THE ASTROPHYSICAL JOURNAL, 581:427–430, 2002 December 10

DESCRIPTION OF PSEUDO-NEWTONIAN POTENTIAL FOR THE RELATIVISTIC ACCRETION DISKS AROUND KERR BLACK HOLES

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Received 2002 June 7; accepted 2002 August 16

ABSTRACT

We present a pseudo-Newtonian potential for accretion disk modeling around a rotating black hole. This potential can describe the general relativistic effects on the accretion disk. As the inclusion of rotation in a proper way is very important at the inner edge of the disk, the potential is derived from the Kerr metric. This potential can reproduce all the essential properties of general relativity within 10% error, even for rapidly rotating black holes.

Subject headings: accretion, accretion disks — black hole physics — gravitation — relativity

1. INTRODUCTION

Most of the theoretical studies of general relativity in astronomy are approached by a Newtonian or pseudo-Newtonian method. To avoid the complexity of full general relativistic equations, it is simpler to use nonrelativistic equations, but with the inclusion of the corresponding (pseudo) potential, which can reproduce some relativistic effects according to the geometry of spacetime. Using this potential, one can get the approximate solutions of the hydrodynamical equations. Shakura & Sunyaev (1973) initiated the modeling of accretion disks around black holes using the Newtonian gravitational potential as

$$V_0 = -\frac{1}{x} , \qquad (1)$$

where $x = r/r_g$, r is the radial coordinate of the disk, and $r_g = GM/c^2$. However, this potential cannot reproduce the properties of the inner region of the disk, where relativistic effects become important. Subsequently, Paczyński & Wiita (1980) proposed a pseudo-Newtonian potential that could approximately reproduce the properties of the inner disk close to the equatorial plane around a nonrotating black hole without using relativistic fluid equations as

$$V_1 = -\frac{1}{(x-2)} . \qquad (2)$$

The beauty of their potential is that it can give the right positions of the marginally stable (x_s) and marginally bound (x_b) orbits in a Schwarzschild metric. It also reproduces the total mechanical energy per unit mass at the last stable circular orbit (E_s) and the total energy dissipation at a given radius (η), in good agreement with those of Schwarzschild geometry (Artemova, Björnsson, & Novikov 1996). The error in both cases is less than 10%. Nowak & Wagoner (1991) proposed another potential for an accretion disk around a nonrotating black hole as

$$V_2 = -\frac{1}{x}\left(1 - \frac{3}{x} + \frac{12}{x^2}\right) , \qquad (3)$$

which can reproduce the values of x_s and the angular velocity (Ω) at that radius in Schwarzschild geometry. So far, this choice of the potential gives the best approximate radial epicyclic frequency. Artemova et al. (1996) proposed two "correct potentials" for describing the accretion disk around a rotating black hole. The form of one of their potentials is given as

$$\frac{dV_3}{dx} = -\frac{1}{x^{2-\beta}(x-x_1)^{\beta}} , \qquad (4)$$

where x_1 is the black hole horizon and β is a constant for a particular specific angular momentum of the black hole, a (for the exact expression, see Artemova et al. 1996). They showed that their potentials can reproduce the value of x_s to be exactly as that for Kerr geometry and can reproduce the values of η at different radii in good agreement with those of general relativistic results. After that, several authors (e.g., Artemova et al. 1996; Lovas 1998; Semerák & Karas 1999) analyzed the efficiency of different pseudopotentials prescribed for accretion disks around black holes. In the context of the accretion-disk corona, which is infalling toward the black hole, Miwa et al. (1998) chose the pseudopotential V_3 (eq. [4]) and discussed radiation flux and the velocity of the infalling corona up to very close to the black hole.

However, while equation (4) is the analytical form for force, other pseudopotentials for nonrotating black holes such as equations (2) and (3) have a simple analytical form for the potential. For the study of parameter space, e.g., sonic point analysis, etc., it is useful to have an analytical expression for the potential that asymptotically varies as $-1/x$ and reduces to the Paczyński-Wiita form (eq. [2]) for zero rotation. Apart from that, equation (15) of Artemova et al. (1996) is only valid for corotating (positive values of the Kerr parameter a) disks. If one takes negative values of a to represent counterrotation, when using that equation (15) to calculate E_s and x_b, the error may be up to 50% and 500%, respectively. Similarly, for negative a, it cannot give the correct value of x_s. The general expression of equation (15) of Artemova et al. (1996) should read $r_{\rm in} = 3 + Z_2 \mp [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2}$ (Bardeen 1973; Novikov & Thorne 1973), where the upper and lower signs are for corotation and counterrotation, respectively.

As the inner region of an accretion disk is very influenced by the rotation of the black hole, rotation should be incorporated in theoretical studies. There are observational indications that black holes could be rotating rapidly, and thus to study the inner properties of an accretion disk, rotation should be incorporated correctly. Iwasawa et al. (1996)

have argued that the variable iron K emission line in MCG -6-30-15 arises from the inner part of an accretion disk and that it is strongly related to the spin of the black hole. It has been argued from other observational points of view (Karas & Kraus 1996; Iwasawa et al. 1996) that central black holes in galactic nuclei are likely to be rapidly rotating. In addition, for the observed gravitomagnetic precession, the inner edge of the accretion disk is responsible (Markovic & Lamb 1998; Stella & Vietri 1998). The temporal properties of the system are expected to depend on the inner edge of disk, which in turn depends on the rotation of the black hole. The predictions of the disk properties will be incorrect if the pseudo-Newtonian modeling does not take into account the spin of the black hole.

The aim of this paper is to present a pseudo-Newtonian potential that can reproduce exactly or in good agreement all the (inner) accretion disk properties close to the equatorial plane in Kerr geometry. The potential should reproduce those features of a rotating black hole geometry that have been reproduced by the Paczyński & Wiita (1980) potential (eq. [2]) for a nonrotating black hole. Thus, we establish our potential in the same spirit as Paczyński & Wiita did for a nonrotating black hole. All other forms of the potential have been introduced without clear relation to the spacetime metric. Here we formulate our pseudopotential from the Kerr metric. As the metric is involved directly in our calculation, many of the features of Kerr geometry are inherent in our potential by design. In § 2 we present the basic equations and derive the pseudopotential. In § 3 we compare a few results of the Kerr geometry with that of the potential, and in § 4 we make our conclusions.

2. BASIC EQUATIONS AND PSEUDOPOTENTIAL

The Lagrangian density for a particle in Kerr spacetime in Boyer-Lindquist coordinates in the equatorial plane ($\theta = \pi/2$) can be written as

$$2\mathscr{L} = -\left(1 - \frac{2GM}{c^2 r}\right)\dot{t}^2 - \frac{4GMa}{c^3 r}\dot{t}\dot{\phi} + \frac{r^2}{\Delta}\dot{r}^2 + \left(r^2 + \frac{a^2}{c^2} + \frac{2GMa^2}{c^4 r}\right)\dot{\phi}^2 , \quad (5)$$

where overdots denote the derivative with respect to the proper time τ and $\Delta = r^2 + a^2/c^2 - 2GMr/c^2$.

The geodesic equations of motion are

$$E = \text{const} = \left(1 - \frac{2GM}{c^2 r}\right)\dot{t} + \frac{2GMa}{c^3 r}\dot{\phi} , \quad (6)$$

$$\lambda = \text{const} = -\frac{2GMa}{cr}\dot{t} + \left(r^2 + \frac{a^2}{c^2} + \frac{2GMa^2}{c^4 r}\right)\dot{\phi} . \quad (7)$$

For a particle with nonzero rest mass, $g_{\mu\nu}p^\mu p^\nu = -m^2$ (where p^μ is the momentum of the particle and $g_{\mu\nu}$ is the metric). Replacing the solution for $\dot{t}$ and $\dot{\phi}$ from equations (6) and (7) into equation (5) gives a differential equation for r:

$$\left(\frac{dr}{d\tau}\right)^2 = \left(1 + \frac{a^2}{c^2 r^2} + \frac{2GMa^2}{c^4 r^3}\right)E^2 - \left(1 - \frac{2GM}{c^2 r}\right)\frac{\lambda^2}{r^2} - \frac{4GMaE\lambda}{c^3 r^3} - \frac{m^2\Delta}{r^2} = \Psi . \quad (8)$$

Here, Ψ can be identified as an effective potential for the radial geodesic motion. The conditions for circular orbits are

$$\Psi = 0, \quad \frac{d\Psi}{dr} = 0 . \quad (9)$$

Solving for E and λ from equation (9), we get

$$\frac{E}{m} = \frac{r^2 - 2GMr/c^2 + a\sqrt{GMr/c^4}}{r\left(r^2 - 3GMr/c^2 + 2a\sqrt{GMr/c^4}\right)^{1/2}} , \quad (10)$$

$$\frac{\lambda}{m} = \frac{\sqrt{GMr/c^2}\left(r^2 - 2a\sqrt{GMr/c^4} + a^2/c^2\right)}{r\left(r^2 - 3GMr/c^2 + 2a\sqrt{GMr/c^4}\right)^{1/2}} . \quad (11)$$

Equations (10) and (11) have been derived by Bardeen (1973). Now as standard practice, we can define the Keplerian angular momentum distribution $\lambda_{\rm K} = \lambda/E$. Therefore, the corresponding centrifugal force in Kerr geometry can be written as

$$\frac{\lambda_{\rm K}^2}{x^3} = \frac{(x^2 - 2a\sqrt{x} + a^2)^2}{x^3[\sqrt{x}(x-2) + a]^2} = F_x . \quad (12)$$

Thus, from the above equation, F_x can be identified as the gravitational force of the black hole at the Keplerian orbit. The above expression reduces to the Paczyński-Wiita form for $a = 0$. Thus, we propose that equation (12) is the most general form of the gravitational force corresponding to the pseudopotential in the accretion disk around a black hole. The general form of the corresponding pseudopotential (which is $V_x = V_4 = \int F_x\,dx$) is algebraically complicated but simplifies for any given value of a. In the Appendix, the general form of the potential and its reduced form for a few particular Kerr parameters are given.

3. COMPARISON OF THE RESULTS FOR KERR GEOMETRY AND PSEUDOPOTENTIAL

To establish the validity of this potential, we raise the following questions: (1) Does this potential (V_4) reproduce the values of x_b and x_s as for in Kerr geometry? (2) Does it give the correct value of E_s as for around a Kerr black hole? (3) How does the corresponding dissipation energy distribution $\eta(x)$ in an accretion disk with this potential match with the pure general relativistic result?

Apart from that, one can ask how simple the form of this potential should be so that it is applicable to other studies in which an analytical form is required. Below, we discuss all the questions one by one. If we can show that our potential tackles all the above issues fairly well, we can conclude that this is one of the best potentials for an accretion disk around a rotating back hole as well as a nonrotating one.

At the marginally bound orbit, the mechanical energy E reduces to 0, and we get

$$\frac{v^2}{2} + V = \frac{x}{2}\frac{dV}{dx} + V = 0 , \quad (13)$$

which can be used to calculate x_b for V_4. For the stability of an orbit, $d\lambda/dx \geq 0$, which for the specific potential V_4 is

$$-3a^4 + 14a^3\sqrt{x} + (x-6)x^3 + 6ax^{3/2}(x+2) - 2a^2x(x+11) \geq 0 . \quad (14)$$

TABLE 1
VALUES OF x_b

a	x_b V_4	x_b Kerr	a	x_b V_4	x_b Kerr
0..........	4.0	4.0			
0.1........	3.788	3.797	−0.1........	4.206	4.198
0.3........	3.347	3.373	−0.3........	4.606	4.580
0.5........	2.870	2.914	−0.5........	4.993	4.949
0.7........	2.333	2.395	−0.7........	5.368	5.308
0.998....	1.037	1.091	−0.998....	5.911	5.825

The solution of equation (14) with an equal sign gives the location of the last stable circular orbit (x_s) for V_4. For any a, the x_s computed from the above equation (14) matches exactly with the radius of the last stable circular orbit in Kerr geometry. We do not report the x_s for various a-values, as it is available in the standard literature. In Tables 1 and 2 we list x_b and E_s for the potential V_4 and Kerr geometry for various values of a.

From Table 1, it is clear that for all values of a, V_4 can reproduce the value of x_b in very good agreement with general relativistic results. The maximum error in x_b is $\sim$5%. Table 2 indicates that V_4 produces E_s in a fairly good agreement with Kerr geometry, with a maximum possible error of $\sim$10%. Thus, the potential V_4 produces a slightly larger luminosity than the general relativistic one in the accretion disk for a particular accretion rate. Note that for counterrotating black holes, the errors are less than those of a corotating one.

Next we compare the total energy dissipation η in the accretion disk for this potential with that of general relativistic result. We choose a simple α-disk model to compute a pseudo-Newtonian $\eta(x)$ (Björnsson & Svensson 1991; Frank, King, & Raine 1992; Artemova et al. 1996), where for different choices of pseudopotential, Ω can be different, which can be reflected in the final profile of $\eta(x)$ (Björnsson & Svensson 1991, 1992). Following Novikov & Thorne (1973), Page & Thorne (1974), and Björnsson (1995), we calculate the corresponding general relativistic η profile. In Figure 1 we show $\eta/\dot{m}$ ($\dot{m}$ is the accretion rate) as a function of x, for different values of the Kerr parameter a. Here also our pseudo-Newtonian results agree within 10% of general relativistic values. For moderate rotation, there is almost no error, while for very rapidly rotating black holes, the deviation increases but is still within 10%.

Apart from that, the analytical expressions for the force (eq. [12]) as well as the potential at given values of a (eqs. [A3] and [A4]) are relatively simple, which will make it easy to implement in other applications such as detailed fluid dynamical studies, analysis of the parameter space in the disk, etc. Thus, this potential satisfies all the criteria for a good pseudopotential that can describe an accretion disk using nonrelativistic equations.

TABLE 2
VALUES OF E_s

a	E_s V_4	E_s Kerr	a	E_s V_4	E_s Kerr
0..........	−0.0625	−0.0571			
0.1.......	−0.0663	−0.0606	−0.1.......	−0.0591	−0.0542
0.3.......	−0.0761	−0.0693	−0.3.......	−0.0536	−0.0492
0.5.......	−0.0904	−0.0821	−0.5.......	−0.0491	−0.0451
0.7.......	−0.1149	−0.1036	−0.7.......	−0.0454	−0.0418
0.998...	−0.3533	−0.3209	−0.998...	−0.0409	−0.0378

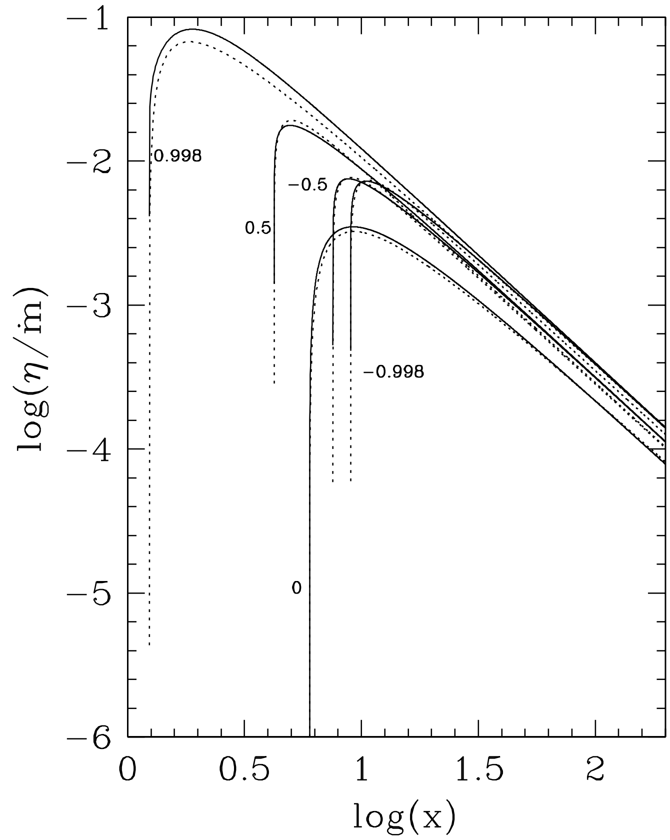

FIG. 1.—Energy dissipation per unit accretion rate ($\eta/\dot{m}$) as a function of radial coordinate (x) for various values of specific angular momentum of the black hole (a), which are indicated at each set. For each set, the solid and dotted curves show the results of general relativity and our potential, respectively.

4. CONCLUSIONS

We have prescribed a general pseudopotential for the modeling of accretion disks around rotating black holes. For the nonrotating case, it reduces to the Paczyński-Wiita potential. Unlike previous works, this potential is derived from the metric (Kerr geometry) at the equatorial plane. Naturally, it exhibits better accuracy, as it is derived from the metric itself. Following the same procedure using the Schwarzschild metric, one can derive the Paczyński-Wiita potential. The detailed calculations of various geodesic equations are available in the standard literature (e.g., Shapiro & Teukolsky 1983). Our potential is valid for both corotating and counterrotating black holes, which is not necessarily the case for earlier potentials (Artemova et al. 1996). In fact, for our pseudo-Newtonian potential, the counterrotating results agree better with the general relativistic ones, presumably because of the larger values of x_b and x_s with respect to those of the corotating cases. However, for our potential, the possible error is still at most 10% for any rotation. Thus, the inference of various observational aspects using our potential may have better accuracy. If the description of a disk property is acceptable within 10% accuracy, our potential should be recommended. It should be mentioned that although this pseudo-Newtonian potential is applicable close to the equatorial plane, it may not be a good approximation for following light rays and orbits far from the equator. Because at the very beginning of our calculation we chose $\theta = \pi/2$, this further constraint arises. Such a pseudo-Newtonian potential for generalized θ has not been proposed before, and hence it might be useful to derive it following the method prescribed in this work.

Next, one can apply this potential to various fluid dynamical problems. It is shown here that the maximum error in calculation of η is 10%, even for rapidly rotating black holes. One should study the following: How does the rotation of the black hole affect the fluid properties in an accretion disk? How does it affect the parameter region of the disk? In a future work, we expect to explore all these issues.

I would like to thank the anonymous referee for various suggestions that improved the presentation of the paper. I also would like to thank Ranjeev Misra for discussions.

APPENDIX

ANALYTICAL EXPRESSIONS OF PSEUDOPOTENTIAL

As we have an analytical form of the gravitational force (eq. [12]), we can calculate the corresponding potential as $V_x = \int F_x\, dx$. Thus, the most general expression for the pseudopotential is

$$\begin{aligned} V_x = &-\frac{a^2}{2x^2} + \frac{4a}{\sqrt{x}} + \frac{2(9a^3x - 10ax + 16\sqrt{x} - 13a^2\sqrt{x} + 6a^3 - 8a)}{(27a^2 - 32)(x^{3/2} - 2\sqrt{x} + a)} - 2\log(x) \\ &+ \frac{2}{27a^2 - 32} \sum_{y=x_1,x_2,x_3} \left[\frac{1}{3y^2 - 2} \log\left(\sqrt{x} - y\right)\left(54a^2y^2 - 64y^2 + 63a^3y - 74ay - 107a^2 + 128\right)\right] , \end{aligned} \tag{A1}$$

where

$$x_1 = \frac{2^{4/3}}{p} + \frac{p}{2^{1/3}3}, \qquad x_2 = -\left(\frac{2^{1/3}q}{p} + \frac{pq^*}{2^{1/3}6}\right), \qquad x_3 = x_2^*, \qquad p = \left(\sqrt{729a^2 - 864} - 27a\right)^{1/3}, \qquad q = \left(1 + i\sqrt{3}\right) . \tag{A2}$$

Here, the asterisk denotes the complex conjugate. Although equation (A1) looks very complicated, if we specify particular values of a, it reduces to a rather simpler expression. For example, if we choose $a = 0$, it reduces to the Paczyński-Wiita potential $[-1/(x-2)]$. For other values of a, the analytical forms of the potential reduce to

$$V_x^{a=\pm 1} = -\frac{1}{2x^2} \pm \frac{4}{\sqrt{x}} - \frac{2}{5}\left\{\frac{(\mp x + 3\sqrt{x} \mp 2)}{(x^{3/2} - 2\sqrt{x} \pm 1)} + \log\left[\frac{(\sqrt{x} \pm B)^A}{(\sqrt{x} \mp D)^C}\right]\right\} - 2\log(x) , \tag{A3}$$

where $A = 2.15542$, $B = 1.61803$, $C = 12.1554$, $D = 0.618034$, and

$$V_x^{a=\pm 0.5} = -\frac{1}{8x^2} \pm \frac{2}{\sqrt{x}} - E\left\{\frac{\mp 7.75x + 25.5\sqrt{x} \mp 6.5}{2x^{3/2} - 4\sqrt{x} \pm 1} + \log\left[\frac{(\sqrt{x} \pm G)^F}{(\sqrt{x} \mp I)^H (\sqrt{x} \mp K)^J}\right]\right\} - 2\log(x) , \tag{A4}$$

where $E = 0.0792079$, $F = 5.64616$, $G = 1.52569$, $H = 5.93863$, $I = 1.26704$, $J = 50.2075$, and $K = 0.258652$. As $a \to 1$, $H \to 0$. One can easily check that both equations (A3) and (A4) asymptotically vary as $-1/x$. The expressions (A3) and (A4) can be reduced to a simpler form if we approximate the values of the decimal numbers in the constants. However, in this manner, the accuracy of the solution would be reduced.

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