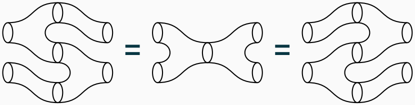
# Classifying extended 2D TQFTs

## XXII Encuentro de Jóvenes Topólogos


Santiago Pareja Pérez

Unizar–UCM


24th October 2024



# Outline: questions and keywords

Questions:

- What are TQFTs?
- How do we classify 2D TQFTs?
- Can we generalize this to higher dimensions?
- How does our generalization relate to the original 2D case?

Keywords:

- Symmetric monoidal functors.
- Frobenius algebras.
- *n*-categories and the Cobordism Hypothesis.
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# Cobordisms and TQFTs

# Cobordisms

We assume everything smooth and compact.

Let $M$ and $N$ be two closed oriented $(n-1)$-manifolds.

A ***cobordism*** $B\colon M \to N$ is an $n$-manifold with boundary endowed with a diffeomorphism $\partial B \cong \overline{M} \sqcup N$.



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$M$ is the ***in-boundary*** and $N$ is the ***out-boundary***.

A cobordism needs not be connected:



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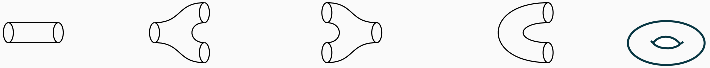

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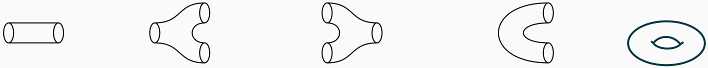

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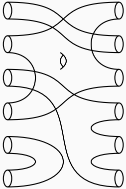

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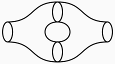
# Gluing and adding cobordisms

We can ***compose*** cobordisms by gluing.



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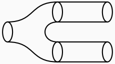

$\mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1$

The identities are the cylinders $M \times [0,1] \colon M \to M$.

We can add cobordisms by taking their disjoint union.



$\mathbb{S}^1 \sqcup (\mathbb{S}^1 \sqcup \mathbb{S}^1) \to (\mathbb{S}^1 \sqcup \mathbb{S}^1) \sqcup \mathbb{S}^1$

This is a ***monoidal structure*** with unit the empty manifold: $M \sqcup \varnothing \cong M$.

We can freely interchange connected components.
These *twist cobordisms* give a ***symmetric structure***.




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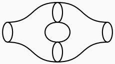

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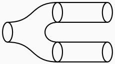

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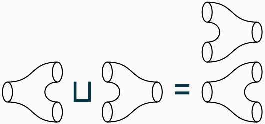


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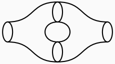

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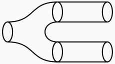

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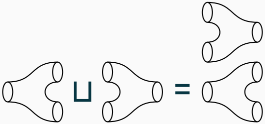

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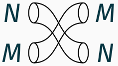


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# $\mathbf{Cob}_n$, the category of $n$-cobordisms

$n$-cobordisms assemble into a symmetric monoidal category, $\mathbf{Cob}_n$:

**Objects** Closed $(n-1)$-manifolds $M$, $N$.

**Morphisms** $n$-cobordisms $B : M \to N$, up to diffeomorphism.

**Identities** "Cylinders" $M \times [0, 1] : M \to M$.

**Composition** Gluing of cobordisms.

**Monoid** Disjoint union $B \sqcup B' : M \sqcup M' \to N \sqcup N'$.

**Unit** Empty manifold $\varnothing$.

**Twists** Twist cobordisms $M \sqcup N \to N \sqcup M$.

One well-understood symmetric monoidal category is $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$, the category of vector spaces equipped with tensor product and the usual interchange of factors.
(i.e., the twist maps $\sigma_{V,W} : V \otimes W \to W \otimes V$ are given by $v \otimes w \mapsto w \otimes v$).

We can use $\mathbf{Vect}_{\Bbbk}$ to study $\mathbf{Cob}_n$, via maps $\mathbf{Cob}_n \to \mathbf{Vect}_{\Bbbk}$.

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# Topological quantum field theories

A ***TQFT*** is a rule $Z\colon \mathbf{Cob}_n \to \mathbf{Vect}_{\Bbbk}$ which assigns

- closed $(n-1)$-manifold $M$ $\rightsquigarrow$ $\Bbbk$-vector space $Z(M)$.
- $n$-cobordism $B\colon M \to N$ $\rightsquigarrow$ $\Bbbk$-linear map $Z(B)\colon Z(M) \to Z(N)$.

According to the following laws.

- Diffeomorphic cobordisms have equal image: $B \cong B' \rightsquigarrow Z(B) = Z(B')$.
- Cylinders go to identities: $Z(\ ) = \mathrm{id}_{Z(M)}$.
- Gluing cobordism is composing functions: $Z(\ ) = Z(\ ) \circ Z(\ )$.

- Disjoint union is tensor product: $Z\left(\ \right) = Z(\ ) \otimes Z(\ )$.
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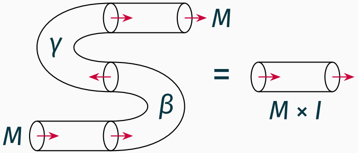
# Zorro's Lemma: decomposing cylinders




$$\begin{tikzcd}
M \arrow[d, equal] & (M \sqcup \overline{M}) \sqcup M \arrow[r, "\beta \sqcup \mathrm{id}"] \arrow[d, phantom, "\cong"] & \varnothing \sqcup M \arrow[d, equal] \\
M \sqcup \varnothing \arrow[r, "\mathrm{id} \sqcup \gamma"'] & M \sqcup (\overline{M} \sqcup M) & M
\end{tikzcd} \quad = \quad M \times I.$$

Now we evaluate a TQFT $Z$ on this diagram.
Let $V \coloneqq Z(M)$ and $W \coloneqq Z(\overline{M})$, and also $\mathrm{ev} \coloneqq Z(\beta)$ and $\mathrm{coev} \coloneqq Z(\gamma)$:

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V \otimes \Bbbk \arrow[r, "\mathrm{id} \otimes \mathrm{coev}"'] & V \otimes (W \otimes V) & V
\end{tikzcd} \quad = \quad \mathrm{id}_V.$$

So $V \xrightarrow{\mathrm{id} \otimes \mathrm{coev}} V \otimes W \otimes V \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} V$ is $\mathrm{id}_V \colon V \to V$.

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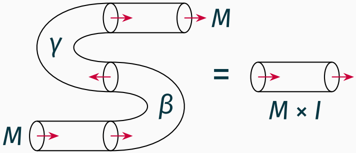


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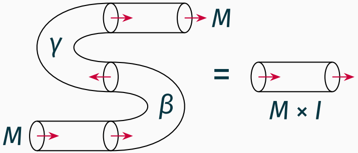


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The composition $V \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} V \otimes W \otimes V \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} V$ is $\mathrm{id}_V$.
This forces $V$ to have finite dimension, as follows.

- $\mathrm{coev}\colon \Bbbk \to W \otimes V$ is determined by its image at 1, say

$$\mathrm{coev}(1) =: \sum_{i=1}^{n} w_i \otimes v_i.$$

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$$v \longmapsto \sum_{i=1}^{n} v \otimes (w_i \otimes v_i) \longmapsto \sum_{i=1}^{n} \mathrm{ev}(v \otimes w_i) \cdot v_i = v.$$

Notice that $\mathrm{ev}(v \otimes w_i) \in \Bbbk$, so $\{v_1, \dots, v_n\}$ is a spanning set for $V$. $\square$

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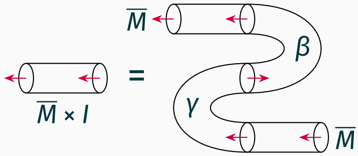
By similar arguments we can identify $W \equiv V^*$.

Here we need the dual diagram (the "Z" to our prior "S"):




An object $X$ in a monoidal category $(\mathcal{C}, \Box, \mathrm{I})$ is called ***dualizable*** if it satisfies the conditions in both Zorro's diagrams.

- There exists a ***dual object*** $X^\vee$,
- along with maps $\mathrm{ev} : X \Box X^\vee \to \mathrm{I}$ and $\mathrm{coev} : \mathrm{I} \to X^\vee \Box X$,
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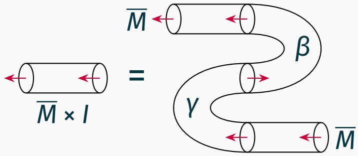


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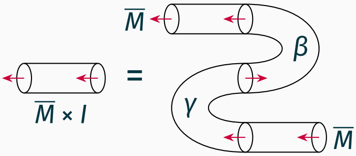


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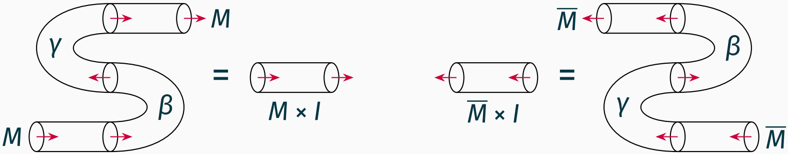


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Every object $M \in \mathbf{Cob}_n$ is dualizable ($\mathbf{Cob}_n$ is ***rigid***).

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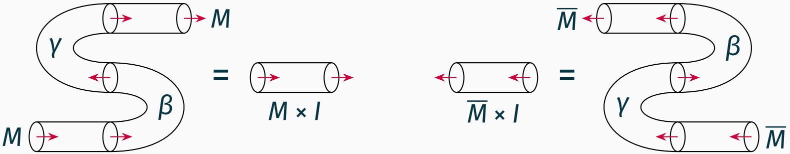


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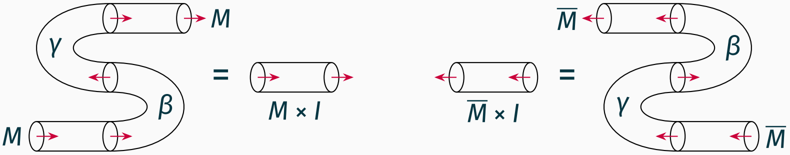


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# 1D TQFTs are determined by their image at the point

Here are all of the connected 1-cobordisms:

As a symmetric monoidal category, $\mathbf{Cob}_1$ is generated by the two objects $\mathrm{pt}_+$, $\mathrm{pt}_-$ and two of the arcs:



These satisfy two aditional relations, which are just Zorro's Lemma.



**Theorem**

$\mathbf{Cob}_1$ *is the free symmetric monoidal category on a dualizable object.*

So oriented 1D TQFTs $Z\colon \mathbf{Cob}_1 \to \mathbf{Vect}_{\Bbbk}$ are determined by the image of the point $Z(\mathrm{pt}_+)$, which must be a finite-dimensional vector space.

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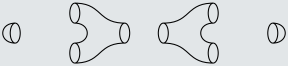
# 2D oriented TQFTs are the same as commutative Frobenius Algebras

# Generators of $\mathbf{Cob}_2$

**Theorem**

*$\mathbf{Cob}_2$ is generated by the object $\mathbb{S}^1$ and the four morphisms below.*



*Proof*:
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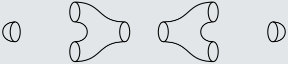

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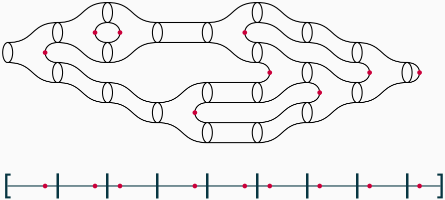

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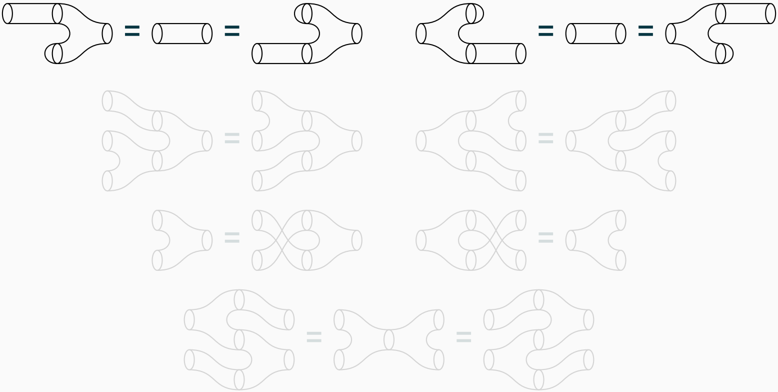

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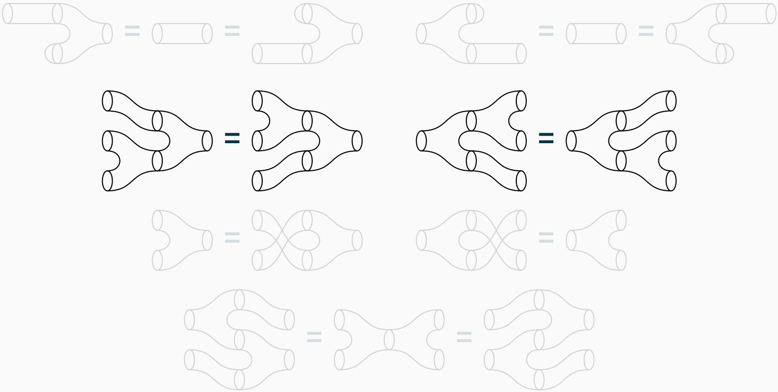

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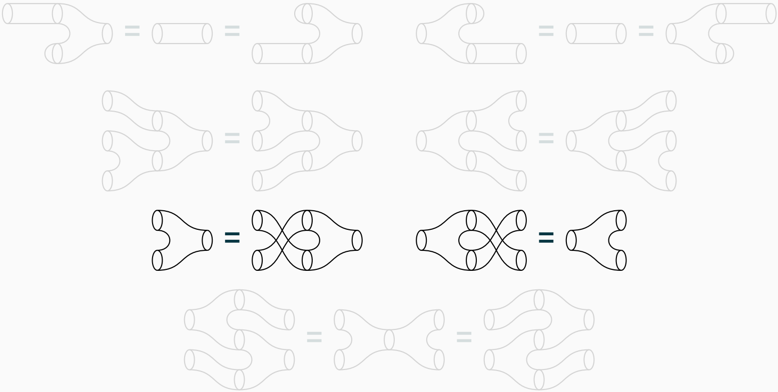

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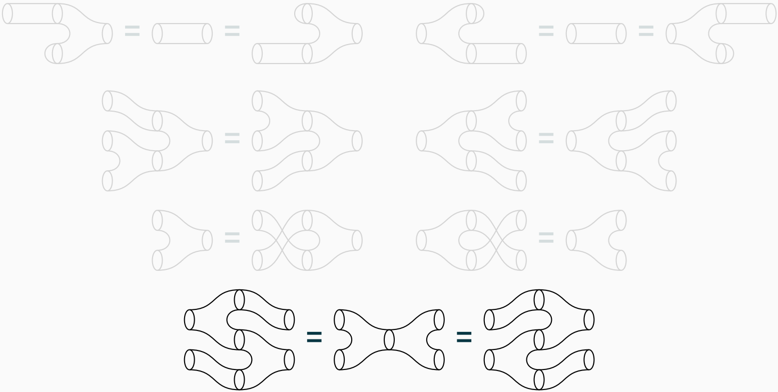

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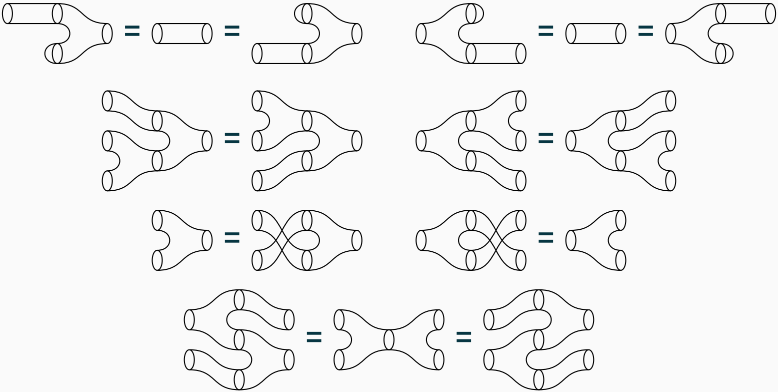

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# Algebras, graphically

We can give algebraic sense to the generators and relations of **Cob**$_2$.
Generators become algebraic structure. Relations become axioms.

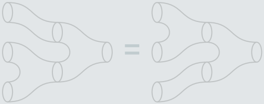
Algebras are unital and associative, but not necessarily commutative.

**Definition**

An ***algebra*** over a field $\Bbbk$ is a $\Bbbk$-vector space $A$ equipped with linear maps

- multiplication $\mu : A \otimes A \to A$ (drawn ),
- unit map $\eta : \Bbbk \to A$ (drawn ),

satisfying



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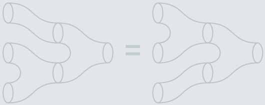

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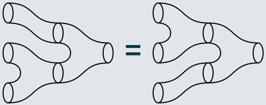

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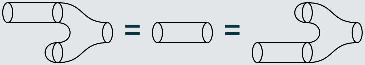

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# Frobenius algebras

**Definition**

A ***Frobenius algebra*** $(A, \varepsilon)$ is a $\Bbbk$-algebra $A$ equipped with a linear "trace" map $\varepsilon : A \to \Bbbk$ whose kernel contains no non-trivial ideals.

Examples:
- Matrices $n \times n$ with the trace $\mathrm{tr} : M_{\Bbbk}(n) \to \Bbbk$.
- Complex numbers with the real part $\mathfrak{Re} : \mathbb{C} \to \mathbb{R}$.

Let $(A, \varepsilon)$ a Frobenius algebra. We draw $\varepsilon$ as .

Define the ***pairing*** $\beta : A \otimes A \to \Bbbk$ (drawn ) as



$$\beta(x \otimes y) \coloneqq \varepsilon(x \cdot y).$$

The pairing is ***associative***:



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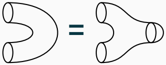

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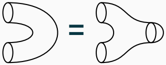

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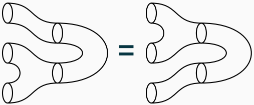

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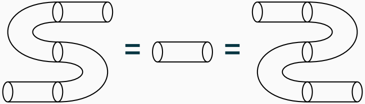
# Frobenius algebras in terms of pairings

The map $\beta$ =  is a ***non-degenerate*** pairing:
there exists a ***copairing*** $\gamma\colon \Bbbk \to A \otimes A$ (drawn ) such that



(Since $\operatorname{Ker} \varepsilon$ contains no non-trivial ideals).

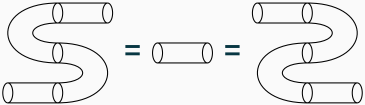
**Characterization**

*A Frobenius algebra $(A, \beta)$ is a $\Bbbk$-algebra $A$ equipped with an associative non-degenerate pairing $\beta\colon A \otimes A \to \Bbbk$.*

(Given $\beta$, we recover the previous definition by setting $\varepsilon = \beta(- \otimes 1_A)$).

# Frobenius algebras in terms of pairings

The map $\beta$ =  is a ***non-degenerate*** pairing:
there exists a ***copairing*** $\gamma\colon \Bbbk \to A \otimes A$ (drawn ) such that



(Since $\operatorname{Ker}\varepsilon$ contains no non-trivial ideals).

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# Coalgebras

The dual concept to “algebra”.

**Definition**

An ***coalgebra*** over a field $\Bbbk$ is a $\Bbbk$-vector space $A$ equipped with linear maps

- comultiplication $\delta \colon A \to A \otimes A$ (drawn ),
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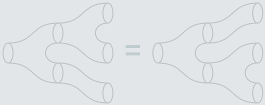

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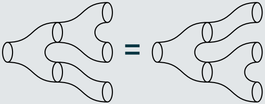

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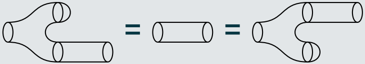

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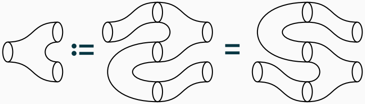

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*A Frobenius algebra $(A, \mu, \eta, \delta, \varepsilon)$ is a $\Bbbk$-algebra $(A, \mu, \eta)$ which is also a $\Bbbk$-coalgebra $(A, \delta, \varepsilon)$, and such that the two structures satisfy the **Frobenius relation**:*



Example:

$$\varepsilon : \mathbb{C} \to \mathbb{R} \qquad \delta : \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$$
$$z \mapsto \mathfrak{Re}(z), \qquad z \mapsto z \otimes 1 - iz \otimes i.$$

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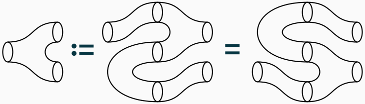

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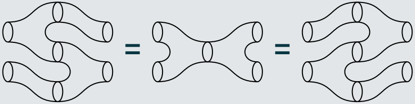

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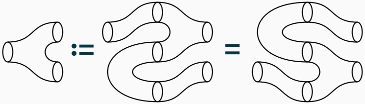

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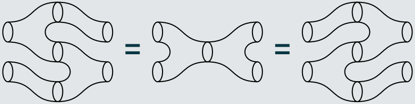

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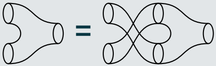
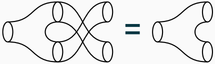
# Commutative and symmetric Frobenius algebras

**Definition**

A Frobenius algebra is ***commutative*** if it is a commutative algebra:



Equivalent to being a ***cocommutative*** coalgebra: 

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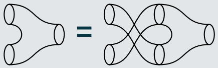
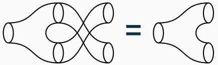


Equivalent to being a ***cocommutative*** coalgebra: 

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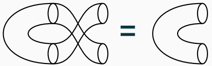
Equivalent to the copairing $\gamma$ being ***symmetric***: 

# The correspondence

An oriented 2D TQFT $Z$ determines a commutative Frobenius algebra.

| | |
|---|---|
| **Vector space** | $\mathbb{S}^1 \xmapsto{Z} A,$ |
| **Multiplication** |  $\longmapsto \mu : A \otimes A \to A,$ |
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Likewise, a commutative Frobenius algebra determines a TQFT.

**Theorem (folklore)**

*There is a natural equivalence between* ***oriented 2D TQFTs*** *and* ***commutative Frobenius algebras****, given by the evaluation functor*

$Z \longmapsto (Z(\mathbb{S}^1), Z($  $), Z($  $), Z($  $), Z($  $)).$

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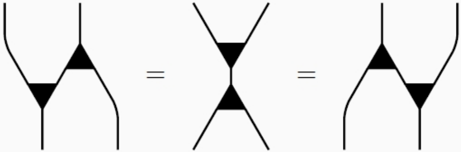
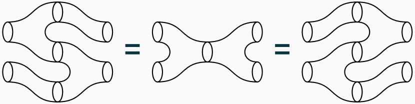
# A note on graphical calculus

We have obtained a system of graphical calculus for commutative Frobenius algebras: any “pants diagram” which holds true topologically will also hold algebraically.

This is usually done with string diagrams (read from top to bottom).

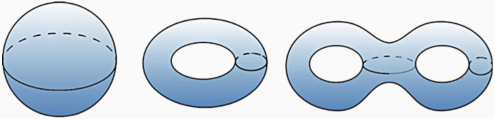


Compare with  .

# Beyond 2D:<br>The Cobordism Hypothesis

# The problem with increasing dimension

We would like to generalize this result for dimensions $n > 2$.
But notice that even the case $n = 3$ is a lot more complex: the category $\mathbf{Cob}_3$ has infinitely many generating objects (the $g$-tori).



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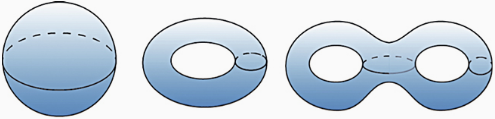

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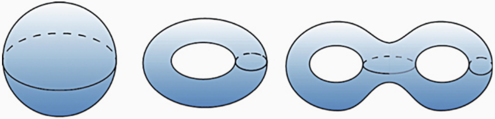

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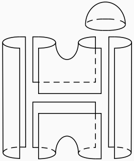
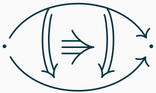


# $n$-categories of cobordisms

An $n$-category has $k$-morfisms between $(k-1)$-morfisms:



We can define an $n$-category of cobordisms, $\mathbf{Cob}_n(n)$:



**Objects** Closed 0-manifolds (finite unions of points).

**1-morphisms** 1-cobordisms between 0-manifolds.



**2-morphisms** 2-cobordisms with corners.

⋮

***n*-morphisms** $n$-cobordisms with corners, up to diffeomorphism.



The compositions are given by gluing, in different directions.

We can cut along submanifold of arbitrary codimension.
Essentially, we can triangulate our cobordisms.

Multiple versions: oriented $\mathbf{Cob}_n^{\mathrm{or}}(n)$, framed $\mathbf{Cob}_n^{\mathrm{fr}}(n)$.

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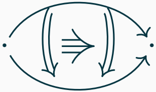

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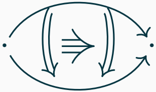

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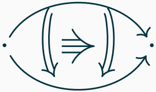

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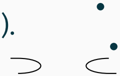

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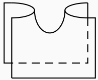

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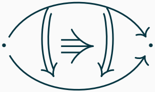

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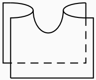

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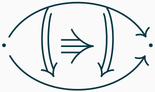

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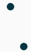

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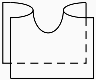

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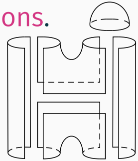

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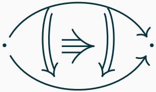

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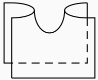

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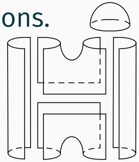

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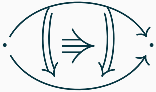

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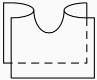

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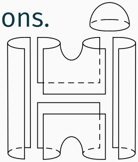

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# Extended TQFTs

An ***extended TQFT*** is a symmetric monoidal functor of $n$-categories

$$\mathbf{Cob}_n(n) \longrightarrow \mathcal{C}.$$

An extended TQFT requires specifying a colossal amount of data. However, this data must be subject to a colossal amount of conditions: no matter how we cut up a given cobordism, after taking images and composing back, the resulting $n$-morphism of $\mathcal{C}$ must be the same.

Framed manifolds are especially rigid. Given that we can decompose as much as we want, it is not unreasonable to think that framed extended TQFTs will be determined by their image at a single point.

Remember that this is what happened in the $n = 1$ case!

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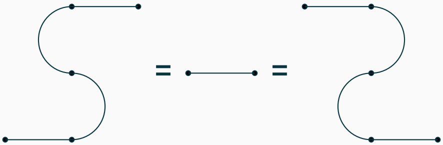

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*A **framed extended TQFT** is determined by its image at a single point.*

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Restricting the image, we should have an equivalence of $n$-categories.

We believe it is true, with some caveats.
We want it to be true, so the definitions are constructed to make it so.
(Similar to the Homotopy Hypothesis in the theory of $\infty$-groupoids).

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# Lurie's approach to the Cobordism Hypothesis

Since the $n$-morphisms of $\mathbf{Cob}^{\mathrm{fr}}_{n+1}(n+1)$ and $\mathbf{Cob}^{\mathrm{fr}}_{n}(n)$ are defined similarly, we hope to proceed by induction on the dimension $n$.

For this to work, we need to be careful with the data we track: taking diffeomorphism classes of $n$-cobordisms will discard important data. Working inside $\mathbf{Cob}^{\mathrm{fr}}_{n}(n)$ will not suffice.

So we work with $(\infty, n)$-***categories***, which have infinite layers of $k$-morphisms — but all $k$-morphisms above layer $n$ are invertible. (For example, $(\infty, 0)$-categories are $\infty$-groupoids).

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# The $(\infty, n)$-category of framed cobordisms

The $(\infty, n)$-category of framed cobordisms, $\mathbf{Bord}_n^{\mathrm{fr}}$, consists of:

| | |
|---|---|
| **Objects** | Closed framed 0-manifolds. |
| **1-morphisms** | Framed 1-cobordisms. |
| $\vdots$ | |
| $n$**-morphisms** | Framed $n$-cobordisms with corners. |
| $(n+1)$**-morphisms** | Diffeomorphisms between $n$-cobordisms. |
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# Fully dualizable objects

The image objects of an extended TQFT must be ***fully dualizable***.

For $n = 1$, these are the dualizable objects:
there exist 1-morphisms ev =  and coev =  such that



For $n = 2$, we require the prior two 1-morphisms to admit adjoints:

there exist 2-morphisms $u$ =  and $v$ =  such that

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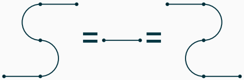

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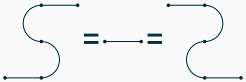

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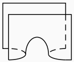
there exist 2-morphisms $u$ =  and $v$ =  such that

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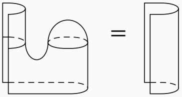

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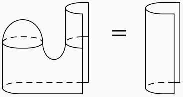

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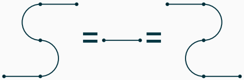

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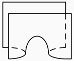
there exist 2-morphisms $u$ =  and $v$ =  such that

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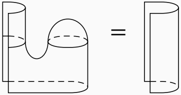

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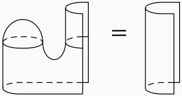

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## More statements of the Cobordism Hypothesis

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Lurie also proves an oriented version, but it is a lot more technical.

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***Oriented extended TQFTs*** *$Z\colon \mathbf{Bord}_n^{\mathrm{or}} \to \mathcal{C}$ correspond to* ***homotopy fixed points*** *of a certain canonical action $\mathrm{SO}(n) \curvearrowright \mathrm{Core}(\mathcal{C}^{\mathrm{fd}})$ on the core ∞-groupoid of the subcategory of fully dualizable objects of $\mathcal{C}$:*

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Oriented TQFTs are still determined by their values on the point...

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# Returning to 2D:
# (0, 1, 2)-TQFTs in the Morita bicategory

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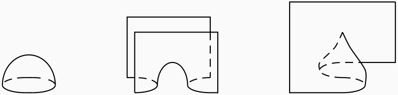
0D:  1D:  2D: 

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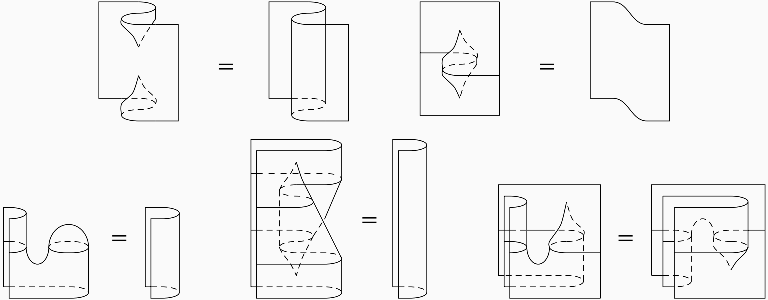

# The Morita bicategory

This is one possible higher-categorical analogue of $\mathbf{Vect}_{\Bbbk}$.

**Definition**

Let $A$ and $B$ be algebras.
A ***B-A-bimodule*** ${}_BM_A$ is a left $B$-module which is also a right $A$-module in a compatible manner: $\quad b \cdot (m \cdot a) = (b \cdot m) \cdot a.$

The ***Morita bicategory*** $\mathbf{Alg}_2$ consists of:

**Objects** $\Bbbk$-algebras $A$.
**1-morphisms** An arrow $A \to B$ is an $B$-$A$-bimodule ${}_BM_A$.
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**Composition** of 1-morphisms: tensor product over the algebra.

$$ {}_CN_B \circ {}_BM_A := {}_C(N \otimes_B M)_A. $$

**Identities** For each algebra $A$, the identity 1-morphism is ${}_AA_A$.

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Let $A$ and $B$ be algebras.
A ***B-A-bimodule*** ${}_BM_A$ is a left $B$-module which is also a right $A$-module in a compatible manner: $b \cdot (m \cdot a) = (b \cdot m) \cdot a$.

The ***Morita bicategory*** $\mathbf{Alg}_2$ consists of:

**Objects** $\Bbbk$-algebras $A$.
**1-morphisms** An arrow $A \to B$ is an $B$-$A$-bimodule ${}_BM_A$.
**2-morphisms** Bimodule homomorphisms.
**Composition** of 1-morphisms: tensor product over the algebra.

$$ {}_CN_B \circ {}_BM_A \coloneqq {}_C(N \otimes_B M)_A. $$

**Identities** For each algebra $A$, the identity 1-morphism is ${}_AA_A$.

It is **symmetric monoidal** with monoid tensor product over $\Bbbk$:

$$ {}_AM_B \otimes {}_CN_D = {}_{A\otimes C}(M \otimes N)_{B\otimes D}. $$

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# The 2D oriented Cobordism Hypothesis

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A $\Bbbk$-algebra is (strongly) ***separable*** if:

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Separable algebras are the fully dualizable objects of $\mathbf{Alg}_2$.

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*There is a natural equivalence between* ***2D oriented extended TQFTs*** $Z\colon \mathbf{Cob}_2^{\mathrm{or}}(2) \to \mathbf{Alg}_2$ *and* ***separable symmetric Frobenius algebras****.*

- The algebra is given by $Z(\mathrm{pt}_+)$.
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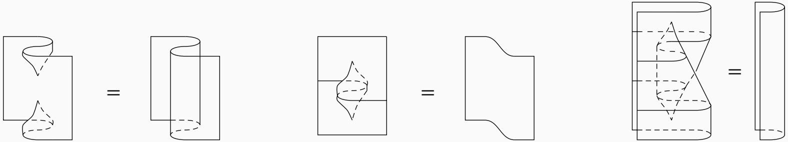
# Identifying the value of the circle

We made multiple identifications which we should detail.

First, assign $Z(\mathrm{pt}_+) =: A$.

The cusp relations allow us to identify $Z(\mathrm{pt}_-) \simeq A^{\mathrm{op}}$:



(This is a ***Morita context***, an adjoint equivalence in $\mathbf{Alg}_2$).

We will abuse notation and treat this as an equality: $Z(\mathrm{pt}_-) =: A^{\mathrm{op}}$.

The two generating 1-morphisms must become bimodules:

$Z($  $) =: {}_{A\otimes A^{\mathrm{op}}}M \qquad Z($  $) =: N_{A\otimes A^{\mathrm{op}}}$

Consider the circle $\mathbb{S}^1 =$  .
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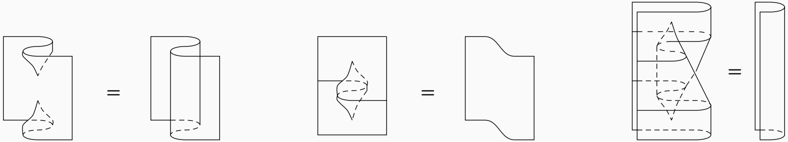

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# The value of the circle is the center

We can identify Center($A$) with the $A$-$A$-bimodule maps $f\colon {}_AA_A \to {}_AA_A$. (These $f$ are of the form $f(a) = c \cdot a = a \cdot c$ for some $c \in$ Center($A$)).

We will prove that $V \cong \mathrm{Hom}_{{}_A\mathbf{Bimod}_A}$. This is done graphically.



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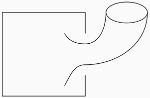

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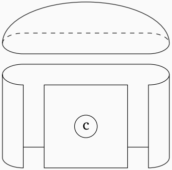


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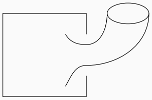

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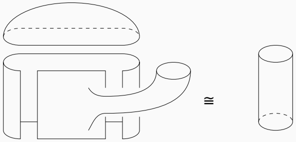


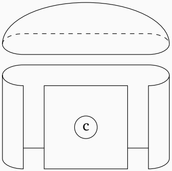


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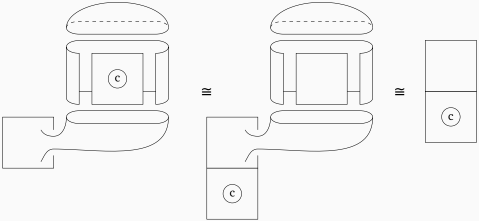

# Relating unextended and extended TQFTs

Given an extended TQFT $Z\colon \mathbf{Cob}_2^{\mathrm{or}}(2) \to \mathbf{Alg}_2$, one can obtain an unextended TQFT $\Omega Z\colon \mathbf{Cob}_2^{\mathrm{or}} \to \mathbf{Vect}_{\Bbbk}$ by taking loops:

$$\Omega\,\mathbf{Cob}_2^{\mathrm{or}}(2) = \mathrm{Map}_{\mathbf{Cob}_2^{\mathrm{or}}(2)}(\varnothing, \varnothing) \simeq \mathbf{Cob}_2, \quad \Omega\,\mathbf{Alg}_2 = \mathrm{Map}_{\mathbf{Alg}_2}(\Bbbk, \Bbbk) \simeq \mathbf{Vect}_{\Bbbk}.$$

This has the effect of taking centers in the Frobenius algebra analogy:

$$(\Omega Z)(\mathbb{S}^1) = \mathrm{Center}\, Z(\mathrm{pt}_Z) = Z(\mathbb{S}^1).$$

An unextended TQFT $W\colon \mathbf{Cob}_2^{\mathrm{or}} \to \mathbf{Vect}_{\Bbbk}$ extends down to points if and only if the Frobenius algebra $W(\mathbb{S}^1)$ is separable.

This extension is in general not unique.

# Final recap

We compare our two kinds of 2D oriented TQFTs.

| **Unextended** | **Extended** |
|---|---|
| 1-categorical | 2-categorical |
| Morse Theory | Cerf Theory |
| $\mathbf{Vect}_{\mathbb{k}}$ | $\mathbf{Alg}_2$ |
| $\mathbb{S}^1$,  and  | $\mathrm{pt}_+$,  ,  ,  and  |
| Commutative Frobenius algebras | Separable symmetric Frobenius algebras |

Thank you!

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