\Gamma \vdash x \Rightarrow T} \qquad\qquad (\text{T-VAR})$$

Type Inference

$$\frac{\dfrac{}{\Gamma,\ b:?^0 \vdash b \Rightarrow ?^0}\ \text{(T-VAR)} \quad \dfrac{}{\Gamma,\ b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\ \text{(T-FALSE)} \quad \dfrac{}{\Gamma,\ b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\ \text{(T-TRUE)} \quad [?^0 = \texttt{Bool}]\ [\texttt{Bool} = \texttt{Bool}]}{\dfrac{new(?^0) \qquad\qquad \Gamma,\ b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \ \Rightarrow\ \texttt{Bool}}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \ \Rightarrow\ ?^0 \to \texttt{Bool}}\ \text{(T-LAM)}}\ \text{(T-IF)}$$

$$\frac{}{\boxed{\Gamma}\ \vdash\ x\ \Rightarrow\ T} \qquad\qquad (\text{T-VAR})$$

Type Inference

$$
\dfrac{\begin{array}{c} new(?^0) \end{array} \quad \dfrac{\dfrac{}{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\ (\text{T-VAR}) \quad \dfrac{}{\Gamma,\, b:?^0 \vdash \mathtt{false} \Rightarrow \mathtt{Bool}}\ (\text{T-FALSE}) \quad \dfrac{}{\Gamma,\, b:?^0 \vdash \mathtt{true} \Rightarrow \mathtt{Bool}}\ (\text{T-TRUE}) \quad [?^0 = \mathtt{Bool}]\ [\mathtt{Bool} = \mathtt{Bool}]}{\Gamma,\, b:?^0 \vdash \mathtt{if}\ b\ \mathtt{then}\ \mathtt{false}\ \mathtt{else}\ \mathtt{true}\ \Rightarrow\ \mathtt{Bool}}\ (\text{T-IF})}{\Gamma \vdash \lambda b.\ \mathtt{if}\ b\ \mathtt{then}\ \mathtt{false}\ \mathtt{else}\ \mathtt{true}\ \Rightarrow\ ?^0 \to \mathtt{Bool}}\ (\text{T-LAM})
$$

$$
\dfrac{}{\Gamma,\, b:\ T\ \vdash\ x\ \Rightarrow\ T}
$$
(T-VAR)

Type Inference

$$\cfrac{new(?^0) \qquad \cfrac{\cfrac{}{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\ \text{(T-VAR)} \qquad \cfrac{}{\Gamma,\, b:?^0 \vdash \text{false} \Rightarrow \text{Bool}}\ \text{(T-FALSE)} \qquad \cfrac{}{\Gamma,\, b:?^0 \vdash \text{true} \Rightarrow \text{Bool}}\ \text{(T-TRUE)} \qquad [?^0 = \text{Bool}]\ [\text{Bool} = \text{Bool}]}{\Gamma,\, b:?^0 \vdash \text{if } b \text{ then false else true} \ \Rightarrow \ \text{Bool}}\ \text{(T-IF)}}{\Gamma \vdash \lambda b.\ \text{if } b \text{ then false else true} \ \Rightarrow \ ?^0 \to \text{Bool}}\ \text{(T-LAM)}$$

$$\cfrac{}{\Gamma,\ b:\ T\ \vdash\ x\ \Rightarrow\ T} \qquad\qquad \text{(T-VAR)}$$

Type Inference

$$\frac{\dfrac{}{\Gamma,\ b:?^0 \vdash b \Rightarrow ?^0}\ (\text{T-var}) \quad \dfrac{}{\Gamma,\ b:?^0 \vdash \mathtt{false} \Rightarrow \mathtt{Bool}}\ (\text{T-false}) \quad \dfrac{}{\Gamma,\ b:?^0 \vdash \mathtt{true} \Rightarrow \mathtt{Bool}} \ (\text{T-true}) \qquad [?^0 = \mathtt{Bool}]\ [\mathtt{Bool} = \mathtt{Bool}]}{\dfrac{new(?^0) \qquad \qquad \qquad \Gamma,\ b:?^0 \vdash \mathtt{if}\ b\ \mathtt{then}\ \mathtt{false}\ \mathtt{else}\ \mathtt{true}\ \Rightarrow\ \mathtt{Bool}}{\Gamma \vdash \lambda b.\ \mathtt{if}\ b\ \mathtt{then}\ \mathtt{false}\ \mathtt{else}\ \mathtt{true}\ \Rightarrow\ ?^0 \rightarrow \mathtt{Bool}}\ (\text{T-lam})}\ (\text{T-if})$$

$$\frac{\rule{4cm}{0.4pt}}{\Gamma,\ b:\ T\ \vdash\ \boxed{x}\ \Rightarrow\ T} \qquad (\text{T-var})$$

Type Inference

$$\cfrac{new(?^0) \quad \cfrac{\overline{\Gamma,\, b : ?^0 \vdash b \Rightarrow ?^0}\ \text{(T-VAR)} \quad \overline{\Gamma,\, b : ?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\ \text{(T-FALSE)} \quad \overline{\Gamma,\, b : ?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\ \text{(T-TRUE)} \quad [?^0 = \texttt{Bool}] \quad [\texttt{Bool} = \texttt{Bool}]}{\Gamma,\, b : ?^0 \vdash \texttt{if } b \texttt{ then false else true} \;\Rightarrow\; \texttt{Bool}}\ \text{(T-IF)}}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \;\Rightarrow\; ?^0 \rightarrow \texttt{Bool}}\ \text{(T-LAM)}$$

$$\cfrac{}{\Gamma,\ b:\ \ T\ \ \vdash\ \boxed{b}\ \Rightarrow\ T} \qquad (\text{T-VAR})$$

Type Inference

$$\cfrac{new(?^0) \quad \cfrac{\cfrac{}{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\;(\text{T-VAR}) \quad \cfrac{}{\Gamma,\, b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\;(\text{T-FALSE}) \quad \cfrac{}{\Gamma,\, b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\;(\text{T-TRUE}) \quad [?^0 = \texttt{Bool}]\;[\texttt{Bool} = \texttt{Bool}]}{\Gamma,\, b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \;\Rightarrow\; \texttt{Bool}}\;(\text{T-IF})}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \;\Rightarrow\; ?^0 \rightarrow \texttt{Bool}}\;(\text{T-LAM})$$

$$\cfrac{\rule{6cm}{0.4pt}}{\Gamma,\, b:\ T\ \vdash\ b\ \Rightarrow\ T}\qquad (\text{T-VAR})$$

Type Inference

$$\cfrac{new(?^0) \qquad \cfrac{\cfrac{}{\Gamma, b:?^0 \vdash b \Rightarrow ?^0}\ (\text{T-VAR}) \quad \cfrac{}{\Gamma, b:?^0 \vdash \mathtt{false} \Rightarrow \mathtt{Bool}}\ (\text{T-FALSE}) \quad \cfrac{}{\Gamma, b:?^0 \vdash \mathtt{true} \Rightarrow \mathtt{Bool}}\ (\text{T-TRUE}) \qquad [?^0 = \mathtt{Bool}] \quad [\mathtt{Bool} = \mathtt{Bool}]}{\Gamma, b:?^0 \vdash \mathtt{if}\ b\ \mathtt{then}\ \mathtt{false}\ \mathtt{else}\ \mathtt{true}\ \Rightarrow\ \mathtt{Bool}}\ (\text{T-IF})}{\Gamma \vdash \lambda b.\ \mathtt{if}\ b\ \mathtt{then}\ \mathtt{false}\ \mathtt{else}\ \mathtt{true}\ \Rightarrow\ ?^0 \to \mathtt{Bool}}\ (\text{T-LAM})$$

$$\cfrac{}{\Gamma,\ b:\ \boxed{T}\ \vdash\ b\ \Rightarrow\ \boxed{T}} \qquad (\text{T-VAR})$$

Type Inference

$$\frac{new(?^0) \qquad \frac{\dfrac{}{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\ \text{(T-VAR)} \qquad \dfrac{}{\Gamma,\, b:?^0 \vdash \text{false} \Rightarrow \text{Bool}}\ \text{(T-FALSE)} \qquad \dfrac{}{\Gamma,\, b:?^0 \vdash \text{true} \Rightarrow \text{Bool}}\ \text{(T-TRUE)} \qquad [?^0 = \text{Bool}]\ [\text{Bool} = \text{Bool}]}{\Gamma,\, b:?^0 \vdash \text{if } b \text{ then false else true } \Rightarrow \text{ Bool}}\ \text{(T-IF)}}{\Gamma \vdash \lambda b.\ \text{if } b \text{ then false else true } \Rightarrow ?^0 \to \text{Bool}}\ \text{(T-LAM)}$$

$$\frac{\rule{3cm}{0.4pt}}{\Gamma,\, b:\ ?^0\ \vdash\ b\ \Rightarrow\ ?^0} \qquad (\text{T-VAR})$$

Type Inference

$$\frac{\dfrac{}{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\;\text{(T-VAR)} \quad \dfrac{}{\Gamma,\, b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\;\text{(T-FALSE)} \quad \dfrac{}{\Gamma,\, b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\;\text{(T-TRUE)} \quad [?^0 = \texttt{Bool}]\;[\texttt{Bool} = \texttt{Bool}]}{\quad}$$

$$\frac{new(?^0) \qquad\qquad \Gamma,\, b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \;\Rightarrow\; \texttt{Bool}}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \;\Rightarrow\; ?^0 \to \texttt{Bool}}\;\text{(T-LAM)}$$

$$\frac{\qquad\qquad\qquad\qquad\qquad}{\Gamma,\, b:\ ?^0\ \vdash\ b\ \Rightarrow\ ?^0}\qquad\text{(T-VAR)}$$

Type Inference

$$\cfrac{\cfrac{}{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\ (\text{T-VAR})\quad \cfrac{}{\Gamma,\, b:?^0 \vdash \text{false} \Rightarrow \text{Bool}}\ (\text{T-FALSE})\quad \cfrac{}{\Gamma,\, b:?^0 \vdash \text{true} \Rightarrow \text{Bool}}\ (\text{T-TRUE})\quad [?^0 = \text{Bool}]\ [\text{Bool} = \text{Bool}]}{\cfrac{new(?^0)\qquad\qquad \Gamma,\, b:?^0 \vdash \text{if } b \text{ then false else true} \Rightarrow \text{Bool}}{\Gamma \vdash \lambda b.\ \text{if } b \text{ then false else true} \Rightarrow ?^0 \to \text{Bool}}\ (\text{T-LAM})}\ (\text{T-IF})$$

$$\cfrac{\Gamma,\, b:?^0 \vdash b \Rightarrow B \qquad\qquad \Gamma,\, b:?^0 \vdash \text{false} \Rightarrow T_1 \qquad \Gamma,\, b:?^0 \vdash \text{true} \Rightarrow T_2 \qquad [\,B = \text{Bool}\,]\ \ [\,T_1 = T_2\,]}{\Gamma,\, b:?^0 \vdash \text{if } b \text{ then false else true} \Rightarrow T_1}\ (\text{T-IF})$$

Type Inference

$$\cfrac{\cfrac{}{\Gamma, b:?^0 \vdash b \Rightarrow ?^0}\text{(T-VAR)} \quad \cfrac{}{\Gamma, b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\text{(T-FALSE)} \quad \cfrac{}{\Gamma, b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\text{(T-TRUE)} \quad [?^0 = \texttt{Bool}] \; [\texttt{Bool} = \texttt{Bool}]}{\cfrac{new(?^0) \qquad \Gamma, b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}}{\Gamma \vdash \lambda b. \texttt{ if } b \texttt{ then false else true} \Rightarrow ?^0 \to \texttt{Bool}}\text{(T-LAM)}}\text{(T-IF)}$$

$$\cfrac{\begin{array}{c}\Gamma, b:?^0 \vdash \texttt{false} \Rightarrow T_1 \qquad \Gamma, b:?^0 \vdash \texttt{true} \Rightarrow T_2 \qquad [\boxed{B} = \texttt{Bool}] \\ \Gamma, b:?^0 \vdash b \Rightarrow \boxed{B} \qquad\qquad\qquad\qquad\qquad [\, T_1 = T_2 \,]\end{array}}{\Gamma, b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow T_1}\text{(T-IF)}$$

Type Inference

$$\frac{new(?^0)}{\dfrac{\overline{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}^{\text{(T-VAR)}} \quad \overline{\Gamma,\, b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}^{\text{(T-FALSE)}} \quad \overline{\Gamma,\, b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}^{\text{(T-TRUE)}} \quad [?^0 = \texttt{Bool}] \; [\texttt{Bool} = \texttt{Bool}]}{\Gamma,\, b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}} \text{(T-IF)}}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \Rightarrow ?^0 \to \texttt{Bool}} \text{(T-LAM)}$$

$$\frac{\Gamma,\, b:?^0 \vdash \texttt{false} \Rightarrow T_1 \qquad \Gamma,\, b:?^0 \vdash \texttt{true} \Rightarrow T_2 \qquad [\,?^0 = \texttt{Bool}\,] \qquad \Gamma,\, b:?^0 \vdash b \Rightarrow ?^0 \qquad\qquad\qquad\qquad [\,T_1 = T_2\,]}{\Gamma,\, b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow T_1} \text{(T-IF)}$$

Type Inference

$$\frac{\displaystyle \frac{new(?^0)}{\overline{\Gamma, b : ?^0 \vdash b \Rightarrow ?^0} \text{ (T-VAR)} \quad \overline{\Gamma, b : ?^0 \vdash \text{false} \Rightarrow \text{Bool}} \text{ (T-FALSE)} \quad \overline{\Gamma, b : ?^0 \vdash \text{true} \Rightarrow \text{Bool}} \text{ (T-TRUE)} \quad [?^0 = \text{Bool}] \; [\text{Bool} = \text{Bool}]}{\Gamma, b : ?^0 \vdash \text{if } b \text{ then false else true} \Rightarrow \text{Bool}} \text{ (T-IF)}}{\Gamma \vdash \lambda b. \text{ if } b \text{ then false else true} \Rightarrow ?^0 \to \text{Bool}} \text{ (T-LAM)}$$

$$\frac{\Gamma, b : ?^0 \vdash \text{false} \Rightarrow T_1 \qquad \Gamma, b : ?^0 \vdash \text{true} \Rightarrow T_2 \qquad [\,?^0 = \text{Bool}\,]}{\Gamma, b : ?^0 \vdash b \Rightarrow ?^0 \qquad\qquad\qquad\qquad\qquad\qquad [\,T_1 = T_2\,]}{\Gamma, b : ?^0 \vdash \text{if } b \text{ then false else true} \Rightarrow T_1} \text{ (T-IF)}$$

Type Inference

$$
\cfrac{
  \cfrac{
    \cfrac{}{\Gamma,\ b:?^0 \vdash b \Rightarrow ?^0}\ \text{(T-\textsc{var})}
    \qquad
    \cfrac{}{\Gamma,\ b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\ \text{(T-\textsc{false})}
    \qquad
    \cfrac{}{\Gamma,\ b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\ \text{(T-\textsc{true})}
    \qquad [?^0 = \texttt{Bool}]\ [\texttt{Bool} = \texttt{Bool}]
  }{
    \Gamma,\ b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \ \Rightarrow\ \texttt{Bool}
  }\ \text{(T-\textsc{if})}
  \qquad new(?^0)
}{
  \Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \ \Rightarrow\ ?^0 \rightarrow \texttt{Bool}
}\ \text{(T-\textsc{lam})}
$$

$$
\cfrac{}{\Gamma \ \vdash \texttt{false} \Rightarrow \texttt{Bool}} \qquad\qquad (\text{T-\textsc{false}})
$$

Type Inference

$$
\cfrac{new(?^0) \quad \cfrac{\cfrac{}{\Gamma, \, b : ?^0 \vdash b \Rightarrow ?^0} \text{ (T-VAR)} \quad \cfrac{}{\Gamma, \, b : ?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}} \text{ (T-FALSE)} \quad \cfrac{}{\Gamma, \, b : ?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}} \text{ (T-TRUE)} \quad [?^0 = \texttt{Bool}] \; [\texttt{Bool} = \texttt{Bool}]}{\Gamma, \, b : ?^0 \vdash \texttt{if } b \texttt{ then false else true } \Rightarrow \texttt{ Bool}} \text{ (T-IF)}}{\Gamma \vdash \lambda b. \texttt{ if } b \texttt{ then false else true } \Rightarrow \; ?^0 \rightarrow \texttt{Bool}} \text{ (T-LAM)}
$$

$$
\cfrac{}{\boxed{\Gamma} \vdash \texttt{false} \Rightarrow \texttt{Bool}} \qquad\qquad \text{(T-FALSE)}
$$

Type Inference

$$
\dfrac{
new(?^0) \qquad
\dfrac{
\dfrac{}{\Gamma,\ b : ?^0 \vdash b \Rightarrow ?^0}\ \text{(T-VAR)} \qquad
\dfrac{}{\Gamma,\ b : ?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\ \text{(T-FALSE)} \qquad
\dfrac{}{\Gamma,\ b : ?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\ \text{(T-TRUE)} \qquad [?^0 = \texttt{Bool}]\quad [\texttt{Bool} = \texttt{Bool}]
}{\Gamma,\ b : ?^0 \vdash \texttt{if } b \texttt{ then false else true} \ \Rightarrow \ \texttt{Bool}}\ \text{(T-IF)}
}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \ \Rightarrow \ ?^0 \to \texttt{Bool}}\ \text{(T-LAM)}
$$

$$
\dfrac{}{\Gamma,\ b : ?^0 \ \vdash \texttt{false} \Rightarrow \texttt{Bool}} \qquad \text{(T-FALSE)}
$$

Type Inference

$$\cfrac{new(?^0) \qquad \cfrac{\cfrac{}{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\text{ (T-VAR)} \quad \cfrac{}{\Gamma,\, b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\text{ (T-FALSE)} \quad \cfrac{}{\Gamma,\, b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\text{ (T-TRUE)} \quad [?^0 = \texttt{Bool}] \quad [\texttt{Bool} = \texttt{Bool}]}{\Gamma,\, b:?^0 \vdash \text{if } b \text{ then false else true} \Rightarrow \texttt{Bool}}\text{ (T-IF)}}{\Gamma \vdash \lambda b.\ \text{if } b \text{ then false else true} \Rightarrow ?^0 \to \texttt{Bool}}\text{ (T-LAM)}$$

$$\cfrac{}{\Gamma,\, b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}} \qquad \text{(T-FALSE)}$$

Type Inference

$$\frac{\Gamma, b:?^0 \vdash \texttt{false} \Rightarrow T_1 \qquad \Gamma, b:?^0 \vdash \texttt{true} \Rightarrow T_2 \qquad [\,?^0 = \texttt{Bool}\,]}{\Gamma, b:?^0 \vdash b \Rightarrow ?^0 \qquad\qquad\qquad\qquad\qquad [\,T_1 = T_2\,]}{\Gamma, b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow T_1} \; (\text{T-IF})$$

Type Inference

$$\cfrac{new(?^0) \qquad \cfrac{\overline{\Gamma, b:?^0 \vdash b \Rightarrow ?^0}\ (\text{T-VAR}) \quad \overline{\Gamma, b:?^0 \vdash \text{false} \Rightarrow \text{Bool}}\ (\text{T-FALSE}) \quad \overline{\Gamma, b:?^0 \vdash \text{true} \Rightarrow \text{Bool}}\ (\text{T-TRUE}) \qquad [?^0 = \text{Bool}] \quad [\text{Bool} = \text{Bool}]}{\Gamma, b:?^0 \vdash \text{if } b \text{ then false else true} \Rightarrow \text{Bool}}\ (\text{T-IF})}{\Gamma \vdash \lambda b.\ \text{if } b \text{ then false else true} \Rightarrow ?^0 \to \text{Bool}}\ (\text{T-LAM})$$

$$\cfrac{\Gamma,\, b:?^0 \vdash \text{false} \Rightarrow \boxed{T_1} \qquad \Gamma,\, b:?^0 \vdash \text{true} \Rightarrow T_2 \qquad [\ ?^0\ = \text{Bool}] \\ \Gamma,\, b:?^0 \vdash b \Rightarrow ?^0 \hspace{6cm} [\ \boxed{T_1}\ =\ T_2\ ]}{\Gamma,\, b:?^0 \vdash \text{if } b \text{ then false else true} \Rightarrow \boxed{T_1}}\ (\text{T-IF})$$

Type Inference

$$\frac{\overline{\Gamma, b:?^0 \vdash b \Rightarrow ?^0} \; \text{(T-VAR)} \quad \overline{\Gamma, b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}} \; \text{(T-FALSE)} \quad \overline{\Gamma, b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}} \; \text{(T-TRUE)} \quad [?^0 = \texttt{Bool}] \; [\texttt{Bool} = \texttt{Bool}]}{\frac{new(?^0) \qquad \Gamma, b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}}{\Gamma \vdash \lambda b.\texttt{ if } b \texttt{ then false else true} \Rightarrow ?^0 \to \texttt{Bool}} \; \text{(T-LAM)}} \; \text{(T-IF)}$$

$$\frac{\Gamma, b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool} \qquad \Gamma, b:?^0 \vdash \texttt{true} \Rightarrow T_2 \qquad [\,?^0 = \texttt{Bool}\,] \\ \Gamma, b:?^0 \vdash b \Rightarrow ?^0 \qquad\qquad\qquad\qquad\qquad\qquad [\,\texttt{Bool} = T_2\,]}{\Gamma, b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}}$$

$$(\text{T-IF})$$

Type Inference

$$\cfrac{\cfrac{new(?^0)}{\cfrac{\overline{\Gamma,\ b:?^0 \vdash b \Rightarrow ?^0}\ \text{(T-VAR)} \quad \overline{\Gamma,\ b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\ \text{(T-FALSE)} \quad \overline{\Gamma,\ b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\ \text{(T-TRUE)} \quad [?^0 = \texttt{Bool}]\ [\texttt{Bool} = \texttt{Bool}]}{\Gamma,\ b:?^0 \vdash \texttt{if}\ b\ \texttt{then false else true} \ \Rightarrow\ \texttt{Bool}}\ \text{(T-IF)}}{\Gamma \vdash \lambda b.\ \texttt{if}\ b.\ \texttt{then false else true} \ \Rightarrow\ ?^0 \to \texttt{Bool}}\ \text{(T-LAM)}$$

$$\cfrac{\begin{array}{cc} \Gamma,\ b:?^0 \ \vdash\ \texttt{false}\ \Rightarrow\ \texttt{Bool} & \Gamma,\ b:?^0 \ \vdash\ \texttt{true}\ \Rightarrow\ T_2 \qquad [\ ?^0 = \texttt{Bool}] \\ \Gamma,\ b:?^0 \ \vdash\ b\ \Rightarrow\ ?^0 & [\ \texttt{Bool}\ =\ T_2\ ] \end{array}}{\Gamma,\ b:?^0 \ \vdash \texttt{if}\ b\ \texttt{then false else true}\ \Rightarrow\ \texttt{Bool}}$$

$$(\text{T-IF})$$

Type Inference

$$\dfrac{new(?^0) \qquad \dfrac{\dfrac{}{\Gamma,\, b : ?^0 \vdash b \Rightarrow ?^0}\ (\text{T-VAR}) \quad \dfrac{}{\Gamma,\, b : ?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\ (\text{T-FALSE}) \quad \dfrac{}{\Gamma,\, b : ?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\ (\text{T-TRUE}) \quad [?^0 = \texttt{Bool}]\ [\texttt{Bool} = \texttt{Bool}]}{\Gamma,\, b : ?^0 \vdash \texttt{if } b \texttt{ then false else true} \ \Rightarrow\ \texttt{Bool}}\ (\text{T-IF})}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \ \Rightarrow\ ?^0 \to \texttt{Bool}}\ (\text{T-LAM})$$

$$\dfrac{}{\Gamma \ \vdash \texttt{true} \Rightarrow \texttt{Bool}} \qquad\qquad (\text{T-TRUE})$$

Type Inference

$$
\frac{new(?^0) \quad \dfrac{\overline{\Gamma, \; b : ?^0 \vdash b \Rightarrow ?^0} \; (\text{T-VAR}) \quad \overline{\Gamma, \; b : ?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}} \; (\text{T-FALSE}) \quad \overline{\Gamma, \; b : ?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}} \; (\text{T-TRUE}) \quad [?^0 = \texttt{Bool}] \; [\texttt{Bool} = \texttt{Bool}]}{\Gamma, \; b : ?^0 \vdash \texttt{if } b \texttt{ then false else true} \; \Rightarrow \; \texttt{Bool}} \; (\text{T-IF})}{\Gamma \vdash \lambda b. \; \texttt{if } b \texttt{ then false else true} \; \Rightarrow \; ?^0 \to \texttt{Bool}} \; (\text{T-LAM})
$$

$$
\overline{\Gamma \vdash \texttt{true} \Rightarrow \texttt{Bool}} \qquad\qquad (\text{T-TRUE})
$$

Type Inference

$$
\frac{
\begin{array}{c}
new(?^0)
\end{array}
\quad
\begin{array}{c}
\dfrac{}{\Gamma, \, b : ?^0 \vdash b \Rightarrow ?^0} \text{ (T-VAR)} \quad
\dfrac{}{\Gamma, \, b : ?^0 \vdash \mathtt{false} \Rightarrow \mathtt{Bool}} \text{ (T-FALSE)} \quad
\dfrac{}{\Gamma, \, b : ?^0 \vdash \mathtt{true} \Rightarrow \mathtt{Bool}} \text{ (T-TRUE)} \quad [?^0 = \mathtt{Bool}] \; [\mathtt{Bool} = \mathtt{Bool}] \\[2mm]
\dfrac{\Gamma, \, b : ?^0 \vdash \mathtt{if}\ b\ \mathtt{then}\ \mathtt{false}\ \mathtt{else}\ \mathtt{true}\ \Rightarrow\ \mathtt{Bool}}{} \text{ (T-IF)}
\end{array}
}{
\Gamma \vdash \lambda b.\ \mathtt{if}\ b\ \mathtt{then}\ \mathtt{false}\ \mathtt{else}\ \mathtt{true}\ \Rightarrow\ ?^0 \rightarrow \mathtt{Bool}
} \text{ (T-LAM)}
$$

$$
\frac{}{\boxed{\Gamma, \, b : ?^0} \vdash \mathtt{true} \Rightarrow \mathtt{Bool}} \qquad \text{(T-TRUE)}
$$

Type Inference

$$\dfrac{\dfrac{new(?^0)}{\dfrac{\dfrac{}{\Gamma,\, b : ?^0 \vdash b \Rightarrow ?^0}\,(\text{T-VAR}) \quad \dfrac{}{\Gamma,\, b : ?^0 \vdash \text{false} \Rightarrow \text{Bool}}\,(\text{T-FALSE}) \quad \dfrac{}{\Gamma,\, b : ?^0 \vdash \text{true} \Rightarrow \text{Bool}}\,(\text{T-TRUE}) \quad [?^0 = \text{Bool}]\ [\text{Bool} = \text{Bool}]}{\Gamma,\, b : ?^0 \vdash \text{if } b \text{ then false else true } \Rightarrow \text{ Bool}}\,(\text{T-IF})}}{\Gamma \vdash \lambda b.\ \text{if } b \text{ then false else true } \Rightarrow\ ?^0 \to \text{Bool}}\,(\text{T-LAM})$$

$$\dfrac{\rule{4cm}{0.4pt}}{\Gamma,\, b : ?^0\ \vdash \text{true} \Rightarrow \text{Bool}} \qquad (\text{T-TRUE})$$

Type Inference

$$
\frac{new(?^0) \quad \dfrac{\overline{\Gamma,\, b : ?^0 \vdash b \Rightarrow ?^0}\ (\text{T-VAR}) \quad \overline{\Gamma,\, b : ?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\ (\text{T-FALSE}) \quad \overline{\Gamma,\, b : ?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\ (\text{T-TRUE}) \quad [?^0 = \texttt{Bool}]\ [\texttt{Bool} = \texttt{Bool}]}{\Gamma,\, b : ?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}}\ (\text{T-IF})}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \Rightarrow ?^0 \rightarrow \texttt{Bool}}\ (\text{T-LAM})
$$

$$
\frac{\Gamma,\, b : ?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool} \qquad \Gamma,\, b : ?^0 \vdash \texttt{true} \Rightarrow T_2 \qquad [\,?^0 = \texttt{Bool}\,]}{\Gamma,\, b : ?^0 \vdash b \Rightarrow ?^0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad [\,\texttt{Bool} = T_2\,]}
$$

$$
\frac{}{\Gamma,\, b : ?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}}
$$

$$(\text{T-IF})$$

Type Inference

$$\frac{\overline{\Gamma, b : ?^0 \vdash b \Rightarrow ?^0}}{(\text{T-VAR})} \quad \overline{\Gamma, b : ?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}^{(\text{T-FALSE})} \quad \overline{\Gamma, b : ?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}^{(\text{T-TRUE})} \quad [?^0 = \texttt{Bool}] \ [\texttt{Bool} = \texttt{Bool}]$$

$$\frac{new(?^0) \qquad\qquad \Gamma, b : ?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}}{\Gamma \vdash \lambda b. \texttt{ if } b \texttt{ then false else true} \Rightarrow ?^0 \to \texttt{Bool}} \ (\text{T-LAM})$$

$$\frac{\Gamma, b : ?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool} \qquad \Gamma, b : ?^0 \vdash \texttt{true} \Rightarrow \boxed{T_2} \qquad [\,?^0 = \texttt{Bool}\,]}{\Gamma, b : ?^0 \vdash b \Rightarrow ?^0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad [\,\texttt{Bool} = \boxed{T_2}\,]}$$

$$\Gamma, b : ?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}$$

$$(\text{T-IF})$$

Type Inference

$$\cfrac{new(?^0) \quad \cfrac{\overline{\Gamma, b:?^0 \vdash b \Rightarrow ?^0} \ (\text{T-VAR}) \quad \overline{\Gamma, b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}} \ (\text{T-FALSE}) \quad \overline{\Gamma, b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}} \ (\text{T-TRUE}) \quad [?^0 = \texttt{Bool}] \ [\texttt{Bool} = \texttt{Bool}]}{\Gamma, b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}} \ (\text{T-IF})}{\Gamma \vdash \lambda b.\ \texttt{if } b \texttt{ then false else true} \Rightarrow ?^0 \rightarrow \texttt{Bool}} \ (\text{T-LAM})$$

$$\cfrac{\Gamma, b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool} \qquad \Gamma, b:?^0 \vdash \texttt{true} \Rightarrow \boxed{\texttt{Bool}} \qquad [\ ?^0 = \texttt{Bool}]}{\Gamma, b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}}$$

$$\Gamma, b:?^0 \vdash b \Rightarrow ?^0 \qquad\qquad\qquad\qquad\qquad [\ \texttt{Bool} = \boxed{\texttt{Bool}}\ ]$$

$$(\text{T-IF})$$

Type Inference

$$
\cfrac{
new(?^0) \quad
\cfrac{
\overline{\Gamma, b : ?^0 \vdash b \Rightarrow ?^0}\ \text{(T-VAR)} \quad
\overline{\Gamma, b : ?^0 \vdash \text{false} \Rightarrow \text{Bool}}\ \text{(T-FALSE)} \quad
\overline{\Gamma, b : ?^0 \vdash \text{true} \Rightarrow \text{Bool}}\ \text{(T-TRUE)} \quad [?^0 = \text{Bool}]\ [\text{Bool} = \text{Bool}]
}{\Gamma, b : ?^0 \vdash \text{if } b \text{ then false else true} \Rightarrow \text{Bool}}\ \text{(T-IF)}
}{\Gamma \vdash \lambda b.\ \text{if } b \text{ then false else true} \Rightarrow ?^0 \rightarrow \text{Bool}}\ \text{(T-LAM)}
$$

$$
\cfrac{
\begin{array}{cc}
\Gamma, b : ?^0 \vdash \text{false} \Rightarrow \text{Bool} \qquad \Gamma, b : ?^0 \vdash \text{true} \Rightarrow \text{Bool} \qquad [\ ?^0 = \text{Bool}] \\
\Gamma, b : ?^0 \vdash b \Rightarrow ?^0 \qquad\qquad\qquad\qquad\qquad\qquad [\ \text{Bool} = \text{Bool}\ ]
\end{array}
}{\Gamma, b : ?^0 \vdash \text{if } b \text{ then false else true} \Rightarrow \text{Bool}}
$$

$$(\text{T-IF})$$

Type Inference

$$\dfrac{\dfrac{new(?^0)}{\Gamma \vdash \lambda b.\ \text{if } b \text{ then false else true } \Rightarrow\ ?^0 \to \text{Bool}}\ \ \dfrac{\dfrac{}{\Gamma,\ b:?^0 \vdash b \Rightarrow ?^0}\ (\text{T-VAR})\quad \dfrac{}{\Gamma,\ b:?^0 \vdash \text{false} \Rightarrow \text{Bool}}\ (\text{T-FALSE})\quad \dfrac{}{\Gamma,\ b:?^0 \vdash \text{true} \Rightarrow \text{Bool}}\ (\text{T-TRUE})\quad [?^0 = \text{Bool}]\ \ [\text{Bool} = \text{Bool}]}{\Gamma,\ b:?^0 \vdash \text{if } b \text{ then false else true } \Rightarrow\ \text{Bool}}\ (\text{T-IF})}{}\ (\text{T-LAM})$$

$$?^0 = \text{Bool}$$

$$\text{Bool} = \text{Bool}$$

Type Inference

$$\dfrac{new(?^0) \quad \dfrac{\overline{\Gamma, \, b : ?^0 \vdash b \Rightarrow ?^0} \;\text{(T-VAR)} \quad \overline{\Gamma, \, b : ?^0 \vdash \text{false} \Rightarrow \text{Bool}} \;\text{(T-FALSE)} \quad \overline{\Gamma, \, b : ?^0 \vdash \text{true} \Rightarrow \text{Bool}} \;\text{(T-TRUE)} \quad [?^0 = \text{Bool}] \;\; [\text{Bool} = \text{Bool}]}{\Gamma, \, b : ?^0 \vdash \text{if } b \text{ then false else true} \; \Rightarrow \; \text{Bool}} \;\text{(T-IF)}}{\Gamma \vdash \lambda b. \text{ if } b \text{ then false else true} \; \Rightarrow \; ?^0 \to \text{Bool}} \;\text{(T-LAM)}$$

$$\dfrac{\begin{array}{cc} \Gamma, \, b : ?^0 \vdash \text{false} \Rightarrow \text{Bool} \qquad \Gamma, \, b : ?^0 \vdash \text{true} \Rightarrow \text{Bool} \qquad [\, ?^0 = \text{Bool}\,] \\[2mm] \Gamma, \, b : ?^0 \vdash b \Rightarrow ?^0 \hspace{12em} [\, \text{Bool} = \text{Bool} \,] \end{array}}{\Gamma, \, b : ?^0 \vdash \text{if } b \text{ then false else true} \; \Rightarrow \; \text{Bool}}$$

$$(\text{T-IF})$$

Type Inference

$$\frac{new(?^0) \qquad \dfrac{\dfrac{}{\Gamma,\, b:?^0 \vdash b \Rightarrow ?^0}\text{ (T-VAR)} \quad \dfrac{}{\Gamma,\, b:?^0 \vdash \mathtt{false} \Rightarrow \mathtt{Bool}}\text{ (T-FALSE)} \quad \dfrac{}{\Gamma,\, b:?^0 \vdash \mathtt{true} \Rightarrow \mathtt{Bool}}\text{ (T-TRUE)} \quad [?^0 = \mathtt{Bool}] \quad [\mathtt{Bool} = \mathtt{Bool}]}{\Gamma,\, b:?^0 \vdash \mathtt{if}\ b\ \mathtt{then\ false\ else\ true}\ \Rightarrow\ \mathtt{Bool}}\text{ (T-IF)}}{\Gamma \vdash \lambda b.\ \mathtt{if}\ b\ \mathtt{then\ false\ else\ true}\ \Rightarrow\ ?^0 \to \mathtt{Bool}}\text{ (T-LAM)}$$

$$\frac{new(\,?^0\,) \qquad \Gamma,\ b\ :\ ?^0\ \vdash\ \mathtt{if}\ b\ \mathtt{then\ false\ else\ true}\ \Rightarrow\ T}{\Gamma \vdash \lambda\ b\ .\ \mathtt{if}\ b\ \mathtt{then\ false\ else\ true}\ \Rightarrow\ ?^0\ \to\ T}\ \text{(T-LAM)}$$

Type Inference

$$
\cfrac{new(?^0) \qquad \cfrac{\cfrac{}{\Gamma,\, b : ?^0 \vdash b \Rightarrow ?^0}\ (\text{T-VAR}) \quad \cfrac{}{\Gamma,\, b : ?^0 \vdash \text{false} \Rightarrow \text{Bool}}\ (\text{T-FALSE}) \quad \cfrac{}{\Gamma,\, b : ?^0 \vdash \text{true} \Rightarrow \text{Bool}}\ (\text{T-TRUE}) \quad [?^0 = \text{Bool}]\ [\text{Bool} = \text{Bool}]}{\Gamma,\, b : ?^0 \vdash \text{if } b \text{ then false else true} \Rightarrow \text{Bool}}\ (\text{T-IF})}{\Gamma \vdash \lambda b.\ \text{if } b \text{ then false else true} \Rightarrow ?^0 \to \text{Bool}}\ (\text{T-LAM})
$$

$$
\cfrac{new(\ ?^0\ ) \qquad \Gamma,\ b\ :\ ?^0\ \vdash\ \text{if } b \text{ then false else true} \Rightarrow \boxed{T}}{\Gamma \vdash \lambda\, b\ .\ \text{if } b \text{ then false else true} \quad \Rightarrow \quad ?^0\ \to\ \boxed{T}}\ (\text{T-LAM})
$$

Type Inference

$$\cfrac{new(?^0) \qquad \cfrac{\overline{\Gamma, b:?^0 \vdash b \Rightarrow ?^0}\ ^{(\text{T-VAR})} \quad \overline{\Gamma, b:?^0 \vdash \texttt{false} \Rightarrow \texttt{Bool}}\ ^{(\text{T-FALSE})} \quad \overline{\Gamma, b:?^0 \vdash \texttt{true} \Rightarrow \texttt{Bool}}\ ^{(\text{T-TRUE})} \quad [?^0 = \texttt{Bool}] \ [\texttt{Bool} = \texttt{Bool}]}{\Gamma, b:?^0 \vdash \texttt{if } b \texttt{ then false else true} \Rightarrow \texttt{Bool}}\ ^{(\text{T-IF})}}{\Gamma \vdash \lambda b. \texttt{ if } b \texttt{ then false else true} \Rightarrow ?^0 \rightarrow \texttt{Bool}}\ ^{(\text{T-LAM})}$$

$$\cfrac{new(\ ?^0\ ) \qquad \Gamma,\ b\ :\ ?^0\ \vdash\ \texttt{if } b \texttt{ then false else true} \Rightarrow \boxed{\texttt{Bool}}}{\Gamma \vdash \lambda\ b\ .\ \texttt{if } b \texttt{ then false else true} \quad \Rightarrow \quad ?^0\ \rightarrow\ \boxed{\texttt{Bool}}}\ (\text{T-LAM})$$

Type Inference

$$
\frac{
\begin{array}{c}
\\
new(?^0)
\end{array}
\qquad
\dfrac{
\dfrac{}{\Gamma, b : ?^0 \vdash b \Rightarrow ?^0}\text{(T-VAR)} \quad
\dfrac{}{\Gamma, b : ?^0 \vdash \mathtt{false} \Rightarrow \mathtt{Bool}}\text{(T-FALSE)} \quad
\dfrac{}{\Gamma, b : ?^0 \vdash \mathtt{true} \Rightarrow \mathtt{Bool}}\text{(T-TRUE)} \quad [?^0 = \mathtt{Bool}] \; [\mathtt{Bool} = \mathtt{Bool}]
}{\Gamma, b : ?^0 \vdash \mathtt{if}\ b\ \mathtt{then\ false\ else\ true}\ \Rightarrow\ \mathtt{Bool}}\text{(T-IF)}
}{
\Gamma \vdash \lambda b.\ \mathtt{if}\ b\ \mathtt{then\ false\ else\ true}\ \Rightarrow\ ?^0 \to \mathtt{Bool}
}\text{(T-LAM)}
$$

$$
\frac{new(\ ?^0\ ) \qquad \Gamma,\ b\ :\ ?^0\ \vdash\ \mathtt{if}\ b\ \mathtt{then\ false\ else\ true}\ \Rightarrow\ \mathtt{Bool}}{\Gamma \vdash \lambda\ b\ .\ \mathtt{if}\ b\ \mathtt{then\ false\ else\ true}\ \Rightarrow\ ?^0\ \to\ \mathtt{Bool}}\ (\text{T-LAM})
$$

Type Inference

$$?^0 \rightarrow \texttt{Bool}$$

$$?^0 = \texttt{Bool}$$

$$\texttt{Bool} = \texttt{Bool}$$

Type Inference

$$\lambda b.\ \text{if } b \text{ then false else true}$$

$$\text{Bool} \rightarrow \text{Bool}$$

Unification

Unification

- Base types

  Bool, Int, String, ...

- Compound types

  $\rightarrow$, List, Either, ...

- Metavariables

  $?^0$, $?^1$, $?^2$, ...

Unification

$$?^0 = \texttt{Bool}$$
$$\texttt{Bool} = \texttt{Bool}$$
$$\texttt{Bool} = \texttt{Bool} \rightarrow ?^1$$
$$?^2 \rightarrow ?^3 = ?^2$$

Unification

Substitutions (Solutions/Unifiers)

$$\begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & ?^2 \to \texttt{Bool} \\ ?^2 \mapsto & ?^2 \\ \ldots \\ ?^n \mapsto & ?^n \end{cases}$$

$$\begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & ?^2 \to \texttt{Bool} \end{cases}$$

Unification

Applying substitutions

$$s = \begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & ?^2 \to \texttt{Bool} \end{cases}$$

Unification

Applying substitutions

$$s = \begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & ?^2 \rightarrow \texttt{Bool} \end{cases}$$

$$s(?^0) \rightsquigarrow \texttt{Bool}$$

Unification

Applying substitutions

$$s = \begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & ?^2 \to \texttt{Bool} \end{cases}$$

$$s(?^0) \rightsquigarrow \texttt{Bool}$$
$$s(?^1) \rightsquigarrow ?^2 \to \texttt{Bool}$$

Unification

Applying substitutions

$$s = \begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & ?^2 \to \texttt{Bool} \end{cases}$$

$s(?^0) \rightsquigarrow \texttt{Bool}$

$s(?^1) \rightsquigarrow ?^2 \to \texttt{Bool}$

$s(?^2) \rightsquigarrow ?^2$

Unification

Applying substitutions

$$s = \begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & ?^2 \to \texttt{Bool} \end{cases}$$

Unification

Applying substitutions

$$s = \begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & ?^2 \to \texttt{Bool} \end{cases}$$

$s(\texttt{Bool}) \rightsquigarrow \texttt{Bool}$

Unification

Applying substitutions

$$s = \begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & ?^2 \to \texttt{Bool} \end{cases}$$

$$s(\texttt{Bool}) \rightsquigarrow \texttt{Bool}$$
$$s(?^0 \to ?^0) \rightsquigarrow \texttt{Bool} \to \texttt{Bool}$$

Unification

Applying substitutions

$$s = \begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & ?^2 \to \texttt{Bool} \end{cases}$$

$$s(\texttt{Bool}) \rightsquigarrow \texttt{Bool}$$
$$s(?^0 \to ?^0) \rightsquigarrow \texttt{Bool} \to \texttt{Bool}$$
$$s(?^1 \to ?^2) \rightsquigarrow (?^2 \to \texttt{Bool}) \to ?^2$$

Unification

$$id = \{$$

$$(s_2 \circ s_1)(x) = s_2(s_1(x))$$

Unification

s is a solution (unifier) of $x = y$ when $s(x) = s(y)$

s unifies x and y

Unification

What's the solution?

$$?^0 \to \mathtt{Bool} = \mathtt{Bool} \to \mathtt{Bool}$$

Unification

What's the solution?

$$?^0 \to \texttt{Bool} = \texttt{Bool} \to \texttt{Bool}$$

$$s = \left\{ ?^0 \mapsto \quad \texttt{Bool} \right.$$

Unification

What's the solution?

$$?^0 \to \texttt{Bool} = \texttt{Bool} \to \texttt{Bool}$$

$$s = \left\{ ?^0 \mapsto \quad \texttt{Bool} \right.$$

$$s' = \left\{ \begin{array}{l} ?^0 \mapsto \quad \texttt{Bool} \\ ?^1 \mapsto \quad \texttt{Bool} \end{array} \right.$$

Unification

What's the solution?

$$?^0 \to \texttt{Bool} = \texttt{Bool} \to \texttt{Bool}$$

$$s = \begin{cases} ?^0 \mapsto & \texttt{Bool} \end{cases}$$

$$s' = \begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & \texttt{Bool} \end{cases}$$

$$s'' = \begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & \texttt{Bool} \\ ?^2 \mapsto & ?^{24} \to ?^{25} \end{cases}$$

Unification

What's the solution?

$$?^0 \to \mathtt{Bool} = \mathtt{Bool} \to \mathtt{Bool}$$

$$s = \left\{ ?^0 \mapsto \quad \mathtt{Bool} \right.$$

$$s' = \begin{cases} ?^0 \mapsto \quad \mathtt{Bool} \\ ?^1 \mapsto \quad \mathtt{Bool} \end{cases}$$

$$s'' = \begin{cases} ?^0 \mapsto \quad \mathtt{Bool} \\ ?^1 \mapsto \quad \mathtt{Bool} \\ ?^2 \mapsto \quad ?^{24} \to ?^{25} \end{cases}$$

$\dots$

Unification

Most General (Solution/Unifier):

s is the most general solution for an equation if all other solutions s' can be written in terms of s composed with some other substitution

$$\frac{solves(s, e)}{\forall s'. \; \exists h. \; solves(s', e) \Rightarrow s' = s \circ h}$$

Unification

$$s = \begin{cases} ?^0 \mapsto & \texttt{Bool} \end{cases}$$

$$s' = \begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & \texttt{Bool} \end{cases}$$

$$s'' = \begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & \texttt{Bool} \\ ?^2 \mapsto & ?^{24} \to ?^{25} \end{cases}$$

Unification

$$s = \begin{cases} ?^0 \mapsto & \texttt{Bool} \end{cases} \qquad = s \circ id$$

$$s' = \begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & \texttt{Bool} \end{cases}$$

$$s'' = \begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & \texttt{Bool} \\ ?^2 \mapsto & ?^{24} \to ?^{25} \end{cases}$$

Unification

$$s = \left\{ ?^0 \mapsto \quad \texttt{Bool} \qquad\qquad = s \circ \mathit{id} \right.$$

$$s' = \begin{cases} ?^0 \mapsto \quad \texttt{Bool} \\ ?^1 \mapsto \quad \texttt{Bool} \end{cases} \quad = s \circ \left\{ ?^1 \mapsto \texttt{Bool} \right.$$

$$s'' = \begin{cases} ?^0 \mapsto \quad \texttt{Bool} \\ ?^1 \mapsto \quad \texttt{Bool} \\ ?^2 \mapsto \quad ?^{24} \to ?^{25} \end{cases}$$

Unification

$$s = \begin{cases} ?^0 \mapsto & \texttt{Bool} \end{cases} \qquad = s \circ id$$

$$s' = \begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & \texttt{Bool} \end{cases} \qquad = s \circ \begin{cases} ?^1 \mapsto \texttt{Bool} \end{cases}$$

$$s'' = \begin{cases} ?^0 \mapsto & \texttt{Bool} \\ ?^1 \mapsto & \texttt{Bool} \\ ?^2 \mapsto & ?^{24} \to ?^{25} \end{cases} \qquad = s \circ \begin{cases} ?^1 \mapsto & \texttt{Bool} \\ ?^2 \mapsto & ?^{24} \to ?^{25} \end{cases}$$

Idempotence:

$$s \circ s = s$$

Unification

unify   : *Set Equation $\rightarrow$ Substitution*

Unification

$$unify \qquad\qquad\qquad : Set\ Equation \rightarrow Substitution$$
$$unify(\{\}) \qquad\qquad\qquad = id$$

Unification

$$unify \qquad\qquad\qquad : Set\ Equation \rightarrow Substitution$$
$$unify(\{\}) \qquad\qquad\qquad = id$$
$$unify(\{B_1 = B_2, rest\}) \qquad\qquad =$$

Unification

$$unify \qquad\qquad\qquad\qquad\qquad\qquad\qquad : Set\ Equation \rightarrow Substitution$$
$$unify(\{\}) \qquad\qquad\qquad\qquad\qquad\qquad = id$$
$$unify(\{B_1 = B_2, rest\}) \qquad\qquad\qquad\qquad =$$

  if $B_1 = B_2$ then $unify(rest)$ else $\perp$

Unification

$$unify \qquad\qquad\qquad\qquad\qquad\qquad\qquad : Set\ Equation \rightarrow Substitution$$
$$unify(\{\}) = id$$
$$unify(\{B_1 = B_2, rest\}) =$$
$$\text{if } B_1 = B_2 \text{ then } unify(rest) \text{ else } \bot$$
$$unify(\{C_1(x_1,\ \ldots,\ x_n) = C_2(y_1,\ \ldots,\ y_m), rest\}) =$$

Unification

$$
\begin{array}{ll}
\textit{unify} & : \textit{Set Equation} \rightarrow \textit{Substitution} \\
\textit{unify}(\{\}) & = \textit{id} \\
\textit{unify}(\{B_1 = B_2, \textit{rest}\}) & = \\
\quad \texttt{if } B_1 = B_2 \texttt{ then } \textit{unify}(\textit{rest}) \texttt{ else } \bot \\
\textit{unify}(\{C_1(x_1, \ldots, x_n) = C_2(y_1, \ldots, y_m), \textit{rest}\}) & = \\
\quad \texttt{if } C_1 = C_2 \texttt{ then } \textit{unify}(\{\, x_i = y_i \mid i \in [1, n] \,\} \cup \textit{rest}) \texttt{ else } \bot
\end{array}
$$

Unification

$$
\begin{aligned}
&unify && : Set\ Equation \rightarrow Substitution \\
&unify(\{\}) && = id \\
&unify(\{B_1 = B_2, rest\}) && = \\
&\quad \texttt{if } B_1 = B_2 \texttt{ then } unify(rest) \texttt{ else } \bot \\
&unify(\{C_1(x_1,\ \ldots,\ x_n) = C_2(y_1,\ \ldots,\ y_m), rest\}) && = \\
&\quad \texttt{if } C_1 = C_2 \texttt{ then } unify(\{\ x_i = y_i \mid i \in [1,\ n]\ \} \cup rest) \texttt{ else } \bot \\
&unify(\{?^n = t, rest\}) && =
\end{aligned}
$$

Unification

$$
\begin{aligned}
&\textit{unify} && : \textit{Set Equation} \rightarrow \textit{Substitution} \\
&\textit{unify}(\{\}) && = \textit{id} \\
&\textit{unify}(\{B_1 = B_2, \textit{rest}\}) && = \\
&\quad \text{if } B_1 = B_2 \text{ then } \textit{unify}(\textit{rest}) \text{ else } \bot \\
&\textit{unify}(\{C_1(x_1, \ldots, x_n) = C_2(y_1, \ldots, y_m), \textit{rest}\}) && = \\
&\quad \text{if } C_1 = C_2 \text{ then } \textit{unify}(\{ x_i = y_i \mid i \in [1, n] \} \cup \textit{rest}) \text{ else } \bot \\
&\textit{unify}(\{?^n = t, \textit{rest}\}) && = \\
&\quad \text{let } s = \left\{ ?^n \mapsto t \right.
\end{aligned}
$$

Unification

$$unify \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad : Set\ Equation \rightarrow Substitution$$
$$unify(\{\}) \qquad\qquad\qquad\qquad\qquad\qquad\qquad = id$$
$$unify(\{B_1 = B_2, rest\}) \qquad\qquad\qquad\qquad =$$
if $B_1 = B_2$ then $unify(rest)$ else $\bot$
$$unify(\{C_1(x_1, \ldots, x_n) = C_2(y_1, \ldots, y_m), rest\}) =$$
if $C_1 = C_2$ then $unify(\{\ x_i = y_i \mid i \in [1,\ n]\ \} \cup rest)$ else $\bot$
$$unify(\{?^n = t, rest\}) \qquad\qquad\qquad\qquad =$$
let $s = \left\{ ?^n \mapsto t \right.$
if $occurs(?^n, t)$ then $\bot$ else $s \circ unify(\{\ s(t) = s(u) \mid (t = u) \in rest\ \})$

Unification

$$unify \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad : Set\ Equation \rightarrow Substitution$$
$$unify(\{\}) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = id$$
$$unify(\{B_1 = B_2, rest\}) \qquad\qquad\qquad\qquad\qquad =$$
$$\quad \texttt{if}\ B_1 = B_2\ \texttt{then}\ unify(rest)\ \texttt{else}\ \bot$$
$$unify(\{C_1(x_1,\ \ldots,\ x_n) = C_2(y_1,\ \ldots,\ y_m), rest\}) \ =$$
$$\quad \texttt{if}\ C_1 = C_2\ \texttt{then}\ unify(\{\ x_i = y_i \mid i \in [1,\ n]\ \} \cup rest)\ \texttt{else}\ \bot$$
$$unify(\{?^n = t, rest\}) \qquad\qquad\qquad\qquad\qquad =$$
$$\quad \texttt{let}\ s = \left\{?^n \mapsto t \right.$$
$$\quad \texttt{if}\ occurs(?^n, t)\ \texttt{then}\ \bot\ \texttt{else}\ s \circ unify(\{\ s(t) = s(u) \mid (t = u) \in rest\ \})$$
$$unify(\{s = t, rest\}) \qquad\qquad\qquad\qquad\qquad\qquad = unify(\{t = s, rest\})$$

Unification

Why the 'occurs' check?

Unification

$$?^n = \texttt{Bool}$$

Unification

$$?^n = \texttt{Bool}$$

$$s = \left\{ ?^n \mapsto \texttt{Bool} \right.$$

Unification

$$?^n = \texttt{Bool}$$

$$s = \left\{ ?^n \mapsto \texttt{Bool} \right.$$

$$s(?^n) = s(\texttt{Bool})$$

Unification

$$?^n = \texttt{Bool}$$

$$s = \left\{ ?^n \mapsto \texttt{Bool} \right.$$

$$
\begin{aligned}
s(?^n) &= s(\texttt{Bool}) \\
\texttt{Bool} &= \texttt{Bool}
\end{aligned}
$$

Unification

$$?^n = ?^n \to \texttt{Bool}$$

Unification

$$?^n = ?^n \to \texttt{Bool}$$

$$s = \left\{ ?^n \mapsto ?^n \to \texttt{Bool} \right.$$

Unification

$$?^n = ?^n \to \texttt{Bool}$$

$$s = \left\{ ?^n \mapsto ?^n \to \texttt{Bool} \right.$$

$$s(?^n) \qquad = s(?^n \to \texttt{Bool})$$

Unification

$$?^n = ?^n \to \texttt{Bool}$$

$$s = \left\{ ?^n \mapsto ?^n \to \texttt{Bool} \right.$$

$$s(?^n) \qquad = s(?^n \to \texttt{Bool})$$
$$?^n \to \texttt{Bool} = (?^n \to \texttt{Bool}) \to \texttt{Bool}$$

Unification

Performance considerations:

Unification

Performance considerations:
- Substitution is slow

Unification

Performance considerations:

- ► Substitution is slow
    - ► Mutable variables for metas

Unification

Performance considerations:

- ► Substitution is slow
    - ► Mutable variables for metas
    - ► Union-Find data structure

Unification

Performance considerations:

- ▶ Substitution is slow
  - ▶ Mutable variables for metas
  - ▶ Union-Find data structure
- ▶ Occurs check is slow

Unification

Performance considerations:

- ▶ Substitution is slow
  - ▶ Mutable variables for metas
  - ▶ Union-Find data structure
- ▶ Occurs check is slow
  - ▶ Ignore it

Unification

Performance considerations:

- ▶ Substitution is slow
    - ▶ Mutable variables for metas
    - ▶ Union-Find data structure
- ▶ Occurs check is slow
    - ▶ Ignore it
    - ▶ Defer it

Polymorphism

Polymorphism

$$\lambda x \, . \, \lambda y \, . \, x$$

Polymorphism

$$\cfrac{new(?^0) \quad \cfrac{new(?^1) \quad \cfrac{}{\Gamma, x : ?^0, y : ?^1 \vdash x \Rightarrow ?^0}\;(\text{T-VAR})}{\Gamma, x : ?^0 \vdash \lambda y.\, x \;\Rightarrow\; ?^1 \rightarrow ?^0}\;(\text{T-LAM})}{\Gamma \vdash \lambda x.\, \lambda y.\, x \;\Rightarrow\; ?^0 \rightarrow ?^1 \rightarrow ?^0}\;(\text{T-LAM})$$

Polymorphism

$$tyvar ::=$$
$$\alpha$$
$$\beta$$
$$\dots$$

$$Type ::=$$
$$\dots$$
$$tyvar$$

Polymorphism

$Scheme ::=$

        $Type$

        $\forall tyvar.\ Scheme$

$Context ::=$

        $\bullet$

        $Context,\ var : Scheme$

Polymorphism

$$\forall \alpha. \ \alpha \to \alpha$$

$$\forall \alpha. \ \forall \beta. \ \alpha \to \beta \to \alpha$$

Polymorphism

$$freetvs_{\text{Type}} \quad : Type \rightarrow Set\ tyvar$$
$$freetvs_{\text{Type}} \quad = \ldots$$

$$freetvs_{\text{Scheme}} \qquad\qquad : Scheme \rightarrow Set\ tyvar$$
$$freetvs_{\text{Scheme}}(\forall \alpha.\,T) \ = freetvs_{\text{Scheme}}(T) - \{\alpha\}$$
$$freetvs_{\text{Scheme}}(T) \qquad = freetvs_{\text{Type}}(T)$$

$$freetvs_{\text{Context}}(\bullet) \qquad\quad = \{\}$$
$$freetvs_{\text{Context}}(\Gamma, x : T) \ = freetvs_{\text{Scheme}}(T) \cup freetvs_{\text{Context}}(\Gamma)$$

$$\begin{aligned}
\textit{instantiate} &: \textit{Scheme} \rightarrow \textit{Type} \\
\textit{instantiate}(\forall \alpha \,.\, T) &= \textit{new}(?^n);\ \textit{instantiate}(T)[\alpha := ?^n] \\
\textit{instantiate}(T) &= T
\end{aligned}$$

Polymorphism

$solve : Type \rightarrow Type$
$solve = \ldots$

$generalize \qquad : Context \rightarrow Type \rightarrow Scheme$
$generalize(\Gamma, T) =$
$\quad \texttt{let } T' = solve(T)$
$\quad \forall(freetvs_{\texttt{Type}}(T') - freetvs_{\texttt{Context}}(\Gamma)).\, T'$

Polymorphism

$$\frac{}{\Gamma,\ x : T \vdash x \Rightarrow \mathit{instantiate}(T)} \quad (\text{T-var})$$

Polymorphism

$$term ::=$$
$$\dots$$
$$\texttt{let } var = term \texttt{ in } term$$

Polymorphism

$$\frac{\Gamma \vdash e \Rightarrow S \qquad \Gamma, x : \textit{generalize}(\Gamma, S) \vdash b \Rightarrow T}{\Gamma \vdash \texttt{let } x = e \texttt{ in } b \; \Rightarrow \; T} \qquad \text{(T-LET)}$$

Polymorphism

$$term ::=$$
$$\dots$$
$$\texttt{letrec } var = term \texttt{ in } term$$

Polymorphism

$$\frac{new(?^n) \qquad \Gamma, x : ?^n \vdash e \Rightarrow S \qquad ?^n = S \qquad \Gamma, x : \textit{generalize}(\Gamma, S) \vdash b \Rightarrow T}{\Gamma \vdash \texttt{letrec } x = e \texttt{ in } b \;\Rightarrow\; T}$$

$$(\text{T-LETREC})$$

Polymorphism

Performance improvements:

Polymorphism

Performance improvements:

- ▶ 'Batched' instantiation/generalization

Polymorphism

Performance improvements:
- 'Batched' instantiation/generalization
  - $\forall \tilde{\alpha}.\, T$

Polymorphism

Performance improvements:

- ▶ 'Batched' instantiation/generalization
  - ▶ $\forall \tilde{\alpha} . T$
- ▶ Generalization is slow

Polymorphism

Performance improvements:

- 'Batched' instantiation/generalization
  - $\forall \tilde{\alpha} . T$
- Generalization is slow
  - Lambda / Let ranking

Further reading

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▶ Martelli, A., & Montanari, U. (1982). An efficient unification algorithm. *ACM Transactions on Programming Languages and Systems (TOPLAS)*, 4(2), 258-282.

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▶ http://dev.stephendiehl.com/fun/006_hindley_milner.html

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- `http://dev.stephendiehl.com/fun/006_hindley_milner.html`
- Kuan, G., & MacQueen, D. (2007, October). Efficient type inference using ranked type variables. In *Proceedings of the 2007 workshop on Workshop on ML* (pp. 3-14). ACM.

blog.ielliott.io/talks/type-inference.pdf