Master's Thesis

2D Topological Quantum Field Theories, Frobenius Structures, and Higher Algebra

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24th June 2024

Abstract

An oriented ***TQFT*** is a symmetric monoidal functor $Z\colon \mathsf{Cob}_n \to \mathsf{Vect}_{\Bbbk}$. Equivalently, this is a rule which assigns to each oriented closed $(n-1)$-manifold M a vector space $Z(M)$ and to each n-cobordism $B\colon M \to N$ a linear map $Z(M) \to Z(N)$, satisfying certain conditions. The classification for 2D oriented TQFTs is a classical "folklore" result; they are equivalent to commutative ***Frobenius algebras*** — $\Bbbk$-algebras A equipped with a linear functional $\varepsilon\colon A \to \Bbbk$ whose kernel contains no non-trivial ideals. This extends to a categorical equivalence $\mathrm{Fun}^{\otimes}(\mathsf{Cob}_2, \mathsf{Vect}_{\Bbbk}) \simeq \mathsf{cFrob}_{\Bbbk}$.

A framed ***extended TQFT*** is a symmetric monoidal functor $Z\colon \mathsf{Bord}_n^{\mathrm{fr}} \to \mathcal{C}$, where here $\mathsf{Bord}_n^{\mathrm{fr}}$ and $\mathcal{C}$ are (∞,n)-categories. The ***Cobordism Hypothesis*** conjectures that framed extended TQFTs are determined by their image at a single point, which must be a ***fully dualizable*** object in the target category $\mathcal{C}$. More precisely, the $(\infty,0)$-category of framed extended TQFTs is equivalent to the core ∞-groupoid of the subcategory of $\mathcal{C}$ spanned by its fully dualizable objects: $\mathrm{Fun}^{\otimes}(\mathsf{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \simeq \mathrm{Core}(\mathcal{C}^{\mathrm{fd}})$.

The (∞,n)-category of framed cobordisms $\mathsf{Bord}_n^{\mathrm{fr}}$ carries an $\mathrm{O}(n)$-action, which by the previous statement determines a canonical $\mathrm{O}(n)$-action on $\mathrm{Core}(\mathcal{C}^{\mathrm{fd}})$ for each (∞,n)-category $\mathcal{C}$. This allows stating a Cobordism Hypothesis for G-structured manifolds: G-structured extended TQFTs are equivalent to the ***homotopy fixed points*** of a certain G-action on $\mathrm{Core}(\mathcal{C}^{\mathrm{fd}})$. Notably, and up to homotopy, an orientation is the same as an $\mathrm{SO}(n)$-structure.

When specializing to 2D oriented TQFTs and choosing as target the bicategory Alg_2 of algebras, bimodules and intertwiners, these $\mathrm{SO}(2)$-homotopy fixed points correspond to ***separable symmetric Frobenius algebras***. Hence, by taking loops, we recover a particular case of the classical correspondence between unextended 2D oriented TQFTs and commutative Frobenius algebras.

Una ***TQFT*** orientada es un funtor monoidal simétrico $Z\colon \mathsf{Cob}_n \to \mathsf{Vect}_{\Bbbk}$. Equivalentemente, es una regla que asigna a cada $(n-1)$-variedad cerrada orientada M un espacio vectorial $Z(M)$, y a cada n-cobordismo $B\colon M \to N$ una aplicación lineal $Z(M) \to Z(N)$, y que satisface ciertas condiciones. La clasificación de TQFTs orientadas 2D es un resultado clásico del «folklore»; son equivalentes a ***álgebras de Frobenius*** conmutativas — $\Bbbk$-álgebras A equipadas con un funcional lineal $\varepsilon\colon A \to \Bbbk$ cuyo núcleo no contiene ideales no triviales. Podemos extender esta clasificación a una equivalencia $\mathrm{Fun}^{\otimes}(\mathsf{Cob}_2, \mathsf{Vect}_{\Bbbk}) \simeq \mathsf{cFrob}_{\Bbbk}$.

Una ***TQFT extendida*** enmarcada es un funtor monoidal simétrico $Z\colon \mathsf{Bord}_n^{\mathrm{fr}} \to \mathcal{C}$, donde ahora $\mathsf{Bord}_n^{\mathrm{fr}}$ y $\mathcal{C}$ son (∞,n)-categorías. La ***Hipótesis del Cobordismo*** conjetura que las TQFTs extendidas enmarcadas están determinadas por su imagen en un único punto, y dicha imagen debe ser un objeto ***completamente dualizable*** de la categoría $\mathcal{C}$. Es más, la $(\infty,0)$-categoría de TQFTs extendidas enmarcadas es equivalente al ∞-grupoide subyacente de la subcategoría de $\mathcal{C}$ generada por sus objetos completamente dualizables: $\mathrm{Fun}^{\otimes}(\mathsf{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \simeq \mathrm{Core}(\mathcal{C}^{\mathrm{fd}})$.

La (∞,n)-categoría de cobordismos enmarcados $\mathsf{Bord}_n^{\mathrm{fr}}$ tiene asignada una acción distinguida por $\mathrm{O}(n)$, y por lo anterior esto determina una $\mathrm{O}(n)$-acción canónica en $\mathrm{Core}(\mathcal{C}^{\mathrm{fd}})$ para cada (∞,n)-categoría $\mathcal{C}$. Esto nos permite obtener una Hipótesis del Cobordismo para variedades equipadas con una G-estructura: las TQFTs extendidas G-estructuradas son equivalentes a los ***puntos fijos homotópicos*** de cierta G-acción en $\mathrm{Core}(\mathcal{C}^{\mathrm{fd}})$. Notar que, salvo homotopía, una orientación es lo mismo que una $\mathrm{SO}(n)$-estructura.

Especializando a TQFTs orientadas 2D y eligiendo como codominio la bicategoría Alg_2 de álgebras, bimódulos y morfismos de bimódulos, los puntos fijos homotópicos de la acción por $\mathrm{SO}(2)$ corresponden a ***álgebras de Frobenius simétricas separables***. Por lo tanto, tomando bucles, recuperamos un caso particular de la correspondencia clásica entre TQFTs orientadas 2D sin extender y álgebras de Frobenius conmutativas.

Acknowledgements

Escribo aquí en español, mi lengua nativa.

Quería agradecer a los participantes del seminario de ∞-categorías UCM–ICMAT que tuvo lugar del 11 de enero al 13 de junio de 2024. Ellos saben quienes son; prefiero evitar una ristra de nombres.

De no haber sido por este seminario, la parte II de este trabajo sería muchísimo más corta — y más importante que eso, estaría muchísimo menos refinada. Principalmente seguimos «el paper de Lurie», [Lur09], pero una clase tomó aspectos de [HTT] y las cuatro últimas clases discutieron resultados de la tesis de Schommer-Pries, [Sch14]. En esto último hubo una influencia mutua; yo había empezado a leer partes de dicha tesis para escribir lo que ahora forma la sección 4.2, y fue mi tutor Ángel quien propuso discutirlo en el seminario — después de varias tutorías aguantando mis insistentes preguntas al respecto.

Disfruté y aprendí bastante del seminario; los jueves eran un día destacado de la semana. Además, pude probar a dar una de las primeras clases — y me habría preparado otras, de haber vivido más meses en Madrid. Espero que al resto de asistentes les guste este trabajo, si encuentran el tiempo para leerlo.

Más allá de eso, me abstengo de listar aquí familiares, amigos y compañeros; no porque no tenga cosas que agradecer, que son muchas, sino porque los involucrados ya lo saben de sobra. Muchas gracias a todos los que hacéis que merezca la pena continuar adelante.

Contents

Summary

English

Part I, chapter 1

Given two closed oriented $(n-1)$-manifolds M and N, we define a ***cobordism*** $B\colon M \to N$ to be a compact n-manifold whose boundary we identify with the two closed manifolds: we equip B with the data of a decomposition $\partial B \cong \overline{M} \sqcup N$. An oriented ***topological quantum field theory***, or TQFT for short, is a rule which assigns to each closed oriented $(n-1)$-manifold a vector space $Z(M)$ and to each n-cobordism a linear map $Z(B)\colon Z(M) \to Z(N)$ between the vector spaces corresponding to its two boundaries. This rule must satisfy some algebraic conditions reminiscent of the theoretical physics origins of the concept — a succinct way to express these is that a TQFT Z is a ***symmetric monoidal functor*** from the category of cobordisms $(\mathtt{Cob}_n, \sqcup, \varnothing, \sigma)$ equipped with disjoint union to the category of vector spaces $(\mathtt{Vect}_{\Bbbk}, \otimes, \Bbbk, \tau)$ equipped with tensor product.

TQFTs have gained notoriety in pure mathematics in the last decades — after Witten's 1989 discovery of a TQFT which recovered the Jones polynomial of a knot [Wit89]. Since then, they have been used to discover new knot and 3-manifold invariants, and to study moduli spaces in algebraic geometry — among other applications. Here we consider TQFTs as objects of study of its own, and concern ourselves with their classification.

Part I, chapter 2

The classification of 2D oriented TQFTs is what is known as a "folklore theorem", a result which was known to be true but of which no complete proof was available in the literature. These correspond to commutative ***Frobenius algebras***, which are $\Bbbk$-algebras equipped with a ***Frobenius form*** $\varepsilon\colon A \to \Bbbk$; a linear functional whose kernel contains no non-trivial ideals. The proof is very intuitive and evocative, ultimately reinterpreting the structure of the algebra A as cobordisms — the multiplication $A \otimes A \to A$ is a pair of pants , the unit $\Bbbk \to A$ is a cap , the aforementioned Frobenius form $A \to \Bbbk$ is another cap , and it turns out that one can then define a comultiplication $A \to A \otimes A$ pictured as . Then one proves that all statements about such cobordisms which are true topologically must also be true algebraically; all of this can be made rigorous through the use of graphical calculus. But properly establishing the structures needed to justify some of these steps takes time, so it took a number of years from its discovery for a detailed account of this 2D classification to appear. We follow the approach of [Koc03], an excellent source providing numerous details. That, along with some generalities on symmetric monoidal categories, makes up the bulk of part I.

Part II, chapter 3

But the more interesting question is what comes after 2: how does one classify higher-dimensional TQFTs? Here it quickly becomes apparent that our methods are insufficient; being able to decompose cobordisms by cutting along one-codimensional submanifolds is enough for dimensions 1 or 2, or even 3 with enough persistence, but trying to obtain a result for general n is hopeless. So we classify ***extended TQFTs***, instead. These are the higher-categorical analogue of TQFTs — where

as before we had closed $(n-1)$-manifolds as objects and n-cobordisms as morphisms, we now consider 0-dimensional closed manifolds (i.e., finite collections of points) as objects, 1-cobordisms between those as 1-morphisms, 2-cobordisms *with corners* between those as 2-morphisms, and so on all the way up to n. Even more, it can be helpful to keep going — $(n+1)$-morphisms are *diffeomorphisms* between n-cobordisms, $(n+2)$-morphisms are *isotopies* between diffeomorphisms, and then we get isotopies between isotopies and so on. There is no natural place to stop our ascent, and we may rightfully not want to stop — thus obtaining what is known as an ***(∞, n)-category***, an ∞-category where each k-morphism is invertible for $k > n$. Any n-category can be seen as an (∞, n)-category whose only k-morphisms for $k > n$ are the required identities, so this definition does not lose generality. The extended TQFTs — now symmetric monoidal functors between the (∞, n)-category of *framed* cobordisms $\mathsf{Bord}_n^{\mathrm{fr}}$ and an arbitrary target (∞, n)-category $\mathcal{C}$ — must preserve all of this data. Effectively, this lets us decompose our cobordisms as much as we want — down to points and lines, as if defining a triangulation or CW decomposition.

But being able to cut cobordisms with such flexibility imposes a great amount of restrictions on the data of extended TQFTs. If we have so much freedom to decompose, essentially being able to triangulate our cobordisms, and our extended TQFTs must return the same value no matter if we evaluate them before or after this process — aren't they actually determined by an extremely small amount of information? Aren't they, say, determined by the value at a single point? That is what Baez and Dolan thought, in 1995 [BD95], when stating their ***Cobordism Hypothesis***. This is the main conjecture around which part II is structured. As many of the long-standing conundrums in higher category theory, this is as much of a guiding principle for *developing* higher category theory as it is a theorem to prove — in a certain sense, we are sure that the Cobordism Hypothesis must be true, and any reasonable theory of higher categories should make it a true theorem. That is to say, a disproval of this hypothesis would reveal a lack of the proper *definitions* — and not necessarily the falsehood of the hypothesis itself. A detailed proof sketch of the Cobordism Hypothesis appeared in [Lur09], which is our other main source. This uses ***n-fold complete Segal spaces*** as a model for (∞, n)-categories.

Part II, chapter 4

After some philosophical discussions of what an (∞, n)-category *should* be (which form their own Interlude), we spend chapter 4 comparing the classical correspondence between 2D oriented TQFTs and commutative Frobenius algebras (the main result of part I) with the Cobordism Hypothesis (the main conundrum of part II). First, we need a version of the Cobordism Hypothesis which takes oriented cobordisms as input; for this, it is convenient to work in the more general framework of ***G-structures*** on manifolds. Roughly speaking, and up to homotopy, we identify $\mathrm{O}(n)$-structures as determining no structure at all, $\mathrm{SO}(n)$-structures with giving an orientation on our manifold and $\{1\}$-structures with a choice of framing.

Given a framed n-manifold M, there is a canonical action by $\mathrm{GL}_{\mathbb{R}}(n)$ on each fiber $\mathrm{Fr}(\mathrm{T}_x M)$ of its frame bundle; this is given by matrix multiplication of bases. Thus, this induces a $\mathrm{GL}(<)\mathbb{R} > n$-action on the space of all possible framings on M, which by homotopy yields an $\mathrm{O}(n)$-action. Since this is possible for each framed n-manifold, as well as for each n-framed m-manifold $(m \leq n)$, we have determined an $\mathrm{O}(n)$-action on $\mathsf{Bord}_n^{\mathrm{fr}}$. But by the Cobordism Hypothesis, given any (∞, n)-category $\mathcal{C}$ we can identify

$$\mathsf{Fun}^{\otimes}(\mathsf{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \simeq \mathsf{Core}(\mathcal{C}^{\mathrm{fd}}),$$

so this in fact determines a canonical $\mathrm{O}(n)$-action on $\mathsf{Core}(\mathcal{C}^{\mathrm{fd}})$! Following similar lines of reasoning, one eventually reaches a version of the Cobordism Hypothesis for G-manifolds. This version claims that G-structured extended TQFTs are still determined by their value at the point — but that value is now considered as living inside the ***homotopy fixed space*** of the action by G, and thus carries extra structure. That is to say, now the correspondence goes

$$\mathrm{Fun}^{\otimes}(\mathsf{Bord}_n^G, \mathcal{C}) \simeq (\mathsf{Core}(\mathcal{C}^{\mathrm{fd}}))^{\mathrm{h}G}.$$

The second half of chapter 4 specializes this result to $n = 2$ and to the target bicategory Alg_2 of algebras, bimodules and intertwiners, following [Sch14]. Here we sketch a proof for the classification result of extended 2D oriented TQFTs $\mathsf{Cob}_2^{\mathrm{or}}(2) \to \mathsf{Alg}_2$, modulo some implicit identifications. The backbone of this proof is an explicit presentation for $\mathsf{Cob}_2^{\mathrm{or}}(2)$ in terms of generators and relations — the main result of [Sch14]. These extended TQFTs correspond to ***separable symmetric Frobenius algebras***. In particular, via the identifications

$$\mathsf{Cob}_2^{\mathrm{or}} \simeq \mathrm{Map}_{\mathsf{Cob}_2^{\mathrm{or}}(2)}(\varnothing, \varnothing), \qquad \mathsf{Vect}_{\Bbbk} \simeq \mathrm{Map}_{\mathsf{Alg}_2, \mathsf{Alg}_2},$$

we recover some of the classical correspondence: non-extended 2D oriented TQFTs which can be extended down to points are those corresponding to *separable* commutative Frobenius algebras.

This ends the main portion of the text, but we also include an appendix in the form of a Coda.

Part II, Coda

In this separate Coda, we explore a particular model for (∞, n)-categories: ***n-fold complete Segal spaces***. In the case $n = 1$, these arise as a homotopy theoretical version of the identification between a 1-category and its ***nerve***, a simplicial set. Roughly speaking, for a simplicial set to be isomorphic to the nerve of a category, a number of commutative diagrams defined from its structure must be pullback diagrams. One obtains Segal spaces by changing simplicial sets to ***simplicial spaces*** (simplicial objects in Top) and the pullback condition to the ***Segal condition***: the same diagrams must now be *homotopy fiber products*. Then one requires completeness, which is an additional technical condition. The construction of n-fold Segal spaces is similar, but here one uses n-fold simplicial spaces — where as a simplicial space is a functor $\Delta^{\mathrm{op}} \to \mathsf{Top}$, an n-fold simplicial space is a functor $\Delta^{\mathrm{op}} \times \cdots \times \Delta^{\mathrm{op}} \to \mathsf{Top}$.

We end the Coda by sketching the construction of the (∞, n)-categories of cobordisms as n-fold complete Segal spaces. The core philosophy is actually very simple — instead of trying to define composition by gluing n-cobordisms with corners, it is easier to start with arbitrary n-manifolds with corners and cut them up in all possible ways. Then, one defines one possible composition for the cobordisms obtained to be the original n-manifold. Everything else consists of technical steps implementing this idea, such as embedding our manifolds into Euclidean space (to facilitate the cutting) and then taking a direct limit so as to undo the effect of this embedding.

Spanish

Parte I, capítulo 1

Dadas dos $(n-1)$-variedades cerradas y orientadas M y N, definimos un ***cobordismo*** $B\colon M \to N$ como una n-variedad compacta cuyo borde está identificado con las dos variedades cerradas: equipamos a B con una descomposición $\partial B \cong \overline{M} \sqcup N$. Una ***teoría topológica de campos cuánticos*** orientada, o TQFT por sus siglas en inglés, es una regla que asigna a cada $(n-1)$-variedad cerrada orientada un espacio vectorial $Z(M)$ y a cada n-cobordismo una aplicación lineal $Z(B)\colon Z(M) \to Z(N)$ entre los espacios vectoriales correspondientes a sus dos fronteras. Esta regla debe satisfacer ciertas condiciones algebraicas, que recuerdan a los orígenes del concepto en la física teórica — un modo rápido de expresar dichas condiciones es que una TQFT Z es un ***funtor monoidal simétrico*** de la categoría de cobordismos $(\mathsf{Cob}_n, \sqcup, \varnothing, \sigma)$ equipada con la unión disjunta a la categoría de espacios vectoriales $(\mathsf{Vect}_{\Bbbk}, \otimes, \Bbbk, \tau)$ equipada con el producto tensorial.

En las últimas décadas, las TQFTs han ganado gran notoriedad en el ámbito de la matemática pura — tras el descubrimiento por Witten, en 1989, de una TQFT que recupera el polinomio de Jones de un nudo [Wit89]. Desde entonces, se han usado para descubrir nuevos invariantes de nudos y 3-variedades y para estudiar espacios de móduli en geometría algebraica — por decir unas pocas aplicaciones. Nosotros consideraremos TQFTs como objetos de estudio en sí mismos, y nos preocupamos especialmente por su clasificación.

Parte I, capítulo 2

La clasificación de TQFTs orientadas 2D es lo que se conoce como un teorema «del folklore», un resultado que se sabía que era cierto pero para el cual no había demostraciones completas en la literatura. Estas TQFTs corresponden a ***álgebras de Frobenius*** conmutativas, que son $\Bbbk$-álgebras equipadas con una ***forma de Frobenius*** $\varepsilon\colon A \to \Bbbk$; un funcional lineal cuyo núcleo no contiene ideales no triviales. La demostración es bastante intuitiva y evocadora, y en el fondo se trata de una reinterpretación de la estructura del álgebra A como cobordismos — la multiplicación $A \otimes A \to A$ es el par de pantalones, la unidad $\Bbbk \to A$ es un capuchón, la antedicha forma de Frobenius $A \to \Bbbk$ es otro capuchón y resulta que uno puede definir una comultiplicación $A \to A \otimes A$, que dibujamos como otro pantalón. Tras esto, se demuestra que todos los enunciados sobre estos cobordismos que son ciertos topológicamente también deben ser ciertos algebraicamente; esto se puede volver riguroso, mediante el uso del cálculo gráfico. Pero desarrollar las estructuras necesarias para justificar algunos de estos pasos lleva tiempo, así que pasaron ciertos años desde el descubrimiento de esta clasificación hasta que apareció un reporte detallado. Nosotros seguimos el enfoque de [Koc03], una fuente excelente y con mucho detalle. Esto, junto a algunas generalidades sobre categorías monoidales simétricas, forma la mayoría de la parte I.

Parte II, capítulo 3

Pero una pregunta más interesante es qué pasa despues del 2: ¿cómo clasificamos TQFTs de dimensión superior? Y es aquí donde se vuelve rápidamente aparente que nuestros métodos son insuficientes; poder descomponer cobordismos cortando a través de subvariedades de codimensión 1 es suficiente para clasificar TQFTs en dimensión 1 o 2, o incluso en dimensión 3 con suficiente persistencia, pero parece imposible obtener un resultado general para todo n. Así que en vez de eso clasificamos ***TQFTs extendidas***; el análogo de las TQFTs para categorías superiores. Mientras que antes teníamos $(n-1)$-variedades cerradas como objetos y n-cobordismos entre ellas como morfismos, ahora consideramos 0-variedades cerradas (es decir, colecciones finitas de puntos) como objetos, 1-cobordismos entre estas como 1-morfismos, 2-cobordismos *con esquinas* entre esto como 2-morfismos, y así hasta llegar a n. Pero no tenemos que parar ahí — los $(n+1)$-morfismos son *difeomorfismos* entre n-cobordismos, los $(n+2)$-morfismos son *isotopías* entre difeomorfismos, y luego pasamos a isotopías entre isotopías. No hay lugar donde resulte natural parar nuestro ascenso, y podemos decidir no querer parar — así obtenemos lo que se conoce como una ***(∞, n)-categoría***, una ∞-categoría donde todos los k-morfismos son invertibles para $k > n$. Cualquier n-categoría se puede ver como una (∞, n)-categoría cuyos únicos k-morfismos para $k > n$ son las identidades, así que esta definición no pierde generalidad. Las TQFTs extendidas — que ahora son funtores monoidales simétricos entre la (∞, n)-categoría de cobordismos *enmarcados* $\mathsf{Bord}_n^{\mathrm{fr}}$ y una (∞, n)-categoría codominio arbitraria $\mathcal{C}$ — deben preservar todos estos datos. A nivel efectivo, esto nos permite descomponer nuestros cobordismos tanto como queramos — bajando hasta el nivel de puntos y líneas, como si estuviéramos definiendo una triangulación o una descomposición como CW-complejo.

Pero ser capaces de cortar nuestros cobordismos con tanta flexibilidad impone una gran cantidad de restricciones en los datos definiendo una TQFT extendida. Si tenemos tanta libertad para descomponer, pudiendo prácticamente triangular nuestros cobordismos, y nuestras TQFTs extendidas deben devolver el mismo valor sin importar que las evaluemos antes o después del proceso — ¿acaso no estarán determinadas por una cantidad extremadamente pequeña de información? ¿Acaso no estarán determinadas, digamos, por su valor en un único punto? Eso es lo que Baez y Dolan pensaron, en 1995 [BD95], cuando enunciaron su ***Hipótesis del Cobordismo***. Esta es la principal conjetura alrededor de la cual se estructura la parte II. Como muchos de los dilemas primordiales de la teoría de categorías superior, esto es tanto un teorema a demostrar como un principio a seguir para *desarrollar* una teoría de categorías superiores — en cierto modo, estamos seguros de que la Hipótesis del Cobordismo debe ser cierta, y cualquier teoría razonable de categorías superiores debería hacer que fuera un teorema cierto. En otras palabras, una demostración de que la hipótesis es falsa revelaría la falta de *definiciones* adecuadas — y no necesariamente la falsedad de la hipótesis. Un esbozo detallado de la demostración de la Hipótesis del Cobordismo se encuentra en [Lur09], que es nuestra otra fuente principal. Como modelo para (∞, n)-categorías usa los llamados ***espacios completos de Segal n-tuples***.

Parte II, capítulo 4

Después de algunas discusiones filosóficas sobre qué *debería ser* una (∞, n)-categoría (que forman su propio Interludio), nos ocupamos en el capítulo 4 de comparar la correspondencia clásica entre TQFTs orientadas 2D y álgebras de Frobenius conmutativas (el resultado principal de la parte I) con la Hipótesis del Cobordismo (el dilema principal de la parte II). Primero necesitamos una versión para la Hipótesis del Cobordismo que trate con variedades orientadas; para esto, es conveniente trabajar en el ámbito más general de ***G-estructuras*** en variedades. A grandes rasgos, y salvo homotopía, identificamos una $\mathrm{O}(n)$-estructura como no dar ninguna estrucutura especial, una $\mathrm{SO}(n)$-estructura como dar una orientación a nuestra variedad y una $\{1\}$-estructura como una paralelización (una elección de marco móvil global).

Dada una n-variedad enmarcada M, existe una acción canónica por $\mathrm{GL}(<)\mathbb{R} > n$ en cada fibra $\mathrm{Fr}(\mathrm{T}_x M)$ de su fibrado de referencias; esta acción viene dada por la multiplicación de las matrices de las bases. Esto induce una $\mathrm{GL}_{\mathbb{R}}(n)$-acción en el espacio de todos los posibles marcos móviles globales en M, que por homotopía se puede ver como una $\mathrm{O}(n)$-acción. Como esto es posible para cada n-variedad enmarcada, y también para cada m-variedad n-enmarcada ($m \leq n$), esto determina una $\mathrm{O}(n)$-acción en $\mathsf{Bord}_n^{\mathrm{fr}}$. Pero por la Hipótesis del Cobordismo, dada cualquier (∞, n)-categoría $\mathcal{C}$ podemos identificar

$$\mathsf{Fun}^{\otimes}(\mathsf{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \simeq \mathsf{Core}(\mathcal{C}^{\mathrm{fd}}),$$

luego esto determina una $\mathrm{O}(n)$-acción canónica en $\mathrm{Core}(\mathcal{C}^{\mathrm{fd}})$. Mediante líneas de razonamiento similares, se llega a una versión de la Hipótesis del Cobordismo para G-variedades. Esta versión aserta que las TQFTs extendidas G-estructuradas siguen estando determinadas por su valor en el punto — pero ahora dicho valor se considera dentro del ***espacio de puntos fijos homotópicos*** de la acción por G, y lleva consigo estructura adicional. Es decir, ahora la correspondencia es

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^G, \mathcal{C}) \simeq (\mathrm{Core}(\mathcal{C}^{\mathrm{fd}}))^{\mathrm{h}G}.$$

La segunda mitad del capítulo 4 especializa este resultado a $n = 2$ y al codominio de la bicategoría Alg_2 de álgebras, bimódulos y morfismos de bimódulos, siguiendo [Sch14]. Aquí esbozamos una demostración de la clasificación de TQFTs orientadas 2D extendidas $\mathsf{Cob}_2^{\mathrm{or}}(2) \to \mathsf{Alg}_2$, módulo algunas identificaciones implícitas. La base de esta demostración es una presentación explícita de $\mathsf{Cob}_2^{\mathrm{or}}(2)$ en términos de generadores y relaciones — es decir, el resultado principal de [Sch14]. Estas TQFTs extendidas se corresponden con ***álgebras de Frobenius simétricas separables***. En particular, mediante la identificación

$$\mathsf{Cob}_2^{\mathrm{or}} \simeq \mathrm{Map}_{\mathsf{Cob}_2^{\mathrm{or}}(2)}(\varnothing, \varnothing), \qquad \mathsf{Vect}_{\Bbbk} \simeq \mathrm{Map}_{\mathsf{Alg}_2, \mathsf{Alg}_2},$$

podemos recuperar ciertos aspectos de la correspondencia clásica: las TQFTs orientadas 2D no extendidas, pero que pueden extenderse hacia abajo hasta puntos, son las que corresponden a álgebras de Frobenius conmutativas *separables*.

Así finaliza la parte principal del texto, pero también incluimos un apéndice en forma de Coda.

Parte II, Coda

En la Coda, exploramos un modelo particular para (∞, n)-categorías: los ***espacios de Segal completos n-tuples***. En el caso $n = 2$, surgen como versiones homotópicas de la identificación entre una 1-categoría y su ***nervio***, un conjunto simplicial. A grandes rasgos, para que un conjunto simplicial sea isomorfo al nervio de una categoría debe cumplir una condición: ciertos diagramas conmutativos definidos a partir de su estructura deben ser diagramas expresando un producto fibrado. Los espacios de Segal se obtienen cuando cambiamos conjuntos simpliciales por ***espacios simpliciales*** (objetos simpliciales en Top) y la condición anterior por la ***condición de Segal***: esos mismos diagramas deben ser ahora *productos fibrados homotópicos*. Luego se les pide completitud, que es una condición técnica adicional. La construcción de espacios de Segal n-tuples es similar, pero aquí empezamos con espacios simpliciales n-tuples — mientras que un espacio simplicial es un funtor $\Delta^{\mathrm{op}} \to \mathsf{Top}$, un espacio simplicial n-tuple es un functor $\Delta^{\mathrm{op}} \times \cdots \times \Delta^{\mathrm{op}} \to \mathsf{Top}$.

Acabamos la Coda esbozando la construcción de las (∞, n)-categorías de cobordismos como espacios de Segal completos n-tuples. La filosofía es, de hecho, bastante simple — en vez de definir la composición intentando pegar n-cobordismos con esquinas, es más sencillo empezar con n-variedades arbitrarias con esquinas, que cortamos de todos los modos posibles. Después, definimos que una posible composición para los cobordismos obtenidos mediante esto cortes sea la n-variedad original. Todo lo demás son pasos técnicos implementando esta idea; por ejemplo, para facilitar el pegado, encajamos nuestras variedades en un espacio Euclídeo. Y después, para eliminar los efectos de este encaje, tomamos un límite directo.

Preface

The Unreasonable Effectiveness of Physics in the Mathematical Sciences

Topological quantum field theories have a curious origin: they are a modern example of physics fundamentally inspiring deep pure mathematical constructs. That was the norm some centuries ago, with the prime example being the calculus of Newton and Leibniz, but the status quo after the 19th century seemed to be the complete opposite: for most of the past century, developments in physics were facilitated by the formal background of mathematics, and not the other way around. Physics required the invention of intricate mathematical tooling, but most areas of mathematics did not benefit, at least directly, from a physicist's tools. Even more, "pure" formal mathematics, in more abstract fields such as topology or algebraic geometry, seemed too hopelessly detached from reality for the tools of physicists to matter.

But here we are, nonetheless: we are forced to acknowledge the fact that one of the most useful tools in modern low-dimensional topology comes from theoretical physics and the study of quantum field theories. As Atiyah, Dijkgraaf, and Hitchin put it in [ADH10], emphasis added:

> What has been described so far is the familiar story of the advance of physics necessitating the use and development of increasingly sophisticated mathematics, to the mutual benefit of both fields. Mathematicians have been driven to investigate new areas; and physicists, to quote Eugene Wigner (1960), have been impressed by "the unreasonable effectiveness of mathematics in physics" — the remarkable universal properties of mathematical structures.
>
> But over the past 30 years a new type of interaction has taken place, probably unique, in which physicists, exploring their new and still speculative theories, have stumbled across a whole range of mathematical "discoveries". These are derived by physical intuition and heuristic arguments, which are beyond the reach, as yet, of mathematical rigour, but which have withstood the tests of time and alternative methods. There is great intellectual excitement in these mutual exchanges.
>
> The impact of these discoveries on mathematics has been profound and widespread. Areas of mathematics such as topology and algebraic geometry, which lie at the heart of pure mathematics and appear very distant from the physics frontier, have been dramatically affected.
>
> This development has led to many hybrid subjects, such as topological quantum field theory, quantum cohomology or quantum groups, which are now central to current research in both mathematics and physics. The meaning of all this is unclear and one may be tempted to invert Wigner's comment and marvel at "**the unreasonable effectiveness of physics in mathematics**".

Or, like a fellow student told me: *one must reason like a physicist, but prove like a mathematician*.

Even though this theory does have its roots in physics, we will approach it from a pure mathematical perspective; I, the author, am no physicist, and pretending otherwise would be unwise. As such, this thesis is purposely light on physical interpretations: there is not much more I can say without treading dangerous territory. The interested reader can check [Ati88; Wit88], among others. A nice and modern introduction to the physics that emphasizes the categorical viewpoint can be found in [Bae06].

One could argue that deep understanding of the physics leading to the invention of TQFTs is necessary to comprehend the whole picture. While we won't refute those claims directly, it is the author's hope that the reader will be able to appreciate the beautiful and elegant picture painted by this mathematical formalism, perhaps devoid of aspirations to physical interpretations, which nicely ties together areas such as abstract algebra, higher category theory, and knot theory.

Concerning the text itself

This text assumes some familiarity with the basic concepts of category theory. To follow the text, it should be enough to know the very basics about categories, functors, and natural transformations, as well as some of the more common universal properties (products and pullbacks, and also their duals, coproducts and pushouts). The author recommends the first chapters of either of [Rie16; Lei16], but there are many phenomenal resources available online.

The main references used for most of the thesis were [Koc03] and [Lur09]. Both of these sources are very well-written and provide plenty of detail, and we encourage the interested reader to consult them — whether shortly after this thesis or in some distant future. They discuss very good and very deep mathematics without compromising style or readability; a remarkable feat. Our goal here is not to supersede these, but to condense at least a subset of the incredible amount of information they contain into a single coherent package, while also maintaining a narrative flow to tie the loose ends together.

Part I is more or less a retelling of Kock's marvelous book [Koc03], with some reordering of contents and some adaptations of style to better connect with Part II. Chapter 1 introduces the basic concepts of cobordisms and TQFTs, as well as the symmetric monoidal category of cobordisms. Chapter 2 is a proof of the famous "folklore theorem" classifying 2D oriented TQFTs: these are in bijection with commutative Frobenius algebras.

Part II, on the other hand, mainly summarizes aspects from the first two sections of Lurie's foundational paper [Lur09]. Here we are a lot less faithful to the source material than we were in part I, but readers familiar with Lurie's paper will surely notice the similarity in structure and content. We spend more time introducing concepts which [Lur09] takes for granted, and less dealing with technicalities which are mainly of interest for Lurie's proof sketch of the Cobordism Hypothesis. Hence we have moved some deeply interesting but — for our purposes — very technical constructions to a separate Coda, and introduced an Interlude treating some of the more philosophical aspects of higher category theory. Chapter 3 explores the need for higher categories and successively builds up to the concepts needed to state the Cobordism Hypothesis in its simpler formulation. Along the way, we will try and fail multiple times to define sufficiently weak n-categories — ultimately culminating in the Interlude, exploring the properties we want a good theory of higher categories to satisfy; as well as the Coda, where we get a bit more technical to define concrete models and constructions. Chapter 4 continues this remixing of Lurie's text, by first discussing its main result (the Cobordism Hypothesis for (X,ζ)-manifolds) and then borrowing from [Sch14] a concrete statement of the Cobordism Hypothesis for oriented $(0,1,2)$-TQFTs with values in the Morita bicategory Alg_2 of algebras, bimodules and intertwiners. This lets us directly compare the Cobordism Hypothesis with the classical correspondence of part I, and conclude that fully extended 2D oriented TQFTs correspond to *separable* symmetric Frobenius algebras.

To those who might feel the page count of this thesis daunting, I would argue that most of the length is due to factors not pertaining the content. There are three main factors.

Firstly, this is a very visual area, with plenty of pictures to make. When we are not drawing 2D cobordisms, that space will usually be filled by commutative diagrams or thorough descriptions.

Secondly, we are very explicit in defining variants of previously-defined notions, even when these could easily be summarized in two to three lines. The reasoning is that this will make the text easier to skim, with the meaning of each concept and piece of notation hopefully being clear at every step. This is most prominent in the multiple higher categories of cobordisms in part II, with each of them occupying about a third of a page; much shorter descriptions would probably have sufficed.

Thirdly, and in a very predictable fashion, this project did undergo scope creep — in multiple occasions. The totality of part II was originally going to occupy no more than 15 to 20 pages.

On a related matter, note that this text is somewhat unusual — as far as mathematical texts go — in that there are very little theorems and proofs. Most of our time is spent on forging the correct definitions for our theory — which here is usually harder than proving things about those definitions! Since we do not skimp on motivation and informal discussion, there are regions of the text where nothing is formally proved for dozens of pages. We are instead concerned with laying the conceptual groundwork for further reading of the sources in the Bibliography. Once again, the worst offender of this tendency is part II, with the only formal proofs being those of section 4.2.

The downside is that this may seem philosophically nebulous and mathematically unsubstantial for some, but I am of the opinion that the core of mathematical practice is in this very process; the rigorous details are better produced after one has observed the big picture, for it is informal reasoning which provides essential guidance. In any case, the Summary should provide a good feel for what is actually proved here.

I personally hope the reader can find joy in this text, just as I did while writing it.

Part I

2D oriented TQFTs, also known as commutative Frobenius algebras

Chapter 1

Cobordisms and TQFTs

Said the straight man to the late man
"Where have you been?"
I've been here and I've been there
And I've been in between.

Greg Lake, *I Talk to the Wind*.

In this opening chapter, we introduce the notion of *topological quantum field theories*, or TQFTs for short. For that, first of all we need to define what *cobordisms* are. And while we could immediately start working on the theory after doing just that, we take our time to develop some categorical language which will make our goals clearer, and the necessary constructions much more amenable. In particular, we define a category Cob_n whose objects are oriented closed $(n-1)$-manifolds M, N and whose morphisms are cobordisms $B\colon M \to N$ between those manifolds: n-manifolds with boundary B equipped with a decomposition of their boundary into an in-boundary identified with M and an out-boundary identified with N. This, along with a short historical note, makes up section 1.1.

In section 1.2 we then define oriented TQFTs. The succinct way is to define these as symmetric monoidal functors $Z\colon \mathsf{Cob}_n \to \mathsf{Vect}_{\mathbb{k}}$, but that hides the physical origins of the theory. So we first state an equivalent axiomatization, inspired by Atiyah's original formulation, and only after having already seen a number of examples do we make sense of the functorial definition. We also comment on one simple but very important result: Zorro's Lemma, a graphical proof of the dualizability of each object in Cob_n. In particular, this implies that the image vector spaces of a TQFT must have finite dimension. The second half of the section, namely the discussion on rigid symmetric monoidal categories, is a bit more thorough than strictly necessary for part I. This is an intentional warm-up for the comparatively more abstract part II.

Finally, to bridge the gap with chapter 2, we spend section 1.3 classifying the simplest kind of TQFTs there is: those of dimension one. This requires introducing a notion of presentations of categories in terms of generators and relations, which we then apply to the case of Cob_1. That presentation immediately yields a classification result for 1D TQFTs, telling us that we can identify a 1D TQFT $Z\colon \mathsf{Cob}_1 \to \mathsf{Vect}_{\mathbb{k}}$ with the image of the point $Z(\mathrm{pt}_+)$. Thus, 1D TQFTs are in bijective correspondence with finite-dimensional vector spaces.

1.1 Cobordisms

We assume all manifolds smooth and compact unless stated otherwise.[1] Remember that a closed manifold is one which is compact and has no boundary.

1.1.1 Cobordisms with and without orientation

Ultimately we are interested in oriented cobordisms, but we first consider the case of unoriented manifolds. Later on, in part II, we will consider cobordisms endowed with further structure; most prominently, framings.

Unoriented cobordisms

A *cobordism* between two closed manifolds M_0, M_1 of the same dimension is a manifold with boundary B, one dimension higher, such that its boundary is made up of our two closed manifolds: $\partial B = M_0 \sqcup M_1$. In order to be able to define operations between cobordisms (such as cutting and gluing), we also keep track of the data embedding our manifolds into the boundary.

Definition 1.1.1. Let M_0 and M_1 be two closed manifolds of dimension $n-1$. An unoriented ***cobordism*** from M_0 to M_1 is an n-dimensional manifold with boundary, B, equipped with two smooth embeddings $M_0 \xhookrightarrow{\iota_0} \partial B \xhookleftarrow{\iota_1} M_1$ such that $\partial B = \iota_0(M_0) \sqcup \iota_1(M_1)$.

Alternatively, we can consider the manifold B to be equipped with a decomposition of its boundary $\partial B = \partial_{\text{in}} B \sqcup \partial_{\text{out}} B$ and diffeomorphisms $M_0 \xrightarrow{\cong} \partial_{\text{in}} B$ and $\partial_{\text{out}} B \xleftarrow{\cong} M_1$. We call $\partial_{\text{in}} B = \iota_0(M_0)$ the ***in-boundary***, and $\partial_{\text{out}} B = \iota_1(M_1)$ the ***out-boundary***. Which manifold is the in-boundary and which one is the out-boundary is part of the data defining a cobordism, as we will soon explore in Remark 1.1.3. We will sometimes say “the in-boundaries” or “the out-boundaries”, in plural, when referring to the connected components of the in- or out-boundary. The singular forms will often, too, refer to each connected component separately; the meaning will be clear by context.

If B is a cobordism between M_0 and M_1, we often write $B\colon M_0 \to M_1$. This is not just a cute notation, but hints at a greater structure: as we will soon see, n-dimensional cobordisms assemble into a *category* where the objects are $(n-1)$-dimensional closed manifolds and the morphisms are cobordisms between them.

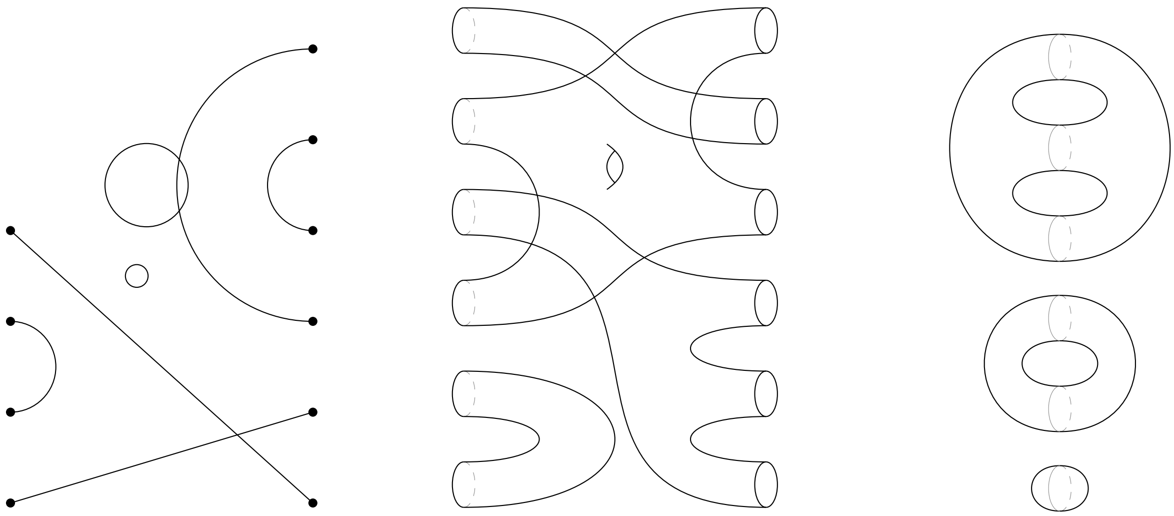
Remark 1.1.2 — The empty set is an *n*-dimensional manifold, for all values of *n*.
For any dimension $n-1$, the empty set $\varnothing$ is a closed $(n-1)$-manifold: the *trivial* $(n-1)$*-manifold*. As such, we can talk about cobordisms from and to the empty set. We will occasionally denote the trivial $(n-1)$-manifold as $\varnothing^{n-1}$, so as to keep track of the dimension.

A cobordism $B\colon \varnothing^{n-1} \to \varnothing^{n-1}$ is just an n-dimensional closed manifold: it is a compact manifold with boundary $\partial B = \varnothing \sqcup \varnothing$; that is, without boundary. We also have an empty n-cobordism, which we will denote as the “cylinder” $\varnothing^{n-1} \times I\colon \varnothing^{n-1} \to \varnothing^{n-1}$.

Let’s draw some pictures! We consistently draw cobordisms from left to right: in-boundaries on the left, out-boundaries on the right. We can think of a one-dimensional cobordism as a collection of lines and circles between two columns of dots; see figure 1.1(a) for an example. In the same way, a two-dimensional cobordism consists of surfaces between disjoint union of circles; see figures 1.1(b) and 1.1(c). Do not be confused into thinking that some connected components are on top or others or that they cross over or under each other: our cobordisms aren’t embedded in any ambient space, so these notions are meaningless.[2] But the order of the boundaries *is* important, as we will see later!

[1] We want our manifolds to be smooth so that we can apply results from Morse theory. Most — if not everything — of what we do here can be adapted to a purely topological setting, but that requires a number of technical results which deviate us from our goal.

[2] Not that we *couldn’t* embed them and still have an interesting theory to study! These are called *tangles*.



(a) A cobordism $\bigsqcup_4 \mathrm{pt} \to \bigsqcup_6 \mathrm{pt}$.
(b) A cobordism $\bigsqcup_6 \mathbb{S}^1 \to \bigsqcup_6 \mathbb{S}^1$.
(c) A cobordism $\varnothing^1 \to \varnothing^1$.

Figure 1.1: Some cobordisms in dimensions one and two.

When we get to three dimensions, though, things get a lot more complex: we have spaces that "interpolate" between disjoint unions of spheres, tori of arbitrary genus, and (in the unoriented case) projective planes and Klein bottles! For these, it is better to adopt a different point of view, since two-dimensional drawings start to be ambiguous and messy. One option is to take advantage of the temporal dimension: we can consider a suitable function $f\colon B \to [0,1]$ from our 3-cobordism to a "time" segment, such that $f^{-1}(0) = \partial_{\mathrm{in}} B$ and $f^{-1}(1) = \partial_{\mathrm{out}} B$. Then, we take "slices" $f^{-1}(t)$ to see how our manifold evolves over time. One of these "films" is pictured in figure 1.2. It depicts a cobordism between a torus and a sphere; topologically, this cobordism is a solid torus with a sphere removed from its interior. Note the critical point, when we pinch part of the torus to a point: a singularity appears and the slice stops being a smooth manifold for a single instant.

In the previous paragraph, "suitable" means that f should be a *Morse function*: a differentiable function where the only singular points are non-degenerate. These are a very useful tool which we will get accustomed with.

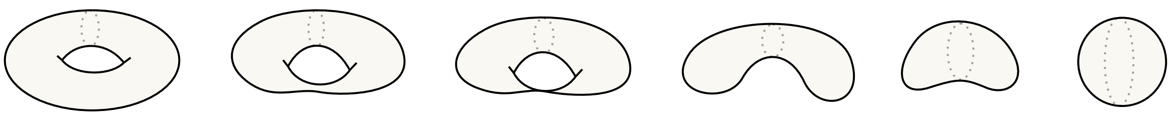
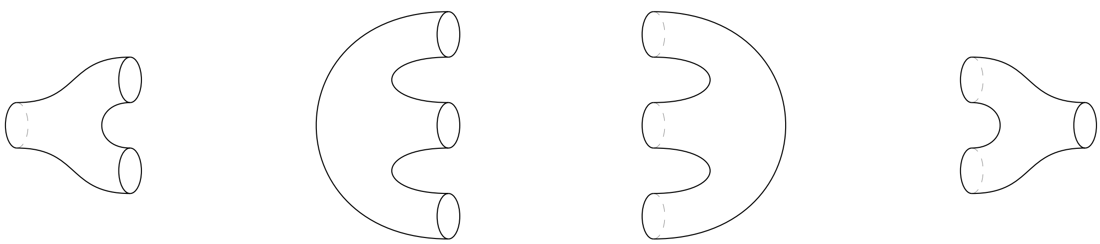


Figure 1.2: Slices of a 3-cobordism $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^2$.



(a) $B\colon \mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1$
(b) $B'\colon \varnothing^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1 \sqcup \mathbb{S}^1$
(c) $B''\colon \mathbb{S}^1 \sqcup \mathbb{S}^1 \sqcup \mathbb{S}^1 \to \varnothing^1$
(d) $\overline{B}\colon \mathbb{S}^1 \to \mathbb{S}^1 \sqcup \mathbb{S}^1$

Figure 1.3: Four different cobordisms which are diffeomorphic as manifolds.

Remark 1.1.3 — Moving in- and out-boundaries around.
Let $B\colon M_0 \to M_1$ be a cobordism, and let $|B|$ be its underlying manifold with boundary. By moving the in-boundary into the out-boundary, we can define a cobordism $B'\colon \varnothing^{n-1} \to M_0 \sqcup M_1$ with the same underlying manifold: we decompose the boundary $\partial|B| = \varnothing^{n-1} \sqcup (\partial_{\text{in}}B \sqcup \partial_{\text{out}}B)$, so that our two equipped embeddings are the trivial one $\varnothing^{n-1} \hookrightarrow \varnothing^{n-1}$ and the disjoint union

$$\iota_0 \sqcup \iota_1 \colon M_0 \sqcup M_1 \hookrightarrow \partial_{\text{in}}B \sqcup \partial_{\text{out}}B.$$

Dually, we can define a cobordism $B''\colon M_0 \sqcup M_1 \to \varnothing^{n-1}$.

In the same way, we can define the opposite cobordism $\overline{B}\colon M_1 \to M_0$ by defining its in-boundary as $\partial_{\text{out}}B$ and its out-boundary as $\partial_{\text{in}}B$. If one prefers, we can write the embeddings $\bar{\iota}_0\colon M_1 \hookrightarrow \overline{B}_0 = \partial_{\text{out}}B$ and $\bar{\iota}_1\colon M_0 \hookrightarrow \overline{B}_1 = \partial_{\text{in}}B$. See figure 1.3 for a particular example of these operations.

Even though the underlying manifolds of B, B', B'' and $\overline{B}$ are diffeomorphic (even more, they are strictly the same manifold!), we don't want to regard these cobordisms as being "the same". The decomposition $\partial B = \partial_{\text{in}}B \sqcup \partial_{\text{out}}B$ of the boundary, as well as the choice of in- and out-boundary in this decomposition, are as much a part of the data of a cobordism as the underlying manifold B is.

That being true, these choices (as well as the embeddings $M_0 \hookrightarrow \partial B \hookleftarrow M_1$) will usually be clear from context, and as such we often won't state them explicitly.

As we often do in mathematics, we want to define maps between our objects, and define when two of our objects should be "the same" in a precise sense. Whatever a morphism between two cobordisms is, it should respect the embeddings of Definition 1.1.1.

Definition 1.1.4. A morphism between two cobordisms $B\colon M_0 \to M_1$ and $B'\colon N_0 \to N_1$ is a smooth map $f\colon B \to B'$ between the underlying manifolds such that $f(\partial_{\text{in}}B) \subset B'_0$ and $f(\partial_{\text{out}}B) \subset B'_1$.

In other words, a morphism between cobordisms maps the in-boundary to the in-boundary and the out-boundary to the out-boundary.

For our purposes, the important notion is that of ***isomorphism*** of cobordisms. As usual, this is just a morphism which admits a two-sided inverse. Equivalently, it is a diffeomorphism $f\colon B \to B'$ between the underlying manifolds such that the following diagram commutes:

$$\begin{tikzcd}
M_0 \cong \partial_{\text{in}}B \arrow[r, hook] \arrow[d, "f|_{\partial_{\text{in}}B}"'] & B \arrow[d, "\cong"', "f"] & \partial_{\text{out}}B \cong M_1 \arrow[l, hook'] \arrow[d, "f|_{\partial_{\text{out}}B}"] \\
N_0 \cong \partial_{\text{in}}B' \arrow[r, hook] & B' & \partial_{\text{out}}B' \cong N_1. \arrow[l, hook']
\end{tikzcd}$$

If we talk about diffeomorphisms between two cobordisms, we are also referring to this notion.

Now we want our manifolds to carry an orientation, and our cobordisms should be compatible with that orientation. To fix the conventions we will work with, we first give a quick reminder.

Orientations of manifolds

Definition 1.1.5. Let V be a finite-dimensional real vector space. Two ordered bases B and B' are said to have ***the same orientation*** if the linear change of basis $P_{B\to B'}\colon V \to V$ has positive determinant, and ***opposite orientation*** if it has negative determinant. This defines an equivalence relation in the set of bases of V.

An ***orientation*** ζ on V is a choice of sign ($+$ or $-$) for each basis such that two bases have the same sign if and only if they belong to the same equivalence class.

A linear map $f\colon (V,\zeta) \to (W,\xi)$ between oriented vector spaces is said to ***preserve orientation*** if it takes every positive basis to a positive basis, and ***reverse orientation*** if it takes every positive basis to a negative basis.

Each finite-dimensional real vector space admits exactly two possible orientations. The reason the definition is worded in a slightly clunky way is to include the case of the trivial vector space $\{0\}$, where an orientation is simply assigning $+$ or $-$ to the empty set (its only basis).

Definition 1.1.6. An ***orientation*** ζ on a smooth manifold M is a smooth choice of orientation for each tangent space. That is, the differential of each change of coordinates should preserve the orientation.

A manifold M is called ***orientable*** if it admits an orientation.

A differentiable map $f\colon (M,\zeta) \to (N,\xi)$ between oriented manifolds is said to ***preserve orientation*** if at each point $x \in M$ its differential $d_x f\colon (\mathrm{T}_x M,\zeta) \to (\mathrm{T}_{f(x)} N,\xi)$ preserves orientation, and to ***reverse orientation*** if the differential reverses orientation.

Proposition 1.1.7. Each orientable manifold M admits 2^m possible orientations, where $m \in [0,\infty]$ is its number of connected components. In particular, each connected orientable manifold M admits exactly 2 orientations.

Proposition 1.1.8: Orientation of a product. Let (X,ζ) and (Y,ξ) be oriented manifolds (one of them without boundary).[3] The product $X \times Y$ inherits an orientation $\zeta \times \xi$, where in each point $(x,y) \in X \times Y$ we declare $\{v_1,\dots,v_n,w_1,\dots,w_m\}$ to be a positive basis for $\mathrm{T}_{(x,y)}(X \times Y)$ if $\{v_1,\dots,v_n\}$ is a positive basis for $\mathrm{T}_x X$ and $\{w_1,\dots,w_m\}$ is a positive basis for $\mathrm{T}_y Y$.

Definition 1.1.9. Let B be a smooth manifold with boundary. Let $x \in \partial B$ be a point in its boundary, and let $w \in \mathrm{T}_x B \smallsetminus \mathrm{T}_x \partial B$ be a tangent vector which is not tangent to the boundary (e.g., a vector normal to the boundary). We say that w ***points inward*** if there is a curve $\gamma\colon [0,\varepsilon) \to B$ such that $\gamma_0 = x$, $\gamma_0' = w$ and $\gamma_t \notin \partial B$ for $t > 0$; and that is ***points outward*** if it doesn't point inward (i.e., $-w$ points inward).

Proposition 1.1.10: Orientation induced on the boundary. Let (B,ζ) be an oriented manifold with boundary. Its boundary ∂B inherits an orientation in a canonical way.

Let $w \in \mathrm{T}_x B \smallsetminus \mathrm{T}_x \partial B$ be a vector pointing outward. Now, for any basis $\{v_1,\dots,v_{n-1}\}$ of $\mathrm{T}_x \partial B$, we declare it to be positive if $\{v_1,\dots,v_{n-1},w\}$ is a positive basis for $\mathrm{T}_x B$, and declare it to be negative otherwise.

This is the "outward normal last" convention. The way an orientation on a manifold with boundary descends to an orientation on the boundary is best exemplified by the cylinder, as in figure 1.4. This figure also exemplifies how a product of oriented manifolds gets its orientation!

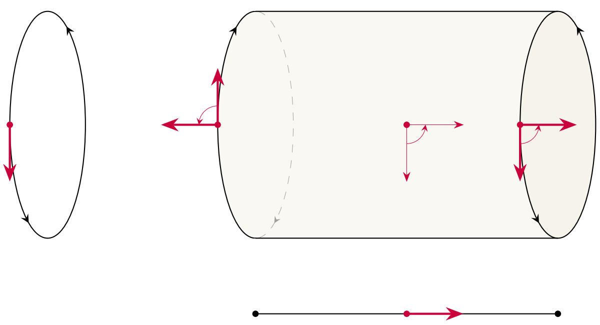


Figure 1.4: A cylinder $\mathbb{S}^1 \times I$ induces opposite orientations on its two boundaries.

Given any oriented manifold $M := (M,\zeta)$, we denote its ***opposite manifold*** by $\overline{M} := (M,\overline{\zeta})$. It is the same underlying manifold, but equipped with the opposite orientation. With this, we end the crash course on orientation of manifolds, and proceed with our theory.

[3] A product of manifolds with non-empty boundary is not a manifold with boundary, but a manifold with *corners*.

Oriented cobordisms

Definition 1.1.11. Let M_0 and M_1 be two oriented closed manifolds of dimension $n-1$. An ***oriented cobordism*** $B\colon M_0 \to M_1$ is an oriented n-dimensional manifold with boundary equipped with two orientation-preserving embeddings $\overline{M}_0 \overset{\iota_0}{\hookrightarrow} \partial B \overset{\iota_1}{\hookleftarrow} M_1$ such that $\partial B = \iota_0(\overline{M}_0) \sqcup \iota_1(M_1)$, where the orientation of the boundary is inherited from the orientation of B.

As in the unoriented case, we may instead require B to be equipped with a decomposition of its boundary $\partial B = \partial_{\mathrm{in}} B \sqcup \partial_{\mathrm{out}} B$ and orientation-preserving diffeomorphisms $\overline{M}_0 \xrightarrow{\cong} \partial_{\mathrm{in}} B$, $M_1 \xrightarrow{\cong} \partial_{\mathrm{out}} B$.

Yet another option is to consider smooth embeddings $M_0 \overset{\iota_0}{\hookrightarrow} \partial B \overset{\iota_1}{\hookleftarrow} M_1$ such that ι_0 *reverses* orientation and ι_1 preserves orientation. This is a perfectly fine characterization, and useful as a mental shortcut. We will employ it from time to time; for example, we use it in the following Remark 1.1.12. This is, however, a very bad definition when working in a categorical framework: the orientation-reversing maps are *not* morphisms in the category of oriented manifolds, so we can't reason about them directly.

But why do we want this map to reverse the orientation, anyway?

Remark 1.1.12 — The reason for reversing the in-boundary's orientation.
At first glance, the fact that we reverse the orientation of one of the boundaries may seem strange. But this is necessary in order to be able to construct our theory; it all stems from the way a product of oriented manifolds carries an orientation.

Consider the case of the cylinder, as in figure 1.4. Let M be an oriented closed manifold, and consider the product $M \times I$, where $I := [0,1]$ is equipped with its standard orientation (a positive vector points from 0 to 1). Intuitively, this should be a cobordism from M to itself; but if we ask for both embeddings to preserve orientation, this would actually be a cobordism from $\overline{M}$ to M! Indeed, the orientation induced on the boundary $M \times \partial I$ by the product $M \times I$ is such that the canonical diffeomorphism $M \hookrightarrow M \times \{0\}$ reverses orientation, while $M \hookrightarrow M \times \{1\}$ preserves orientation. So if we ask for our embeddings to respect orientation — as we should — we actually have embeddings $\overline{M} \hookrightarrow M \times \partial I \hookleftarrow M$.

Other authors (including our main source, [Koc03]) do consider orientation-preserving maps $M_0 \hookrightarrow \partial B \hookleftarrow M_1$, but instead let ∂B be oriented independently from B. That is, they don't require the orientation of ∂B to coincide with the one inherited by B. Then, the components of ∂B whose orientation coincide with the one induced by B form the out-boundary $\partial_{\mathrm{out}} B$, while the ones where the orientation is reversed form the in-boundary $\partial_{\mathrm{in}} B$. It is easy to see that this definition is equivalent to the one we gave.

Also note that, as per the prior discussion, whether a manifold is an in- or out-boundary is completely determined by the orientation.

Remark 1.1.13 — Specifying in- and out-boundaries.
Let M and N be two closed $(n-1)$-manifolds and let B be an n-manifold with boundary. Assume there exists embeddings $M \overset{\iota_M}{\hookrightarrow} B \overset{\iota_N}{\hookleftarrow} N$ such that the boundary ∂B decomposes into their images: $\partial B = \iota_M(M) \sqcup \iota_N(N)$.

In the unoriented case, this is not enough information to define a cobordism: we need to specify which one of the manifolds is the in-boundary and which is the out-boundary. If we didn't specify in- and out-boundaries, we would have no way of telling apart one cobordism $B\colon M \to N$ from its opposite $\overline{B}\colon N \to M$. (cf. Remark 1.1.3).

In the oriented case, however, this *is* enough information. However, we need the embeddings to satisfy some further properties: one of them (let's say ι_N) should preserve orientations, while the other (in this case ι_M) should reverse them. So M is the in-boundary and N is the out-boundary, and we have completely determined a cobordism $B\colon M \to N$.

In other words, specifying which of the two manifolds is the in-boundary is necessary for unoriented cobordisms (it is a *structure*), but redundant for oriented ones (it is a *property*).

Remark 1.1.14 — Moving in- and out-boundaries around, again.
In the oriented world, the opposite cobordism $\overline{B}: \overline{N} \to \overline{M}$ still exists, but it also reverses the orientation of the boundaries. The same is true for the other two cobordisms of Remark 1.1.3: using the same underlying manifold and only changing the embeddings, we can define cobordisms $B': \varnothing \to \overline{M} \sqcup N$ and $B'': M \sqcup \overline{N} \to \varnothing$.

From now on we assume every manifold and cobordism to be oriented, unless specified otherwise.

Finally, the notion of "morphism of oriented cobordisms" is almost exactly the same as in the unoriented case [Def. 1.1.4], with the only difference being that the map $f: B \to B'$ should preserve orientations. So we can talk about isomorphism classes of oriented cobordisms, which is precisely what we need in order to accomplish our next immediate goal: as foreshadowed, we are going to assemble our cobordisms into a category. For that, we need to define a way of composing cobordisms together: *gluing*. But this operation will be defined only up to isomorphism!

1.1.2 Cob_n, the category of oriented n-cobordisms

Gluing topological cobordisms

For a change of pace, let M_0, M_1 and M_2 be closed *topological* manifolds of the same dimension $n-1$. Let $B: M_0 \to M_1$ and $B': M_1 \to M_2$ be topological cobordisms; that is, the embeddings

$$M_0 \xhookrightarrow{\iota_0} \partial B \xhookleftarrow{\iota_1} M_1 \xhookrightarrow{\iota'_0} \partial B' \xhookleftarrow{\iota'_1} M_2$$

are continuous instead of smooth. Gluing these together is easy: since $\partial_{\text{out}} B \cong M_1 \cong \partial_{\text{in}} B'$, we just take their *pushout*

$$B \sqcup_{\partial_{\text{out}} B \sqcup \partial_{\text{in}} B'} B' := \frac{B \sqcup B'}{\partial_{\text{out}} B \sqcup \partial_{\text{in}} B'} = \frac{B \sqcup B'}{\iota_1(M_1) \sqcup \iota'_0(M_1)},$$

which is a cobordism from M_0 to M_2. We call this the ***composition*** (or sometimes just *gluing*) of the cobordisms B and B', and write it as $B' \circ B$. See figure 1.5 for an example.

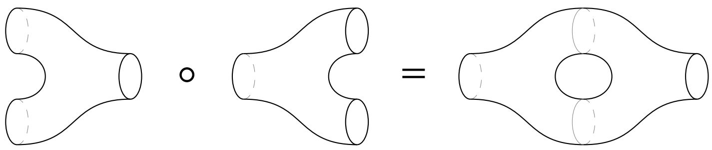


Figure 1.5: Gluing cobordisms $\mathbb{S}^1 \to (\mathbb{S}^1 \sqcup \mathbb{S}^1) \to \mathbb{S}^1$.

Notation 1.1.15: Writing order. Given cobordisms $M_0 \xrightarrow{B} M_1 \xrightarrow{B'} M_2$, we write their composition as $B' \circ B: M_0 \to M_2$. Note that this order is reverse to that in which both our pictures and arrows are drawn, just as one writes functions $X \xrightarrow{f} Y \xrightarrow{g} Z$ but their composition as $g \circ f: X \to Z$. Sadly, the alternatives are just as confusing, so we adopt the standard convention found in the literature.

This is great, now we have a composition for our category! Well, not exactly.

Remark 1.1.16 — The problem with strict associativity.
This operation isn't strictly associative; even if we restrict ourselves to the underlying product of sets, $A \times (B \times C)$ is technically a different set from $(A \times B) \times C$. But these are obviously very closely related, so we choose to identify them canonically through an *unique* isomorphism; namely, we identify $(a,(b,c)) \mapsto ((a,b),c)$. This is so natural that we do not usually think twice about it, as this identification carries little harm. But we can not afford to ignore this identification, canonical as it may be, when trying to formally define the composition for our category.

In the smooth case, it gets even worse: pushouts do *not* generally exist in the category of smooth manifolds. We can still glue two manifolds with diffeomorphic boundary together, but the process involves choosing a collar for each boundary in order to smooth the glued edge. All manifolds obtained in this way are diffeomorphic, but there is no single canonical diffeomorphism relating any two choices of collars.

So we can't define a strictly associative composition of cobordisms. We can however define *multiple* possible compositions for any two cobordisms, one for each choice of collars; this is the approach taken by higher category theory, which we will discuss in part II. These compositions will carry additional information, a *witness* 2*-morphism* — in this case, we add a choice of diffeomorphism between two ways of gluing our cobordisms. Those witness 2-morphisms will have to satisfy some higher coherence restraints (expressed through 3-morphisms), lest our structure be too weak to prove anything.

For the time being, however, we will take the comparatively less refined approach of accepting some loss of information and collapsing all of our isomorphism classes together, so that we can apply regular "lower" category theory instead.

Remark 1.1.17 — The problem with strict identities.
Note that we also lack a canonical way of identifying the composition $M \xrightarrow{M\times[0,1]} M \xrightarrow{B} N$ with B. Even in the simplest non-empty case, when B itself is a cylinder $M \times [1,2]$, we require a choice of homeomorphism from $[0,1]$ to $[0,2] \cong [0,1] \sqcup_1 [1,2]$. In other words, we also lack an identity cobordism, since this should really be a cylinder "of height zero"; otherwise, it would not be a unit for our composition.

The easiest way to fix this part of the definition of our category, without resorting to taking isomorphism classes, is to accept the "zero-height cylinder" $M \times \{0\}$ as a proper cobordism — even when it is not an n-manifold. This carries some amount of technical inconveniences, but is a valid approach nonetheless.

The fancy way of solving this problem, through higher category theory, is by having multiple possible identity cobordisms — along with additional coherence 2-morphisms forming part of the categorical structure, a recurring motif.

And, once again, we will sidestep this issue by brutally truncating all this higher information and just taking isomorphism classes of cobordisms.

This discussion might seem all too pedant to be insightful, but is an important thing to keep in mind; what are we referring to when we say that two mathematical objects (or in this case, morphisms) are "equal"? Should we weaken our definitions, so as to permit different levels of "sameness"? After all, we already do this when introducing categories — on which we usually ask for objects to be *isomorphic*, rather than strictly *equal*. And we then introduce a further level of weak equality, since that is what a natural isomorphism between two categories is — we ask not for the categories to be equal, nor even isomorphic, but *equivalent*. This is realized through an isomorphism of *functors* between the two categories. Why stop there?[4]

But we are getting ahead of ourselves. Let's slow down and look in more detail at the smooth case. As mentioned, this is very similar to the topological case, but requires a choice a *collar* for each boundary. First, we introduce some of the Morse-theoretic tools which we will use.

Basic concepts from Morse theory

We will provide no proof for statements from Morse theory, and instead refer the reader to the classics [Mil73; Hir76] for further information. The main objects of study of this theory are well-behaved smooth functions from a manifold to an interval $I = [0,1]$ (or to the real line $\mathbb{R}$).

Definition 1.1.18. Let M be a compact smooth manifold. A ***Morse function*** is a smooth function $f\colon M \to I$ such that all of its critical points are non-degenerate, i.e. have non-singular Hessian.

[4] Because things become increasingly complicated.

The Hessian matrix depends on the coordinate system choosen, but its regularity does not. The ***index*** of the critical point is defined as the number of negative eigenvalues of the Hessian, counted with multiplicity. In the case of surfaces, points with index 0 are local minimums, points with index 2 are local maximums and points with index 1 are saddle points.

Almost all smooth functions $f\colon M \to I$ are Morse functions; more precisely, these form an open and dense subset of $\mathscr{C}^\infty(M, I)$ equipped with the $\mathscr{C}^\infty$ topology [Hir76, Thm. 6.1.2]. In particular, Morse functions always exist, and can be choosen such that all the critical points have different image. Furthermore, when our domain is a cobordism B, we can choose Morse functions which are compatible with the in- and out-boundaries; i.e., such that 0 and 1 are regular values of f satisfying $f^{-1}(0) = \partial_{\text{in}} B$ and $f^{-1}(1) = \partial_{\text{out}} B$. (This includes the case when either boundary is empty). We assume that all of our Morse functions are choosen in this way, unless stated otherwise.

As mentioned when discussing the "time evolution" interpretation of cobordisms [fig. 1.2], we can think of the codomain I as a temporal dimension, and of Morse functions as well-behaved projections into this dimension. So we can take slices $f^{-1}(t) \subset B$, which will either be smooth manifolds or have a single singularity.

This lets us decompose our cobordism into simpler parts, by taking preimages $f^{-1}[a, b] \subset B$ of intervals; here, a and b should be regular values. Simply put, we can cut our cobordisms at nice points to get more (and simpler) cobordisms: Morse theory provides us of the scissors and guidelines necessary for doing so. Of course, we want the cutting and gluing operations to be inverses, at least up to diffeomorphism: if we cut a cobordism in half and then glue the two pieces, we should get a cobordism equivalent to the one we started with.

The operation of cutting is very well-behaved; better behaved than our composition, in fact, since if doesn't suffer from the bane of "uniqueness up to non-unique diffeomorphism"! This is actually a philosophy at the core of Morse theory: *If you have something complicated, it is easier to split it apart than to assemble it from its parts* [Koc03, p. 63]. We will return to this again in the Coda, where we will use it to *define* composition of cobordisms in a more nuanced manner!

In any case, let's end this section by stating a useful theorem. We paraphrase a version from [Koc03, Thm. 1.3.8], which itself is a minor adaptation of [Hir76, Thm. 6.2.2].

Theorem 1.1.19: Regular Interval Theorem. Let $B\colon M_0 \to M_1$ be a cobordism and let $f\colon M \to I$ be a smooth map with no critical points and such that $f^{-1}(0) = M_0$ and $f^{-1}(1) = M_1$. Then, there are diffeomorphisms from the cylinders $M_i \times I$ to B which are compatible with the projection to I:

$$\begin{tikzcd} M_0 \times I \arrow[r, dashed, "\cong"'] \arrow[dr] & B \arrow[d, "f"] & M_1 \times I \arrow[l, dashed, "\cong"] \arrow[dl] \\ & I. & \end{tikzcd}$$

This is enough Morse theory for now, until section 2.1.1.

The smooth case: gluing smooth cobordisms

Since we already discussed the strategy for gluing smooth cobordisms in Remark 1.1.16, let us go straight to the construction. Let $B\colon M_0 \to M_1$ and $B'\colon M_1 \to M_2$ be smooth cobordisms. We can choose Morse functions $f\colon B \to I$ and $g\colon B' \to I$ satisfying the assumptions mentioned after Definition 1.1.18; in particular, such that $f^{-1}(1) = M_1 = g^{-1}(0)$. These lets us choose *collars*, which for our purposes are submanifolds $f^{-1}(1-\varepsilon, 1] \subset B$ and $g^{-1}[0, \varepsilon) \subset B'$; in particular, f and g do not have critical points in the two open intervals $(1-\varepsilon, 1)$ and $(0, \varepsilon)$. Since the set of critical points of a smooth function is discrete, this is always possible. Furthermore, by the Regular Interval Theorem [1.1.19], both of these collars will be diffeomorphic to a cylinder $M_1 \times I$.

Now, we can "overlap" the collars of B and B' and smooth the resulting intersection by "interpolating" the smooth structures through an isotopy. The definition depends on the choice of collars and on the choice of smoothing isotopy, but any two choices of collars yield diffeomorphic manifolds.

Some authors require smooth cobordisms to (by definition) come equipped with a choice of collars for the in- and out-boundaries, so as not to have to make arbitrary choices for the composition. This is yet another approach to sidestep strictness woes, but one which (for our purposes) bloats the category with way too many redundant morphisms.

Remark 1.1.20 — The category of smooth manifolds behaves very poorly.
The category of smooth manifolds is interesting in a very particular way: its objects are very well behaved, but the category itself is a terrible place to work in.

This can be stated as a general phenomenon: nice objects tend to form non-nice categories, while nice categories tend to contain non-nice objects.[5] The more we restrict ourselves to nice objects with very good structural properties, the less those properties are preserved by more general maps. The category of fields is another common example, lacking even an initial or terminal object. This makes its younger sibling Ring a better place to work in, for developing a theory.

Smooth manifolds are very well-behaved as objects; to list a few remarkable properties, they are paracompact, they are Euclidean Neighbourhood Retracts, and they always admit essentially unique PL-structures and Riemannian metrics. But common categorical tools, such as pushouts, pullouts and limits, do not generally exist. We can still define objects approximating a proper definition (by proceeding analogously to the topological case and then smoothing any singularities we create), but we lack any kind of universal property.

That poses a greater problem, and one of much bigger scope: a detailed account of the consequences of this lack of coherent choice of isomorphism, as well as the ways of trying to fix the theory to still be able to apply categorical tools, would take rivers of ink and a couple of lifetimes. This is not that much of an exaggeration; we will see a small glimpse of this in part II, when trying to define weak ∞-categories.

Another alternative is to generalize the notion of “smooth manifold” so that these universal properties always exist, even if the objects themselves become worse behaved as a consequence. See, for example, the introductory paper [BH11]. This is not relevant for our purposes, so we only mention it briefly.

Of course, differential topology has historically been able to progress smoothly without use of these fancy algebraic tools. But it is always good to keep an updated toolkit available.

As mentioned, we now take the radical approach of collapsing each isomorphism class of cobordisms to a single morphism; we lose a lot of information, but that loss of information is tolerable in low dimensions. Note that two diffeomorphic closed manifolds are cobordant through an invertible cobordism (namely the cylinder $M \times I$ equipped with diffeomorphisms $\overline{M} \hookrightarrow M \times \{0\}$ and $M' \cong M \hookrightarrow M \times \{1\}$), so this isn't a problem.

The category of oriented n-cobordisms

Without further ado, here is the definition we have been building up to.

Definition 1.1.21. The ***category of oriented n-cobordisms***, Cob_n, is the category consisting of:

- **Objects:** Closed oriented $(n-1)$-manifolds.
- **Morphisms:** Classes of equivalence of oriented n-cobordisms up to isomorphism.
- **Composition:** Gluing of (classes of equivalence of) n-cobordisms along the common boundary.
- **Identities:** Cylinders $M \times I \colon M \to M$.

[5] This quote is directly taken from the nLab: `https://ncatlab.org/nlab/revision/dichotomy+between+nice+objects+and+nice+categories/16`.

Note that although categories are usually named after their objects, the category of cobordisms is named after its *morphisms*. We will occasionally consider the category of *unoriented* cobordisms, which we denote by $\mathsf{Cob}_n^{\mathrm{un}}$, and whose construction is completely analogous.

We will spend quite a bit of time (chapter 2) studying the structure of Cob_2 in depth, ultimately providing a complete description of all objects and morphisms in the category. For that, it is necessary to address a particular kind of construction common in category theory: a *monoidal* structure. The axiomatic definitions have to wait until section 1.2.2, but the idea is simple: a lot of objects in mathematics accept a binary operation that acts not only on objects, but also on morphisms between the objects. If this operation is associative and has a unit, then it endows the category with a monoidal structure.

Monoidal and symmetric structures

One large class of examples of monoidal structures is that of products and coproducts (e.g., cartesian product and disjoint union of sets), but the main motivating example is the tensor product of vector spaces: given two $\Bbbk$-linear maps $f\colon V \to W$ and $g\colon A \to B$, we can define a map $f \otimes g\colon V \otimes A \to W \otimes B$. Furthermore, the ground field acts as a unit, since $\Bbbk \otimes V \cong V \cong V \otimes \Bbbk$ for any $\Bbbk$-vector space V. These three vector spaces aren't *strictly* the same object, but they are related through a distinguished isomorphism in a *coherent* manner, so there's usually no harm in pretending they are.[6] Furthermore, given three different vector spaces U, V and W, the two ways of tensoring them in order are coherently isomorphic: $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$. So this operation is unital (up to coherent isomorphism), and also associative (up to coherent isomorphism). And that is what a monoidal structure is!

This example is so fundamental that some authors use the term *tensor category* to refer to a monoidal category. However, this term usually carries an implication of some further structure shared with vector spaces. For instance, it usually implies a *symmetric structure*.

The tensor product of vector spaces is symmetric, since we can change factors around in a particularly nice manner: $U \otimes V \cong V \otimes U$ in such a way that the diagram

$$\begin{tikzcd}
U \otimes V \otimes W \arrow[rr, "U\leftrightarrow V\otimes W"] \arrow[dr, "U\leftrightarrow V"'] & & V \otimes W \otimes U \\
& V \otimes U \otimes W \arrow[ur, "U\leftrightarrow W"'] &
\end{tikzcd}$$

commutes. These maps which interchange the order of the factors are called *twist maps* and written as $\tau_{U,V}\colon U \otimes V \to V \otimes U$. They are defined elementwise in the intuitive way: $(u,v) \mapsto (v,u)$. The twist maps endow $\mathsf{Vect}_\Bbbk$ with the structure of a *symmetric* monoidal category. Again, we will give the precise definition in section 1.2.2; for now, it is enough to trust that we can shift factors around without much careful consideration.

But let us stop talking about vector spaces and return to the topic at hand: cobordisms.

The category of cobordisms has a monoidal structure given by disjoint union of manifolds. This extends to cobordisms: given $B\colon M \to N$ and $B'\colon M' \to N'$, we can define $B \sqcup B'\colon M \sqcup M' \to N \sqcup N'$. In our drawings, we depict this as putting B' on top of B. Same goes for the boundaries: we draw a manifold $M \sqcup N$ with M below N. See figure 1.6 for an example. Note that even when the manifolds M and N are diffeomorphic (which is always the case for connected closed manifolds in dimensions 1 and 2), the order of the different copies matters: remember that an isomorphism of cobordisms must preserve both boundaries.

We also have a symmetric structure: for any two manifolds M and N, we can define the *twist cobordism* $\sigma_{M,N}\colon M \sqcup N \to N \sqcup M$ as the cylinder $(M \sqcup N) \times I$ equipped with embeddings

[6]This rather hand-wavy claim can be stated formally. Essentially, this is an interpretation of Mac Lane's famous *Coherence* and *Strictification* Theorems for monoidal categories. More on that in section 1.2.2, once we give concrete definitions. As the reader probably surmises, "coherent" has a precise technical meaning.

$M \sqcup N \hookrightarrow (M \sqcup N) \times \{0\}$ (the natural one) and $(M \sqcup N) \times \{1\} \hookleftarrow N \sqcup M$ (the composition of the natural embedding and the twist map of sets $N \sqcup M \to M \sqcup N$.) We depict this as two interlaced cylinders, as in figure 1.7. Importantly, the twist map $\sigma_{M,N}$ is *not* diffeomorphic to the cylinder $(M \sqcup N) \times I = (M \times I) \sqcup (N \times I)$, for non-empty manifolds M and N. Once again, remember that an isomorphism of cobordisms must preserve both boundaries.

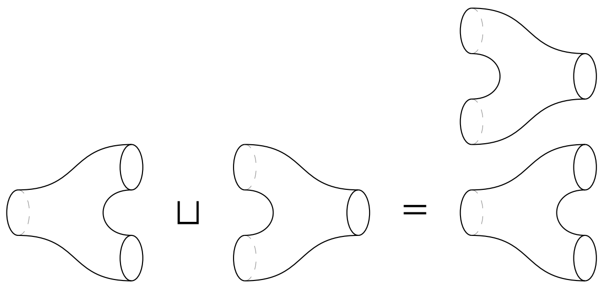


Figure 1.6: Disjoint union of cobordisms.

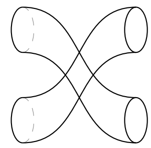


Figure 1.7: A twist cobordism.

Permutation cobordisms

Given any n-cobordism $B\colon M \to N$, we can decompose it into its topological connected components. First, we decompose its underlying manifold $|B|$ into connected components $|B_i|$, so $B \cong \bigsqcup_i |B_i|$. Each connected component $|B_i|$ contains a number of in- and out-boundaries, and no two components contain the same boundary. So we can define connected cobordisms $B_i\colon M_i \to N_i$, such that $M \cong \bigsqcup_i M_i$ and $N \cong \bigsqcup_i N_i$.[7] However, this is not a true decomposition, in the sense that in general $B \ncong \bigsqcup_i B_i$; the order of the boundaries might not be the same. Remember our discussion of the twist cobordisms, and also see the following figure 1.8.

But these should be closely related, shouldn't they? And yes they are, and yes there is a precise way to relate the two: the twist cobordisms let us freely interchange the order of the boundaries. In order to better articulate the specifics, let us introduce the notion of *permutation cobordism* and state a small lemma.

Definition 1.1.22. We say that a cobordism is a ***permutation cobordism*** if, up to isomorphism, it can be written using composition and disjoint union of cylinders $M \times I$ and twist cobordisms $\sigma_{M,N}\colon M \sqcup N \to N \sqcup M$.

(This means that it is an element of the monoidal subcategory of $\mathtt{Cob}_n$ generated by the twist cobordisms, but we haven't properly defined most of these terms yet). In other words, a permutation cobordism $\sigma\colon M_1 \sqcup \cdots \sqcup M_k \to M_{\rho(1)} \sqcup \cdots \sqcup M_{\rho(k)}$ interchanges the boundaries without incurring change in the topology; its underlying manifold is a disjoint union of cylinders $M_i \times I$.

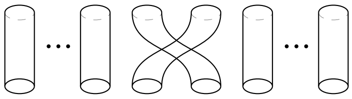
Let $\mathrm{Sym}(k)$ denote the symmetric group of order k, whose elements are permutations of the set $\{1, \dots, k\}$ (i.e., bijective functions $\{1, \dots, k\} \to \{1, \dots, k\}$) and whose operation is composition of these functions. Any permutation cobordism determines a permutation $\rho \in \mathrm{Sym}(k)$: the in-boundaries $M_1, \dots, M_k$ get reordered into the out-boundaries $M_{\rho(1)}, \dots, M_{\rho(k)}$, thus defining a permutation under the identification $M_i \leftrightarrow i$. Conversely, any permutation $\rho \in \mathrm{Sym}(k)$ (along with a choice of connected manifolds $M_1, \dots, M_k$) determines a permutation cobordism

$$\sigma_\rho\colon M_1 \sqcup \cdots \sqcup M_k \to M_{\rho(1)} \sqcup \cdots \sqcup M_{\rho(k)}.$$

This last fact requires proof; it stems from the fact that the symmetric group $\mathrm{Sym}(k)$ is generated by transpositions τ_i, where $\tau_i(j) = j$ for $j \neq i$, $i+1$ and $\tau_i(i) = i+1$, $\tau_i(i+1) = i$. That is, any permutation $\rho \in \mathrm{Sym}(k)$ can be written as the composition of certain transpositions.

[7]Do not confuse the manifolds M_i and N_i for the connected components of M and N, since these need not be connected. See the multiple drawings we have already made.

The permutation cobordisms σ_i realizing these transpositions are twist cobordisms “padded” by cylinders $\mathrm{id}_j := M_j \times I$:



$$\begin{array}{c} (M_1 \sqcup \cdots \sqcup M_{i-1}) \sqcup M_i \sqcup M_{i+1} \sqcup (M_{i+2} \sqcup \cdots \sqcup M_k) \\ \big\downarrow \scriptstyle \sigma_i := (\mathrm{id}_1 \sqcup \cdots \sqcup \mathrm{id}_{i-1}) \sqcup \sigma_{M_i, M_{i+1}} \sqcup (\mathrm{id}_{i+2} \sqcup \cdots \sqcup \mathrm{id}_k) \\ (M_1 \sqcup \cdots \sqcup M_{i-1}) \sqcup M_{i+1} \sqcup M_i \sqcup (M_{i+2} \sqcup \cdots \sqcup M_k). \end{array}$$

Since these *are* permutation cobordisms, and we know that transpositions generate the symmetric group, we can compose these to realize any $\rho \in \mathrm{Sym}(k)$ into a permutation cobordism. Note that every permutation cobordism is invertible, since $\sigma_\rho \circ \sigma_{\rho^{-1}} = \sigma_{\mathrm{id}_{\mathrm{Sym}(k)}}$ is a cylinder!

All of this discussion is related to the fact that symmetric monoidal categories are a *categorification* of the symmetric groups, in a certain sense; cf. Example 1.2.15.

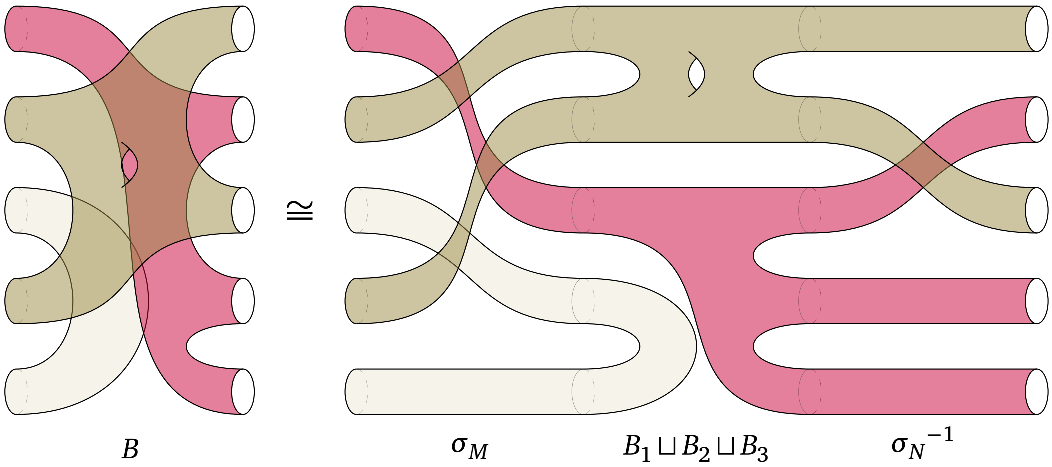


Figure 1.8: Example of Lemma 1.1.23.

As promised, we will now properly relate a cobordism B with the disjoint union of its connected components $\bigsqcup_i B_i$. See figure 1.8 for an example. In the words of [Koc03, Lem. 1.4.19]:

Lemma 1.1.23. Every cobordism factors as a permutation cobordism, followed by a disjoint union of connected cobordisms, followed by a permutation cobordism.

Proof. Let $B\colon M \to N$ be a cobordism, and let us decompose both boundaries into their connected components: $M = \bigsqcup_{p=1}^m M_p$, $N = \bigsqcup_{q=1}^n N_q$. These strict equalities should actually be diffeomorphisms; but since two diffeomorphic manifolds are cobordant through an invertible cobordism, we can proceed without loss of generality. There is a finite number of each, since our manifolds are compact. On the other hand, each connected component B_i of our cobordism B will contain some in-boundaries $\{M_{p_1^i}, \dots, M_{p_{r_i}^i}\}$ and some out-boundaries $\{N_{q_1^i}, \dots, N_{q_{s_i}^i}\}$. This induces partitions on the sets of connected components:

$$M = \bigsqcup_i \Big(\bigsqcup_r M_{p_r^i} \Big), \qquad N = \bigsqcup_i \Big(\bigsqcup_s N_{q_s^i} \Big).$$

Now, consider the permutation $\rho_M \in \mathrm{Sym}(m)$ given by

$$\{1, \dots, m\} \longmapsto \{p_1^1, \dots, p_{r_1}^1, p_1^2, \dots, p_{r_m}^m\}.$$

That is, we group together the indices corresponding to each connected component. Similarly, consider the permutation $\rho_N \in \mathrm{Sym}(n)$ mapping $\{1, \dots, n\}$ into $\{q_1^1, \dots, q_{s_1}^1, q_1^2, \dots, q_{s_n}^n\}$. By the previous discussion, these induce permutation cobordisms

$$\sigma_M\colon \bigsqcup_{p=1}^m M_p \longrightarrow \bigsqcup_i \Big(\bigsqcup_r M_{p_r^i} \Big), \qquad \sigma_N\colon \bigsqcup_{q=1}^n N_q \longrightarrow \bigsqcup_i \Big(\bigsqcup_s N_{q_s^i} \Big).$$

Composing these with B, we obtain

$$\sigma_N \circ B \circ \sigma_M^{-1} = \bigsqcup_i B_i,$$

where each $B_i \colon \bigsqcup_{r=1}^{r_i} M_{p_r^i} \to \bigsqcup_{s=1}^{s_i} N_{q_s^i}$ is a connected cobordism. So, by reversing the permutation cobordisms, we write

$$B = \sigma_N^{-1} \circ \Big(\bigsqcup_i B_i\Big) \circ \sigma_M,$$

thus factorizing B as wanted. □

Our next goal is to finally define the titular topological quantum field theories. After that, in section 1.2, we will properly discuss some of the categorical concepts we only briefly outlined here. But first we take the time to clarify some historical aspects, since the terminology used in the literature varies from author to author.

1.1.3 A quick historical note about nomenclature

We consistently use "cobordism" to refer to what some authors call a "bordism". The history of the terms is an interesting one, which we broadly sketch here. For the following discussion, assume all manifolds unoriented and compact.

First, the concept of *bordism* was born, along with the following problem: when is a given closed manifold the boundary of a bigger one? So a bordism for X was just a manifold (with boundary) Σ such that $\partial\Sigma = X$. ("Bord" is French for boundary, and shares its etymology with the English synonym "border").

Counter-intuitively, not all manifolds admit bordisms: apart from the 0-dimensional point, the first example is the real projective plane $\mathbb{R}\mathrm{P}^2$. If our manifold admits a bordism, we say that it is *null-bordant*.

In order to study the bordism problem, the notion of *cobordism* is introduced: now we have two different closed manifolds M and N of the same dimension, and we want to study whether there is a bigger manifold Σ such that $\partial\Sigma = M \sqcup N$. Note that a cobordism between M and N is just a bordism of $M \sqcup N$. If there exists such a cobordism, we say that our two manifolds are *cobordant*. When dealing with orientations, as the reader might suspect, a cobordism between M and N is a bordism of $\overline{M} \sqcup N$.

The first example of an oriented manifold which does not accept a bordism is in dimension 4, with the complex projective plane $\mathbb{C}\mathrm{P}^2$. As a consequence, $\mathbb{C}\mathrm{P}^2$ is not cobordant to its opposite $\overline{\mathbb{C}\mathrm{P}^2}$. So not every given pair of two non-empty manifolds are cobordant, either.

Here the prefix "co" means "together with" (as in *co*operation, *co*mmunity, etc), and not the usual meaning in mathematics (which is that of a dual notion, as in *co*homology). A lot of authors use both of these terms interchangeably, as one should do when you consider the empty set to be a bona-fide manifold; a null-bordant manifold is just one which is cobordant with $\varnothing$.

We won't cover the specifics here, but the study of the bordism problem saw the development of very interesting mathematics.

Remark 1.1.24 — Solving the unoriented bordism problem.
The relationship of cobordism between manifolds is an equivalence relation, since any manifold is cobordant to itself (by taking the cylinder $M \times I$) and you can glue two cobordisms $\Sigma \colon M \to M'$ and $\Sigma' \colon M' \to M''$ along the common boundary M' to obtain a cobordism $\Sigma'' \colon M \to M''$. So we can try to classify manifolds up to cobordism, and compute *cobordism invariants*. These cobordism classes assemble into a graded ring (Thom's *cobordism ring*), where addition is induced by the *disjoint union* of manifolds and multiplication is induced by the *cartesian product* of manifolds:

$$\Omega_* = \bigoplus_{n\geq 0} \Omega_n, \qquad -\sqcup - \colon \Omega_n \times \Omega_n \to \Omega_n, \qquad -\times - \colon \Omega_n \times \Omega_m \to \Omega_{n+m}.$$

In his PhD thesis [Tho54], Thom proves that the unoriented cobordism ring is isomorphic to a certain infinite polynomial ring with coefficients in $\mathbb{Z}/2\mathbb{Z}$. See [Fre13a, Thm. 1.37]. In particular, the groups Ω_n of homogeneous components of degree n are completely known, so we do in fact have a classification of unoriented manifolds up to cobordism!

It turns out that the bordism problem can be completely characterized in terms of cohomology, by defining a cobordism invariant known as the *Stiefel–Whitney number*; two manifolds M and N are cobordant if and only if the Stiefel–Whitney number of $M \sqcup N$ vanishes. See [Fre13a, Thm. 1.40]. As mentioned, the first non-trivial example of a closed manifold which is not null-bordant is in dimension 2, where $\Omega_2 \simeq \mathbb{Z}/2\mathbb{Z}$: the cobordism class of $\mathbb{R}\mathrm{P}^2$ generates the group.

The oriented version was solved in 1960 [Wal60]. See also [Gwy16] for a modern account. The oriented cobordism ring is usually denoted Ω_*^{SO}; see Remark 1.2.23 (and much later on, section 4.1) for an explanation of this notation.

In any case, the situation in the 1950s was that most authors used both terms interchangeably, and that is still the norm for a number of authors today; notably, [BD95], whose Cobordism Hypothesis we will discuss in chapter 3.

But then, some amount of years later, Atiyah noticed [Ati61] that we can define a notion which is dual to “bordism” in a precise sense, and as such should reserve the term “cobordism” for this new concept! That is *bordism homology* and *cobordism cohomology*, which are deeply interesting topics we unfortunately won’t cover in this text past this short note.

Remark 1.1.25 — A synopsis of bordism homology.
The gist of it is that Thom’s bordism ring can be “upgraded” into a structure known as a *spectrum*, and spectra are in bijection with extraordinary homology theories! A “extraordinary homology theory” is one which satisfies all Eilenberg–Steenrod axioms except Dimension, and it is completely determined by the values it takes on the point. So “bordism” is a homology theory, and its dual *co*homology theory should therefore be called “*co*bordism”.

As mentioned, we will consistently use “cobordism” to refer to what Atiyah calls a bordism — following the style of [Koc03], and in order to reduce confusion when talking about the Cobordism Hypothesis — but the convention varies within the literature. One notable exception is chapter 3, where we use the naming conventions established in [Lur09]: when discussing higher categories of (framed) cobordisms, we denote the n-categorical version as $\mathsf{Cob}_n^{\mathrm{fr}}(n)$ and the (∞, n)-categorical version by $\mathsf{Bord}_n^{\mathrm{fr}}$.[8] As far as we have been able to infer from the text, this seems to be an arbitrary choice without deeper meaning.

Note that this short note is not at all comprehensive — any reader familiar with the history will no doubt notice Pontrjagrin’s absence, among many other glaring omissions. For more information on (co)bordism theory, we recommend [Fre13a; Rud98].

[8] We do not follow Lurie’s conventions strictly, however; the meaning of $\mathsf{Cob}_n(m)$ differs from that in [Lur09].

1.2 Topological quantum field theories

Finally, we get to our main object of study: topological quantum field theories. First we will briefly explore — from afar — some of the quantum physics which birthed the concept. Then, in section 1.2.2, we will properly examine some of the categorical notions we already touched on in prior sections; namely, the definition of rigid symmetric monoidal categories.

1.2.1 The axioms of a TQFT

The quickest way to define the concept of a TQFT, as we will see in section 1.2.2, is as a symmetric monoidal functor from the category Cob_n of n-cobordisms (with monoid "disjoint union" $\sqcup$) to the category $\mathsf{Vect}_{\Bbbk}$ of $\Bbbk$-vector spaces (with monoid "tensor product" $\otimes$). But while that is succinct (and dare I say, very elegant), it requires quite a bit of unraveling in order to get to the essence. To shed some light upon the concept, we will begin with the classical Atiyah axioms — which still retain a lot of flavor from the theoretical physics origins of this theory — and then see that these are fully equivalent to the functorial definition we just sketched.

Let's do a brief summary of the quantum mechanics motivation, then. The scope of this analysis will be limited, to prevent having to apologize to any physicists in the audience, but we can at least scratch some of the surface.

A quick rundown of the physical viewpoint

We think of *spaces* (physical systems) as compact smooth oriented $(n-1)$-manifolds, and of *spacetimes* (that is, the evolution of a system) as a (compact, smooth, oriented) n-cobordism between two such manifolds. We regard the additional dimension the cobordism has as a time dimension. Since it lacks a metric, we can't measure time; but we have a notion of "past" space and "future" space, and (if we equip our cobordisms with a Morse function) we can take slices in order to watch the intermediate stages. Once again, remember the "film interpretation" of figure 1.2 — even without a metric, we can choose a "time" direction. So we can think of particles *birthing* and *dying*, and *splitting* or *colliding*. These processes are often described by Feynman diagrams, as in table 1.1.

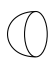
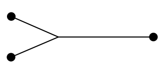
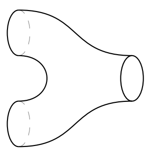
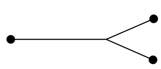
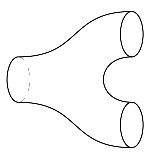
Principle	Feynman diagram	2D cobordism	Algebraic operation	
creation			unit	$\Bbbk \to A$
annihilation			counit	$A \to \Bbbk$
merging			multiplication	$A \otimes A \to A$
splitting			comultiplication	$A \to A \otimes A$

Table 1.1: Dictionary between physics, topology and algebra [Koc03, p. 4].

States of quantum systems are modelled as vectors in a Hilbert space — a particular kind of vector space. *Processes* are modelled as (bounded) linear operators from a Hilbert space to another. For simplicity, we will consider regular vector spaces and linear maps between them.[1]

A quantum field theory assigns to each space Σ a vector space V (of states), and to each spacetime $M\colon \Sigma \to \Sigma'$ a linear map (or process) $\varphi\colon V \to V'$ between the vector spaces corresponding to the "past" and "future" spaces. This is sometimes known as the *Schrödinger picture* of quantum mechanics: state vectors propagate (or evolve) through time, but the observables are held fixed. That is, we always associate the same vector space to each manifold, and we always associate the same linear operator to each cobordism. That assignment will need to satisfy certain niceness conditions, which model our physical intuition. So far, we have described a *functorial* quantum field theory, or FQFT.

Since we don't care about additional metric structures, and care only about global changes in the topology of our system, this is a *topological* quantum field theory; TQFT, for short. In physics terminology, this is a quantum field theory without *local degrees of freedom*.

The first axiomatization of TQFTs was formulated by Atiyah in 1988 [Ati88], inspired by Segal's prior axiomatization of *conformal* quantum field theories.[2] We state a slight variation of Atiyah's axioms — this is the version found in [Koc03, 1.2.23], to which we have added an extra axiom A6 for completeness.[3]

Definition 1.2.1: Atiyah's Axiomatization.
An n-dimensional ***topological quantum field theory*** (TQFT) consists of the following data:

- For every closed $(n-1)$-manifold M, a $\Bbbk$-vector space $Z(M)$;
- For every n-cobordism $B\colon M \to N$, a $\Bbbk$-linear map $Z(B)\colon Z(M) \to Z(N)$.

This data is subject to the following axioms:

A1. Two isomorphic cobordisms have the same image:

$$B \cong B' \text{ implies } Z(B) = Z(B').$$

A2. The cylinder cobordism $M \times I\colon M \to M$ is mapped to the identity $\mathrm{id}_{Z(M)}\colon Z(M) \to Z(M)$:

$$Z(M \times I) = \mathrm{id}_{Z(M)}.$$

A3. Gluing of cobordisms is mapped to composition of linear maps:

$$Z(B' \circ B) = Z(B') \circ Z(B).$$

A4. Disjoint union goes to tensor product, in both manifolds and cobordisms:

$$Z(M \sqcup N) = Z(M) \otimes Z(N), \qquad Z(B \sqcup B') = Z(B) \otimes Z(B').$$

A5. The empty manifold goes to the ground field (and the empty cobordism to its identity):

$$Z(\varnothing) = \Bbbk, \qquad Z(\varnothing \times I) = \mathrm{id}_{\Bbbk}.$$

A6. The twist cobordism goes to the twist map of vector spaces:

$$Z(\sigma_{M,N}) = \tau_{Z(M),Z(N)}.$$

[1] Actually, the axioms for a TQFT will imply that the Hilbert spaces associated to manifolds are of finite dimension. Since every finite-dimensional vector space accepts an inner product, and is complete with respect to that inner product, this simplification doesn't lose generality.

[2] Segal's axiomatiozation of CQFTs remained unpublished until 2004; see [Seg04].

[3] Axiom A6 does not appear in either formulation, but it is necessary. See [Koc03, 1.3.32, 3.3.3].

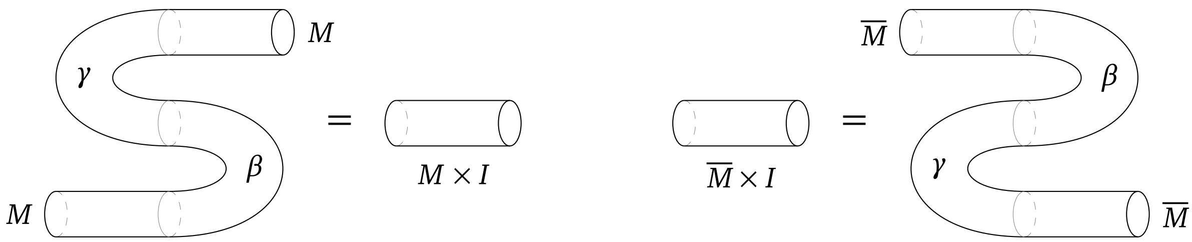
Mathematically, the first three axioms encode functoriality, the following two encode monoidality, and the last one encodes symmetry.

Physically, the first two axioms mean that the theory is *topological*, or that it *lacks local degrees of freedom*: it does not depend on further structure such as a Riemannian metric or a symplectic structure, but only on the shape of the spacetimes. The fourth axiom imposes the tensor product postulate of quantum mechanics: the state space of two independent systems is the tensor product of the individual state spaces.

We shall speak no more of physics from here on out; for the remainder of this thesis, we will work with TQFTs as pure mathematical constructs. We will soon see some examples.

The following construction is very important; even though it may appear so simple that it could qualify as "trivial", it has very deep implications in the theory. We will revisit it multiple times throughout the text, with each time marking our descend into further depths. Ultimately, in section 3.2.2, it will form the backbone for the notion of *fully dualizable object*, needed in order to state the *Cobordism Hypothesis*. But that is far in the future! Let us focus on the matter at hand.

Zorro's Lemma



(a) The snake decomposition of a cylinder. (b) Decomposition of the opposite cylinder.

Figure 1.9: Zorro's Lemma.

Example 1.2.2 — Snake decomposition.
Consider an $(n-1)$-manifold $M \in \mathsf{Cob}_n$. The cylinder $M \times I$ can be decomposed topologically as in figure 1.9(a): namely,

$$M = M \sqcup \varnothing \xrightarrow{\mathrm{id} \sqcup \gamma} M \sqcup \overline{M} \sqcup M \xrightarrow{\beta \sqcup \mathrm{id}} \varnothing \sqcup M = M,$$

where id_M is the cylinder $M \times I$ while $\gamma \colon \varnothing \to \overline{M} \sqcup M$ and $\beta \colon M \sqcup \overline{M} \to \varnothing$ are the pieces which bend to the side. Of course, these pieces are diffeomorphic (as manifolds) to a cylinder $M \times I$, but they are not isomorphic as cobordisms; remember Remarks 1.1.3 and 1.1.14. In conclusion,

$$\mathrm{id}_M \cong (\beta \sqcup \mathrm{id}_M) \circ (\mathrm{id}_M \sqcup \gamma).$$

Both halves represent strictly the same morphism in our category, since we collapsed each isomorphism class. Meanwhile, the equation of figure 1.9(b) says

$$\mathrm{id}_{\overline{M}} \cong (\mathrm{id}_{\overline{M}} \sqcup \beta) \circ (\gamma \sqcup \mathrm{id}_{\overline{M}}) \colon \overline{M} \xrightarrow{\gamma \sqcup \mathrm{id}} \overline{M} \sqcup M \sqcup \overline{M} \xrightarrow{\mathrm{id} \sqcup \beta} \overline{M}.$$

Let's evaluate our TQFT Z in these equations! We denote $V := Z(M)$ and $W := Z(\overline{M})$. For the cylinders, by Atiyah's axiom A2 we have $Z(\mathrm{id}_M) = \mathrm{id}_V$ and $Z(\mathrm{id}_{\overline{M}}) = \mathrm{id}_W$. Now, disjoint union goes to tensor product (axiom A4) while the empty manifold goes to the ground field (axiom A5), so β and γ map to $\mathrm{ev} \colon V \otimes W \to \Bbbk$ and $\mathrm{coev} \colon \Bbbk \to W \otimes V$ respectively. By axiom A1, any two isomorphic cobordisms get mapped to the same linear morphism. And by axiom A3, gluing cobordisms is the same as composing linear maps. So, in conclusion, we have:

$$\begin{aligned} \mathrm{id}_V &= (\mathrm{ev} \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \mathrm{coev}) \colon V \cong V \otimes \Bbbk \xrightarrow{\mathrm{id} \otimes \mathrm{coev}} V \otimes W \otimes V \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} \Bbbk \otimes V \cong V, \\ \mathrm{id}_W &= (\mathrm{id}_W \otimes \mathrm{ev}) \circ (\mathrm{coev} \otimes \mathrm{id}_W) \colon W \cong \Bbbk \otimes W \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} W \otimes V \otimes W \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} W \otimes \Bbbk \cong W. \end{aligned} \tag{1.1}$$

But these are two very constraining identities. In particular, they force the target vector spaces to have finite dimension! Let's state it properly.

Lemma 1.2.3: Zorro's Lemma. The image vector spaces of a TQFT are of finite dimension.

Proof. Let M be an $(n-1)$-manifold and $Z\colon \mathsf{Bord}_n \to \mathsf{Vect}_{\Bbbk}$ be a TQFT. Consider the images $V := Z(M)$ and $W := Z(\overline{M})$. By applying the snake decomposition [Ex. 1.2.2] to the cylinder $M \times I$ and evaluating our TQFT, we define linear maps $\mathrm{ev}\colon V \otimes W \to \Bbbk$ and $\mathrm{coev}\colon \Bbbk \to W \otimes V$.

Notice that coev is determined by a single vector in the codomain $W \otimes V$, namely the image of the unit $1 \in \Bbbk$. We write $\mathrm{coev}(1) = \sum_{i=1}^n w_i \otimes v_i$, where $w_i \in W$ and $v_i \in V$; it is always possible to write an element of $W \otimes V$ as a finite sum of this form. Now, we evaluate our composition $W \to W \otimes V \otimes W \to W$ on an arbitrary element $w \in W$:

$$w \longmapsto \sum_{i=1}^n w_i \otimes v_i \otimes w \longmapsto \sum_{i=1}^n \mathrm{ev}(v_i \otimes w) \cdot w_i.$$

Note that we have used the distributivity of the tensor product over sums, and that each $\mathrm{ev}(v_i \otimes w)$ is a scalar of $\Bbbk$. We also identify $1 \otimes w \equiv w \equiv w \otimes 1$, as usual.

This composition is the identity map on W, by eq. (1.1). So, for every $w \in W$, we can write $w = \sum_{i=1}^n \mathrm{ev}(v_i \otimes w) \cdot w_i$. In particular, this means that $\{w_1, \dots, w_n\}$ is a spanning set for W. Therefore, W is a vector space of finite dimension.

For finite-dimensionality of V, repeat the same process with the manifold $\overline{M}$, since $\overline{\overline{M}} = M$. $\square$

This argument is known as *Zorro's Lemma*, since it is proven by carving a big Z into the paper: that of figure 1.9(b)! This is an instance of the concept of *duality* in category theory: two objects are said to be duals if there exist maps which compose just like γ and β (or ev and coev) do [Def. 1.2.19]. Think, for example, of finite-dimensional vector spaces. So the same argument tells us that every object $M \in \mathsf{Cob}_n$ has a dual, namely $\overline{M}$! That is the core essence of Zorro's Lemma.

The map ev is called a *pairing*, while coev is a *copairing* [Defs. 2.2.8 and 2.2.9]. The compatibility condition of eq. (1.1) is called *non-degeneracy* [Def. 2.2.10].

We also deduce another big structural constraint for TQFTs: we can canonically identify $W = Z(\overline{M})$ with the dual space of $V = Z(M)$. This, too, is often called Zorro's Lemma.

Corollary 1.2.4. Let M be a closed $(n-1)$-manifold and $Z\colon \mathsf{Cob}_n \to \mathsf{Vect}_{\Bbbk}$ a TQFT. Let $V = Z(M)$ and $W = Z(\overline{M})$ be the image vector spaces of M and its opposite. Then:

(i) W is canonically isomorphic to V^* through the map $\mathrm{ev}_{\mathrm{left}}\colon W \to V^*$ mapping $w \mapsto \mathrm{ev}(- \otimes w)$.

(ii) V is canonically isomorphic to W^* through the map $\mathrm{ev}_{\mathrm{right}}\colon V \to W^*$ mapping $v \mapsto \mathrm{ev}(v \otimes -)$.

Proof. Let $\mathrm{coev}(1) = \sum_{i=1}^n w_i \otimes v_i$, as in the prior proof. Consider the linear map $\mathrm{ev}_{\mathrm{left}}\colon W \to V^*$ given by $w \mapsto \mathrm{ev}(- \otimes w)$. Assume $\mathrm{ev}(- \otimes w)$ is the zero functional of V^*. Then, for each v_i we have $\mathrm{ev}(v_i \otimes w) = 0$. But remember that we can write $w = \sum_{i=1}^n \mathrm{ev}(v_i \otimes w) \cdot w_i$, so necessarily $w = 0$. In conclusion, $\mathrm{ev}_{\mathrm{left}}$ is injective.

Similarly for the linear map $\mathrm{ev}_{\mathrm{right}}\colon V \to W^*$ given by $v \mapsto \mathrm{ev}(v \otimes -)$: if $\mathrm{ev}(v \otimes -) = 0 \in W^*$, then $\mathrm{ev}(v \otimes w_i) = 0$ for each w_i; and since $v = \sum_{i=1}^n \mathrm{ev}(v \otimes w_i) \cdot v_i$, we have $v = 0$ and $\mathrm{ev}_{\mathrm{right}}$ is injective. The identity $v = \sum_{i=1}^n \mathrm{ev}(v \otimes w_i) \cdot v_i$ follows from the argument dual to that constructed in the proof of Lemma 1.2.3, by considering the composition $V \to V \otimes W \otimes V \to V$ of eq. (1.1).

We have linear injections $\mathrm{ev}_{\mathrm{left}}\colon W \hookrightarrow V^*$ and $\mathrm{ev}_{\mathrm{right}}\colon V \hookrightarrow W^*$, so $\dim V^* = \dim V \geq W$ and $\dim W^* = \dim W \geq V$. Each of the four vector spaces has the same dimension, which must be finite by Lemma 1.2.3. Therefore, the two injections $\mathrm{ev}_{\mathrm{left}}$ and $\mathrm{ev}_{\mathrm{right}}$ are isomorphisms. $\square$

The two maps $\mathrm{ev}_{\mathrm{left}}$ and $\mathrm{ev}_{\mathrm{right}}$ are duals: $\mathrm{ev}_{\mathrm{right}} = \mathrm{ev}_{\mathrm{left}}^* \circ \varphi_V$, where $\varphi_V\colon V \to V^{**}$ is the canonical identification of finite-dimensional vector spaces given by $v \mapsto [\mathrm{ev}_v\colon \Lambda \mapsto \Lambda(v)]$. The dual map $\mathrm{ev}_{\mathrm{left}}^*\colon V^{**} \to W^*$ is given by $T \mapsto T \circ \mathrm{ev}_{\mathrm{left}}\colon W \to V^* \to \Bbbk$. Of course, as one can quickly check, we also have $\mathrm{ev}_{\mathrm{left}} = \mathrm{ev}_{\mathrm{right}}^* \circ \varphi_W$.

Now, on the promised examples!

Examples of TQFTs

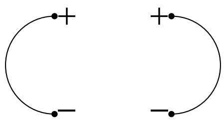
The first example is not very exciting.

Example 1.2.5 — The trivial TQFT.
Let $\mathbb{k}$ be a field. Then, the assignment $Z(M) = \mathbb{k}$ and $Z(B) = \mathrm{id}_{\mathbb{k}}$ defines a TQFT, for any $(n-1)$-dimensional closed manifolds M and any n-cobordism $B\colon M \to N$. This is the *trivial* or *identity* n-dimensional TQFT.

In dimension 1, TQFTs are very simple. We will see this in more detail in section 1.3.

Example 1.2.6 — 1D TQFTs are finite-dimensional vector spaces.
Let V be a finite-dimensional $\mathbb{k}$-vector space. We define the 1D oriented TQFT $Z\colon \mathsf{Cob}_1 \to \mathsf{Vect}_{\mathbb{k}}$ given by $Z(\mathrm{pt}_+) = V$, where pt_+ is the positively oriented point. It turns out that this determines the rest of the values, and we in fact have a correspondence between 1D oriented TQFTs and finite-dimensional vector spaces.

As we will see, that is because Cob_1 is (symmetric monoidally) generated by the *arcs*:



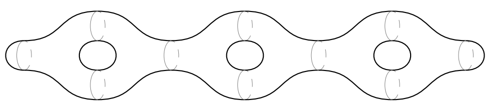
By Zorro's Lemma [fig. 1.9], the value of our TQFT Z on these two arcs is determined by the value on the point. Every other 1-cobordism can be obtained by gluing and taking disjoint union of the two arcs and the permutation cobordisms — and Atiyah's axioms let us determine the value of our TQFT after these operations.

The following 2-dimensional TQFT is used in the definition of *Khovanov homology*, a knot invariant "categorifying" the Jones polynomial and the Kauffman bracket. See [Kho00; Bar02].

Example 1.2.7 — A TQFT from Khovanov homology.
Let $V = \langle v_+, v_- \rangle$ be a $\mathbb{k}$-vector space on two generators. We can define a 2D oriented TQFT $Z\colon \mathsf{Cob}_2 \to \mathsf{Vect}_{\mathbb{k}}$ by defining the following values.

$$
\begin{aligned}
&Z(\mathbb{S}^1) = V. \\
&\quad Z(\mathbb{O})\colon \mathbb{k} \to V, && 1 \mapsto v_+. \\
&\quad Z(\mathbb{O})\colon V \to \mathbb{k}, && \begin{cases} v_+ \mapsto 0, \\ v_- \mapsto 1. \end{cases} \\
&\quad Z(\text{⊃})\colon V \otimes V \to V, && \begin{cases} v_+ \otimes v_- \mapsto v_-, \quad v_+ \otimes v_+ \mapsto v_+, \\ v_- \otimes v_+ \mapsto v_-, \quad v_- \otimes v_- \mapsto 0. \end{cases} \\
&\quad Z(\text{⊂})\colon V \to V \otimes V, && \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+, \\ v_- \mapsto v_- \otimes v_-. \end{cases}
\end{aligned}
$$

It turns out that this is enough to determine the totality of the TQFT, as we will prove in Theorem 2.3.3. Furthermore, we can compute its value on a closed surface of genus g. First, we cut it into pairs of pants, depicted here for $g = 3$:



Now, we use Atiyah's axiom A3 (compositionality) to compute the value. This will be

$$Z(\mathbb{O}) \circ \big(Z(\text{⊃}) \circ Z(\text{⊂})\big)^g \circ Z(\mathbb{O}),$$

which is a linear function $\Bbbk \to \Bbbk$. These are given by multiplication with an element of $\Bbbk$, so we can identify the set of $\Bbbk$-linear functions $\Bbbk \to \Bbbk$ with the set of elements of $\Bbbk$.

We compute the composite $Z(\succ) \circ Z(\prec)$; this is a linear map $V \to V$. First, the image of v_+:

$$v_+ \longmapsto v_+ \otimes v_- + v_- \otimes v_+ \longmapsto v_- + v_- = 2v_-.$$

And now, the image of v_-:

$$v_- \longmapsto v_- \otimes v_- \longmapsto 0.$$

Let's call this map $f = Z(\succ) \circ Z(\prec)$. So, if we iterate, the value of the composition becomes 0. In conclusion,

$$\big(Z(\succ) \circ Z(\prec)\big)^g = \begin{cases} \mathrm{id}, & g = 0; \\ f, & g = 1; \\ 0, & g \geq 2. \end{cases}$$

Now, precompose by $Z(\mathbb{C})$ and postcompose by $Z(\mathbb{D})$:

$$1 \longmapsto v_+ \longmapsto \begin{cases} v_+, & g = 0; \\ 2v_-, & g = 1; \\ 0, & g \geq 2. \end{cases} \longmapsto \begin{cases} 0, & g = 0; \\ 2, & g = 1; \\ 0, & g \geq 2. \end{cases}$$

So, assuming that $2 \neq 0$ in $\Bbbk$, this TQFT distinguishes the torus from every other compact oriented surface! Not a very interesting TQFT, when put it that way, but useful in practice.

Generally, although the definition of a TQFT requires specifying a very big amount of information, they are often fully determined by a very small amount of data. This is analogous to how (e.g.) a group homomorphism is completely determined by the image of a generating set for the domain.

By modifying Example 1.2.7, we can get a TQFT which computes the genus of our surface.

Example 1.2.8 — A TQFT which computes the genus.
Let us consider $V = \langle v_+, v_- \rangle$ again. This time, we define a TQFT as follows.

$$\begin{aligned} Z(\mathbb{S}^1) &= V. \\ Z(\mathbb{C})\colon \Bbbk \to V, &\qquad 1 \mapsto v_+. \\ Z(\mathbb{D})\colon V \to \Bbbk, &\qquad \begin{cases} v_+ \mapsto 1, \\ v_- \mapsto 0. \end{cases} \\ Z(\succ)\colon V \otimes V \to V, &\qquad \begin{cases} v_+ \otimes v_- \mapsto v_-, & v_+ \otimes v_+ \mapsto v_+, \\ v_- \otimes v_+ \mapsto v_-, & v_- \otimes v_- \mapsto v_+. \end{cases} \\ Z(\prec)\colon V \to V \otimes V, &\qquad \begin{cases} v_+ \mapsto v_+ \otimes v_+ + v_- \otimes v_-, \\ v_- \mapsto v_- \otimes v_+ + v_+ \otimes v_-. \end{cases} \end{aligned}$$

Again, let's compute the composite $f = Z(\succ) \circ Z(\prec)$. First, v_+:

$$v_+ \longmapsto v_+ \otimes v_+ + v_- \otimes v_- \longmapsto v_+ + v_+ = 2v_+.$$

And now, v_-:

$$v_- \longmapsto v_- \otimes v_+ + v_+ \otimes v_- \longmapsto v_- + v_- = 2v_-.$$

So now, our composite f is multiplication by two! Therefore, the gth iteration is multiplication by 2^g. Composing with the two caps, we can see that this TQFT assigns to a surface the invariant $2^g \in \Bbbk$, where g is its genus:

$$1 \longmapsto v_+ \longmapsto 2^g v_+ \longmapsto 2^g.$$

So, if our base field $\Bbbk$ has characteristic zero, this TQFT precisely detects the genus of our surfaces.

Examples 1.2.7 and 1.2.8 are relabellings of [Koc03, Exers. 3.3.2, 3.3.3] for the case $n=2$, as we will see in Example 2.2.27. As the dimension of our manifolds increases, it gets harder to give explicit examples of TQFTs. So we will stop here.

Our next short-order goal, as we promised, is to unravel the functorial definition of a TQFT. We need quite a bit of definitions, first.

1.2.2 Rigid symmetric monoidal categories

As we already mentioned, we can give the following succinct definition:

Definition 1.2.9. A TQFT is a symmetric monoidal functor $Z\colon \mathsf{Cob}_n \to \mathsf{Vect}_{\Bbbk}$.

Since we already assembled our cobordisms into a category [Def. 1.1.21], we know what a functor from Cob_n to $\mathsf{Vect}_{\Bbbk}$ is. But what does it mean for a functor to be monoidal and symmetric?

Monoidal categories

Firstly, we shall provide precise definitions of *monoidal* and *symmetric monoidal* categories, since we only sketched them before. Let $\mathbb{1}=\{*\}$ denote the category with a single object and no arrows other than the identity. Note that specifying a functor $F\colon \mathbb{1}\to\mathcal{C}$ is the same as selecting an object $F(*)\in\mathcal{C}$ (but has the advantage of not referring to *objects* directly, only to *morphisms*).

Definition 1.2.10. A (strict) ***monoidal category*** is a category $\mathcal{V}$ equipped with two functors $\mu\colon \mathcal{V}\times\mathcal{V}\to\mathcal{V}$ and $\eta\colon \mathbb{1}\to\mathcal{V}$ such that the following two diagrams both commute.

$$\begin{tikzcd}[column sep=small] & \mathcal{V}\times\mathcal{V}\times\mathcal{V} \arrow[dl, "\mu\times\mathrm{id}_{\mathcal{V}}"'] \arrow[dr, "\mathrm{id}_{\mathcal{V}}\times\mu"] & \\ \mathcal{V}\times\mathcal{V} \arrow[dr, "\mu"'] & & \mathcal{V}\times\mathcal{V} \arrow[dl, "\mu"] \\ & \mathcal{V} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{V}\times\mathcal{V} \arrow[dr, "\mu"] & & \mathcal{V}\times\mathcal{V} \arrow[dl, "\mu"'] \\ \mathbb{1}\times\mathcal{V} \arrow[u, "\eta\times\mathrm{id}_{\mathcal{V}}"] \arrow[r] & \mathcal{V} & \mathcal{V}\times\mathbb{1} \arrow[u, "\mathrm{id}_{\mathcal{V}}\times\eta"'] \arrow[l] \end{tikzcd}$$

(associativity) (unit laws).

We will usually use infix notation for the functor μ: we write $X\star Y := \mu(X,Y)$. Likewise, we write $\mathrm{I} := \eta(*)$ for the neutral object (the image under η of the single object $*\in\mathbb{1}$).

Why use commutative diagrams, instead of a traditional elementwise definition? Well, for one, this definition is portable: if we change Cat to Set (and functors to functions, and $\mathbb{1}$ to $\{*\}$), we recover the definition of a *monoid*: a set equipped with a binary operation which is unital and associative. This leads right into the definition of *internal monoid* in a monoidal category, which we let the reader deduce for themselves. But there are other benefits, and one particularly interesting one is that a functorial definition makes further generalizations easier. For instance, it lets us *weaken* our definition while still capturing the essence.

In any case, that is what a monoidal category is. As the reader might suspect, a ***monoidal functor*** is one which respects this monoidal structure: $F\colon (\mathcal{C},\star)\to(\mathcal{D},\diamond)$ must satisfy $F(X\star Y) = F(X)\diamond F(Y)$. So we are requiring the commutativity of this diagram:

$$\begin{tikzcd} \mathcal{V}\times\mathcal{V} \arrow[r, "-\star-"] \arrow[d, "F\times F"'] & \mathcal{V} \arrow[d, "F"] \\ \mathcal{W}\times\mathcal{W} \arrow[r, "-\diamond-"'] & \mathcal{W} \end{tikzcd}$$

Now we must define what "symmetric monoidal" means. But before that, a bit of an aside on the aforementioned slightly weaker notion, to sate the inquisitive reader.

Weak monoidal categories

This weaker notion is actually at the correct level of generality for our purposes, but we can also ignore it altogether thanks to the powerful General Theory of Abstract Nonsense [Thm. 1.2.13]! But we won't ignore it forever: it forms a precedent for the discussion that will constitute most of chapter 3. Essentially, it is the same as definition 1.2.10, changing each instance of "equality of functors" (i.e., the commutativity of each diagram) with suitable natural equivalences between those functors. These natural equivalences must themselves satisfy some extra *coherence relations*.

Definition 1.2.11. A ***weak monoidal category*** is a sextuple $(\mathcal{V},\mu,\eta,\alpha,\lambda,\rho)$ where:

- $\mathcal{V}$ is a category.
- $\mu\colon \mathcal{V}\times\mathcal{V}\to\mathcal{V}$ and $\eta\colon \mathbb{1}\to\mathcal{V}$ are functors.
- α, λ and ρ are natural equivalences, with domain and codomain given by

$$\begin{tikzcd}
 & \mathcal{V}\times\mathcal{V}\times\mathcal{V} \arrow[dl, "\mu\times\mathrm{id}_{\mathcal{V}}"'] \arrow[dr, "\mathrm{id}_{\mathcal{V}}\times\mu"] & \\
\mathcal{V}\times\mathcal{V} \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, "\mu"'] & & \mathcal{V}\times\mathcal{V} \arrow[dl, "\mu"] \\
 & \mathcal{V} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{V}\times\mathcal{V} \arrow[dr, "\mu"] \arrow[d, phantom, "\Downarrow\lambda"] & & \mathcal{V}\times\mathcal{V} \arrow[dl, "\mu"'] \arrow[d, phantom, "\Downarrow\rho"] \\
\mathbb{1}\times\mathcal{V} \arrow[u, "\eta\times\mathrm{id}_{\mathcal{V}}"] \arrow[r, "\pi_2"'] & \mathcal{V} & \mathcal{V}\times\mathbb{1} \arrow[u, "\mathrm{id}_{\mathcal{V}}\times\eta"'] \arrow[l, "\pi_1"]
\end{tikzcd}$$

$$\big(\alpha\colon \mu\circ(\mu\times\mathrm{id}_{\mathcal{V}})\Longrightarrow\mu\circ(\mathrm{id}_{\mathcal{V}}\times\mu)\big)
\qquad
\begin{pmatrix}\lambda\colon \mu\circ(\eta\times\mathrm{id}_{\mathcal{V}})\Longrightarrow\pi_2\\ \rho\colon \mu\circ(\mathrm{id}_{\mathcal{V}}\times\eta)\Longrightarrow\pi_1\end{pmatrix}.$$

This data must satisfy certain coherence constraints: for all objects A, B, C, $D\in\mathcal{V}$, the following diagrams must commute.

$$\begin{tikzcd}
 & (A\star B)\star(C\star D) \arrow[dr, "\alpha"] & \\
((A\star B)\star C)\star D \arrow[ur, "\alpha"] \arrow[d, "\alpha\star\mathrm{id}_D"'] & & A\star(B\star(C\star D)) \\
(A\star(B\star C))\star D \arrow[rr, "\alpha"'] & & A\star((B\star C)\star D) \arrow[u, "\mathrm{id}_A\star\alpha"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(A\star \mathrm{I})\star B \arrow[rr, "\alpha"] \arrow[dr, "\rho_A\star\mathrm{id}_B"'] & & A\star(\mathrm{I}\star B) \arrow[dl, "\mathrm{id}_A\star\lambda_B"] \\
 & A\star B &
\end{tikzcd}$$

(pentagon identity) (triangle identity).

The natural transformation α is called the *associator*, while λ and ρ are called the *left unitor* and *right unitor*. Of course, these names reference the diagrams which they make commute. We can consider a strict monoidal category to be a weak monoidal category where all these structural natural transformations are identities. Note how we have weakened *properties* (the fact that the three diagrams of Definition 1.2.10 commute strictly) into *structure* (natural transformations *making* these diagrams commute)! This is a common pattern in category theory.

Monoidal categories are our first encounter with the weakening of a prior categorical notion, other than the very short foray made in Remarks 1.1.16 and 1.1.17. These kinds of weakenings are plentiful in the theory: sometimes, the axioms we might want a category to satisfy are too strict for any reasonable example to comply. That is to say, the categories one finds "in the wild" which should perfectly exemplify our notion are off, if only by a little bit, so the definitions must be weakened in order to include them. For example, $(\mathsf{Vect}_{\mathbb{k}},\otimes,\mathbb{k})$ — the category of vector spaces equipped with tensor product — is *not* strictly monoidal: $V\otimes\mathbb{k}$ and V are different objects, although coherently isomorphic nonetheless. Remember, this is our main motivating example! So the need for such weaker notions should be clear.

Monoidal categories are also our first encounter with higher categorical concepts, even in their strict version, but that is not so obvious at first glance. Wait until the Interlude for the big twist!

Weak monoidal categories are often called *monoidal categories*, without qualifiers. This might seem confusing, but most of the time there is no harm in pretending a (weak) monoidal category is strictly monoidal — as long as we don't insist in monoidal *functors* also being strict. In fact, we can make this precise, through Mac Lane's Coherence and Strictification Theorems. These particular statements are quoted verbatim from [Koc03, 3.2.19, 3.2.18], and proofs can be found in [Mac78, §§ VII.2, XII.3].

Theorem 1.2.12: Mac Lane's Coherence Theorem. Let $(\mathcal{C},\star,\mathrm{I},\alpha,\lambda,\rho)$ be a monoidal category. Every diagram that can be built out of the components of α, λ, ρ, and identity maps, using composition and monoidal operations, commutes.

So it does not matter in which order we apply the coherence isomorphisms, and we often ignore them altogether without incurring loss of generality. Even more:

Theorem 1.2.13: Strictification Theorem. Every monoidal category is monoidally equivalent to a strict monoidal category.

A (weak) monoidal natural equivalence, as the reader probably suspects, is a natural equivalence between (weak) monoidal functors which respects the (weak) monoidal structure. We haven't defined most of these terms, and neither will we define them later; they are not really relevant for our purposes. One can consult [Mac78, § XI.2].

The main takeaway from the Strictification Theorem is that we can treat weak monoidal categories as if they were strictly monoidal, without loss of generality. As such, we will not mention weak monoidal categories for most of our discussion, and act as if all our monoidal categories are strict (even when, we have already seen, they are not).

It would be nice if all weakened categorical concepts satisfied such powerful coherence theorems, but alas, that road only leads to disappointment.

Let us get back to the matter at hand. What is a symmetric monoidal category?

Symmetric monoidal categories

Given any two categories $\mathcal{C}$, $\mathcal{D}$, define the ***twist functor*** $\mathrm{twist}_{\mathcal{C},\mathcal{D}}\colon \mathcal{C}\times\mathcal{D}\to\mathcal{D}\times\mathcal{C}$ as the functor that interchanges the order of each pair; i.e., $(X,Y)\mapsto(Y,X)$, and $(F,G)\mapsto(G,F)$.

Definition 1.2.14. A ***symmetric monoidal category*** is a (strict) monoidal category $(\mathcal{V},\mu,\eta)$ equipped, for each pair of objects X, $Y\in\mathcal{V}$, with a *twist map* $\tau\colon X\star Y\to Y\star X$. These twist maps must satisfy the following axioms:

(i) The twist maps are the components of a natural transformation $\tau\colon \mu\Rightarrow\mu\circ\mathrm{twist}_{\mathcal{V},\mathcal{V}}$:

$$\begin{tikzcd} \mathcal{V}\times\mathcal{V} \arrow[r, "\mathrm{twist}_{\mathcal{V},\mathcal{V}}"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\overset{\tau}{\Nearrow}"] & \mathcal{V}\times\mathcal{V} \arrow[d, "\mu"] \\ \mathcal{V} \arrow[r, equal] & \mathcal{V} \end{tikzcd}$$

(ii) For each triple of objects X, Y, $Z\in\mathcal{V}$, the following diagrams commute.

$$\begin{tikzcd} X\star Y\star Z \arrow[rr, "\tau_{X,Y\star Z}"] \arrow[dr, "\tau_{X,Y}\star\mathrm{id}_Z"'] & & Y\star Z\star X \\ & Y\star X\star Z \arrow[ur, "\mathrm{id}_Y\star\tau_{X,Z}"'] & \end{tikzcd} \qquad \begin{tikzcd} X\star Y\star Z \arrow[rr, "\tau_{X\star Y,Z}"] \arrow[dr, "\mathrm{id}_X\star\tau_{Y,Z}"'] & & Z\star X\star Y \\ & X\star Z\star Y \arrow[ur, "\tau_{X,Z}\star\mathrm{id}_Y"'] & \end{tikzcd}$$

(iii) For each pair of objects X, $Y\in\mathcal{V}$, we have $\tau_{Y,X}\circ\tau_{X,Y}=\mathrm{id}_{X\star Y}$.

Essentially, we can treat twist maps as if they were elements of the symmetric group. In this analogy, the monoidal operation corresponds to concatenation: an element $\tau \in \mathrm{Sym}(n)$ plus an element $\sigma \in \mathrm{Sym}(m)$ gives us an element $\tau \oplus \sigma \in \mathrm{Sym}(n+m)$. In fact, these groups assemble into a category Sym, which perfectly exemplifies the core structure of symmetric monoidal categories! We can make some pretty pictures, too.

Example 1.2.15.
The ***symmetric groupoid*** Sym is the category described as follows.

- **Objects:** Sets of natural numbers $[\boldsymbol{n}] := \{0, \dots, n\}$, as well as the empty set $[\boldsymbol{-1}] := \varnothing$.
- **Morphisms:** For each object $[\boldsymbol{n}]$, bijections $[\boldsymbol{n}] \to [\boldsymbol{n}]$. If $n \neq m$, there are no morphisms from $[\boldsymbol{n}]$ to $[\boldsymbol{m}]$.

With composition and identity of functions.

That is, the objects are natural numbers and the endomorphisms $\mathrm{Hom}_{\mathsf{Sym}}([\boldsymbol{n}],[\boldsymbol{n}])$ of each object are the elements the symmetric group $\mathrm{Sym}(n+1)$. In the notation of part II, this is $\mathsf{Sym} := \bigsqcup_n \mathcal{B}\,\mathrm{Sym}(n)$, the disjoint union of the deloopings of the symmetric groups. The Interlude will make the notion of "delooping" precise; the disjoint union is just the coproduct in Cat.

The category Sym is actually symmetric and monoidal, and the monoidal operation $+$ is given by juxtaposing both sets of numbers. For example, $[\boldsymbol{m}]+[\boldsymbol{n}]$ is $[\boldsymbol{m}+\boldsymbol{n}]$, where we identify

$$\{1_m, \dots, m_m, 1_n, \dots, n_n\} \longmapsto \{1, \dots, m, m+1, \dots, m+n\}.$$

The action on morphisms is defined by applying this identification in the intuitive way; given functions $f\colon [\boldsymbol{m}] \to [\boldsymbol{n}]$ and $f'\colon [\boldsymbol{m}'] \to [\boldsymbol{n}']$, we define

$$f+f'\colon [\boldsymbol{m}+\boldsymbol{m}'] \longrightarrow [\boldsymbol{n}+\boldsymbol{n}'], \qquad i \longmapsto \begin{cases} f(i), & i \leq m; \\ n+f'(i-m), & i > m. \end{cases}$$

The unit is the empty set. The symmetric structure is given by interchanging both factors, in the intuitive way: we go from $[\boldsymbol{m}]+[\boldsymbol{n}]$ to $[\boldsymbol{n}]+[\boldsymbol{m}]$ by applying

$$\{1_m, \dots, m_m, 1_n, \dots, n_n\} \longmapsto \{1_n, \dots, n_n, 1_m, \dots, m_m\}.$$

Now, let's make the promised pictures. We draw each set as a column of dots, ordered from bottom to top, and permutations of sets as lines between the columns of dots. Then, axioms (ɪɪ) and (ɪɪɪ) correspond to the diagrams of figures 1.10 and 1.11.

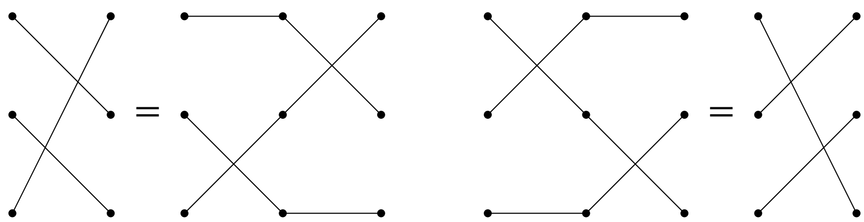


Figure 1.10: Axiom (ɪɪ) of the definition of symmetric monoidal category.

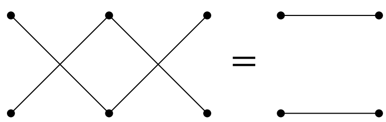


Figure 1.11: Axiom (ɪɪɪ) of the definition of symmetric monoidal category.

We could define our category just by using these pictures, if we wanted to! In any case, Sym is the *free* symmetric monoidal category on a single object. We postpone discussion of freely generated categories until section 2.3.2, although this notion will appear multiple times in passing.

Remark 1.2.16 — Φ, the category of finite cardinals.
In Example 1.2.15 we could have defined the morphisms to be any kind of functions from $[\boldsymbol{n}]$ to $[\boldsymbol{m}]$, instead of just bijections. This yields the category of *finite cardinals*, usually denoted by Φ or $\mathbb{F}$. It is a skeleton for the category of finite sets; i.e., a equivalent category which has a single object in each isomorphism class. There is a sense in which this is a *categorification* of the natural numbers since it recovers the sum (here the coproduct) and multiplication (here the product) of $\mathbb{N}$! See, for example, [BD01].

This category, Φ, is the free symmetric monoidal category on a single *commutative monoid object*; compare with the later discussion in section 2.3.2.

Remark 1.2.17 — Braided monoidal categories.
These pictures also hint at another twisty (and twisted) kind of category — *braided* monoidal categories. We ask ourselves: what would happen if the strands could go over or under each other? We will explore these a bit more properly in the Interlude, but their definition is easy enough to give: just remove axiom (iii) from Definition 1.2.14. That is, we do not ask for the twist maps (now named *braidings*) to be idempotent.

The reader already familiar with the Artin braid groups $\mathrm{Braid}(n)$ might want to think about applying to them the same concatenation operation we defined on the symmetric groups $\mathrm{Sym}(n)$.

Just as a category can admit several distinct monoidal structures, a monoidal category can admit several distinct symmetric structures. These examples can be a bit hard to find; when the monoidal structure is given by the product or coproduct, there is only one symmetry, the canonical interchange of factors (this is an easy exercise using the universal properties). But that is not true of every category.

Example 1.2.18 — The Koszul sign change.
Let $\mathsf{grVect}_{\Bbbk}$ be the category of *graded* vector spaces, i.e. direct sums of vector spaces $V = \bigoplus_{n\in\mathbb{Z}} V_n$ along with graded linear maps. These are linear maps $f\colon V \to W$ which are compatible with the grading, so equivalently $f = \bigoplus_{n\in\mathbb{Z}} f_n$ with $f_n\colon V_n \to W_n$. As an example, most cohomology rings are actually graded vector spaces.

The tensor product of vector spaces restricts to the grading, $(V\otimes W)_n = V_n \otimes W_n$, and endows $\mathsf{grVect}_{\Bbbk}$ with the structure of a monoidal category. The unit is the base field $\Bbbk$, equipped with the grading $\Bbbk_0 = \Bbbk$ and $\Bbbk_n = \{0\}$ for any $n \neq 0$.

We have the canonical symmetry $\sigma\colon v\otimes w \mapsto w\otimes v$. But we also have ***Koszul's sign change***,

$$\kappa\colon v\otimes w \longmapsto (-1)^{pq} w\otimes v,$$

where p and q are the degrees of v and w. It is quick to check that this operation satisfies the axioms of a symmetric monoidal category.

We will briefly come back to this example, in section 2.3.2, when discussing symmetric Frobenius objects [Def. 2.3.15].

Note that symmetry is a *weak* notion, in the sense that it concerns itself with natural equivalences between functors: it is more similar to definition 1.2.11 than to definition 1.2.10. Is there a strictified version of this concept, where we change natural equivalences to identities? Well, yes. And there are even some coherence results — but these are a lot less powerful than the Coherence Theorem for monoidal categories! In particular, there is a Coherence Theorem, but not an Strictification Theorem. Thus, we cannot ignore the twist maps entirely — they are an intrinsic part of the structure. See [Mac78, § XI].

In any case, a monoidal functor $F\colon (\mathcal{V},\star,\mathrm{I},\tau) \to (\mathcal{W},\diamond,\mathrm{J},\sigma)$ is ***symmetric*** if it respects the twist maps: $F\circ\tau_{X,Y} = \sigma_{F(X),F(Y)}$. This is the last definition we needed.

The functorial definition of TQFT, again

So, we can finally restate our functorial definition 1.2.9 of TQFT: an oriented n-TQFT is a symmetric monoidal functor $Z\colon (\mathsf{Cob}_n, \sqcup, \varnothing, \sigma) \to (\mathsf{Vect}_{\Bbbk}, \otimes, \Bbbk, \tau)$. Compare, again, with the axioms of definition 1.2.1!

In essence, topological quantum field theories serve as a connection between the category of cobordisms and the category of vector spaces. One of the takeaways from their study, as nicely explored in [Bae01], is the number of striking structural similarities these two categories share. In short, both Cob_n and $\mathsf{Vect}_{\Bbbk}$ are rigid symmetric monoidal categories where the monoidal structure isn't a cartesian product. We will now define rigid categories, but it is not the first time we mention them. These essentially formalize the notion of having dual objects.

Rigid categories

We will spend comparatively less time talking about these, but since we have had our first encounter with duality in Zorro's Lemma [1.2.3] we might as well spell it out. And this, too, will be relevant come chapter 3!

Definition 1.2.19. Let $(\mathcal{V}, \star, \mathrm{I})$ be a monoidal category.

An object $X \in \mathcal{V}$ is said to have a ***right dual*** $X^\vee$ if there exists morphisms $\mathrm{ev}_X\colon X \star X^\vee \to \mathrm{I}$ and $\mathrm{coev}_X\colon \mathrm{I} \to X^\vee \star X$ such that the composites are the identities:

$$\mathrm{id}_X\colon X \xrightarrow{\mathrm{id}_X \star \mathrm{coev}_X} X \star X^\vee \star X \xrightarrow{\mathrm{ev}_X \star \mathrm{id}_X} X,$$
$$\mathrm{id}_{X^\vee}\colon X^\vee \xrightarrow{\mathrm{coev}_X \star \mathrm{id}_{X^\vee}} X^\vee \star X \star X^\vee \xrightarrow{\mathrm{id}_{X^\vee} \star \mathrm{ev}_X} X^\vee.$$

Dually, it has a ***left dual*** ${}^\vee X$ if there exist morphisms $\mathrm{ev}_X\colon {}^\vee X \star X \to \mathrm{I}$ and $\mathrm{coev}_X\colon \mathrm{I} \to X \star {}^\vee X$ satisfying the dual conditions:

$$\mathrm{id}_{{}^\vee X}\colon {}^\vee X \xrightarrow{\mathrm{id}_{{}^\vee X} \star \mathrm{coev}_X} {}^\vee X \star X \star {}^\vee X \xrightarrow{\mathrm{ev}_X \star \mathrm{id}_{{}^\vee X}} X^\vee,$$
$$\mathrm{id}_X\colon X \xrightarrow{\mathrm{coev}_X \star \mathrm{id}_X} X \star {}^\vee X \star X \xrightarrow{\mathrm{id}_X \star \mathrm{ev}_X} X.$$

If $\mathcal{V}$ is symmetric, then left and right duals coincide, and we call them simply ***duals***.

This is just Zorro's Lemma again! Take care with the nomenclature, however: not every vector space has a dual, in this sense. While for every vector space we can define a dual $V^* := \mathrm{Hom}(V, \Bbbk)$, and an evaluation map $\mathrm{ev}_V\colon V \otimes V^* \to \Bbbk$ (by $(x, \varphi) \mapsto \varphi(x)$), a compatible coevaluation $\mathrm{coev}_V\colon \Bbbk \to V^* \otimes V$ only exists for finite-dimensional spaces. For clarity, we will call objects which admit categorical duals ***dualizable***. So every object $V \in \mathsf{Vect}_{\Bbbk}^{\mathrm{Fin}}$ is dualizable!

Definition 1.2.20. A monoidal category $\mathcal{V}$ is called ***rigid***, or said to ***have duals for objects***, if every of its objects is dualizable.

Other synonym common in the literature is "autonomous category". There are also *left rigid* and *right rigid* categories, where we only require one-sided duals.

So both Cob_n and $\mathsf{Vect}_{\Bbbk}^{\mathrm{Fin}}$ are *rigid* symmetric monoidal categories. Shouldn't we ask for TQFTs to preserve this structure? Well, they already do! As it turns out, this axiom — for TQFTs to be *rigid* functors, i.e. satisfy $Z(X^\vee) = Z(X)^\vee$ — was part of the original definition of TQFT, but it is redundant. In fact, Zorro's Lemma *is* the reason for this redundancy, as we have already seen — remember Corollary 1.2.4.

To finish this section, we now briefly explore unoriented and framed TQFTs. This discussion will be very relevant come section 4.1.

Unoriented and framed TQFTs

Remark 1.2.21 — Unoriented TQFTs.
As the reader surely expects by now, an unoriented TQFT is just a symmetric monoidal functor from $\mathsf{Cob}_n^{\mathrm{un}}$ to $\mathsf{Vect}_{\mathbb{k}}$, where $\mathsf{Cob}_n^{\mathrm{un}}$ is the category of *unoriented* cobordisms. But notice that we have a forgetful functor $\mathsf{Cob}_n \to \mathsf{Cob}_n^{\mathrm{un}}$, namely the one which forgets that our manifolds and cobordisms ever had an orientation. As such, by precomposing with this functor, any unoriented TQFT Z gives rise to an oriented TQFT:

$$\mathsf{Cob}_n \longrightarrow \mathsf{Cob}_n^{\mathrm{un}} \xrightarrow{Z} \mathsf{Vect}_{\mathbb{k}}.$$

So every unoriented TQFT can be made into an oriented TQFT — but the reverse isn't true, as there are oriented TQFTs which don't factorize through an unoriented TQFT.

Another way of thinking about this forgetful functor is as a (sort of) quotient map. Consider the binary group $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$, acting on Cob_n by reversing the orientation; i.e., we have a functor[4] $\Gamma\colon \mathbb{Z}/2\mathbb{Z} \times \mathsf{Cob}_n \to \mathsf{Cob}_n$ such that Γ_0 is the identity while $\Gamma_1(M) = \overline{M}$ and $\Gamma_1(B) = \overline{B}$ for every object and morphism. Passing to orbits, this has the effect of identifying every manifold with its opposite: when we take the quotient by this action, we are effectively forgetting the orientation of our manifolds. This functor is not surjective on objects for $n > 1$, of course, since there are non-orientable manifolds. So the quotient $\mathsf{Cob}_n/(\mathbb{Z}/2\mathbb{Z}) \subset \mathsf{Cob}_n^{\mathrm{un}}$ selects the full subcategory of $\mathsf{Cob}_n^{\mathrm{un}}$ composed of *orientable* unoriented manifolds.

Most of part II will be spent discussing *framed* TQFTs. We can get a head start by informally introducing them here!

Remark 1.2.22 — Framed TQFTs.
Another more rigid structure which we can endow (some) manifolds with is a *framing*, which is a smooth choice of basis for each tangent space. Wait until section 3.1.2 for more details on this category; here, we are concerned with three key facts.

First, there is a category of framed n-cobordisms, denoted $\mathsf{Cob}_n^{\mathrm{fr}}$. Second, a framed manifold inherits an orientation, by taking the distinguished basis to be positive. Finally, given a framed manifold, the set of bases $\mathrm{Fr}(\mathrm{T}_x M)$ of each tangent space $\mathrm{T}_x M$ is endowed with a canonical action by $\mathbf{GL}_{\mathbb{R}}(n)$. We write this action as

$$\mathrm{Fr}(\mathrm{T}_x M) \times \mathbf{GL}_{\mathbb{R}}(n) \to \mathrm{Fr}(\mathrm{T}_x M), \qquad (\xi, P) \mapsto \xi \cdot P,$$

where $\cdot$ denotes matrix multiplication. Here we are writing each basis $\xi \in \mathrm{T}_x M$ as a matrix by considering its coordinates with respect to the distinguished basis (i.e., the basis selected by the framing). Note that this action has a single orbit, since every two bases are related by a change of basis (i.e., a matrix $P \in \mathbf{GL}_{\mathbb{R}}(n)$).

These actions induce a group action on the space $\mathrm{Fr}\,M$ of possible framings of the manifold (by acting on each fiber), and therefore on the entire category of framed n-cobordisms. Here, each $P \in \mathbf{GL}_{\mathbb{R}}(n)$ acts on a framed manifold (M, ξ) by sending it to $(M, \xi \cdot P)$, a different framing on the same underlying manifold. The orbits are all possible framings on a given manifold, so quotienting by this action has the effect of completely forgetting the framing.

Something more interesting happens if we consider a subgroup of $\mathbf{GL}_{\mathbb{R}}(n)$ instead. If we restrict the canonical action to matrices with positive determinant, denoted by $\mathbf{GL}_{\mathbb{R}}^{+}(n) := \det^{-1}(0,\infty)$, the orbit of each framing ξ now consists of all framings which induce the same orientation as ξ! This is by the definition of "orientation" [Def. 1.1.5]. So now there are two orbits,[5] and the quotient $\mathsf{Cob}_n^{\mathrm{fr}}/\mathrm{SO}(n) \subset \mathsf{Cob}_n$ selects the full subcategory of Cob_n composed of *parallelizable* oriented manifolds.

[4]Notice that here we are viewing the group as a category with a single object. Do not fret if you are unfamiliar with this viewpoint: we will soon discuss it further, in Remark 1.3.2.

Once again, an oriented TQFT can always be restricted into a framed TQFT, but not conversely.

But why write all of this in terms of group actions, instead of just taking forgetful functors — if we even wanted to do that? We will eventually answer these inquiries, but you will have to wait until section 4.1.2.

One last quick remark, before the eternal worm devours this digression.

Remark 1.2.23 — Restricting the actions to compact subgroups.
By endowing our manifold with a metric — which is always possible, as a consequence of Whitney's Embedding Theorem — we can consider orthonormal bases instead. Then, the set of *orthonormal* frames $\mathrm{Fr}_{\mathrm{O}} M$ on our manifold comes equipped with an action by $\mathrm{O}(n)$ — defined just like the prior action by $\mathrm{GL}_{\mathbb{R}}(n)$. Since $\mathrm{O}(n)$ is a *compact* Lie group (the maximal compact subgroup of $\mathrm{GL}_{\mathbb{R}}(n)$, in fact), this has some technical advantages.

We recover the forgetful functor $\mathsf{Cob}_n^{\mathrm{fr}} \to \mathsf{Cob}_n$ by instead quotienting by the equivalent action by $\mathrm{SO}(n)$ (the maximal compact subgroup of $\mathrm{GL}_{\mathbb{R}}^{+}(n)$).

All of this depends on a choice of metric, but since the space of all metrics on a manifold is contractible, this process is well-defined up to homotopy. A slight variation on this theme (using the language of *G-structures*) lets us use this to *define* the concept of orientation in a manifold — even for manifolds which do not admit a framing! Hence why Thom's *oriented* cobordism ring was denoted Ω_*^{SO} in section 1.1.3.

We will see this in more detail when we more properly talk about G-structures, on section 4.1 — that is, almost at the very end of our journey! These will let us tackle the precise way in which framed, oriented and unoriented TQFTs relate to each other.

[5]Actually, framings on 0-dimensional manifolds still have a single orbit, since these form the empty set. That is because we lied: we do not equip our manifolds with a framing, but rather with an *n-framing*. This is a stable generalization of framings: given a manifold of dimension m, an n-framing is a smooth choice of basis for each *stabilized* tangent space $\mathrm{T}_x M \oplus \mathbb{R}^{n-m}$. This has a number of benefits, which we will explore in section 3.1.2.

1.3 The classification of 1D TQFTs

Our main goal for chapter 2 is to classify TQFTs in dimension two (up to isomorphism)[1]. We will then — in chapter 3 — spend some time near the current frontiers of research, discussing the general case and its nuances. Logic dictates that we should start our study in the simplest case: one-dimensional cobordisms.

1.3.1 Presentations of categories

Categories with a single object

What should our classification look like, and how do we find it? Let us take inspiration from some thoroughly-studied examples of categories: those with only one object.

Definition 1.3.1. A ***monoid*** is a set M equipped with a unit 1 and a closed binary operation $\star$ which is unital and associative:

$$(a \star b) \star c = a \star (b \star c), \qquad a \star 1 = a = 1 \star a.$$

That is, a monoid is a "group without inverses".

Remark 1.3.2 — Categories with a single object are monoids.
For any category $\mathcal{C}$, and any object $X \in \mathrm{ob}(\mathcal{C})$, its set of endomorphisms $\mathrm{Hom}_{\mathcal{C}}(X,X)$ carries the structure of a monoid: indeed, the composition in the category is associative and unital, and every endomorphism $f\colon X \to X$ is composable with any other endomorphism $g\colon X \to X$.

So, in this sense, a category with only one object is just a monoid: the singular object provides no information at all, and the only important algebraic structure lies within the composition of endomorphisms. This category is called the *delooping* of the monoid M, and denoted $\mathcal{B}M$, for reasons that we will explore in the Interlude.

Functors between categories with a single object correspond, in this analogy, to homomorphisms of monoids; these are required to preserve the monoids' operations and take one identity to the other identity.

For the sake of exposition, let us restrict our view to *groups*, since these are a structure the reader will probably be more deeply familiar with. A group is just a monoid where every element admits an inverse, so by analogy it is a one-object category where every morphism is invertible: a one object *groupoid*. Again, functors between these one-object categories are homomorphisms of groups. On the other hand, a functor $G \to \mathsf{Vect}_{\mathbb{k}}$ is a (linear) ***representation*** of the group: it is a group homomorphism $G \to \mathrm{GL}(V) := \mathrm{Hom}_{\mathsf{Vect}_{\mathbb{k}}}(V,V)$ with codomain the group of endomorphisms of some vector space. (This is also a (linear) group *action* $G \curvearrowright V$, and the terminology used depends mainly on context).

Any group G admits a ***presentation*** in terms of generators and relations, $G = \langle a_\alpha \mid r_\beta \rangle$, where the indices range over some (possibly uncountable) set. Each generator a_α is a separate character, while each relation r_β is a finite *word* formed with those characters (i.e., a formal string $a_{\alpha_1}, \cdots, a_{\alpha_r}$) that we define to equal the identity e (which in this context is just the empty word). The core property is that G is isomorphic to the free group generated by the generators a_α, quotiented by the subgroup generated by the relations r_β; that is, we identify each element $g \in G$ with a finite word $a_{\alpha_1} \cdots a_{\alpha_r}$, with two words representing the same element if we can get from one to the other by applying the relations.

Any homomorphism of groups $G \to G'$, and indeed any functor $G \to \mathcal{C}$, is completely determined by the images of the generators a_α. So this provides a useful framework in which to classify homomorphisms from G to any other group.

(Do not confuse presentations with *re*presentations; these are two very different concepts).

[1]Since a TQFT is just a (symmetric monoidal) functor $Z\colon \mathsf{Cob}_n \to \mathsf{Vect}_{\mathbb{k}}$, a morphism between TQFTs is just a (monoidal) natural transformation $u\colon Z \Rightarrow Z'$; i.e., a natural transformation satisfying $u_{M \sqcup N} = u_M \otimes u_N$ and $u_\varnothing = \mathrm{id}_{\mathbb{k}}$.

Example 1.3.3 — Presentations for the symmetric and braid groups.
As an example, the symmetric group $\mathrm{Sym}(n)$ admits a presentation

$$\mathrm{Sym}(n) \cong \left\langle \tau_1, \dots, \tau_{k-1} \;\middle|\; \begin{array}{cl} \tau_i\tau_j\tau_i = \tau_j\tau_i\tau_j & (j = i+1) \\ \tau_i\tau_j = \tau_j\tau_i & (j > i) \\ \tau_i\tau_i = \mathrm{id} & \end{array} \right\rangle.$$

Here, each τ_i is identified with the transposition which interchanges $(i+1)$ and i, as we saw in the discussion before Lemma 1.1.23.

Artin's braid group $\mathrm{Braid}(n)$ (mentioned in Remark 1.2.17) admits a similar presentation, where we instead think of σ_i as the braid which crosses the $(i+1)$th strand *over* the ith strand:

$$\mathrm{Braid}(n) \cong \left\langle \sigma_1, \dots, \sigma_{k-1} \;\middle|\; \begin{array}{cl} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & (j = i+1) \\ \sigma_i\sigma_j = \sigma_j\sigma_i & (j > i+1) \end{array} \right\rangle.$$

Once again, the core difference between the two is removing the idempotency axiom.

In analogy with group representations, we can define a linear representation of a symmetric monoidal category $(\mathcal{C}, \star, \tau)$ to be a symmetric monoidal functor $\mathcal{C} \to \mathsf{Vect}_{\Bbbk}$. If we do, we arrive at the following slogan: *a TQFT is a linear representation of the category of cobordisms*.

It is usually straightforward to study the group representations of a finitely presented group: you have only a finite number of generators and relations, and you know that any map $G \to \mathrm{GL}(V)$ is determined by the images of the generators, so you just need to use the relations in order to determine when any such two maps are the same.

Can we do the same for symmetric monoidal categories? Yes, in fact.

Presentations of categories

Any category can be given a ***presentation*** in terms of generators and relations, in the same way one presents a group. Here we can take the same word approach as with groups, but now we have both generating objects and generating *morphisms*. We can think of generating morphisms as arrows $f_\alpha \colon A_\alpha \to B_\alpha$, and a word is then a tuple of *composable* arrows. Now, our category is equivalent (not necessarily *isomorphic*) to the free category generated by the arrows f_α quotiented by the subcategory generated by the relations.[2]

Remark 1.3.4 — The skeleton of a category.
As it often happens in category theory, isomorphism of categories is not the correct level of "sameness", and we would rather prefer to talk about *equivalence* of categories.

In our case, in order to reduce cluttering our category with unnecessary objects, we will try to find presentations which are isomorphic to the *skeleton* of Cob_n: a full subcategory formed by taking one (and only one) object from each isomorphism class, as well as all morphisms between the selected objects. Any category is equivalent to any of their skeletons, so this poses no problem.

For example, a groupoid (a category where each morphism is an isomorphism) has as skeleton a group (a groupoid with a single object); and $\mathsf{Vect}_{\Bbbk}^{\mathrm{Fin}}$ has as skeleton the full subcategory formed by the objects $\Bbbk^n$ (since any finite-dimensional vector space V is isomorphic to $\Bbbk^{\dim(V)}$).

If our category has a monoidal structure, we can form words not only by composing arrows, but also by *multiplying* arrows using the monoidal operation $\star$. And if it is symmetric, we can also take twist maps. If it is rigid, i.e. has duals for objects, we can take duals and evaluation and coevaluation maps!

In any of these cases, a functor $\mathcal{C} \to \mathcal{D}$ preserving the additional structure (monoidal, symmetric and/or rigid) is fully determined by the images of the generating set of arrows. So in particular,

[2] An alternative approach, perhaps more elegant, is to use universal properties to define the presented category.

a symmetric monoidal functor $\mathsf{Cob}_n \to \mathsf{Vect}_{\mathbb{k}}$ is determined by the images of a generating set of Cob_n (whatever that generating set is), and we "just" need to find an amenable presentation of Cob_n.

Representations of Cob_n become more and more complicated as one goes up in dimension. The following section 1.3.2 will tackle the case $n = 1$, while section 2.1 will be devoted to $n = 2$. For dimension 3, see the very recent paper [NPZ24], where the authors describe Cob_3 through planar diagrams. For arbitrary dimension, see [Juh18, Thm. 1.7].

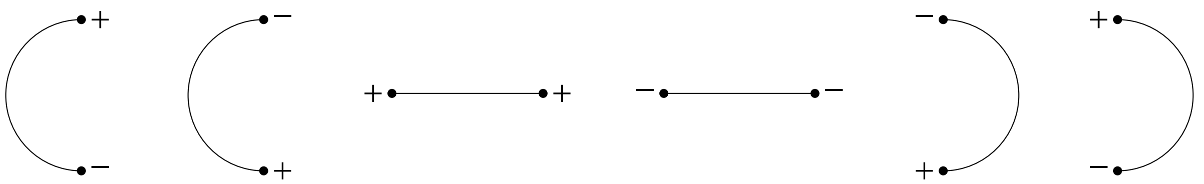
1.3.2 Generators and relations for Cob_1

Classifying oriented 1-cobordisms

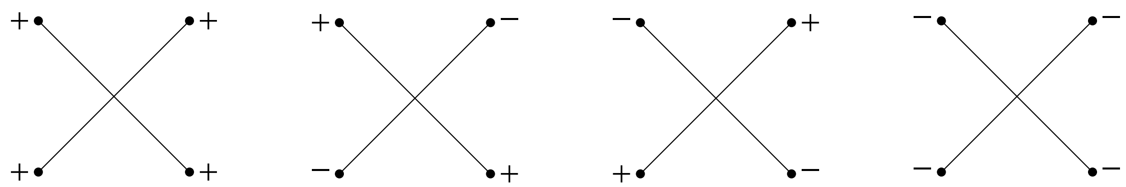
We are now fully equipped with the machinery needed in order to classify 1D TQFTs. Once again, these are linear representations of Cob_1: symmetric monoidal functors $Z\colon \mathsf{Cob}_1 \to \mathsf{Vect}_{\mathbb{k}}$. The objects of Cob_1 are 0-dimensional oriented closed manifolds; that is, a finite collection of points each equipped with positive (pt_+) or negative (pt_-) orientation. In other words, the objects are generated by the set $\{\mathrm{pt}_+, \mathrm{pt}_-\}$, using disjoint union; the empty 0-manifold $\varnothing^0$ corresponds to the empty disjoint union.

The morphisms are 1-dimensional cobordisms (up to diffeomorphism), which topologically are just a finite union of circumferences and closed line segments. And, remarkably, a segment has only two ends.[3] These can either be all in-boundaries (one positive and the other negative), all out-boundaries (same situation), or one of each (both positive or both negative).

So we have the situation of figures 1.12(a) to 1.12(e), and one might conjecture that all one-dimensional cobordisms are disjoint unions of compositions[4] of these basic generating arrows. (Again, the empty 1-cobordism $\varnothing^0 \times I\colon \varnothing^0 \to \varnothing^0$ corresponds to the empty disjoint union).



(a) Out-arc $\substack{+\\-}$. (b) Out-arc $\substack{-\\+}$. (c) Positive cylinder. (d) Negative cylinder. (e) In-arc $\substack{-\\+}$. (f) In-arc $\substack{+\\-}$.



(g) Twist cobordism $\substack{+\\+}$. (h) Twist cobordism $\substack{+\\-}$. (i) Twist cobordism $\substack{-\\+}$. (j) Twist cobordism $\substack{-\\-}$.

Figure 1.12: A monoidal generating set of morphisms for Cob_1.

Except, not quite: these are the only connected 1-cobordisms, yes, but we should not forget our symmetric structure. The four cobordisms of figures 1.12(g) to 1.12(j) are *not* the disjoint union of two cylinders — even though, topologically, they are — since they permute the order of the two components in the boundary. Those four cobordism are *twist maps* of our symmetric monoidal category Cob_1, and every other possible twist map can be obtained from these.

Now, we can either include them to get a generating set of Cob_1 as a *monoidal* category, or exclude them and still have a generating set of Cob_1 as a *symmetric* monoidal category.

[3] We will provide no proof of such a fact.

[4] Notice that the only composition which is not already isomorphic to a line segment is the circumference, which is obtained by composing an out-arc with an in-arc.

Regardless of the type of generating set we choose, we can remove half of the four arcs — the $^{-}_{+}$ out-arc 1.12(b) is obtained by composing 1.12(a) with the twist 1.12(h), and likewise the $^{-}_{+}$ in-arc 1.12(e) is obtained by composing 1.12(f) and 1.12(i). Similarly, we can exclude the two cylinders; they are redundant, since the identity arrows *always* form part of a category.

If we admit taking duals, then we have no generating morphisms at all, since the existence of in- and out-arcs is guaranteed by Zorro's Lemma! Remember Definition 1.2.19 and figure 1.9.

Since these approaches are equivalent, and for clarity of exposition, let us use as little additional operations as possible and admit a little redundancy. Let us argue why these ten cobordisms, together with the operations of composition and disjoint union, suffice to construct any cobordism in $\mathrm{Hom}_{\mathtt{Cob}_1}$ (up to isomorphism, that is). Later, when discussing the relations, we will need to include extra relations encoding the axioms of symmetry and rigidity — but doing this is a good exercise.

We first reduce to the connected case, by applying Lemma 1.1.23.

Theorem 1.3.5. The ten cobordisms of figure 1.12 form a monoidal generating set for $\mathtt{Cob}_1$.

Proof. Let B be a one-dimensional cobordism. We want to see that B is isomorphic to some disjoint union of compositions of the ten cobordisms in figure 1.12.

First, applying Lemma 1.1.23, compose B with some permutation cobordisms in order to obtain a disjoint union of connected cobordisms: $\rho \circ B \circ \sigma = B_1 \sqcup \cdots \sqcup B_k$. These permutation cobordisms can be written as a composition of disjoint unions of cylinders and twists, as do their inverses.

Then, let us tackle each connected component on its own. Note that there are only two non-empty connected compact 1-manifolds with (perhaps empty) boundary: the interval I and the circumference $\mathbb{S}^1$. So our connected 1-cobordism is, topologically, one of these two. If it is a circumference, then it is the composition of 1.12(a) with 1.12(f).[5] On the other hand, if it is the interval, it must be one of the six cobordisms in figures 1.12(a) to 1.12(e). In either case, we can write each connected component as the composition of some of our generators.

Taking the disjoint union of these circles and intervals, we obtain a 1-cobordism B' which is isomorphic to $B_1 \sqcup \cdots \sqcup B_k = \rho \circ B \circ \sigma$. By composing B' with the inverse permutation cobordisms, i.e. $\rho^{-1} \circ B' \circ \sigma^{-1}$, we obtain a cobordism that is isomorphic to B. □

Relations between the generators

What about the relations between the generators? Well, these will depend on whether we consider them as a monoidal, symmetric or rigid generating set; in each case, we will need to add relations to encode the axioms of the other structures. And it turns out that $\mathtt{Cob}_1$ is free on a single generating object when regarded as a rigid symmetric monoidal category! That is, its objects are $\bigsqcup_m \mathrm{pt}_+ \sqcup \bigsqcup_n \mathrm{pt}_+^\vee$ (where $\mathrm{pt}_+^\vee = \mathrm{pt}_-$ and m and n may be zero), and the only morphisms are the ones necessary for the definition of "rigid symmetric monoidal category": identities, twist maps, evaluation and coevaluation maps, and disjoints unions of all these. Compare with our list of generators in figure 1.12.

However, it is quite enlightening to consider the relations these generators must satisfy when regarding them as just a plain *monoidal* generating set. We will ignore orientation for the following arguments, to avoid boring ourselves with unnecessary repetition.

First, and most trivially, including the cylinders in our set of generators require us to specify that these are identities. See figure 1.13.

More interesting is the case of the twist cobordisms. The symmetry axioms of Definition 1.2.14 come down to the pictures in figure 1.14. Compare these with the relations for the symmetric group

[5]It is also the composition of 1.12(b) with 1.12(e). Crucially, this does not depend on the orientation, since there is an orientation-reversing diffeomorphism $\mathbb{S}^1 \to \mathbb{S}^1$.

given in Example 1.3.3. Of note, we can write the symmetric groupoid Sym of Example 1.2.15 as the monoidal category generated by the twist cobordism and satisfying the relations of figure 1.14.

We also need to specify that arcs commute with twist maps; see figure 1.15. This is a side effect of including the two redundant arcs in our list of generators.

Finally, the rigidity axiom of Definition 1.2.20 is just Zorro's Lemma [fig. 1.9] for the case where M is the point pt_+; see figure 1.16.

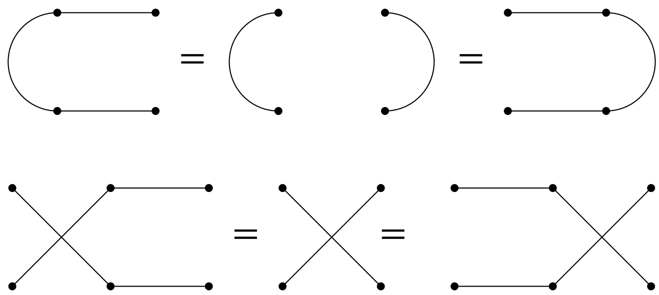


Figure 1.13: The identity relations for the cylinders of Cob_1.

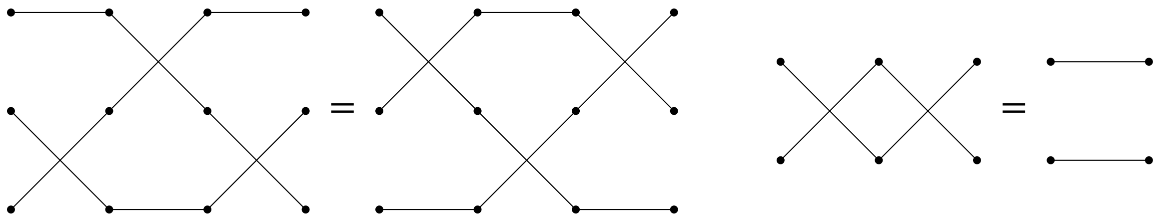


Figure 1.14: Relations for the twist maps in any symmetric monoidal category.

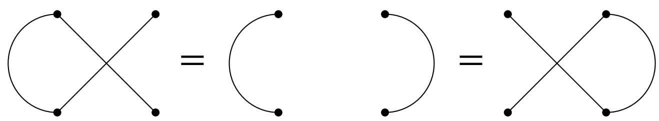


Figure 1.15: Relations between twists and arcs in Cob_1.

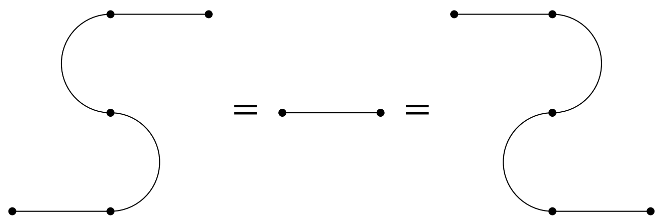


Figure 1.16: 1D Zorro's Lemma.

1.3.3 1D TQFTs are finite-dimensional vector spaces

Now, our classification result is almost immediate. But first, let us properly assemble our TQFTs into a category.

The category of 1D oriented TQFTs

As mentioned before, since TQFTs are symmetric monoidal functors $Z, Z' \colon \mathsf{Cob}_n \to \mathsf{Vect}_{\Bbbk}$, we can define a morphism of TQFTs to be a monoidal natural transformation $Z \Rightarrow Z'$.

Definition 1.3.6. Given two monoidal functors $F, G \colon (\mathcal{V}, \star, \mathrm{I}) \to (\mathcal{W}, \diamond, \mathrm{J})$, a ***monoidal natural transformation*** is a natural transformation $u \colon F \Rightarrow G$ such that $u_{X\star Y} = u_X \diamond u_Y$ (for each pair of objects $X, Y \in \mathcal{V}$) and $u_{\mathrm{I}} = \mathrm{J}$.

Definition 1.3.7. The ***category of n-dimensional oriented TQFTs***, $\mathsf{TQFT}_n^{\mathrm{or}}(\mathsf{Vect}_{\Bbbk})$, is the category with n-dimensional oriented TQFTs $\mathsf{Cob}_n \to \mathsf{Vect}_{\Bbbk}$ as objects and monoidal natural transformations as morphisms.

Another common notation is $\mathrm{Fun}^{\otimes}(\mathsf{Cob}_n, \mathsf{Vect}_{\Bbbk})$ or $\mathrm{Hom}_{\mathsf{SymMonCat}}(\mathsf{Cob}_n, \mathsf{Vect}_{\Bbbk})$.

One can check that all monoidal natural transformations between TQFTs are invertible. In other words, TQFTs form a groupoid.

The classification of 1D oriented TQFTs

In conclusion: 1-dimensional oriented TQFTs are fully determined by their image at the oriented point, which is a finite-dimensional $\Bbbk$-vector space. Conversely, any $\Bbbk$-vector space V of finite dimension determines a 1D oriented TQFT, by defining the TQFT which maps the oriented point to V; every other value is forced by monoidality, symmetry and rigidity. This is deduced from the generators and relations of Cob_1: the in- and out-arcs must go to the canonical evaluation $V \otimes V^* \to \Bbbk$ and coevaluation $\Bbbk \to V^* \otimes V$, the cylinders to the identities, and the twist cobordisms to the twist maps of vector spaces.

We formally state this classification theorem.

Theorem 1.3.8. There is a natural equivalence between 1D oriented TQFTs Z with values in $\mathsf{Vect}_{\Bbbk}$ and finite-dimensional $\Bbbk$-vector spaces (V), which is given by evaluation of the TQFT on the positively oriented point:

$$\mathsf{TQFT}_1^{\mathrm{or}}(\mathsf{Vect}_{\Bbbk}) \xrightarrow{\simeq} (\mathsf{Vect}_{\Bbbk}^{\mathrm{Fin}})^{\sim}, \qquad Z \longmapsto Z(\mathrm{pt}_+).$$

Once again, the fact that the image vector spaces have finite dimension is a consequence of Zorro's Lemma [1.2.3]. We can think of this dualizability as a *finiteness* condition, which is expressed by the four equations of Definition 1.2.19.

Remark 1.3.9 — The groupoid of finite-dimensional $\Bbbk$-vector spaces.
To be slightly more precise, we are defining $(\mathsf{Vect}_{\Bbbk}^{\mathrm{Fin}})^{\sim}$ to be the category of finite-dimensional $\Bbbk$-vector spaces and *isomorphisms* between vector spaces. In the nomenclature of chapter 4, this is the *core groupoid* $\mathrm{Core}(\mathsf{Vect}_{\Bbbk}^{\mathrm{Fin}})$. This is necessary to make the assignment of Theorem 1.3.8 an equivalence of categories, since TQFTs form a groupoid.

1D TQFTs with values in other categories

There is no a priori reason for why TQFTs should take values in $\mathsf{Vect}_{\Bbbk}$; that is just a combination of historical inertia and the fact that vector spaces are very well understood. We can generalize TQFTs to be symmetric monoidal functors with values in any symmetric monoidal category $(\mathcal{C}, \star, \mathrm{I}, \tau)$. If you check the arguments done before, you will notice they all translate to this more general framework; we just replace finite-dimensionality by *dualizability* [Def. 1.2.19].

Theorem 1.3.10. There is a natural equivalence between 1D oriented TQFTs Z with values in a symmetric monoidal category $(\mathcal{C},\star,\mathrm{I},\tau)$ and dualizable objects X in $\mathcal{C}$, which is given by evaluation of the TQFT on the positively oriented point:

$$\mathsf{TQFT}_1^{\mathrm{or}}(\mathcal{C}) \xrightarrow{\simeq} \mathsf{Dualizable}(\mathcal{C})^{\sim}, \qquad Z \longmapsto Z(\mathrm{pt}_+).$$

Here $\mathsf{Dualizable}(\mathcal{C})^{\sim}$ is the subcategory of $\mathcal{C}$ obtained by selecting only those objects which are dualizable, as well as all *invertible* morphisms between these.[6] Note that $\mathsf{Dualizable}(\mathcal{C})^{\sim}$ is always non-empty, since the unit $\mathrm{I}\in\mathcal{C}$ is always dualizable: the isomorphism $\mathrm{I}\star\mathrm{I}\to\mathrm{I}$ determines the evaluation and coevaluation maps. So there is always at least one TQFT from Cob_1 to $\mathcal{C}$, namely the trivial TQFT of Example 1.2.5.

Another way of stating this theorem is that Cob_1 is the *free* symmetric monoidal category on a single dualizable object. This is the approach we will take in Theorem 2.3.17 (and later on, in Theses 3.2.2 and 3.2.18).

This particular classification theorem will be extremely relevant for chapter 3. It is not hard to also give a classification of 1D *unoriented* TQFTs, which we will come back to in chapter 4.

Classifying 1D unoriented TQFTs

An unoriented TQFT is a symmetric monoidal functor $\mathsf{Cob}_1^{\mathrm{un}}\to\mathcal{C}$; remember Remark 1.2.21. Since every 1D manifold is orientable, we can reuse our generators of figure 1.12: we just need to forget their orientation and remove any duplicates. Excluding cylinders and twists, we have a single generating object pt, an evaluation $\mathrm{ev}\colon \mathrm{pt}\sqcup\mathrm{pt}\to\varnothing$, and a coevaluation $\mathrm{coev}\colon\varnothing\to\mathrm{pt}\sqcup\mathrm{pt}$. And the relations these must satisfy (as a symmetric monoidal generating set) are exactly those of the prior figures 1.15 and 1.16. Note that we have found an use for the relations of figure 1.15, which previously spawned from including redundant cobordisms in our generating set! This relation is necessary for the unoriented case, and in fact this is where all of the additional structure of 1D unoriented TQFTs stems from — remember that each unoriented TQFT restricts to an oriented TQFT, but not conversely.

If we wanted to take a *rigid* symmetric monoidal generating set, then we would have a single generating object pt, no generating morphisms, and the relations given by $\mathrm{pt}=\mathrm{pt}^{\vee}$ and by figure 1.15.

In any case, now a symmetric monoidal functor $\mathsf{Cob}_1^{\mathrm{un}}\to\mathcal{C}$ will be determined by the image of its generators (the point, the evaluation, and the coevaluation) restricted to the relations. This comes down to specifying, along with the object $X=Z(\mathrm{pt})$, (co)evaluations $Z(\mathrm{ev})\colon X\star X\to\mathrm{I}$ and $Z(\mathrm{coev})\colon\mathrm{I}\to X\star X$ which exhibit X as an object dual to itself. Furthermore, by the relations of figure 1.15, these (co)evaluations must be *symmetric* — they must commute with the twist maps of $\mathcal{C}$. It suffices to specify just the value at the evaluation $Z(\mathrm{ev})$, since the conditions in play will enforce that the value $Z(\mathrm{coev})$ is uniquely determined. We will see this in Lemma 2.2.30, modulo some differences in language.

In any case, let us call an object $X\in\mathcal{C}$ equipped with a symmetric map $X\star X\to\mathrm{I}$ satisfying the prior conditions a ***symmetric self-dual object*** in $\mathcal{C}$, and define a subcategory $\mathsf{SymSelfDual}(\mathcal{C})\subset\mathcal{C}$ in the obvious way. A morphism of self-dual objects should commute with the equipped evaluations; we leave the details for the reader.

Theorem 1.3.11. There is a natural equivalence between 1D unoriented TQFTs Z with values in a symmetric monoidal category $(\mathcal{C},\star,\mathrm{I},\tau)$ and symmetric self-dual objects X in $\mathcal{C}$, which is given by evaluation of the TQFT on the point and evaluation cobordism:

$$\mathsf{TQFT}_1^{\mathrm{un}}(\mathcal{C}) \xrightarrow{\simeq} \mathsf{SymSelfDual}(\mathcal{C}), \qquad Z \longmapsto (Z(\mathrm{pt}),Z(\mathrm{ev})).$$

[6] Same remark as before: this is the core groupoid of $\mathsf{Dualizable}(\mathcal{C})$, the full subcategory of dualizable objects of $\mathcal{C}$. In chapter 4 we will utilize the more explicit notation $\mathsf{Core}(\mathcal{C}^{\mathrm{fd}})$ (replacing dualizability with *full dualizability*), but here we prefer the slightly more compact version so as to make it easier to ignore.

In the case where $\mathcal{C} = \mathsf{Vect}_{\Bbbk}$ (equipped with tensor product and the usual twists), this determines a *non-degenerate symmetric pairing* $\beta\colon V \otimes V \to \Bbbk$ with associated *copairing* $\gamma\colon \Bbbk \to V \otimes V$. This is equivalent to specifying a linear isomorphism $V \to V^*$. Given a prior identificaton $V \cong V^*$ (e.g. a choice of basis on V), we can think of this map as an *involution* $V \to V$ — the "$*$" part in the definition of $*$-*algebras*. We will discuss these (co)pairings in depth in section 2.2, when considering the 2D classification.

The other kind of TQFTs we will revisit throughout the text are *framed* TQFTs; remember Remark 1.2.22. In dimension 1, framed TQFTs coincide with oriented TQFTs: any orientation on a curve induces a (unique up to homotopy) framing on that curve. Therefore, Theorem 1.3.10 also serves as the framed classification.

What's next?

We will spend the remainder of this thesis building up the tools necessary to generalize these results to arbitrary dimension. First, in chapter 2, we will study with great detail the case of 2 dimensional TQFTs. This will culminate in section 2.3, with the proof of Theorem 2.3.3.

Then, in part II, we will discuss the higher-ordered category theory needed to fully state the **Cobordism Hypothesis**: the aforementioned arbitrary-dimensional generalization of the equivalence between 1D TQFTs (with values in $\mathsf{Vect}_{\Bbbk}$) and finite-dimensional $\Bbbk$-vector spaces.

Chapter 2

The classification of 2D TQFTs

You see, a circle is a flawless shape
Every part equidistant from the center
On the arc without a start or a finish.

Shing02, *Luv(sic) Grand Finale*.

In this chapter, we will prove the theorem that part I is named after: the classification of 2D oriented TQFTs. First, in section 2.1, we spend quite a bit of time discussing a presentation for Cob_2 in terms of relations and generators. After some quick lemmas from Morse theory, we provide a complete set of generators, together with an easy proof of that fact. The second part involves the comparatively more tedious process of providing a complete set of relations for the generators. This involves defining a normal form for each connected cobordism and describing an algorithm which uses the aforementioned relations to convert any combination of the generators to a normal form.

section 2.2 then concerns itself with the other object: Frobenius algebras, and eventually *commutative* Frobenius algebras. Here we start by giving a classical definition, in terms of finite-dimensional algebras equipped with a *Frobenius form* or “trace map”, and then slowly morphing it into equivalent characterizations. Each of these characterizations will emphasize different aspects of the structure, and we ultimately need most of them — particularly, a characterization as an algebra equipped with a *non-degenerate associative pairing*, and a characterization as an algebra which is also a *coalgebra* and such that the two structures are related through the *Frobenius relation*. Then, we familiarize ourselves with a system of graphical notation (*graphical calculus*) which makes the connection between Frobenius algebras and 2D oriented TQFTs clearer, by assigning a concrete algebraic meaning to each of the previously found generators of Cob_2. Here we prove that each of the relations for Cob_2 also holds when interpretted algebraically, and that (in general) no more relations holds. Thus, the classification result for 2D oriented TQFTs follows — as an immediate corollary.

Finally, section 2.3 is a victory lap, where we quickly summarize our findings before — in preparation for part II — generalizing our target category. The key insight is that $\mathsf{Vect}_{\Bbbk}$ is not special as a symmetric monoidal category, except in the fact that we know it exceptionally well. As such, we should not artificially restrict our study to TQFTs with values in $\mathsf{Vect}_{\Bbbk}$ — there are plenty more options to consider, and plenty of contexts in which a given construction naturally fits into another category. It would be a shame to dismiss these, due to a fear of the abstract. Taking this step is important for part II, where we will lack a natural choice of target category.

2.1 Generators and relations for Cob_2

As we already mentioned before, our main goal in this section is to find a suitable presentation of the category of two dimensional oriented cobordisms. In other to do so, we want to define a *normal form* for any 2D cobordism, so that we can easily compare any two cobordisms.[1] The first part of the process will involve cutting our cobordism into smaller pieces, whose topology we will identify by using the tools of Morse theory. Then, it's just a matter of checking what operations we can do to reassemble the pieces into our normal form.

2.1.1 Generators for Cob_2

With that out of the way, let's start by stating the two lemmas we borrow from Morse theory. As mentioned before, we will not develop this theory here, and refer the reader to [Mil73; Hir76].

Preliminary lemmas from Morse theory

The following lemma is paraphrased from [Koc03, Lem. 1.4.22], which cites [Hir76, Thm. 9.3.4].[2]

Lemma 2.1.1. Let M be a compact connected oriented surface equipped with a Morse function $M \to I$. If there is a unique critical point x, and x has index 1 (i.e. it is a saddle point), then M is diffeomorphic to a pair of pants: a disc with two discs missing. Furthermore, the preimages of 0 and 1 are not empty and correspond to the three boundaries of the pair of pants.

In [Hir76, Thm. 3.5], this is used as part of a proof of the classification of compact oriented surfaces; each critical point of index 1 coresponds to attaching a handle to (what is initially) a sphere, thus increasing the genus of the surface by 1. So it should be no surprise that our presentation of Cob_2 will also require the classification of compact oriented surfaces.

On the other hand, it is not hard to see that a Morse function with no critical points at all must correspond to a cylinder. In fact, this follows as a corollary of the Regular Interval Theorem [1.1.19].

Lemma 2.1.2. Let M be a compact connected surface equipped with a Morse function $f\colon M \to I$. If f has no critical points, then M is equivalent to a cylinder.

With these two lemmas, we have the tools needed for our goal. Ultimately, we want to classify our morphisms (the 2D oriented cobordisms), but we should first start by classifying our objects (the closed oriented curves).

Generating objects of Cob_2

A closed and connected oriented curve is diffeomorphic to the circle, $\mathbb{S}^1$, in one of two orientations. But note that there exists a diffeomorphism $\mathbb{S}^1 \to \mathbb{S}^1$ which inverts the orientation: for example, any involution obtained by reflection along a line, such as $e^{it} \mapsto e^{i(2\pi - t)}$. So our objects are, up to orientation-preserving diffeomorphism, a finite disjoint union of circles all with the same orientation. This is in contrast with our previous analysis of Cob_1, where the points were equipped with one of two different orientations. We will denote a disjoint union of circles as $(\boldsymbol{n}) := \bigsqcup_n \mathbb{S}^1 = \mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1$.

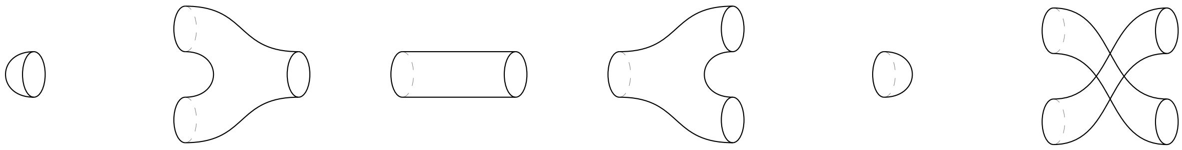
Remember that two oriento-diffeomorphic closed $(n-1)$-manifolds are isomorphic as objects of Cob_n, since an orientation-preserving diffeomorphism $f\colon M \to N$ induces an invertible oriented cobordism $B_f\colon M \to N$. So we can collapse these equivalence classes of objects together to obtain a skeleton of our category.

[1] This is analogous to fixing a normal form for the elements of a finitely presented group, when attempting to solve the word problem for that presentation, via (e.g.) the Knuth–Bendix completion algorithm.

[2] In the book, Theorem 4.4.2 is wrongly cited instead. This is acknowledged in the errata, available at Kock's homepage: `https://mat.uab.cat/~kock/TQFT.html#errata`.

Now, we want to propose a set of pieces from which every cobordism can be constructed; think of these as Legos, or any other kind of building blocks. So we want to cut up our cobordisms into parts which are as simple as possible, and try to identify each of these pieces with an isomorphic representative from our list. Our two lemmas 2.1.1 and 2.1.2 will be very useful for this.

A set of generating morphisms for Cob_2



(a) Cap. (b) Copair of pants. (c) Cylinder. (d) Pair of pants. (e) Cocap (or cup). (f) Twist.

Figure 2.1: A monoidal generating set of morphisms for Cob_2.

Let us start this section by stating the theorem we seek to prove.

Theorem 2.1.3. The six cobordisms of figure 2.1 form a monoidal generating set for Cob_2.

Again, the cylinder and the twist cobordism are redundant, but including them makes our argument clearer. If we wanted to give a *rigid* symmetric generating set, we would only need the cobordisms of figures 2.1(a) and 2.1(b), since these have obvious duals.

How do we go about proving this? Our approach will be that of [Koc03, 1.4.23]. We want to prove that, if a non-empty cobordism is not already isomorphic to one of our distinguished six, then we can decompose it into two simpler pieces. Then, we iterate[3] this process until no extraneous pieces remain; and for this process to terminate in a finite number of steps, we need a way of measuring complexity that is guaranteed to decrease at each step. This is where Morse theory comes in; our "measure of complexity" will be *the number of critical points* in a previously chosen Morse function. So, other than the cylinder and the twist cobordism, we only need to classify cobordisms with a single critical point! And this is where we apply Lemma 2.1.1, to get the other four cobordisms of figure 2.1.

Now that we have explained the strategic outline, let us give a proper proof. Figure 2.2 depicts the core essence of the argument.

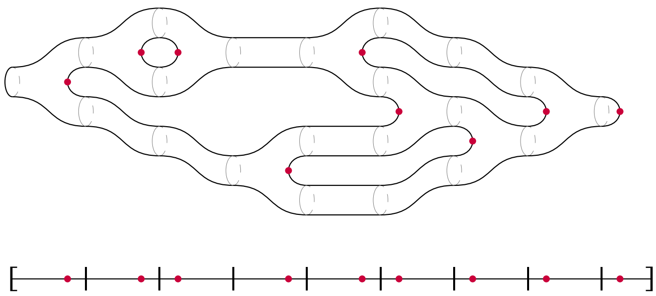


Figure 2.2: A cobordism decomposition with critical points isolated.

Proof of Theorem 2.1.3. Let $B\colon (\boldsymbol{m}) \to (\boldsymbol{n})$ be a two-dimensional cobordism. Choose a Morse function $f\colon B \to I$, that we assume such that any two critical points have different image. We take regular values $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$ such that each interval $[t_i, t_{i+1}]$ contains at most one critical value. Since each t_i is a regular value, its preimage $M_i := f^{-1}(t_i)$ is a smooth manifold.

[3]We can also do the process all at once, instead of iteratively. And in fact, our proof will be written that way, since it is shorter. But the core essence lies in being able to decompose any cobordism not already accounted for into at least two pieces!

Furthermore, the preimages of the intervals $B_i \coloneqq f^{-1}[t_i, t_{i+1}]$ are cobordisms $B_i\colon M_i \to M_{i+1}$ such that $B \cong B_{k+1} \circ \ldots \circ B_0$, where $B_0 = \partial_{\text{in}} B$ and $B_{k+1} = \partial_{\text{out}} B$.

In short, we have cut up our cobordism into smaller pieces, each with at most one critical point; see figure 2.2 for an example.

We want to generate each B_i from our six cobordisms. By the same arguments as in the one-dimensional case [Thm. 1.3.5], we reduce the problem to the connected case, as follows.

If B_i is any two-dimensional cobordism, we can apply Lemma 1.1.23 to factorize it as a composition $\rho \circ (\bigsqcup_j B_{i,j}) \circ \sigma$ of some permutation cobordisms and a disjoint union of connected cobordisms. By definition the permutation cobordisms are generated monoidally from the twist cobordism 2.1(f), so we can construct these. So if we can generate each connected component $B_{i,j}$, then we can apply disjoint union to generate $\bigsqcup_j B_{i,j}$ and from there B_i. Note that this process preserves the Morse functions, since $\bigsqcup_j B_{i,j}$ and B_i share the same underlying manifold.

Now, these smaller pieces $B_{i,j}$ will be connected cobordisms equipped with a Morse function with at most one critical point. But we have already classified these: the ones without critical points are cylinders [Lem. 2.1.2], while the ones with a single critical point are either discs or pairs of pants. More precisely: a critical point of index 0 is a local minimum, so this case corresponds to the cap. If the index is 2 then the point is a local maximum, so this is the cocap. And if the index is 1, we have a saddle point and this is either of the pairs of pants and, by Lemma 2.1.1.

All of these cobordisms are in our generating set, so we can reverse the previous process and write each $B_i = \rho \circ (\bigsqcup_j B_{i,j}) \circ \sigma$ using composition and disjoint union of our generators. By composing these B_i together, we obtain a cobordism isomorphic to our starting B. $\square$

One small remark about the proof, in which we also embed a small example of a decomposition.

Remark 2.1.4 — The other two pairs of pants.
The acute reader might remember that there are two other cobordisms whose underlying manifold is a disc with two discs removed. We discussed them all the way back in Remark 1.1.3, and we illustrated them in figures 1.3(b) and 1.3(c): they are $B'\colon \varnothing \to (\mathbf{3})$ and $B''\colon (\mathbf{3}) \to \varnothing$. Haven't we missed these on our proof?

Not at all. For one, it is easy to write an explicit decomposition into our generators, as we show in figure 2.3. But most importantly, since our Morse functions must satisfy $f^{-1}(0) = \partial_{\text{in}} B$ and $f^{-1}(1) = \partial_{\text{out}} B$, there is no way of constructing a smooth function from either cobordism to the interval I with only one critical point. This is why Lemma 2.1.1 specified that both preimages $f^{-1}(0)$ and $f^{-1}(1)$ must be non-empty.

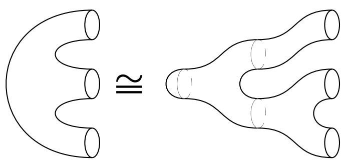


Figure 2.3: Decomposition of the "pair of pants" with three out-boundaries.

Now that we know that any 2D cobordism can be written using only our six building blocks, we want to provide a sufficient set of relations. For that, the strategy is to define a unique *normal form* for each connected cobordism in Cob_2, prove that two connected cobordisms are isomorphic if and only if their normal form coincide, and then construct a set of relations which lets us convert any one connected cobordism into its normal form. One must then extend this result to account for nonconnected cobordisms; that is a routine exercise for which we will spare the details.

2.1.2 Relations for Cob_2

The spider form of a connected 2D cobordism

So, what should our normal form look like? We factorize each connected cobordism $B\colon (\boldsymbol{m}) \to (\boldsymbol{n})$ as a composition of an *in-part* $(\boldsymbol{m}) \to (\boldsymbol{1})$, a *topological part* $(\boldsymbol{1}) \to (\boldsymbol{1})$ and an *out-part* $(\boldsymbol{1}) \to (\boldsymbol{n})$. Each of these must be of a certain form; this is easier to describe with a visual example, as in figure 2.4.

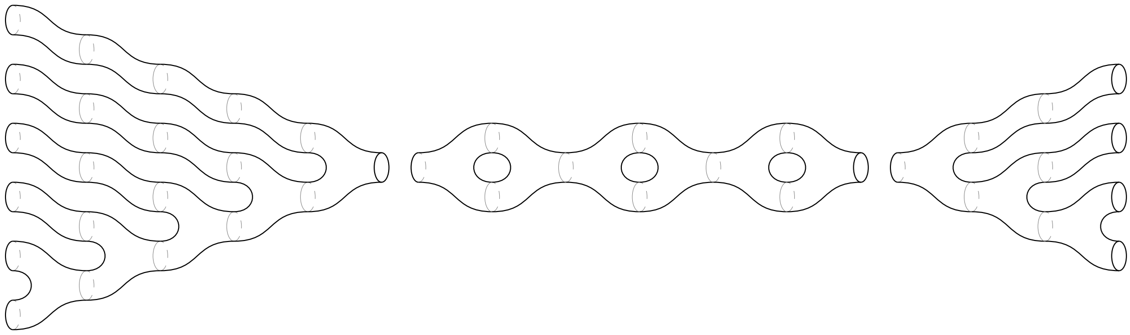


Figure 2.4: The spider form of a cobordism with $m = 5$, $g = 3$, $n = 3$.

The in-part consists of only copairs of pants ( ) and cylinders, arranged such that each pair of pants attaches to the lower leg of the next, and each cylinder attaches either to the top leg of a pair of pants or to the next cylinder. The out-part is the dual construction, consisting of pairs of pants ( ) and cylinders. If $m = 0$, the in-part consists of a single cap ( ). Likewise, if $n = 0$, the out-part consists of a single cocap ( ).

The topological part consists of a composition of g copies of a building block which is obtained by composing both types of pairs of pants.

We call this the ***spider form*** or *normal form* of a connected 2D cobordism. Any connected 2D cobordism is isomorphic to one of this form (as we will prove), and it is already clear that this form is unique: the in-part is characterized by its number of in-boundaries, the out-part by its number of out-boundaries, and the topological part by the *genus* of the surface. And, clearly, two cobordisms with different number of in- or out-boundaries can't be isomorphic. Of course, the genus invariance is the content of the classification of surfaces.

Furthermore, these three numbers are a complete invariant: that is, two connected cobordisms for which these three numbers coincide are isomorphic.

We could have used the classification of surfaces to provide a very short proof of Theorem 2.1.3, by comparing any cobordism with its normal form. However, the tools of Morse theory are generalizable to higher dimensions, where we do not have the luxury of invoking a classification of manifolds.[4]

Now, we want to find a set of relations which let us transform any combination of the six generators of figure 2.1 into its normal form. Which relations should we use? Let us play with our building blocks for a bit and see which interesting combinations we find.

The main list of relations

We take all of these relations as topologically evident. They are also a immediate consequence of the classification of compact connected surfaces, since one can just "count holes" (i.e., compute the genus) and keep track of the in- and out-boundaries. Nevertheless, the reader wanting a visual proof can consult [Koc03, 1.4.29–1.4.34].

[4]To obtain this result without ultimately appealing to the classification of surfaces, one can use *Cerf theory*. Essentially, Cerf theory studies *families* of Morse functions and *paths* between Morse functions; this is nicely explained in the introduction of [Sch14].

Given a pair of pants, we can "sew shut" one of the legs by attaching a cap. This yields a cylinder, as in figure 2.5.

Figure 2.5: Sewing shut the legs of a pair of pants by attaching a cap.

Given a pair of pants with a cylinder and another pair of pants attached to its legs, it does not matter which leg gets attached what; see figure 2.6 We call these the *(co)associativity* axioms, since our (co)pants are analogous to (co)multiplication.

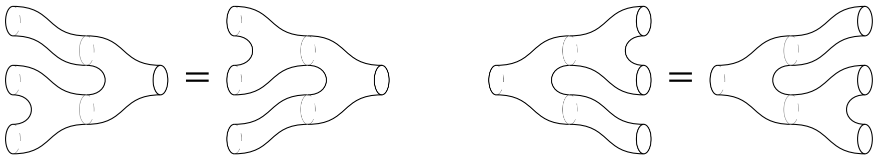


Figure 2.6: Associativity and coassociativity.

In this same analogy, our operation (co)commutes: sewing a twist cobordism to a pair of pants does nothing. See figure 2.7.

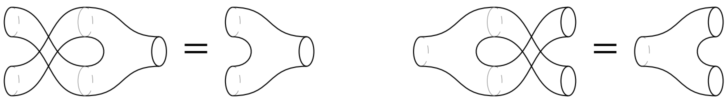


Figure 2.7: Commutativity and cocommutativity.

Finally, our last relation has a proper name attached: it is the *Frobenius relation* of figure 2.8. This is very similar to Zorro's Lemma [fig. 1.9], but changing the "arched" cylinders  and  by pairs of pants  and .

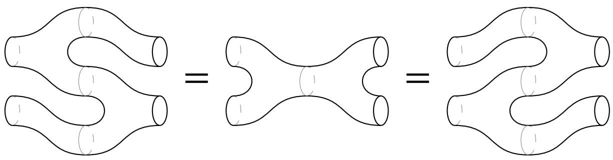


Figure 2.8: The Frobenius relation.

These relations are sufficient when considering only four (symmetric monoidal) generators; that is, ignoring the cylinder and the twist cobordism. And if we only consider two (rigid symmetric monoidal) generators, then the left-hand side of each figure suffices.

Since we have already talked enough about the axioms of symmetric categories, we will not give explicit relations for the twists (apart from the required figure 2.7). Even more so for the cylinders, which are just identities! The reader may consult [Koc03, 1.4.24, 1.4.35] for pictures of these, or try to deduce them from figure 1.14.

Before proving the sufficiency of our relations, we need to remember the properties of the Euler characteristic of a surface.

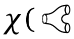
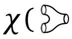
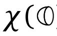
Recollections on the Euler characteristic

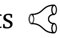
Remember that the Euler characteristic of an oriented surface with boundary is

$$\chi(S) = 2 - 2g - p,$$

where g is its genus and p is the number of connected componentes of its boundary. Of course, this is invariant by diffeomorphism. For 2D cobordisms, we can write $p = m + n$, where m is the number of in-boundaries and n is the number of out-boundaries.

Note that

$\chi($  $) = \chi($  $) = 1, \qquad \chi($  $) = \chi($  $) = -1,$

while the cylinders (and therefore the twist cobordism) have characteristic 0. So we have two ways of computing the Euler characteristic of a decomposed cobordism. The first way is the topological argument made above, while the second way is combinatorial: if our decomposition consists of a pairs of pants , b copairs of pants , p caps  and q cocaps , then its Euler characteristic must be $\chi(B) = p + q - a - b$.

Therefore,

$$2 - 2g - m - n = \chi(B) = p + q - a - b.$$

On the other hand, we can sum the total number of in- and out- boundaries to get $2a+b+q+n = a + 2b + p + m$, or equivalently

$$a + q + n = b + p + m.$$

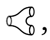
In conclusion,

$$a = m + g - 1 + p, \qquad b = n + g - 1 + q.$$

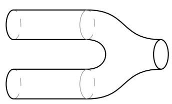
Proving sufficiency of the relations

We will now describe an algorithm for converting any decomposed connected cobordism to normal form.

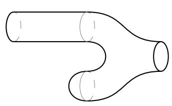
The argument proceeds by induction on the number of twist cobordisms () in our decomposition. First, let's prove the base case, with no twists.

Moving all copants left. We want to move each  leftwards, as much as it can go. Some of them will end up to the left of every , to form the in-part of the spider form. Others will become "locked" in place, and will form the topological part of the spider form.

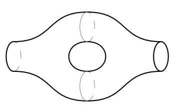
We will use a different relation depending on what we encounter to the left of our pants. The first case is the most obvious: if we encounter two cylinders then we just remove the cylinders to get an equivalent cobordism.



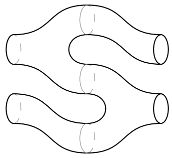
The second most simple case is encountering a cap, meaning either the picture



or its vertical reflection. Here we apply the relations for sewing in caps [fig. 2.5], to get a cylinder. The third and final case is encountering the opposite pair of pants, which can go in either of two ways. First, the case



is the “lock” condition we mentioned before. We do not do anything in this case; this will eventually form our topological part. On the other hand, the case



is solvable: it suffices to apply the Frobenius relation of figure 2.8.

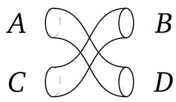
If we encounter another copair of pants (the ones we are moving, oriented as), then we move that one as left as it can go before continuing our process. If it is locked in place by an opposing , then we can use associativity [fig. 2.6] and the Frobenius relation [fig. 2.8] again in order to pass it safely.

We start the process with $a = m + g - 1 + p$ copies of . Of them, p vanish by meeting a cap ; another $m - 1$ pass all the way left to form the in-part; and the final g are locked, stuck against a in the position, waiting until they can form the topological part.

Moving pants right. Now we do the analogous process for the dual pairs of pants, the pieces, forming the out-part of the spider form.

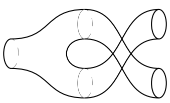
Finally, we take all which are neither in the in- nor the out-part and move them left until they collide with an opposite and stop in place. This forms the topological part, with exactly g copies of each type of pair of pants.

Dealing with twists. Pick any twist cobordism in our decomposition and label its boundaries.



Assume that all pieces parallel to the twist are cylinders. Then, we want to consider the topology of our piece after cutting out the portion corresponding to our twist (but not the parallel cylinders).[5]

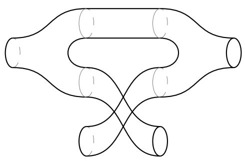
Since our surface is connected, some of the regions A, B, C and D must remain connected with each other after the cut. First, assume A and C are connected. Then they formed a connected surface with strictly less twists than our original cobordism, so by induction we can assume it can be brought to spider form. Then only the out-part touches our twist, and we can until we have exactly the situation in the picture below.



Now, by cocommutativity 2.7, we can remove our twist.

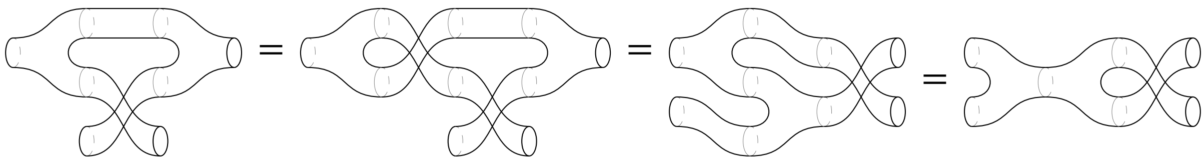
The same argument applies to the case in which B and D are connected, so it only remains to consider the case in which A and B are connected.

Assuming A and B are connected, we can disconnect them by taking out the cylinders parallel to our twist. Then, both of them have less twists than our original, so we can bring them to spider form. Adding back the pieces we cut out, near we must have the situation of the following picture.



[5]The cut can be done by choosing a Morse function $f\colon B \to I$ and taking the preimage of an interval, as usual; and then gluing back together the parts where we cutted out cylinders. In our drawings, we implicitly assume the Morse function is the projection to the horizontal line.

This twist is eliminated by the following argument.



One then uses cocommutativity, of course. The first step is also cocommutativity and the third step is the Frobenius relation. The second step comes from the properties of the twist maps in any symmetric monoidal category.

About the non-connected case

Now, in reality we have only proved the connected case; to be thorough, we still would need to prove that our relations encode all relations that can occur between non-connected cobordisms. But the connected case was the interesting part!

For the non-connected case, we can define a normal form which also incorporates a "permutation in-part" and "permutation out-part", akin to Lemma 1.1.23 (the lemma which let us factorize every cobordism into a permutation, a disjoint union of connected cobordisms and another permutation). Then, we move twists as far as they can go in either direction (until we obtain an in- and out-part consisting of only twists and cylinders), and process each inner connected spider form as we did before. We won't bore the reader with the details — after all, most proofs in a symmetric monoidal category proceed by first considering the "connected" objects and then applying routine arguments to generalize to every other object, a process which we have already repeated multiple times.

So, with our presentation of Cob_2 done, let's take a break from cobordisms and spend some time talking about Frobenius algebras.

2.2 Frobenius algebras

Here we introduce the notion of Frobenius algebras, in a fairly standard manner, and then slowly morph it into a number of equivalent formulations which make their connection to TQFTs clear.

2.2.1 Four different characterizations of Frobenius algebras

There are multiple equivalent definitions of "Frobenius algebra", and they all emphasize different aspects important for our classification. We don't strictly need them all, but it doesn't hurt to be aware of their existence. First, it helps to fix an appropriate definition of "algebra".

Algebras over a field

We assume all algebras to be associative and unital, but not necessarily commutative, unless stated otherwise.

Definition 2.2.1. An ***algebra*** A over a field $\Bbbk$ is a $\Bbbk$-vector space equipped with linear maps $\mu\colon A\otimes A\to A$ (multiplication) and $\eta\colon \Bbbk\to A$ (unit map) such that the following diagrams commute.

$$\begin{tikzcd} & A\otimes A\otimes A \arrow[dl, "\mu\otimes\mathrm{id}_A"'] \arrow[dr, "\mathrm{id}_A\otimes\mu"] & \\ A\otimes A \arrow[dr, "\mu"'] & & A\otimes A \arrow[dl, "\mu"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} A\otimes A \arrow[dr, "\mu"] & & A\otimes A \arrow[dl, "\mu"'] \\ \Bbbk\otimes A \arrow[u, "\eta\otimes\mathrm{id}_A"] \arrow[r] & A & A\otimes\Bbbk \arrow[l] \arrow[u, "\mathrm{id}_A\otimes\eta"'] \end{tikzcd}$$

(associativity) (unit laws).

This is equivalent to the more standard definition of $\Bbbk$-algebra: a $\Bbbk$-vector space equipped a product $A\times A\to A$ which distributes over addition, is associative and unital, and is compatible with scalar product. In our definition, the unit is the image of $1\in\Bbbk$ under the embedding η, while the distributivity is encoded by requiring our map $\mu\colon A\otimes A\to A$ to be linear (i.e., μ is bilinear when regarded as a map $A\times A\to A$). The axioms encode associativity and compatibility with scalars; note that the maps $\Bbbk\otimes A\to A$ and $A\otimes\Bbbk\to A$ are the canonical scalar multiplication.

The observant reader might notice that these are the same axioms as those for a monoidal category [Def. 1.2.10], but changing categories to vector spaces and product of categories to tensor product of vector spaces. The proper way to state this is to say that $\Bbbk$-algebras are monoids *internal* to the category $(\mathsf{Vect}_{\Bbbk},\otimes,\Bbbk)$ of $\Bbbk$-vector spaces equipped with tensor product. We will define "internal monoid" properly in section 2.3.2.

Homomorphisms of $\Bbbk$-algebras, too, admit a definition in terms of commutative diagrams.

Definition 2.2.2. A $\Bbbk$***-algebra homomorphism*** $\varphi\colon (A,\mu,\eta)\to(A',\mu',\eta')$ is a linear map which commutes with the multiplication and unit. That is, the following diagrams commute.

$$\begin{tikzcd} A\otimes A \arrow[r, "\varphi\otimes\varphi"] \arrow[d, "\mu"'] & A'\otimes A' \arrow[d, "\mu'"] \\ A \arrow[r, "\varphi"'] & A' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\varphi"] & A' \\ \Bbbk \arrow[u, "\eta"] \arrow[r, equal] & \Bbbk \arrow[u, "\eta'"'] \end{tikzcd}$$

Now, we want to define modules over our algebra A. This isn't strictly necessary for most of our definitions, but they will play a crucial role in section 4.2. In this chapter they are mostly used for Characterization 2.2.17, a conceptually nice but nonessential characterization.

Modules over an algebra

Once again, we state most definitions in terms of commutative diagrams.

Definition 2.2.3. A ***right module*** over an algebra A is a vector space M equipped with a right A-action $\alpha\colon M \otimes A \to M$ such that both of the following diagrams commute.

$$
\begin{tikzcd}
M \otimes A \otimes A \arrow[r, "\alpha \otimes \mathrm{id}_A"] \arrow[d, "\mathrm{id}_M \otimes \mu"'] & M \otimes A \arrow[d, "\alpha"] \\
M \otimes A \arrow[r, "\alpha"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
M \otimes A \arrow[d, "\alpha"'] & M \otimes \Bbbk \arrow[l, "\mathrm{id}_M \otimes \eta"'] \arrow[dl] \\
M &
\end{tikzcd}
$$

(associativity) (unity)

If we write $x \cdot a := \alpha(x \otimes a)$ — to differentiate from the algebra multiplication $ab := \mu(a \otimes b)$ — then elementwise we have $(x \cdot a) \cdot b = x \cdot (ab)$ and $x \cdot 1 = x$. And, once again, distributivity is encoded in the tensor product $M \otimes A$.

Definition 2.2.4. Let M and N be right A-modules. A ***right A-linear map***, or *right A-module homomorphism*, is a $\Bbbk$-linear map $\varphi\colon M \to N$ for which the following diagram commutes.

$$
\begin{tikzcd}
M \otimes A \arrow[r, "\varphi \otimes \mathrm{id}_A"] \arrow[d] & N \otimes A \arrow[d] \\
M \arrow[r, "\varphi"'] & N
\end{tikzcd}
$$

Elementwise, this is the condition $\varphi(x \cdot a) = \varphi(x) \cdot a$.

We also need left-handed versions of these concepts. These are given by a left A-action $\alpha\colon A \otimes N \to N$ instead, and satisfy analogous axioms. We state them explicitly, for completeness.

Definition 2.2.5. A ***left module*** over an algebra A is a vector space N equipped with a left A-action $\alpha\colon A \otimes N \to M$ such that both of the following diagrams commute.

$$
\begin{tikzcd}
A \otimes N \arrow[d, "\alpha"'] & A \otimes A \otimes N \arrow[l, "\mathrm{id}_A \otimes \alpha"'] \arrow[d, "\mu \otimes \mathrm{id}_N"] \\
N & A \otimes N \arrow[l, "\alpha"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes N \arrow[r, "\eta \otimes \mathrm{id}_N"] \arrow[dr] & A \otimes N \arrow[d, "\alpha"] \\
& N
\end{tikzcd}
$$

(associativity) (unity)

Definition 2.2.6. Let M and N be left A-modules. A ***left A-linear map***, or *left A-module homomorphism*, is a $\Bbbk$-linear map $\varphi\colon N \to M$ for which the following diagram commutes.

$$
\begin{tikzcd}
A \otimes N \arrow[d] & A \otimes M \arrow[l, "\mathrm{id}_A \otimes \varphi"'] \arrow[d] \\
N & M \arrow[l, "\varphi"]
\end{tikzcd}
$$

Note that an algebra A canonically carries a right and left A-module structure, where the action $\alpha\colon A \otimes A \to A$ is given by the multiplication μ. The fact that these two actions are compatible (i.e., $a \cdot (x \cdot b) = (a \cdot x) \cdot b$) makes it an instance of something called a *bimodule structure*, which we will define in section 4.2.

Finally, note that the dual vector space of a right A-module M has the structure of a left A-module: we define $A \otimes M^*$ as $a \otimes \Lambda \mapsto a \cdot \Lambda := [x \mapsto \Lambda(x \cdot a)]$. Dually, the dual vector space of a left A-module carries a canonical right A-module structure.

Now, let's state our first definition of Frobenius algebra.

In terms of Frobenius forms

Definition 2.2.7. A ***Frobenius algebra*** is a finite-dimensional $\Bbbk$-algebra A equipped with a linear functional $\varepsilon\colon A \to \Bbbk$ whose kernel $\operatorname{Ker}\varepsilon$ contains no nontrivial left ideals.

The functional $\varepsilon \in A^*$ is called the ***Frobenius form*** or *Frobenius structure*. A useful interpretation is thinking of ε as a trace map; indeed, one of the more elementary examples of Frobenius algebras is that of square matrices $\mathrm{M}_{\Bbbk}(n)$ equipped with the trace $\mathrm{tr}\colon \mathrm{M}_{\Bbbk}(n) \to \Bbbk$.

Note that the kernel $\operatorname{Ker}\varepsilon$ is generally *not* an ideal, since ε is generally *not* a $\Bbbk$-algebra homomorphism. The condition of $\operatorname{Ker}\varepsilon$ having no nontrivial left ideals is equivalent to saying that $\varepsilon(Ax) := \{\varepsilon(ax) \mid a \in A\} = 0$ implies $x = 0$, which is the main characterization that we will use. Furthermore, it is equivalent to having no nontrivial *right* ideals [Koc03, 2.2.4].

A given $\Bbbk$-algebra can admit multiple distinct Frobenius structures, and it must be specified. As usual, we will sometimes abuse nomenclature and leave it unstated, when the context is enough to imply it.

We leave the examples until the end of this section, when we will have already deduced a number of equivalent ways of defining a Frobenius structure.

In terms of pairings and copairings

A *pairing* of vector spaces is just a linear map with certain domain and codomain.

Definition 2.2.8. A ***pairing*** of two $\Bbbk$-vector spaces V and W is a linear map $\beta\colon V \otimes W \to \Bbbk$.

As all things tensorial, we can equivalently rephrase the definition of "pairing" as a bilinear map $V \times W \to \Bbbk$. We often write its action on elements using bra-ket notation: $\langle v \mid w\rangle := \beta(v \otimes w)$.

A *copairing* is just the dual concept.

Definition 2.2.9. A ***copairing*** of two $\Bbbk$-vector spaces V and W is a linear map $\gamma\colon \Bbbk \to W \otimes V$.

Note that we invert the order of the two spaces. That is done since we are interested in the following definition.

Definition 2.2.10. A pairing $\beta\colon V \otimes W \to \Bbbk$ is ***non-degenerate*** if there exists a compatible copairing $\gamma\colon \Bbbk \to W \otimes V$ in the sense that the two following compositions equal the identities:

$$\mathrm{id}_V = (\beta \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \gamma)\colon\ V \cong V \otimes \Bbbk \xrightarrow{\mathrm{id}\otimes\gamma} V \otimes W \otimes V \xrightarrow{\beta\otimes\mathrm{id}} \Bbbk \otimes V \cong V,$$
$$\mathrm{id}_W = (\mathrm{id}_W \otimes \beta) \circ (\gamma \otimes \mathrm{id}_W)\colon\ W \cong \Bbbk \otimes W \xrightarrow{\gamma\otimes\mathrm{id}} W \otimes V \otimes W \xrightarrow{\mathrm{id}\otimes\beta} W \otimes \Bbbk \cong W.$$

Our old friend Zorro appears once again, with a new disguise! As we have mentioned before, the notion of duality [Def. 1.2.19] is pervasive in category theory, and even more so in the study of TQFTs.

Just as was argued in Zorro's Lemma [1.2.3], a vector space equipped with a non-degenerate pairing is necessarily of finite dimension. Furthermore, a non-degenerate pairing determines a linear isomorphism $W \to V^*$, by mapping $w \mapsto \langle - \mid w\rangle$ (the adjoint map). We can also fix v instead, and get the dual linear isomorphism $V \to W^*$ given by $v \mapsto \langle v \mid -\rangle$. These two facts are analogous to Corollary 1.2.4. If we assume our vector spaces to have finite dimension, then we also have the converse: if one of these maps is an isomorphism, then the pairing is non-degenerate.

Lemma 2.2.11: Zorro's Lemma for pairings. A pairing $\beta\colon V \otimes W \to \Bbbk$ is non-degenerate if and only if V and W have finite dimension and the induced maps $\beta_{\mathrm{left}}\colon W \to V^*$ and $\beta_{\mathrm{right}}\colon V \to W^*$ are injective.

Proof. The proof of the direct implication is analogous to those of Lemma 1.2.3 and Corollary 1.2.4, so we only need to prove the reverse implication. Let $\beta\colon V \otimes W \to \Bbbk$ be a pairing between vector spaces of finite dimension such that the two induced maps β_{left} and β_{right} are injective.

Suppose that W is finite-dimensional and β_{left} is injective. We can choose a basis $w_1, \ldots, w_n$, and by injectivity the images $\langle - \mid w_1 \rangle, \ldots, \langle - \mid w_n \rangle$ must also be linearly independent. So there exist vectors $v_1, \ldots, v_n \in V$ such that $\langle v_i \mid w_i \rangle = 1$ and $\langle v_j \mid w_i \rangle = 0$ for $i \neq j$.

We define a copairing $\gamma \colon \Bbbk \to W \otimes V$ by mapping $1 \mapsto \sum_{i=1}^{n} w_i \otimes v_i$. Evaluating the composition $W \to W \otimes V \otimes W \to W$ on an arbitrary element $w = \sum_{j=1}^{n} \lambda_j w_j$ in W yields

$$\sum_{j=1}^{n} \lambda_j w_j \longmapsto \sum_{i,j=1}^{n} w_i \otimes v_i \otimes \lambda_j w_j \longmapsto \sum_{i,j=1}^{n} \lambda_j w_i \langle v_i \mid w_j \rangle = \sum_{j=1}^{n} \lambda_j w_j,$$

so this composition equals the identity on W.

This is one of the two compatibility conditions needed for non-degeneracy. For the other condition, we do the analogous process for the map β_{right} and the vector space V, thus proving that the composition $V \to V \otimes W \otimes V \to V$ is the identity and the pairing β is non-degenerate. $\square$

Now note that in the finite-dimensional case the maps β_{left} and β_{right} are duals, so requiring the injectivity of both is equivalent to requiring a single one of them to be an isomorphism. Indeed, the dual of β_{right} is the map $W^{**} \to V^*$ which maps $T \colon W^* \to \Bbbk$ into $T \circ \beta_{\text{right}} \colon V \to W^* \to \Bbbk$. By identifying $W \cong W^{**}$ through $w \mapsto T_w := [\Lambda \mapsto \Lambda(w)]$, this dual β^*_{right} maps w to $T_w \circ \beta_{\text{right}}$, which is the map $v \mapsto \beta_{\text{right}}(v)(w) = \langle v \mid w \rangle$. So β^*_{right} maps $w \mapsto [v \mapsto \langle v \mid w \rangle]$, and therefore coincides with β_{left}. (cf. the remark after Corollary 1.2.4). In summary, we have proven the following lemma.

Lemma 2.2.12. Let $\beta \colon V \otimes W \to \Bbbk$ be a pairing between finite-dimensional vector spaces. The following are equivalent:

(I) The pairing β is non-degenerate.

(II) The map $\beta_{\text{left}} \colon W \to V^*$ given by $w \mapsto \langle - \mid w \rangle$ is a linear isomorphism.

(III) The map $\beta_{\text{right}} \colon V \to W^*$ given by $v \mapsto \langle v \mid - \rangle$ is a linear isomorphism.

If $\dim V = \dim W < \infty$, then we also deduce the two equivalent conditions

$$\begin{aligned} \langle v \mid w \rangle = 0 \ \ \forall v \in V &\implies w = 0; \\ \langle v \mid w \rangle = 0 \ \forall w \in W &\implies v = 0. \end{aligned}$$

This justifies the name "non-degenerate". Of course, when $\Bbbk = \mathbb{R}$ and β is positive-definite, this is just an inner product.

Note that given a non-degenerate pairing β, there is a *unique* copairing γ realizing the equations of Definition 2.2.10. This follows from a standard argument; we include a graphical proof in Lemma 2.2.30.

One last definition, before we relate these to Frobenius algebras. We state it in the general case, in terms of A-modules, but note that in practice both M and N will be A!

Definition 2.2.13. Let A be a $\Bbbk$-algebra, M a right A-module and N a left A-module. A pairing $\beta \colon M \otimes N \to \Bbbk$ is *associative* if the following diagram commutes.

$$\begin{tikzcd} & M \otimes A \otimes N \arrow[dl, "\alpha_M \otimes \mathrm{id}_N"'] \arrow[dr, "\mathrm{id}_M \otimes \alpha_N"] & \\ M \otimes N \arrow[dr, "\beta"'] & & M \otimes N \arrow[dl, "\beta"] \\ & \Bbbk & \end{tikzcd}$$

That is, $\langle xa \mid y \rangle = \langle x \mid ay \rangle$ for any $x \in M$, $a \in A$ and $y \in N$. We can characterize associative pairings in a manner similar to Lemma 2.2.12.

Lemma 2.2.14. Let $\beta\colon M \otimes N \to \Bbbk$ be a pairing between A-modules. The following are equivalent:

(I) The pairing β is associative.

(II) The map $\beta_{\text{left}}\colon N \to M^*$ given by $a \mapsto \langle - \mid a\rangle$ is left A-linear.

(III) The map $\beta_{\text{right}}\colon M \to N^*$ given by $a \mapsto \langle a \mid -\rangle$ is right A-linear.

PROOF. Right A-linearity of $\beta_{\text{right}}\colon M \to N^*$ is defined by the commutativity of the diagram

$$\begin{tikzcd}
M \otimes A \arrow[r, "\beta_{\text{right}} \otimes \mathrm{id}_A"] \arrow[d] & N^* \otimes A \arrow[d] \\
M \arrow[r, "\beta_{\text{right}}"'] & N^*.
\end{tikzcd}$$

Elementwise, this means $x \otimes a \mapsto [y \mapsto \langle x \mid ay\rangle]$ and $x \otimes a \mapsto [y \mapsto \langle xa \mid y\rangle]$ must be the same map. But that yields precisely the definition for associativity of β: $\langle x \mid ay\rangle = \langle xa \mid y\rangle$ for all $x \in M$, $a \in A$ and $y \in N$.

Analogously for the left A-linearity of β_{left}. □

Now for the Frobenius characterization. Note that, given any linear functional $\lambda\colon A \to \Bbbk$, we can define a pairing $A \otimes A \to \Bbbk$ by taking products: $\langle x \mid y\rangle := \lambda(xy)$. As such, our Frobenius form $\varepsilon\colon A \to \Bbbk$ canonically determines a pairing $\beta\colon A \otimes A \to \Bbbk$. This pairing is associative, of course, since $\langle xa \mid y\rangle = \lambda(xay) = \langle x \mid ay\rangle$.

Conversely, given an associative pairing $A \otimes A \to \Bbbk$, we can construct a functional $A \to \Bbbk$ by setting one of the coordinates to the unit of A: $a \mapsto \langle 1 \mid a\rangle = \langle a \mid 1\rangle$.

Now, the important result.

Lemma 2.2.15. Let A be a finite-dimensional $\Bbbk$-algebra. An associative pairing $\beta\colon A \otimes A \to \Bbbk$ is non-degenerate if and only if the functional $\varepsilon\colon A \to \Bbbk$ given by $a \mapsto \langle 1 \mid a\rangle$ is a Frobenius form.

PROOF. Let $\beta\colon A \otimes A \to \Bbbk$ be an associative pairing. Since the dimensions of both paired modules A and A coincide and are finite, we get from the corollary of Lemma 2.2.12 that β is non-degenerate if and only if $\langle A \mid y\rangle = 0$ implies $y = 0$.

But $\varepsilon\colon A \to \Bbbk$ is a Frobenius form if and only if its kernel contains no nontrivial left ideals; equivalently, if and only if $\varepsilon(Ay) = 0$ implies $y = 0$. □

Characterization 2.2.16: In terms of pairings. A Frobenius $\Bbbk$-algebra A is a $\Bbbk$-algebra equipped with an associative and non-degenerate pairing $\beta\colon A \otimes A \to \Bbbk$. (Therefore, it is also canonically equipped with a copairing $\gamma\colon \Bbbk \to A \otimes A$).

The equivalence is obtained by setting $\varepsilon(a) := \beta(1 \otimes a)$ and $\beta(x \otimes y) := \varepsilon(xy)$.

Note that this definition doesn't specify A to be of finite dimension, since that follows by Lemma 2.2.11. And by applying Lemmas 2.2.12 and 2.2.14, we immediately deduce another characterization.

Characterization 2.2.17. A Frobenius $\Bbbk$-algebra A is a finite-dimensional $\Bbbk$-algebra equipped with a left A-linear isomorphism to its dual $A \to A^*$.

This characterization is less relevant for our purposes, but it can be used to give easy examples of Frobenius algebras. Of course, by symmetry we can instead equip A with a *right* A-linear isomorphism $A \to A^*$.

In terms of coalgebras

We started this section by defining a (unital, associative) $\Bbbk$-algebra as a vector space equipped with maps $\mu\colon A\otimes A\to A$ and $\eta\colon \Bbbk\to A$, satisfying certain axioms [Def. 2.2.1]. We can invert all arrows to get the dual concept; that of a (counital, coassociative) $\Bbbk$-*coalgebra*.

Definition 2.2.18. A ***coalgebra*** A over a field $\Bbbk$ is a $\Bbbk$-vector space equipped with linear maps $\delta\colon A\to A\otimes A$ (comultiplication) and $\varepsilon\colon A\to\Bbbk$ (counit map) such that the following diagrams commute.

$$\begin{tikzcd} & A\otimes A\otimes A & \\ A\otimes A \arrow[ur, "\delta\otimes\mathrm{id}_A"] & & A\otimes A \arrow[ul, "\mathrm{id}_A\otimes\delta"'] \\ & A \arrow[ul, "\delta"] \arrow[ur, "\delta"'] & \end{tikzcd} \qquad \begin{tikzcd} A\otimes A \arrow[d, "\varepsilon\otimes\mathrm{id}_A"'] & & A\otimes A \arrow[d, "\mathrm{id}_A\otimes\varepsilon"] \\ \Bbbk\otimes A & A \arrow[l] \arrow[r] \arrow[ul, "\delta"'] \arrow[ur, "\delta"] & A\otimes\Bbbk \end{tikzcd}$$

(coassociativity) (counit laws).

In the same manner, coalgebra morphisms are obtained by reversing all arrows in Definition 2.2.2.

Definition 2.2.19. A $\Bbbk$-***coalgebra homomorphism*** $\varphi\colon (A,\delta,\varepsilon)\to(A',\delta',\varepsilon')$ is a linear map which commutes with the comultiplication and counit:

$$\begin{tikzcd} A'\otimes A' & A\otimes A \arrow[l, "\varphi\otimes\varphi"'] \\ A' \arrow[u, "\delta'"] & A \arrow[l, "\varphi"] \arrow[u, "\delta"'] \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[d, "\varepsilon'"'] & A \arrow[l, "\varphi"'] \arrow[d, "\varepsilon"] \\ \Bbbk \arrow[r, equal] & \Bbbk \end{tikzcd}$$

Coalgebras are clunky to work with elementwise, and as such we won't give too many explicit examples. But we can spare time for one or two. We can think of comultiplication as a sort of generalized diagonal map, as the following example shows.

Example 2.2.20 — The free coalgebra on a set.
Let S be a set and let $A=\Bbbk^S$ be the free $\Bbbk$-vector space on S; this is the set of functions $S\to\Bbbk$ with finite support (i.e., such that only finitely many elements get mapped to values other than zero). We can identify an element $s\in S$ with the function $s\mapsto 1$ and $r\mapsto 0$ for $r\neq s$, and then define $\delta(s)=s\otimes s$ and $\varepsilon(s)=1$ for each $s\in S$. This endows A with the structure of a coalgebra.

Example 2.2.21 — The divided power coalgebra.
A similar example comes from the vector space $\Bbbk[t]$ of polynomials in one variable. Here, we can define the *divided power coalgebra*:

$$\delta(t^n)=\sum_{k=0}^{n}\binom{n}{k}t^k\otimes t^{n-k}, \qquad \varepsilon(t^n)=\begin{cases}1, & n=0;\\ 0, & n>0.\end{cases}$$

This is a *bialgebra*: an algebra which is also a coalgebra, and such that the two structures satisfy a certain compatibility condition we will not specify. Similarly, the tensor and exterior algebras admit a bialgebra structure.

Example 2.2.22 — The trigonometric coalgebra.
Let $V=\langle s,c\rangle$ be a $\Bbbk$-vector space on two generators. We define

$$\delta(s)=s\otimes c+c\otimes s, \qquad \delta(c)=c\otimes c-s\otimes s, \qquad \varepsilon(s)=0, \qquad \varepsilon(c)=1.$$

The name comes from the analogous trigonometric identities:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta, \qquad \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta.$$

We now state our final characterization.

Characterization 2.2.23: In terms of coalgebras. A Frobenius $\Bbbk$-algebra A is a $\Bbbk$-vector space equipped with both an algebra structure $\mu\colon A\otimes A\to A$, $\eta\colon \Bbbk\to A$ and a coalgebra structure $\delta\colon A\to A\otimes A$, $\varepsilon\colon A\to\Bbbk$ such that the Frobenius relation holds:

$$(\mu\otimes \mathrm{id}_A)\circ(\mathrm{id}_A\otimes\delta)=\delta\circ\mu=(\mathrm{id}_A\otimes\mu)\circ(\delta\otimes\mathrm{id}_A).$$

The equivalence is obtained by setting the Frobenius form to be the counit ε. If you instead have a Frobenius form ε, you construct the comultiplication by defining $\delta:=(\mathrm{id}_A\otimes\mu)\circ(\gamma\otimes\mathrm{id}_A)$, where γ is the copairing induced by ε (see Characterization 2.2.16).

In fact, in this characterization we don't even need our algebra to be associative nor our coalgebra to be coassociative; those are consequences of the Frobenius relation, as we will see. If we require (co)associativity, then it suffices to require $(\mu\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\delta)=(\mathrm{id}_A\otimes\mu)\circ(\delta\otimes\mathrm{id}_A)$; the middle identity can be deduced from the other axioms. See Lemma 2.2.39 below.

Proving this characterization is very messy when using usual mathematical notation (or even with commutative diagrams), so we postpone the proofs until section 2.2.2. There we will accustom ourselves with a curious form of notation: *graphical calculus*.

Take care that the notion of *bialgebra* commonly found in the literature also refers to an algebra which is simultaneously a coalgebra, but where the two structures satisfy a radically different compatibility condition.

Summary of the characterizations

Characterizations 2.2.24. A Frobenius $\Bbbk$-algebra is a $\Bbbk$-algebra A equipped with either of the following.

(i) A linear functional $\varepsilon\colon A\to\Bbbk$ whose kernel $\operatorname{Ker}\varepsilon$ contains no nontrivial left ideals, together with the property of A being of finite dimension.

(ii) An associative non-degenerate pairing $\beta\colon A\otimes A\to\Bbbk$. (As well as a copairing $\gamma\colon\Bbbk\to A\otimes A$).

(iii) A coalgebra structure $\delta\colon A\to A\otimes A$, $\varepsilon\colon A\to\Bbbk$ such that the Frobenius relation holds:

$$(\mu\otimes \mathrm{id}_A)\circ(\mathrm{id}_A\otimes\delta)=\delta\circ\mu=(\mathrm{id}_A\otimes\mu)\circ(\delta\otimes\mathrm{id}_A).$$

These are only a tiny slice of the plethora of characterizations there exist for Frobenius algebras. The curious reader might want to consult [Sch14, Prop. 3.64].

Now we want to define what it means for a Frobenius algebra to be *symmetric*; these will play a prominent role in chapter 4.

Definition 2.2.25. A Frobenius algebra A is ***symmetric*** if any of the following equivalent conditions holds:

(i) The Frobenius form $\varepsilon\colon A\to\Bbbk$ is *central*: $\varepsilon(ab)=\varepsilon(ba)$ for all $a,b\in A$.

(ii) The pairing β is *symmetric*: $\langle a\mid b\rangle=\langle b\mid a\rangle$ for all $a,b\in A$.

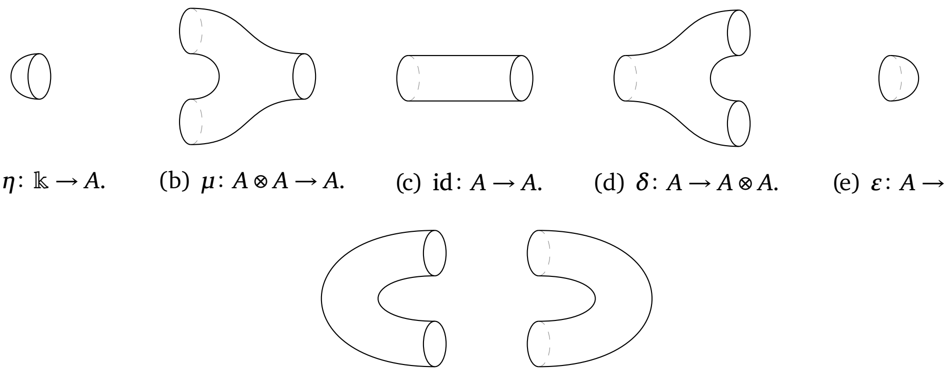
(iii) The counit $\varepsilon\colon A\to\Bbbk$ is central.

Of course, the third of these is redundant; the counit *is* the Frobenius form. But ignoring it would (quite ironically!) break symmetry with Characterizations 2.2.24, so we choose to state it regardless.

Do not confuse symmetric Frobenius algebras with *commutative* Frobenius algebras. These are exactly what one might rightfully expect, and are the deuteragonist of part I.

Definition 2.2.26. A ***commutative*** Frobenius algebra A is a commutative algebra equipped with the structure of a Frobenius algebra.

That is, a commutative Frobenius algebra is just a Frobenius algebra which happens to be commutative as an algebra. Note that any commutative (Frobenius) algebra is symmetric, but not every symmetric Frobenius algebra must be commutative! We will soon see one such example.



(a) $\eta\colon \Bbbk \to A$. (b) $\mu\colon A\otimes A \to A$. (c) $\mathrm{id}\colon A \to A$. (d) $\delta\colon A \to A\otimes A$. (e) $\varepsilon\colon A \to \Bbbk$.

(f) $\gamma\colon \Bbbk \to A\otimes A$. (g) $\beta\colon A\otimes A \to \Bbbk$.

Figure 2.9: Some of the maps determined by the structure of a Frobenius algebra.

If we draw the maps which appear in our characterizations, using circles for each instance of our algebra A and empty space for the ground field $\Bbbk$, we get figure 2.9. Some very familiar pictures, indeed! But these would just be glorified notation if there weren't also identities justifying a topological picture. How lucky we are, for such identities are the topic of our next section!

But first, the promised examples.

Some examples of Frobenius algebras

Example 2.2.27 — Examples we have already used, secretly.
Our two examples of 2D TQFTs, Examples 1.2.7 and 1.2.8, were constructed by considering the following two commutative Frobenius algebras.

- The algebra $A = \Bbbk[t]/t^2$, equipped with the Frobenius form $\varepsilon\colon A \to \Bbbk$ given by

$$1 \longmapsto 0, \qquad t \longmapsto 1.$$

- The algebra $A = \Bbbk[t]/(t^2-1)$, equipped with the Frobenius form $\varepsilon\colon A \to \Bbbk$ given by

$$1 \longmapsto 1, \qquad t \longmapsto 0.$$

Both of these are easily generalizable from 2 to n, in which case their associated TQFTs return n (instead of 2) and n^g (instead of 2^g) as invariants of manifolds. See [Koc03, Exers. 3.3.2, 3.3.3].

Example 2.2.28 — The associative normed division algebras.
"The associative normed division algebras" is just a more cryptic way of talking about $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$.

Trivially, the real numbers $\mathbb{R}$ form a Frobenius $\mathbb{R}$-algebra, with Frobenius form $\mathrm{id}\colon \mathbb{R} \to \mathbb{R}$. Of course, any field $\Bbbk$ becomes a Frobenius $\Bbbk$-algebra in the same way.

Similarly to the previous Example 2.2.27, the complex numbers $\mathbb{C} \cong \mathbb{R}[t]/(t^2+1)$ become a Frobenius $\mathbb{R}$-algebra when equipped with the Frobenius form given by taking the real part:

$$\varepsilon\colon \mathbb{C} \longrightarrow \mathbb{R}, \qquad a+bi \longmapsto a.$$

This is also true for the quaternions $\mathbb{H}$, where now

$$\varepsilon\colon \mathbb{H} \longrightarrow \mathbb{R}, \qquad a+bi+cj+dk \longmapsto a.$$

Example 2.2.29 — The algebra of matrices equipped with the trace.
As mentioned before, one of the motivating examples of Frobenius algebras is that of square matrices $\mathrm{M}_{\Bbbk}(n)$ equipped with the trace map $\mathrm{tr}\colon \mathrm{M}_{\Bbbk}(n) \to \Bbbk$. Note that this algebra is symmetric, but not commutative; indeed, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ but $AB \neq BA$ in general.

See [Koc03, § 2.2] for some more specialized examples. Among others, this includes regular representations of groups (which is what initially lead to the study of Frobenius algebras), the ring of group characters, and cohomology rings. There are plenty of contexts where Frobenius algebras naturally appear, but that would be too much of a digression for our purposes — we prefer not to artificially enlarge this already lengthy text.

2.2.2 The joy of graphical calculus

The pictures of figure 2.9 aren't an informal analogy, or a cute gimmick; they are a system of formal notation, where each symbol has a precise meaning and is subject to certain composition rules. It just happens that all of these composition rules are topological in nature (i.e., give us identities which are true when interpreting each drawing topologically), but that is *a theorem*. In fact, that is a consequence of the main theorem of this chapter, which the totality of part I revolves around: the classification of 2D TQFTs. We could easily have defined a system of notation whose axioms don't reflect this fact, as one can do in the case of bialgebras and Hopf algebras; see [Koc03, 2.4.9, 2.4.11].

First of all, we can state the axioms we already known to be true. These are the axioms for the algebra structure and the two correspondences of Characterization 2.2.16.

Our axioms, in graphical language

The axioms for a $\Bbbk$-algebra are easy: they state that μ is associative and that η is its unit. Graphically, we draw them as in figure 2.10. Compare these with the commutative diagrams of Definition 2.2.1, which encode the same information.

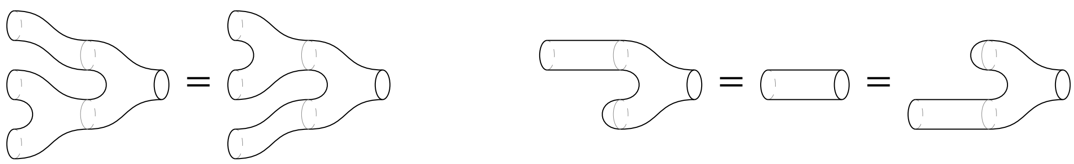


Figure 2.10: Associativity and unit axioms.

Both of these pictures make sense topologically. In fact, we already saw them in section 2.1, when discussing the relations for Cob_2.

Remember that the Frobenius form ε determines a pairing β and viceversa [Char. 2.2.16]. We can draw these relationships as in figure 2.11.

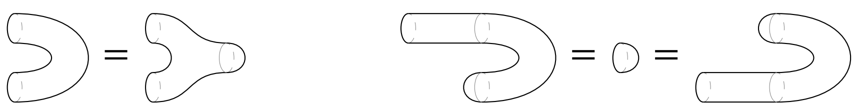


Figure 2.11: Relation between the Frobenius form ε and the pairing β.

In traditional notation, these just say $\langle x \mid y \rangle = \varepsilon(xy)$ and $\langle 1 \mid x \rangle = \varepsilon(x) = \langle x \mid 1 \rangle$. When expressed in this graphical way, the way a pairing determines a Frobenius form is very similar to the unit axioms of an algebra!

Next up, in figure 2.12, is the associativity of the pairing [Def. 2.2.13]. This follows from associativity of the multiplication and the left relation of figure 2.11.

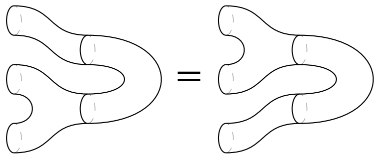


Figure 2.12: Associativity of the pairing.

Finally, non-degeneracy! Remember Definition 2.2.10. As we have already mentioned, this is just Zorro's Lemma; compare figures 1.9 and 2.13. They are the same picture!

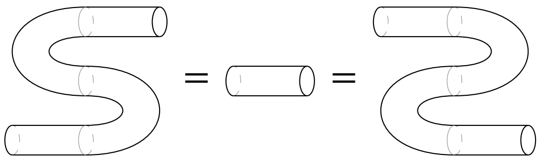
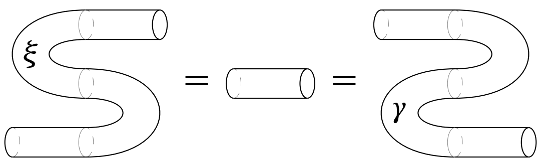


Figure 2.13: Non-degeneracy of the pairing (Zorro's Lemma).

We will now prove that this copairing is unique. This is our first proof in graphical language, but it should be intuitive and short enough to stand by itself. As an exercise, the reader may try expressing it in usual mathematical notation.

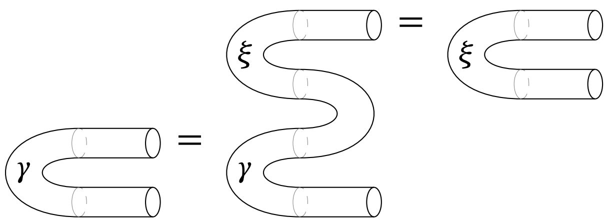
Lemma 2.2.30. Let β be a non-degenerate pairing. If γ and ξ are two copairings satisfying the non-degeneracy relation of figure 2.13, then $\gamma = \xi$.

Proof. Our starting assumption is the following:



(Technically we have two equations more, but these will suffice).

Now we just apply the relation multiple times. Remember that the cylinders are identities.

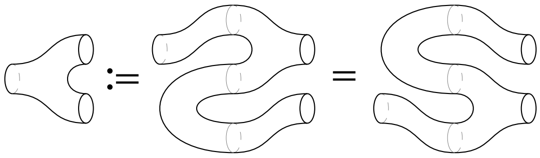


And that's it! ∎

Now, we want to use these axioms to prove additional facts about Frobenius algebras. Our main goal here is to prove Characterization 2.2.23, which will later become our main characterization of Frobenius algebras.

Obtaining the comultiplication

Now, if multiplication was , it only makes sense that comultiplication should be . We define this piece as follows.



(2.1)

Compare with the ad hoc written form in Characterization 2.2.23:

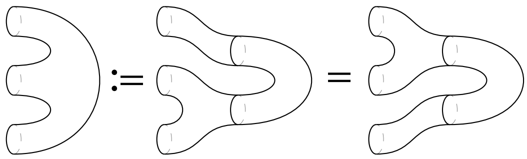
$$\delta := (\mathrm{id}_A \otimes \mu) \circ (\gamma \otimes \mathrm{id}_A) = (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \gamma).$$

The graphical form should feel intrinsically motivated by the picture. But we still need to check that this endows our vector space with a coalgebra structure, since otherwise this is just a cute drawing! First, we want to check that both definitions coincide. Following [Koc03, 2.3.12], we introduce the *three-point function*.

Definition 2.2.31. The ***three-point function*** $\phi\colon A \otimes A \otimes A \to \Bbbk$ is defined as

$$\phi := \beta \circ (\mu \otimes \mathrm{id}_A) = \beta \circ (\mathrm{id}_A \otimes \mu).$$

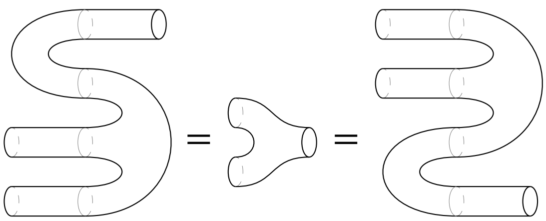
In graphical language, this is written as



and the two expressions at the right coincide by the associativity of the pairing [fig. 2.12].

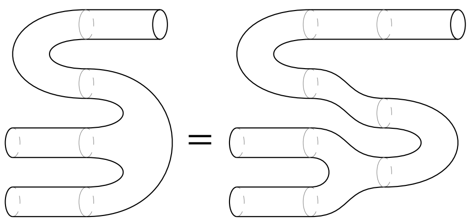
Now, let us proceed to our second proof in graphical language! We will format each step in two columns, with both a commutative diagram and a picture expressing the same information. The following lemma will immediately let us deduce that both sides of equation (2.1) coincide.

Lemma 2.2.32. We can express the multiplication using the three-point function:



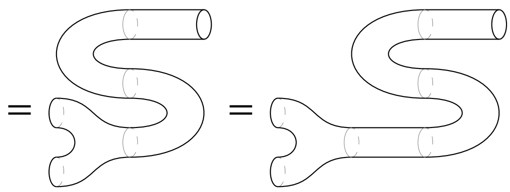
$$\begin{tikzcd}
 & A \otimes A \otimes A \otimes A \arrow[dr, "\phi\otimes\mathrm{id}"] & \\
A \otimes A \arrow[ur, "\mathrm{id}\otimes\mathrm{id}\otimes\gamma"] \arrow[rr, "\mu"'] & & A.
\end{tikzcd}$$

Proof. First, apply the definition of the three-point function.



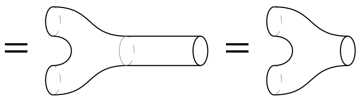
$$\begin{tikzcd}
 & A \otimes A \otimes A \otimes A \arrow[d, dashed, "\mu\otimes\mathrm{id}\otimes\mathrm{id}"] \arrow[ddr, "\phi\otimes\mathrm{id}"] & \\
 & A \otimes A \otimes A \arrow[dr, dashed, "\beta\otimes\mathrm{id}"] & \\
A \otimes A \arrow[uur, "\mathrm{id}\otimes\mathrm{id}\otimes\gamma"] & & A.
\end{tikzcd}$$

Remove most of the identities, and add one after the multiplication to line things up neatly.



$$\begin{tikzcd}
 & & A\otimes A\otimes A\otimes A \arrow[d, dashed, "\mu\otimes\mathrm{id}\otimes\mathrm{id}"] \arrow[ddrr, "\phi\otimes\mathrm{id}"] & & \\
 & & A\otimes A\otimes A \arrow[ddrr, "\beta\otimes\mathrm{id}"] & & \\
A\otimes A \arrow[uurr, "\mathrm{id}\otimes\mathrm{id}\otimes\gamma"] \arrow[urr, dashed, "\mu\otimes\gamma"] \arrow[rr, dashed, "\mu"'] & & A \arrow[u, dashed, "\mathrm{id}\otimes\gamma"] & & A.
\end{tikzcd}$$

Finally, use non-degeneracy of the pairing [fig. 2.13] and remove the superfluous identity.

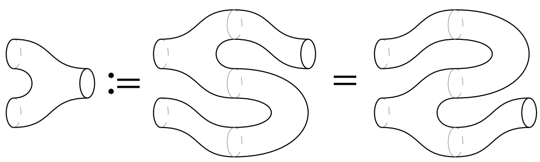


$$\begin{tikzcd}
 & & A\otimes A\otimes A\otimes A \arrow[d, dashed, "\mu\otimes\mathrm{id}\otimes\mathrm{id}"] \arrow[ddrr, "\phi\otimes\mathrm{id}"] & & \\
 & & A\otimes A\otimes A \arrow[ddrr, "\beta\otimes\mathrm{id}"] & & \\
A\otimes A \arrow[uurr, "\mathrm{id}\otimes\mathrm{id}\otimes\gamma"] \arrow[urr, "\mu\otimes\gamma"] \arrow[rr, "\mu"'] \arrow[rrrr, dashed, bend right, "\mu"'] & & A \arrow[u, dashed, "\mathrm{id}\otimes\gamma"] \arrow[rr, dashed, "\mathrm{id}"'] & & A.
\end{tikzcd}$$

The other half of the equation is analogous. □

So we deduce that both sides of equation (2.1) are equal, and thus we can unambiguously define our comultiplication .

Remark 2.2.33 — Obtaining multiplication from comultiplication.
We can also perform the dual process. Assume that we start with a comultiplication , a counit , and a unit candidate . Furthermore, assume that the copairing obtained by attaching a cap to is non-degenerate. Then, there exists an associated pairing , and we can use our pieces to construct a multiplication :

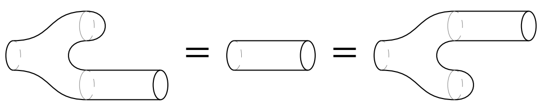


(2.2)

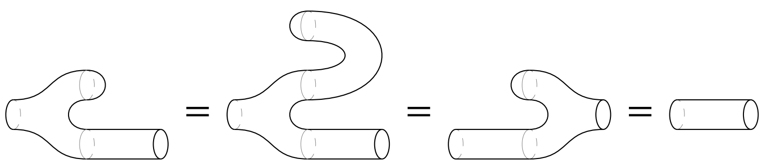
Proving the axioms for comultiplication

Now, let's start with our promised proof of Characterization 2.2.23! First, let's prove that (i.e., ε) is a counit for (i.e., δ).

Lemma 2.2.34. The Frobenius form ε is a counit for δ:



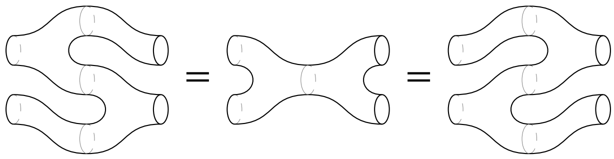
Proof. The argument is as follows.



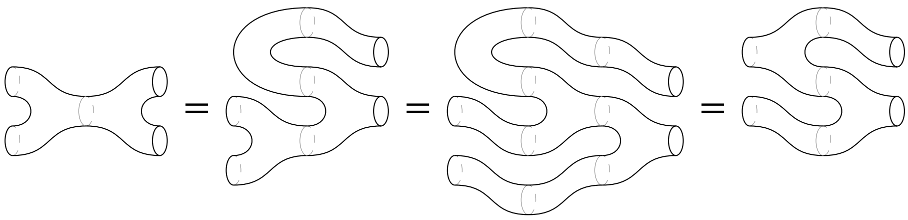
We used the axioms of the pairing [fig. 2.11], the definition of multiplication in terms of comultiplication [eq. (2.2)], and finally the fact that is a unit for [fig. 2.10]. □

And now for the main course: proving the Frobenius relation.

Lemma 2.2.35. The comultiplication  satisfies the Frobenius relation:



Proof. We use the “redefinition” of multiplication [eq. (2.2)], then associativity [fig. 2.10], and finally the redefinition of multiplication again.

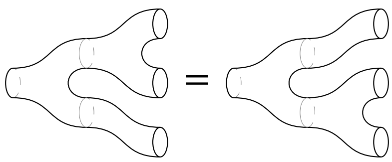


The other hand of the equation is analogous. □

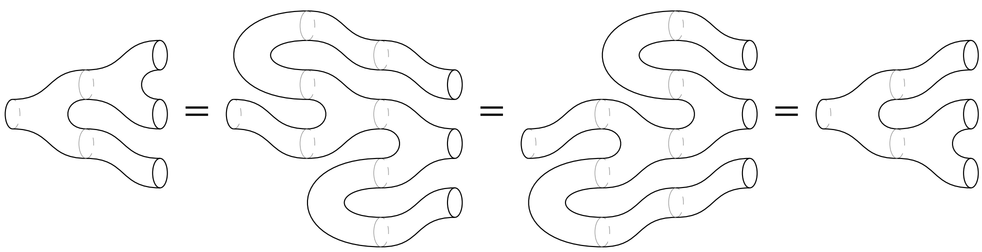
The Frobenius relation is a kind of duality constraint on algebras which are also coalgebras, akin to Zorro’s Lemma [1.2.3]. When we generalize our notion of Frobenius algebra to arbitrary categories [Def. 2.3.13], this will be the characterization we will use. It avoids terms which are specific to vector spaces, such as kernels or ideals, and only requires a notion analogous to that of “tensor product”; any monoidal operation can do the job.

But we are getting ahead of ourselves. Let us continue our quest towards proving Characterization 2.2.23. We are at the halfway point.

Lemma 2.2.36. The comultiplication is coassociative:



Proof.



We used the definition of comultiplication [eq. (2.1)], followed by associativity [fig. 2.10], followed by the definition of comultiplication again. □

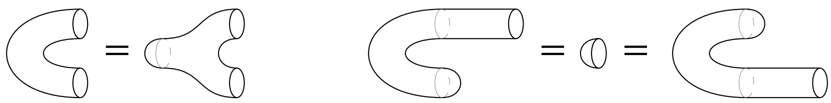
Notice that the dual argument is also true. Namely, in the situation of Remark 2.2.33, starting from a coassociative comultiplication  produces an associative multiplication  [eq. (2.2)].

In any case, given a Frobenius algebra as in Definition 2.2.7 and Characterization 2.2.16, we have constructed an algebra, which is also a coalgebra, which satisfies the Frobenius relation. So we have proven one of the two implications of Characterization 2.2.23. Before proceeding with the other implication, note that this comultiplication is unique.

Uniqueness of the comultiplication

To simplify the argument, it is useful to state some easy relations first. These are the duals of figure 2.11.

Lemma 2.2.37. The following relations hold.

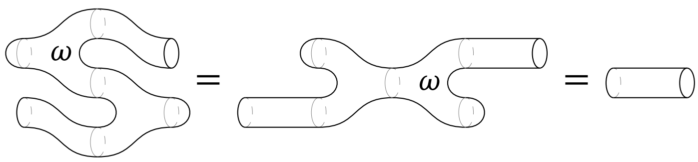


Proof. The first relation comes from the definition of the comultiplication [eq. (2.1)], by attaching a cap and applying unit laws [fig. 2.10]. Once that is proven, the second relation follows by the counit laws [Lem. 2.2.34]. □

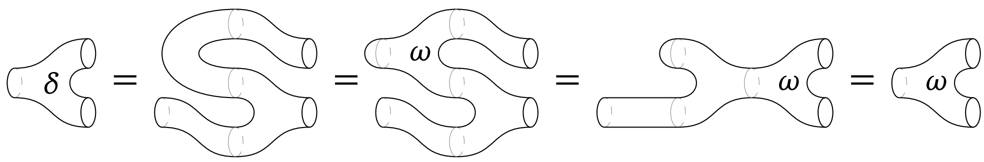
This lets us use the uniqueness of the copairing  associated to a non-degenerate pairing  [Lem. 2.2.30] to prove the uniqueness of the comultiplication .

Lemma 2.2.38. Given a Frobenius algebra (A, ε), there exists a unique comultiplication  satisfying the Frobenius relation which has ε as counit.

Proof. Let β be the non-degenerate pairing induced from ε by Characterization 2.2.16. Let δ be the comultiplication constructed in equation (2.1), and let ω be another comultiplication satisfying the requisites of the statement. By the previous Lemma 2.2.37, these determine two copairings, both of which satisfy the non-degeneracy conditions with β:



By Lemma 2.2.30, the two copairings coincide. But this implies that δ and ω coincide, as well:



So the comultiplication is unique. □

Proving the characterization in terms of coalgebras

And now, we have a wonderful surprise for you! A proof that for Characterization 2.2.23 we don't need to specify that our vector space is of finite dimension, nor that our (co)algebra is (co)associative. First, let us see that if we require (co)associativity, then it suffices to specify the "outer" half of the Frobenius relation. And conversely, requiring the entire Frobenius relation immediately implies (co)associativity.

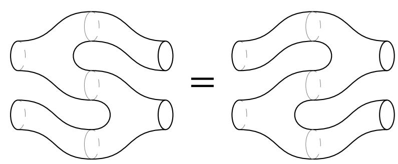


Figure 2.14: The outer half of the Frobenius relation.

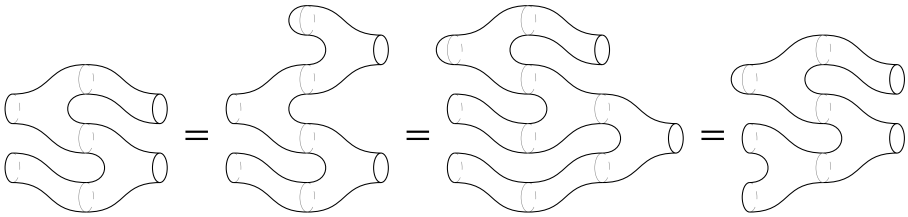
Lemma 2.2.39. $(A, \,,\,,\,,\,)$ be a not necessarily associative algebra which is also a not necessarily coassociative coalgebra, and such that the outer half of the Frobenius relation holds, as pictured in figure 2.14. Then, the following are equivalent:

(i) The multiplication is associative.

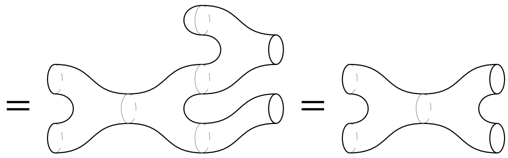
(ii) The comultiplication is coassociative.

(iii) The entire Frobenius relation holds.

Proof. We have already shown that (i) and (ii) imply each other [Lem. 2.2.36]. We prove that (i) implies (iii), as follows.

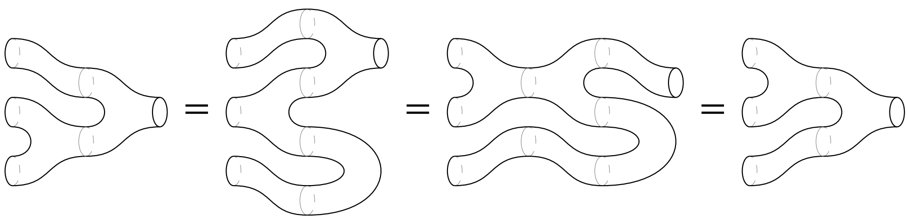


First we apply the unit law [fig. 2.10] and then the Frobenius relation, followed by associativity [fig. 2.10].



Then we undo the Frobenius relation, and finally we undo the unit law.

Proving that (iii) implies (i) is similar:



Here we used the definition of comultiplication [eq. (2.1)], followed by the "inner" Frobenius relation [fig. 2.8], followed by the outer part of the Frobenius relation. □

Finally, a proof that the axioms of Characterization 2.2.23 implies finite-dimensionality of the vector space. This was the last piece we needed — this will end the proof of our characterization.

Lemma 2.2.40. Let $(A, \,,\,,\,,\,) = (A, \mu, \eta, \delta, \varepsilon)$ be a not necessarily associative algebra which is also a not necessarily coassociative coalgebra, and such that the structures are related through the Frobenius relation.

(i) The vector space A is finite dimensional.

(ii) The multiplication μ is associative, and the comultiplication δ is coassociative.

(iii) The counit ε is a Frobenius form.

In conclusion, (A, ε) is a Frobenius algebra.

Proof. Associativity was proven in the previous Lemma 2.2.39.

Define the pairing $\beta := \varepsilon \circ \mu$, as in figure 2.11. This pairing is non-degenerate, with associated copairing $\gamma := \delta \circ \eta$ (as in Lemma 2.2.37), by the same argument of the prior Lemma 2.2.30;

putting caps into the Frobenius relation yields the non-degeneracy condition of figure 2.13. Since our multiplication is associative, the pairing β is associative too.

This implies a correspondence with Characterization 2.2.16 (in terms of a non-degenerate associative pairing), and in particular forces the vector space to have finite dimension by Zorro's Lemma [2.2.11]. $\square$

This ends the main bulk of the section; it has been a while since we discussed the generators and relations of Cob_2, in the previous section 2.1. Just one short comment more, before proceeding to the main result of part I.

The commutative and symmetric axioms, graphically

A commutative Frobenius algebra [Def. 2.2.26] satisfies commutativity and cocommutativity,

while a *symmetric* Frobenius algebra [Def. 2.2.25] satisfies the analogous condition replacing (co)multiplication by the (co)pairings.

As mentioned before, commutative algebras will be the protagonist for now, but symmetric algebras will get their time in the spotlight in section 4.2!

Let us finally state our main result: the classical correspondence between 2D oriented TQFTs and commutative Frobenius algebras. After all our hard work, this is almost a tautology.

2.3 The main result: classification of 2D TQFTs

We have reached the end of part I, appropriately named after the classification theorem we are about to state. It should not come as a surprise, after all of our discussion. After properly stating the things we already know, we will take an important step by generalizing the target category — after all, $\mathsf{Vect}_{\Bbbk}$ is just one of a multitude of symmetric monoidal categories, and there is no a priori reason for it to hold a privileged place in our theory. It just happens that we know $\mathsf{Vect}_{\Bbbk}$ specially well.

Without further ado, let us recollect everything we have seen.

2.3.1 2D TQFTs are commutative Frobenius algebras

First, let us define the relevant categories.

The category of 2D TQFTs

We already defined it! Remember Definition 1.3.7, and remember that this is a *groupoid*.

The category of Frobenius algebras

Even though we spent the previous section familiarizing ourselves with Frobenius algebras, we didn't define a category for them. We will now correct the error of our ways. We want to preserve the algebra and coalgebra structures of Characterization 2.2.23.

Definition 2.3.1. A ***Frobenius algebra homomorphism*** between two Frobenius $\Bbbk$-algebras $(A,\mu,\eta,\delta,\varepsilon)$ and $(A',\mu',\eta',\delta',\varepsilon')$ is a $\Bbbk$-algebra homomorphism which is also a $\Bbbk$-coalgebra homomorphism.

In particular, a Frobenius algebra homomorphism preserves the counit ε; i.e., the Frobenius form. But not all $\Bbbk$-algebra maps which preserve the Frobenius form preserve the comultiplication δ, hence why we stated the definition in this way.

Frobenius algebras form a category $\mathsf{Frob}_{\Bbbk}$ — and in fact, this is a *groupoid*: every Frobenius algebra homomorphisms is bijective, and its inverse is another Frobenius algebra homomorphism.

Lemma 2.3.2. A Frobenius algebra homomorphism $\phi\colon (A,\varepsilon) \to (A',\varepsilon')$ is always bijective, and its inverse is another Frobenius algebra homomorphism.

Proof. Since ϕ is multiplicative and respects the units and counits, the kernel of ϕ must be an ideal of A contained in the set $\operatorname{Ker}(\varepsilon)$: we have $\varepsilon' \circ \phi = \varepsilon$. But $\operatorname{Ker}(\varepsilon)$ contains no non-trivial ideals, so $\operatorname{Ker}(\phi) = 0$ and therefore ϕ is injective.

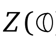
Now, since ϕ is comultiplicative (and respects units and counits), the "linear dual" $\phi^*\colon A'^* \to A^*$ is multiplicative, and also respects units and counits. By the same argument as before, ϕ^* must be injective. But A has finite dimension, so this implies that ϕ is surjective.

So ϕ has an inverse ϕ^{-1}, which is (co)multiplicative and respects (co)units; i.e., it is a Frobenius algebra homomorphism. □

Commutative Frobenius algebras form a (full) subcategory $\mathsf{cFrob}_{\Bbbk} \hookrightarrow \mathsf{Frob}_{\Bbbk}$.

Finally: the main result

Theorem 2.3.3. There is a natural equivalence between 2D oriented TQFTs Z with values in $\mathsf{Vect}_{\Bbbk}$ and commutative Frobenius $\Bbbk$-algebras $(A,\mu,\eta,\delta,\varepsilon)$, which is given by evaluation of the TQFT on the circle, pair of pants and cup cobordisms:

$\mathsf{TQFT}_2^{\mathrm{or}}(\mathsf{Vect}_{\Bbbk}) \xrightarrow{\simeq} \mathsf{cFrob}_{\Bbbk}, \qquad Z \longmapsto (Z(\mathbb{S}^1), Z($  $), Z($  $), Z($  $), Z($  $)).$

It suffices to specify three of the four values on morphisms, since the other can be obtained by the methods of section 2.2.2. One can get away with just two, if they are either the pair composed of and or the pair composed of and .[1]

We will refer to this result as the *classical correspondence*. Its history is rooted in mathematical folklore, with the theorem regarded as true for a number of years before the first rigorous proof appeared in [Abr96].

For us, its proof will be extremely simple: one just needs to take another look at our relations for Cob_2 in figures 2.5 to 2.8 and notice that these are exactly the axioms for a commutative Frobenius algebra. So the assignment

$$A = Z(\mathbb{S}^1) \quad \mu = Z(\;) \quad \eta = Z(\;) \quad \delta = Z(\;) \quad \varepsilon = Z(\;)$$

is a bijection! Now, for the slightly stronger result that this is a natural equivalence, one just needs to check that a morphism of TQFTs transforms to a morphism of Frobenius algebras and viceversa. But this is once again immediate, since Frobenius algebra homomorphisms preserve the four structural maps.

But $\mathsf{Vect}_{\mathbb{k}}$ should not have the monopoly of accepting functors from Cob_2.

2.3.2 Generalizing the target category

Why should TQFTs take values in $\mathsf{Vect}_{\mathbb{k}}$, instead of another category? Well, the historical reason lies (as we already discussed in the Preface and in section 1.2.1) in the quantum-physical origins of the theory; but mathematically, there is no reason we should not be able to generalize our definitions to accept values in an arbitrary symmetric monoidal category. In fact, the further we go in the study of TQFTs, the more important it is to discern what properties of $\mathsf{Vect}_{\mathbb{k}}$ it is that we are actually using, and which are just extraneous structure. We already saw a glimpse of this in section 1.3.3, when we stated Theorem 1.3.10. So let us indulge in the dangerous joy of abstraction for a while, but stop before we, too, become infatuated lotophages.

Internal monoids and comonoids

As promised, we first define (co)monoids *internal* to a monoidal category. These are completely analogous to our previous definitions of (co)algebras [Defs. 2.2.1 and 2.2.18].

Definition 2.3.4. An ***internal monoid*** M in a monoidal category $(\mathcal{V}, \star, \mathrm{I})$ is an object equipped with morphisms $\mu\colon M \star M \to M$ (multiplication) and $\eta\colon \mathrm{I} \to M$ (unit map) such that the following diagrams commute.

$$\begin{tikzcd} & M \star M \star M \arrow[dl, "\mu \star \mathrm{id}_M"'] \arrow[dr, "\mathrm{id}_M \star \mu"] & \\ M \star M \arrow[dr, "\mu"'] & & M \star M \arrow[dl, "\mu"] \\ & M & \end{tikzcd} \qquad \begin{tikzcd} M \star M \arrow[dr, "\mu"] & & M \star M \arrow[dl, "\mu"'] \\ \mathrm{I} \star M \arrow[u, "\eta \star \mathrm{id}_M"] \arrow[r] & M & M \star \mathrm{I} \arrow[l] \arrow[u, "\mathrm{id}_M \star \eta"'] \end{tikzcd}$$

Same for the dual concept.

[1] From these pairs one can construct either the pairing or the copairing , and the opposite is obtained from non-degeneracy. Finally, from the (co)pairing and one pair of pants one can obtain the opposite pair of pants, thus reducing to the case of having specified three of the four generators.

Definition 2.3.5. An ***internal comonoid*** M in a monoidal category $(\mathcal{V},\star,\mathrm{I})$ is an object equipped with morphisms $\delta\colon M\to M\star M$ (comultiplication) and $\varepsilon\colon M\to\mathrm{I}$ (counit map) such that the following diagrams commute.

$$\begin{tikzcd}
 & M\star M\star M & \\
M\star M \arrow[ur, "\delta\star\mathrm{id}_M"] & & M\star M \arrow[ul, "\mathrm{id}_M\star\delta"'] \\
 & M \arrow[ul, "\delta"] \arrow[ur, "\delta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
M\star M \arrow[d, "\varepsilon\star\mathrm{id}_M"'] & & M\star M \arrow[d, "\mathrm{id}_M\star\varepsilon"] \\
\mathrm{I}\star M & M \arrow[l] \arrow[r] \arrow[ul, "\delta"'] \arrow[ur, "\delta"] & M\star\mathrm{I}
\end{tikzcd}$$

Our first examples should be clear: as previously stated, $\Bbbk$-(co)algebras are (co)monoids internal to the category $(\mathsf{Vect}_{\Bbbk},\otimes,\Bbbk)$ of $\Bbbk$-vector spaces equipped with tensor product. Monoids are monoids internal to Set, while comonoids (which haven't appeared yet) are comonoids internal to Set. And note that a (small, strict) monoidal category can be seen as a monoid internal to $(\mathsf{Cat},\times,\mathbb{1})$! Compare the diagrams of Definitions 1.2.10 and 2.3.4. Of course, this is circular reasoning, and as such does not serve as a definition of monoidal category.

Just as was done for (co)algebras [Defs. 2.2.2 and 2.2.19], one can easily define morphisms between internal monoids and comonoids.

Definition 2.3.6. A ***homomorphism of internal monoids*** in a monoidal category $(\mathcal{V},\star,\mathrm{I})$ is a morphism $\varphi\colon(M,\mu,\eta)\to(M',\mu',\eta')$ which commutes with the multiplication and unit:

$$\begin{tikzcd}
M\star M \arrow[r, "\varphi\star\varphi"] \arrow[d, "\mu"'] & M'\star M' \arrow[d, "\mu'"] \\
M \arrow[r, "\varphi"'] & M'
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "\varphi"] & M' \\
\mathrm{I} \arrow[u, "\eta"] \arrow[r, equal] & \mathrm{I} \arrow[u, "\eta'"']
\end{tikzcd}$$

Definition 2.3.7. A ***homomorphism of internal comonoids*** in a monoidal category $(\mathcal{V},\star,\mathrm{I})$ is a morphism $\varphi\colon(M,\delta,\varepsilon)\to(M',\delta',\varepsilon')$ which commutes with the comultiplication and counit:

$$\begin{tikzcd}
M'\star M' & M\star M \arrow[l, "\varphi\star\varphi"'] \\
M' \arrow[u, "\delta'"] & M \arrow[l, "\varphi"] \arrow[u, "\delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
M' \arrow[d, "\varepsilon'"'] & M \arrow[l, "\varphi"'] \arrow[d, "\varepsilon"] \\
\mathrm{I} \arrow[r, equal] & \mathrm{I}
\end{tikzcd}$$

Equipped with these morphisms, the set of (co)monoids internal to a monoidal category $(\mathcal{V},\star,\mathrm{I})$ form a category. We will denote these by $\mathsf{Mon}(\mathcal{V})$ and $\mathsf{Comon}(\mathcal{V})$, although this notation will only be used for Lemma 2.3.12.

The augmented simplex category Δ_a is the free monoidal category on a single monoid

Before generalizing our correspondence, let's see an analogous result for internal monoids. We need to define a closely related category for the discussion in the Coda, anyway.

Definition 2.3.8. The ***augmented simplex category*** Δ_a is the category defined by the following data.

- **Objects:** Sets of natural numbers $[\boldsymbol{n}]:=\{0,1,\dots,n\}$, as well as the empty set $[\boldsymbol{-1}]:=\varnothing$.
- **Morphisms:** Non-decreasing functions $[\boldsymbol{n}]\to[\boldsymbol{m}]$.

With function composition and identities.

Note that some authors define $[\boldsymbol{n}]:=\{0,1,\dots,n-1\}$ instead, so in particular $[\boldsymbol{0}]:=\varnothing$. These are the finite von Neumann ordinals, with $[\boldsymbol{n}]$ having cardinality n.

We follow the "geometric" convention, where $[\boldsymbol{n}]$ has cardinality $n+1$; we think of it as n consequent *arrows*:

$$0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n.$$

In fact, this is a drawing of the partially ordered set $[\boldsymbol{n}]$ when viewed as a category: its objects are the elements of $[\boldsymbol{n}]$, and there is an arrow $i \to j$ if and only if $i \leq j$. Furthermore, these arrows are unique: between each pair of objects $i, j \in [\boldsymbol{n}]$ there are either no arrows ($i > j$) or one single arrow ($i \leq j$).

The word "augmented" implies that there is a "base" or non-augmented version, and indeed there is. This is the full subcategory of Δ_a obtained by removing the empty set.

Definition 2.3.9. The ***simplex category*** Δ is the category defined by the following data.

- **Objects:** Sets of natural numbers $[\boldsymbol{n}] := \{0, 1, \dots, n\}$.
- **Morphisms:** Non-decreasing functions $[\boldsymbol{n}] \to [\boldsymbol{m}]$.

With function composition and identities.

Why allocate two different definitions to concepts which are almost identical? Well, we need both. The augmented category is needed to make Lemma 2.3.10 true, while the plain simplex category models simplicial sets correctly. We postpone discussion of simplicial sets — as closely related to Δ as they are — until Definition C.1.1, since they won't be strictly relevant until then.

Note that Δ_a accepts a monoidal structure where the operation $+$ is given by ordinal sum, denoted $[\boldsymbol{m}] + [\boldsymbol{n}] = [\boldsymbol{m}+\boldsymbol{n}]$. This is the disjoint union of the underlying sets, equipped with the ordering extending those of $[\boldsymbol{m}]$ and $[\boldsymbol{n}]$ my making all elements in the second set be greater than all elements in the first. In other words, we consider the following bijection between $[\boldsymbol{m}] \sqcup [\boldsymbol{n}]$ and $[\boldsymbol{m}+\boldsymbol{n}]$:

$$\{1_m, \dots, m_m, 1_n, \dots, n_n\} \longmapsto \{1, \dots, m, m+1, \dots, m+n\}.$$

The action on morphisms is defined by applying this identification in the intuitive way; given functions $f\colon [\boldsymbol{m}] \to [\boldsymbol{n}]$ and $f'\colon [\boldsymbol{m}'] \to [\boldsymbol{n}']$, we define

$$f + f'\colon [\boldsymbol{m}+\boldsymbol{m}'] \longrightarrow [\boldsymbol{n}+\boldsymbol{n}'], \qquad i \longmapsto \begin{cases} f(i), & i \leq m; \\ n + f'(i-m), & i > m. \end{cases}$$

The operation of ordinal sum has a unit in Δ_a: the empty set $\varnothing$. Note that the non-augmented simplex category Δ does not include the empty set, and as such this operation does not define a monoidal structure there.

All of this will surely be reminiscent of the category Sym of Example 1.2.15.

In any case, let's state why we are interested in Δ_a and explain the details along the way. The proofs are left as an exercise for the reader.

Lemma 2.3.10. The augmented simplex category, Δ_a, is the free monoidal category on a single monoid.

There are multiple ways of defining "free", in this context. The core idea is that we can write a presentation of the category in terms of generators and relations, and the only relations which appear are those characterizing the stated properties — in this case, the relations needed for the single object generator M to be monoidal. Compare with the previous mentions of freely generated categories; namely, Remark 1.2.16 as well as the discussion preceding it, and the short comment following Theorem 1.3.10.

A more precise statement is the following.

Lemma 2.3.11. Any monoid (M, μ, η) on any monoidal category $(\mathcal{V}, \star, \mathrm{I})$ is the image under $[\boldsymbol{0}]$ of a unique monoidal functor $\Delta_a \to \mathcal{V}$.

Yet another rephrasing:

Lemma 2.3.12. There is a natural equivalence between monoidal functors $F\colon \Delta_a \to \mathcal{V}$ and monoids (M,μ,η) internal to $(\mathcal{V},\star,\mathrm{I})$, which is given by evaluation on $[\mathbf{0}]$:

$$\mathsf{Fun}^{\otimes}(\Delta_a,\mathcal{V}) \xrightarrow{\simeq} \mathsf{Mon}(\mathcal{V}), \qquad F \longmapsto F[\mathbf{0}].$$

Hey, this looks familiar! Isn't this the same as Theorem 2.3.3? Indeed it is: the category of 2D oriented cobordisms is the free symmetric monoidal category on a single *Frobenius object*. But what are those?

Frobenius objects in a category

Just as a Frobenius algebra was (ultimately) an algebra which was also a coalgebra in a compatible way [Char. 2.2.23], a *Frobenius object* internal to a monoidal category will be an internal monoid which is also an internal comonoid, and such that the two structures satisfy the Frobenius relation.

Definition 2.3.13. A ***Frobenius object*** in a monoidal category $(\mathcal{V},\star,\mathrm{I})$ is a monoid internal to $\mathcal{V}$ which is also a comonoid internal to $\mathcal{V}$, and such that the Frobenius relation holds:

$$(\mu \star \mathrm{id}_M)\circ(\mathrm{id}_M \star \delta) = \delta\circ\mu = (\mathrm{id}_M \star \mu)\circ(\delta \star \mathrm{id}_M).$$

In particular, a Frobenius $\Bbbk$-algebra is a Frobenius object internal to $(\mathsf{Vect}_{\Bbbk},\otimes,\Bbbk)$. Once again, a Frobenius object homomorphism will be a homomorphism of internal monoids which is also a homomorphism of internal comonoids. Every Frobenius object homomorphism is invertible,[2] so we can define a groupoid $\mathsf{Frob}(\mathcal{V})$ of Frobenius objects in a given monoidal category.

If we work in a symmetric monoidal category, it makes sense to ask for our Frobenius objects to be commutative; i.e., for μ and δ to (co)commute with the twist maps τ.

Definition 2.3.14. A Frobenius object $(M,\mu,\eta,\delta,\varepsilon)$ in a symmetric monoidal category $(\mathcal{V},\star,\mathrm{I},\tau)$ is ***commutative*** if the twist τ commutes with multiplication and cocommutes with comultiplication; that is, $\mu\circ\tau_{M,M} = \mu$ and $\tau_{M,M}\circ\delta = \delta$.

We actually only need one of the two compatibility conditions, since the other can be deduced. Commutative Frobenius objects form their own subcategory, which we will denote $\mathsf{cFrob}(\mathcal{V})$

Once again, do not confuse these with *symmetric* Frobenius objects, where only the pairing $\beta = \eta\circ\mu$ (and the copairing $\gamma = \delta\circ\varepsilon$, by duality) is required to commute with the twist maps.

Definition 2.3.15. A Frobenius object $(M,\mu,\eta,\delta,\varepsilon)$ in a symmetric monoidal category $(\mathcal{V},\star,\mathrm{I},\tau)$ is ***symmetric*** if the twist τ commutes with the pairing $\beta := \mu\circ\varepsilon$; that is, $\beta\circ\tau_{M,M} = \beta$.

Commutative (resp. symmetric) Frobenius objects in the category $(\mathsf{Vect}_{\Bbbk},\otimes,\Bbbk,\tau)$ are just commutative (resp. symmetric) Frobenius $\Bbbk$-algebras. Note, however, that the specific symmetry one chooses matters; if we work in the category of graded vector spaces equipped with tensor product and the Koszul sign change $\kappa\colon v\otimes w \mapsto (-1)^{pq} w\otimes v$ instead (cf. Example 1.2.18), we reach a different definition, which we might call "graded-symmetric" Frobenius algebras. So, even when discussing TQFTs whose target category has vector spaces as objects, this generalization is useful — sometimes "graded commutativity" is the correct notion to consider.

A universal property for Cob_2

If we now go back to our proofs, one can see that the crux of our arguments only used the properties of commutative Frobenius *objects*. The exceptions are some proofs which use kernels of linear maps, but these were not relevant for the main argument. So we can immediately deduce an analogue for Theorem 2.3.3.

[2] This is a bit harder than the linear case, since we can't take kernels. See [HV19, Lem. 5.19] for a graphical proof.

Theorem 2.3.16. There is a natural equivalence between 2D oriented TQFTs Z with values in a symmetric monoidal category $(\mathcal{C}, \star, \mathrm{I}, \tau)$ and commutative Frobenius objects $(M, \mu, \eta, \delta, \varepsilon)$ in $\mathcal{C}$, which is given by evaluation of the TQFT on the circle, pair of pants and cup cobordisms:

$$\mathsf{TQFT}_2^{\mathrm{or}}(\mathcal{C}) \xrightarrow{\simeq} \mathsf{cFrob}(\mathcal{C}), \qquad Z \longmapsto (Z(\mathbb{S}^1), Z(\ ), Z(\ ), Z(\ ), Z(\ )).$$

We can rephrase this as a universal property, like we did for Δ and internal monoids.

Theorem 2.3.17. The category of oriented 2-cobordisms, Cob_2, is the free symmetric monoidal category on a single commutative Frobenius object.

This might feel like unnecessary abstraction, but it is an important conceptual aid to keep in mind. Even more so when we discuss the generalization of this correspondence to arbitrary dimension, which is the topic of the following part II.

As we did in section 1.3, we now take a very quick look at *unoriented* 2D TQFTs. For simplicity, here we will take values in $\mathsf{Vect}_{\Bbbk}$. The reader should now know how to adapt these structures into an "internal object" approach.

Classifying 2D unoriented with values in $\mathsf{Vect}_{\Bbbk}$

In the 1D case [Thm. 1.3.11] we could reuse the generators from the oriented category to serve as generators of $\mathsf{Cob}_1^{\mathrm{un}}$, which greatly simplified the classification. That will not be the case here: non-orientable surfaces exists. It is harder to find detailed accounts of this classification in the literature, since this too was folklore, but the recent preprint [Cze23] seems to have filled that gap. We also refer to [TT06], as well as to a set of lecture notes by You Qi of a course taught by Khovanov [QK20, § 2]. Both sources [Cze23; QK20] follow a graphical approach similar to the one in [Koc03] (and therefore to ours), with the lecture notes being more concise.

Remark 2.3.18 — Classifying 2D unoriented TQFTs.
We get two new generators: one cobordism $\mathbb{S}^1 \to \mathbb{S}^1$ which can be constructed as two cylinders glued along their boundary *by a reflection*, and another cobordism $\varnothing \to \mathbb{S}^1$ which is given by the Möbius band — this is sometimes called a *cross-cap*. There is also the opposite Möbius band $\mathbb{S}^1 \to \varnothing$, which we ignore; this is obtained by "inverting" the two boundaries via the pairing .

In summary, the new cobordism $\mathbb{S}^1 \to \mathbb{S}^1$ maps to an involution[3] $\varphi\colon A \to A$, while the Möbius band corresponds to a map $\theta\colon \Bbbk \to A$ which selects an element $\theta(1)$.

All in all, this comes down to the structure of a commutative ***extended*** Frobenius algebra: a commutative Frobenius $\Bbbk$-algebra equipped with an involution $\varphi\colon A \to A$ and a distinguished element $\theta\colon \Bbbk \to A$ such that the following diagrams commute.

$$\begin{tikzcd}
\Bbbk \otimes A \arrow[r, "\theta\otimes\mathrm{id}"] \arrow[d, "\theta\otimes\mathrm{id}"'] & & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"'] & A \arrow[r, "\varphi"'] & A,
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \arrow[r, "\eta"] \arrow[d, "\cong"] & A \arrow[rr, "\delta"] & & A \otimes A \arrow[r, "\varphi\otimes\mathrm{id}"] & A \otimes A \arrow[d, "\mu"] \\
\Bbbk \otimes \Bbbk \arrow[rr, "\theta\otimes\theta"'] & & A \otimes A \arrow[rr, "\mu"'] & & A.
\end{tikzcd}$$

Elementwise, this means that for all $a \in A$ one must have

$$\varphi(\Theta \cdot a) = \Theta \cdot a, \qquad \varphi(\Delta) \cdot \Delta' = \Theta^2,$$

where $\Theta := \theta(1)$ and $\Delta \otimes \Delta' := \delta(1)$.

[3] This is actually an *anti*-involution, a map $\varphi\colon A \to A^{\mathrm{op}}$ to the opposite algebra [Def. 4.2.3]. In other words, this reverses the order of multiplication: $\varphi(xy) = \varphi(y) \cdot \varphi(x)$. Since our classification selects only the *commutative* Frobenius algebras, this is not relevant yet; but it will be, in chapter 4.

The involution $\varphi \colon A \to A$ makes this an instance of an $*$-*algebra*, but the distinguished element θ lacks an analogue. As mentioned, we refer the reader to the sources above for more details — including the graphical interpretations of both commutative diagrams.

The framed classification is left for the reader to ponder. Note that most of the surfaces in our generating set are *not* parallelizable; in fact, the only parallelizable compact surfaces are the cylinder and the torus, since they are the only compact orientable surfaces of Euler characteristic zero. So topologically every cobordism comes from  and , in a similar manner to the classification of 1D unoriented TQFTs of Theorem 1.3.11. But here we must also take the framing into account, which will affect the classification.

What's next?

And thus ends part I. Now, we will take our notions and intuitions, hold on to them with all our might, and take a daring leap as we climb higher and higher in the mountain of abstraction. We want to understand the statement of the Cobordism Hypothesis, and understand it deeply enough that it feels natural and obvious. For that, there are a lot of conceptual gaps we must fill — gaps which require time and perseverance. To keep us from falling down, we will need a healthy dose of philosophical motivation and heuristic reasoning.

There are very few things to prove in part II, but there are a lot of interesting things to discuss, and a lot of dreams to have. That is not due to everything being trivial — on the contrary, it is a symptom of the difficulty of reaching the correct definitions. I consider this sort of discussion to be the true main goal of this thesis — in that regard, the very title is a slight misdirection, and all of part I can be considered an unreasonably lengthy prologue. I hope you will pardon this lie; some things are better kept hidden, you see.

With that out of the way, let's start our higher-categorical journey.

> And if your head explodes with dark forebodings too
> I'll see you on the dark side of the moon.
>
> **Roger Waters, *Brain Damage*.**

Part II

Beyond 2D: The Cobordism Hypothesis

Chapter 3

A first approximation to the Cobordism Hypothesis

> The fact is…
> No matter how closely I study it,
> No matter how I take it apart,
> No matter how I break it down,
> It remains consistent.
> I wish you were here to see it!
>
> **Adrian Belew, *Indiscipline*.**

In this second and final part we familiarize ourselves with the notions needed to state (but not prove) the **Cobordism Hypothesis**: the higher-dimensional generalization of the equivalence between 2D TQFTs and Frobenius Algebras we saw in part I. In order to do so, we need to keep track of more structure; we need to expand our categories into ∞-categories.

The main motivating principle for higher category theory is, put simply, that equality is not nuanced enough — it's too strict of a notion to enforce on objects *or* morphisms. We can apply the core philosophy of category theory, that of prioritizing relationships between things over the things themselves, to the morphisms. In this manner, we get 2-*morphisms* between morphisms, which shall give rise to 3-*morphisms* between 2-morphisms, and so on all the way up to infinity.

This chapter summarizes aspects of [Lur09], the 2009 article by Lurie where the Cobordism Hypothesis was (mostly) solved.[1] After developing some intuitive notions for higher categories (and particularly those of cobordisms), and realizing the unfitness of some naïve approaches, the Interlude will explore the fundamental pillars upon which higher category theory is built.

Then, chapter 4 compares the Cobordism Hypothesis with the classical correspondence of Theorem 2.3.3 in a more concrete way. For this, we use the main result from [Sch14]. Finally, the Coda discusses one model for (∞, n)-categories — n-fold complete Segal spaces.

We will give but a small glimpse of the vast theory of ∞-categories; the adventurous reader shall thus be encouraged to venture further into these matters. We refer them to the "foundational texts" of $(\infty, 1)$-category theory: [HTT; Kerodon]. Note that both of these sources use the term "∞-category" as a synonym for $(\infty, 1)$-category — a common convention in the literature. For other surveys on the Cobordism Hypothesis, see the excellent [Fre13b; Tel16].

[1] As of the writing of this document, Lurie refers to this paper as "an informal account of a proof of the Baez–Dolan cobordism hypothesis and related matters" (cited from his webpage, `https://www.math.ias.edu/~lurie/`). He adds, "a more detailed account will appear here *eventually*" [emphasis added]. Simply put, [Lur09] can be considered the outline of a program for proving the Cobordism Hypothesis. As of the writing of this document, a full proof detailing the technical gaps in Lurie's program has yet to appear in a peer-reviewed publication. However, preprints claiming a reduction to another conjecture [AF17] or a full proof [GP22] have been written in the past several years.

3.1 A tour through higher category theory

This section intends to introduce the reader to the informal notion of *weak n-category*, which will then be used to construct an n-category of cobordisms and state a version of the Cobordism Hypothesis. As usual, we ignore size issues and treat all categories as small (i.e., having a set of objects and a set of morphisms, instead of bigger kinds of collections).

3.1.1 Weak and strict n-categories

Even though this section will focus on n-categories, the natural motivating example is actually the *truncation* of an ∞-category.

The ∞-category of topological spaces

Consider the category Top of topological spaces. As structural basis, it has spaces (objects or 0-morphisms X, Y) and continuous functions between those (1-morphisms $f, g\colon X \to Y$), but we can climb to ever-higher heights: we can construct homotopies between functions (2-morphisms $H\colon X \times I \to Y$ with $H(-,0) = f$ and $H(-,1) = g$, which we can succinctly write as $H\colon f \rightsquigarrow g$), homotopies between homotopies between functions (3-morphisms $\Phi\colon X \times I \times I \to Y$ such that $\Phi\colon H \rightsquigarrow H'$), and so on and so forth.

We have, thus, a hierarchy of higher and higher ordered morphisms, which give us the structure necessary to conduct more nuanced arguments. For example, the kind of arguments which are usually carried over in HoTop by necessity (where HoTop is the category of topological spaces and *homotopy classes* of continuous functions between them) can be carried over in our ∞-category Top. To be more precise, HoTop is the *homotopy category* $\mathrm{h}_1\,\mathsf{Top}$ of Top when regarded as an ∞-category: it consists of the same objects as Top, but the arrows between them are the isomorphism classes of 1-morphisms in Top.

Note that this process loses a great amount of information. Consider, for example, the fundamental group functor π_1. Although two homotopically equivalent path-connected spaces $X \cong Y$ will have isomorphic fundamental groups $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$, this isomorphism of groups depends on the concrete homotopy equivalence $H\colon X \times I \to Y$ selected. This important information is lost when taking equivalence classes of objects in HoTop. Similarly, important information about 1-morphisms will be lost when passing from the ∞-category Top to its homotopy category HoTop — and this higher-ordered information can be tracked effectively through use of higher category theory.

We can either continue all the way up to infinity, or truncate our ∞-category into an n-category by forgetting everything past the nth layer; that is, we take the n-category $\mathrm{h}_n\,\mathsf{Top}$ with the same k-morphisms for $0 \le k \le n-1$ and whose n-morphisms are isomorphism classes of the n-morphisms in Top. For example, we can take the 2-category $\mathrm{h}_2\,\mathsf{Top}$ of topological spaces, continuous functions between them, and classes of equivalence of homotopies between these functions. So we can choose the level of nuance our arguments require, to avoid tracking an overwhelmingly large amount of layers. As the reader probably knows, infinity is *very* large.

Sometimes, however, the naïve definition one could imagine from the previous paragraphs has *too much structure* to be useful, since it doesn't model the kind of relations we're interested in: we need to *weaken* our notion of category. In fact, we have the same conundrum as we had with monoidal categories and $(\mathsf{Vect}_{\Bbbk}, \otimes, \Bbbk)$: our motivating example does *not* satisfy our axioms strictly, and we instead apply identifications implicitly. In this case, Top is *not* a strict ∞-category, since the composition of homotopies is not associative. Given three continuous functions $f, g, h\colon X \to Y$ and two homotopies $H\colon f \rightsquigarrow g$ and $H'\colon g \rightsquigarrow h$, our way of defining a composite is through *concatenation*: we define the homotopy $H' * H\colon X \times I \to Y$ given by

$$(H' * H)(x,t) = \begin{cases} H(x, 2t), & 0 \le t \le 1/2; \\ H'(x, 2t-1), & 1/2 \le t \le 1. \end{cases}$$

But note that the operation is not strictly associative: the composite[1] $H''*(H'*H)$ is a different function from $(H''*H')*H$. More explicitly, these are:

$$(H''*(H'*H))(x,t) = \begin{cases} H(x,4t), & 0 \le t \le 1/4; \\ H'(x,4t-1), & 1/4 \le t \le 1/2; \\ H''(x,2t-1), & 1/2 \le t \le 1. \end{cases}$$

$$((H''*H')*H)(x,t) = \begin{cases} H(x,2t), & 0 \le t \le 1/2; \\ H'(x,4t-2), & 1/2 \le t \le 3/4; \\ H''(x,4t-3), & 3/4 \le t \le 1. \end{cases}$$

We are "squishing" the intervals in a different manner, in each case. Furthermore, one could think of directly defining a "trinary composition" $H''*H'*H$ by dividing the interval into thirds; or any other way one desires, really. These are all "the same", in a certain way, *but that certain way must be specified*. So we consider *weak ∞-categories*, where composition is only defined "up to coherent higher-ordered morphism" — in this case, *up to higher homotopy*.

But note that this is not the only possible way of composing homotopies, either; this is the *vertical* composition. Given two pairs of continuous functions $f, g\colon X \to Y$ and $\tilde{f}, \tilde{g}\colon Y \to Z$, we can compose homotopies $H\colon f \rightsquigarrow g$ and $\tilde{H}\colon \tilde{f} \rightsquigarrow \tilde{g}$ into an homotopy $\tilde{H} \circ H\colon \tilde{f} \circ f \rightsquigarrow \tilde{g} \circ g$ — *horizontal* composition, which in this case is already strictly associative.

$$(\tilde{H} \circ H)(x,t) = \tilde{H}(H(x,t),t).$$

We draw these two different compositions, one vertical and another horizontal, as follows.

$$\begin{tikzcd} X \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow H"] \arrow[r, phantom, shift right=3, "\Downarrow H'"] & Y, \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, "\Downarrow H"] & Y \arrow[r, bend left=60, "\tilde{f}"] \arrow[r, bend right=60, "\tilde{g}"'] \arrow[r, phantom, "\Downarrow \tilde{H}"] & Z. \end{tikzcd}$$

These obey an interchange law — horizontal composition followed by vertical composition is the same, up to coherent higher homotopy, as vertical composition followed by horizontal composition.

$$\begin{tikzcd} X \arrow[r, bend left=60] \arrow[r] \arrow[r, phantom, shift left=3, "\Downarrow"] & Y \arrow[r, bend left=60] \arrow[r] \arrow[r, phantom, shift left=3, "\Downarrow"] & Z \\ & \circ & \\ X \arrow[r, bend right=60] \arrow[r] \arrow[r, phantom, shift right=3, "\Downarrow"] & Y \arrow[r, bend right=60] \arrow[r] \arrow[r, phantom, shift right=3, "\Downarrow"] & Z \end{tikzcd} \simeq \begin{tikzcd} X \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] & Y * Y \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] & Z. \end{tikzcd}$$

Finally, every homotopy has an inverse, so every k-morphism of our category will be invertible when $k > 1$. We will later see that this means Top is an instance of an $(\infty,1)$*-category*.

With this in mind, let us first discuss a strict definition of n-category, before weakening it into a more suitable form. Heed a warning: the latter of these will take a long time to formalize. We won't finish it until the Coda — and even then, the formalism employed might prove unsatisfactory. But this is one of those areas where the ideas and core philosophy are perhaps more important than the nitty-gritty details; at least, for our purposes, they sure are.

Strict n-categories

First, let's give the intuitive idea of what an n-category *should* be. We will be intentionally vague in some aspects.

[1]Here $H''\colon h \rightsquigarrow w$ is another arbitrary homotopy from h to a fourth arbitrary continuous function $w\colon X \to Y$.

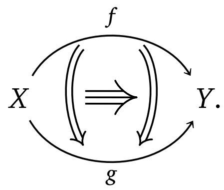
Idea 3.1.1. An ***n-category*** $\mathcal{C}$ consists of:

- A set of objects $\mathrm{ob}(\mathcal{C})$.
- For each pair of objects $(X,Y) \in \mathrm{ob}(\mathcal{C}) \times \mathrm{ob}(\mathcal{C})$, a set of 1-morphisms $\mathrm{Mor}^1_{\mathcal{C}}(X,Y)$.
- For each pair of 1-morphisms $(f,g) \in \mathrm{Mor}^1_{\mathcal{C}}(X,Y) \times \mathrm{Mor}^1_{\mathcal{C}}(X,Y)$ between the same objects, a set of 2-morphisms $\mathrm{Mor}^2_{\mathcal{C}}(f,g)$.

  $\vdots$
- For each pair of $(n-1)$-morphisms $(f_{n-1}, g_{n-1}) \in \mathrm{Mor}^{n-1}_{\mathcal{C}}(f_{n-2}, g_{n-2}) \times \mathrm{Mor}^{n-1}_{\mathcal{C}}(f_{n-2}, g_{n-2})$ between the same $(n-2)$-morphisms, a set of n-morphisms $\mathrm{Mor}^n_{\mathcal{C}}(f_{n-1}, g_{n-1})$.

Along with identity k-morphisms and laws of composition of k-morphisms in k different directions, for each $1 \leq k \leq n$. These must satisfy associativity, unit and interchange laws.

This is known as the *globular* approach to n-categories — in contrast with the *simplicial* or *cubical* approaches — since we can draw the source and target of an n-morphism as in the following diagram:



In this case, $n = 3$. To reduce wordiness in further descriptions of n-categories, we will call two k-morphisms ***parallel*** if they share the same source and target.

Just as we had two different ways to compose homotopies, one horizontal and one vertical, there are k different "dimensions" through which we can compose k-morphisms. It can be a helpful visual to think of each k-morphism as a cube I^k, whose faces are $(k-1)$-morphisms. For example, a 2-morphism $H\colon f \rightsquigarrow g$ can be regarded as the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow H"] & Y \arrow[d, equal] \\ X \arrow[r, "g"'] & Y \end{tikzcd} \tag{3.1}$$

between $\mathrm{id}_X \circ f$ and $g \circ \mathrm{id}_Y$. Composition of 2-morphism is realized by gluing these squares along matching edges — either horizontally or vertically. In the globular approach to higher categories we fill some of the edges with identities, so some of the corners coincide. The previously mentioned cubical approach lets them vary: here a general 2-morphism has source $f\colon X \to Y$ and target $g\colon Z \to W$.

Remark 3.1.2 — The guiding principles of higher category theory.
As we just implied, there are multiple possible approaches when defining either strict or weak higher categories. It is not always the case that two such approaches lead to an equivalent theory, but there exist *guiding principles* to prevent them from straying too far from one another. We will discuss some of these fundamental pillars in the Interlude; there, we will discuss the *Homotopy*, *Stabilization* and *Delooping* Hypotheses [Ths. I.1.7, I.2.7 and I.2.9].

A good theory of higher categories should lay the foundations so that these Hypotheses can be made concrete in a truthful way; these principles model our intuition of how the patterns we have observed in lower-dimensional cases should generalize. So, even if the technical aspects vary, this core philosophy ties the different approaches together.

This is similar to how, in topology, simplicial sets are not the same as simplicial *complexes*, which are not the same as *semi*simplicial sets. But they are all variations on the same core idea, and most theorems proved about one variant can be adapted to another. For informal work guided by intuition, the different approaches can essentially be taken to be equivalent without much harm done; it is only when filling in the painstaking details that one should fix a particular model which is well-suited for their purposes.

In a sense, even the Cobordism Hypothesis is not a theorem to prove, but rather something that we desperately want to be true — and the definitions necessary to state it are made in a way which makes it true. It just happens that, in this case, the subsequent proof is also highly non-trivial.

The approach we will discuss, *enrichment*, is conceptually very simple. When defining a (small) category $\mathcal{C}$, we use sets in four places: the objects $\mathrm{ob}\,\mathcal{C}$ of the category form a set, the morphisms $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ between two objects also form a set, the composition laws $-\circ-\colon \mathrm{Hom}_{\mathcal{C}}(Y,Z)\times\mathrm{Hom}_{\mathcal{C}}(X,Y)\to\mathrm{Hom}_{\mathcal{C}}(X,Z)$ are functions between sets, and the identities $\mathrm{id}_X\in\mathrm{Hom}_{\mathcal{C}}(X,X)$ are elements of a set. Aren't we giving a privileged place to $(\mathsf{Set},\times,\{*\})$, when one of the core ideas of category theory is to abstract away from concrete set-theoretical properties and constructions? Enrichment replaces all of these (except the first!) with properties of an arbitrary monoidal category $(\mathcal{V},\star,\mathrm{I})$.

Definition 3.1.3. Let $(\mathcal{V},\star,\mathrm{I})$ be a monoidal category. A ***$\mathcal{V}$-enriched category*** $\mathcal{C}$ consists of the data of:

- A set of objects $\mathrm{ob}(\mathcal{C})$.
- For each pair of objects $(X,Y)\in\mathrm{ob}(\mathcal{C})\times\mathrm{ob}(\mathcal{C})$, an *object of morphisms* $\mathrm{Map}_{\mathcal{C}}(X,Y)\in\mathcal{V}$.
- For each triple of objects $(X,Y,Z)\in\mathrm{ob}(\mathcal{C})\times\mathrm{ob}(\mathcal{C})\times\mathrm{ob}(\mathcal{C})$, a morphism in $\mathcal{V}$ giving a *composition law* $\circ_{XYZ}\colon \mathrm{Map}_{\mathcal{C}}(Y,Z)\star\mathrm{Map}_{\mathcal{C}}(X,Y)\to\mathrm{Map}_{\mathcal{C}}(X,Z)$.
- For each object $X\in\mathrm{ob}(\mathcal{C})$, a morphism in $\mathcal{V}$ called an *identity element* $\mathrm{j}_X\colon \mathrm{I}\to\mathrm{Map}_{\mathcal{C}}(X,X)$.

This data must satisfy associativity and unit laws, expressed by the commutativity of the following diagrams:

$$\begin{tikzcd}
 & \mathrm{Map}_{\mathcal{C}}(Z,W)\star\mathrm{Map}_{\mathcal{C}}(Y,Z)\star\mathrm{Map}_{\mathcal{C}}(X,Y) \arrow[dl, "\circ_{YZW}\star\mathrm{id}"'] \arrow[dr, "\mathrm{id}\star\circ_{XYZ}"] & \\
\mathrm{Map}_{\mathcal{C}}(Y,W)\star\mathrm{Map}_{\mathcal{C}}(X,Y) \arrow[dr, "\circ_{XYW}"'] & & \mathrm{Map}_{\mathcal{C}}(Z,W)\star\mathrm{Map}_{\mathcal{C}}(X,Z) \arrow[dl, "\circ_{XZW}"] \\
 & \mathrm{Map}_{\mathcal{C}}(X,W) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(X,X)\star\mathrm{Map}_{\mathcal{C}}(X,Y) \arrow[dr, "\circ_{XXY}"] & & \mathrm{Map}_{\mathcal{C}}(X,Y)\star\mathrm{Map}_{\mathcal{C}}(Y,Y) \arrow[dl, "\circ_{XYY}"'] \\
\mathrm{I}\star\mathrm{Map}_{\mathcal{C}}(X,Y) \arrow[u, "\mathrm{j}_X\star\mathrm{id}"] \arrow[r] & \mathrm{Map}_{\mathcal{C}}(X,Y) & \mathrm{Map}_{\mathcal{C}}(X,Y)\star\mathrm{I} \arrow[u, "\mathrm{id}\star\mathrm{j}_Y"'] \arrow[l]
\end{tikzcd}$$

These diagrams certainly look messy, since there is a lot of notational overhead, but they just express the core two properties we constantly return to: associativity and unit laws.

Remark 3.1.4 — Generalized elements.
Note that we have replaced the identity morphisms, which are elements of a set, with a morphism from the unit of the monoidal structure. This is common in category theory: note that a function of sets $\{*\}\to X$ determines an element in X (the image of $*$), so when replacing $(\mathsf{Set},\times,\{*\})$ with another monoidal category, we usually replace set membership $x\in X$ with morphisms $x\colon \mathrm{I}\to X$. This is often called a *generalized element*.

We use Map instead of the usual Hom to emphasize that these aren't sets. With this definition in hand, it is very easy to define 2-categories.

Definition 3.1.5. A (small, strict) ***2-category*** is a category enriched over $(\mathsf{Cat}, \times, \mathbb{1})$.

We then apply a "dictionary" so that different aspects of the structure correspond to the informal of objects, 1-morphisms and 2-morphisms we discussed before.

- The objects of $\mathcal{C}$ are the elements of $\mathrm{ob}(\mathcal{C})$, as usual.
- The 1-morphisms are the objects of each hom-category $\mathrm{Map}_{\mathcal{C}}(X,Y)$. We denote the set of 1-morphisms between two objects as $\mathrm{Mor}^1_{\mathcal{C}}(X,Y) := \mathrm{ob}(\mathrm{Map}_{\mathcal{C}}(X,Y))$.
- The 2-morphisms are the arrows of each hom-category $\mathrm{Map}_{\mathcal{C}}(X,Y)$. We denote the set of 2-morphisms between two 1-morphisms as $\mathrm{Mor}^2_{\mathcal{C}}(f,g)$.
- Composition of 1-morphisms is the action of $\circ_{XYZ}$ on objects: $(g,f) \mapsto g \circ f$.
- Horizontal composition of 2-morphisms is the action of $\circ_{XYZ}$ on arrows: $(\beta,\alpha) \mapsto \beta * \alpha$.
- Vertical composition of 2-morphisms is composition in the hom-category $\mathrm{Map}_{\mathcal{C}}(X,Y)$.

It is easy to check that (small) 2-categories admit products and thus form a monoidal category $(\mathsf{Cat}_2, \times, \mathbb{1})$, where $\mathbb{1}$ is the trivial 2-category consisting of a single object and the identity 1- and 2-morphisms. So we can iterate our definition.

Definition 3.1.6. A (small, strict) ***n-category*** is a category enriched over $(\mathsf{Cat}_{n-1}, \times, \mathbb{1})$.

Before expanding on the issues arising from strictness, let us quickly comment on another similar but distinct approach to n-categories: *internalization*.

Remark 3.1.7 — Internalization and *n*-fold categories.
Do not confuse enrichment with *internalization*, where the whole structure of a category is expressed in terms of commutative diagrams inside Set. Then, these diagrams are transported to another category $\mathcal{D}$ — just like we did for internal monoids [Def. 2.3.4]. Both approaches use iteration in order to define n-categories, but the specifics vary greatly.

In particular, internalization also replaces the *set of objects* with an object of $\mathcal{D}$. If we internalize over Cat, we get the notion of *double categories*. In a double category, there are two different kinds of morphisms: the morphisms of the category of objects $\mathrm{Map}_{\mathrm{ob}\,\mathcal{C}}(X,Y)$, and the objects of the category of morphisms $\mathrm{ob}\,\mathrm{Map}_{\mathcal{C}}(X,Y)$. We think of these as *vertical* and *horizontal* morphisms, respectively, and a 2-morphism can be drawn as in the following diagram.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha_0"'] \arrow[dr, phantom, "\Downarrow H"] & Y \arrow[d, "\alpha_1"] \\ Z \arrow[r, "g"'] & W \end{tikzcd} \tag{3.2}$$

As the reader might notice, this is the aforementioned *cubical* approach to higher categories. Iterating this process, we get what are called *n-fold categories*. These often appear on the literature since they more closely model certain aspects of higher homotopy theory.

Notice, in diagram (3.2), that if all vertical arrows α_i are identities we basically obtain a globular 2-category: we are now in the situation of diagram (3.1). So we can also use n-fold categories to define globular n-categories — but not every n-fold category is equivalent to a globular n-category. Ultimately our model of (∞,n)-categories (called *n-fold complete Segal spaces*, see section C.1) will actually be n-fold categories where part of the structure is trivial.

Weak *n*-categories

Our first instinct, when trying to weaken the definition of an n-category, is to do the same process we did for weakening monoidal categories in Definition 1.2.11: just replace every instance of equality of k-morphism with a $(k+1)$-morphism making the appropriate diagram commute, and add some coherence relations between these higher morphisms so that they behave as we want them to. This instinct will carry some problems, as we will soon see. The definition for $n=2$ is already a bit unwieldy, but it is still useful for certain work. We include it only for reference, since we will not use it; the reader may skip it without harm, after a glance in its general direction.

Definition 3.1.8. A (weak) ***bicategory*** $\mathcal{C}$ consists of the following data.

- A collection of *objects* $\mathrm{ob}\,\mathcal{C}$.
- For each pair of objects $X, Y \in \mathrm{ob}\,\mathcal{C}$, a category of 1-morphisms $\mathrm{Map}_{\mathcal{C}}(X,Y)$.
- Composition functors $\circ_{XYZ}\colon \mathrm{Map}_{\mathcal{C}}(Z,Y) \times \mathrm{Map}_{\mathcal{C}}(X,Y) \to \mathrm{Map}_{\mathcal{C}}(X,Z)$ acting on objects (1-morphisms) as $(g,f) \mapsto g \circ f$ and on arrows (2-morphisms) as $(\beta,\alpha) \mapsto \beta * \alpha$.
- Identity functors $\mathrm{j}_X\colon \mathbb{1} \to \mathrm{Map}_{\mathcal{C}}(X,X)$, that we identify with their image.
- Natural isomorphisms α_{XYZW}, λ_{XY}, ρ_{XY} with domain and codomain as follows.

$$\begin{tikzcd}
& \mathrm{Map}_{\mathcal{C}}(Z,W) \times \mathrm{Map}_{\mathcal{C}}(Y,Z) \times \mathrm{Map}_{\mathcal{C}}(X,Y) \arrow[dl, "\circ_{YZW}\times\mathrm{id}"'] \arrow[dr, "\mathrm{id}\times\circ_{XYZ}"] & \\
\mathrm{Map}_{\mathcal{C}}(Y,W) \times \mathrm{Map}_{\mathcal{C}}(X,Y) \arrow[rr, Rightarrow, "\alpha_{XYZW}"] \arrow[dr, "\circ_{XYW}"'] & & \mathrm{Map}_{\mathcal{C}}(Z,W) \times \mathrm{Map}_{\mathcal{C}}(X,Z) \arrow[dl, "\circ_{XZW}"] \\
& \mathrm{Map}_{\mathcal{C}}(X,W) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(X,X) \times \mathrm{Map}_{\mathcal{C}}(X,Y) \arrow[dr, "\circ_{XXY}"] \arrow[d, phantom, "\Downarrow\lambda_{XY}"] & & \mathrm{Map}_{\mathcal{C}}(X,Y) \times \mathrm{Map}_{\mathcal{C}}(Y,Y) \arrow[dl, "\circ_{XYY}"'] \arrow[d, phantom, "\Downarrow\rho_{XY}"] \\
\mathrm{I} \times \mathrm{Map}_{\mathcal{C}}(X,Y) \arrow[u, "\mathrm{j}_X\times\mathrm{id}"] \arrow[r] & \mathrm{Map}_{\mathcal{C}}(X,Y) & \mathrm{Map}_{\mathcal{C}}(X,Y) \times \mathrm{I} \arrow[u, "\mathrm{id}\times\mathrm{j}_Y"'] \arrow[l]
\end{tikzcd}$$

The components of these natural transformations are invertible 2-morphisms we denote by $\alpha_{hgf}\colon (h\circ g)\circ f \to h\circ(g\circ f)$, $\lambda_f\colon f\circ \mathrm{j}_X \to f$ and $\rho_f\colon \mathrm{j}_Y \circ f \to f$.

This data must satisfy certain coherence constraints: for all 4-tuples of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \xrightarrow{k} V$, the following diagrams must commute.

$$\begin{tikzcd}
& (k\circ h)\circ(g\circ f) \arrow[dr, "\alpha_{kh(g\circ f)}"] & \\
((k\circ h)\circ g)\circ f \arrow[ur, "\alpha_{(k\circ h)gf}"] \arrow[d, "\alpha_{khg}*\mathrm{id}_f"'] & & k\circ(h\circ(g\circ f)) \\
(k\circ(h\circ g))\circ f \arrow[r, "\alpha_{k(h\circ g)f}"'] & k\circ((h\circ g)\circ f) \arrow[ur, "\mathrm{id}_k*\alpha_{hgf}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(g\circ \mathrm{j})\circ f \arrow[rr, "\alpha"] \arrow[dr, "\rho_g*\mathrm{id}_f"'] & & g\circ(\mathrm{j}\circ f) \arrow[dl, "\mathrm{id}_g*\lambda_f"] \\
& g\circ f &
\end{tikzcd}$$

Some authors use the terms bicategory for $n=2$ and tricategory for $n=3$, with the connotation usually being that they refer specifically to weak categories. We will occasionally use these terms, too.

Now, if we wanted to give an analogous definition for tricategories, we would need to take the coherence relations between the associators (the α_{hgf}) and the unitors (the λ_f and ρ_f) and weaken them into coherence 3-morphisms making the diagrams commute. These coherence 3-morphisms would need to satisfy coherence relations of their own. As we go up in dimension, we need more and more coherence data satisfying more and more coherence relations — and explicitly specifying all of this structure is a problem in its own right. All of these complicated diagrams must be verified whenever we want to prove that something is a weak n-category, in the absence of theorems letting us take shortcuts. So this is a big problem.

The combinatorics required to state a definition of n-categories are nightmarishly complex, even for n as low as 4.[2] We have no hope of constructing an explicit definition for, say, $n = 10$ — and even less so for general n.

Remark 3.1.9 — A note on circularity.
Notice that there is another issue with this approach, one that is a bit more subtle. As part of the definition of bicategory, we need to specify certain natural transformations; that is, we need to specify certain 2-*morphisms in* Cat. In order to define bicategories, we are seeing Cat as a bicategory! This is not quite a circular argument, but it comes dangerously close to being one.

But notice that in Definition 3.1.8 we only use natural *isomorphisms*; that is, we only use the *invertible* natural transformations. So we can first properly define $(2,1)$-*categories*, i.e. bicategories where every 2-morphism is invertible, and then use the $(2,1)$-category structure on Cat to define general bicategories. This avoids all traces of circularity, and is the method implemented in (e.g.) [Lur09b].

It should be clear that we will need a better approach. Weirdly, the solution to our problems will be adding *more* structure. An infinite amount of structure, even: it turns out that a detour through (∞, n)-categories will be the key which allows us to develop the theory. But until the Coda, let's move along as if we had a good definition of higher categories. It is important to dream, from time to time.

From now on, unless stated otherwise, all of our n-categories will be weak. This is not a problem, since strict n-categories can be seen as weak n-categories where every associator and unitor is an identity. Equivalently, these are n-categories where every associativity and unit law holds strictly.

3.1.2 $\mathsf{Cob}_n^{\mathrm{fr}}(n)$, the n-category of framed n-cobordisms

Now that we have a general idea of how weak n-categories should behave, we return to our main focus: the Cobordism Hypothesis. Ultimately, we want to give a result analogous to the classical correspondence between 2D oriented TQFTs and commutative Frobenius algebras [Thm. 2.3.3] which applies to n-dimensional TQFTs.

The main problem is that, as we go up in dimension, manifolds become more and more complicated on a global scale. This didn't manifest itself on the 2D case, since we could decompose our cobordisms into pieces by cutting along closed curves — and curves are very simple objects. So we ended up reducing our classification to the values at the circle, the pairs of pants, and the caps: the rest of the TQFT was fully determined. Even more so in the 1D case, where we reduced everything to the incredibly simple case of the value at the *point*!

This was done by specifying generators and relations for Cob_1 and Cob_2, of which we had a finite amount. But note that trying to do the same for Cob_3 ends up with an infinite number of generators for the *objects*: each compact surface of genus g, those being the sphere $\mathbb{S}^2$ and the g-tori, is a distinct generator. This will get worse for larger and larger n, quickly hitting the point where we do not even have a classification for compact $(n-1)$-manifolds.

So, for large n, cutting along $(n-1)$-submanifolds doesn't simplify our problem enough. We need finer decompositions; we want to be able to cut along arbitrary submanifolds with corners, as if giving a triangulation or cellular decomposition of our cobordism. If we could do that, then we would have the tools to greatly simplify our arguments. In the limiting case, going all the way

[2] See the infamous hand-drawn diagrams by Todd Trimble, which are contained in five PDF files hosted at Baez's homepage: `https://math.ucr.edu/home/baez/trimble/tetracategories.html`. Of course, as the accompanying note explains, these are not ad hoc. Other algebraic definitions of higher categories, using gadgets such as operads, also exist. We will ultimately give a *geometric* definition of higher categories, in the Coda.

down to the point, our situation would not be that much more complicated than in the 1D case; if we asked our TQFTs to preserve enough of this structure, one could conjecture that the image of the point would in fact determine the whole TQFT. That is the Cobordism Hypothesis.

But in order to decompose with such granularity, we need manifolds which are more rigid, since merely smooth manifolds are too flexible for this purpose. One way of doing this is to equip our manifolds and cobordisms with a *framing*.

Extending down

Roughly, we want to define an n-category whose top two layers (those of $(n-1)$-morphisms and n-morphisms) are closed $(n-1)$-manifolds and n-cobordisms between these manifolds, generalizing our 1-category Cob_n. The natural choice is for k-morphisms to be k-cobordisms *with corners* whose boundaries are $(k-1)$-morphisms of the category.

Remark 3.1.10 — Triviality along the boundary.
Since we will have multiple layers of decompositions in play, we should ask for higher-dimensional cobordisms to preserve the boundaries of its lower-ordered components. That is, we want them to be *trivial along the boundary*: to restrict to cylinders in the corners. We exemplify for the case $n = 2$, following [Sch14, § 3.2].

Assume we have two 0-manifolds P_0, P_1 and two parallel 1-cobordisms $C_0, C_1 \colon P_0 \to P_1$. As part of their definition, we will have specified certain diffeomorphisms $\partial C_0 \cong P_0 \sqcup P_1 \cong \partial C_1$. Now, we want to define a 2-cobordism $S \colon C_0 \to C_1$.

Note that we don't have a decomposition $\partial S := \partial_{\text{in}} S \sqcup \partial_{\text{out}} S$ of the boundary, since our manifold can now have corners. As such, topologically both C_0 and C_1 might belong to the same connected component of the boundary; we need a different kind of stratification. One easy approach is to first remove the corners and then take connected components; a formalization uses the theory of *$\langle n\rangle$-manifolds*, which are a kind of manifold *with faces* [Sch14, Def. 3.5]. In any case, for our mostly-informal purposes let's just assume that we can decompose

$$\partial S := \partial_{0,\text{in}} S \cup \partial_{0,\text{out}} S \cup \partial_{1,\text{in}} S \cup \partial_{1,\text{out}} S$$

as in figure 3.1. The components numbered with a 0 will correspond to C_0 and C_1, while the components numbered with a 1 will correspond to $P_0 \times I$ and $P_1 \times I$. Hence, we realize our "trivial along the boundary" condition.

So we need to specify two different diffeomorphisms:

$$\begin{aligned} \partial_0 S &= \partial_{0,\text{in}} S \sqcup \partial_{0,\text{out}} S \xrightarrow{f} C_0 \sqcup C_1, \\ \partial_1 S &= \partial_{1,\text{in}} S \sqcup \partial_{1,\text{out}} S \xrightarrow{g} (P_0 \times I) \sqcup (P_1 \times I). \end{aligned}$$

We want for everything to commute nicely, so it should not matter in which order we take our decompositions. That is, we want for the diffeomorphisms induced on the corners

$$\begin{aligned} (g \circ f^{-1})_0 \colon \partial_{\text{in}} C_0 \sqcup \partial_{\text{out}} C_0 &\longrightarrow (P_0 \times \{0\}) \sqcup (P_1 \times \{1\}) \\ (g \circ f^{-1})_1 \colon \partial_{\text{in}} C_1 \sqcup \partial_{\text{out}} C_1 &\longrightarrow (P_0 \times \{0\}) \sqcup (P_1 \times \{1\}) \end{aligned}$$

to coincide with the restriction of the structural maps $\partial C_0 \cong P_0 \sqcup P_1 \cong \partial C_1$.

The previous construction can be performed with arbitrary closed $(d-2)$-manifolds instead of P_0 and P_1. Some authors call these 2-*fold* cobordisms or just 2-*cobordisms*, but the latter term tends to confuse itself with the dimension d of the underlying manifold. In any case, these will be the 2-morphisms of our n-category.

We leave implicit the definition of "trivial along the boundary" for greater dimensions. Ultimately, our construction of a good higher category of cobordisms [Def. C.2.7] won't explicitly

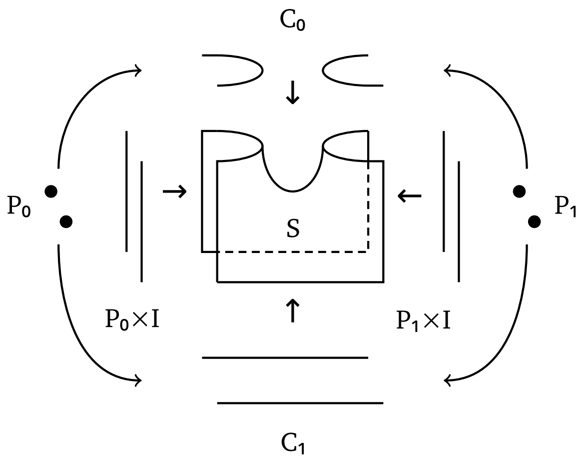


Figure 3.1: The structural decomposition defining a 2-cobordism [Sch14, fig. 3.2].

use this notion, since we would encounter the same combinatorial problems we had when trying to define weak n-categories. This is not much of a problem for low dimensions, such as $n = 2$ or $n = 3$. It is much less useful as a definition for the general case, but it is still a good conceptual tool for describing our structure.

With that, we can give our first of many higher-categorical versions of the category of cobordisms. For maximum clarity, we will be equally explicit in our description of subsequent variations, even when a shorter form would suffice. Thankfully, digital paper does not consume trees.

Idea 3.1.11. The ***n*-category of cobordisms**, $\mathsf{Cob}_n(n)$, is the category described by:

- **Objects:** Closed 0-manifolds (i.e., possibly empty finite unions of points).
- **1-morphisms:** For each pair of 0-manifolds P_0 and P_1, 1-cobordisms $C\colon P_0 \to P_1$.
- **2-morphisms:** For each pair of parallel 1-cobordisms C_0 and C_1, 2-cobordisms with corners $S\colon C_0 \to C_1$ which reduce to the trivial cobordism along the corners.

$\vdots$

- **$(n-1)$-morphisms:** For each pair of parallel $(n-2)$-cobordisms N_0 and N_1, $(n-1)$-cobordisms with corners $M\colon N_0 \to N_1$ which reduce to the trivial cobordism along the corners.
- ***n*-morphisms:** For each pair of parallel $(n-1)$-cobordisms M_0 and M_1, n-cobordisms with corners $B\colon M_0 \to M_1$ which reduce to the trivial cobordism along the corners, *up to diffeomorphism*. The diffeomorphisms must preserve every layer of decompositions.

The identities are cylinders, and the compositions are realized through gluing of cobordisms.

We need to take diffeomorphism classes in the nth level in order to solve the same associativity issues we discussed in Remark 1.1.16. Note that this is intrinsically a weak n-category: we can't take diffeomorphism classes of the lower levels without losing our n-categorical structure, and these compositions are associative only up to higher k-morphism. And unlike weak monoidal categories [Thm. 1.2.13], there are in general no ways of strictifying weak n-categories. This is impossible even for "nice" categories such as n-groupoids, which are n-categories where every k-morphism is invertible. For example, the fundamental 3-groupoid [Def. I.1.3] of the 2-sphere, $\pi_{\leq 3}(\mathbb{S}^2)$, is *not* equivalent to any strict 3-groupoid [Sim98]. There *is* a strictification result for bicategories, so this is only a problem for $n \geq 3$.

Of course, we haven't defined weak n-categories — and it will be a while until we do — but we can still reason about them informally.

We can also consider the k-category of n-cobordisms, $\mathsf{Cob}_n(k)$, for $k < n$. For example, the 1-category of n-cobordisms is just $\mathsf{Cob}_n^{\mathrm{un}}$ [Def. 1.1.21],[3] and the 2-category is composed of closed $(n-2)$-manifolds as objects, $(n-1)$-cobordisms as 1-morphisms and n-cobordisms with corners as 2-morphisms.

Idea 3.1.12. The ***k-category of n-cobordisms*** for $k \leq n$, $\mathsf{Cob}_n(k)$, is the category described by:

- **Objects:** Closed $(n-k)$-manifolds.
- **1-morphisms:** For each pair of $(n-k)$-manifolds K_0 and K_1, $(n-k+1)$-cobordisms $C\colon K_0 \to K_1$.
- **2-morphisms:** For each pair of parallel $(n-k+1)$-cobordisms C_0 and C_1, $(n-k+2)$-cobordisms with corners $S\colon C_0 \to C_1$ which reduce to the trivial cobordism along the corners.

$\vdots$

- **$(n-1)$-morphisms:** For each pair of parallel $(n-2)$-cobordisms N_0 and N_1, $(n-1)$-cobordisms with corners $M\colon N_0 \to N_1$ which reduce to the trivial cobordism along the corners.
- ***n*-morphisms:** For each pair of parallel $(n-1)$-cobordisms M_0 and M_1, n-cobordisms with corners $B\colon M_0 \to M_1$ which reduce to the trivial cobordism along the corners, *up to diffeomorphism*. The diffeomorphisms must preserve every layer of decompositions.

The identities are cylinders, and the compositions are realized through gluing of cobordisms.

Having a notation for these will be useful for the discussion of chapter 4.

Note that in both cases these are just manifolds, without any additional structure such as a choice of orientation or Riemannian metric. It turns out that it is useful for developing the theory to endow our cobordisms with a very rigid kind of structure: a *framing*.

Framed *n*-cobordisms

First and foremost, let's define what we mean by framed manifolds.

Definition 3.1.13. Let M be a manifold of dimension m, and let $n \geq m$. An ***n-framing*** of M is a trivialization of its stabilized tangent bundle,

$$\Phi\colon \mathrm{T}M \oplus \underline{\mathbb{R}}^{n-m} \xrightarrow{\cong} \underline{\mathbb{R}}^{n},$$

where $\underline{\mathbb{R}}^k$ denotes the trivial vector bundle over M with constant fiber $\mathbb{R}^k$.

If $n = m$, we call it a ***framing***. Note that giving a trivialization of the stabilized tangent bundle is the same as giving a smoothly varying choice of basis $\{x_1, \ldots, x_m, v_{m+1}, \ldots, v_n\}$ of $\mathrm{T}_xM \oplus \mathbb{R}^{n-m}$ for every point $x \in M$.

The reason why we want to consider trivializations of the *stabilized* tangent bundle is that this helps us define the n-category of framed cobordisms. Here, objects will be compact 0-manifolds, the 1-morphisms will be cobordisms between these, the 2-morphisms will be cobordisms with *corners* between the 1-morphisms, and so on up to level n. We will want the boundary of each framed m-cobordism to coincide with the $(m-1)$-cobordisms, and for the framings of both to also coincide; so it helps that the fibers of each vector bundle all have fixed dimension n.

[3] In part I, we used the notation Cob_n to refer to the *oriented* version of the category of cobordisms, since that was our main object of study. Now, in part II, omitting any superscripts will usually imply the unoriented version. In cases where the distinction is relevant, such as chapter 4, we will be explicit with our use of notation.

However, not every manifold admits a framing (i.e., is parallelizable): among the closed 2-manifolds, only the torus is.[4] So our category will accept less manifolds than the oriented or unoriented versions.

The *n*-category of framed *n*-cobordisms

The informal definition of the n-category of framed n-cobordisms is almost exactly the same as that of Idea 3.1.11.

Idea 3.1.14. The ***n*-category of framed cobordisms**, $\mathsf{Cob}_n^{\mathrm{fr}}(n)$, is the category described by:

- **Objects:** Closed n-framed 0-manifolds.
- **1-morphisms:** For each pair of 0-manifolds P_0 and P_1, n-framed 1-cobordisms $C\colon P_0 \to P_1$.
- **2-morphisms:** For each pair of parallel 1-cobordisms C_0 and C_1, n-framed 2-cobordisms with corners $S\colon C_0 \to C_1$ which reduce to the trivial cobordism along the corners.

$\vdots$

- **$(n-1)$-morphisms:** For each pair of parallel $(n-2)$-cobordisms N_0 and N_1, n-framed $(n-1)$-cobordisms with corners $M\colon N_0 \to N_1$ which reduce to the trivial cobordism along the corners.
- ***n*-morphisms:** For each pair of parallel $(n-1)$-cobordisms M_0 and M_1, framed n-cobordisms with corners $B\colon M_0 \to M_1$ which reduce to the trivial cobordism along the corners, *up to diffeomorphism*. The diffeomorphisms must preserve every layer of decompositions and be compatible with the framings.

The identities are cylinders, and the compositions are realized through gluing of cobordisms.

As mentioned before, we ask for everything to respect the framings. In particular, a k-cobordism $B\colon M_0 \to M_1$ between two $(k-1)$-manifolds should induce compatible framings on its boundaries. The framing induced in the out-boundary is equal, while the one induced in the in-boundary is opposite: $\partial B \cong \overline{M}_0 \sqcup M_1$. Here, the *opposite framing* is defined as the framing obtained by negating the last vector of the base. This is analogous to Remark 1.1.12 and figure 1.4.

Likewise, one can only compose cobordisms for which the framing on the common boundary coincides; and the diffeomorphisms considered in the top layer, when taking equivalence classes of n-cobordisms, should preserve the framings.

One interpretation of these n-framings is that we are equipping our cobordisms with n different "time" *directions* along which we can glue them, just as we conceptualized the n-cobordisms of the 1-category Cob_n to have $n-1$ spatial dimensions and one time dimension along which our $(n-1)$-manifold evolved. For non-framed cobordisms, this is realized by taking Morse functions in the 1-categorical case and generalized Morse functions (with codomain $\mathbb{R}^k$) in the k-categorical case. See [Fre13b, § 5; CS19, § 5.1] for some pretty pictures exploring this interpretation.

Fully extended TQFTs

Now, an (n-dimensional, framed) fully extended TQFT will be a symmetric monoidal functor from $\mathsf{Cob}_n^{\mathrm{fr}}(n)$ to some other symmetric monoidal n-category $(\mathcal{C}, \star, \mathrm{I}, \tau)$. We will not try to define a suitable n-categorical generalization for $\mathsf{Vect}_{\Bbbk}$ here, since that would distract us from our goals. However, in section 4.2 we will see a bicategory which we can conceptualize as being a *categorification* of $\mathsf{Vect}_{\Bbbk}$.

[4] But it turns out that every orientable 2-manifold is 3-parallelizable, so $\mathsf{Cob}_3^{\mathrm{fr}}(3)$ will have all orientable 2-manifolds as 2-morphisms! Not every orientable manifold is stably parallelizable either, though; as it is usually the case for these kinds of problems, the first counterexamples appear at dimension 4.

We can also define "once extended" TQFTs, as functors $Z\colon \mathsf{Cob}_n^{\mathrm{fr}}(2) \to \mathcal{C}$. In the same manner, "$k$-fold extended" TQFTs are functors $Z\colon \mathsf{Cob}_n^{\mathrm{fr}}(k+1) \to \mathcal{C}$. Note that these terms can be ambiguous — for example, some authors use "once extended" to refer to TQFTs extended *upwards*, where objects are closed manifolds, 1-morphisms are cobordisms, and 2-morphisms are diffeomorphisms.

Notation 3.1.15. A common convention is to call unextended TQFTs ***$(n-1, n)$-TQFTs***, and fully extended TQFTs ***$(0, \dots, n)$-TQFTs***. In general, an ***$(n-k, \dots, n)$-TQFT*** is a symmetric monoidal functor $Z\colon \mathsf{Cob}_n(k) \to \mathcal{C}$. With the previous nomenclature, this is a $(k-1)$-fold extended TQFT.

We will use this notation in chapter 4.

Extending up: the m-category of framed n-cobordisms

In $\mathsf{Cob}_n^{\mathrm{fr}}(n)$, our top layer of morphisms — the n-morphisms — consist of *diffeomorphism classes* of framed cobordisms. But we would rather not do this identification, since there is loss of information in this process. Instead, we want to consider the diffeomorphisms between these cobordisms as $(n+1)$-morphisms, the diffeotopies between two diffeomorphisms as $(n+2)$-morphisms, and so on — all the way up to some layer $m > n$, where we say "well, that's enough" and take equivalence classes again. We get the m-category denoted as $\mathsf{Cob}_n^{\mathrm{fr}}(m)$.

Idea 3.1.16. The ***m-category of framed n-cobordisms*** for $m \geq n$, $\mathsf{Cob}_n^{\mathrm{fr}}(m)$, is the m-category composed of the following data.

- **Objects:** Closed n-framed 0-manifolds.
- **1-morphisms:** For each pair of 0-manifolds P_0 and P_1, n-framed 1-cobordisms $C\colon P_0 \to P_1$.
- **2-morphisms:** For each pair of parallel 1-cobordisms C_0 and C_1, n-framed 2-cobordisms with corners $S\colon C_0 \to C_1$ which reduce to the trivial cobordism along the corners.

$\vdots$

- ***n*-morphisms:** For each pair of parallel $(n-1)$-cobordisms M_0 and M_1, framed n-cobordisms with corners $B\colon M_0 \to M_1$ which reduce to the trivial cobordism along the corners.
- **$(n+1)$-morphisms:** For each pair of parallel n-cobordisms B and B', diffeomorphisms $f\colon B \to B'$ respecting the framings.
- **$(n+2)$-morphisms:** For each pair of parallel diffeomorphisms f and g, diffeotopies $H\colon f \rightsquigarrow g$.

$\vdots$

- **$(m-1)$-morphisms:** For each pair of parallel $(m-n-3)$-diffeotopies H and H', $(m-n-2)$-diffeotopies $\Phi\colon H \rightsquigarrow H'$.
- ***m*-morphisms:** For each pair of parallel $(m-n-2)$-diffeotopies H and H', $(m-n-1)$-diffeotopies $\Phi\colon H \rightsquigarrow H'$, *up to higher diffeotopy*.

But there is no reason (other than complexity) to stop after a finite number of steps; if we had a way of making sense of the structure, we could "take the limit" $m \to \infty$ and obtain an *∞-category* of framed n-cobordisms. That will be the goal of section C.2, where we will define $\mathsf{Bord}_n^{\mathrm{fr}}$: the (∞, n)-category of framed n-cobordisms. But what is an (∞, n)-category? Well, note that all morphisms above layer n are invertible, since diffeomorphisms and diffeotopies always have an inverse. This will simplify the theory, since (∞, n)-categories are easier to formally define — we don't have "infinite layers of unbounded complexity".

For the time being, let's content ourselves with the n-categorical version. We now proceed to our first of many statement of the Cobordism Hypothesis.

3.2 Statements of the Cobordism Hypothesis

We now state the Cobordism Hypothesis, the main conundrum throughout part I. First we give an n-categorical version, with some definitions pending. We fill those gaps in section 3.2.2, which also introduces some (∞, n)-categorical language. In the later section 4.1, we will give yet another version — for cobordisms with structure other than framings.

3.2.1 The Cobordism Hypothesis for n-categories

We have enough theory to be able to state the Cobordism Hypothesis, except for one detail. Note that in our previous classification results we did not have an equivalence with the target category, but only with a good-behaved subcategory. For 1D oriented TQFTs it was *finite-dimensional* vector spaces [Thm. 1.3.8], while for 2D oriented TQFTs ir was either *commutative Frobenius* algebras [Thm. 2.3.3]. In the more general statements with arbitrary target category, these where *dualizable* objects [Thm. 1.3.10] and *commutative Frobenius* objects [Thm. 2.3.16]. What should the analogue of these "finiteness" conditions be? Those will be *fully dualizable* objects.

For now, let's state the Hypothesis anyway; we will fill in the gaps as we go. In a sense, the definition of "fully dualizable" will be the one which makes our statement true.

Our first of many statements of the Cobordism Hypothesis

Without further ado, let's see what the Cobordism Hypothesis is all about. We will later discuss why this should be true.

Thesis 3.2.1: Cobordism Hypothesis for n-categories. There is a bijection between (isomorphism classes of) n-dimensional framed fully extended TQFTs $Z\colon \mathsf{Cob}_n^{\mathrm{fr}}(n) \to \mathcal{C}$ with values in a symmetric monoidal n-category $(\mathcal{C}, \star, \mathrm{I}, \tau)$ and (isomorphism classes of) fully dualizable objects M in $\mathcal{C}$, which is given by evaluation of the TQFT on the positively framed point:

$$\mathsf{TQFT}_n^{\mathrm{fr}}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{C}^{\mathrm{fd}}, \qquad Z \longmapsto Z(\mathrm{pt}_+).$$

This, too, can be stated as a universal property.

Thesis 3.2.2. The n-category of framed n-cobordisms, $\mathsf{Cob}_n^{\mathrm{fr}}(n)$, is the free symmetric monoidal n-category on a single fully dualizable object.

Remark 3.2.3 — Symmetric monoidal n-categories.
We have not defined "symmetric monoidal n-categories", and will not do so formally — at least not in a way reminiscent of Definition 1.2.14, which is what one would hope to obtain. Soon, in Remark 3.2.8, we will sketch one possible approach. Then, in the Interlude, we will explore some of the philosophy involved.

Note that our perhaps naïve first approximation for the problem, that of 1D oriented TQFTs [Thm. 1.3.10], *is* the Cobordism Hypothesis for $n = 1$! Here framed manifolds and oriented manifolds coincide (up to equivalence), since orienting a curve also determines a framing. So it should not be *that* surprising that, once we get down to the level of points and ask for our TQFTs to preserve *all* of this structure, we get an analogous result — even in higher dimensions. Even if specifying a fully extended TQFT means specifying a colossal amount of information, we are asking for it to satisfy an equally colossally big amount of coherence relations. And we are also asking to be able to decompose our cobordisms by what is effectively a triangulation, take the value of our TQFT on that decomposition, and then assemble these values back to get the same result. As such, we should actually expect that these TQFTs are fully determined by the tiniest amount of information possible — the image at a single point.

Now that we are hopefully convinced that this is not utter nonsense, let's fill in the gaps in our statement. What are fully dualizable objects? And most importantly — what *should* they be? We should first fix some of the ideas we are building up to; we can no longer postpone talk of (∞, n)-categories. We will not properly define (∞, n)-categories on the main text. Rather, we include an Interlude discussing what (∞, n)-categories *should be*, and a Coda as a lengthy technical appendix detailing a particular model for (∞, n)-categories. These should work in tandem to clear the thick fog which has been accumulating around these concepts — at least partially.

3.2.2 Letting n tend to ∞: from n-categories to (∞, n)-categories

Here we properly start the discussion of (∞, n)-categories we have been building up to. We do not technically need them until section 4.1.1, but it will be helpful to start familiarizing ourselves with the language.

(n, m)-categories

Although the name can be intimidating, the concept of an (n, m)-category is actually extremely simple, once one knows about n-categories.

Idea 3.2.4. An ***(n, m)-category*** for $0 \le m \le n \le \infty$ is a category with n layers of morphisms and such that all k-morphisms are invertible for $k > m$.

We assume every (n, m)-category to be weak; this does not lose generality, since we can define a strict (n, m)-category as one where all associativity and unit laws hold strictly.

Remark 3.2.5 — Some (n, m)-categories we have already seen, in secret.
In all of this discussion, n is allowed to take ∞ as a value.

With this notation, an $(n, 0)$-category is the same as an ***n-groupoid***: an n-category where every k-morphism is invertible, for all k. On the other hand, an (n, n)-category is a general n-category.

The ∞-category of topological spaces, Top, is an example of an $(n, 1)$-category: while not all 1-morphisms (continuous functions) are invertible, all k-morphisms for $k > 1$ (homotopies) have an inverse. The m-categories $\mathsf{Cob}_n^{\mathrm{fr}}(m)$ of Idea 3.1.16, for $m \ge n$, are examples of (m, n)-categories.

We can see any n-category as an (∞, n)-category where the only k-morphisms for $k > n$ are the identity morphisms. In fact, this will be our *definition* of n-category.

Another way of thinking about the hierarchy of higher categories is the following. An (∞, n)-category is an ∞-category where “not much of interest” happens after layer n, so they are easier to study than arbitrary ∞-categories. Then, n-categories are ∞-categories (or (∞, n)-categories) where “nothing of interest” happens after layer n; every k-morphism for $k > n$ is an identity, which is the bare minimum structure needed for something to be an ∞-category.

The reason we are interested in (∞, n)-categories, rather than the even more general ∞-categories, is that they are much more tractable — while still carrying all of the information we are interested in. Each set of morphisms $\mathrm{Map}_{\mathcal{C}}(X, Y)$ of an (∞, n)-category forms an $(\infty, n-1)$-category, which is simpler, so we can often do proofs by induction or iteration. This is reflected on most of our definitions, too. The sets of morphisms of an ∞-category, however, form another ∞-category — so there is no reduction in complexity, and we need to employ a different set of techniques.

For the remainder of this thesis, we will act as if we already had a good definition of (∞, n)-category. As mentioned, we will properly define these in the Coda.

$\mathsf{Bord}_n^{\mathrm{fr}}$, the (∞, n)-category of framed n-cobordisms

For example, we can now sketch a definition for the (∞, n)-categorical analogue to Idea 3.1.16. This (and some of its variants) will be our main object of study from now on.

Idea 3.2.6. The ***(∞, n)-category of framed n-cobordisms***, $\mathsf{Bord}_n^{\mathrm{fr}}$, is the n-category composed of the following data.

- **Objects:** Closed n-framed 0-manifolds.
- **1-morphisms:** For each pair of 0-manifolds P_0 and P_1, n-framed 1-cobordisms $C\colon P_0 \to P_1$.
- **2-morphisms:** For each pair of parallel 1-cobordisms C_0 and C_1, n-framed 2-cobordisms with corners $S\colon C_0 \to C_1$ which reduce to the trivial cobordism along the corners.

$\vdots$

- ***n*-morphisms:** For each pair of parallel $(n-1)$-cobordisms M_0 and M_1, framed n-cobordisms with corners $B\colon M_0 \to M_1$ which reduce to the trivial cobordism along the corners.
- **$(n+1)$-morphisms:** For each pair of parallel n-cobordisms B and B', diffeomorphisms $f\colon B \to B'$ respecting the framings.
- **$(n+2)$-morphisms:** For each pair of parallel diffeomorphisms f and g, diffeotopies $H\colon f \rightsquigarrow g$.

$\vdots$

- ***m*-morphisms:** For each pair of parallel $(m-n-2)$-diffeotopies H and H', $(m-n-1)$-diffeotopies $\Phi\colon H \rightsquigarrow H'$.

$\vdots$

The aforementioned Coda also contains a detailed construction for Bord_n, the unoriented sibling of $\mathsf{Bord}_n^{\mathrm{fr}}$.

In Lurie's case, the use of (∞, n)-categories — and the study of $\mathsf{Bord}_n^{\mathrm{fr}}$, instead of the n-category $\mathsf{Cob}_n^{\mathrm{fr}}(n)$ for which the Cobordism Hypothesis was originally formulated — arised as a necessity: the proof sketch in [Lur09] proceeds by induction on n, and proper understanding of the $(n+1)$-morphisms of $\mathsf{Bord}_{n+1}^{\mathrm{fr}}$ requires an understanding of the $(n+1)$-morphisms of $\mathsf{Bord}_n^{\mathrm{fr}}$. This information is completely lost when passing to $\mathsf{Cob}_n^{\mathrm{fr}}(n)$ — which we can recover as the *homotopy n-category* [Def. 3.2.9] $\mathrm{h}_n \mathsf{Bord}_n^{\mathrm{fr}}$. This does not mean that our previous statements of the Cobordism Hypothesis [Ths. 3.2.1 and 3.2.2] are false due to this lack of higher information, but that Lurie's method for proving them passes through a more general (∞, n)-categorical version [Ths. 3.2.17 and 3.2.18] before forgetting the higher information (if one wants to do so).

Now, let's continue our journey towards fully dualizable objects.

Towards a definition of fully dualizable objects

Remember how, for Cob_1, the Cobordism Hypothesis amounts to Theorem 1.3.10: 1D framed fully extended TQFTs are the same as 1D oriented TQFTs, and the latter is in bijective correspondence with dualizable objects in the target category $\mathcal{C}$. Since Cob_1 and $\mathcal{C}$ are 1-categories here we only had objects and 1-morphisms, and this notion of duality was enough. But for higher categories, we want to extend the notion of "dualizability" from objects (also known as 0-morphisms) to k-morphisms, for all $0 \leq k < n$.

Remember, of course, the definitions of dualizability [Def. 1.2.19] and of categories with duals for objects [Def. 1.2.20]. These require to work within a monoidal category, but the reader might rightfully complain that we haven't defined monoidal higher categories at all. We can apply the same strategy as we did in Remark 1.3.2, and *define* a monoidal (∞, n)-category as an $(\infty, n+1)$-category with a single object.[1] This is done through the operation of *looping*.

[1]Here we are implicitly employing the *Delooping Hypothesis* [Ths. I.2.9], which we will discuss in the Interlude.

Looping and truncation of (∞, n)-categories

Definition 3.2.7. Let $\mathcal{C}$ be an $(\infty, n+1)$-category with a single object $*$. Its ***looping category*** $\Omega\mathcal{C}$ is the (∞, n)-category of endomorphisms $\mathrm{Map}_{\mathcal{C}}(*,*)$, equipped with the monoidal structure $\star$ induced by composition of 1-morphisms in $\mathcal{C}$.

The idea is that $\mathrm{Map}_{\mathcal{C}}(*,*)$ remembers an additional piece of structure from $\mathcal{C}$ which isn't part of the definition of an (∞, n)-category — we can compose *objects*, since these were just the 1-morphisms in our prior category! So given two 1-morphisms in $\mathcal{C}$, say X, $Y\colon * \to *$, we define $X \star Y = X \circ Y$, and similarly for $(k+1)$-morphisms in $\mathcal{C}$ (which are k-morphisms in $\Omega\mathcal{C}$). Since our category $\mathcal{C}$ satisfied all relevant coherence laws for composition, our monoidal category $\Omega\mathcal{C}$ will satisfy all relevant coherence laws for the monoidal operation.

Remark 3.2.8 — Symmetric monoidal (∞, n)-categories.
A *symmetric monoidal* (∞, n)-category is defined as an $(\infty, 2n+2)$-category with a single k-morphism for $0 \le k \le n+2$, but that is harder to justify — we will take their definition for granted, and explore some lower-dimensional cases in the Interlude. Once again, the different compositions of k-morphisms in $\mathcal{C}$, for $k \le n+2$, end up becoming the symmetric structure of the $(n+2)$-times looped category $\Omega^{n+2}\mathcal{C}$. The key concept here is that a symmetric monoidal (∞, n)-category is something called a $(n+2)$-*tuply monoidal* (∞, n)-*category*.

We need another quick definition, before being able to iterate as we want.

Definition 3.2.9. Let $\mathcal{C}$ be an (∞, n)-category. We describe $\mathrm{h}_m\mathcal{C}$, the ***homotopy m-category*** of $\mathcal{C}$.

- **Objects:** Objects of $\mathcal{C}$.
- **1-morphisms:** 1-morphisms of $\mathcal{C}$.

$\vdots$

- **$(m-1)$-morphisms:** $(m-1)$-morphisms of $\mathcal{C}$.
- ***m*-morphisms:** *Equivalence classes* of m-morphisms of $\mathcal{C}$ up to $(m+1)$-isomorphism.

So we are "chopping up" our category at the mth level, just as we did at the start of this chapter: we *truncated* the $(\infty, 1)$-category Top at the nth level, to get an $(n, 1)$-category $\mathsf{Ho}_{\le n}\mathsf{Top}$. When $m = 1$, we usually omit the subscript and write $\mathrm{h}\mathcal{C}$.

When $m \ge n$, the category described in the same way is usually called the *m-truncation* of $\mathcal{C}$ and denoted $\tau_{\le m}\mathcal{C}$. This is done for formal reasons; we will not distinguish between the two.

Fully dualizable objects

In any case, now that we have all of these operations, we can iterate the definition of dualizability without much hassle. First, we define adjunctions for 1-morphisms in a bicategory. This generalizes the notion of adjunction between functors, which can be seen as adjoint 1-morphisms in the bicategory Cat.

Definition 3.2.10. Let $\mathcal{C}$ be a 2-category, let X, $Y \in \mathcal{C}$ be objects and let $f\colon X \to Y$, $g\colon Y \to X$ be morphisms. A 2-morphism

$$u\colon \mathrm{id}_X \Longrightarrow g \circ f$$

is the ***unit of an adjunction*** between f and g if there exists another 2-morphism

$$v\colon f \circ g \Longrightarrow \mathrm{id}_Y,$$

called the ***counit*** of the adjunction, such that the following compositions equal the identities.

$$\mathrm{id}_f = (v \times \mathrm{id}_f) \circ (\mathrm{id}_f \times u)\colon f \cong f \circ \mathrm{id}_X \xRightarrow{\mathrm{id}\otimes u} f \circ g \circ f \xRightarrow{v\otimes \mathrm{id}} \mathrm{id}_Y \circ f \cong f,$$
$$\mathrm{id}_g = (\mathrm{id}_g \times v) \circ (u \times \mathrm{id}_g)\colon g \cong \mathrm{id}_X \circ g \xRightarrow{u\otimes \mathrm{id}} g \circ f \circ g \xRightarrow{\mathrm{id}\otimes v} g \circ \mathrm{id}_Y \cong g.$$

Compare this with Definition 1.2.19 (dual object) and with equations (1.1) (of Zorro's Lemma). One often says that the unit u *exhibits* f *as a left adjoint to* g and that the counit v *exhibits* g *as a right adjoint to* f.

A bicategory has adjoints for 1-morphisms if every 1-morphism is part of an adjunction.

Definition 3.2.11. A 2-category $\mathcal{C}$ ***admits adjoints for 1-morphisms*** if:

- For every 1-morphism $f\colon X \to Y$, there exists a 1-morphism $g\colon Y \to X$ and a 2-morphism $u\colon \mathrm{id}_X \Rightarrow g \circ f$ which is the unit of an adjunction.
- For every 1-morphism $g\colon Y \to X$, there exists a 1-morphism $f\colon X \to Y$ and a 2-morphism $u\colon \mathrm{id}_X \Rightarrow g \circ f$ which is the unit of an adjunction.

Now, the general definition is obtained through the homotopy 2-categories, and we generalize for k-morphisms by iteratively taking hom-categories.

Definition 3.2.12. An (∞, n)-category $\mathcal{C}$ ***admits adjoints for 1-morphisms*** if $\mathrm{h}_2\mathcal{C}$ admits adjoints for morphisms. An (∞, n)-category $\mathcal{C}$ ***admits adjoints for k-morphisms*** if, for all objects $X, Y \in \mathcal{C}$, the $(\infty, n-1)$-category $\mathrm{Map}_{\mathcal{C}}(X, Y)$ admits adjoints for $(k-1)$-morphisms.

Definition 3.2.13. An (∞, n)-category $\mathcal{C}$ ***has adjoints*** if it admits adjoints for k-morphisms for all $0 < k < n$. A monoidal (∞, n)-category $\mathcal{C}$ ***has duals*** if it has duals for objects and has adjoints.

Note that this depends on the choice of n! Viewing an (∞, n)-category as an $(\infty, n+1)$-category will change this notion, and the two properties will not coincide in general.[2] In our case, the meaning will be clear by context.

Also notice that $\mathcal{C}$ is an $(\infty, 1)$-category with duals if and only if its homotopy category $\mathrm{h}_1\mathcal{C}$ is a 1-category with duals for objects (i.e., a rigid category).

Now, a fully dualizable object will be an object which admits higher-ordered adjoints. However, specifying the properties these adjoints should is subject to combinatorial problems — the same ones we encountered when trying to define weak n-categories in section 3.1.1. To avoid these pitfalls, the approach we will take is to first define the maximal subcategory with duals $\mathcal{C}^{\mathrm{fd}}$ of $\mathcal{C}$, and then define a fully dualizable object as an object belonging to $\mathcal{C}^{\mathrm{fd}}$. The actual definition will be slightly different, to preserve the principle of equivalence.

Claim 3.2.14. Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. There is a symmetric monoidal (∞, n)-category $\mathcal{C}^{\mathrm{fd}}$ equipped with a symmetric monoidal functor $i\colon \mathcal{C}^{\mathrm{fd}} \hookrightarrow \mathcal{C}$ such that

(i) the symmetric monoidal (∞, n)-category $\mathcal{C}^{\mathrm{fd}}$ has duals;

(ii) the map $i\colon \mathcal{C}^{\mathrm{fd}} \hookrightarrow \mathcal{C}$ is universal with respect to (i).

The universality mentioned in condition (ii) can be stated as follows: for any other symmetric monoidal (∞, n)-category with duals $\mathcal{D}$ equipped with a symmetric monoidal functor $F\colon \mathcal{D} \to \mathcal{C}$, there exists a unique (up to isomorphism) $f\colon \mathcal{D} \to \mathcal{C}^{\mathrm{fd}}$ such that $F \cong i \circ f$.

We identify $\mathcal{C}^{\mathrm{fd}}$ with the maximal full subcategory with duals of $\mathcal{C}$; in particular, if $\mathcal{C}$ already has duals then $\mathcal{C} = \mathcal{C}^{\mathrm{fd}}$. Now, we use the inclusion $i\colon \mathcal{C}^{\mathrm{fd}} \hookrightarrow \mathcal{C}$ to *define* full dualizability.

Definition 3.2.15. An object $X \in \mathcal{C}$ is ***fully dualizable*** if it belongs to the essential image of the functor $i\colon \mathcal{C}^{\mathrm{fd}} \hookrightarrow \mathcal{C}$.

Remember that the *essential image* of a functor is composed by all objects in the target category which are equivalent to those in the strict image. Informally, we pass from $\mathcal{C}$ to $\mathcal{C}^{\mathrm{fd}}$ by discarding every object which does not admit a dual and every k-morphism which does not admit (both left and right) adjoints. We can make this a bit more precise. First, if $\mathcal{C}$ is any (∞, n)-category, we define $\mathbf{G}_{>k}\mathcal{C}$ to be the (∞, k)-category obtained by discarding each non-invertible m-morphism for $m > k$.

[2] In fact, an (∞, n)-category will have adjoints for n-morphisms if and only if $\mathcal{C}$ is an ∞-groupoid.

Remark 3.2.16 — An intuitive interpretation of full dualizability.
An object X in an (∞, n)-category $\mathcal{C}$ is fully dualizable if the following holds.

- X is dualizable, as an object of the 1-category $\mathbf{G}_{>1}\mathcal{C}$.
- The unit and counit of the previous adjunction are dualizable, as morphisms of $\mathbf{G}_{>2}(\mathcal{C})$.
- $\vdots$
- The unit and counit of the previous adjunctions are dualizable, as $(n-1)$-morphisms of $\mathbf{G}_{>n}(\mathcal{C})$.

We stop at $(n-1)$-morphisms, since otherwise we would force all of the morphisms implementing the adjunctions to be invertible. This would leave out a lot of the interesting examples we want to study.

So we can now make sense of Theses 3.2.1 and 3.2.2, and see that they do in fact extend the 1-categorical Theorem 1.3.10.

More statements of the Cobordism Hypothesis

Now that we have warmed ourselves up to some even higher category theory, we can give statements of the Cobordism Hypothesis for (∞, n)-categories. analogously to Theses 3.2.1 and 3.2.2.

Thesis 3.2.17: Cobordism Hypothesis for (∞, n)-categories. There is a bijection between (isomorphism classes of) n-dimensional framed fully extended TQFTs $Z\colon \mathsf{Bord}_n^{\mathrm{fr}} \to \mathcal{C}$ with values in a symmetric monoidal (∞, n)-category $(\mathcal{C}, \star, \mathrm{I}, \tau)$ and (isomorphism classes of) fully dualizable objects M in $\mathcal{C}$, which is given by evaluation of the TQFT on the positively framed point:

$$\mathsf{Fun}^{\otimes}(\mathsf{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \xrightarrow{\simeq} \mathcal{C}^{\mathrm{fd}}, \qquad Z \longmapsto Z(\mathrm{pt}_+).$$

In particular, any fully extended TQFT $Z\colon \mathsf{Bord}_n^{\mathrm{fr}} \to \mathcal{C}$ factors through $\mathcal{C}^{\mathrm{fd}}$:

$$\begin{tikzcd} & \mathcal{C}^{\mathrm{fd}} \arrow[d, "i"] \\ \mathsf{Bord}_n^{\mathrm{fr}} \arrow[ur, dashed] \arrow[r, "Z"'] & \mathcal{C}. \end{tikzcd}$$

We also state the universal property, for good measure.

Thesis 3.2.18. The (∞, n)-category of framed n-cobordisms, $\mathsf{Bord}_n^{\mathrm{fr}}$, is the free symmetric monoidal (∞, n)-category on a single fully dualizable object.

Note that the n-categorical versions of Theses 3.2.1 and 3.2.2 follow as a corollary, since we can identify $\mathsf{Cob}_n^{\mathrm{fr}}(n)$ with the homotopy n-category $\mathrm{h}_n \mathsf{Bord}_n^{\mathrm{fr}}$ [Def. 3.2.9] and any n-category $\mathcal{C}$ with an (∞, n)-category where the only k-morphisms for $k > n$ are the identities.

We now take a break from cobordisms with a short Interlude, where we will detail some of the more philosophical aspects of higher category theory mentioned in Remark 3.1.2. Then, two different paths will be presented to the reader: they can either proceed to chapter 4, where we directly compare the Cobordism Hypothesis with the classical correspondence of part I; or read the more technical Coda first, where we give a suitable model for (∞, n)-categories and construct the (∞, n)-category of (unstructured) cobordisms Bord_n.

Interlude

The guiding principles of higher category theory

> Everything I'm going to tell you *should* be true, once we really understand what is going on. Right now it's more in the nature of dreams and speculations, but I don't think we'll be able to prove the theorems until we dream enough.
>
> **John C. Baez, "Lectures on *n*-Categories and Cohomology".**

I.1 Grothendieck's Dream: The Homotopy Hypothesis

As it turns out, to talk about higher categories you just need to develop enough homotopy theory.

Weak homotopy equivalences, *n*-types and *n*-groupoids

For our purposes, we are interested in topological spaces up to *weak* homotopy equivalence.

Definition I.1.1. A continuous function $f\colon X \to Y$ is a ***weak homotopy equivalence*** if it induces isomorphisms in each homotopy group, for each choice of basepoint $x \in X$:

$$f_*\colon \pi_0(X) \xrightarrow{\cong} \pi_0(Y), \qquad f_*\colon \pi_n(X,x) \xrightarrow{\cong} \pi_n(Y,f(x)) \quad \forall n>0, \forall x \in X.$$

Two topological spaces X and Y are said to be *weak homotopy equivalent* if there exists a weak homotopy equivalence between them.

Since $\pi_0(X)$ is only a set (of path components), there the appropriate notion of "isomorphism" is for f_* to be a bijection. And note that it suffices to check that f induces isomorphisms for each choice of *path component* of X.

Recall that a groupoid is a category where every morphism is invertible. To any topological space we can assign a groupoid, its *fundamental groupoid*, analogous to the fundamental group $\pi_1(X,x)$.

Definition I.1.2. The ***fundamental groupoid*** $\pi_{\leq 1}(X)$ of a topological space X is the groupoid described as follows.

- **Objects:** Points $x \in X$.
- **Morphisms:** For each pair of points x, $y \in X$, equivalence classes of paths $\gamma\colon x \rightsquigarrow y$ up to homotopy.
- **Composition:** Concatenation of parallel paths $\delta * \gamma\colon x \rightsquigarrow y$.
- **Identities:** Constant paths c_x.

This contains the information for the fundamental group at each point: $\pi_1(X,x)$ is the set of endomorphisms $\mathrm{Hom}_{\pi_{\leq 1}}(x,x)$, homotopy classes of paths starting and ending at x. But it also contains information for the set of path components π_0: the isomorphism classes of the objects are precisely the path components of the space.

This definition easily generalizes to higher category theory, assuming we define (weak) n-categories properly. Remember that an *n-groupoid* is an n-category where each k-morphism is invertible.

Definition I.1.3. The ***fundamental n-groupoid*** $\pi_{\leq n}(X)$ of a topological space X is the n-groupoid described as follows.

- **Objects:** Points $x \in X$.
- **1-morphisms:** For each pair of points x and y, paths $\gamma\colon x \rightsquigarrow y$.
- **2-morphisms:** For each pair of parallel paths γ and δ, homotopies $H\colon \gamma \rightsquigarrow \delta$ which are constant along the common boundary $\partial I \times I$.

$\vdots$

- **$(n-1)$-morphisms:** For each pair of parallel $(n-3)$-homotopies h and h', $(n-2)$-homotopies $H\colon h \rightsquigarrow h'$ which are constant along the common boundary $\partial I^{n-2} \times I$.
- **n-morphisms:** For each pair of parallel $(n-2)$-homotopies H and H', equivalence classes of $(n-1)$-homotopies $\Phi\colon H \rightsquigarrow H'$ which are constant along the common boundary $\partial I^{n-1} \times I$, up to n-homotopy.

The identity k-morphisms are the constant paths/homotopies $\mathrm{c}_H\colon I^{k-1} \times I \to X$ defined as $\mathrm{c}_H(-,t) = H(-)$, while the compositions are given by concatenation in each dimension.

The fundamental n-groupoid $\pi_{\leq n}(X)$ contains the information of all homotopy groups of our space up to level n.

Remark I.1.4 — Recovering the homotopy groups from the fundamental n-groupoid. Let's see, first, how we can recover the nth homotopy group $\pi_n(X,x)$ from the n-groupoid $\pi_{\leq n}(X)$.

By taking the hom-set $\mathrm{Map}_{\pi_{\leq n}}(x,x)$, we get an $(n-1)$-category whose objects are loops on x; this is called the *looping* of the category [Defs. I.2.2 and 3.2.7]. We can once again take the looping of this category, this time on the constant path c_x, to get an $(n-2)$-category of homotopies starting and ending at c_x. By successively taking loops, we eventually reach a 0-category (or set) of equivalence classes of $(n-1)$-homotopies from the constant $(n-2)$-homotopy c_x to itself. These are functions $H\colon I^n \to X$ which are constant on the whole boundary ∂I^n, so by collapsing the boundary to a point we get functions $H\colon \mathbb{S}^n \to X$, which are precisely the elements of the nth homotopy group $\pi_n(X,x)$.

Another useful operation on n-categories is *truncation* at the level m [Def. 3.2.9], which just means removing every k-morphism for $k > m$ and taking equivalence classes of m-morphisms to obtain a new m-category. The k-morphisms for $k < m$ are left unchanged. In this case, the m-truncation of $\pi_{\leq n}(X)$ is the fundamental m-groupoid $\pi_{\leq m}(X)$. Therefore, by applying the operations of looping and truncation, we can recover each homotopy group.

So if we are only interested in the n first homotopy groups, we can discard the higher-ordered information without loss of generality.

Definition I.1.5. A topological space X is called an ***homotopy n-type*** if its homotopy groups $\pi_k(X,x)$ vanish for all $k > n$.

When talking about fundamental n-groupoids, the proper notion of equivalence between topological spaces is equivalence of n-types; we want to "truncate" our spaces into n-types before deciding if they are homotopy equivalent. This can always be done, by first passing to a CW-complex through a weak homotopy equivalence and then attaching cells to kill the higher homotopy groups.

Plus, just as for any group G one can construct a connected CW-complex $K(G,1)$ with $\pi_1(K(G,1)) \cong G$ and $\pi_i(K(G,1)) \cong 0$ (when $i > 1$), for any 1-groupoid $\mathcal{G}$ one can construct a space $\mathrm{B}\mathcal{G}$ whose fundamental groupoid $\pi_{\leq 1}(\mathrm{B}\mathcal{G})$ is equivalent to $\mathcal{G}$ and whose higher homotopy groups $\pi_i(\mathrm{B}\mathcal{G})$ vanishes. The spaces $K(G,1)$ are called *Eilenberg–Mac Lane spaces*; as the notation implies, if G is abelian then for any $n > 0$ one can construct a space $K(G,n)$ with $\pi_n(K(G,n)) \cong G$ and $\pi_i(K(G,n)) \cong 0$ for $i \neq n$. On the other hand, $\mathrm{B}\mathcal{G}$ is called the *classifying space* of the groupoid $\mathcal{G}$; see [Fre13a, § 18] for a construction.[1] So we have a bijection between homotopy 1-types and 1-groupoids. The Homotopy Hypothesis conjectures that this is also true for n-groupoids.

The Homotopy Hypothesis

Here is one particular formulation of the Homotopy Hypothesis:

Thesis I.1.6: Homotopy Hypothesis for n-groupoids. The fundamental n-groupoid functor, $\pi_{\leq n}$, determines a bijection between weak homotopy classes of n-types and equivalence classes of n-groupoids:

$$\left\{\begin{array}{c}\text{homotopy } n\text{-types}\\ \text{up to weak homotopy equivalence}\end{array}\right\} \xrightarrow{\pi_{\leq n}} \left\{\begin{array}{c}n\text{-groupoids}\\ \text{up to equivalence}\end{array}\right\}.$$

The Homotopy Hypothesis has been proved for low dimensions, such as $n = 1$ and $n = 2$. For higher dimensions, as we discussed in Remark 3.1.2, it should be regarded as a guiding principle for what we want an n-groupoid to be. This hypothesis also reinforces our insistence in weakening our definitions — we know there are fundamental n-groupoids which cannot be strictified, such as $\pi_{\leq 3}(\mathbb{S}^2)$ [Sim98].

As we go to larger and larger n, the truncation to n-types preserves more and more of the homotopy information of our space. Taking the limit as n goes to ∞, we should hopefully get a suitable notion for ∞-groupoids.

Thesis I.1.7: Homotopy Hypothesis for ∞-groupoids. The fundamental ∞-groupoid functor, $\pi_{\leq\infty}$, determines a bijection between weak homotopy classes of topological spaces and equivalence classes of ∞-groupoids:

$$\left\{\begin{array}{c}\text{topological spaces}\\ \text{up to weak homotopy equivalence}\end{array}\right\} \xrightarrow{\pi_{\leq \infty}} \left\{\begin{array}{c}\infty\text{-groupoids}\\ \text{up to equivalence}\end{array}\right\}.$$

This is one of the fundamental principles of higher category theory; it is the basis upon which all the theory of ∞-groupoids is built. As such, it is often taken as a *definition*.

Definition I.1.8. An ***∞-groupoid*** is a topological space.

So it turns out that ∞-groupoids are thoroughly studied in pretty much every Mathematics undergrad program out there! We just didn't know them by this name until now.

There are similar alternatives; for example, we can take simplicial sets instead, since every topological space is weak homotopy equivalent to the geometric realization of a simplicial set (see Remark C.1.10).

Grothendieck's Dream

The Homotopy Hypothesis is sometimes [BS10] called *Grothendieck's Dream*. This name is a reference to *Pursuing Stacks*, the $(12+593)$-paged letter Grothendieck sent to Quillen in 1983.[2]

[1] In fact, the classifying space $\mathrm{B}\mathcal{G}$ is defined as $|\mathrm{N}(\mathcal{G})_\bullet|$, the geometric realization [Rem. C.1.10] of the nerve [Def. C.1.4] of the category $\mathcal{G}$. See also Remark 4.1.8 discussing the classifying space of a *group*.

[2] A modern typesetted version of *Pursuing Stacks* can be found in the arXiv: `https://arxiv.org/abs/2111.01000`. The typesetting is credited to GitHub user `thescrivener`, Mateo Carmona and Ulrik Buchholtz.

We quote (pp. 1–2):

> Last year Ronnie Brown from Bangor sent me a heap of reprints and preprints by him and a group of friends, on various foundational matters of homotopical algebra.
>
> [...]
>
> At first sight it has seemed to me that the Bangor group had indeed come to work out (quite independently) one basic intuition of the program I had envisioned in those letters to Larry Breen — namely that the study of n-truncated homotopy types (of semisimplicial sets, or of topological spaces) was essentially equivalent to the study of so-called n-groupoids (where n is any natural integer). This is expected to be achieved by associating to any space (say) X its "fundamental n-groupoid" $\Pi_n(X)$, generalizing the familiar Poincaré fundamental groupoid for $n = 1$.
>
> [...]
>
> But all this kind of thing for the time being is pure heuristics — I never so far sat down to try to make explicit at least a definition of n-categories and n-groupoids, of n-functors between these etc. When I got the Bangor reprints I at once had the feeling that this kind of work had been done and the homotopy category expressed in terms of ∞-groupoids. But finally it appears this is not so, they have been working throughout with a notion of ∞-groupoid too restrictive for the purposes I had in mind (probably because they insist I guess on strict associativity of compositions, rather than associativity up to a (given) isomorphism, or rather, homotopy) — to the effect that the simply connected homotopy types they obtain are merely products of Eilenberg–Mac Lane spaces, too bad! They do not seem to have realized yet that this makes their set-up wholly inadequate to a sweeping foundational set-up for homotopy.

Should this short fragment catch their interest, the reader is encouraged to read the entire 12-paged portion of *Pursuing Stacks* — this constitutes the actual letter to Quillen, with the other 593 pages being an "appendix" of research notes. This is a historical source of tremendous value, providing plenty of motivation for higher categories while also exploring many of the pitfalls one can inadvertently fall into.

Weak (∞, n)-categories

As mentioned in Idea 3.2.4, an (∞, n)-category is an ∞-category where every k-morphism is invertible for $k > n$. Since a topological space is an $(\infty, 0)$-category (or groupoid), we could do an inductive process similar to Definition 3.1.6: we could define an $(\infty, 1)$-category as a topological category (a category enriched over Top), and an (∞, n)-category as a category enriched over the $(\infty, n-1)$-category of all (small) $(\infty, n-1)$-categories. Then we could weaken this definition, in a manner similar to bicategories [Def. 3.1.8]. Of course, we encounter the same issues we did there. They are a bit more salvageable in this case, due to the properties of topological spaces, but even then they prove to be technically inconvenient.

So we will spend some time in the Coda constructing a good model for $(\infty, 1)$-categories (*complete Segal spaces*), that we will then be able to iterate in order to construct a good model for (∞, n)-categories (*n-fold complete Segal spaces*). All tools necessary for doing so are contained in chapter 3, in the Coda itself, or in this very Interlude. Therefore, readers interested in this technical construction can — if they so desire — read it before proceeding to chapter 4.

For now, let us tackle the other conceptual gap we have yet to fill: assuming we have a good definition of (∞, n)-categories, we still need to construct a good definition of *symmetric monoidal* (∞, n)-categories. Perhaps surprisingly, the techniques necessary for solving both of our problems will come from higher homotopy theory. But in light of Grothendieck's Dream, this should not be that much of a surprise.

And now, to see the next part of the dream.

I.2 The other Baez–Dolan hypotheses

In the 1995 paper by Baez and Dolan where the Cobordism Hypothesis was originally formulated, [BD95], the authors conjecture a total of three hypotheses of higher-categorical nature.

We discuss them in their original formulation, as questions about n-categories. But as the reader is probably aware by now, most questions about n-categories can be reformulated as questions about (∞, n)-categories.

The first one, the **Stabilization Hypothesis**, asserts the stabilization of the columns of the so-called *Periodic Table of n-categories*. We have already discussed the second, the **Cobordism Hypothesis**, in-depth: this provides an n-categorical description of cobordisms. The last of the three conjectures, the **Tangle Hypothesis**, does the same for tangles and their higher-dimensional counterparts.[1]

In order to talk about these, we first need to discuss *The Periodic Table of k-tuply monoidal n-categories*. What is that table, and what kind of structures are we talking about?

The Periodic Table of k-tuply monoidal n-categories

A ***0-tuply monoidal n-category*** is just a pointed n-category — an n-category $\mathcal{C}$ equipped with a base object $X \in \mathrm{ob}(\mathcal{C})$.[2] A ***k-tuply monoidal n-category*** is an n-category $\mathcal{C}$ equipped with a distinguished *unit* object $\mathrm{I} \in \mathrm{ob}(\mathcal{C})$ and k different ways (or "directions") of composing objects (and also morphisms between these objects), which must be coherent in some way.

Like most concepts in higher category theory, there is no one generally accepted definition for all k and n — coming up with a good definition is a whole field of study. Similarly to the Homotopy Hypothesis [Ths. I.1.7], an empirically observed pattern (the Delooping Hypothesis [Ths. I.2.9], which we will soon see) is often used to *define* k-tuply monoidal n-categories.

But we do know what a k-tuply monoidal n-category is for small values of n ($n \leq 2$) or small values of k ($k \leq 3$), since after all these are the "initial observations" we are trying to generalize. We assemble them into The Periodic Table of k-tuply monoidal n-categories, Table I.1.

k \ n	0	1	2	$\cdots$
0	pointed set	pointed category	pointed bicategory	$\cdots$
1	monoid	monoidal category	monoidal bicategory	$\cdots$
2	commutative monoid	braided monoidal category	braided monoidal bicategory	$\cdots$
3	commutative monoid	symmetric monoidal category	sylleptic monoidal bicategory	$\cdots$
4	commutative monoid	symmetric monoidal category	symmetric monoidal bicategory	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table I.1: The Periodic Table of k-tuply monoidal n-categories.

The Periodic Table of n-categories is named this way not because it is actually periodic, but because it lets us predict new structures by analyzing its patterns — just like its chemical counterpart. Let's see some of the ways of moving around the table, and why these constructions

[1]Here we paraphrase the abstract of this talk by Baez: `https://math.ucr.edu/home/baez/cat/`. This also mentions Grothendieck's **Homotopy Hypothesis**, which we already discussed in section I.1. As mentioned, nowadays the formal definitions of n-categories are chosen such that this hypothesis is true; it is a general principal of higher category theory, which guides intuition.

[2]The original formulation [BD95, § V] did not endow categories with a choice of basepoint. This is relevant for technical reasons; in sort, we will define operations of *looping* and *delooping*, which — like their eponymous analogues in homotopy theory — require a basepoint in order to satisfy nice properties. See [BS10, § 5.6] for more details.

have merit. We already saw our first example back on Remark 1.3.2: we identified a category with a single object as a *monoid*. The actual process is called *looping*, and its proper source is pointed categories. We gave a definition in Definition 3.2.7, but there we artificially limited ourselves to categories with a single object — which are intrinsically pointed.

Looping of categories

Definition I.2.1. The ***looping*** of a pointed category $(\mathcal{C}, X)$ is the monoid

$$\Omega(\mathcal{C}, X) \coloneqq (\mathrm{Hom}_{\mathcal{C}}(X, X), \circ).$$

That is, it is the set $\mathrm{End}_{\mathcal{C}}(X)$ of endomorphisms of the basepoint equipped with composition of morphisms $\circ\colon \mathrm{End}_{\mathcal{C}}(X) \times \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{C}}(X)$.

This process is named after the analogous operation in algebraic topology, which works on pointed topological spaces (X, x) by taking the *loop space* defined as the set

$$\Omega(X, x) = \{\gamma\colon I \to X \mid \gamma_0 = \gamma_1 = x\}$$

with the compact-open topology.

Note that concatenation of loops induces a continuous group operation on $\Omega(X, x)$, so this is canonically a topological group: Ω is a functor $\mathsf{Top}_* \to \mathsf{GrTop}$. Equivalently, one can define $\Omega(X, x)$ as the space $\mathrm{Hom}_{\mathsf{Top}_*}((\mathbb{S}^1, \{*\}), (X, x))$ of basepoint-preserving continuous functions from the circle $\mathbb{S}^1$ to X. Of course, the path-connected components of this space form the fundamental group $\pi_1(X)$.

We can define an analogous looping operation for any k-tuply monoidal n-category with $n \geq 1$. In fact, we did the case $k = 0$ in Remark 3.2.8. Note that a monoidal category has a distinguished object which we can take as basepoint: namely, the monoidal unit I.

Definition I.2.2. The ***looping*** of a k-tuply monoidal n-category $(\mathcal{C}, X)$ is the $(k+1)$-tuply monoidal $(n-1)$-category

$$\Omega(\mathcal{C}, X) \coloneqq (\mathrm{Map}_{\mathcal{C}}(X, X), \circ).$$

So this moves us diagonally on the table, in the southwest direction.

Note that this is a functor. More precisely, this is a family of functors $\Omega_n^{\otimes k}\colon \mathsf{Cat}_n^{\otimes k} \to \mathsf{Cat}_{n-1}^{\otimes(k+1)}$ from the category of k-tuply monoidal n-categories to the category of $(k+1)$-tuply monoidal $(n-1)$-categories.[3]

Let's see some examples, since there is a number of implicit identifications in play. The process of making this identifications is called the *Eckmann–Hilton argument*, and it is very geometric in nature.

The Eckmann–Hilton argument

Let's start by actually defining k-tuply monoidal n-categories — but only for the very particular case of $k = 2$ and $n = 0$!

Definition I.2.3. A ***2-tuply monoidal* 0*-category*** is a set X equipped with a distinguished unit $1 \in X$ and with two associative and unital binary operations $\star, \diamond\colon X \times X \to X$ which satisfy the *interchange law*:

$$(a \diamond p) \star (q \diamond b) = (a \star q) \diamond (p \star b).$$

This interchange law is just mutual distributivity, of course.

As it turns out, we have just redefined the concept of a commutative monoid! That is a consequence of the slightly more general Eckmann–Hilton Theorem.

[3]This notation is completely ad hoc; do not confuse it with iterated tensor product.

Theorem I.2.4: The Eckmann–Hilton Theorem. Let X be a set equipped with two unital binary operations $\star$ and $\diamond$ which distribute over each other:

$$(a \diamond p) \star (q \diamond b) = (a \star q) \diamond (p \star b),$$
$$1_\star \star a = a = a \star 1_\star, \qquad 1_\diamond \diamond a = a = a \diamond 1_\diamond.$$

Then, $\star = \diamond$ and $1_\star = 1_\diamond$. Furthermore, this operation is associative and commutative.

Proof. The two units $1_\star$ and $1_\diamond$ coincide, since

$$1_\star = 1_\star \star 1_\star = (1_\diamond \diamond 1_\star) \star (1_\star \diamond 1_\diamond) = (1_\diamond \star 1_\star) \diamond (1_\star \star 1_\diamond) = 1_\diamond \diamond 1_\diamond = 1_\diamond.$$

Now, let $a, b \in X$, and write

$$a \star b = (a \diamond 1) \star (1 \diamond b) = (a \star 1) \diamond (1 \star b) = a \diamond b = (1 \star a) \diamond (b \star 1) = (a \diamond 1) \star (1 \diamond b) = b \star a.$$

So the two operations coincide, and they are commutative.

Finally, for associativity, we write

$$(a \star b) \star c = (a \star b) \star (1 \star c) = (a \star 1) \star (b \star c) = a \star (b \star c). \qquad \square$$

The step we are interested in is the middle one, establishing commutativity and equality. We can interpret it geometrically, as manipulating 0-dimensional points in a 2-dimensional space. We write one of the operations vertically, and the other horizontally.

$$\boxed{\begin{array}{c|c} a & b \end{array}} = \boxed{\begin{array}{c|c} a & 1 \\ \hline 1 & b \end{array}} = \boxed{\begin{array}{c} a \\ \hline b \end{array}} = \boxed{\begin{array}{c|c} 1 & a \\ \hline b & 1 \end{array}} = \boxed{\begin{array}{c|c} b & a \end{array}}$$

$$\begin{tikzcd}
a \star b \arrow[r, equal] & (a \diamond 1) \star (1 \diamond b) \arrow[d, equal] & & (a \diamond 1) \star (1 \diamond b) \arrow[r, equal] & b \star a. \\
& (a \star 1) \diamond (1 \star b) \arrow[r, equal] & a \diamond b \arrow[r, equal] & (1 \star a) \diamond (b \star 1) \arrow[u, equal] &
\end{tikzcd}$$

As some readers will recognize, this is analogous to the argument commonly used for proving the commutativity of the higher homotopy groups of a topological space![4]

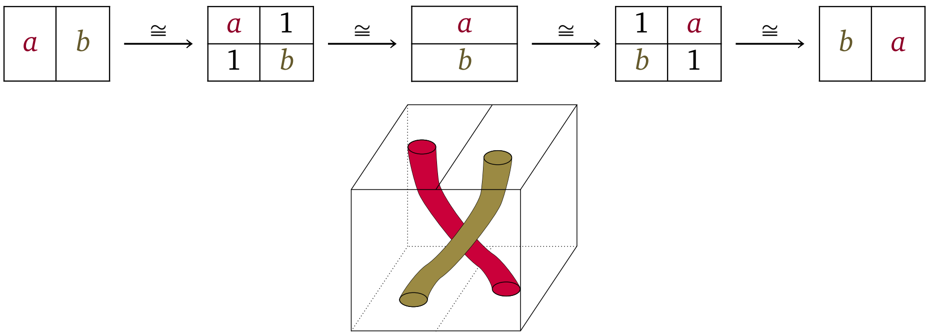
We started with a set ($k = 0$ and $n = 0$), then got the *structure* of a monoid ($k = 1$) and then got the *property* of being commutative ($k = 2$)! What will happen when we go to $k = 3$, and consider 3-tuply monoidal 0-categories? Well, as it turns out, nothing more; we get commutative monoids once again, and the third operation is completely redundant. The column stabilizes here.

The geometric interpretation is that we have already "maxed out" our codimensional degrees of freedom — moving 0-dimensional points in a 3-dimensional space is exactly as restrictive as manipulating them inside a 2-dimensional space, for the purpose of interchanging their position. This will be clearer by considering our next example, 1-tuply monoidal 2-categories! Here we think of moving 1-dimensional *lines* (1-morphisms) inside a 3-dimensional space — we can interchange them, but they form a *braid*.

Where as commutativity is a strict notion, $a \otimes b = b \otimes a$, a braiding is weaker — we identify $a \otimes b \simeq b \otimes a$, but we keep track of *how* that identification is made. In particular, we can cross a *over* or *under* b, and the two ways are *not* equivalent. We replace strict equalities with canonical

[4]In fact, the Periodic Table explains why $\pi_0(X)$ is a *set*, $\pi_1(X, x)$ is a *group* and $\pi_k(X, x)$ are *abelian groups* for $k \geq 2$ — each of these is a "k-tuply monoidal 0-*groupoid*".

isomorphisms:



$$\sigma_{a,b} \colon a \otimes b \xrightarrow{\cong} b \otimes a.$$

We identify 2-tuply monoidal 1-categories with *braided monoidal categories* [Rem. 1.2.17]. Remember that the twist maps of a symmetric monoidal category behave just like the elements of the symmetric groups, as we explored in Example 1.2.15. The braiding maps of a braided monoidal category behave like the elements of the Artin braid groups! We gave a presentation of both families of groups in Example 1.3.3. In this jump, we have added *structure* to a monoidal category.

Let's keep increasing the codimension k. We now consider 3-tuply monoidal 1-categories, whose morphisms we think of as 1D lines in 4D space. As the reader probably is aware of, knots and braids do not exist in four dimensional space — there is always an isotopy to the trivial knot or braid. So here the Eckmann–Hilton argument works, once again, and we get back *symmetric* monoidal categories! Here there is no crossing over or crossing under — just as there wasn't when we considered the category of cobordisms, all the way back in section 1.1.2. Note that a symmetric monoidal category is a braided monoidal category satisfying the *property* of all braidings being idempotent; this is axiom (iii) of Definition 1.2.14!

The Periodic Table explains why both braided and symmetric monoidal categories appear naturally throughout various fields of mathematics — the latter can be regarded as a slightly higher-dimensional, or more commutative, version of the former.

What happens if we repeat the process, one dimension up, and consider 4-tuply monoidal 1-categories? Do we get "super-symmetric" monoidal categories, in some way? Well, as it turns out, we just get symmetric monoidal categories, again: the added structure obtained by the dimension jump is completely redundant!

In general, we think of the k different compositions in a k-tuply monoidal n-category as letting us manipulate n-dimensional objects in an $(n+k)$-dimensional space. The maximum degrees of freedom are obtained for codimension $k \geq n+2$ — or at least that is what the low-dimensional examples lead us to believe. This conjecture is known as the *Stabilization Hypothesis*, which relates to the *Freudenthal Suspension Theorem* in homotopy theory. Let us borrow for a bit the tools needed for the statement of this theorem.

Looping and suspension in homotopy theory

In topology, there is a sort-of inverse operation to taking loop spaces — that of taking *classifying spaces* of a topological group G. We will later on sketch one construction [Rem. 4.1.8].

$$\mathrm{B}G = \mathrm{E}G/G.$$

The functor B is left-adjoint to Ω, and under certain niceness conditions is an equivalence of categories.

However, looping is also related to *suspension* in homotopy theory: if we forget that $\Omega(X,x)$ is a group and regard it as just a (pointed) topological space, its left-adjoint is the (reduced) suspension functor.

$$\Sigma(X,x) = \left(\frac{X \times I}{[X \times \{0\}, X \times \{1\}, \{x\} \times I]}, \{x\} \times I \right).$$

Alternatively, this is the pointed smash product with the circle: $(X,x) \vee (\mathbb{S}^1, \{*\})$. The classical example motivating this operation is its relation with spheres: $\Sigma\,\mathbb{S}^n \cong \mathbb{S}^{n+1}$.

Note the effect that these operations have on the homotopy groups of our space: one can prove that $\pi_{k-1}(\Omega X) \cong \pi_k(X)$. By iteration, we get $\pi_{k-i}(\Omega^i X) \cong \pi_k(X)$, where $\Omega^i := \Omega \overbrace{\circ \cdots \circ}^{i} \Omega$. But what about suspension? Is it always true that $\pi_k(X) \cong \pi_{k+i}(\Sigma^i X)$? Well, it turns out that we need some conditions on our space X.

Recall that a space X is ***n-connected*** if its homotopy groups $\pi_k(X)$ are trivial for $0 \le k \le n$.

Theorem I.2.5: Freudenthal Suspension Theorem. If X is an n-connected CW-complex, then the group homomorphism $\pi_k(X) \to \pi_k(\Omega \Sigma X)$ induced by suspension and looping is an isomorphism for $k \le 2n$.

Corollary I.2.6. The homotopy groups of the spheres $\pi_{n+k}(\mathbb{S}^n)$ stabilize for $n \ge k+2$.

This explains the diagonal patterns one finds when tabulating the homotopy groups of spheres: each diagonal is constant for big enough values of n.

Since for any topological space we have inclusions $\pi_k(X) \hookrightarrow \pi_{k+1}(\Sigma X)$, we can define its ***stable homotopy groups*** as

$$\pi_k^{\mathrm{S}}(X) := \varinjlim_{k} \pi_{k+i}(\Sigma^i X).$$

The Freudenthal Suspension Theorem says that, for sufficiently nice topological spaces such as CW-complexes, the sequence of inclusions $\pi_{k+i}(\Sigma^i X) \hookrightarrow \pi_{k+i+1}(\Sigma^{i+1} X)$ eventually stabilizes. So these limits are particularly nice to compute.

The study of the looping and suspension operations (and their generalization to spectra), when thus iterated, form the backbone of stable homotopy theory. In this realm, the adjunction $\Sigma \dashv \Omega$ becomes a categorical equivalence.

In any case, given the patterns observed in Table I.1 — as well as the relationship between higher categories and topological spaces — it is not too far-fetched to conjecture the following.

Thesis I.2.7: Stabilization Hypothesis. The nth column stabilizes at the $(n+2)$nd row.

Like the other hypotheses mentioned in this text, this should not be taken as a theorem to prove. Rather, this is best employed as a "litmus test" on whether a given approach to higher categories is "good", in the sense that it makes true the things we intuitively want to be true. In that sense, we can think of the Homotopy, Stabilization and Delooping Hypotheses as *guiding principles* on which to build a coherent theory of higher categories.

We will name the categorical version of suspension (or more accurately, of taking classifying spaces) *delooping*, to simplify nomenclature.

Delooping of categories

We denote delooping of categories by $\mathcal{B}$, to distinguish it from the classifying space construction BG. No such considerations were taken to distinguish the looping Ω of categories and topological spaces, since from now on we will only consider loops of categories.

Definition I.2.8. Let $\mathcal{C}$ be a k-tuply monoidal n-category. Its ***delooping*** $\mathcal{B}\mathcal{C}$ is the $(k-1)$-tuply monoidal $(n+1)$-category with a single object and with m-morphisms corresponding to the $(m-1)$-morphisms of $\mathcal{C}$.

That is, we add a new layer at the bottom, and reindex everything by one. This is what we did when we saw a monoid as a category with a single object: we applied the delooping functor from an 1-tuply monoidal 0-category to a 0-tuply monoidal 1-category. For brevity, we will call n-categories with a single object ***degenerate***. Similarly, we will say an n-category is ***i-fold degenerate*** if it has a single k-morphism for $0 \le k \le i$. So applying the delooping functor i times yields us an i-fold degenerate $(k-i)$-tuply monoidal $(n+i)$-category. This moves us diagonally on the table, in the direction opposite to looping (i.e., northeast).

More recently, in Definition 3.2.7, we *defined* a 1-tuply monoidal n-category as a degenerate 0-tuply monoidal $(n+1)$-category. And in Remark 3.2.8, we discussed that this same process could be used to define a symmetric monoidal n-category as an $(n+2)$-fold degenerate 0-tuply monoidal $(2n+2)$-category!

Here we have two different hypotheses in play. First, the Stabilization Hypothesis [Ths. I.2.7] tells us that the "maximum degree of symmetry" is obtained when $k = n+2$, so $(n+2)$-tuply monoidal n-categories should be the good notion for symmetric monoidal n-categories. But then, we are implicitly asking for looping and delooping to be inverse operations, when restricting to degenerate $(k-1)$-tuply monoidal $(n+1)$-categories. This is known as the *Delooping Hypothesis*.

Thesis I.2.9: Delooping Hypothesis. There is an adjoint pair $\mathcal{B}^i \dashv \Omega^i$ between

$$\{k\text{-tuply monoidal } n\text{-categories}\} \underset{\Omega^i}{\overset{\mathcal{B}^i}{\rightleftarrows}} \{(k-i)\text{-tuply monoidal } (n+i)\text{-categories}\}.$$

This adjunction restricts to an equivalence when restricting the right-hand side to i-fold degenerate $(k-i)$-tuply monoidal $(n+i)$-categories.

This is the reason we wanted 0-tuply monoidal n-categories to be equipped with a basepoint; this could not be an adjunction otherwise.

The first mention (by name) of this hypothesis is in [BS10, Hyp. 22], much later than the other ones. It was used implicitly before then, just as we did when assembling our table.

Now we will discuss another pair of operations for moving around the table: *forgetting structure* (the easy one), and *freely adjoining structure* (the hard one).

Moving vertically: forgetting structure and stabilization

Forgetting things is easy: we see a $(k+1)$-tuply monoidal n-category as a k-tuply monoidal n-category. So this moves us north in the table. Note that in some cases this forgets *structure* (such as moving from a monoid to a set, or from a monoidal category to a pointed category), where as in other cases this forgets *properties* (moving from a commutative monoid to a monoid, or from a symmetric monoidal category to a braided monoidal category). We will denote this forgetful functor by U.

The hard part is defining a functor which is left adjoint to U, so that we can freely complete a k-tuply monoidal n-category to a $(k+1)$-tuply monoidal n-category. We will not construct it here, but this functor is called *stabilization* and denoted by S.[5] Of course, it moves us in the opposite direction to forgetting structure: it moves us *south*.

The final two operations we will consider move us *horizontally* across the table. We have already seen them before, although not by these names.

Moving horizontally: discrete categorification and decategorification

We can move east in the table through *discrete categorification*, denoted D. This is just a formalization of the process of viewing an n-category as an $(n+1)$-category where the only $(n+1)$-morphisms are the identities, like we have been doing for a couple of sections.

[5] This was named "suspension" in [BD95], but is called "stabilization" in [BD98].

Its right adjoint is *decategorification*, K, which obtains an $(n-1)$-category from an n-category by taking isomorphism classes of the $(n-1)$-morphisms. That is, this is the homotopy $(n-1)$-category $\mathrm{h}_{n-1}\mathcal{C}$ of Definition 3.2.9.

$k \backslash n$	0	1	2	⋯
0	pointed set	pointed category $\mathrm{S}\downarrow\uparrow\mathrm{U}$	$\xrightarrow{\mathcal{B}}$ pointed bicategory	⋯
1	monoid $\underset{\mathrm{K}}{\overset{\mathrm{D}}{\rightleftarrows}}$	monoidal category $\xleftarrow{\Omega}$	monoidal bicategory	⋯
2	commutative monoid	braided monoidal category	braided monoidal bicategory	⋯
3	commutative monoid	symmetric monoidal category	sylleptic monoidal bicategory	⋯
4	commutative monoid	symmetric monoidal category	symmetric monoidal bicategory	⋯
⋮	⋮	⋮	⋮	⋱

Table I.2: Operations on the Periodic Table.

With this, we end our short discussion of the Periodic Table. Our six different ways of moving around the table are summarized in Table I.2. There are other ways of moving around the table, of course, but with just these six we already have a number of different ways of obtaining a k-tuply monoidal n-category from an l-tuply monoidal m-category. Some will carry more loss of information than others, and will be relevant in different contexts.

Now, only one of the Baez–Dolan hypotheses remain: the *Tangle Hypothesis*.

The Tangle Hypothesis

Finally, we get to the last of the Baez–Dolan hypotheses. This is a generalization of the Cobordism Hypothesis, dealing with another kind of topological object: framed n-tangles embedded in $n+k$ dimensions. Roughly, these are n-manifolds with corners embedded in a cube I^{n+k} in such a way that the corners of codimension j are mapped into the corners of the cube where the last j coordinates are all zeroes or all ones. In particular, for $n=1$ and $k=2$, we recover the usual tangles from knot theory.

We refer the reader to [BD95, § 7; Lur09, § 4.4] for further details and pretty pictures, and limit ourselves to the statement of the hypothesis.

Thesis I.2.10: Tangle Hypothesis. The n-category of framed n-tangles in $n+k$ dimensions is $(n+k)$-equivalent to the free k-tuply monoidal n-category with duals on one object.

For big enough codimension k, and if our definitions are correct, we should recover the Cobordism Hypothesis — at least in some shape or form. One way of formalizing this is that we can obtain the category of framed n-cobordisms as a direct limit of the categories of (n,k)-tangles:

$$\mathsf{Bord}_n^{\mathrm{fr}} \simeq \varinjlim_k \mathsf{Tang}_{n,k}^{\mathrm{fr}}.$$

There has been tremendous progress on this hypothesis, to the point where some authors consider it a proven theorem, but a full proof has yet to appear in a peer-reviewed publication.[6] Of note, Lurie's approach in [Lur09, § 4.4] claims that the Tangle Hypothesis can be derived from a version of the Cobordism Hypothesis admitting singularities [Lur09, Thm. 4.3.11], so it suffices to fill in the gaps in the latter's proof.

This ends our short Interlude. We have deliberately kept it short, since the topics touched on here are far-reaching and potentially very deep. Some sources for further reading are listed on the following page.

[6] Remember that Lurie claims that a number of proofs in [Lur09] are informal proof *sketches*.

Further reading

Before moving on to chapter 4, I would like to take the end of this Interlude as a chance to break out of the *pluralis modestiae* for a short while.

If you enjoyed the sort of mathematics explored here, I deeply recommend reading some of the papers written by Baez and/or Dolan, such as those listed in the Bibliography. Also note that a lot of really interesting but not formally published mathematics can be found on Baez's home page `https://math.ucr.edu/home/baez/`, including his 1993–2010 column *This Week's Finds in Mathematical Physics* available at `https://math.ucr.edu/home/baez/TWF.html` [TWF].[7] This contains a "Tale of n-Categories", contained in Weeks 73–80, 83–84, 89, 92 and 99–100. Other relevant numbers include Week 49, discussing the periodic table; and Week 121, discussing categorification. Weeks 117–118 explore the nerve of a category (which we will discuss in the Coda), while Weeks 173–174 include a very pretty visual representation of internal monoids and adjunctions. Of note, Week 275 (from June 14, 2009) talks about progress on the Cobordism Hypothesis — in particular, about (prior iterations of) our two main sources for part II, [Lur09; Sch14]! This list is by no means comprehensive; this really is a treasure trove of interesting mathematical and physical viewpoints. And every post has plenty of references, too!

For topics relating to the Periodic Table, see also lectures 11–13 of Baez's homonymous seminar, available at `https://math.ucr.edu/home/baez/twf/`.

[7] Some people call this "the world's first blog"!

Chapter 4

Comparisons with the classical 2D correspondence

> *Do not be seduced by the lotus-eaters into infatuation with untethered abstraction.* Hold tight to your geometric motivation as you learn the formal structures which have proved to be so effective in studying fundamental questions. When introduced to a new idea, always ask why you should care. Do not expect an answer right away, but demand an answer eventually. Try at least to apply any new abstraction to some concrete example you can understand well. See if you can make a rough picture to capture the essence of the idea.
>
> **Ravi Vakil, "The Rising Sea: Foundations of Algebraic Geometry".**

Now that we have finished stating the Cobordism Hypothesis (in multiple formulations!), a question remains: what is the exact relationship between it and the classical correspondence between 2D oriented TQFTs and commutative Frobenius algebras?

On the one hand, it *is* a generalization: it relates fully extended framed TQFTs to fully dualizable objects in the target category, just as the classical correspondence related 2D oriented TQFTs with Frobenius objects in the target category. And the definitions of "fully dualizable" and of "Frobenius object" share striking resemblances, too: in the graphical versions, Zorro's relation [fig. 1.9] and the Frobenius relation [fig. 2.8] are very similar, as their only difference is whether you put on or remove some caps (which correspond to units and counits).

But on the other hand, it *isn't* a generalization: as we will show in this chapter, you can't completely recover the classical correspondence from the Cobordism Hypothesis.

For disambiguation, we will call "classical" 1-categorical 2D TQFTs **$(1,2)$-*TQFTs***, as they oftentimes are in the literature. In the same manner, we will use **$(0,1,2)$-*TQFTs*** to refer to 2D fully extended TQFTs. This is the nomenclature we introduced in Notation 3.1.15.

So far we have discussed the classification of oriented $(1,2)$-TQFTs and of framed $(0,1,\dots,n)$-TQFTs, but note that classification results also exist for $(1,2,3)$-TQFTs [BDSV15] and even for their unextended relatives, $(2,3)$-TQFTs [Juh18, Thm. 1.10]. As the discussion of this chapter shows, non-extended TQFTs are harder to classify — they are much more numerous than "fully extended" TQFTs.

We highlight three main obstructions for trying to recover Theorems 2.3.3 and 2.3.17 from Theses 3.2.1 and 3.2.2.

- Firstly, there's the problem of the higher categorical structure. Fully extended TQFTs, as we previously discussed, impose very constraining conditions: even for a dimension as low as 2, most $(1,2)$-TQFTs can't be extended down to form a $(0,1,2)$-TQFT. This will, in fact, be the most important problem, as well as the only one we won't be able to overcome.

- The Cobordism Hypothesis (in the formulations we have stated) deals with framed cobordisms, while the classical correspondence only orients them. Disturbingly, most surfaces aren't parallelizable; so while every oriented TQFT can be restricted to a framed TQFT, the converse isn't true: there exists framed TQFTs which don't have an oriented analogue. Every orientable surface is 3-stably parallelizable, but even that will not be enough. We will sidestep this problem by using an *oriented* version of the Cobordism Hypothesis, as stated in [Lur09, Thm. 2.4.26]. This uses the language of *G-manifolds*.

- Thirdly, if we want to explicitly recover the correspondence with commutative Frobenius algebras, we need to figure out a good bicategory $\mathcal{C}$ in which to take values. This is a bit disorientating, but thankfully someone already figured it out before us: [Sch14, Thm. 3.52] proves a very explicit version of the 2D oriented Cobordism Hypothesis, stated for TQFTs taking values in a certain bicategory called Alg_2. So this won't be a problem.

This chapter is divided in two sections. In section 4.1 we will explain Lurie's general result [Lur09, Thm. 2.4.18], which generalizes the Cobordism Hypothesis to manifolds equipped with an (X,ζ)-*structure*. We are mainly interested in G-structures, where G is a topological group; the framed, oriented and unoriented cases follow from particular cases of this result, as do others such as manifolds endowed with a spin structure. This requires a number of prerequisites from the general theory of tangential structures of manifolds, for which we loosely follow a series of lecture notes by Freed [Fre13a, § 9].

In section 4.2 we specialize to the more explicit statement that appears in Schommer-Pries's 2009 PhD thesis [Sch14, Thm. 3.52]. Here we will need to introduce a number of algebraic constructions; of particular interest are *bimodules* between algebras, as well as *Morita contexts*. Then we define the *Morita bicategory* Alg_2 to be the bicategory consisting of algebras, bimodules, and intertwiners — which are just morphisms of bimodules. Assuming a presentation for the bicategory $\mathsf{Cob}_2^{\mathrm{or}}(2)$ in terms of generators and relations, we follow [Sch14, § 3.8.5] to prove that oriented $(0,1,2)$-TQFTs with values in Alg_2 are in correspondence with separable symmetric Frobenius algebras. Taking loops let us conclude that the oriented $(1,2)$-TQFTs which can be extended down to points correspond to *separable* commutative Frobenius algebras — hence recovering some of the classical correspondence of Theorem 2.3.3.

4.1 The Cobordism Hypothesis for G-manifolds

In short, while framed fully extended TQFTs correspond to fully dualizable objects in the target category, one needs more data to classify oriented or unoriented theories. These carry the additional information of either an $\mathrm{SO}(n)$- or an $\mathrm{O}(n)$-action on the fully dualizable objects of the target category, and are classified by the *homotopy fixed points* of such.

4.1.1 Principal bundles and G-manifolds

Frame bundles

Let M be an n-manifold and let $V \to M$ be a rank k vector bundle over M. Usually, we will take V to be $\mathrm{T}M$, the tangent bundle. Now, as we mentioned in Definition 3.1.13, a trivialization $V \cong \underline{\mathbb{R}^k}$ of the bundle is equivalent to selecting a basis for each fiber V_x in a smooth way. This is a bit clunky to write out, and it would be nice if we could define a structure to hold the information of all possible bases of each fiber V_x. Of course, these sets of bases do not form a vector space, so this will not be a vector bundle. The structure we obtained, called the *frame bundle* $\mathrm{Fr}\,V \to M$ of V, is an example of a $\mathrm{GL}_{\mathbb{R}}(k)$-*principal bundle* (or $\mathrm{GL}_{\mathbb{R}}(k)$-bundle, for short).

Definition 4.1.1. Let G be a Lie group and let M be a manifold. A ***G-principal bundle*** over M is a fiber bundle $\pi\colon P \to M$ equipped with a right G-action $P \times G \to P$ which is simply transitive on each fiber.

This means that the action restricts to each fiber, $P_x \times G \to P_x$, and that for each $p, q \in P_x$ there is a unique $g \in G$ such that $pg = q$. Of course, every map in the previous definition must be smooth.

Remark 4.1.2 — A more classical definition of G-principal bundles.
Alternatively, a G-principal bundle over M is a topological space P equipped with a projection $\pi\colon P \to M$, an open cover $\{U_\alpha\}$, local trivializations $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$ such that the diagram

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha", "\cong"'] \arrow[dr, "\pi"'] & & U_\alpha \times G \arrow[dl, "p_1"] \\
& U_\alpha &
\end{tikzcd}$$

commutes, and a free continuous G-action $P \times G \to P$ such that all of the relevant maps π and φ_α are G-invariant. This definition is actually slightly better for our purposes — we will want to consider bundles over general topological spaces, instead of restricting our view to manifolds.

Intuitively, we identify the "changes of charts" $\varphi_\beta \circ \varphi_\alpha^{-1}\colon (U_\alpha \cap U_\beta) \times G \to (U_\alpha \cap U_\beta) \times G$ with functions to G: we define the ***transition functions*** $\varphi_{\alpha\to\beta}\colon U_\alpha \cap U_\beta \to G$ via the equation

$$(\varphi_\beta \circ \varphi_\alpha^{-1})(x,g) = (x, \varphi_{\alpha\to\beta}(x)\cdot g) \in (U_\alpha \cap U_\beta) \times G, \quad \forall (x,g) \in (U_\alpha \cap U_\beta) \times G.$$

In other words, if $p_2\colon (U_\alpha \cap U_\beta) \times G \to G$ is the projection,

$$\varphi_{\alpha\to\beta}(x) = (p_2 \circ \varphi_\beta \circ \varphi_\alpha^{-1})(x,g)\cdot g^{-1}, \quad \forall g \in G.$$

We can recover a principal bundle from its transition functions, by a gluing procedure.

Now, we can easily define the ***frame bundle*** $\mathrm{Fr}\,V$. The main information we want to keep track of is that of the changes of bases; given two bases ξ, ξ' of V_x, there is a unique matrix $G \in \mathrm{GL}_{\mathbb{R}}(k)$ such that $\xi G = \xi'$. So we can take the disjoint union $\mathrm{Fr}\,V := \sqcup_{x\in M}(\mathrm{Fr}\,V)_x$ and define a right $\mathrm{GL}_{\mathbb{R}}(k)$-action $\mathrm{Fr}\,V \times \mathrm{GL}_{\mathbb{R}}(k) \to \mathrm{Fr}\,V$, which restricts to each fiber to a simply transitive action. This is a $\mathrm{GL}_{\mathbb{R}}(k)$-principal bundle over M. (We omit the details of how the maps in play induce a smooth structure on $\mathrm{Fr}\,V$; this is a routine exercise).

One can also check that we have the converse: a $\mathrm{GL}_{\mathbb{R}}(k)$-bundle $P \to M$ induces a rank k vector bundle $V \to M$ such that $\mathrm{Fr}\,V \cong P$.

In any case, when applying this construction to the tangent bundle $\mathrm{T}M \to M$, we obtain the frame bundle of M, denoted as $\mathrm{Fr}\,M$. And we can now restate the characterization for framings: a framing on M is a smooth choice of basis for each tangent space $\mathrm{T}_x M$; i.e., it is a smooth section $M \hookrightarrow \mathrm{Fr}\,M$ of the frame bundle $\mathrm{Fr}\,M$.

$\mathbf{O}(n)$-structures on manifolds

Since $\mathrm{GL}_{\mathbb{R}}(n)$ is not a compact manifold, we will want to replace it by its maximal compact subgroup, namely $\mathrm{O}(n)$. In turn, we will replace $\mathrm{Fr}\,M$ by the *orthonormal* frame bundle $\mathrm{Fr}_{\mathrm{O}}\,M$. This process is actually equivalent to choosing a Riemannian metric on M; since every manifold admits a metric, it can always be done. And since the space of Riemannian metrics on M is contractible, this is well-defined up to homotopy: if we choose any two Riemannian metrics on M and perform the (yet undisclosed) process, the two $\mathrm{O}(n)$-bundles obtained will be homotopy equivalent. So, up to homotopy, this doesn't endow our manifolds with any extra data — the newly attached data is discarded when passing to homotopy classes.

This mysterious process is called a *reduction of structure group* from $\mathrm{GL}_{\mathbb{R}}(n)$ to $\mathrm{O}(n)$.

Reductions of structure groups

Let H, G be Lie groups and let $\rho\colon H \to G$ be a homomorphism; usually, this will be an inclusion.

Definition 4.1.3. Let $Q \to M$ be a principal H-bundle. The ***associated G-bundle*** $Q_\rho \to M$ is the quotient

$$Q_\rho = (Q \times G)/H$$

by the free right H-action

$$(Q \times G) \times H \to Q \times G, \qquad (q,g)\cdot h = (q \cdot h, \rho(h)^{-1} g).$$

Now, given a G-bundle $P \to M$, a reduction of structure group from G to H is an H-bundle such that its associated G-bundle is isomorphic to P.

Definition 4.1.4. Let $P \to M$ be a principal G-bundle. A ***reduction of structure group*** to H is a pair (Q,θ) consisting of a principal H-bundle $Q \to M$ and an isomorphism of principal G-bundles

$$\begin{tikzcd}
Q_\rho \arrow[rr, "\theta"] \arrow[dr] & & P \arrow[dl] \\
& M. &
\end{tikzcd}$$

Intuitively, this means that the transition functions $\varphi_{\alpha\to\beta}\colon U_\alpha \cap U_\beta \to G$ of Remark 4.1.2 can be chosen so as to take values in $\rho(H) \subset G$: we write $\varphi_{\alpha\to\beta}\colon U_\alpha \cap U_\beta \to \rho(H)$.

So a Riemannian metric on M is a reduction of structure group from $\mathrm{GL}_{\mathbb{R}}(n)$ to $\mathrm{O}(n)$ of the frame bundle $\mathrm{Fr}\,M$, which yields the orthonormal frame bundle $\mathrm{Fr}_{\mathrm{O}}\,M$. In the same way, we can define an *orientation* on M to be a reduction of structure group from $\mathrm{GL}_{\mathbb{R}}(n)$ to $\mathrm{GL}^+_{\mathbb{R}}(n) = \det^{-1}(\mathbb{R}_+)$ (this is essentially the same as Definition 1.1.5).[1] In general, a reduction of the frame bundle $\mathrm{Fr}\,M$ to a subgroup $G \subset \mathrm{GL}_{\mathbb{R}}(n)$ is called a *G-structure* on M.

As mentioned, for our purposes it is more convenient to first define an (unique up to homotopy) Riemannian metric on M and then consider reductions of the orthonormal frame bundle $\mathrm{Fr}_{\mathrm{O}}\,M$.

[1]This does not work for 0-dimensional manifolds, but in practice we will consider reductions of the *stabilized* frame bundle $\mathrm{Fr}(\mathrm{T}M \oplus \underline{\mathbb{R}}^{n-m})$ instead.

Definition 4.1.5. Let $G \hookrightarrow \mathrm{O}(n)$ be a compact Lie group. A ***G-structure*** on a Riemannian manifold (M, g) is a reduction of structure group of its orthonormal frame bundle $\mathrm{Fr_O}\, M$ from $\mathrm{O}(n)$ to G; that is, a G-bundle $Q \to M$ equipped with an isomorphism $Q \to \mathrm{Fr_O}\, M$.

A ***G-structure*** on a smooth manifold M, well-defined up to a contractible space of choices, is a reduction of structure group of (M, g), where g is any Riemannian metric on M.

Note, however, that we will usually consider reductions of the *stabilized* orthonormal frame bundle $\mathrm{Fr_O}(\mathrm{T}M \oplus \underline{\mathbb{R}}^{n-m})$ instead, just like we did for framings (cf. Definition 3.1.13).

We will end this section by defining G-structures in another equivalent way, via *classifying spaces* of compact topological groups. So we first need to define those.

Classifying spaces for groups

For the following discussion, we will need one particularly useful universal object: the *classifying space* $\mathrm{B}G$ of a compact topological group G.

Definition 4.1.6. Let G be a compact group. The ***universal G-principal bundle*** is the unique (up to G-equivariant homotopy equivalence) G-bundle $\mathrm{E}G \to \mathrm{B}G$ such that $\mathrm{E}G$ is weakly contractible.

Weakly contractible, of course, meaning that it is weak homotopy equivalent to the point: connected and with vanishing homotopy.

We call $\mathrm{E}G$ the *total space* and $\mathrm{B}G$ the *base space*, just like one usually does for arbitrary fiber bundles, but $\mathrm{B}G$ is also called the ***classifying space*** for the group G. The reason for calling such a bundle "universal" is that it satisfies the following universal property.

Property 4.1.7. For any manifold M and any G-principal bundle $P \to M$, there exists a unique (up to G-equivariant homotopy) map $f\colon M \to \mathrm{B}G$ such that P is isomorphic to the pullback $f^*(\mathrm{E}G)$:

$$\begin{tikzcd} P \arrow[r, dotted, "\cong"'] \arrow[d] & f^*(\mathrm{E}G) \arrow[r, dotted] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{E}G \arrow[d] \\ M \arrow[r, equal] & M \arrow[r, dashed, "\exists!", "f"'] & \mathrm{B}G. \end{tikzcd}$$

We will neither prove the universal property nor the fact that a construction of the universal bundle exists. As an aside, we can use some concepts explored in the closing Coda to sketch a particularly interesting construction in the case of groups equipped with the discrete topology.

Remark 4.1.8 — A construction of the classifying space of a discrete group.
We can construct $\mathrm{B}G$ as $|\mathrm{N}(\mathcal{B}G)_\bullet|$, the geometric realization [Rem. C.1.10] of the nerve [Def. C.1.4] of the group G when viewed as a category with a single object $\mathcal{B}G$ [Rem. 1.3.2].

The total space $\mathrm{E}G$ is then $|\mathrm{N}(\mathcal{E}G)_\bullet|$, the geometric realization of the nerve of the category $\mathcal{E}G$ whose objects are the elements of G and where there is a unique morphism between any two objects g and h, which we identify with gh^{-1} (that is, the unique element $k \in G$ such that $g = kh$).

The map $\mathrm{E}G \to \mathrm{B}G$ is obtained by applying the two functors

$$\mathsf{Cat} \xrightarrow{\mathrm{N}(-)_\bullet} \mathsf{sSet} \xrightarrow{|-|} \mathsf{Top}$$

of nerve and geometric realization to the functor $\mathcal{E}G \to \mathcal{B}G$ obtained by mapping each object $g \in \mathcal{E}G$ to the unique object $* \in \mathcal{B}G$ and each arrow $gh^{-1} \in \mathrm{Hom}_{\mathcal{E}G}(g, h)$ to $gh^{-1} \in \mathrm{Hom}_{\mathcal{B}G}(*, *)$.

For groups equipped with a non-discrete topology, a similar construction can be made using simplicial *spaces* (i.e. simplicial objects in Top, see Definition C.1.1) and the geometric realization of simplicial spaces; this takes into account the topology of each simplex. This is sometimes called Milgram's *bar construction*, in contrast with Milnor's *join construction*. See [BS09, § 5.1] for historical references.

Of course, there are things left to check, and we have used some yet-undefined concepts without worry. The reason we spared some time for discussing this particular construction is that it easily generalizes to groupoids with more than one object, filling some gaps in the Interlude. And we have yet another connection: when G is a discrete group, $\mathrm{B}G$ is an Eilenberg–Mac Lane space $K(G,1)$; i.e., $\pi_1(\mathrm{B}G,x) \cong G$ and $\pi_i(\mathrm{B}G,x) = 0$ for $i > 1$.

For a more explicit description of the construction of Remark 4.1.8, see [Hat02, Ex. 1B.7]. As an example, one can check that $\mathrm{E}(\mathbb{Z}/2\mathbb{Z}) = \mathbb{S}^\infty$ (the *infinite-dimensional sphere* $\mathbb{S}^\infty = \varinjlim_n \mathbb{S}^n$) and that $\mathrm{B}(\mathbb{Z}/2\mathbb{Z}) = \mathbb{R}\mathrm{P}^\infty$ (the *infinite-dimensional projective space* $\mathbb{R}\mathrm{P}^\infty = \varinjlim_n \mathbb{R}\mathrm{P}^n$). The action $\mathbb{Z}/2\mathbb{Z} \curvearrowright \mathbb{S}^\infty$ is the *antipodal action*, which maps each point to its opposite. We will use this later on, in Remark 4.1.19.

In any case, let us move on. Now that we have the classifying space, we can exploit its universal property in order to simplify further constructions. We can see a G-structure as an instance of a more general notion, that of (X,ζ)-*structures*.

(X,ζ)-structures on manifolds

Definition 4.1.9. Let X be a topological space and let $\zeta \to X$ be a rank n real vector bundle. Let M be a manifold of dimension $m \leq n$. An ***(X,ζ)-structure*** on M is a continuous map $f\colon M \to X$ equipped with an isomorphism of vector bundles $\mathrm{T}M \oplus \underline{\mathbb{R}}^{n-m} \cong f^*\zeta$.

For us, this will just be a useful generalization for theorem-stating; our true interest lies in G-structures. We can restate Definition 4.1.5 as an instance of an (X,ζ)-structure.

Definition 4.1.10. Let G be a topological group equipped with a homomorphism $\chi\colon G \to \mathrm{O}(n)$. A ***$G$-structure*** on a manifold M is a $(\mathrm{B}G,\zeta_\chi)$-structure on M, where $\mathrm{B}G$ is a classifying space for G and $\zeta_\chi = (\mathbb{R}^n \times \mathrm{E}G)/G$ is the vector bundle over $\mathrm{B}G$ determined by χ.

Here we are seeing $\mathrm{O}(n)$ canonically as a bundle over $\mathbb{R}^n$ — its orthonormal frame bundle $\mathrm{Fr}_{\mathrm{O}}\,\mathbb{R}^n$, in fact — and defining an action by G on the product bundle $\mathbb{R}^n \times \mathrm{E}G$:

$$(\mathbb{R}^n \times \mathrm{E}G) \times G \to \mathbb{R}^n \times \mathrm{E}G, \qquad (x,y)\cdot g = (x \cdot \chi(g), y \cdot g).$$

By using the universal property 4.1.7, it is not hard to see that Definition 4.1.10 is equivalent to the prior Definition 4.1.5.

Strictly following our statement of Property 4.1.7 would mean that $\mathrm{B}G$ should be a manifold, but that is only because we preferred simple but very constrained statements over technical but more general statements. We will admit some inaccuracies in our treatment, and instead refer the reader to the more specialized sources previously cited — in general, $\mathrm{B}G$ should be assumed to be a CW-complex.

Let us give our main three examples.

$\{1\}$-structures, $\mathrm{SO}(n)$-structures and $\mathrm{O}(n)$-structures

To cut the chase short: we make the following identifications.

- An $\{1\}$-structure on a manifold M is an n-framing, since giving a reduction from $\mathrm{O}(n)$ to $\{1\}$ of the orthonormal frame bundle $\mathrm{Fr}_{\mathrm{O}}\,M$ is equivalent to giving a trivialization of the tangent bundle $\mathrm{T}M$ (modulo determining a Riemannian structure on M, which as always we ignore due to depending on a contractible space of choices).
- An $\mathrm{SO}(n)$-structure on M is equivalent to giving an orientation of M, since a reduction from $\mathrm{O}(n)$ to $\mathrm{SO}(n)$ is the same as restricting the orbits of the $\mathrm{O}(n)$-action on each fiber $(\mathrm{Fr}_{\mathrm{O}}\,M)_x$ to only the positive changes of basis (as in Definition 1.1.5).
- An $\mathrm{O}(n)$-structure on M is equivalent to giving no structure at all, since there is always a reduction from $\mathrm{O}(n)$ to $\mathrm{O}(n)$.

Compare with the discussion of Remarks 1.2.21 to 1.2.23. One can also define (e.g.) almost-complex, almost-symplectic or spin structures in this way, using the appropriate matrix groups.

Now, we will state a version of the Cobordism Hypothesis for cobordisms equipped with an (X,ζ)-structure.

4.1.2 The Cobordism Hypothesis for (X,ζ)-manifolds

In this section we want to state a version of the Cobordism Hypothesis for oriented manifolds. This is a particular case of the Cobordism Hypothesis for (X,ζ)-manifolds, which is ultimately the version for which [Lur09] sketches a proof. For the same low price, this also includes cases such as unoriented manifolds and manifolds equipped with a spin structure.

First of all, let us denote by $\mathsf{Bord}_n^{(X,\zeta)}$ the (∞,n)-category of cobordisms equipped with an (X,ζ)-structure, and denote the G-structured version by

$$\mathsf{Bord}_n^G := \mathsf{Bord}_n^{(\mathrm{B}G,\zeta_\chi)}.$$

We start by rephrasing Thesis 3.2.17.

The Cobordism Hypothesis for framed manifolds, again

Remember from Claim 3.2.14 the definition of the maximal subcategory of fully dualizable objects, $\mathcal{C}^{\mathrm{fd}}$. we will now define the maximal ∞-groupoid (or *core*), $\mathrm{Core}(\mathcal{C})$, in a similar way.

Claim 4.1.11. Let $\mathcal{C}$ be an (∞,n)-category. There is an (∞,n)-category $\mathrm{Core}(\mathcal{C})$ equipped with a homomorphism $j\colon \mathrm{Core}(\mathcal{C}) \hookrightarrow \mathcal{C}$ such that

(I) the (∞,n)-category $\mathrm{Core}(\mathcal{C})$ is an ∞-groupoid;

(II) the map $j\colon \mathrm{Core}(\mathcal{C}) \hookrightarrow \mathcal{C}$ is universal with respect to (I).

Informally, we pass from $\mathcal{C}$ to $\mathrm{Core}(\mathcal{C})$ by discarding every non-invertible k-morphism.

A refinement of Thesis 3.2.17 reads as follows.

Thesis 4.1.12. Let $\mathcal{C}$ be a symmetric monoidal (∞,n)-category. Then, the evaluation functor $Z \mapsto Z(\mathrm{pt}_+)$ induces an equivalence of ∞-groupoids

$$\mathrm{Fun}^{\otimes}(\mathsf{Bord}_n^{\mathrm{fr}},\mathcal{C}) \longrightarrow \mathrm{Core}(\mathcal{C}^{\mathrm{fd}}).$$

In particular, the category $\mathrm{Fun}^{\otimes}(\mathsf{Bord}_n^{\mathrm{fr}},\mathcal{C})$ of fully extended TQFTs is an ∞-groupoid.

The canonical $\mathrm{O}(n)$-action on a symmetric monoidal (∞,n)-category

Let M be an m-manifold. Note that the space of all possible n-framings on M carries an action by $\mathrm{O}(n)$; namely, an element $P \in \mathrm{O}(n)$ acts on a trivialization $\mathrm{T}M \oplus \underline{\mathbb{R}}^{n-m} \to \underline{\mathbb{R}}^n$ by postcomposition, yielding the trivialization

$$\mathrm{T}M \oplus \underline{\mathbb{R}}^{n-m} \longrightarrow \underline{\mathbb{R}}^n \xrightarrow{-\cdot P} \underline{\mathbb{R}}^n.$$

This determines an action by $\mathrm{O}(n)$ on the symmetric monoidal (∞,n)-category $\mathsf{Bord}_n^{\mathrm{fr}}$. In particular, the action maps each k-morphism (for $0 \le k \le n$) to the same manifold equipped with the framing obtained by postcomposing with $P \in \mathrm{O}(n)$. The higher k-morphisms are invariant by the action.

Remark 4.1.13 — Group actions on (higher) categories.
In general, an action of a group G on an object $S \in \mathcal{D}$ is a group homomorphism $G \to \mathrm{Aut}_{\mathcal{D}}(S)$ from our group to the group of automorphisms of the object S. By viewing G as a groupoid with a single object, this is the same as a functor $\mathcal{B}G \to \mathcal{D}$ such that the unique object $* \in \mathcal{B}G$ goes to $S \in \mathcal{D}$: the isomorphisms $\mathrm{Aut}_{\mathcal{B}G} = \mathrm{Hom}_{\mathcal{B}G}(*,*)$ will go to isomorphisms $\mathrm{Aut}_{\mathcal{D}}(S)$.

Similarly, we can define a group action of G on an (∞,n)-category $\mathcal{C}$ as a functor $\mathcal{B}G \to \mathsf{Cat}_{(\infty,n)}$ such that $* \mapsto \mathcal{C}$. As usual, here we are viewing the 1-groupoid $\mathcal{B}G$ as an $(\infty,0)$-category where the only k-morphisms for $k>1$ are the identities, and that in turn is seen as an (∞,n)-category. (In the Interlude, we named this the "discrete categorification").

So far this is equivalent to a group homomorphism $G \to \mathrm{Aut}_{\mathsf{Cat}_{(\infty,n)}}(\mathcal{C})$, but the functorial approach is more easily generalizable to groups endowed with a topology. When G is a topological group, we define a ***group action*** on an (∞,n)-category as a functor $\rho\colon \mathcal{B}(\pi_{\leq\infty}G) \to \mathsf{Cat}_{(\infty,n)}$ such that $* \mapsto \mathcal{C}$. This consists of the following data.

- For each group element $g \in G$, an equivalence of (∞,n)-categories $F_g \coloneqq \rho(g)\colon \mathcal{C} \to \mathcal{C}$.
- For each path $\gamma\colon g \rightsquigarrow h$ between two elements, a natural isomorphism $\rho(\gamma)\colon F_g \Rightarrow F_h$.
- For each homotopy $H\colon \gamma \rightsquigarrow \gamma'$, an invertible modification $\rho(m)\colon \rho(\gamma) \Rrightarrow \rho(\gamma')$.

$\vdots$

As always, here the notions of equivalence, natural transformation, modification[2] and so on should be as weak as required by the definition of (∞,n)-categories.

For further details (in the case of bicategories), see [HSV17, § 3].

Now, for any symmetric monoidal (∞,n)-category $\mathcal{C}$, this group action $\mathrm{O}(n) \curvearrowright \mathsf{Bord}_n^{\mathrm{fr}}$ induces an action of $\mathrm{O}(n)$ on the ∞-groupoid $\mathrm{Fun}^{\otimes}(\mathsf{Bord}_n^{\mathrm{fr}},\mathcal{C})$. But applying Thesis 4.1.12, this is equivalent to $\mathrm{Core}(\mathcal{C}^{\mathrm{fd}})$. So as a corollary, we have quite an striking statement.

Corollary 4.1.14. For each symmetric monoidal (∞,n)-category $\mathcal{C}$, there is a canonical action

$$\mathrm{O}(n) \curvearrowright \mathrm{Core}(\mathcal{C}^{\mathrm{fd}}).$$

So each symmetric monoidal (∞,n)-category with duals (i.e. such that $\mathcal{C} \simeq \mathcal{C}^{\mathrm{fd}}$), just by existing, carries the information of an $\mathrm{O}(n)$-action on its core ∞-groupoid!

Remark 4.1.15 — The canonical $\mathrm{O}(n)$-actions, for very small n.
Let $n=1$, and consider the group $\mathrm{O}(1) \cong \mathbb{Z}/2\mathbb{Z}$. Remember that an $(\infty,1)$-category $\mathcal{C}$ has duals if and only if its homotopy 1-category $\mathrm{h}_1\mathcal{C}$ is rigid. Now, the action of $\mathrm{O}(1)$ on the core $\mathrm{Core}(\mathcal{C})$ is the involution mapping each object to its dual, $X \mapsto X^{\vee}$. So the dual $X^{\vee}$ of an object $X \in \mathcal{C}$ is defined up to isomorphism in the homotopy category $\mathrm{h}_1\mathcal{C}$, and up to a contractible space of choices in the actual $(\infty,1)$-category $\mathcal{C}$.

And notice that this is the same as the action we described in Remark 1.2.21, when considering the forgetful functor $\mathsf{Cob}_n^{\mathrm{or}} \to \mathsf{Cob}_n^{\mathrm{un}}$! Indeed, since $\mathsf{Bord}_1^{\mathrm{fr}} \simeq \mathsf{Bord}_1^{\mathrm{or}}$, for $n=1$ this is also the forgetful functor $\mathsf{Cob}_1^{\mathrm{fr}} \to \mathsf{Cob}_1^{\mathrm{un}}$.

Now, for $n=2$, the action $\mathrm{O}(2) \curvearrowright \mathcal{D}$ induces an automorphism $S_X\colon X \to X$ for every object, known as the *Serre automorphism* of X. In the category $\mathsf{Bord}_2^{\mathrm{fr}}$, this automorphism acts as a clockwise twist in the framing; see [Tel16, § 4; Hes17, § 4] for further information.

The Cobordism Hypothesis for (X,ζ)-manifolds

As usual, let $\mathcal{C}^{\mathrm{fd}}$ be a symmetric monoidal (∞,n)-category with duals. By Corollary 4.1.14 along with the Homotopy Hypothesis [Ths. I.1.7], we can identify its core ∞-groupoid $\mathrm{Core}(\mathcal{C}^{\mathrm{fd}})$ as a topological space equipped with an action by $\mathrm{O}(n)$.

Now, let X be a CW-complex and $\zeta \to X$ a rank n vector bundle (as in Definition 4.1.9). A choice of inner product on ζ determines a principal $\mathrm{O}(n)$-bundle $\mathrm{Fr}_{\mathrm{O}}\,\zeta \to X$, the bundle of orthonormal frames on ζ.

[2] A modification, or 2-*transfor*, is to natural transformations as natural transformations are to functors: these are the 1-morphisms in the category of natural transformations. The general term for these is *k-transfor*, with functors accounting for the case $k=0$. In analogy with homotopy theory, we can see these as functors $\mathcal{C} \times \mathbb{I}^k \to \mathcal{D}$, where $\mathbb{I} = \{0 \mapsto 1\}$ is the *interval category* (also called the "walking arrow").

With these conventions, we can now state yet another version of the Cobordism Hypothesis.

Thesis 4.1.16: Cobordism Hypothesis for (X,ζ)-manifolds.
Evaluation at the point induces an equivalence of ∞-groupoids

$$\mathrm{Fun}^{\otimes}(\mathsf{Bord}_n^{(X,\zeta)}, \mathcal{C}) \simeq \mathrm{Hom}_{\mathrm{O}(n)}(\mathrm{Fr}_{\mathrm{O}}\,\zeta, \mathrm{Core}(\mathcal{C}^{\mathrm{fd}})).$$

Here $\mathrm{Hom}_{\mathrm{O}(n)}$ denotes $\mathrm{O}(n)$-equivariant continuous functions. If we wanted, we could write $\mathrm{Hom}_{\mathsf{Top}_{\mathrm{O}(n)}}$ instead, by defining $\mathsf{Top}_{\mathrm{O}(n)}$ to be the category of topological spaces equipped with an $\mathrm{O}(n)$ action and demanding the morphisms to be $\mathrm{O}(n)$-equivariant

Applying this thesis to groups will yield the final piece of our puzzle: the notion of *homotopy fixed points* of a group action.

Homotopy fixed points of a group action on an (∞, n)-category

Definition 4.1.17. Let G be a topological group acting continuously on a space X. The ***homotopy fixed space*** $X^{\mathrm{h}G}$ is the space of G-equivariant maps $\mathrm{Hom}_G(\mathrm{E}G, X)$.

As one would expect, we call points of the homotopy fixed space ***homotopy fixed points*** of the action by G. Where as a strict fixed point would satisfy $x = xg$ for every $g \in G$, we can think of a homotopy fixed point as being a collection of paths $\gamma^g \colon x \rightsquigarrow xg$ for each $g \in G$. (This is due to EG being pointed and weakly contractible, by construction). These paths must satisfy some coherence constraints — weakened forms of the axioms of associativity, unity and inverses — and the witness homotopies must satisfy further coherence constraints, and so on.[3] In this way, we can identify each homotopy fixed point with a point in our space X, since we have a projection $\mathrm{Hom}_G(\mathrm{E}G, X) \to X$ given by $\gamma^g \mapsto \gamma^g_0$. Similarly, the set of strict fixed points injects into $X^{\mathrm{h}G}$, by mapping each $x = xg$ to the constant path c_x. Note that under this identification, being a homotopy fixed point is *not* a property, but a *structure*.

The Cobordism Hypothesis for G-manifolds

Remember that $\mathsf{Bord}_n^G := \mathsf{Bord}_n^{(\mathrm{B}G, \zeta_\chi)}$, with $\chi \colon G \to \mathrm{O}(n)$ a continuous homomorphism as in Definition 4.1.10. The notion of homotopy fixed points lets us adapt Thesis 4.1.16 to the case of G-manifolds.

Thesis 4.1.18: Cobordism Hypothesis for G-manifolds.
Evaluation at the point induces an equivalence of ∞-groupoids

$$\mathrm{Fun}^{\otimes}(\mathsf{Bord}_n^G, \mathcal{C}) \simeq (\mathrm{Core}(\mathcal{C}^{\mathrm{fd}}))^{\mathrm{h}G}.$$

It might not be obvious how the identifications with Thesis 4.1.16 are made.

Proof. Let $\mathcal{D} := \mathrm{Core}(\mathcal{C}^{\mathrm{fd}})$ and let $\tilde{\mathrm{B}G}$ be the $\mathrm{O}(n)$-bundle $(\mathrm{E}G \times \mathrm{O}(n))/G$ over $\mathrm{B}G$ determined by the homomorphism χ. We can identify:

$$\mathrm{Fun}^{\otimes}(\mathsf{Bord}_n^G, \mathcal{C}) \simeq \mathrm{Hom}_{\mathrm{O}(n)}(\tilde{\mathrm{B}G}, \mathcal{D}) \simeq \mathrm{Hom}_G(\mathrm{E}G, \mathcal{D}) \simeq \mathcal{D}^{\mathrm{h}G}. \qquad \square$$

So G-structured TQFTs are still determined by their value on the point; but that value is now regarded as being inside the homotopy fixed space of the group action by G, and thus carries more information than the objects of the target category. This data is encoded in the higher-ordered morphisms of the category.

We already saw our first example, when classifying 1D unoriented TQFTs all the way back in Theorem 1.3.11! After all, the classification of 1D TQFTs is just the Cobordism Hypothesis for $n = 1$. We can now express that result in our new language.

[3]For a simplified discussion which highlights a similar interpretation, see the following blog post by Baez as well as its listed references: `https://golem.ph.utexas.edu/category/2020/04/crossed_homomorphisms_part_2.html`.

Remark 4.1.19 — The classification of 1D unoriented TQFTs, again.
We mentioned in Theorem 1.3.11 that 1D unoriented TQFTs where in bijection with what we called *symmetric self-dual objects* $X \in \mathcal{C}$; objects equipped with a symmetric evaluation $X \star X \to \mathrm{I}$ which exhibited $X^\vee = X$. By our newly-stated Thesis 4.1.18, this structure must be encoded in the homotopy fixed points of the action $\mathrm{O}(1) \cong \mathbb{Z}/2\mathbb{Z} \curvearrowright \mathrm{Core}(\mathcal{C}^{\mathrm{fd}})$ of Remark 4.1.15, which we thought of as the involution $X \mapsto X^\vee$ mapping each object to its dual.

Remember that $\mathrm{E}(\mathbb{Z}/2\mathbb{Z}) = \mathbb{S}^\infty$, with the action by $\mathbb{Z}/2\mathbb{Z}$ being the antipodal action. The homotopy fixed points will be points in the space $\mathrm{Hom}_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{S}^\infty, \mathrm{Core}(\mathcal{C}^{\mathrm{fd}}))$, which are $(\mathbb{Z}/2\mathbb{Z})$-equivariant maps from $\mathbb{S}^\infty$ to our category. So the action by $\mathbb{Z}/2\mathbb{Z}$ must be preserved — the antipodal action must become the action identifying each object with its dual. And in preserving that action, we are fixing a particular way of identifying X with $X^\vee$: this becomes the symmetric evaluation $X \star X \to \mathrm{I}$.

Now that we understand most aspects of the general statement, we will specialize to oriented TQFTs in dimension 2 with a very particular target bicategory: the *Morita bicategory* Alg_2. Our goal is to recover some of the classical correspondence [Thm. 2.3.3] between 2D oriented TQFTs and commutative Frobenius algebras, by identifying $\mathsf{Cob}_2^{\mathrm{or}} \cong \Omega\,\mathsf{Bord}_2^{\mathrm{or}} = \mathrm{End}_{\mathsf{Bord}_2^{\mathrm{or}}}(\varnothing)$. This lets us directly compare both statements, and in doing so we can better assess the difficulties of producing classifications for unextended TQFTs in higher dimensions.

4.2 The oriented Cobordism Hypothesis in the Morita bicategory

The goal of this closing section is to explain the statement of the following theorem, as it appears in [Sch14, Thm. 3.52], as well as some of its consequences.

Theorem 4.2.1. The bicategory of 2-dimensional oriented fully extended TQFTs with values in Alg_2 is equivalent to the bicategory Frob_2.

Per Thesis 4.1.18, this amounts to say that Frob_2 is equivalent to $(\mathsf{Core}(\mathsf{Alg}_2^{\mathrm{fd}}))^{h\,\mathrm{SO}(2)}$, the bigroupoid of homotopy fixed points of an SO(2)-action on the core of the subcategory of fully dualizable objects of Alg_2. For more information on this action, see [HSV17].[1]

That is a lot to keep track of, but we will take our time to gradually explore this statement. There is no rush to the finish line. We will follow the self-contained proof in [Sch14, § 3.8.5], but a succinct proof of Theorem 4.2.1 using the Cobordism Hypothesis for G-manifolds [Ths. 4.1.18] can be found in [FHLT10, Ex. 2.13].

4.2.1 Preliminaries: the Morita bicategory and a presentation of $\mathsf{Cob}_2^{\mathrm{or}}(2)$

First we reproduce a set of generators and relations for $\mathsf{Cob}_2^{\mathrm{or}}(2)$, and then we will explain what the *Morita bicategory* Alg_2 and the bigroupoid Frob_2 are.

Generators and relations for $\mathsf{Cob}_2^{\mathrm{or}}(2)$

Here we reproduce various figures from [Sch14]. Firstly, in figure 4.1, we include a set of symmetric monoidal generators for the bicategory of unoriented cobordisms $\mathsf{Cob}_2^{\mathrm{un}}(2)$. In figure 4.2, we give a set of relations.

We read 1-cobordisms from left to right, and read 2-cobordisms between these from top to bottom. Note that the symmetry generators are *not* orientable.

Note that now Zorro's Lemma [fig. 1.9] for objects does *not* hold strictly, but rather up to witness 2-morphism; namely, the two cusp generators. Full dualizability [Def. 3.2.15] is encoded in the two relations at the top right of figure 4.2, stating that the vertical composition of the two cusp generators (in either order) equals the identity (i.e., a cylinder). In other words, the cusp generators are the unit and counit of an adjunction [Def. 3.2.10] between the straight line and the "Z" shape; the two sides of figure 1.16.

We take for granted that the bicategory presented by figures 4.1 and 4.2 is equivalent to $\mathsf{Cob}_2^{\mathrm{un}}(2)$; a full proof is out of our scope, and is the main result of [Sch14, Thm. 3.44].

Orienting each possible generator (that is, every one except for the symmetry generators) yields a presentation of $\mathsf{Cob}_2^{\mathrm{or}}(2)$, which we reproduce in figure 4.4. Compare these with the generators for the 1-category $\mathsf{Cob}_2^{\mathrm{or}}$, in figure 2.1. Note how we can now decompose handles! In particular, the pair of pants  is obtained by stitching together the 2D Morse generators and some cylinders; see figure 4.3. The cusp generators do not have a direct analogue; as we previously said, they "categorify" Zorro's Lemma for $\mathsf{Cob}_1^{\mathrm{or}}$, converting a strict equality (of isomorphism classes of 1-cobordisms) into a coherent 2-isomorphism which relates the two sides of that equality.

We will use this presentation of $\mathsf{Cob}_2^{\mathrm{or}}(2)$ to prove a number of things about extended 2D oriented TQFTs. For now, let us define our target bicategory. First we need a notion of *bimodules*.

[1]This also includes the case of another common target bicategory for 2D extended TQFTs, the bicategory Vect_2 of linear abelian categories, linear functors and natural transformations. Here, the oriented cobordism hypothesis recovers the notion of *Calabi–Yau categories*.

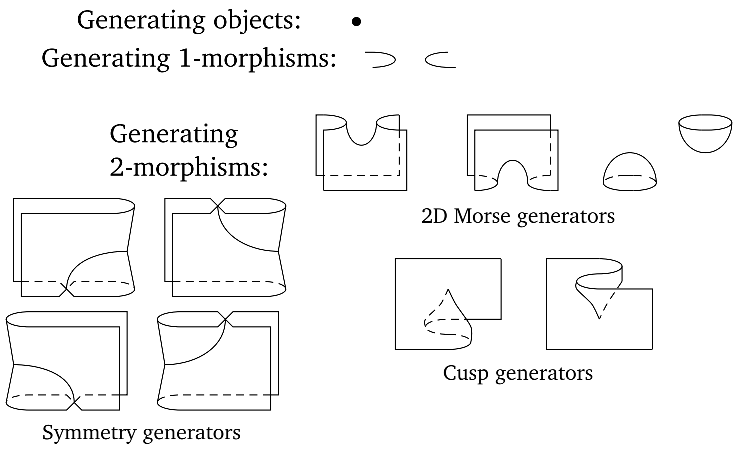


Figure 4.1: Generators for the bicategory $\mathtt{Cob}_2^{\mathrm{un}}(2)$ [Sch14, figs. 0.2, 3.7].

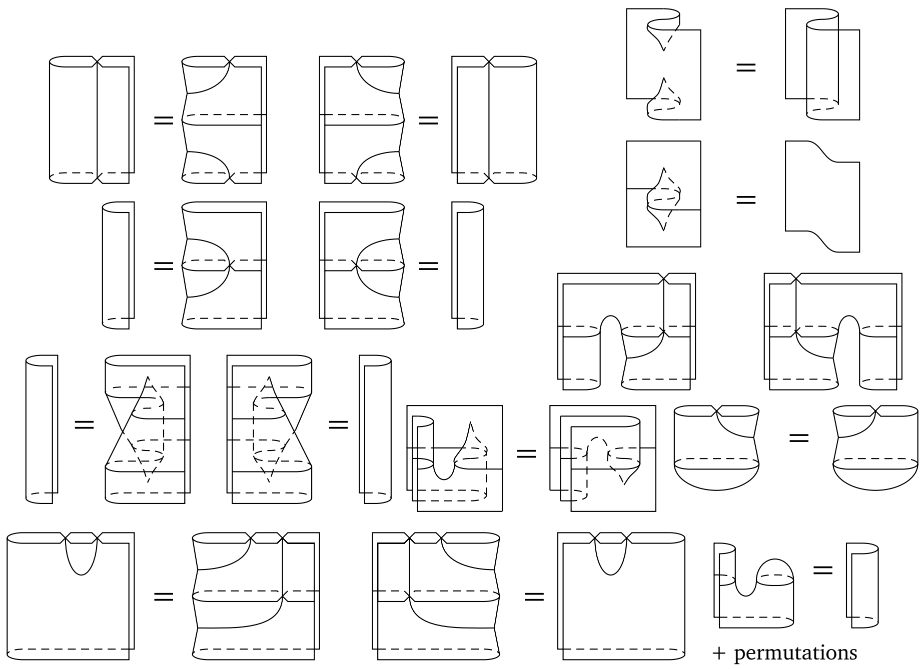


Figure 4.2: Relations among 2-morphisms in the bicategory $\mathtt{Cob}_2^{\mathrm{un}}(2)$ [Sch14, figs. 0.3, 3.8].

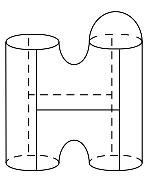


Figure 4.3: The pair of pants written using the generators of $\mathtt{Cob}_2(2)$.

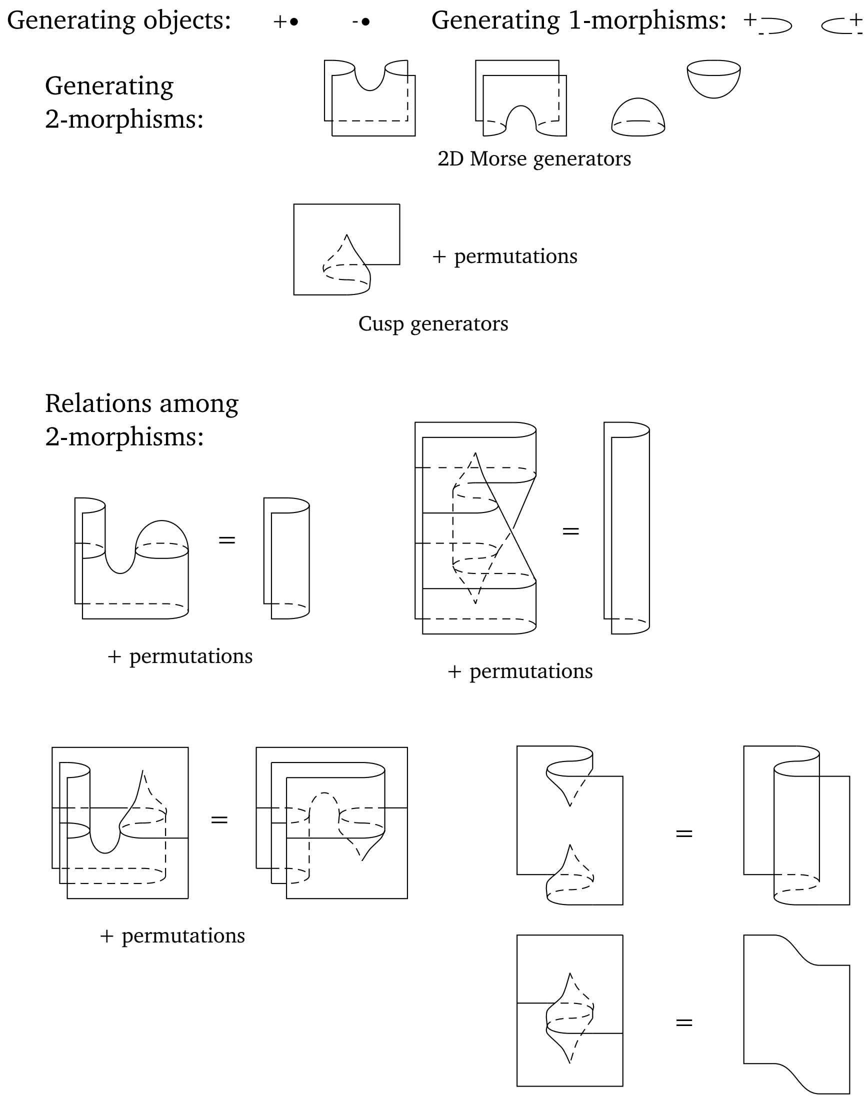


Figure 4.4: Generators and relations for the bicategory $\mathrm{Cob}_2^{\mathrm{or}}(2)$ [Sch14, fig. 3.13].

Bimodules over algebras

Remember from section 2.2.1 the definition of left and right A-modules [Defs. 2.2.3 and 2.2.5]. We assume $\Bbbk$ to be a field and algebras to be associative and unital, but not necessarily commutative.

Definition 4.2.2. A ***bimodule*** over two $\Bbbk$-algebras A and B is a left A-module M which is simultaneously a right B-module, and such that the two structures $\alpha\colon A \otimes M \to M$ and $\beta\colon M \otimes B \to M$ satisfy the compatibility condition expressed by the commutativity of the following diagram.

$$\begin{tikzcd}
 & A \otimes M \otimes B \arrow[dl, "\alpha \otimes \mathrm{id}_B"'] \arrow[dr, "\mathrm{id}_A \otimes \beta"] & \\
M \otimes B \arrow[dr, "\beta"'] & & A \otimes M \arrow[dl, "\alpha"] \\
 & M &
\end{tikzcd}$$

So $a \cdot (x \cdot b) = (a \cdot x) \cdot b$, which is very similar to associativity but differs crucially in that here there are two maps in play: one $A \otimes M \to M$ and another $M \otimes B \to M$. Therefore, we call this condition *mixed associativity*. Morphisms of A-B-bimodules, called ***intertwiners***, are just maps which are simultaneously left A-module and right B-module homomorphisms.

In analogy with how we moved the in- and out-boundaries of cobordisms around [Fig. 1.3], we can consider any A-B-bimodule as a B^{op}-A^{op}-bimodule instead. These are the *opposite* algebras of A and B. The definition is simple: we just invert the order of multiplication through the twist map $\tau_{A,A}\colon a \otimes b \mapsto b \otimes a$.

Definition 4.2.3. The ***opposite algebra*** A^{op} of a $\Bbbk$-algebra (A, μ, η) is the algebra with unit $\eta\colon \Bbbk \to A$ and multiplication $\tau_{A,A} \circ \mu\colon A \otimes A \to A$.

That is, if $a \otimes b \mapsto ab$ in A, then $a \otimes b \mapsto ba$ in A^{op}. Of course, if A is commutative then $A = A^{\mathrm{op}}$.

So an A-B-bimodule M can also be canonically regarded as an B^{op}-A^{op}-bimodule, or as an $(A \otimes_{\Bbbk} B^{\mathrm{op}})$-$\Bbbk$-bimodule, or as an $\Bbbk$-$(A^{\mathrm{op}} \otimes_{\Bbbk} B)$-bimodule. Let's fix some clearer notation!

Notation 4.2.4. We write an A-B-bimodule M as ${}_AM_B$, so that the two associated algebras are clear. The canonical bimodule structures mentioned above may thus be denoted by ${}_{B^{\mathrm{op}}}M_{A^{\mathrm{op}}}$, ${}_{A \otimes B^{\mathrm{op}}}M$ and $M_{A^{\mathrm{op}} \otimes B}$.

Note that we omit the base field $\Bbbk$, since this does not carry any extra information: we identify a $\Bbbk$-A-bimodule as a right A-module, an A-$\Bbbk$-bimodule as a left A-module and a $\Bbbk$-$\Bbbk$-bimodule as a $\Bbbk$-module (that is, a $\Bbbk$-vector space).

Similarly, when writing a tensor product ${}_AM_B \otimes_B {}_BN_C$ (which yields an A-C-bimodule), we will declutter the notation and leave the inner algebra B implied by the subscript of $\otimes$.[2] That is, we write ${}_AM \otimes_B N_C$. If a tensor product lacks a subscript, then it is taken over the base field $\Bbbk$:

$$A \otimes B := A \otimes_{\Bbbk} B.$$

We also have the converse: given a tensor product $M \otimes_A N$, it always makes sense to see to see M as a right A-module and N as a left A-module. For example, as we will see in equation (4.1), the notation $A \otimes_{A \otimes A^{\mathrm{op}}} A$ implicitly sees A as a left or right $(A \otimes A^{\mathrm{op}})$-module.

Note that any $\Bbbk$-algebra A carries a canonical A-A-bimodule structure, where the left and right multiplications are defined as the internal multiplication $\mu\colon A \otimes A \to A$. Equivalently, A can be seen as a left $(A \otimes A^{\mathrm{op}})$-module, or as a right $(A^{\mathrm{op}} \otimes A)$-module. We call $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$ the ***enveloping algebra*** of A.

Remark 4.2.5 — Bimodules generalize homomorphisms of $\Bbbk$-algebras.
In general, given an algebra homomorphism $h\colon A \to B$ we can define the B-A-bimodule ${}_BBh_A$ where $Bh = B$ (as a vector space), $B \otimes Bh \to Bh$ is given by multiplication in B and $Bh \otimes A \to Bh$ is given by $b \otimes a \mapsto b \cdot h(a)$.

The Morita bicategory

Why do we want to consider bimodules? Well, they let us "categorify" the category of vector spaces; that is, we can define a bicategory Alg_2 such that $\mathsf{Vect}_{\Bbbk} \simeq \Omega\,\mathsf{Alg}_2$ (using Definition I.2.2).

Definition 4.2.6. The ***Morita bicategory***, Alg_2, is the following symmetric monoidal bicategory.

- **Objects:** $\Bbbk$-algebras.
- **1-morphisms:** For each pair of $\Bbbk$-algebras A and B, B-A-bimodules ${}_BM_A\colon A \to B$.
- **2-morphisms:** Intertwiners of B-A-bimodules.
- **Composition:** Given two 1-morphisms ${}_BM_A$ and ${}_CN_B$, we compose them as

$${}_C(N \circ M)_A := {}_CN \otimes_B M_A.$$

- **Identities:** For each $\Bbbk$-algebra A, $\mathrm{id}_A = {}_AA_A$.

[2] The tensor product $\otimes_B$ can be constructed as ${}_AM \otimes_B N_C := ({}_AM_B \otimes_{\Bbbk} {}_BN_C)/\langle mb \otimes n - m \otimes bn \mid m \in M, b \in B, n \in N\rangle$.

Both vertical and horizontal composition of 2-morphism should be clear; these are given by composition and tensor product of maps between bimodules. Similarly, the monoidal operation is given by tensor product of $\Bbbk$-algebras over $\Bbbk$ (with unit $\Bbbk$), and the symmetric structure by the usual twist maps $\tau\colon A\otimes B\to B\otimes A$ regarded as $(B\otimes A)$-$(A\otimes B)$-bimodules ${}_{B\otimes A}\tau_{A\otimes B}$ through Remark 4.2.5. In 1-morphisms, these act as follows:

$$ {}_BM_A\otimes {}_DN_C = {}_{B\otimes D}(M\otimes N)_{A\otimes C}, \qquad {}_CN\otimes_{A\otimes B}\tau_{B\otimes A} = {}_C(N\circ\tau)_{B\otimes A}. $$

Remark 4.2.7 — Another notation for the Morita bicategory.
The Morita bicategory is also denoted $\mathsf{Alg}_1(\mathsf{Vect}_{\Bbbk})$ in the literature; see (e.g.) [Lur09, Def. 4.1.11]. As this notation implies, the construction makes sense in any category in which one can define internal algebras and internal bimodules. Furthermore, one can iterate the process by considering an n-category $\mathsf{Alg}_{n-1}(\mathsf{Vect}_{\Bbbk})$ of *E_{n-1}-algebras*, bimodules between E_{n-1}-algebras, bimodules *between bimodules*, and so on.

Also note that some authors follow the opposite convention for 1-morphisms: they regard an A-B-bimodule as a morphism from A to B.

As we said, a $\Bbbk$-$\Bbbk$-bimodule is the same as a $\Bbbk$-vector space, so we can indeed identify

$$ \mathsf{Map}_{\mathsf{Alg}_2}(\Bbbk,\Bbbk)\simeq \mathsf{Vect}_{\Bbbk}. $$

Since $\Omega\,\mathsf{Alg}_2\simeq\mathsf{Vect}_{\Bbbk}$ and $\Omega\,\mathsf{Cob}_2^{\mathrm{or}}(2)\simeq\mathsf{Cob}_2^{\mathrm{or}}$, the correspondence in the statement of the Cobordism Hypothesis [Ths. 4.1.18] should recover at least some of the classical correspondence [Thm. 2.3.3]. But how much does it recover? Answering this question will also answer which 2D oriented TQFTs extend down to points, and which don't.

In order to clarify these matters, we need some more algebraic definitions. We now describe Frob_2: the category of *separable* symmetric Frobenius algebras, *Morita contexts*, and morphisms between Morita contexts.

Yet another characterization of Frobenius algebras

First of all, it will be convenient to introduce yet another characterization of Frobenius algebras, following [Sch14, Def. 3.61].

Definition 4.2.8. Let A be a $\Bbbk$-algebra. An element $e=\sum_i x_i\otimes y_i\in A\otimes_{\Bbbk}A$ is called ***A-central*** if

$$ \sum_i (wx_i)\otimes y_i=\sum_i x_i\otimes(y_iw), \qquad \forall\, w\in A. $$

Characterization 4.2.9. A ***Frobenius algebra*** is a triple (A,ε,e) where A is a $\Bbbk$-algebra, $\varepsilon\colon A\to\Bbbk$ is a linear functional and $e=\sum_i x_i\otimes y_i\in A\otimes A$ is an A-central element. This triple must satisfy the following Frobenius normalization condition:

$$ \sum_i \varepsilon(x_i)y_i=1_A=\sum_i x_i\varepsilon(y_i). $$

This is equivalent to Characterization 2.2.23. One direction of the equivalence defines the comultiplication as $\delta\colon w\mapsto\sum_i x_i\otimes(y_iz)$. In the other direction, one defines the A-central element as $e=\delta(1_A)$. Further details of the proof are left as an exercise for the reader.

Under this characterization, symmetric Frobenius algebras are those satisfying $\varepsilon(xy)=\varepsilon(yx)$. But this is also equivalent to the A-central element e being *A-bicentral*; see [Sch14, Prop. 3.66] for a proof.

Definition 4.2.10. Let A be a $\Bbbk$-algebra. An element $e=\sum_i x_i\otimes y_i\in A\otimes A$ is called ***A-bicentral*** if

$$ \sum_i (wx_iz)\otimes y_i=\sum_i x_i\otimes(zy_iw), \qquad \forall\, w,z\in A. $$

Remark 4.2.11 — Alternative characterization of A-central elements.
Let A be a $\Bbbk$-algebra, and let $A^{\mathrm{e}} = A \otimes A^{\mathrm{op}}$ be its enveloping algebra. Now, we can consider the multiplication map $\mu\colon A^{\mathrm{e}} \to A$ as a map of left A^{e}-modules, with its kernel being the ideal $J \lhd A^{\mathrm{e}}$ generated by elements of the form $a \otimes 1 - 1 \otimes a$. An element $e \in A \otimes A$ is A-central if and only if $Je = 0$, where we are viewing e as an element of $A \otimes A^{\mathrm{op}}$.

It turns out that our classification will select only the *separable* algebras; these are the fully dualizable objects of Alg_2. Again, our definition will be a bit unusual; we follow [Sch14, Def. 3.67].

Definition 4.2.12. A $\Bbbk$-algebra A is ***separable*** if there exists an A-central element $e = \sum_i x_i \otimes y_i$ such that the separability normalization condition is satisfied:

$$\sum_i x_i y_i = 1_A.$$

This is sometimes called *strong separability*; see [Hat65]. Definition 4.2.12 is equivalent to A being finite dimensional and "classically" separable; that is, such that $A \otimes \mathbb{K}$ is semisimple for each field extension $\mathbb{K}$ of $\Bbbk$. The notions of separability and semisimplicity coincide for finite-dimensional algebras over perfect fields (such as those which are finite or have zero characteristic).

As the reader probably expects, the objects of our bigroupoid Frob_2 will be separable symmetric Frobenius algebras.[3] Now, it remains to describe the 1- and 2-morphisms.

Morita contexts

Let us consider the Morita bicategory Alg_2 of Definition 4.2.6. A *Morita equivalence* between two algebras A, $B \in \mathsf{Alg}_2$ is an equivalence in Alg_2, while a *Morita context* is an *adjoint equivalence* $({}_BM_A, {}_AN_B, u, v)$; that is, an adjunction [Def. 3.2.10] where the 2-morphisms u and v are invertible.

We will now untangle both of these definitions.

Definition 4.2.13. Two $\Bbbk$-algebras A and B are said to be ***Morita equivalent*** if there exists bimodules ${}_BM_A$ and ${}_AN_B$ such that

$$\begin{aligned} {}_AN \otimes_B M_A &\cong {}_AA_A, \\ {}_BM \otimes_A N_B &\cong {}_BB_B. \end{aligned}$$

Per Remark 4.2.5, this generalizes (and is weaker than) isomorphisms of $\Bbbk$-algebras. This is similar to how invertible cobordisms between manifolds generalize diffeomorphisms (although these two notions coincide for low dimension).

Definition 4.2.14. A ***Morita context*** between two $\Bbbk$-algebras A and B is a tetrad $({}_BM_A, {}_AN_B, u, v)$ where

- ${}_BM_A$ and ${}_AN_B$ are bimodules;
- $u\colon {}_AA_A \to {}_AN \otimes_B M_A$ and $v\colon {}_BM \otimes_A N_B \to {}_BB_B$ are isomorphisms of bimodules.

This data must determine an adjunction:

$$\begin{aligned} \mathrm{id}_{{}_BM_A}\colon {}_BM_A \cong {}_BM \otimes_A A_A &\xrightarrow{\mathrm{id}\otimes u} {}_BM \otimes_A N \otimes_B M_A \xrightarrow{v\otimes \mathrm{id}} {}_BB \otimes_B M_A \cong {}_BM_A, \\ \mathrm{id}_{{}_AN_B}\colon {}_AN_B \cong {}_AA \otimes_A N_B &\xrightarrow{u\otimes \mathrm{id}} {}_AN \otimes_B M \otimes_A N_B \xrightarrow{\mathrm{id}\otimes v} {}_AN \otimes_B B_B \cong {}_AN_B. \end{aligned}$$

This is the notion we are most interested in, since these will be the 1-morphisms of our bigroupoid Frob_2. The 2-morphisms will be isomorphisms of Morita contexts; as usual, these are just the invertible morphisms. And a morphism of Morita contexts is a pair of maps between the relevant bimodules which commute with the unit and counit of the adjunction.

[3]Note that the A-central elements of Characterization 4.2.9 and Definition 4.2.12 need not coincide.

Definition 4.2.15. Let $({}_BM_A, {}_AN_B, u, v)$ and $({}_BM'_A, {}_AN'_B, u', v')$ be two Morita contexts between the same $\Bbbk$-algebras. A ***morphism of Morita contexts*** is a pair of bimodule morphisms $f\colon {}_BM_A \to {}_BM'_A$ and $g\colon {}_AN_B \to {}_AN'_B$ such that the following diagrams commute.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[dr, "u'"'] & {}_AN \otimes_B M_A \arrow[d, "g\otimes f"] \\ & {}_AN' \otimes_B M'_A \end{tikzcd} \qquad \begin{tikzcd} {}_BM \otimes_A N_B \arrow[r, "v"] \arrow[d, "f\otimes g"'] & B \\ {}_BM' \otimes_A N'_B \arrow[ur, "v'"'] & \end{tikzcd}$$

But not all Morita contexts respect the symmetric Frobenius structure, so we will want to restrict ourselves to those which satisfy a certain compatibility condition. We need a lemma, in order to transport symmetric Frobenius forms along Morita contexts.

First, notice that for any algebra A, we have a linear isomorphism

$$A/[A,A] \longrightarrow A \otimes_{A\otimes A^{\mathrm{op}}} A, \qquad [a] \longmapsto a \otimes 1, \tag{4.1}$$

with inverse $a \otimes b \mapsto [ab]$. Here, $[A,A] = \{ab - ba \mid a, b \in A\}$ is the commutator of the algebra. Note that $A/[A,A]$ is the largest abelian quotient of A; this can be made precise through an universal property. If A is separable [Def. 4.2.12], we can identify this quotient with the center $\mathrm{Center}(A)$; we will return to this point in Remark 4.2.29. As an aside, the tensor product $A\otimes_{A\otimes A^{\mathrm{op}}}A$ is called the *zeroth Hochschild homology* of A.

On the other hand, a Morita context $({}_BM_A, {}_AN_B, u, v)$ induces a canonical linear isomorphism

$$\begin{aligned} \tau\colon (N \otimes_B M) \otimes_{A\otimes A^{\mathrm{op}}} (N \otimes_B M) &\longrightarrow (M \otimes_A N) \otimes_{B\otimes B^{\mathrm{op}}} (M \otimes_A N) \\ n \otimes m \otimes n' \otimes m' &\longmapsto m \otimes n' \otimes m' \otimes n. \end{aligned} \tag{4.2}$$

Now we state our lemma [Sch14, Lem. 3.71; HSV17, Lem. 2.3].

Lemma 4.2.16. Let A and B be $\Bbbk$-algebras and let $f = ({}_BM_A, {}_AN_B, u, v)$ be a Morita context. Then, there is a canonical isomorphism of $\Bbbk$-vector spaces

$$\begin{aligned} f_*\colon A/[A,A] &\longrightarrow B/[B,B] \\ [a] &\longmapsto \sum_{i,j} [u^{-1}(m_j \cdot a \otimes n_i)], \end{aligned}$$

where n_i and m_j are given by $v^{-1}(1_A) = \sum_{i,j}(n_i \otimes m_j) \in {}_AN \otimes_B M_A$.

Proof. We have canonical isomorphisms

$$\begin{aligned} A/[A,A] &\cong A \otimes_{A\otimes A^{\mathrm{op}}} A && (4.1) \\ &\cong (N \otimes_B M) \otimes_{A\otimes A^{\mathrm{op}}} (N \otimes_B M) && (u \otimes u) \\ &\cong (M \otimes_A N) \otimes_{B\otimes B^{\mathrm{op}}} (M \otimes_A N) && (4.2) \\ &\cong B \otimes_{B\otimes B^{\mathrm{op}}} B && (v \otimes v) \\ &\cong B/[B,B] && (4.1). \end{aligned}$$

Chasing through the isomorphisms, we conclude $f_*([a]) = \sum_{i,j}[u^{-1}(m_j \cdot a \otimes n_i)]$. □

Note that, if $\varepsilon\colon A \to \Bbbk$ is a symmetric Frobenius form, then we can factorize it through the maximal abelian quotient $A/[A,A]$:

$$\varepsilon\colon A \longrightarrow A/[A,A] \longrightarrow \Bbbk.$$

Thus, we can transport *symmetric* Frobenius forms through Morita contexts:

$$\varepsilon \circ f_*^{-1}\colon B \to B/[B,B] \xrightarrow{f_*^{-1}} A/[A,A] \xrightarrow{\varepsilon} \Bbbk.$$

Now we can state our compatibility condition: for both Frobenius forms to coincide, when comparing them through the Morita context.

Definition 4.2.17. Let (A, ε^A, e^A) and (B, ε^B, e^B) be two symmetric Frobenius algebras. We say that a Morita context $f = ({}_BM_A, {}_AN_B, u, v)$ between A and B is ***compatible*** with the symmetric Frobenius algebra structures if the following diagram commutes.

$$\begin{tikzcd} A/[A,A] \arrow[rr, "f_*"] \arrow[dr, "\varepsilon^A"'] & & B/[B,B] \arrow[dl, "\varepsilon^B"] \\ & \Bbbk & \end{tikzcd}$$

Shouldn't we also transport the central elements e^A and e^B through the Morita contexts, in order to compare them? Yes, and [Sch14, Def. 3.72] does. But this is already implied by our definition; see [HSV17, Rem. 2.5].

And thus, we can finally define our bigroupoid Frob_2.

Definition 4.2.18. The ***bigroupoid of separable symmetric Frobenius algebras***, Frob_2, is described below.

- **Objects:** Separable symmetric Frobenius algebras (A, ε, e).
- **1-morphisms:** For any two Frobenius algebras A and B, compatible Morita contexts $f\colon A \to B$.
- **2-morphisms:** For any two parallel contexts f and g, isomorphisms of Morita contexts $F\colon f \to b$.

We have now finished up our prerequisites. In the following section 4.2.2, we will prove some easier subtheorems of Theorem 4.2.1.

4.2.2 Extended 2D oriented TQFTs are separable symmetric Frobenius algebras

Here we follow [Sch14, § 3.8.5].

The data required to define an extended 2D oriented TQFT

An extended 2D TQFT $Z\colon \mathsf{Cob}_2^{\mathrm{or}}(2) \to \mathsf{Alg}_2$ must be determined by its values on the generators of $\mathsf{Cob}_2^{\mathrm{or}}(2)$, pictured in figure 4.4. First, the objects pt_+ and pt_- must be assigned two $\Bbbk$-algebras, say

$$Z(\mathrm{pt}_+) =: A, \qquad Z(\mathrm{pt}_-) =: B.$$

By definition, the identity "cylinders" $\mathrm{pt}_+ \times I$ and $\mathrm{pt}_- \times I$ must go to the identity bimodules:

$$Z(\mathrm{pt}_+ \times I) = {}_AA_A, \qquad Z(\mathrm{pt}_- \times I) = {}_BB_B.$$

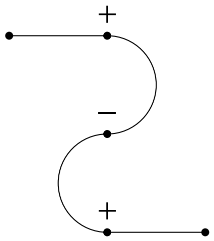
Now we assign bimodules to the two generating 1-morphisms. Write $C_{\mathrm{in}}\colon \mathrm{pt}_+ \sqcup \mathrm{pt}_- \to \varnothing$ for the $\pm$ in-arc [fig. 1.12(f)] and $C_{\mathrm{out}}\colon \varnothing \to \mathrm{pt}_+ \sqcup \mathrm{pt}_-$ for the $\pm$ out-arc [fig. 1.12(a)]. Then, we define[4]

$$Z(C_{\mathrm{in}}) =: {}_{A\otimes B}M, \qquad Z(C_{\mathrm{out}}) =: N_{A\otimes B}.$$

Disclaimer 4.2.19. There are a number of implicit identifications in [Sch14] which we have tried to give more explicit arguments for, in Remarks 4.2.20 to 4.2.24, 4.2.26 and 4.2.29. Determining whether we succeeded in that task or not is left to the discretion of the reader.

[4] We follow [Sch14]'s conventions, and in particular draw the monoidal operation *from top to bottom*. This is opposite to the convention we held throughout all of part I, and we can only apologize.

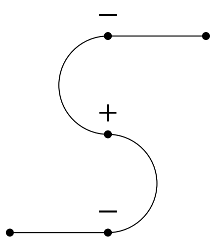
Remark 4.2.20 — Moving in- and out-algebras around.
As mentioned in the discussion following Definition 4.2.3, we can canonically see these as A-B^{op}- and B^{op}-A-bimodules instead, which we write as ${}_A\tilde{M}_{B^{\mathrm{op}}}$, ${}_{B^{\mathrm{op}}}\tilde{N}_A$. Write $N\tau_{B\otimes A}$ for the composition of $N_{A\otimes B}$ with a twist map $\tau\colon A\otimes B \to B\otimes A$. This lets us write the “Z” shape



$$\xmapsto{Z} {}_{A\otimes\Bbbk}(A\otimes N\tau)\otimes_{A\otimes B\otimes A}(M\otimes A)_{\Bbbk\otimes A},$$

an $(A\otimes\Bbbk)$-$(\Bbbk\otimes A)$-bimodule, as the related A-A-module ${}_A\tilde{M}\otimes_{B^{\mathrm{op}}}\tilde{N}_A$ instead.

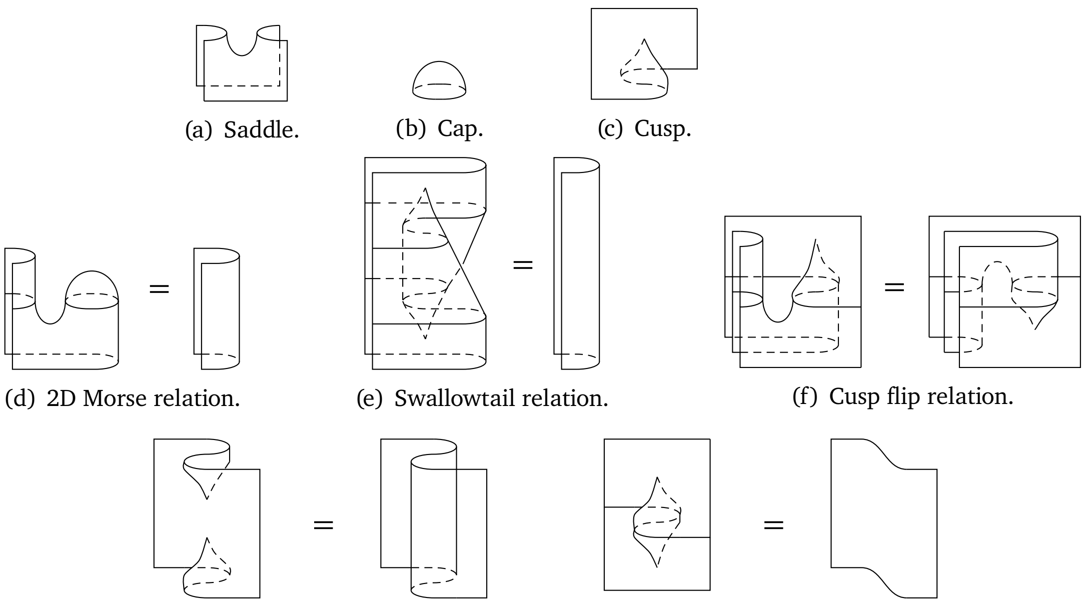
Similarly, we write the “S” shape



$$\xmapsto{Z} {}_{\Bbbk\otimes B}(N\tau\otimes B)\otimes_{B\otimes A\otimes B}(B\otimes M)_{B\otimes\Bbbk},$$

a $(\Bbbk\otimes B)$-$(B\otimes\Bbbk)$-bimodule, as the related B^{op}-B^{op}-bimodule ${}_{B^{\mathrm{op}}}\tilde{N}\otimes_A\tilde{M}_{B^{\mathrm{op}}}$.

We will walk through the generating 2-morphisms slowly, since here we will need to consider the relations between them. It will be helpful to assign names to these generators and relations. We call the generators *saddles*, *caps* and *cusps* — and it should be clear which is which. Then, we have the *2D Morse* relations, the *swallowtail* relations, the *cusp flip* relations and the *cusp inversion* relations. See figure 4.5.



(a) Saddle. (b) Cap. (c) Cusp.

(d) 2D Morse relation. (e) Swallowtail relation. (f) Cusp flip relation.

(g) Cusp inversion relations.

Figure 4.5: 2D generators and relations of $\mathrm{Cob}_2^{\mathrm{or}}(2)$, excluding permutations [Sch14, fig. 3.13].

The four different cusp generators — accounting for shape, "Z" or "S"; and direction, "from" or "to" the cylinder —- give rise, using the identifications of Remark 4.2.20, to four bimodule maps:

$$\begin{aligned} f_1\colon & {}_A\tilde{M} \otimes_{B^{\mathrm{op}}} \tilde{N}_A \longrightarrow {}_A A_A, \\ f_2\colon & {}_A A_A \longrightarrow {}_A\tilde{M} \otimes_{B^{\mathrm{op}}} \tilde{N}_A, \\ f_3\colon & {}_{B^{\mathrm{op}}}\tilde{N} \otimes_A \tilde{M}_{B^{\mathrm{op}}} \longrightarrow {}_{B^{\mathrm{op}}}B^{\mathrm{op}}{}_{B^{\mathrm{op}}}, \\ f_4\colon & {}_{B^{\mathrm{op}}}B^{\mathrm{op}}{}_{B^{\mathrm{op}}} \longrightarrow {}_{B^{\mathrm{op}}}\tilde{N} \otimes_A \tilde{M}_{B^{\mathrm{op}}}. \end{aligned} \tag{4.3}$$

By the cusp inversion relations [fig. 4.5(g)], f_1 is inverse to f_2 and f_3 is inverse to f_4. Furthermore, by the swallowtail relations [fig. 4.5(e)], f_1 and f_4 are adjoint. That is to say, $({}_A\tilde{M}_{B^{\mathrm{op}}}, {}_{B^{\mathrm{op}}}\tilde{N}_A, f_1, f_4)$ form a Morita context between A and B^{op}.

Remark 4.2.21 — Turning B^{op}-modules into A-modules.
We can use this Morita context to turn B^{op}-modules into A-modules, by tensoring with ${}_A\tilde{M}_{B^{\mathrm{op}}}$ or ${}_{B^{\mathrm{op}}}\tilde{N}_A$. For example, given an B^{op}-B^{op}-bimodule ${}_{B^{\mathrm{op}}}P_{B^{\mathrm{op}}}$, we tensor

$$ {}_A\tilde{M} \otimes_{B^{\mathrm{op}}} P \otimes_{B^{\mathrm{op}}} \tilde{N}_A. $$

In the particular cases of having an "unpaired" ${}_AM_{B^{\mathrm{op}}}$ or ${}_{B^{\mathrm{op}}}N_A$, tensoring by the other one lets us apply the isomorphism to ${}_AA_A$.

$$ {}_AM_{B^{\mathrm{op}}} \xrightarrow{-\otimes_{B^{\mathrm{op}}}N_A} {}_AM \otimes_{B^{\mathrm{op}}} N_A \xrightarrow{f_1} {}_AA_A. $$

Our goal will be expressing as much of the structure as possible in terms of A; that is, we want to make the identifications necessary to remove every instance of B, M or N from our equations.

In summary, so far we have specified two $\Bbbk$-algebras $A = Z(\mathrm{pt}_+)$ and $B = Z(\mathrm{pt}_-)$ and a Morita context between A and B^{op}. The remaining structure is given by the saddles and caps, subject to the 2D Morse and cusp flip relations.

Structure determined by the saddles

First, the saddles. These correspond to bimodule maps

$$\begin{aligned} s\colon & {}_{A\otimes B}M \otimes N_{A\otimes B} \longrightarrow {}_{A\otimes B}(A\otimes B)_{A\otimes B}, \\ s'\colon & {}_{A\otimes B}(A\otimes B)_{A\otimes B} \longrightarrow {}_{A\otimes B}M \otimes N_{A\otimes B}. \end{aligned}$$

These determine a pair of A-bicentral elements, as we will show.

For reasons that will be clear later, we want to morph s and s' into maps ${}_AA_A \otimes {}_AA_A \to {}_AA_A \otimes {}_AA_A$. First, we flip the algebra B, as usual.

$$\begin{aligned} \tilde{s}\colon & {}_{A\otimes B^{\mathrm{op}}}(\tilde{M} \otimes \tilde{N})_{A\otimes B^{\mathrm{op}}} \longrightarrow {}_{A\otimes B^{\mathrm{op}}}(A\otimes B^{\mathrm{op}})_{A\otimes B^{\mathrm{op}}}, \\ \tilde{s}'\colon & {}_{A\otimes B^{\mathrm{op}}}(A\otimes B^{\mathrm{op}})_{A\otimes B^{\mathrm{op}}} \longrightarrow {}_{A\otimes B^{\mathrm{op}}}(\tilde{M} \otimes \tilde{N})_{A\otimes B^{\mathrm{op}}}. \end{aligned}$$

Note that now the left B^{op}-module structure on ${}_{A\otimes B^{\mathrm{op}}}(\tilde{M} \otimes \tilde{N})_{A\otimes B^{\mathrm{op}}}$ comes from $\tilde{N}$, while the right B^{op}-module structure comes from $\tilde{M}$!

Now that we have flipped B, we can replace each B^{op}-module structure with A-module structure via our Morita context. This process is inconvenient to write out, but having a concrete goal in mind will help follow the argument. So let's first skip directly to the answer: the two maps s and s' ultimately become maps of $(A\otimes A)$-$(A\otimes A)$-bimodules

$$ g, g'\colon {}_{A_1}A_{A_2} \otimes {}_{A_3}A_{A_4} \longrightarrow {}_{A_1}A_{A_4} \otimes {}_{A_3}A_{A_2}, \tag{4.4}$$

where each A_i is an instance of A which is numbered so as to keep track of the twist maps involved.

Maybe some readers are perspicacious enough to see this as obvious — Schommer-Pries's refusal to elaborate further in [Sch14, pp. 240–241] would indicate that this should be an easy exercise. Alas, the author is not one of those fortunate persons, and spent many hours scratching his head in search for an answer. Hopefully the argument of Remarks 4.2.23 and 4.2.24 will satisfy the consciousness of any restless souls suffering from similar mental struggles.

But first, how does one get an A-bicentral element from here? Well, maps such as g and g' are in fact in bijection with A-bicentral elements! As is usual with bijections between morphisms and elements in algebra, this is realized by multiplication by the element (in one direction) and by evaluation on the unit (on the other direction). Thus, let $e = g(1 \otimes 1)$ and $e' = g'(1 \otimes 1)$; the reader is encouraged to check that these are, in fact, A-bicentral. And as it turns out, the cusp flip relations [fig. 4.5(f)] can be interpreted as saying, precisely, that $e = e'$!

Now, let's fill the details of the argument. While we are at it, let's also give a very hand-wavy graphical interpretation for why we should expect something similar to the maps of equation (4.4).

Remark 4.2.22 — A very hand-wavy graphical interpretation.
Graphically, this identification corresponds to labelling each of the 4 vertical edges of the saddle with a number from 1 to 4, and identifying $Z(C_{\text{in}})$ and $Z(C_{\text{out}})$ with ${}_A A_A$. The "horizontal" disjoint union $C_{\text{in}} \sqcup C_{\text{out}}$ gets mapped to a "vertical" disjoint union $(\text{pt} \times I) \sqcup (\text{pt} \times I)$, so the "horizontal" tensor product ${}_{A_1}A_{A_2} \otimes {}_{A_3}A_{A_4}$ should get mapped to the "vertical" tensor product ${}_{A_1}A_{A_4} \otimes {}_{A_3}A_{A_2}$. See figure 4.6.

Of course, we are being way too careless with our identifications — this is only for intuition. In particular, note that we are applying different conventions when writing the two terms of the domain; we write ${}_{A_1}A_{A_2}$, as established before, but also ${}_{A_3}A_{A_4}$ following the opposite order. Similarly for the codomain. The reason is hidden in the details of the argument — there are some twist maps involved. It is important that we make this argument at least a bit more precise!

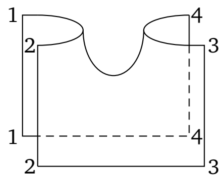


Figure 4.6: Intuition for the identification of the value at the saddle with an A-bicentral element.

Remark 4.2.23 — From $(A \otimes B^{\text{op}})$-$(A \otimes B^{\text{op}})$-bimodules to $(A \otimes A)$-$(A \otimes A)$-bimodules.
We ultimately want to have a map of $(A \otimes A)$-$(A \otimes A)$-bimodules, so we have to tensor with an $(A \otimes A)$-$(A \otimes B^{\text{op}})$-bimodule on the left and with an $(A \otimes B^{\text{op}})$-$(A \otimes A)$-bimodule on the right. For clarity, let's do this one side at a time. First we tensor by ${}_{A\otimes A}(A \otimes \tilde{M})_{A \otimes B^{\text{op}}}$ on the left. We label this instance of $\tilde{M}$ with a prime ($'$), just to keep track of where it goes:

$$ {}_{A\otimes A}(A \otimes \tilde{M}') \otimes_{A \otimes B^{\text{op}}} (\tilde{M} \otimes \tilde{N})_{A \otimes B^{\text{op}}}. $$

Now, we can apply our isomorphism $f_1 \colon {}_A\tilde{M}' \otimes_{B^{\text{op}}} \tilde{N}_A \to {}_A A_A$. In particular, tensoring[5] with the identification $\alpha_M \colon A \otimes_A \tilde{M} \cong \tilde{M}$ (given by the A-action, i.e. multiplication) gives us an isomorphism

$$ \begin{aligned} \alpha_M \otimes f_1 \colon {}_{A\otimes A}(A \otimes \tilde{M}') \otimes_{A \otimes B^{\text{op}}} (\tilde{M} \otimes \tilde{N})_{A \otimes B^{\text{op}}} &\longrightarrow {}_A\tilde{M}_{B^{\text{op}}} \otimes {}_A A_A \\ (a \otimes m') \otimes (m \otimes n) &\longmapsto a \cdot m \otimes f_1(m' \otimes n). \end{aligned} $$

Note that we have an $(A \otimes A)$-$(B^{\text{op}} \otimes A)$-bimodule, instead of the desired $(A \otimes A)$-$(A \otimes B^{\text{op}})$-bimodule. This is no problem; in fact, this is convenient for us! But we could always compose with a twist map, if we so desired, yielding the comparatively bulkier expression

$$ {}_{A\otimes A}(\tilde{M} \otimes A) \otimes_{B^{\text{op}} \otimes A} \tau_{A \otimes B^{\text{op}}}. $$

Also, remember that the B^{op} action we have removed came from N. Otherwise, applying f_1 here would have been an illegal move!

Now, for the other action. The process is similar:

$$\begin{aligned} f_1 \otimes \alpha_A \colon {}_{A\otimes A}(\tilde{M} \otimes A) \otimes_{B^{\mathrm{op}}\otimes A} (\tilde{N} \otimes A)_{A\otimes A} &\longrightarrow {}_A A_A \otimes {}_A A_A \\ (m \otimes a) \otimes (n \otimes a') &\longmapsto f_1(m,n) \otimes a \cdot a'. \end{aligned}$$

We can summarize the process of Remark 4.2.23 by writing it out all at once:

$$\begin{aligned} {}_{A\otimes A}(A \otimes \tilde{M}) \otimes_{A\otimes B^{\mathrm{op}}} (\tilde{M} \otimes \tilde{N}) \otimes_{A\otimes B^{\mathrm{op}}} (A \otimes \tilde{N})_{A\otimes A} &\longrightarrow {}_A A_A \otimes {}_A A_A \\ (a \otimes m') \otimes (m \otimes n) \otimes (a' \otimes n') &\longmapsto a \cdot f_1(m',n) \otimes f_1(m,n') \cdot a'. \end{aligned}$$

Note that we tensor by ${}_{A\otimes B^{\mathrm{op}}}(A \otimes \tilde{N})_{A\otimes A}$ instead of ${}_{B^{\mathrm{op}}\otimes A}(\tilde{N} \otimes A)_{A\otimes A}$, since here we are being more careful with the twist maps.

Of course, the other bimodule involved in the maps s and s' must also undergo this kind of process. This will be a lot quicker.

Remark 4.2.24 — From $(A\otimes B^{\mathrm{op}})$-$(A\otimes B^{\mathrm{op}})$-bimodules to $(A\otimes A)$-$(A\otimes A)$-bimodules, continued. For the bimodule ${}_{A\otimes B^{\mathrm{op}}}(A \otimes B^{\mathrm{op}})_{A\otimes B^{\mathrm{op}}}$, first we apply the map $f_4 \colon {}_{B^{\mathrm{op}}}B^{\mathrm{op}}{}_{B^{\mathrm{op}}} \to {}_{B^{\mathrm{op}}}\tilde{N} \otimes_A \tilde{M}_{B^{\mathrm{op}}}$.

$$\begin{aligned} \mathrm{id}_A \otimes f_4 \colon {}_{A\otimes B^{\mathrm{op}}}(A \otimes B^{\mathrm{op}})_{A\otimes B^{\mathrm{op}}} &\longrightarrow {}_{A\otimes B^{\mathrm{op}}}(A \otimes \tilde{N} \otimes_A \tilde{M})_{A\otimes B^{\mathrm{op}}} \\ a \otimes b &\longmapsto a \otimes f_4(b). \end{aligned}$$

Then, after tensoring with the relevant bimodules, this gets mapped to

$$\begin{aligned} {}_{A\otimes A}(A \otimes \tilde{M}) \otimes_{A\otimes B^{\mathrm{op}}} (A \otimes \tilde{N} \otimes_A \tilde{M}) \otimes_{A\otimes B^{\mathrm{op}}} (A \otimes \tilde{N})_{A\otimes A} &\longrightarrow {}_A A_A \otimes {}_A A_A \\ (a' \otimes m') \otimes (a \otimes n \otimes m) \otimes (a'' \otimes n') &\longmapsto a' \cdot a \cdot a'' \otimes f_1(m',n) \cdot f_1(m,n'). \end{aligned}$$

The chasing of isomorphisms necessary to conclude that the final maps are of the previously stated form, ${}_{A_1}A_{A_2} \otimes {}_{A_3}A_{A_4} \longrightarrow {}_{A_1}A_{A_4} \otimes {}_{A_3}A_{A_2}$, is left to the reader. Hopefully, after this detailed example, the missing details are straightforward enough.

Structure determined by the caps

Finally, the caps. The circle $\mathbb{S}^1$ is obtained by gluing the in- and out-arcs, so its image is

$$Z(\mathbb{S}^1) = Z(C_{\mathrm{in}}) \circ Z(C_{\mathrm{out}}) = M \otimes_{A\otimes B} N =: V.$$

This is a $\Bbbk$-$\Bbbk$-bimodule, i.e. a $\Bbbk$-vector space. The two caps give us linear maps

$$\eta \colon \Bbbk \to V, \qquad \varepsilon \colon V \to \Bbbk.$$

The choice of notation is no coincidence — once again, these will be the (co)units of our Frobenius algebra, once we identify V with the maximal abelian quotient $A/[A,A]$.

By our usual methods [Rems. 4.2.20 to 4.2.24], we can identify $V = M \otimes_{A\otimes B} N$ with the vector space $A \otimes_{A\otimes A^{\mathrm{op}}} A$. But remember that, by equation (4.1), this is canonically isomorphic to the maximal abelian subgroup $A/[A,A]$. It should be no surprise, then, that this can in turn be identified with Center(A), the center of our algebra A!

[5]This is an abuse of terminology and notation; technically speaking, the map we are referring to is *not* a tensor product of maps. The actions by B^{op} in the bimodule ${}_{A\otimes B^{\mathrm{op}}}(\tilde{M} \otimes \tilde{N})_{A\otimes B^{\mathrm{op}}}$ are intertwined, in a way which does not let us decompose the bimodule ${}_{A\otimes A}(A \otimes \tilde{M}') \otimes_{A\otimes B^{\mathrm{op}}} (\tilde{M} \otimes \tilde{N})_{A\otimes B^{\mathrm{op}}}$ as a product of the form ${}_A P_A \otimes {}_A Q_{B^{\mathrm{op}}}$. That is to say, there are some implicit twist maps in play.

This is the same situation we had in Lemma 1.1.23, when we discussed how a cobordism is not always the disjoint union of its components, since twist cobordisms exist.

First, note that we can identify $\mathrm{Center}(A)$ with A-A-bimodule morphisms $f\colon {}_AA_A \to {}_AA_A$. As usual, on one direction this is done through evaluation on the unit; $f(1) \in \mathrm{Center}(A)$, since

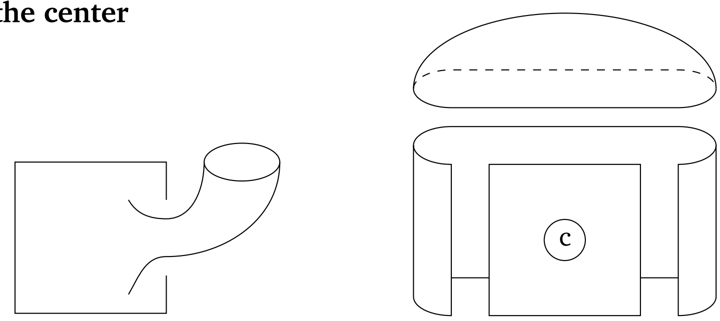
$$a \cdot f(1) = a \cdot f(1) \cdot 1 = f(a \cdot 1 \cdot 1) = f(1 \cdot 1 \cdot a) = 1 \cdot f(1) \cdot a = f(1) \cdot a.$$

And in the inverse direction, an element of the center $c \in \mathrm{Center}(A)$ determines a map ${}_AA_A \to {}_AA_A$ by multiplication: $f(a) = f(c \cdot a) = f(a \cdot c)$.

Now we want to identify $A/[A,A] \cong \mathrm{Hom}_{{}_A\mathsf{Bimod}_A}({}_AA_A, {}_AA_A)$, to provide the missing link:

$$V \cong A \otimes_{A \otimes A^{\mathrm{op}}} A \cong A/[A,A] \cong \mathrm{Hom}_{{}_A\mathsf{Bimod}_A}({}_AA_A, {}_AA_A) \cong \mathrm{Center}(A).$$

Identification with the center



(a) $V \otimes {}_AA_A \to {}_AA_A$. (b) $\mathrm{Center}(A) \to V$.

Figure 4.7: Maps between $V = Z(\mathbb{S}^1)$ and $\mathrm{Center}(A) = \mathrm{Hom}_{{}_A\mathsf{Bimod}_A}({}_AA_A, {}_AA_A)$ [Sch14, fig. 3.23].

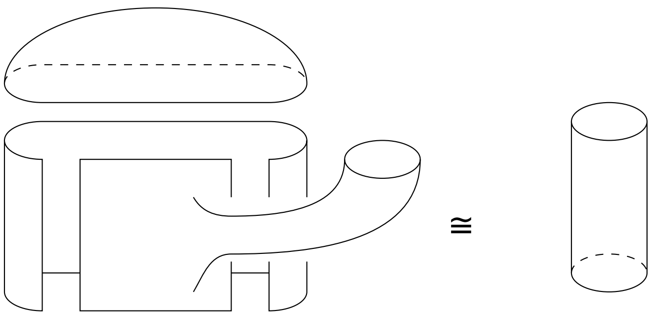


Figure 4.8: Identification $V \cong \mathrm{Center}(A)$, part 1 [Sch14, fig. 3.24].

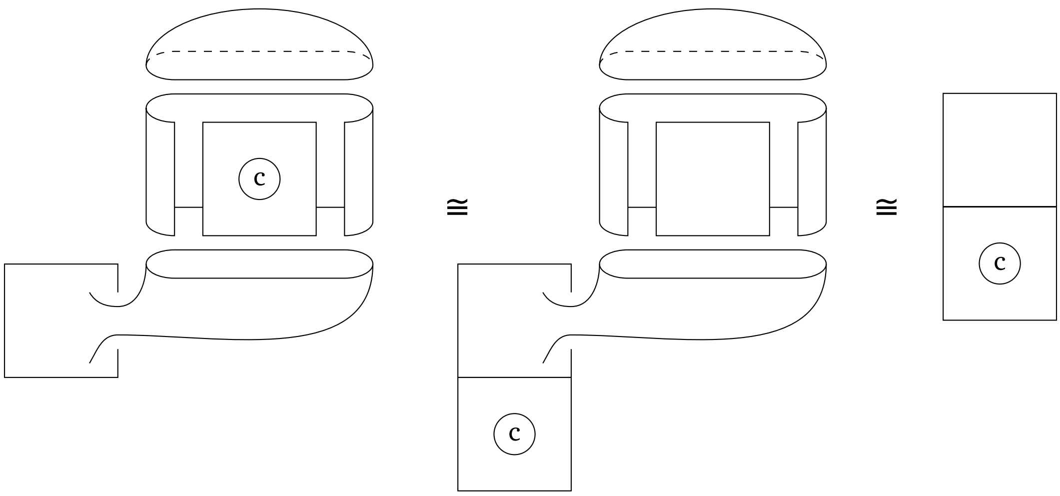


Figure 4.9: Identification $V \cong \mathrm{Center}(A)$, part 2 [Sch14, fig. 3.25].

Lemma 4.2.25. Let $A = Z(\mathrm{pt}_+)$. There is a canonical isomorphism $A/[A,A] \cong \mathrm{Center}(A)$.

Proof. As before, we identify $V = Z(\mathbb{S}^1) \cong A/[A,A]$ and $\mathrm{Center}(A) \cong \mathrm{Hom}_{{}_A\mathsf{Bimod}_A}({}_AA_A, {}_AA_A)$.

Define a map $\tilde{f}\colon V \otimes {}_AA_A \to {}_AA_A$ by evaluating our TQFT Z at the cobordism of figure 4.7(a). We identify this with a map $f\colon V \to \mathrm{Hom}_{{}_A\mathsf{Bimod}_A}({}_AA_A, {}_AA_A)$, namely $v \mapsto f(v) := \tilde{f}(v,-)$.

On the other hand, we define a map $g\colon \mathrm{Hom}_{{}_A\mathsf{Bimod}_A}({}_AA_A, {}_AA_A) \to V$ by composing any intertwiner $c\colon {}_AA_A \to {}_AA_A$ with the maps obtained by evaluating Z in the cobordisms of figure 4.7(b).

The proof that f and g are inverses is conducted graphically, in figures 4.8 and 4.9. □

This proof is quite magical — the reader is encouraged to try to make precise sense of figures 4.7 to 4.9 before reading Remark 4.2.26. Come back after the magic wears off!

In the name of clarity, we will now butcher the illusion.

Remark 4.2.26 — About the figures defining these maps.
Figure 4.7(b) can be misleading, if taken at face value. Most importantly, note that c here is NOT a cobordism! It is "empty space"; a placeholder indicating that something *should* go there.

Given any morphism $c\colon {}_AA_A \to {}_AA_A$, there may or may not be a cobordism $B\colon \mathrm{pt}_+ \times I \to \mathrm{pt}_+ \times I$ such that $Z(B) = c$; that is irrelevant for the argument. The important part is that the maps corresponding to the three cylinders $C_{\mathrm{in}} \times I$, $C_{\mathrm{out}} \times I$ and $\mathrm{pt}_+ \times I \times I$, as well as the map $\eta\colon \Bbbk \to V$ corresponding to the cap, are composable with c in the manner depicted by figure 1.5.

Remember that $V = N \otimes_{A\otimes B} M$. More explicitly, the map of figure 4.7(b) is

$$\begin{aligned} \mathrm{Hom}_{{}_A\mathsf{Bimod}_A}({}_AA_A, {}_AA_A) &\longrightarrow \mathrm{Hom}_{\mathsf{Vect}_\Bbbk}(\Bbbk, V) \\ c &\longmapsto (\mathrm{id}_N \otimes_{A\otimes B} (c \otimes \mathrm{id}_B) \otimes_{A\otimes B} \mathrm{id}_M) \circ \eta, \end{aligned}$$

and we then identify a morphism $\Bbbk \to V$ with its image on the unit 1.

Similarly, Figures 4.8 and 4.9 depict a *process* which one could carry in order to obtain the correct identifications. This is a bicategorical generalization of the graphical calculus we introduced in section 2.2.2.

The left-hand side of figure 4.8 is the composition

$$V \xrightarrow{f} \mathrm{Hom}_{{}_A\mathsf{Bimod}_A}({}_AA_A, {}_AA_A) \xrightarrow{g} V,$$

which we then identify with the identity $V \to V$.

The left-hand side of figure 4.9 is the inverse composition,

$$\mathrm{Hom}_{{}_A\mathsf{Bimod}_A}({}_AA_A, {}_AA_A) \xrightarrow{g} V \xrightarrow{f} \mathrm{Hom}_{{}_A\mathsf{Bimod}_A}({}_AA_A, {}_AA_A),$$

and the "homotopies" pictured are actually a process which first decomposes the cobordism of figure 4.7(a) in terms of the elementary generators, and then uses operations available in any bicategory to move the morphism $c\colon {}_AA_A \to {}_AA_A$ to the bottom. This basically uses the fact that the identity 2-morphisms commute with other 2-morphisms — we can "carve" a path of squares through which c can be safely transported to the bottom.

As mentioned, figure 4.7(a) is a cobordism and should be interpreted as such. This is obtained by gluing the generators as in figure 4.10. We omit a cylinder joining the "back" of the two cusps, for clarity — the two saddles are drawn directly over it.

We will not do this additional post-processing for the last two remaining proofs — hopefully the reader is now equipped with all the tools necessary for filling in the details.

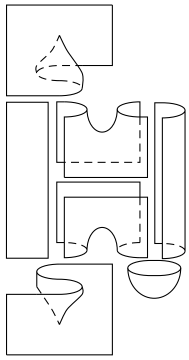


Figure 4.10: Decomposition of the cobordism in figure 4.7(a).

Separability

Lemma 4.2.27. *A is a separable algebra.*

Proof. In terms of the A-bicentral element $e = \sum_i a_i \otimes b_i$, the map of figure 4.7(a) is given by

$$[x] \otimes y \longmapsto \sum_i y a_i x b_i.$$

This map was used in order to define an isomorphism

$$V \longrightarrow \mathrm{Center}(A), \qquad [x] \longmapsto \sum_i a_i x b_i.$$

Now, since this is an isomorphism, there exists $[z] \in V$ such that $\sum_i a_i z b_i = 1$. Therefore, $\sum_i a_i \otimes (z b_i) \in A \otimes A$ is an A-central element realizing the definition of separable algebra [Def. 4.2.12]. □

Frobenius structure on the image

Lemma 4.2.28. *The A-bicentral element e given by the saddle, together with the linear functional $\varepsilon\colon A \to A/[A,A] \cong V \to \Bbbk$ induced by the cap, equip A with the structure of a symmetric Frobenius $\Bbbk$-algebra.*

Proof. We have already seen that ε factors through $A/[A,A]$ (and that e is A-bicentral), so the structure will be symmetric.

Let $e = \sum_i a_i \otimes b_i \in A \otimes A$. We can rewrite the 2D Morse relations of figure 4.5(d) as

$$\sum_i \varepsilon(a_i) b_i = 1, \qquad \sum_i a_i \varepsilon(b_i) = 1,$$

which are the two equations of Characterization 4.2.9. □

The non-degenerate trace $\varepsilon\colon A \to \Bbbk$ lets us identify $Z(\mathbb{S}^1)$ with $\mathrm{Center}(A)$ in a different way.

Remark 4.2.29 — An alternative way of identifying $Z(\mathbb{S}^1)$ with the center.
We already mentioned how we can identify $V \cong A \otimes_{A\otimes A^{\mathrm{op}}} A \cong A/[A,A]$. Now, the trace $\varepsilon\colon A \to \Bbbk$ induces a non-degenerate symmetric pairing $\beta\colon A \otimes A \to \Bbbk$, defined as $\langle x \mid y\rangle = \beta(x \otimes y) = \varepsilon(xy)$ [Char. 2.2.16]. This lets us define orthogonal subspaces:

$$[A,A]^{\perp} := \{x \in A \mid \langle x \mid y\rangle = 0 = \langle x \mid y\rangle \ \forall y \in [A,A]\}.$$

But then it is clear that $A/[A,A] \cong [A,A]^{\perp}$, and one can check that $[A,A]^{\perp} \cong \mathrm{Center}(A)$.

Yet another alternative is to directly decompose $A \cong \mathrm{Center}(A) \oplus [A,A]$, using the fact that A is a (strongly) separable algebra [Def. 4.2.12]; for more details, see [Hat65].

Not all TQFTs extend down

Given a $(0,1,2)$-TQFT $Z\colon \mathsf{Cob}_2^{\mathrm{or}}(2) \to \mathsf{Alg}_2$, we can recover a $(1,2)$-TQFT $\Omega Z\colon \mathsf{Cob}_2^{\mathrm{or}} \to \mathsf{Vect}_{\Bbbk}$ by applying loops; as we mentioned earlier, we can identify

$$\mathsf{Cob}_2^{\mathrm{or}} \simeq \Omega\,\mathsf{Cob}_2^{\mathrm{or}}(2) = \mathrm{Map}_{\mathsf{Cob}_2^{\mathrm{or}}(2)}(\varnothing,\varnothing), \qquad \mathsf{Vect}_{\Bbbk} \simeq \Omega\,\mathsf{Alg}_2 = \mathrm{Map}_{\mathsf{Alg}_2}(\Bbbk,\Bbbk).$$

The $(1,2)$-TQFT ΩZ is defined as one expects: for any closed 1-manifold M, we see it as a cobordism from $\varnothing$ to $\varnothing$ (in the unique way possible) and take the value of Z at that cobordism. Likewise for the 2-cobordisms between closed 1-manifolds.

Now, let A be the separable symmetric Frobenius algebra associated to the $(0,1,2)$-TQFT Z. What is the commutative Frobenius algebra corresponding to the $(1,2)$-TQFT ΩZ? Well, the underlying vector space must be the value of ΩZ at the circle, $(\Omega Z)(\mathbb{S}^1) := Z(\mathbb{S}^1)$. But as we just saw, we can identify this with the center of A, a *commutative* algebra. That identification extends to the other parts of the Frobenius structure (the (co)multiplication and (co)units), and as such gives a separable *commutative* Frobenius algebra. So we have recovered *some* of the classical correspondence of Theorem 2.3.3; namely, the restriction to $(1,2)$-TQFTs which can be extended down to points, which are in correspondence with *separable* commutative Frobenius algebras!

In view of this result, it is not difficult to explicitly construct a $(1,2)$-TQFT which doesn't extend down to a $(0,1,2)$-TQFT. In fact, we already discussed it — it was our very first example of a 2D TQFT.

Example 4.2.30 — Constructing a 2D TQFT which doesn't extend down to points.
As we've already mentioned, the $(1,2)$-TQFTs obtained by looping $(0,1,2)$-TQFTs correspond to *separable* commutative Frobenius algebras. Since separability implies semisimplicity (by taking the trivial field extension $\Bbbk \to \Bbbk$), in order to construct a 2D TQFT which can't be extended down to points it suffices to consider our favorite non-semisimple commutative Frobenius algebra.

Consider the 2D TQFT of Example 1.2.7, used in the definition of Khovanov homology. Per Example 2.2.27, this is the algebra $A = \Bbbk[t]/t^2$ equipped with Frobenius form $1 \mapsto 0$, $t \mapsto 1$. And this algebra has nilpotent elements, since $t^2 = 0$. In particular, it is not semisimple, so it can not extend down to points.

On the other hand, the 2D TQFT which detected genus [Ex. 1.2.8] corresponds to the separable Frobenius algebra $\Bbbk[t]/(t^2-1)$ (equipped with Frobenius form $1 \mapsto 1$, $t \mapsto 0$), so it does extend down to points.

So the classical correspondence does in fact classify strictly more 2D TQFTs than the Cobordism Hypothesis! But of course, it is not at all trivial to classify general $(n-1,n)$-TQFTs — this is an open problem for dimensions as low as $n = 4$.

There are also uniqueness issues.[6]

Remark 4.2.31 — Two $(0,1,2)$-TQFTs with the same underlying $(1,2)$-TQFT.
Consider the Frobenius $\mathbb{R}$-algebra $\mathbb{R}$, with trace $\mathrm{id}\colon \mathbb{R} \to \mathbb{R}$ given by the identity. Consider also the Frobenius $\mathbb{R}$-algebra given by the quaternions $\mathbb{H}$, with trace $\varepsilon\colon \mathbb{H} \to \mathbb{R}$ given by taking the real part.

Both of these are semisimple symmetric Frobenius algebras with the same center $(\mathbb{R}, \mathrm{id})$, so they determine two oriented $(0,1,2)$-TQFTs such that taking loops yields the same oriented $(1,2)$-TQFT. Since $\mathbb{R}$ is not Morita equivalent to $\mathbb{H}$ (although proving so is out of our scope), the two $(0,1,2)$-TQFTs are not isomorphic.

By contrasting the Cobordism Hypothesis with the classical "folklore" Theorem 2.3.3, we hope to shed light on what the intrinsic (∞,n)-categorical nature of the former means, in practice. It is a wonderful result, even one that might seem reasonably obvious to conjecture after proper motivation: having the colossal amount of data necessary to define a fully extended TQFT

[6] This example is also from Chris Schommer-Pries, on MathOverflow: https://mathoverflow.net/a/165891.

constrained to another colossally big amount of coherence relations means that fully extended TQFTs are actually extremely simple to describe. The fact that they are determined by their values on a single point could lead one to believe that these aren't particularly useful for applications of the theory, and that the TQFTs one might encounter "in the wild" are often *intrinsically* lower-categorical in nature — as was the case for the TQFT behind Khovanov homology, in Example 4.2.30. But this is not exactly true, in higher dimensions.

Remark 4.2.32 — The models of 3D TQFTs.
Most known examples of 3D TQFTs are constructed in one of two ways: as Turaev–Viro theories, or as Reshetikhin–Turaev theories. The latter includes the seminar Chern–Simons theories, which Witten famously showed can be used to recover the Jones polynomial of a knot [Wit89].

Lurie's groundbreaking work on the Cobordism Hypothesis brought new life into an already open question — that of determining which TQFTs extend down to points. Both Turaev–Viro and Reshetikhin–Turaev theories have been shown to be extendable to $(1,2,3)$-TQFTs. And both Turaev–Viro theories and the particular case of Chern–Simons theories have long been *conjectured* to be extendable to points — that is, to be examples of $(0,1,2,3)$-TQFTs. There has been tremendous progress on this front, but some necessary conjectures still remain unproven. For more information, see [Fre09; FHLT10; Hen17].

Take care that we have only discussed "pure" TQFTs, while a lot of interesting examples in the literature come from TQFTs accepting *anomalies* or satisfying weaker algebraic conditions (such as *lax monoidality*, instead of strong monoidality).

Final comments

Before finally ending chapter 4 — and with it, the main portion of this thesis — let us quickly comment on the other two kinds of 2D TQFTs we met along our journey.

Remark 4.2.33 — Unoriented and framed 2D extended TQFTs.
Fully dualizable objects in Alg_2 correspond to separable algebras; see [Sch14, Def. 3.70]. As mentioned, the homotopy fixed points of the $\mathrm{SO}(2)$ action on $\mathrm{Core}({\mathsf{Alg}_2}^{\mathrm{fd}})$ are separable algebras equipped with a symmetric Frobenius structure. Finally, homotopy fixed points of the $\mathrm{O}(2)$ action correspond to separable symmetric Frobenius algebras further equipped with an *stellar* structure; see [Sch14, § 3.8.6].

In summary:

- Framed TQFTs $Z\colon \mathsf{Cob}_2^{\mathrm{fr}}(2) \to \mathsf{Alg}_2$ are separable algebras.
- Oriented TQFTs $Z\colon \mathsf{Cob}_2^{\mathrm{or}}(2) \to \mathsf{Alg}_2$ are separable symmetric Frobenius algebras.
- Unoriented TQFTs $Z\colon \mathsf{Cob}_2^{\mathrm{un}}(2) \to \mathsf{Alg}_2$ are separable stellar symmetric Frobenius algebras.

For further details, we refer to [Sch14; Hes17; HSV17].

Remark 4.2.34 — About stellar algebras.
Stellar algebras, defined in [Sch14, Def. 3.78], are a Morita-invariant version of the $*$-algebras of Remark 2.3.18: an algebra A equipped with a Morita context to its opposite algebra A^{op}, denoted $s = ({}_{A^{\mathrm{op}}}M_A, {}_AN_{A^{\mathrm{op}}}, u, v)$, together with an isomorphism of Morita contexts $\sigma\colon s \cong \underline{s}$ such that $\sigma \circ \underline{\sigma}$ is the identity map.

Here the underlines denote *conjugation*: flipping the left- and right-actions to obtain the opposite bimodule, as in Notation 4.2.4. In particular, $\underline{s} = ({}_A\underline{M}_{A^{\mathrm{op}}}, {}_{A^{\mathrm{op}}}\underline{N}_A, \underline{u}, \underline{v})$ is a Morita context from A^{op} to A.

Stellar algebras must satisfy a compatibility condition with the symmetric Frobenius structure, although this can be deduced from the other relations. See [Sch14, fig. 3.29].

This ends our main tale. Now the only thing remaining is to properly define (∞, n)-categories; this will be our Coda. This is a lengthy appendix discussing a concrete model for (∞, n)-categories, *n-fold complete Segal spaces*, as well as a concrete construction of the (∞, n)-category of cobordisms Bord_n. We will take our shared dreams and our guiding hypotheses and put them in the crucible, heated by the fire of intuition, so as to form a strong alloy. After a long process of persistent and rigorous hammering, the resulting metal will be forged into a sharp and tangible reality. This is an arduous process, and one must endure the temptation of trying to end it quickly and haphazardly. Insufficient hammering produces a frail blade — a great decoration, or even an intimidating prop, but unfit as a tool for cutting or slashing. After having endured so much, surely the reader can withstand a little heat.

Coda

Some technical details about (∞, n)-categories

> We have reached a pinnacle of abstraction. Or perhaps the heart of abstraction, or its deepest roots. Perhaps the pinnacle and the heart and the deepest roots are the same, and that, to me, is the joy of abstraction.
>
> **Eugenia Cheng, "The Joy of Abstraction".**

This Coda explores some technical constructions which were originally a part of chapter 3, closely following the order of [Lur09]. Lurie used them extensively, of course, but we did not — as such, they broke the flow of the text, and had to be cut for clarity. However, these constructions are interesting and relevant enough to warrant inclusion, so they were reworked into an appendix of sorts. The reason for the unusual naming convention is simple; merely naming this "Appendix" would make more readers want to skip it, thinking that it was written in a more terse and direct style than the main text — as appendices often are. That is not the case here. Of course, readers are still free to skip it, should they desire to do so! We hold no hostages here.

C.1 (∞, n)-categories as n-fold complete Segal spaces

This section is a crash course on one model of (∞, n)-categories: complete n-fold Segal spaces. We loosely follow [Lur09, § 2.1], and primarily focus on the case $n = 1$.

The core lesson of higher category theory is the same as that of weak monoidal categories [Def. 1.2.11]: it is often useful to replace the strict commutativity of a diagram of n-morphisms with the existence of a *witness* $(n+1)$-morphism which makes the diagram commute, and these $(n+1)$-morphisms should themselves satisfy coherence relations expressed by $(n+2)$-morphisms. Since there is no limit to the height one can reach in such a manner, we need a better way of determining the structural morphisms, lest we be unable to define any concrete examples whatsoever.

That's the function a good model of (∞, n)-categories should serve; it should make obvious to prove the things which are obvious to state. It is the opinion of some mathematicians that determining whether that "good model" actually exists is still an open problem, since every known model proves itself inconvenient in the long run — for one reason or another. We can at least hope to give a good approximation.

First let us see another way of defining a plain 1-category, which we will then generalize for higher dimensions. This uses the language of simplicial sets.

Simplicial sets as functors from Δ^{op}

Remember, from Definition 2.3.9, the definition of the simplex category Δ. We will finally talk about its namesake: simplicial sets.

Definition C.1.1. A ***simplicial object*** in a category $\mathcal{C}$ is a (contravariant) functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. A ***simplicial set*** is a simplicial object in Set.

Notation C.1.2. We denote a simplicial object as $S_\bullet \colon \Delta^{\mathrm{op}} \to \mathcal{C}$, and the image of each $[\boldsymbol{n}] \in \Delta$ as $S_n \in \mathcal{C}$. In the case $\mathcal{C} = \mathrm{Set}$, the elements of S_n are called *n-simplices* or n-cells.

The usual definitions of simplicial sets describe them in term of the "layers" (or *levels*) S_n and of *face* and *degeneracy* maps $d_i \colon S_n \to S_{n-1}$ and $s_j \colon S_n \to S_{n+1}$ satisfying certain combinatorial axioms. We won't go into detail about the axioms these must satisfy, but it is useful to see how the face and degeneracy maps are hidden in our definition. The geometric picture is very important for developing intuition!

Remark C.1.3 — Face and degeneracy maps.
We should first give a presentation of Δ. The generators of this presentation will be mapped into the face and degeneracy maps of each simplicial set. Hence, we will call these *coface* and *codegeneracy* maps. We use Greek letters to distinguish them.

Let $\delta_i^n \colon [\boldsymbol{n-1}] \to [\boldsymbol{n}]$ be the coface map described by

$$\{0, 1, \dots, n-1\} \longmapsto \{0, 1, \dots, i-1, i+1, \dots, n\}.$$

That is, it is the injection $[\boldsymbol{n-1}] \hookrightarrow [\boldsymbol{n}]$ whose image leaves out i. On the other hand, the degeneracy maps $\sigma_i^n \colon [\boldsymbol{n+1}] \to [\boldsymbol{n}]$ are obtained by repeating the ith index:

$$\{0, 1, \dots, n+1\} \longmapsto \{0, 1, \dots, i-1, i, i, i+1, \dots, n\}.$$

The relations these satisfy can be summarized as follows.

- $\delta_i^{n+1} \circ \delta_j^n = \delta_{j+1}^{n+1} \circ \delta_i^n, \qquad i \le j.$
- $\sigma_j^n \circ \sigma_i^{n+1} = \sigma_i^n \circ \sigma_{j+1}^{n+1}, \qquad i \le j.$
- $\sigma_j^n \circ \delta_i^{n+1} = \begin{cases} \delta_i^n \circ \sigma_{j-1}^{n-1}, & i < j; \\ \mathrm{id}_n, & j \le i \le j+1; \\ \delta_{i-1}^n \circ \sigma_j^{n-1}, & j+1 < i. \end{cases}$

Every morphism $[\boldsymbol{n}] \to [\boldsymbol{m}]$ factorizes as a surjection followed by an injection, which in turn are composition of codegeneracy and coface maps. In other words, we have described generators and relations for the simplicial category Δ.

Therefore, each functor $S_\bullet \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ is determined by the images of these maps. The images $d_i^n := S_\bullet(\delta_i^n) \colon S_i \to S_{i-1}$ are the face maps, while the images $s_i^n := S_\bullet(\sigma_i^n) \colon S_i \to S_{i+1}$ are the degeneracy maps. These satisfy the usual simplicial identities:

- $d_j^n \circ d_i^{n+1} = d_i^n \circ d_{j+1}^{n+1}, \qquad i \le j.$
- $s_i^{n+1} \circ s_j^n = s_{j+1}^{n+1} \circ s_i^n, \qquad i \le j.$
- $d_i^{n+1} \circ s_j^n = \begin{cases} s_{j-1}^{n-1} \circ d_i^n, & i < j; \\ \mathrm{id}_n, & j \le i \le j+1; \\ s_j^{n-1} \circ d_{i-1}^n, & j+1 < i. \end{cases}$

If $S_\bullet \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ is a simplicial set, we picture each element $x \in S_n$ as a geometric n-simplex. For $n = 1$ this is a line segment, for $n = 2$ a filled triangle, and for $n = 3$ a filled tetrahedron. See a depiction of the face maps in figure C.1. We think of degeneracy maps as mapping (say) a line segment to a degenerate triangle — one which has two vertices at the same position, and an edge of length zero between them. This is graphically indistinguishable from the original line segment, so we will not try to depict it.

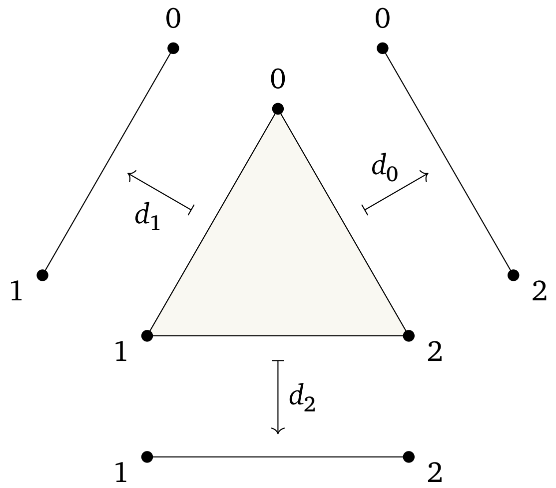


Figure C.1: A 2-simplex with face maps drawn.

The nerve of a category

We can assign a simplicial set to every category, called its *nerve*, which encodes the composition of morphisms.

Definition C.1.4. The ***nerve*** of a category $\mathcal{C}$ is the simplicial set $\mathrm{N}(\mathcal{C})_\bullet = \mathrm{Fun}(-, \mathcal{C})$.

In particular, the set of n-simplices is $\mathrm{N}(\mathcal{C})_n = \mathrm{Fun}([\boldsymbol{n}], \mathcal{C})$. Here we are viewing each partially ordered set $[\boldsymbol{n}] = \{0, 1, \ldots, n\}$ as a category; there is a unique arrow $i \to j$ if and only if $i \leq j$, and there are no other arrows. We can draw this category in the following way, omitting composite arrows and identities:

$$0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n.$$

With this notation, $[\mathbf{0}]$ is isomorphic to $\mathbb{1}$, the category with a single object and only the identity arrow (used, for example, when defining strict monoidal categories in Definition 1.2.10).

We can identify a functor $[\boldsymbol{n}] = \{0 \to 1 \to \cdots \to n\} \longrightarrow \mathcal{C}$ with its image; that is, a selection of n composable morphisms in $\mathcal{C}$. Under this identification, we can give a more detailed description of the nerve.

Remark C.1.5 — A more detailed description of the nerve.
First, the sets of n-simplices.

- The 0-simplices in $\mathrm{N}(\mathcal{C})_0$ are *objects* $X_0 \in \mathcal{C}$.
- The 1-simplices in $\mathrm{N}(\mathcal{C})_1$ are *morphisms* $X_0 \xrightarrow{f_1} X_1$.
- The 2-simplices in $\mathrm{N}(\mathcal{C})_2$ are *pairs of composable morphisms* $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2$.

$\vdots$

- The n-simplices in $\mathrm{N}(\mathcal{C})_n$ are *n-tuples of composable morphisms* $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n$.

$\vdots$

Now, it remains to describe the structural maps.

The face maps d_i are *composition of two adjacent morphisms*:

$$
\begin{array}{ccc}
\mathrm{N}(\mathcal{C})_n & & (X_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} X_{i-1} \xrightarrow{f_i} X_i \xrightarrow{f_{i+1}} X_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} X_n) \\
{\scriptstyle d_i}\big\downarrow & & \big\downarrow \\
\mathrm{N}(\mathcal{C})_{n-1} & & (X_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} X_{i-1} \xrightarrow{f_{i+1} \circ f_i} X_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} X_n).
\end{array}
$$

…Except for the cases $i = 0$ or n, which just remove the first or last morphism respectively:

$$
\begin{array}{ccc}
(X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n) & & (X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n) \\
{\scriptstyle d_0}\big\downarrow & & \big\downarrow{\scriptstyle d_n} \\
(X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n), & & (X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1}).
\end{array}
$$

The degeneracy maps s_j are *insertion of identities*:

$$
\begin{array}{ccc}
\mathrm{N}(\mathcal{C})_n & & (X_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_j} X_j \xrightarrow{f_{j+1}} X_{j+1} \xrightarrow{f_{j+2}} \cdots \xrightarrow{f_n} X_n) \\
{\scriptstyle s_j}\big\downarrow & & \big\downarrow \\
\mathrm{N}(\mathcal{C})_{n+1} & & (X_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_j} X_j \xrightarrow{\mathrm{id}} X_j \xrightarrow{f_{j+1}} X_{j+1} \xrightarrow{f_{j+2}} \cdots \xrightarrow{f_n} X_n).
\end{array}
$$

It seems natural that all relevant information about our category is encoded in its nerve, so we can conjecture that there is a correspondence between (some subset of) simplicial sets and (small) categories. We can make this precise.

Recovering a category from its nerve

Let $\mathcal{C}$ be a category, and let $\mathrm{N}(\mathcal{C})_\bullet$ be its nerve. As we have already seen, we can identify the 0-simplices in $\mathrm{N}(\mathcal{C})_0$ with the objects of $\mathcal{C}$, while the morphisms are the 1-simplices in $\mathrm{N}(\mathcal{C})_1$. We want to recover our composition law and identities, which we previously encoded into the face and degeneracy maps.

The identities are determined by the unique degeneracy map of the 0-simplices:

$$s_0^0 \colon \mathrm{N}(\mathcal{C})_0 \ni X \longmapsto (X \xrightarrow{\mathrm{id}} X) \in \mathrm{N}(\mathcal{C})_1.$$

Now, for the composition. Writing down some of the face maps, we reach the following square.

$$
\begin{array}{ccc}
\mathrm{N}(\mathcal{C})_2 & \xrightarrow{d_2^2} & \mathrm{N}(\mathcal{C})_1 \\
{\scriptstyle d_0^2}\big\downarrow & & \big\downarrow{\scriptstyle d_0^1} \\
\mathrm{N}(\mathcal{C})_1 & \xrightarrow[d_1^1]{} & \mathrm{N}(\mathcal{C})_0
\end{array}
$$

This square commutes, and is in fact a fiber product (or pullback). The top path maps

$$(X \xrightarrow{f} Y \xrightarrow{g} Z) \longmapsto (X \xrightarrow{f} Y) \longmapsto Y,$$

while the bottom path maps

$$(X \xrightarrow{f} Y \xrightarrow{g} Z) \longmapsto (Y \xrightarrow{g} Z) \longmapsto Y.$$

Note that the fiber product is just

$$\begin{aligned}
\mathrm{N}(\mathcal{C})_1 \times_{\mathrm{N}(\mathcal{C})_0} \mathrm{N}(\mathcal{C})_1 &\coloneqq \{(X \xrightarrow{f} Y, W \xrightarrow{g} Z) \in \mathrm{N}(\mathcal{C})_1 \times \mathrm{N}(\mathcal{C})_1 \mid Y = d_0^1(f) = d_1^1(g) = W\} \\
&= \{(X \xrightarrow{f} Y,\ Y \xrightarrow{g} Z) \in \mathrm{N}(\mathcal{C})_1 \times \mathrm{N}(\mathcal{C})_1\},
\end{aligned}$$

and the fact that the commutative square is a pullback means that we have a bijection of sets

$$\phi\colon \mathrm{N}(\mathcal{C})_2 \xrightarrow{\cong} \mathrm{N}(\mathcal{C})_1 \times_{\mathrm{N}(\mathcal{C})_0} \mathrm{N}(\mathcal{C})_1, \qquad (X \xrightarrow{f} Y \xrightarrow{g} Z) \longmapsto (X \xrightarrow{f} Y, Y \xrightarrow{g} Z).$$

This is just a more formal restatement of our previous discussion: two maps f and g are composable (in that order) if and only if the target of the former coincides with the source of the latter. In the same manner, we can identify $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \{X\} \times_{\mathrm{N}(\mathcal{C})_0} \mathrm{N}(\mathcal{C})_1 \times_{\mathrm{N}(\mathcal{C})_0} \{Y\}$.

Now, to recover the composition of our category, we map a pair of composable maps through

$$\begin{array}{ccccc}
\mathrm{N}(\mathcal{C})_1 \times_{\mathrm{N}(\mathcal{C})_0} \mathrm{N}(\mathcal{C})_1 & \xrightarrow{\ \phi^{-1}\ } & \mathrm{N}(\mathcal{C})_2 & \xrightarrow{\ d_1^2\ } & \mathrm{N}(\mathcal{C})_1 \\
(X \xrightarrow{f} Y, Y \xrightarrow{g} Z) & \longmapsto & (X \xrightarrow{f} Y \xrightarrow{g} Z) & \longmapsto & (X \xrightarrow{g \circ f} Z).
\end{array}$$

Remark C.1.6 — Geometric interpretation.
This process admits a geometric interpretation, which is at the core of several different models of higher categories. We can draw a 2-simplex $X \xrightarrow{f} Y \xrightarrow{g} Z$ as a triangle where each edge is the image of the corresponding face map:

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & Z.
\end{tikzcd}$$

So composition of arrows corresponds to 2-cells in the nerve: it is a way of filling up the *horn*

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] & & Z.
\end{tikzcd}$$

For 1-categories, each horn determines a composition uniquely: the only morphism $g \circ f$ such that the resulting diagram commutes. The main motivating principle behind complete Segal spaces — and other geometric models of ∞-categories, such as weak Kan complexes — will be admitting several such 2-cells to serve as a composition of two 1-morphisms. The resulting diagram will commute *up to witness 2-morphism* — and said 2-morphism is precisely identified with the 2-cell expressing the diagram!

Notice that we have found a necessary condition for a simplicial set $S_\bullet$ to be isomorphic to the nerve of a category: the diagram

$$\begin{tikzcd}
S_2 \arrow[r, "d_2^2"] \arrow[d, "d_0^2"'] & S_1 \arrow[d, "d_0^1"] \\
S_1 \arrow[r, "d_1^1"'] & S_0
\end{tikzcd}$$

must be a pullback. We can give a necessary and sufficient condition, but we must first define some notation.

Notation C.1.7. Let $0 \leq i_0 \leq i_1 \leq \cdots \leq i_m \leq n$ be a non-decreasing tuple of integers. We define the map $p_{i_0,i_1,\dots,i_m} \colon [\boldsymbol{m}] \to [\boldsymbol{n}]$ by mapping $j \mapsto i_j$. This induces maps on each simplicial set $S_\bullet \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$, by functoriality:

$$(p^*_{i_0,i_1,\dots,i_m} \colon S_n \longrightarrow S_m) := (S_\bullet p_{i_0,i_1,\dots,i_m} \colon S_\bullet[\boldsymbol{n}] \longrightarrow S_\bullet[\boldsymbol{m}]).$$

In particular, we can write our face maps as $d_i^n := p^*_{0,\dots,\hat{i},\dots,n}$, where $\hat{i}$ denotes an omitted index.

Now we state the following characterization, without proof. It is not too difficult to convince oneself of this fact.

Characterization C.1.8. A simplicial set $S_\bullet$ is isomorphic to the nerve of a category if and only if the following diagram is a pullback for each pair of integers n, $m \geq 0$.

$$\begin{tikzcd} S_{m+n} \arrow[r, "p^*_{0,1,\dots,m}"] \arrow[d, "p^*_{m,m+1,\dots,m+n}"'] & S_m \arrow[d, "p^*_m"] \\ S_n \arrow[r, "p^*_0"'] & S_0 \end{tikzcd} \tag{C.5}$$

That is, if and only is the induced maps $S_{m+n} \to S_m \times_{S_0} S_n$ are bijective. This encodes the fact that, for the nerve of a category, giving an $(m+n)$-tuple of composable maps

$$X_0 \xrightarrow{f_1} \cdots \xrightarrow{f_m} X_m \xrightarrow{f_{m+1}} \cdots \xrightarrow{f_n} X_n$$

is the same as giving an m-tuple and an n-tuple of composable maps

$$X_0 \xrightarrow{f_1} \cdots \xrightarrow{f_m} X_m \qquad X_m \xrightarrow{f_{m+1}} \cdots \xrightarrow{f_n} X_n$$

such that the final object of the first is the same as the first object of the second.

This, in turn, is equivalent to requiring each map $S_n \to S_1 \times_{S_0} \cdots \times_{S_0} S_1$ to be a bijection, where this is the map induced by the functions $g_1, \dots, g_n$ given by

$$g_i \colon [\boldsymbol{1}] \longrightarrow [\boldsymbol{n}], \quad (0 \to 1) \longmapsto (i-1 \to i).$$

We have been using these maps implicitly, all along, in order to identify each n-cell of a nerve $\mathrm{N}(\mathcal{C})_\bullet$ as an n-tuple of composable maps!

In summary:

Definition C.1.9. Let $S_\bullet$ be a simplicial set such that all the diagrams (C.5) are pullbacks, and let $\psi \colon S_1 \times_{S_0} S_1 \to S_2$ be the isomorphism induced by the pullback for $m = n = 1$. We define a category $|S_\bullet|_{\mathsf{Cat}}$, the ***categorical realization*** of $S_\bullet$, as follows.

- **Objects:** Elements of S_0.
- **Morphisms:** For each pair of objects $X, Y \in S_0$, we define $\mathrm{Hom}_{\mathcal{C}}(X,Y) = \{X\} \times_{S_0} S_1 \times_{S_0} \{Y\}$.
- **Composition:** For each triple of objects $X, Y, Z \in S_0$, the composition law

$$\circ_{XYZ} \colon \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

is defined as the function of sets

$$\begin{aligned}
(\{Y\} \times_{S_0} S_1 \times_{S_0} \{Z\}) \times (\{X\} \times_{S_0} S_1 \times_{S_0} \{Y\}) &\overset{\tau}{\simeq} (\{X\} \times_{S_0} S_1 \times_{S_0} \{Y\}) \times (\{Y\} \times_{S_0} S_1 \times_{S_0} \{Z\}) \\
&\simeq \{X\} \times_{S_0} S_1 \times_{S_0} S_1 \times_{S_0} \{Z\} \\
&\overset{\psi}{\simeq} \{X\} \times_{S_0} S_2 \times_{S_0} \{Z\} \\
&\xrightarrow{d_1^2} \{X\} \times_{S_0} S_1 \times_{S_0} \{Z\}.
\end{aligned}$$

- **Identities:** For each object $X \in S_0$, we define $\mathrm{id}_X := s_0(X) \in S_1$.

If our simplicial set $S_\bullet$ was the nerve of a category $\mathcal{C}$, the category $|\mathrm{N}(\mathcal{C})_\bullet|_{\mathsf{Cat}}$ obtained this way is isomorphic to $\mathcal{C}$. And viceversa: if we first construct a category from $S_\bullet$ and then take its nerve, the simplicial set $\mathrm{N}(|S_\bullet|_{\mathsf{Cat}})_\bullet$ obtained will be isomorphic to $S_\bullet$.

This gives an adjunction between categories and simplicial sets,[1]

$$\mathsf{Cat} \underset{\mathrm{N}(-)_\bullet}{\overset{|-|_{\mathsf{Cat}}}{\leftrightarrows}} \mathsf{sSet}.$$

Even more: the nerve functor is full and faithful, so Cat is equivalent to its essential image! For more information, see [Kerodon, Tag 002L].

Note that all information above layer 2 is redundant; we use only properties of $[\mathbf{0}]$, $[\mathbf{1}]$ and $[\mathbf{2}]$. So we could change the definition of the nerve to yield a simplicial complex such that all layers above the second are trivial, and this would still be an adjunction. Of course, this would require modifying Characterization C.1.8.

The fact that Cat is equivalent to its essential image by $\mathrm{N}(-)_\bullet$ means that there is no harm in identifying a category with its nerve. So we can develop 1-category theory as a theory of simplicial sets satisfying Characterization C.1.8; this characterization can be used as a *definition*. Even more: we can *define* a category as a simplicial set $S_\bullet$ such that all the diagrams (C.5) are pullbacks. That is what we will do with (∞, n)-categories: even if we haven't defined what an (∞, n)-category *is*, we intuitively know what its nerve *should* be. We will use that knowledge to forge an actual definition.

The nerve of an $(\infty, 1)$-category

Say we had an object $\mathcal{C}$ which we want to call an $(\infty, 1)$-category, for some definition of that. What should its nerve be? Remember that for 1-categories we had defined $\mathrm{N}(\mathcal{C})_n = \mathrm{Fun}([\boldsymbol{n}], \mathcal{C})$, before identifying these with tuples of composable morphisms. In the case of $(\infty, 1)$-categories, we can see the 1-category $[\boldsymbol{n}]$ as an $(\infty, 1)$-category where all k-morphisms for $k \geq 2$ are identities; then, each of these functor categories is an $(\infty, 1)$-category in its own right. We can then discard all 1-morphisms which are not invertible, and we get an $(\infty, 0)$-category. And we have a model for these: these are topological spaces, per Thesis I.1.7, identified with their fundamental ∞-groupoid. So the nerve of an $(\infty, 1)$-category should not be a simplicial *set*, but a simplicial *space*: a simplicial object in Top, by definition a functor $\Delta^{\mathrm{op}} \to \mathsf{Top}$.

Remark C.1.10 — About simplicial spaces.
Some authors define simplicial spaces as simplicial objects in sSet, the category of simplicial sets; or in an appropriate subcategory of sSet, such as (strong) Kan complexes. This is perhaps a more suitable definition, since the combinatorial nature of simplicial sets oftentimes makes them easier to work with than topological spaces. However, since every topological space is weak homotopy equivalent to (the realization of) a simplicial set, this approach is not fundamentally different from ours.

[1] We have only described the functor $\mathsf{sSet} \to \mathsf{Cat}$ for simplicial sets satisfying Characterization C.1.8. The general construction takes the *homotopy category* $\mathrm{h}S_\bullet$ of each simplicial set $S_\bullet$; see [Kerodon, Tag 004M].

This is the classical Quillen equivalence, which we sketch below. To each topological space $X \in \mathsf{Top}$ we associate its *singular complex* $\mathrm{Sing}_\bullet X$ given by $\mathrm{Sing}_n X := \mathrm{Hom}_{\mathsf{Top}}(\Delta^n, X)$, where

$$\Delta^n := \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}_{\geq 0} \mid x_0 + \cdots + x_n = 1\}$$

is the topological n-simplex. This defines a functor $\mathrm{Sing}_\bullet \colon \mathsf{Top} \to \mathsf{sSet}$, which has a left adjoint $|-| \colon \mathsf{sSet} \to \mathsf{Top}$ called the *geometric realization* functor. As the name implies, this geometric realization is obtained by taking a disjoint union of topological simplices Δ^n, one for each n-cell in our simplicial set $S_\bullet$, and gluing them using the information of the face and degeneracy maps.

The important aspect is that the counit $|\mathrm{Sing}_\bullet X| \to X$ is a weak homotopy equivalence, so we do not lose homotopy information. So one can develop classical homotopy theory without using topological spaces at all, replacing them by simplicial sets. These are purely combinatorial in nature! And, in particular, this lets us identify an $(\infty, 0)$-category with a simplicial set, if we so desire; this approach is taken in [HTT; Kerodon], among others, which define ∞-groupoids as (strong) Kan complexes and $(\infty, 1)$-categories as weak Kan complexes. Both of these are particularly well-behaved simplicial sets, satisfying a horn-filling property akin to Remark C.1.6 [Kerodon, Tag 0039].

The sSet approach requires more work in the initial definitions, but it simplifies proofs about the properties of $(\infty, 1)$-categories. Since we will not prove much ourselves, we have opted for the conceptually easier approach of Top.

Remark C.1.11 — The classifying space of a category.
The geometric realization of the nerve of a category, $|\mathrm{N}(\mathcal{C})_\bullet|$, is called its *classifying space* $B\mathcal{C}$. We mentioned it (for 1-groupoids) in section I.1. If we view a group as a one-object groupoid, this process recovers the usual classifying space; cf. Remark 4.1.8.

We define $\mathrm{N}^\infty(\mathcal{C})_\bullet \colon \Delta^{\mathrm{op}} \to \mathsf{Top}$ by assigning to $\mathrm{N}^\infty(\mathcal{C})_n$ the topological space obtained by the process outlined above. It is not too difficult to see that this is indeed a functor, even though we haven't described its action on morphisms. We use a different notation for this "∞-nerve", since — for 1-categories — it does not coincide with the nerve of Definition C.1.4.

In any case, the simplicial space $\mathrm{N}^\infty(\mathcal{C})_\bullet$ we have defined should behave similarly to the usual nerve $\mathrm{N}(\mathcal{C})_\bullet$ of a 1-category. We are mostly interested in obtaining a condition similar to Characterization C.1.8, so that we can use that condition in order to *define* what an $(\infty, 1)$-category is. But as always, strict commutativity of those diagrams — in this case, of the diagrams expressing the universal property of the pullback — is too rigid of a condition for our purposes. Thankfully, since we now have topological spaces instead of mere sets, we can borrow the tools from homotopy theory in order to weaken it.

Homotopy fiber products

Definition C.1.12. Let us consider the diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$ of continuous functions between topological spaces. The ***homotopy fiber product*** of this diagram is the topological space

$$X \times^{\mathrm{R}}_Z Y := X \times_Z Z^I \times_Z Y = \{(x, \gamma, y) \in X \times Z^I \times Y \mid \gamma_0 = f(x), \gamma_1 = g(y)\},$$

where $Z^I = \{\gamma \colon I \to Z \mid \gamma \text{ continuous}\}$ is equipped with the compact-open topology.

That is, it consists of paths in Z along with the preimages of a starting point $x \mapsto f(x) = \gamma_0$ and a final point $y \mapsto g(y) = \gamma_1$. Why is it defined in this way? Well, that requires a bit of a digression. But that digression should be helpful in order to better understand the philosophy behind numerous constructions in (∞, n)-category theory — including the homotopy fixed points of Definition 4.1.17.

Remark C.1.13 — Digression on fibrations.
We can distinguish three particularly nice kinds of maps in Top: *weak (homotopy) equivalences*, *fibrations* and *cofibrations*. We already know the first, and the third can be summarized as "dual to fibrations", so we will only define the second.

A map $E \xrightarrow{p} B$ is called a ***fibration*** if it satisfies the *homotopy lifting property* for any space X: for every homotopy $H\colon X \times I \to B$ and every map $\varphi\colon X \to E$ lifting $H(-,0)$ (i.e., such that $\varphi = p \circ \tilde{H}(-,0)$), there exists a homotopy $\tilde{H}\colon X \times I \to E$ lifting H (i.e., $H = p \circ \tilde{H}$) and extending φ (i.e., such that $\tilde{H}(-,0) = \varphi$). This generalizes the homotopy lifting property for paths, where $X = \{0\}$. Note, however, that the lift needs not be unique. In diagrammatic terms:

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\varphi"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B. \end{tikzcd}$$

What is interesting is than, in Top, we can replace any map $X \xrightarrow{f} Z$ with the composition of a weak equivalence and a fibration: we factorize $X \xhookrightarrow{\iota_f} P_f \xrightarrow{p_f} Z$, as follows.

- P_f is the *mapping path space* $P_f \coloneqq \{(x,\gamma) \in X \times Z^I \mid \gamma_0 = f(x)\}$.
- $\iota_f\colon X \hookrightarrow P_f$ is the inclusion $x \mapsto (x, \mathrm{c}_{f(x)})$, where $\mathrm{c}_{f(x)}\colon I \to Z$ is the constant path at $f(x)$.
- $p_f\colon P_f \twoheadrightarrow Z$ is the fibration given by $(x,\gamma) \mapsto \gamma_1$.

We can also do the same with cofibrations; any map $X \xrightarrow{f} Z$ can be replaced with a cofibration followed by a weak equivalence. This endows Top with the structure of a *model category*, which we won't treat here. Roughly, these are interesting because most arguments one carries in classical homotopy theory can be carried in any other model category; for instance, sSet can be given the structure of a model category.

Now we can more naturally reach Definition C.1.12.

Remark C.1.14 — Defining the homotopy fiber product.
Consider a diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$ of continuous maps between topological spaces.

Remember that the pullback satisfies the following universal property: for each diagram $X \xleftarrow{p_1} W \xrightarrow{p_2} Y$ such that $f \circ p_1 = g \circ p_2$, there exists a unique morphism $u\colon X \times_Z Y$ making the following diagram commute.

$$\begin{tikzcd} W \arrow[dr, dashed, "u"] \arrow[drr, bend left, "p_1"] \arrow[ddr, bend right, "p_2"'] & & \\ & X \times_Z Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"'] & Z. \end{tikzcd}$$

Equivalently, $X \times_Z Y$ is the limit over the diagram $X \xrightarrow{f} Z \xleftarrow{g} Y$.

We want to take the *homotopy limit* instead: we want this universal property to be satisfied *up to homotopy*. This is so as to make it *homotopy invariant*.

For that, we replace f and g with compositions of weak equivalences and fibrations (via the path space fibration of Remark C.1.13), and take the usual "strict" pullback over the fibrations.

That is to say:

$$\begin{tikzcd}
 & & X \arrow[d, hook, "\iota_f", "\simeq"'] \\
 & X \times^{\mathrm{R}}_{Z} Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & P_f \arrow[d, two heads, "p_f"] \\
Y \arrow[r, hook, "\simeq", "\iota_g"'] & P_g \arrow[r, two heads, "p_g"'] & Z.
\end{tikzcd}$$

Note that, by the universal properties in play, there is a canonical map $X \times_Z Y \to X \times^{\mathrm{R}}_{Z} Y$. This is easier to see if we assemble everything into a big diagram:

$$\begin{tikzcd}
X \times_Z Y \arrow[rr] \arrow[dd] \arrow[dr, dashed] \arrow[drr, dotted, bend left] \arrow[ddr, dotted, bend right] & & X \arrow[d, hook, "\iota_f", "\simeq"'] \\
 & X \times^{\mathrm{R}}_{Z} Y \arrow[r] \arrow[d] & P_f \arrow[d, two heads, "p_f"] \\
Y \arrow[r, hook, "\simeq", "\iota_g"'] & P_g \arrow[r, two heads, "p_g"'] & Z.
\end{tikzcd}$$

This map is given by $(x, y) \mapsto (x, \mathrm{c}_{f(x)}, y)$, where $\mathrm{c}_{f(x)} \colon I \to Z$ is the constant path on $f(x) = g(y)$.

The main reason for introducing homotopy fiber products is that these preserve the property of being invariant under weak homotopy equivalences. That is to say, given weak homotopies $X \simeq X'$, $Y \simeq Y'$, $Z \simeq Z'$, the homotopy fiber product over $X \to Z \leftarrow Y$ is weak homotopy equivalent to the homotopy fiber product over $X' \to Z' \leftarrow Y'$ obtained by composing with these equivalences.

$$\begin{tikzcd}
 & X \arrow[r] \arrow[d] & X' \arrow[d] & \\
X \times^{\mathrm{R}}_{Z} Y \arrow[ur] \arrow[dr] \arrow[rrr, dashed, bend left, "\simeq"] & Z \arrow[r] & Z' & X' \times^{\mathrm{R}}_{Z'} Y' \arrow[ul] \arrow[dl] \\
 & Y \arrow[r] \arrow[u] & Y' \arrow[u] &
\end{tikzcd}$$

This is also true when replacing the two maps $X \xrightarrow{f} Z \xleftarrow{g} Y$ by homotopic ones. Of course, the strict fiber product $X \times_Z Y$ does not satisfy these properties in the slightest.

So, now that we know everything there is to know about homotopy pullbacks, where does that put us? As it turns out, we have our goal just in sight; only one prior definition remains. Recall from Remark C.1.14 that there is a canonical map $X \times_Z Y \to X \times^{\mathrm{R}}_{Z} Y$ given by $(x, y) \mapsto (x, \mathrm{c}_{f(x)}, y)$.

Definition C.1.15. A strictly commutative diagram of topological spaces

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

is called a ***homotopy pullback square*** if the composition $W \to X \times_Z Y \to X \times^{\mathrm{R}}_{Z} Y$ is a weak homotopy equivalence.

Segal spaces

Now, we can state the equivalent of Characterization C.1.8.

Definition C.1.16. A simplicial space $S_\bullet$ is a ***Segal space*** if, for each pair of integers n, $m \geq 0$, the following diagram is a homotopy pullback square.

$$\begin{tikzcd} S_{m+n} \arrow[r, "p^*_{0,1,\dots,m}"] \arrow[d, "p^*_{m,m+1,\dots,m+n}"'] & S_m \arrow[d, "p^*_m"] \\ S_n \arrow[r, "p^*_0"'] & S_0 \end{tikzcd} \tag{C.6}$$

The fact that each diagram (C.6) is a homotopy pullback square is called the ***Segal condition***. As we just said, this boils down to requiring the induced maps $S_{m+n} \to S_m \times^{\mathrm{R}}_{S_0} S_n$ to be weak homotopy equivalences. This, in turn, is equivalent to requiring the ***Segal maps*** $S_n \to S_1 \times^{\mathrm{R}}_{S_0} \cdots \times^{\mathrm{R}}_{S_0} S_1$ to be weak homotopy equivalences. These maps are induced by the functions $g_i\colon [\mathbf{1}] \to [\boldsymbol{n}]$ mapping $(0 \to 1) \mapsto (i-1 \to i)$, for $i = 1, \dots, n$ (cf. Characterization C.1.8).

Some authors, including the seminal paper [Rez01], impose the technical condition of *Reedy fibrancy*; see [Lur09, Warn. 2.1.16].

Now, let's recover the usual notion of $(\infty, 1)$-category, in analogy with Definition C.1.9.

Remark C.1.17 — Recovering an $(\infty, 1)$-category from a Segal space.
From a Segal space $S_\bullet$, we recover the informal description of an $(\infty, 1)$-category $\mathcal{C}$ as follows. Denote by $\psi\colon S_1 \times^{\mathrm{R}}_{S_0} S_1 \to S_2$ the homotopy inverse of the map induced by the Segal condition (C.6).

- **Objects:** Points of the topological space S_0.
- **1-morphisms:** For each pair of points x, $y \in S_0$, we define the hom-space $\mathrm{Map}_{\mathcal{C}}(x, y)$ to be the topological space $\{x\} \times^{\mathrm{R}}_{S_0} S_1 \times^{\mathrm{R}}_{S_0}$.
- **Composition:** For each triple of points x, y, $z \in S_0$, the composition law

  $$\circ_{xyz}\colon \mathrm{Map}_{\mathcal{C}}(y, z) \times \mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{C}}(x, z),$$

  which is a continuous function, is defined as the composition

$$\begin{aligned} (\{y\} \times^{\mathrm{R}}_{S_0} S_1 \times^{\mathrm{R}}_{S_0} \{z\}) \times (\{x\} \times^{\mathrm{R}}_{S_0} S_1 \times^{\mathrm{R}}_{S_0} \{y\}) &\overset{\tau}{\simeq} (\{x\} \times^{\mathrm{R}}_{S_0} S_1 \times^{\mathrm{R}}_{S_0} \{y\}) \times (\{y\} \times^{\mathrm{R}}_{S_0} S_1 \times^{\mathrm{R}}_{S_0} \{z\}) \\ &\simeq \{x\} \times^{\mathrm{R}}_{S_0} S_1 \times^{\mathrm{R}}_{S_0} S_1 \times^{\mathrm{R}}_{S_0} \{z\} \\ &\overset{\psi}{\simeq} \{x\} \times^{\mathrm{R}}_{S_0} S_2 \times^{\mathrm{R}}_{S_0} \{z\} \\ &\xrightarrow{d^2_1} \{x\} \times^{\mathrm{R}}_{S_0} S_1 \times^{\mathrm{R}}_{S_0} \{z\}. \end{aligned}$$

- **Identities:** For each point $x \in S_0$, we define $\mathrm{id}_x := s_0(X) \in S_1$.

Remember that a topological space is the same as an $(\infty, 0)$-category.

The 0-morphisms (or objects) are encoded in the points of S_0, while the 1-morphisms are encoded in the points of S_1. This pattern immediately breaks: the 2-morphisms are *not* encoded in the points of S_2 (although the 2-cells remember some of the information), but rather on the 1-morphisms of the fundamental ∞-groupoid of each hom-space $\mathrm{Map}_{\mathcal{C}}(x, y)$. In general, the k-morphisms for $k \geq 1$ are encoded as $(k-1)$-morphisms of $\pi_{\leq\infty}\mathrm{Map}_{\mathcal{C}}(x, y)$.

Thanks to the Segal condition of Definition C.1.16, we can think of the points in the space of n-cells as n-tuples of composable morphisms, just as we did for the nerve of a category. But unlike the case for 1-categories, where all information above 2-cells was redundant, here the higher layers are needed: they encode the coherence relations of our composition law. Under the previous interpretation, the maps $p_{0,n}^* \colon S_n \to S_1$ determine the possible compositions of the n-tuple of morphisms. Furthermore, this space of possible compositions is *contractible*: the Segal condition (C.6) implies that each fiber of the Segal map $S_n \to S_1 \times^{\mathrm{R}}_{S_0} \cdots \times^{\mathrm{R}}_{S_0} S_1$ is contractible. We say that composition is defined only *up to a contractible space of choices*, rather than "on the nose".

We now discuss the notion of weak associativity we wanted to obtain — this is the culmination of all of our hard work! We should take a moment to rejoice.

Interpreting weak associativity

Remark C.1.18 — The graphical interpretation of weak associativity.
Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ be a string of three composable morphisms in our category; this is a point in S_3. Graphically, this is a 3-cell; a tetrahedron:

$$\begin{tikzcd}
& W & \\
X \arrow[ur] \arrow[rr, "f"] \arrow[dr] & & Y \arrow[ul] \arrow[dl, "g"] \\
& Z. \arrow[uu, "h"] &
\end{tikzcd}$$

This diagram expresses weak associativity of composition! More concretely, consider the faces of the 3-cell for which we already know one of the maps:

$$\begin{tikzcd}
& W & \\
X \arrow[ur, dotted] \arrow[rr, "f"] \arrow[dr, "g\circ f"'] & & Y \arrow[ul, dotted] \arrow[dl, "g"] \\
& Z, \arrow[uu, dotted] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& W & \\
X \arrow[ur, dotted] \arrow[rr, dotted] \arrow[dr, dotted] & & Y \arrow[ul, "h\circ g"'] \arrow[dl, "g"] \\
& Z. \arrow[uu, "h"] &
\end{tikzcd}$$

We think of these 2-cells as witnessing a composition between the relevant maps. Very importantly, note that identifying the arrow $X \to Z$ with $g \circ f$ means taking the appropriate face map of the 2-cell spanned by X, Y and Z.

Now, the other faces of the 3-cells are

$$\begin{tikzcd}
& W & \\
X \arrow[ur, "h\circ(g\circ f)"] \arrow[rr, dotted] \arrow[dr, "g\circ f"'] & & Y \arrow[ul, dotted] \arrow[dl, dotted] \\
& Z, \arrow[uu, "h"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& W & \\
X \arrow[ur, "(h\circ g)\circ f"] \arrow[rr, "f"'] \arrow[dr, dotted] & & Y. \arrow[ul, "h\circ g"'] \arrow[dl, dotted] \\
& Z \arrow[uu, dotted] &
\end{tikzcd}$$

The arrow from X to W gets two different meanings, depending on the order of composition; this is not a surprise. But now, the structure of Segal spaces guarantees that both values are part of the same contractible space — a form of weak associativity!

Now let us move in the other direction: from $S_1 \times^{\mathrm{R}}_{S_0} S_1 \times^{\mathrm{R}}_{S_0} S_1$ to S_3.

Remark C.1.19 — The graphical interpretation of weak associativity, continued.
Consider a tuple of three composable morphisms, $(X \xrightarrow{f} Y, Y \xrightarrow{g} Z, Z \xrightarrow{h} W)$, living in $S_1 \times^{\mathrm{R}}_{S_0} S_1 \times^{\mathrm{R}}_{S_0} S_1$. This is a collection of three consecutive 1-cells, the *spine* of a tetrahedron:

$$\begin{tikzcd}
& W & \\
X \arrow[rr, "f"] & & Y \arrow[dl, "g"] \\
& Z. \arrow[uu, "h"] &
\end{tikzcd}$$

The Segal condition (C.6) guarantees that $S_1 \times^{\mathrm{R}}_{S_0} S_1 \times^{\mathrm{R}}_{S_0} S_1 \simeq S_3$, so there exists a 3-cell witnessing the composition of the triple:

$$\begin{tikzcd}
& W & \\
X \arrow[rr, "f"] \arrow[ur, dashed] \arrow[dr, dashed] & & Y \arrow[dl, "g"] \arrow[ul, dashed] \\
& Z. \arrow[uu, "h"] &
\end{tikzcd}$$

Furthermore, this 3-cell is unique up to weak homotopy equivalence.

Compare Remarks C.1.18 and C.1.19 with [Kerodon]'s logo!

Remark C.1.20 — About the uniqueness of composition.
The tetrahedra of Remarks C.1.18 and C.1.19 is similar to the usual associators required by the more algebraic definitions of higher categories (such as Definition 3.1.8), but there is a crucial difference: where as in bicategories we had a strictly well-defined composition (although only weakly associative), in the model of complete Segal spaces the composition itself is only well-defined up to homotopy. Crucially, for the case of higher categories of cobordisms, we do not need to endow our objects with additional collaring data in order to be able to compose them (as we briefly discussed just before Remark 1.1.20) — those choices can instead be made when defining the multiple possible compositions.

This passage from a discrete number of choices to (possibly) a continuum of choices is a common motif when comparing n-categories with (∞, n)-categories.

So, in a certain sense, we can think of each S_n as a *classifying space* for composable chains of morphisms of length n.

Now, we could define a morphism between Segal spaces to be a morphism of the underlying simplicial spaces, to serve as a notion of $(\infty, 1)$-functors between $(\infty, 1)$-categories. But there is one last technical problem. Although the structure of a Segal space is enough to induce the structure of an $(\infty, 1)$-category (as per our informal description), it turns out that multiple non-equivalent Segal spaces will induce strictly the same $(\infty, 1)$-category. Note that here "$(\infty, 1)$-category" is a heuristical concept we have not defined yet — but we have an intuitive notion for what "strict equality" between these should mean.

For the sake of preventing the technical dilemmas this entails, ultimately our good model for $(\infty, 1)$-categories will be something called a *complete* Segal space. The completeness condition serves as a bridge between the "categorical" groupoid and the "homotopical" groupoid. These are two different kinds of ∞-groupoids encoded in the data of $S_\bullet$: the first is the core ∞-groupoid $\mathrm{Core}(\mathcal{C})$ obtained from $\mathcal{C}$ by discarding all non-invertible 1-morphisms, while the second is the fundamental ∞-groupoid $\pi_{\leq\infty} S_0$ of the space S_0. Even though the objects of both ∞-groupoids are canonically identified, the morphisms are not: the equivalence classes of the objects can differ between both groupoids. This is inconvenient, as we will see.

Complete Segal spaces

Let's exemplify why we require this additional completeness condition. We have not defined equivalences between Segal spaces, so let's do that first. This is probably what the reader already expects.

Definition C.1.21. Two Segal spaces $S_\bullet$ and $R_\bullet$ are said to be ***weakly equivalent*** if they are levelwise weakly equivalent as simplicial spaces; i.e., there exists a morphism $f\colon S_\bullet \to R_\bullet$ which restricts to a weak homotopy equivalence $f_n\colon S_n \to R_n$ on each level.

We borrow our example from [Ras18, Cons. 2.54].

Example C.1.22 — Two equivalent categories whose nerves are not equivalent Segal spaces. Let $\mathcal{C} = \{x \rightleftharpoons y\}$ be the category with two objects and a single invertible arrow between them.[2] Consider the Segal space $S_\bullet$ obtained by taking the nerve $\mathrm{N}(\mathcal{C})_\bullet$ and endowing each set $\mathrm{N}(\mathcal{C})_n$ with the discrete topology. Note that $\mathcal{C}$ has a single arrow between any two objects: the arrows are $\mathrm{id}_x\colon x \to x$, $\mathrm{id}_y\colon y \to y$, $f\colon x \to y$ and $f^{-1}\colon y \to x$. Therefore, each level $S_n = \mathrm{Fun}([\boldsymbol{n}], \mathcal{C})$ is completely determined by the value of the functors at the objects. That is, we can write $S_n \cong \{x,y\}^{[n]} \cong \{x,y\}^{n+1}$: we identify each level with the ordered $(n+1)$-tuples of elements in $\{x,y\}$. In particular, $S_0 = \{x,y\}$ and $S_1 = \{xx, xy, yx, yy\}$.

Now, $\mathcal{C} \simeq [\mathbf{0}]$, the trivial category with a single object and only the identity arrow. But the discrete Segal space corresponding to the nerve of $[\mathbf{0}]$ is the trivial Segal space $P_\bullet$, where each level consists of a single point: $P_n = \{*\}$. And this is clearly not equivalent to $S_\bullet$, since the sets S_n are not weakly contractible.

The core problem is that the elements x and y of the discrete space $S_0 = \{x,y\}$ are not path-connected, but the $(\infty,1)$-category constructed from $S_\bullet$ sees them as equivalent via the two morphisms $xy = f$ and $yx = f^{-1}$ of S_1.

In order to compare Segal spaces and the yet-undefined $(\infty,1)$-categories, we first introduce a way of extracting a 1-category from a Segal space.

Definition C.1.23. The ***homotopy category*** $\mathrm{h}S_\bullet$ of a Segal space $S_\bullet$ is the 1-category described by:

- **Objects:** Points $x \in S_0$.
- **Morphisms:** For each pair of points $x,\, y \in S_0$, the hom-set $\mathrm{Hom}_{\mathrm{h}S_\bullet}(x,y)$ is the set of path components $\pi_0(\{x\} \times^{\mathrm{R}}_{S_0} S_1 \times^{\mathrm{R}}_{S_0} \{y\})$.

With composition and identities defined analogously to Remark C.1.17.

Of course, once we have a correspondence between Segal spaces and $(\infty,1)$-categories, we will want this to coincide with our prior notion of homotopy categories [Def. 3.2.9]. That is our main motivating force behind the concept of completeness.

Definition C.1.24. Let $S_\bullet$ be a Segal space. Let $f \in S_1$ be a 1-cell, and let $x = p_0^*(f)$, $y = p_1^*(f)$. The composition

$$\{f\} \to \{x\} \times_{S_0} S_1 \times_{S_0} \{y\} \to \{x\} \times^{\mathrm{R}}_{S_0} S_1 \times^{\mathrm{R}}_{S_0} \{y\}$$

determines a morphism $[f]\colon x \to y$ in the homotopy category $\mathrm{h}S_\bullet$:

$$[f] \in \mathrm{Hom}_{\mathrm{h}S_\bullet} = \pi_0(\{x\} \times^{\mathrm{R}}_{S_0} S_1 \times^{\mathrm{R}}_{S_0} \{y\}).$$

We say that the 1-cell f is ***invertible*** if $[f]$ is an isomorphism in the homotopy category $\mathrm{h}S_\bullet$.

[2] This is the category $\mathrm{I}[\mathbf{1}]$ obtained from $[\mathbf{1}] = \{0 \to 1\}$ by adding a formal inverse to its sole non-invertible morphism. Using the notation of Remark 4.1.8, this is $\mathcal{E}(\mathbb{Z}/2\mathbb{Z})$.

Now, note that the degeneracy map $s_0 \colon S_0 \to S_1$ has the property that $s_0(x)$ is invertible for each $x \in X_0$; in fact, $[s_0(x)]$ equals the identity id_x in the homotopy category $\mathrm{h}S_0$. We denote the set of invertible elements as $Z \subset S_1$, which we regard as endowed with the subspace topology.

Definition C.1.25. A Segal space $S_\bullet$ is called ***complete*** if the degeneracy map $s_0 \colon S_0 \to Z$ is a weak homotopy equivalence.

More plainly stated, every 1-isomorphism in the associated $(\infty, 1)$-category $\mathcal{C}$ arises from an essentially unique path in the space S_0. So we can identify the fundamental ∞-groupoid of each layer S_n with the ∞-groupoid of the topological space $\mathrm{Fun}([\boldsymbol{n}], \mathcal{C})$.

Using this model, we can finally state a proper definition for $(\infty, 1)$-categories.

Definition C.1.26. An ***$(\infty, 1)$-category*** is a complete Segal space.

When using a different model for $(\infty, 1)$-categories, this becomes a *theorem* to prove — we want any good model for $(\infty, 1)$-categories to be (Quillen) equivalent to complete Segal spaces. See [Ber10] for a survey discussing multiple such models.

In section C.2, when we formally construct the (∞, n)-category of cobordisms, the (n-fold) Segal space obtained will actually be an incomplete one. This is not a problem: any Segal space can be completed in an essentially unique way through a universal property. Furthermore, this assignment is *functorial*: it determines a functor $\mathsf{Segal} \to \mathsf{cSegal}$, where the morphisms are morphisms of the underlying simplicial spaces. A Segal space and its completion induce strictly the same (heuristical) $(\infty, 1)$-category, so we can alternatively think of of complete Segal spaces as a "normal form" which encode exactly the data we want an $(\infty, 1)$-category to have. This is conceptually similar to how one constructs a sheaf from any presheaf, through *sheafification*.

For more information on the completion of a Segal space, see [Rez01, § 14; CS19, § 1.4]. Both of these sources define the completion functor as the *fibrant replacement functor* in a certain model category of Segal spaces, but the model structure differs slightly between the two.

n-fold Segal spaces

Now that we have a good model of $(\infty, 1)$-categories, we want to obtain a good model of (∞, n)-categories. We will be very brief here, and follow [Lur09, § 2.1] very closely. For more details, see [CS19, § 2; BS21, § 14].

As we mentioned in Remark 3.1.7, we will first define a kind of cubical (or *n-fold*) category, which has a notion of n different "directions" for its 1-morphisms. Then we will restrict all but one of these directions to be identities, to recover globular higher categories.

Where as a Segal space was a simplicial space, an n-fold Segal space will be an n-fold simplicial space — a functor from $(\Delta^{\mathrm{op}})^n := \Delta^{\mathrm{op}} \overbrace{\times \cdots \times}^{n} \Delta^{\mathrm{op}}$ to Top. These kind of functors will model our n different "spatial" ways of composing n-morphisms.

Definition C.1.27. An ***n-fold simplicial object*** in a category $\mathcal{C}$ is a functor $(\Delta^{\mathrm{op}})^n \to \mathcal{C}$.

We use $\mathcal{C}^{(n)}$ to denote the category of n-fold simplicial objects in $\mathcal{C}$. Very importantly, $(\mathcal{C}^{(m)})^{(n)} \simeq \mathcal{C}^{(m+n)}$, and in particular we can identify n-fold simplicial objects in $\mathcal{C}$ with simplicial objects in $\mathcal{C}^{(n-1)}$. This will let us give multiple recursive definitions.

Now we need to define conditions analogous to the Segal condition of Definition C.1.16 and the completeness condition of Definition C.1.25. In addition to this, we will need to encode the *interchange laws* between k-morphisms. Let's begin by specializing to Top, before discussing some more general definitions.

Definition C.1.28. An ***n-fold simplicial space*** is an n-fold simplicial object in Top.

Where as a simplicial space $S_\bullet$ has a topological space for each integer $k \in \mathbb{N}$ (the levels S_k), an n-fold simplicial space $S_{\bullet^n}$ is composed of levels $S_{k_1,\dots,k_n}$ indexed by n-tuples of integers in $\mathbb{N}^n$.

Definition C.1.29. A map $S_{\bullet^n} \to Y_{\bullet^n}$ of n-fold simplicial spaces is a ***weak homotopy equivalence*** if it restricts to a weak homotopy equivalences $S_{k_1,\dots,k_n} \to Y_{k_1,\dots,k_n}$ at each level.

Definition C.1.30. A diagram

$$\begin{tikzcd} W_{\bullet^n} \arrow[r] \arrow[d] & X_{\bullet^n} \arrow[d] \\ Y_{\bullet^n} \arrow[r] & Z_{\bullet^n} \end{tikzcd}$$

is called a ***homotopy pullback square*** if its restriction

$$\begin{tikzcd} W_{k_1,\dots,k_n} \arrow[r] \arrow[d] & X_{k_1,\dots,k_n} \arrow[d] \\ Y_{k_1,\dots,k_n} \arrow[r] & Z_{k_1,\dots,k_n} \end{tikzcd}$$

to each level is a homotopy pullback square of topological spaces.

Now, an important definition which we did not need when discussing (1-fold) Segal spaces.

Definition C.1.31. An n-fold simplicial space $S_{\bullet^n}$ is ***essentially constant*** if there exists a weak homotopy equivalence $\mathrm{c}_{\bullet^n} \simeq S_{\bullet^n}$, where $\mathrm{c}_{\bullet^n} \colon (\Delta^{\mathrm{op}})^n \to \mathsf{Top}$ is a constant functor.

This will be the way of imposing the condition of some maps being trivial. Alternatively, we can require each canonical map $S_{0,\dots,0} \to S_{k_1,\dots,k_n}$ to be a weak homotopy equivalence; this is the same as choosing a functor $\mathrm{c}_{\bullet^n}$ which has constant value $S_{0,\dots,0}$.

Now we state our final definitions.

Definition C.1.32. An ***n-fold Segal space*** is an n-fold simplicial space which, when regarded as a simplicial space $S_\bullet$ in the category of $(n-1)$-simplicial spaces, satisfies the following conditions.

(I) For each $0 \le k \le m$, the diagram

$$\begin{tikzcd} S_m \arrow[r] \arrow[d] & S_k \arrow[d] \\ S_{m-k} \arrow[r] & S_0 \end{tikzcd}$$

is a homotopy pullback square.

(II) The $(n-1)$-simplicial space S_0 is essentially constant.

(III) Each of the $(n-1)$-fold simplicial spaces S_k is an $(n-1)$-fold Segal space.

Definition C.1.33. An n-fold Segal space $S_{\bullet^n}$ is called ***complete*** if it satisfies two further conditions.

(IV) Each of the $(n-1)$-fold Segal spaces S_n is complete.

(V) The Segal space $Y_\bullet = X_{\bullet,0,\dots,0}$ is complete.

Definition C.1.34. An ***(∞, n)-category*** is a complete n-fold Segal space.

As before, this definition becomes a theorem to prove when using a different model for (∞, n)-categories.

We will not detail the argument recovering the heuristical description of (∞, n)-categories, as we did with complete Segal spaces in Remark C.1.17. Instead, we give a very interesting example: the construction of the (∞, n)-category of n-cobordisms, Bord_n.

C.2 Constructing Bord_n, the (∞, n)-category of n-cobordisms

I know the pieces fit
'Cause I watched them fall away.

Maynard James Keenan, *Schism*.

Remember the philosophy we mentioned in section 1.1.2: *If you have something complicated, it is easier to split it apart than to assemble it from its parts*. This is in fact the trick Lurie uses in order to properly define composition of cobordisms! It is a similar concept to how one recovers a category from its nerve; cf. Remark C.1.6.

We can summarize the construction as follows: instead of trying to glue k-cobordisms with corners together, consider all possible n-manifolds with corners and cut them in all possible ways to obtain certain arrangements of k-cobordisms. Then, define a composition of the obtained k-cobordisms to be the original n-manifold.

To do this, we first embed the n-manifolds in an Euclidean space $\mathbb{R}^d$ — which is always possible for sufficiently high d, thanks to Whitney's Embedding Theorem — to define auxiliary (∞, n)-categories $\mathsf{Bord}_n(\mathbb{R}^d)$. Then, by embedding one Euclidean space into another in all possible ways, we can take the direct limit $\varinjlim_d \mathsf{Bord}_n(\mathbb{R}^d)$ to obtain our (∞, n)-category of unembedded cobordisms Bord_n. For categories of cobordisms endowed with tangential structure, such as framings or orientations, we do a completely analogous process starting from manifolds endowed with that additional structure; all of our operations preserve the additional structure.

The details are rather more technical, of course, but this underlying simple idea is what makes everything work nicely. We closely follow [Lur09, § 2.2]; in particular, specifying the monoidal and symmetric structures is out of our scope. For more details on the construction, including the matter of the symmetric monoidal structure, see the very comprehensive technical note [CS19].

Constructing $\mathsf{Cob}_n^{(\infty,1)}$, the $(\infty, 1)$-category of n-cobordisms

We will first construct an $(\infty, 1)$-category of n-cobordisms, $\mathsf{Cob}_n^{(\infty,1)}$, where the objects are closed $(n-1)$-manifolds and the 1-morphisms are n-cobordisms. Then 2-morphisms are diffeomorphisms, and 3-morphisms are isotopies, and so on. In other words, this is an $(\infty, 1)$-categorical version of $\mathsf{Cob}_n^{\mathrm{un}}(1)$,[1] where we do not identify the different diffeomorphism classes of cobordisms together. Taking homotopy classes recovers the usual 1-category: $\mathrm{h}\,\mathsf{Cob}_n^{(\infty,1)} \simeq \mathsf{Cob}_n^{\mathrm{un}}(1)$.

Alternatively, $\mathsf{Cob}_n^{(\infty,1)}$ is the iterated looping [Def. I.2.2] $\Omega^{n-1}\,\mathsf{Bord}_n$, obtained by successively taking hom-sets on the empty manifold object; for example,

$$\mathsf{Cob}_2^{(\infty,1)} = \Omega\,\mathsf{Bord}_2 = \mathrm{Hom}_{\mathsf{Bord}_2}(\varnothing, \varnothing).$$

Let V be a real vector space of (finite) dimension d. Let $\mathrm{Sub}_0(V)$ denote the collection of smooth closed $(n-1)$-submanifolds $M \subset V$, and $\mathrm{Sub}(V)$ the collection of all smooth compact n-manifolds with boundary *properly embedded*[2] in $V \times I$.

We can endow $\mathrm{Sub}_0(V)$ and $\mathrm{Sub}(V)$ with topologies, as follows.

Remark C.2.1 — Topologies on the spaces of submanifolds.
Let M be an (unembedded) n-manifold and let $\mathrm{Emb}(M, V \times I)$ be the set of all smooth proper embeddings $M \hookrightarrow V \times I$, endowed with the $\mathscr{C}^\infty$ topology. Note that there is a canonical action on $\mathrm{Emb}(M, V \times I)$ by the group $\mathrm{Diff}(M)$ of diffeomorphisms $M \to M$, given by precomposition. And note that we have a bijection

$$\bigsqcup_M \mathrm{Emb}(M, V \times I)/\mathrm{Diff}\,M \longrightarrow \mathrm{Sub}(V),$$

[1] The unoriented version of the usual 1-category of Definition 1.3.7, with the notation of Idea 3.1.12.
[2] An embedding $M \hookrightarrow V \times I$ is ***proper*** if $\partial M = M \cap (V \times I)$.

where the disjoint union is taken over all diffeomorphism classes of n-manifolds. This is almost by definition: an element of $\mathrm{Sub}(V)$ is an embedded submanifold $M \subset V \times I$, or equivalently an abstract smooth manifold equipped with a proper embedding $(M \hookrightarrow V \times I) \in \mathrm{Emb}(M, V \times I)$. The quotient by $\mathrm{Diff}\, M$ identifies all such embeddings whose image is equal as a subset of $V \times I$.

Thus, we can endow $\mathrm{Sub}(V)$ with the quotient topology. More concretely, a subset $U \subset \mathrm{Sub}(V)$ is open if and only if its inverse image in $\mathrm{Emb}(M, V \times I)$ is open for every n-manifold M. On a slightly more technical note, each of the quotient maps $\mathrm{Emb}(M, V \times I) \to \mathrm{Sub}(V)$ exhibits $\mathrm{Emb}(M, V \times [a,b])$ as a principal $\mathrm{Diff}(M)$-bundle over the corresponding summand of $\mathrm{Sub}(V)$; this is relevant for the "classifying space" interpretation mentioned after Remark C.1.20.

Similarly, $\mathrm{Sub}_0(V)$ gets its topology from the quotient

$$\coprod_M \mathrm{Emb}(M, V)/\mathrm{Diff}\, M \longrightarrow \mathrm{Sub}_0(V),$$

where here M are smooth closed $(n-1)$-manifolds.

Now, let's start the construction of our Segal space! We lied a little — first we consider a *semi*Segal space, which is a *semi*simplicial space. That is, it has face maps but no degeneracy maps [Rem. C.1.3] — it is a functor $\Delta_0^{\mathrm{op}} \to \mathsf{Top}$, where Δ_0 is the subcategory of Δ [Def. 2.3.9] composed of only the *strictly* increasing maps. We will denote this semiSegal space by SCob_n^V, since it depends on the vector space V. Its levels will be denoted by $(\mathsf{SCob}_n)_k$.

The semiSegal spaces SCob_n^V

Since the elements of the kth layer S_k of a Segal space are identified with composable k-tuples of morphisms, here the points of the kth layer $(\mathsf{SCob}_n^V)_k$ will be n-manifolds $M \subset V \times I$ along with $k+1$ marked points $0 = t_0 < t_1 < \cdots < t_k = 1$ chosen in a way which makes each section $M \cap (V \times \{t_i\})$ a smooth manifold. The correct way is for each t_i to be a regular point of the projection $M \to I$, or equivalently for M and $V \times \{t_i\}$ to intersect transversely: $M \pitchfork (V \times \{t_i\})$. This is similar to figure 2.2, when we decomposed 2D cobordisms using Morse functions — and here the Morse function *is* a projection onto the last coordinate, instead of being merely identified with one. The case $k = 0$ requires special attention, since here M should be a closed $(n-1)$-manifold embedded in $V \times \{t_0\}$.

We also need to specify the face maps, but those are easy: just compose two adjacent cobordisms, i.e. remove the point t_i from the list of points!

Definition C.2.2. We describe the ***semiSegal space of n-cobordisms embedded in V***, SCob_n^V.

- **Layers:** The zeroth layer $(\mathsf{SCob}_n^V)_0$ is defined as the set

$$(\mathsf{SCob}_n^V)_0 = \{(t_0; M) \mid t_0 \in \mathbb{R}, M \subset V \times \{t_0\} \text{ is a closed } (n-1)\text{-manifold}\}.$$

Each layer $(\mathsf{SCob}_n^V)_k$ for $k > 0$ is defined as the set

$$(\mathsf{SCob}_n^V)_k = \{(t_0 < \cdots < t_k; M) \mid t_0, \cdots, t_k \in \mathbb{R}, M \subset V \times [t_0, t_k], M \pitchfork (V \times \{t_i\})\},$$

where here $M \subset V \times [t_0, t_k]$ are properly embedded compact n-manifolds.

To endow each layer with a topology, we identify $(\mathsf{SCob}_n^V)_0$ with $\mathrm{Sub}_0(V)$ and $(\mathsf{SCob}_n^V)_k$ with an open subset of $\mathrm{Sub}(V) \times \{(t_0, \ldots, t_k) \in \mathbb{R}^{k+1} \mid t_0 < \cdots < t_k\}$. This identifies $V \times [t_0, t_k]$ with $V \times I$ through a linear change of coordinates.

- **Face maps:** The face maps are given by eliminating one of the indices:

$$(t_0 < \cdots < t_i < \cdots < t_k; M) \longmapsto (t_0 < \cdots < t_{i-1} < t_{i+1} < \cdots < t_k; M).$$

If $i = 0$ or $i = k$, then we must also take the intersection $M \cap (V \times [t_1, t_k])$ (or $M \cap (V \times [t_0, t_{k-1}])$).

Now, pick your favourite infinite dimensional real vector space and denote it by $\mathbb{R}^\infty$ — for example, $\mathbb{R}^\infty := \bigoplus_d \mathbb{R}^d$. The process we have just finished with V can be repeated for every finite-dimensional subspace $W \subset \mathbb{R}^\infty$, and the embeddings $V \hookrightarrow W$ between any two such vector spaces determine embeddings $\mathsf{SCob}_n^V \hookrightarrow \mathsf{SCob}_n^W$ between the corresponding semisimplicial spaces. As such, we can take the direct limit.

Definition C.2.3. The ***semiSegal space of n-cobordisms***, SCob_n, is the direct limit

$$\mathsf{SCob}_n := \varinjlim_{V \subset \mathbb{R}^\infty} \mathsf{SCob}_n^V.$$

To be precise, we have just defined several different semisimplicial spaces; claiming that each of these is a semiSegal space requires a proof, which we will not provide. One should also prove that Definition C.2.3 does not depend (up to homotopy equivalence) on the choice of infinite-dimensional vector space $\mathbb{R}^\infty$.

Now, we have defined a composition for our $(\infty,1)$-category, which is great — we have finally obtained closure on the discussion we started in Remark 1.1.16! But since we don't have degeneracy maps, we don't have identities yet; remember Remark C.1.17. These should be "cylinders of height zero" — intuitively, we want to repeat one of the marked points t_i in our list. But of course, cylinders of height zero are not n-manifolds, so we need something a little more technical; we want any cylinder $M \times [a,b]$ to be a unit, since these are homotopy equivalent to M.

For doing so, it is more convenient to take n-manifolds $M \subset V \times \mathbb{R}$ *without boundary* and *not necessarily compact*. The core intuition is that taking a transversal intersection of M with $V \times [t_0, t_k]$ will recover compact manifolds with (possibly empty) boundary.

The Segal spaces PCob_n^V

Similar to Definition C.2.2, we define the level sets of PCob_n^V as

$$(\mathsf{PCob}_n^V)_k = \{(t_0 \le \cdots \le t_k; M) \mid t_0, \cdots, t_k \in \mathbb{R}, M \subset V \times \mathbb{R}, M \pitchfork (V \times \{t_i\})\},$$

but we also require the projection $M \to \mathbb{R}$ to be a proper map; the preimage of each compact interval $[a,b]$ must be compact. Defining the topology of each level set takes a bit more work — we refer to [CS19, § 5.2.1] as well as the references listed there.

Importantly, here we do not need to distinguish between the $k=0$ case and the $k>0$ case! Since we will be taking the intersection $M \cap (V \times [t_0, t_k])$, for any compact $(n-1)$-manifold M we can consider the infinite cylinder $M \times \mathbb{R} \subset V \times \mathbb{R}$ — and then $M \cap (V \times \{t_0\}) = M \times \{t_0\}$ can be canonically identified with M.

And, most importantly, now it makes sense for multiple indices t_i to coincide — M is still an n-manifold, no matter if we later cut it down to an $(n-1)$-manifold. So here we can define the degeneracy maps without problems: we just map

$$(t_0 \le \cdots \le t_i \le \cdots \le t_k; M) \longmapsto (t_0 \le \cdots \le t_i \le t_i \le \cdots \le t_k; M).$$

Thus, we can define the Segal spaces PCob_n^V.

If we forget these degeneracy maps, then we get a semiSegal space weakly equivalent to SCob_n^V; see the discussion preceding [Lur09, Def. 2.2.7].

Once again, we take direct limits over finite-dimensional vector spaces $V \subset \mathbb{R}^\infty$.

Definition C.2.4. The ***Segal space of n-cobordisms*** is the direct limit $\mathsf{PCob}_n := \varinjlim_{V \subset \mathbb{R}^\infty} \mathsf{PCob}_n^V$.

And finally, at long last, we have our $(\infty,1)$-category.

Definition C.2.5. We define the **$(\infty,1)$*-category of n-cobordisms***, $\mathsf{Cob}_n^{(\infty,1)}$, to be the completion of the Segal space PCob_n.

Now for the general case!

Constructing Bord_n, the (∞, n)-category of n-cobordisms

Generalizing from 1 to n, we would want to properly embed n-manifolds with corners in a vector space $V \times I^n$ in a way which makes the corners match up with the corners of I^n (as in the definition of tangles). This can be done, but it is easier to instead consider *all* possible submanifolds without boundary $M \subset V \times I^n$, compact or not — taking the required intersections will yield compact manifolds with corners, anyway! We did this before, when passing from semiSegal to Segal spaces.

The idea is similar: to form each level set $(\mathsf{PBord}_n^V)_{k_1,\ldots,k_n}$, we consider manifolds $M \subset V \times I^n$ and mark k_1 points in the first direction, k_2 points in the second direction, and so on. We will not specify how the topology is constructed; again, we refer to [CS19, § 5.2.1].

Definition C.2.6. The level set $(\mathsf{PBord}_n^V)_{k_1,\ldots,k_n}$ is defined as the collection

$$(\mathsf{PBord}_n^V)_{k_1,\ldots,k_n} = \left\{ \left(\{t_0^1 \leq \cdots \leq t_{k_1}^1\}, \ldots, \{t_0^n \leq \cdots \leq t_{k_n}^n\}; M\right) \,\middle|\, t_j^i \in \mathbb{R}^{\{i\}} \subset \mathbb{R}^n, M \subset V \times \mathbb{R}^n \right\},$$

and where we also require the following conditions:

(I) $M \subset V \times \mathbb{R}^n$ is an n-submanifold without boundary, and not necessarily compact.

(II) The projection $M \to \mathbb{R}^n$ is proper.

(III) For every subset $S \subseteq \{1, \ldots, n\}$ and every collection of integers $\{j_i\}_{i \in S}$ with $0 \leq j_i \leq k_i$, the projection $M \to \mathbb{R}^n \to \mathbb{R}^S$ does not have $(t_{j_i}^i)_{i \in S}$ as a critical value.

(IV) The projection $M \to \mathbb{R}^{\{i+1,\ldots,n\}}$ is a submersion at every point $x \in M$ belonging to the preimage of the marked points $\{t_0^i, \ldots, t_{k_i}^i\}$ of $\mathbb{R}^{\{i\}}$.

Condition (III) is the multidimensional generalization of the transversality condition — we want every possible cut to be "good", in the sense that we do not cut through any singularities. On the other hand, the new condition (IV) is there to make each $(n-1)$-fold simplicial space $(\mathsf{PBord}_n^V)_{0,\bullet,\ldots,\bullet}$ essentially constant [Def. C.1.31]; otherwise, this would not be an n-fold Segal space. This is what the "trivial along the boundary" condition of Remark 3.1.10 comes down to.

The rest of the process is analogous to the $n = 1$ case: we assemble the level sets into a Segal space PBord_n^V and take the direct limit $\mathsf{PBord}_n := \varinjlim_{V \subset \mathbb{R}^\infty} \mathsf{PBord}_n^V$.

Definition C.2.7. We define the ***(∞, n)-category of n-cobordisms***, Bord_n, to be the completion of the n-fold Segal space PBord_n.

Of course, some of the steps can be better formalized, and we still need to define an important part of the structure — the symmetric and monoidal structure is nowhere to be seen! But we have seen the core idea, and even some of the implementation details.

Now the tale has finished, and we must go home. This time it really is the end — there will be no further encores. The insatiable reader will have to proceed to one of the sources listed in the Bibliography. If you read this far, thank you — I hope you enjoyed the ride.

There is no dark side in the moon, really.
Matter of fact, it's all dark.
The only thing that makes it look light is the sun.

Gerry O'Driscoll, *Eclipse*.

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