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Pseudo-Newtonian potentials and the radiation emitted by a source swirling around a stellar object

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ABSTRACT

We use pseudo-Newtonian potentials to compute the scalar radiation emitted by a source orbiting a stellar object. We compare the results obtained in this approach with the ones obtained via quantum field theory in Schwarzschild spacetime. We find that, up to the marginally stable circular orbit, the potential that better reproduces the Schwarzschild results is the Nowak–Wagoner one. For unstable circular orbits, none of the pseudo-Newtonian potentials considered in our analysis produces satisfactory results. We show that the Paczyński–Wiita potential, the most used in the literature to analyze accretion disks, generates the least satisfactory results for the scalar radiation emitted by the source in circular orbit around a Schwarzschild black hole.

1. Introduction

In the late 1970s B. Paczyński and P.J. Wiita [1] proposed a way to reproduce, without the use of general relativity, some relativistic features of the Schwarzschild spacetime. This kind of approach is carried out assuming that the gravitational interaction between bodies occurs by means of a force in flat spacetime, described by potentials that became known as *pseudo-Newtonian*. These potentials, which are essentially modifications of the Newtonian potential, allowed the analysis of accretion disks around Schwarzschild black holes in the context of Newtonian gravity. Subsequent extensions of this approach have been related to several physical situations, as runaway instability [2], spacetimes with rotation [3], gravitational wave emission [4], spacetimes with cosmological constant [5], spherical accretion onto compact objects [6], acoustic perturbations on steady spherical accretion [7], coalescence of black hole-neutron star binary systems [8], chaotic phenomena [9], and others.

Among the pseudo-Newtonian potentials proposed along the last 30 years to reproduce the structure of circular orbits in black hole spacetimes, the Paczyński–Wiita one still stands out for its simplicity and accuracy. This potential can reproduce exactly the marginally stable circular orbit ($r_{ms} \equiv 6M$), as well as the marginally bounded circular orbit ($r_{mb} \equiv 4M$), for a particle rotating around a Schwarzschild black hole. Even if incoherent radiation generated by interactions within the disk may play a very important role in the energy radiated from accretion disks, it is also interesting to study the radiation emission processes related to particles in circular motion around he black hole.

In this Letter, we present a different test to the pseudo-Newtonian potentials, with an original and independent way to verify their applicability, using them in the context of radiation emission processes, in contrast to the usual studies based in classical mechanics and fluid dynamics.

We use the pseudo-Newtonian potentials proposed by Paczyński and Wiita, Nowak and Wagoner [10], and Artemova et al. [11], as well as the Newtonian potential, to compute the synchrotron scalar radiation emitted by a source orbiting a stellar object in Minkowski spacetime. We compare these results with the power emitted by a source swirling around a Schwarzschild black hole, obtained in the framework of quantum field theory in curved spacetimes. We obtain that, among the pseudo-Newtonian potentials considered in this study, the Paczyński–Wiita one gives the least satisfactory results for this radiation emission problem.

We assume natural units $c = \hbar = G = 1$ and signature $(+, -, -, -)$ throughout this Letter.

2. Pseudo-Newtonian potentials

Pseudo-Newtonian potentials are modifications of the Newtonian potential

$$\varphi_1(r) = -\frac{M}{r} \tag{1}$$

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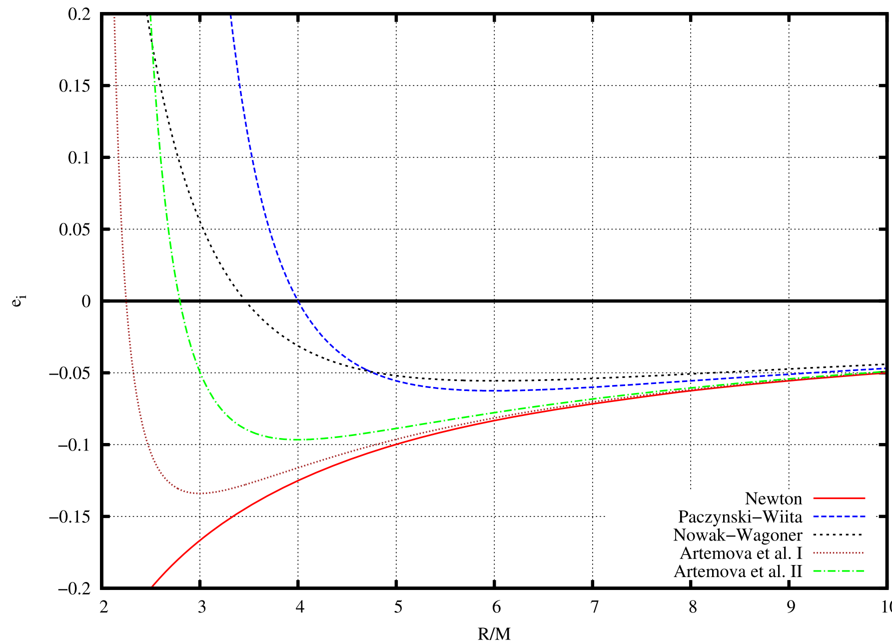

Fig. 1. The specific mechanical energy, as a function of R, for a particle in uniform circular motion around a stellar object, computed using the potentials (1)–(5).

(where M is mass of the central object), done to reproduce some relativistic aspects of particles orbiting central objects.

Here we will be concerned with the following pseudo-Newtonian potentials:

(i) Paczyński–Wiita potential [1]

$$\varphi_2(r) = -\frac{M}{r-2M}, \tag{2}$$

(ii) Nowak–Wagoner potential [10]

$$\varphi_3(r) = -\frac{M}{r}\left[1 - 3\frac{M}{r} + 12\frac{M^2}{r^2}\right], \tag{3}$$

(iii) Artemova et al. potentials [11]

$$\varphi_4(r) = -1 + \left(1 - \frac{2M}{r}\right)^{1/2}, \tag{4}$$

which we will refer to as Artemova et al. I, and

$$\varphi_5(r) = \frac{1}{2}\ln\left(1 - \frac{2M}{r}\right), \tag{5}$$

which will be referred to as Artemova et al. II.

From these potentials, we can find the orbital structure of a particle in uniform circular motion around a stellar object in Minkowski spacetime for each case. With this aim, we use the definition of the specific mechanical energy [1]

$$e_i(r) \equiv \frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{l_i^2}{r^2} + \varphi_i(r),$$

where $i = 1, 2, 3, 4, 5$ stands for each of the potentials given by Eqs. (1), (2), (3), (4), and (5), respectively. The specific angular momentum, l_i, is given by

$$l_i(r) = \left(r^3\frac{d\varphi_i}{dr}\right)^{1/2}. \tag{6}$$

Therefore, the specific mechanical energy for each potential (1)–(5) can be written as

$$e_1(R) = -\frac{M}{2R},$$

$$e_2(R) = \left(-\frac{M}{2R}\right)\left[\frac{(R-4M)R}{(R-2M)^2}\right],$$

$$e_3(R) = \left(-\frac{M}{2R}\right)\left(1 - \frac{12M^2}{R^2}\right),$$

$$e_4(R) = -1 + \frac{M}{2R}\left(1 - \frac{2M}{R}\right)^{-1/2} + \left(1 - \frac{2M}{R}\right)^{1/2},$$

and

$$e_5(R) = \frac{M}{2R}\left(1 - \frac{2M}{R}\right)^{-1} + \frac{1}{2}\ln\left(1 - \frac{2M}{R}\right),$$

respectively. The circular bounded orbits are obtained by imposing $e_i(R) < 0$ [1], where R is the radius of the circular orbit. The marginally bounded orbit is defined as the orbit for which the mechanical energy vanishes.

In Fig. 1 we plot the specific mechanical energy for the potentials (1)–(5). We see that only the Paczyński–Wiita potential reproduces the exact value of the marginally bounded circular orbit provided by general relativity ($r_{mb} = 4M$) [12]. We also see that all orbits are bounded in the Newtonian case, as expected.

The stable orbits are such that $\frac{dl_i}{dR} > 0$ [13,14]. Using Eqs. (1)–(6), we find

$$\frac{dl_1}{dR} = \frac{1}{2}\left(\frac{M}{R}\right)^{1/2},$$

$$\frac{dl_2}{dR} = \frac{1}{2}\left(\frac{M}{R}\right)^{1/2}\left[\frac{(R-6M)R}{(R-2M)^2}\right],$$

$$\frac{dl_3}{dR} = \frac{1}{2}\left(\frac{M}{R}\right)^{1/2}\left(1 - \frac{36M^2}{R^2}\right)\left(1 - \frac{6M}{R} + \frac{36M^2}{R^2}\right)^{-1/2},$$

$$\frac{dl_4}{dR} = \frac{1}{2}\left(\frac{M}{R}\right)^{1/2}\frac{R-3M}{R-2M}\left(1 - \frac{2M}{R}\right)^{-1/4},$$

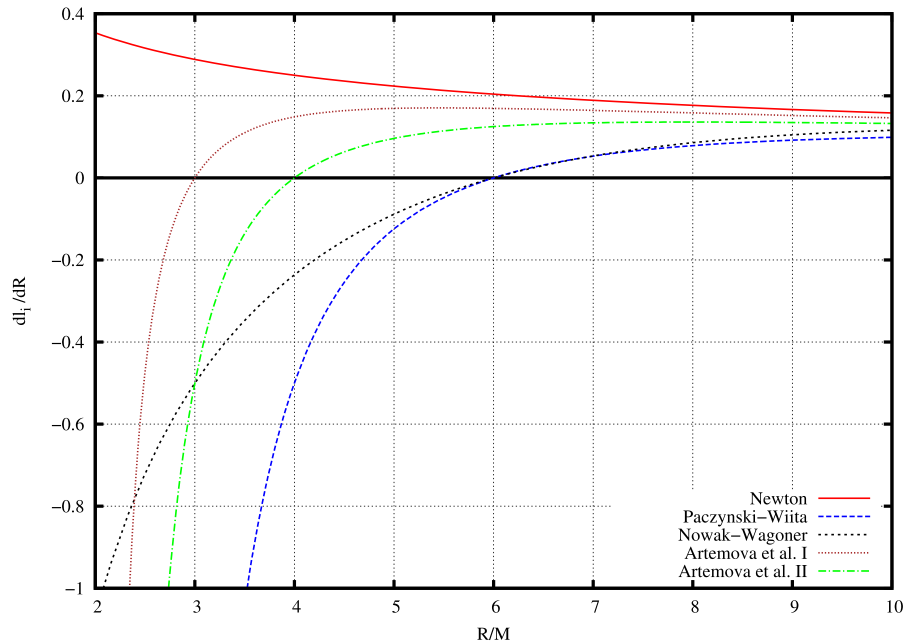

Fig. 2. The radial derivative of the angular momentum, as a function of R, for a particle in uniform circular motion around a stellar object, considering the pseudo-Newtonian potentials and the Newtonian potential.

Table 1
Values of the marginally bounded orbit (r_{mb}), marginally stable orbit (r_{ms}), and efficiency (E), obtained assuming different pseudo-Newtonian and Newtonian potentials, as well as assuming general relativity (GR).

Case	r_{mb}	r_{ms}	E
GR	$4M$	$6M$	0.0572
Newton	–	–	0.0833
Paczyński–Wiita	$4M$	$6M$	0.0625
Nowak–Wagoner	$3.46M$	$6M$	0.0556
Artemova et al. I	$2.25M$	$3M$	0.0814
Artemova et al. II	$2.81M$	$4M$	0.0777

and

$$\frac{dl_5}{dR} = \frac{1}{2}\left(\frac{M}{R}\right)^{1/2} \frac{R-4M}{R-2M}\left(1-\frac{2M}{R}\right)^{-1/2}.$$

In Fig. 2 we plot the radial derivative of the angular momentum, dl_i/dR, for each case. We see that the Paczyński–Wiita potential, as well as the Nowak–Wagoner potential, reproduce the exact value of the marginally stable circular orbit provided by general relativity ($r_{ms} = 6M$). We also see that all orbits are stable in the Newtonian case, as expected.

Let us now define the efficiency E as the specific mechanical energy of the particle in circular orbit computed at $R = 6M$ [11], namely

$$E_i \equiv \left|e_i(R = 6M)\right|.$$

In Table 1 we summarize the results obtained for marginally bounded orbits, marginally stable orbits, and efficiencies for all cases analyzed here. We note that the smallest error ($\approx 3\%$) for the efficiency with respect to Schwarzschild occurs for the Nowak–Wagoner potential. We emphasize that the correct value of the Nowak–Wagoner potential efficiency is 0.0556 (and not the one previously reported in the literature [6,11,15,16]).[1]

[1] The wrong value for the efficiency of the Nowak–Wagoner pseudo-Newtonian potential originally reported in Ref. [11] has been propagated to other works in the literature, e.g., in Refs. [6,15,16].

3. Emitted power in Minkowski spacetime

Next we compute the power emitted by a rotating source in Minkowski spacetime in orbit around a central object. In polar coordinates, the Minkowski line element is given by

$$ds^2 = dt^2 - dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\varphi^2.$$

The 4-velocity and the current of a point source following a circular orbit in the plane $\theta = \pi/2$, with radius R and angular velocity $\Omega > 0$ (as measured by static observers), are

$$u^\alpha(\Omega, R) = \gamma(1, 0, 0, \Omega),$$
$$J(x^\mu) = \frac{q}{R^2\gamma}\delta(r-R)\delta(\theta-\pi/2)\delta(\varphi-\Omega t), \tag{7}$$

respectively, where $\gamma = 1/\sqrt{1-(R\Omega)^2}$ is the Lorentz factor.

We will consider the source $J(x^\mu)$ minimally coupled to a massless scalar field $\hat{\Phi}(x^\mu)$, with q being the magnitude of the source-field coupling, so that the total action is given by

$$\hat{S} = \int d^4x\sqrt{-\eta}\left[\frac{1}{2}\nabla^\alpha\hat{\Phi}(x^\mu)\nabla_\alpha\hat{\Phi}(x^\mu) + J(x^\mu)\hat{\Phi}(x^\mu)\right],$$

with η being the modulus of the determinant of Minkowski metric $\eta_{\alpha\beta}$. The scalar field operator $\hat{\Phi}(x^\mu)$ can be written as

$$\hat{\Phi}(x^\mu) = \sum_{l,m}\int_0^\infty d\omega\left[u_{\omega lm}(x^\mu)\hat{a}_{\omega lm} + u^*_{\omega lm}(x^\mu)\hat{a}^\dagger_{\omega lm}\right],$$

with the positive-frequency modes being expanded in the form

$$u_{\omega lm}(x^\mu) = \sqrt{\frac{\omega}{\pi}}\frac{\psi_{\omega l}(r)}{r}Y_{ml}(\theta,\varphi)e^{-i\omega t}. \tag{8}$$

Here $\omega > 0$ is the frequency of the modes and (l, m) are their angular momentum quantum numbers. The radial functions of the modes obey the following equation

$$\left(-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}\right)\psi_{\omega l}(r)=\omega^2\psi_{\omega l}(r),$$

whose solution is $\psi_{\omega l}(r)\propto r j_l(\omega r)$, with $j_l(\omega r)$ being the spherical Bessel function [17].

We choose the initial state $|0\rangle$ to be the Minkowski vacuum, i.e., $\hat{a}_{\omega lm}|0\rangle=0$, so that we can write the emission amplitude as

$$\begin{aligned}A_{\omega lm}&=\langle 1,\omega lm|i\int d^4x\sqrt{-\eta}\,J\left(x^\mu\right)\hat{\Phi}\left(x^\mu\right)|0\rangle\\&=i\int d^4x\sqrt{-\eta}\,J\left(x^\mu\right)u^*_{\omega lm}\left(x^\mu\right).\end{aligned}\tag{9}$$

The emitted power, as measured by static observers, for a fixed angular momentum, is given by

$$W^M_{lm}=\int_0^{+\infty}d\omega\,\omega\frac{|A_{\omega lm}|^2}{T},\tag{10}$$

with $T=2\pi\delta(0)$ being the total time measured by these observers [18]. Using Eqs. (7)–(10), we obtain the total power irradiated by the source, namely

$$W^M=\sum_{l,m}\frac{2q^2m^2\Omega^2}{\gamma^2}\left|j_l(m\Omega R)\right|^2\left|Y_{ml}(\pi/2,\Omega t)\right|^2.$$

Using the following relation [19]:

$$\sum_{l,m}m^2\left[j_l(m\xi)\right]^2\left|Y_{ml}(\pi/2,\phi)\right|^2=\frac{1}{24\pi}\frac{\xi^2}{(1-\xi^2)^3},$$

we find

$$W^M=\frac{q^2}{12\pi}\alpha^2,$$

where $\alpha=\gamma^2\Omega^2R$ is the 4-acceleration of the source. We note that this result can also be obtained using Green's function in the classical field theory framework.

We shall write the power irradiated by the point source in uniform circular motion as a function of M and Ω (i.e., asymptotic variables). For this purpose, we need to write $R=R(M,\Omega)$, and therefore we need to specify the interaction between the stellar object and the swirling source, assuming Kepler's third law. In the Newtonian case, we have

$$R_1=\left(\frac{M}{\Omega^2}\right)^{1/3}.$$

For the Paczyński–Wiita $\varphi_2(r)$ and the Artemova et al. $\varphi_5(r)$ potentials, the relations $R=R(M,\Omega)$ are, respectively, given by

$$R_2=\frac{4M}{3}+\frac{2^{1/3}4M^2\Omega^2}{3\chi_+^{1/3}}+\frac{\chi_+^{1/3}}{2^{1/3}3\Omega^2},$$

$$R_5=\frac{2M}{3}-\frac{2^{1/3}4M^2\Omega^2}{3\chi_-^{1/3}}-\frac{\chi_-^{1/3}}{2^{1/3}3\Omega^2},$$

with

$$\chi_\pm\equiv M\Omega^4\left(\pm 27-16M^2\Omega^2+3\sqrt{3}\sqrt{27\mp 32M^2\Omega^2}\right).$$

For the Nowak–Wagoner $\varphi_3(r)$ and Artemova et al. $\varphi_4(r)$ potentials, we need to find the numerical solutions of the following equations

$$(R_3)^5-\frac{M}{\Omega^2}(R_3)^2+\frac{6M^2}{\Omega^2}R_3=\frac{36M^3}{\Omega^2},$$

and

$$(R_4)^6-2M(R_4)^5=\frac{M^2}{\Omega^4},$$

respectively.

The power emitted by the particle following uniform circular motion around the stellar object for the different potentials can be written as

$$W^M_i=\frac{q^2\Omega^4\gamma_i^4R_i^2}{12\pi},\tag{11}$$

where $\gamma_i=1/\sqrt{1-(R_i\Omega)^2}$.

4. Emitted power in Schwarzschild spacetime

Let us now compare the emitted power in Minkowski spacetime, obtained using pseudo-Newtonian and Newtonian potentials, with the emitted power in Schwarzschild spacetime, obtained in the framework of quantum field theory in curved spacetimes. The total power, as measured by asymptotic static observers, emitted by a particle circularly moving on the plane $\theta=\pi/2$, at $r=R_S$, with constant angular velocity $\Omega>0$, is [19,20]:

$$\begin{aligned}W^S=\sum_{n,l,m}&2q^2m^2\Omega^2\left[f(R_S)-R_S^2\Omega^2\right]\\&\times\frac{|\psi^n_{\omega l}(R_S)|^2}{R_S^2}\left|Y_{lm}\left(\frac{\pi}{2},\Omega t\right)\right|^2.\end{aligned}\tag{12}$$

Here the set (n,l,m) stands for the field quantum numbers, q is the coupling constant, $f(R_S)=1-2M/R_S$, and $\psi^n_{\omega l}(r)$ obeys the equation

$$\left[-f(r)\frac{d}{dr}\left(f(r)\frac{d}{dr}\right)+V_S\right]\psi^n_{\omega l}(r)=\omega^2\psi^n_{\omega l}(r),$$

with the following scattering potential

$$V_S=\left(1-\frac{2M}{r}\right)\left[\frac{2M}{r^3}+\frac{l(l+1)}{r^2}\right].$$

The index n stands for the two independent sets of solutions, which are the incoming modes from the past horizon $\mathcal{H}^-$ and from the past null infinity $\mathcal{J}^-$, labeled by $n=\rightarrow$ and $n=\leftarrow$, respectively.

The ratio W^S/W^M_i is plotted in Fig. 3, with the angular momentum summation in Eq. (12) performed up to $l=5$. We see that, up to this angular momentum contribution, in almost all cases the emitted power obtained with the use of the potentials (1)–(5), is bigger than in the Schwarzschild case. The exception happens only for the power emitted in the case of the Nowak–Wagoner pseudo-Newtonian potential, from orbits asymptotically far from the central object up to the orbit with $M\Omega\approx 0.02$. Also, we see that, for all stable circular orbits according to general relativity ($0<M\Omega<0.068$), the Nowak–Wagoner pseudo-Newtonian potential is the one that gives the closest results to the Schwarzschild case (for orbits with $0<M\Omega<0.02$, the powers emitted in these two cases are numerically very close). However, for unstable orbits according to general relativity ($0.068<M\Omega<0.192$), none of the pseudo-Newtonian potentials studied here gives satisfactory results. We also see that the Paczyński–Wiita pseudo-Newtonian potential, the most used in literature to analyze accretion disks, happens to be the worst approximation to the Schwarzschild case, among the potentials in this study. The fall of the ratios W^S/W^M_i in Fig. 3 close to $\Omega M=0.192$ ($R=3M$), is a consequence of the fact that the summations in the angular momentum have been

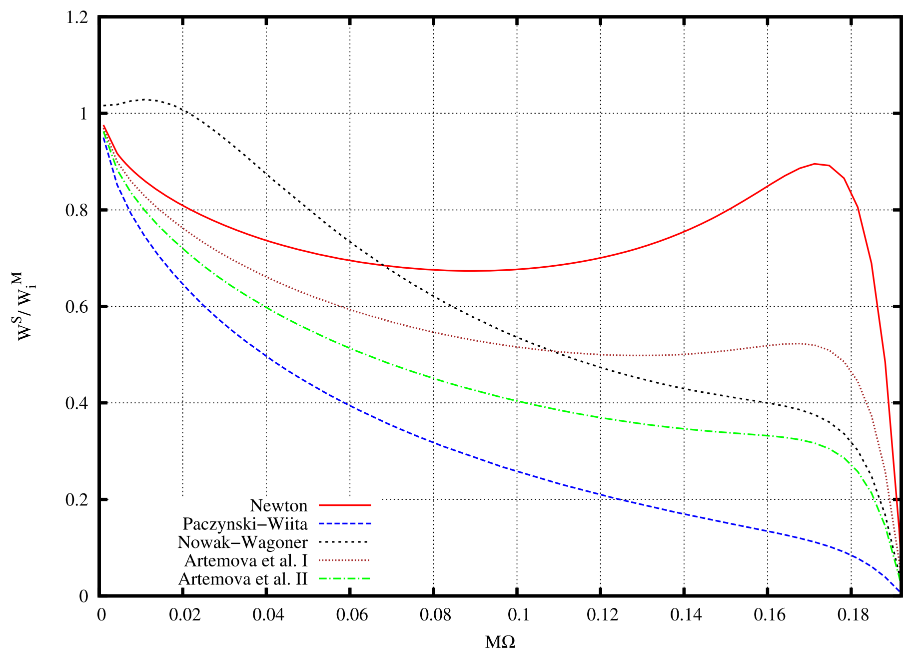

Fig. 3. The ratio between the emitted power in Schwarzschild spacetime [W^S, given by Eq. (12)] and in Minkowski spacetime, obtained using Newtonian and pseudo-Newtonian potentials [W_i^M, given by Eq. (11)], is plotted as a function of Ω, considering angular momentum contributions in Eq. (12) up to $l = 5$.

performed up to a maximum value of l [20]. As the source approaches $R = 3M$ (rotating around the black hole in unstable circular orbits according to general relativity), the emitted particles can be very energetic, with small wavelengths compared to the black hole radius, and hence higher contributions of the angular momentum have to be considered.

The scalar field provides the simplest radiation emission case which allows to capture some of the essential features of the radiation emission problem for the electromagnetic and gravitational fields. It has been obtained numerically that the total electromagnetic power emitted by a charge rotating around a Schwarzschild black hole is basically twice the result for a source coupled to a massless scalar field in the same situation [21]. Therefore, the same conclusions of the present study, regarding the use of the pseudo-Newtonian potentials to investigate the radiation emission from a source rotating around a Schwarzschild black hole, shall hold for the electromagnetic case.

5. Conclusion

In this Letter we investigated the use of pseudo-Newtonian potentials in the computation of the radiation emitted by a particle in circular motion around a stellar object. We computed the power emitted by a scalar source swirling around a stellar object in Minkowski spacetime, assuming the gravitational interaction to be given by the pseudo-Newtonian potentials proposed by Paczyński–Wiita, Nowak–Wagoner, and Artemova et al., as well as by the Newtonian potential. The Paczyński–Wiita and the Nowak–Wagoner potentials reproduce the same radial value obtained by general relativity for the marginally stable circular orbit ($r_{ms} = 6M$), and the Paczyński–Wiita is the only pseudo-Newtonian potential that reproduces the same radial value (as in general relativity) for the marginally bounded circular orbit ($r_{mb} = 4M$).

We compared the results of the emitted power obtained using the pseudo-Newtonian potentials with the power emitted by a scalar source orbiting a Schwarzschild black hole, obtained using the framework of quantum field theory in curved spacetimes. We found that, for stable circular orbits, the Nowak–Wagoner pseudo-Newtonian potential results are the ones that better resemble the Schwarzschild results. For unstable circular orbits, none of the pseudo-Newtonian potentials considered in our study produces satisfactory results. The Paczyński–Wiita potential, the most used pseudo-Newtonian potential in the literature to analyze accretion disks, generates the least satisfactory results for the radiation emission problem investigated here.

Although being eventually good approximations for some situations in the context of accretion disks, we have shown in the present study that pseudo-Newtonian potentials do not produce satisfactory results for the radiation emitted by relativistic particles orbiting black holes.

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