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PSEUDO-NEWTONIAN POTENTIALS TO DESCRIBE THE TEMPORAL EFFECTS ON RELATIVISTIC ACCRETION DISKS AROUND ROTATING BLACK HOLES AND NEUTRON STARS

Banibrata Mukhopadhyay and Ranjeev Misra
Inter-University Center for Astronomy and Astrophysics, Post Bag 4, Ganeshkhind, Pune 411007, India;
bm@iucaa.ernet.in, rmisra@iucaa.ernet.in
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ABSTRACT

Two pseudo-Newtonian potentials, which approximate the angular and epicyclic frequencies of the relativistic accretion disk around rotating (and counter-rotating) compact objects, are presented. One of them, the logarithmically modified potential, is a better approximation for the frequencies while the other, the second-order expanded potential, also reproduces the specific energy for circular orbits in close agreement with the general relativistic values. These potentials may be included in time-dependent hydrodynamic simulations to study the temporal behavior of such accretion disks.

Subject headings: accretion, accretion disks — black hole physics — gravitation — relativity

1. INTRODUCTION

X-ray binaries are known to be powered by accretion disks around neutron stars and black holes. The rapid variability of these sources indicates that the X-ray emission arises from the inner accretion disk where the effects of strong gravity are important. High-frequency ($\approx$kHz) quasi-periodic oscillations (QPOs) have been observed in neutron star systems (see van der Klis 2000 for a review), while slightly lower frequency ($\approx$450 Hz) QPOs have been detected in black hole systems (Strohmayer 2001). For neutron star systems the kHz QPO tends to be observed in pairs. Associating these frequencies with a Keplerian frequency in the disk leads to the conclusion that the phenomena originate at radii of less than 20 gravitational radius ($r_g \equiv GM/c^2$).

A number of theoretical ideas have been proposed to explain the phenomenology of kilohertz QPO. In all these models, one of the two frequencies observed in neutron star systems is identified as the Keplerian frequency of the innermost orbit of an accretion disk. The sonic point model (Miller, Lamb, & Psaltis 1998) identifies the second frequency as the beat of the primary QPO with the spin of the neutron star, while according to the two oscillators model (Osherovich & Titarchuk 1999), the secondary frequency is due to the transformation of the primary (Keplerian) frequency in the rotating frame of the neutron star magnetosphere. On the other hand, Stella & Vietri (1999) have proposed a general relativistic (GR) precession/apsidal motion model, wherein the primary frequency is the Keplerian frequency of a slightly eccentric orbit and the secondary is due to the relativistic apsidal motion of this orbit, i.e., the secondary frequency is the Keplerian frequency minus the epicyclic one. These models in general are based on identifying the characteristic frequencies of the system with observed ones and often do not address the issue of how such oscillations occur in the accreting flow.

A complete understanding of the QPO phenomena would require a self-consistent hydrodynamic simulation of the accreting flow in general relativity. While such an ambitious endeavor has been impeded for several reasons, the main difficulties can be identified to be (*a*) the development of a self-consistent turbulent viscosity and (*b*) the inclusion of GR effects. In hydrodynamic simulations, turbulent viscosity has typically been introduced in a parametric form like the α-parameterization (e.g., Taam & Lin 1984). Since the temporal behavior of accretion disks is expected to depend on the form of the viscosity law, the results of such simulations were not conclusive. A promising mechanism for driving the turbulence responsible for angular momentum and energy transport is the action of the magnetorotational instability (MRI) that is expected to take place in such disks (Balbus & Hawley 1991). Recent three-dimensional magnetohydrodynamic (MHD) simulations have shown that indeed the MRI can give rise to a turbulent viscosity that leads to the accretion flow in a Keplerian disk (Hawley, Balbus & Stone 2001). While presently such simulations do not include radiation (and hence do not describe optically thick accretion flow), it is expected that self-consistent simulations will be possible in the near future, and the temporal behavior of accretion disks can be studied with confidence.

Despite these recent advances, it is still extremely difficult to simulate realistic accretion flows in a complete GR framework. However, relativistic effects may be approximately simulated by using modified Newtonian (or pseudo-Newtonian) potentials in the nonrelativistic radial-momentum equation. Paczyński & Wiita (1980) proposed such a pseudo-Newtonian potential, which has been frequently used in simulations (e.g., Milsom & Taam 1997; Hawley & Balbus 2002). Here the Newtonian potential has been replaced by $\phi = GM/(r - 2r_g)$. The attractive feature of the potential is that it reproduces the last stable orbit exactly and the specific energies of circular orbits within 10% of the GR values (i.e., for Schwarzschild geometry). Several other pseudo-Newtonian potentials have been proposed and used in the literature (e.g., Chakrabarti & Khanna 1992). Artemova et al. (1996) have considered several such potentials and concluded that the Paczyński-Wiita potential is better than the rest based on the above criteria for nonrotating compact objects. Recently, Mukhopadhyay (2002) has proposed a pseudo-potential that is valid for rotating compact objects. This potential reproduces the GR values of the last stable orbit exactly and is a good approximation (<10% error) for the specific energy at the last stable circular orbit in the case of Kerr geometry. It also reduces to the Paczyński-Wiita potential when the spin of the black hole is set to zero.

However, these potentials are not good approximations (with error >50%) for the angular and epicyclic frequencies for radii less than $20r_g$. Thus, while they are adequate to approximate the relativistic effects for a steady state accretion disk, they cannot quantitatively reproduce the temporal behavior of a disk since that is expected to depend on the disk's characteristic (i.e., the angular and epicyclic) frequencies. Nowak & Wagoner (1991) have proposed a potential for a nonrotating black hole that reproduces the Keplerian frequencies (with deviations <15%) and the epicyclic frequencies (with deviations less than 45%) and hence is better than the Paczyński-Wiita potential for such applications.

In this paper, we present two pseudo-Newtonian potentials that may be used to simulate the relativistic time-varying effects in accretion disks around a compact object that may be corotating or counter-rotating with respect to the disk with the spin parameter $a < 0.99$. For faster spin rates the predictions of these potentials deviate pronouncedly (with errors >200%) and hence are no longer good approximations. The first has been named the *second-order expansion potential* (SEP) since it contains terms up to $(r_{\rm ms}/r)^2$, where $r_{\rm ms}$ is the marginally stable orbit. This potential reproduces the specific energy and the angular frequency with deviations less than 10% and less than 25%, respectively, from GR values (i.e., for Kerr geometry). The deviations in epicyclic frequency range from 25% to 170% (for $a \leq 0.9$) depending on the spin rate of the compact object. When the object is not rotating, the potential reduces to the one proposed by Nowak & Wagoner (1991). The second has been named the *logarithmically modified potential* (LMP) since it contains a logarithmic term. This potential reproduces well the angular (with deviations <20% for corotating and <40% for counter-rotating flows) and epicyclic frequencies (with deviations <60%) but predicts specific energies that are around 30% different from the GR values.

2. PSEUDO-NEWTONIAN POTENTIALS

Since hydrodynamic code directly requires the gravitational acceleration, it is practical to modify the Newtonian force instead of the potential. In terms of such a modified force per unit mass (F), the angular (Ω) and epicyclic (κ) frequencies are given by

$$\Omega^2 = \frac{F}{R} \ , \tag{1}$$

$$\kappa^2 = \frac{2\Omega}{R}\frac{d}{dR}(\Omega R^2) = \frac{1}{R^3}\frac{d}{dR}(FR^3) \ . \tag{2}$$

These Newtonian (or pseudo-Newtonian) frequencies should match the GR ones ($\Omega_{\rm GR}$ and $\kappa_{\rm GR}$) as seen by an observer at infinity. In terms of the dimensionless radial coordinate ($r = R/r_g$) and spin parameter (a) these frequencies are given by (e.g., Semerák & Zácek 2000)

$$\Omega_{\rm GR} = \frac{1}{r^{3/2} + a} \ , \tag{3}$$

$$\kappa^2_{\rm GR} = \left(\frac{\Omega_{\rm GR}}{r}\right)^2 [\Delta - 4(\sqrt{r} - a)^2] \ , \tag{4}$$

where $\Delta = r^2 - 2r + a^2$. For the above equations and rest of the paper we have used dimensionless quantities by setting G, M, and c to be unity.

The modified force should also reproduce the $r_{\rm ms}$ given by (Bardeen 1973)

$$\begin{aligned} r_{\rm ms} &= 3 + Z_2 \pm [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2} \ , \\ Z_1 &= 1 + (1 - a^2)^{1/3}[(1 + a)^{1/3} + (1 - a)^{1/3}] \ , \\ Z_2 &= (3a^2 + Z_1^2)^{1/2} \ , \end{aligned} \tag{5}$$

where the minus (plus) sign is for the corotating (counter-rotating) flow.

Since either one of the relativistic frequencies ($\Omega_{\rm GR}$ and $\kappa_{\rm GR}$) can specify the form of the required modified force, F (from eqs. [1] or [2]), it is clear that a modified Newtonian force cannot reproduce both the frequencies exactly. Hence, an appropriately chosen modified force should correspond to frequencies that have minimal deviation from the GR values. Here we present two such modified (or pseudo-Newtonian) forces. Both the potentials are constructed in such a manner that the $r_{\rm ms}$ is always same as the relativistic value (i.e., in Kerr geometry).

2.1. *Second-Order Expansion Potential* (*SEP*)

For this potential the Newtonian force per unit mass is modified to be

$$F = \Omega^2 r = \frac{1}{r^2}\left[1 - \left(\frac{r_{\rm ms}}{r}\right) + \left(\frac{r_{\rm ms}}{r}\right)^2\right] \ , \tag{6}$$

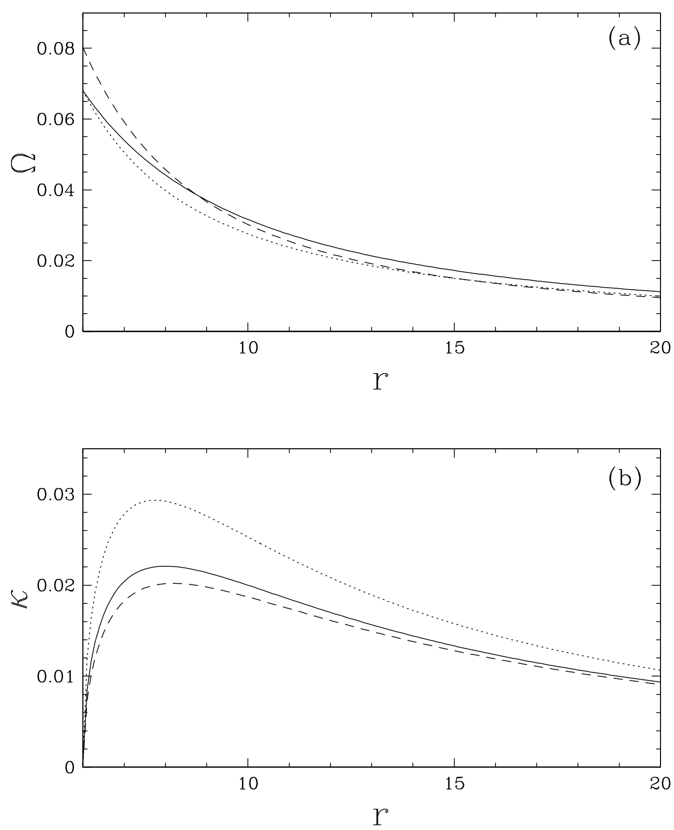

FIG. 1.—Variation of (*a*) angular and (*b*) epicyclic frequencies with radii for a nonrotating compact object ($a = 0$). The solid line is for general relativity, the dotted line is for the SEP (in this case same as the potential given by Nowak & Wagoner 1991), and the dashed line is for the LMP.

where $r_{\rm ms}$ is given by equation (5). The corresponding epicyclic frequency is

$$\kappa = \frac{1}{r^{3/2}}\left[1 - \left(\frac{r_{\rm ms}}{r}\right)^2\right]^{1/2} . \tag{7}$$

In Figures 1, 2, and 3, the variation of angular and epicyclic frequencies with radii are compared with GR values for three different values of the spin parameters $a = 0$, 0.5, and 0.9. Figure 4 shows the variations of these frequencies for spin parameter $a = 0.99$, where the deviations from the GR values are large and the potentials described in the work are no longer good approximations, particularly for the epicyclic frequency. The main advantage of this potential is its relative simplicity and that the angular frequencies deviate from the GR values by less than or equal to 25%. The specific energy (i.e., the energy per unit mass for a circular orbit) is also close (the error is at most ∼10% for all values of the Kerr parameter including $a = 1$) to the relativistic values (see Fig. 5). Its disadvantage is that κ deviates from $\kappa_{\rm GR}$ by around 40% for low-spin values and by nearly 150% for high-spin values ($a \approx 0.9$) of the compact object (Fig. 3). However, for higher counter-rotation of the compact object the error in κ reduces to ∼25%.

2.2. *Logarithmically Modified Potential* (*LMP*)

Here the Newtonian force is modified to be

$$F = \frac{1}{r^2}\left(1 + r_{\rm ms}\left\{\frac{9}{20}\frac{(r_{\rm ms}-1)}{r} - \frac{3}{2r}\log\left[\frac{r}{(3r - r_{\rm ms})^{2/9}}\right]\right\}\right) , \tag{8}$$

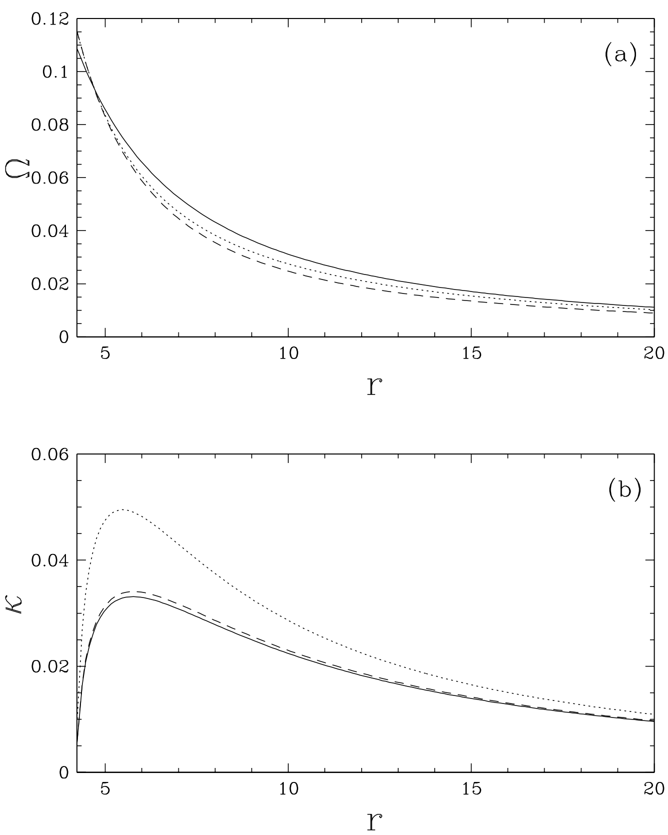

FIG. 2.—Same as Fig. 1, but for $a = 0.5$

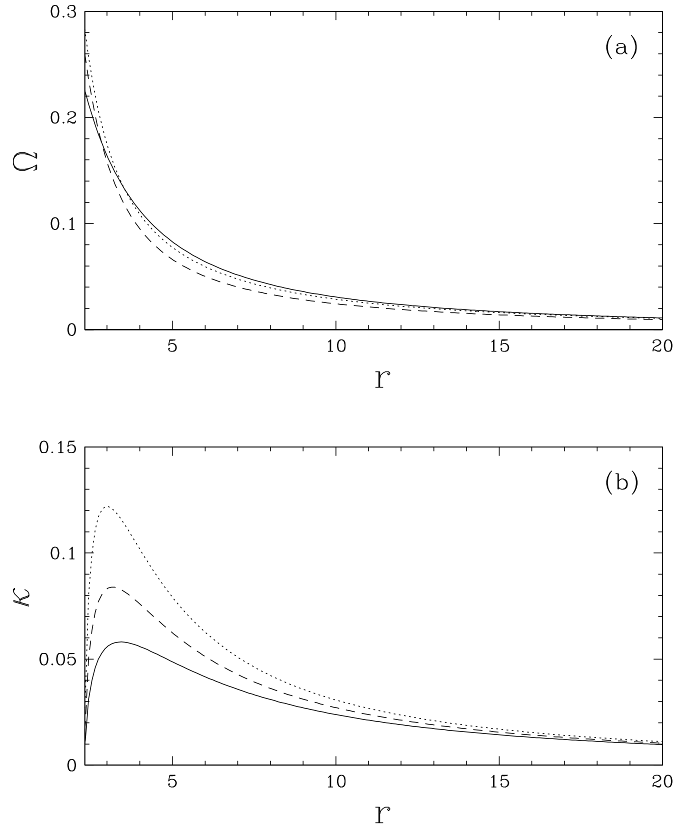

FIG. 3.—Same as Fig. 1, but for $a = 0.9$

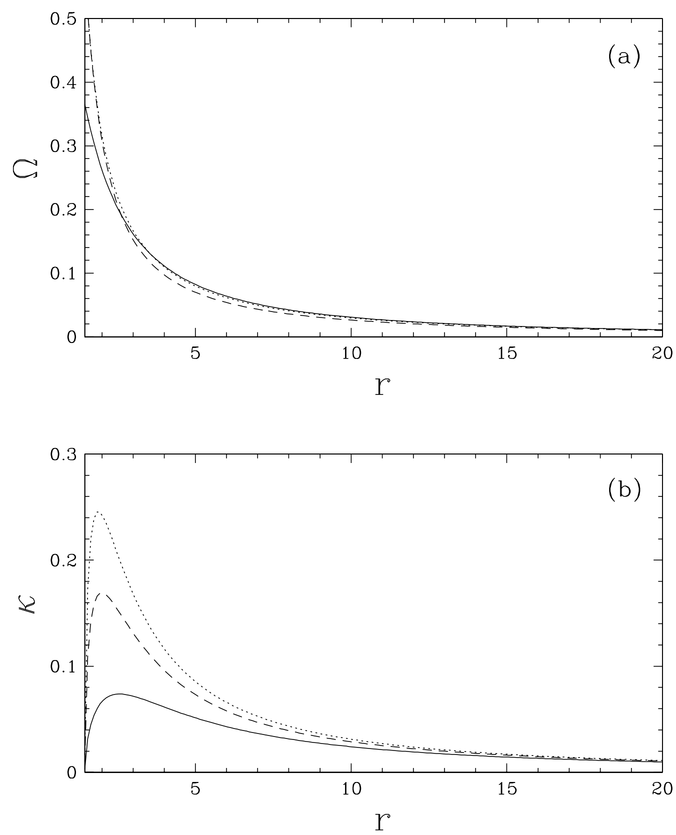

FIG. 4.—Same as Fig. 1, but for $a = 0.99$

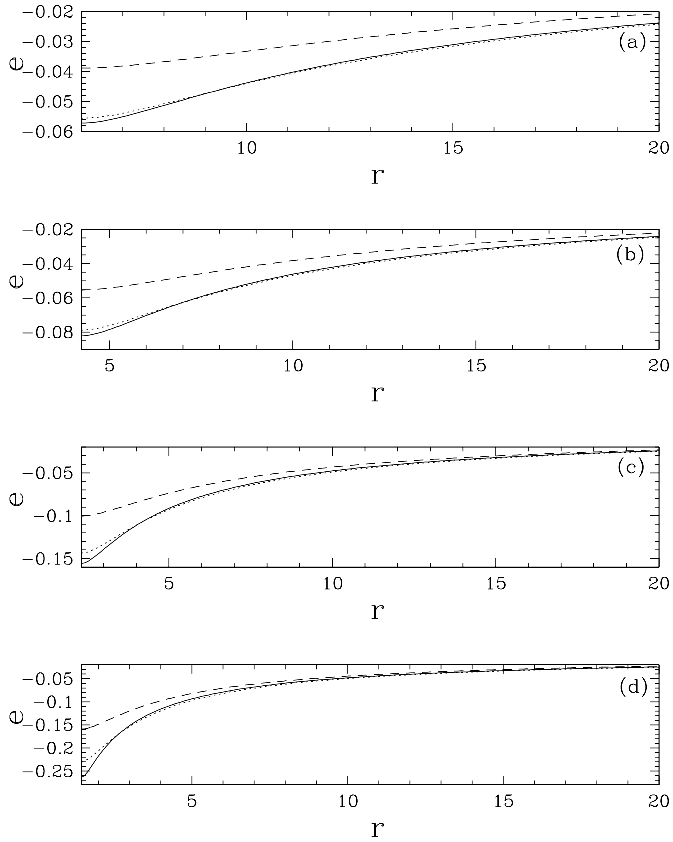

FIG. 5.—Variation of specific energy for circular orbits with radii for (*a*) nonrotating compact object as $a=0$, (*b*) rotating compact object as $a=0.5$, (*c*) rotating compact object as $a=0.9$, and (*d*) rotating compact object as $a=0.99$. The solid line is for general relativity, the dotted line is for the SEP, while the dashed line is for the LMP.

and the corresponding epicyclic frequency is given by

$$\kappa^2 = \frac{3}{2r^4}\frac{(r-r_{\rm ms})(2r-r_{\rm ms})}{(3r-r_{\rm ms})}\ . \tag{9}$$

Note that the term in the force that depends on r^{-3} does not contribute to κ (eq. [2]). The logarithmic form of the modified force (eq. [8]) was obtained by integrating the epicyclic frequency expression (eq. [9]) whose form was guessed to be a good approximation. The advantage of this potential is that both the angular and epicyclic frequencies are generally better comparable with the GR values than the SEP (Figs. 1, 2, and 3). Its disadvantage is that the specific energy deviates by more than 30% from the GR values, which is substantially larger than the deviation for SEP (Fig. 5).

3. SUMMARY AND DISCUSSION

In this work, we have presented two pseudo-Newtonian potentials that approximate the general relativistic effects on an accretion disk around rotating compact objects. These two potentials are designed particularly to approximate the angular and epicyclic frequencies of the accretion disk as seen by an observer at infinity. Table 1 summarizes the results by comparing the maximum percentage deviations from relativistic values (in Kerr geometry) for the two potentials and comparing them with those of another standard pseudo-potential.

TABLE 1

MAXIMUM PERCENTAGE DEVIATION FOR THE PSEUDO-NEWTONIAN POTENTIALS SEP, LMP, AND M

Pseudo-Newtonian Potential	$\Delta\Omega$ (%)	$\Delta\kappa$ (%)	Δe (%)	$r_{\rm ms}$
$a=0.99$				
SEP	57	445	13	1.45
LMP	57	240	40	1.45
M	180	800	14	1.45
$a=0.9$				
SEP	25	168	8	2.32
LMP	22	64	35	2.32
M	100	300	12	2.32
$a=0.5$				
SEP	12	65	4	4.23
LMP	22	17	32	4.23
M	63	122	10	4.23
$a=0.0$				
SEP[a]	13	42	3	6.00
LMP	18	13	32	6.00
M[b]	50	84	9	6.00
$a=-0.5$				
SEP	14	31	2	7.55
LMP	31	20	31	7.55
M	44	67	9	7.55
$a=-0.9$				
SEP	14	26	2	8.72
LMP	42	23	31	8.72
M	42	60	9	8.72
$a=-0.99$				
SEP	15	25	2	8.97
LMP	45	23	44	8.97
M	41	57	8.5	8.97

NOTES.—Maximum percentage deviation of angular frequency ($\Delta\Omega$), epicyclic frequency ($\Delta\kappa$), and specific energy (Δe) for SEP, LMP, and M (the potential given by Mukhopadhyay 2002); a is the spin parameter, with positive sign indicating corotation and negative sign indicating counter-rotation, and $r_{\rm ms}$ is the radius of the marginally stable orbit.

[a] SEP reduces to the one given by Nowak & Wagoner 1991 for $a=0$.

[b] The potential given by Mukhopadhyay 2002 reduces to the Paczyński-Wiita potential for $a=0$.

The SEP not only approximates the frequencies well, but also the specific energies for circular orbits turn out to be remarkably close to the relativistic values. Thus, based on such criteria, this potential is better than other pseudo-Newtonian potentials given in the literature and can be used to simulate both the steady state and time-varying accretion disks. The LMP, while being a better approximation to the frequencies than SEP, gives a rather large (≈30%) deviation from the GR results for the specific energies. Hence, its utility is perhaps limited to the time-dependent studies of accretion disks.

Which one of these two potentials should be used in a hydrodynamic simulation depends on problem being addressed. Acoustic waves (which depend on the epicyclic frequencies) would perhaps be better simulated by the LMP, while the SEP may be more suited for the long-term temporal behavior (which may depend also on the energy dissipation). Moreover, a temporal behavior detected in a simulation could be an artifact of the pseudo-Newtonian potential rather than true GR effects. Hence, it will be prudent to confirm the behavior using both the potentials. Since the mathematical forms of the two potentials are quite different any temporal behavior detected for both the potentials would imply that the behavior is indeed caused by relativistic effects. Use of these potentials in hydrodynamic simulations of accretion disks will help in the understanding of relativistic effects and may serve as a guideline for advanced simulations in general relativity.

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