Indian Institute of Technology, Bombay — Chemical Engineering

CL686: Advanced Process Control — **Assignment 2, 2025**

Date Due: 2 Feb 2025, 11.59 PM,(Note that time is not 12 PM midnight since that is confusing to set.)

Mode of Submission: Moodle submission

Late Submission Policy: Submissions late by 0-24 hrs will incur 25% penalty, submissions late by 24-48 hrs will incur 50% penalty. Submissions later than 48 hrs will not be considered.

Topic: Model Predictive Control using Quadratic Programming

AIM: To implement Linear MPC in regulatory control framework for Quadruple Tank System.

Task at Hand

Our controller will be linear MPC i.e. MPC using a discrete-time linear model, without considering any disturbance within the MPC model. However, the plant may be linear or nonlinear. For this assignment, you don't have to consider disturbance in the plant as well. We want to implement and test our controller for controlling the four states (four levels) for the following cases.

- Case 1: Implement MPC to control the liquid level in tanks at steady state when the plant is also a discrete-time linear plant (same model as MPC).
- Case 2: Repeat Case 1 but now the plant is nonlinear, i.e. represented as coupled nonlinear ODEs thereby requiring integration between successive time-instants.

Compare the MPC performance for Cases 1 and 2, using performance indices given towards the end of the assignment.

System Description

The 4-tank system is governed by the following set of coupled nonlinear ODEs,

$$\frac{dh_1}{dt} = -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1}v_1 \tag{1}$$

$$\frac{dh_2}{dt} = -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2}v_2 \tag{2}$$

$$\frac{dh_3}{dt} = -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2)k_2}{A_3}v_2 + \frac{\gamma_3}{A_3}F_d \tag{3}$$

$$\frac{dh_4}{dt} = -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1)k_1}{A_4}v_1 + \frac{(1-\gamma_3)}{A_4}F_d \tag{4}$$

The parameters of the system are listed in Table 1.

- **System states:** h_1, h_2, h_3, h_4 are the liquid levels in the quadruple tank system.
- **Process inputs:** v_1 and v_2 are the voltages applied to the two pumps.
- **Disturbance input:** F_d acts as the disturbance input to the process.

Variable	Optimum operating point
A_1, A_3 $[cm^2]$	28
A_2, A_4 $[cm^2]$	32
a_1, a_3 $[cm^2]$	0.071
a_2, a_4 $[cm^2]$	0.057
g $[cm/s^2]$	981
k_1, k_2 $[cm^3/Vs]$	3.14, 3.29
$\gamma_1, \gamma_2, \gamma_3$	0.43, 0.34 , 0.4

Table 1: Four tank system parameters

Condition for Simulation: Absence of disturbance

Keep $F_d = 0$ throughout the simulation.

- **Steady state inputs:** $U_s = [3.15 \quad 3.15]^T$; $D_s = 0$
- **Equilibrium/Steady State Operating Point:**
  $X_s = [12.44 \quad 13.17 \quad 4.73 \quad 4.98]^T$
- **Sampling interval**, $T = 4$ sec
- **Number of samples**, $N_s = 150$
- **Initial state**, $X(0) = X_s + [2 \quad 2 \quad 2 \quad 2]^T$.
- **Initial input**, $U(0) = U_s$. This initial input will be needed to implement ΔU constraints in the first window. Towards this end, use this initial input and the initial state to generate state at time instant 1. Then apply MPC starting from time instant 1.
  Note: In matlab, the starting index is 1 and not 0. So accordingly initial time instant will be 1, and the time instant for the first MPC implementation will be 2.
- **Setpoint**, $R = X_s$
- **Input bound constraints**, $U_L = [0 \quad 0]^T$, $U_H = [5 \quad 5]^T$
- **Input rate constraints**, $\Delta U_L = [-3 \quad -3]^T$, $\Delta U_H = [3 \;\; 3]^T$
- **State bound constraints**, $X_L = [10 \;\; 10 \;\; 2 \;\; 2]^T$, $X_H = [16 \;\; 16 \;\; 8 \;\; 8]^T$

MPC Parameters

- **Prediction horizon:** $p = 50$
- **Control horizon:** $q = 20$
- **Weighting matrices:**
  $W_x = I_{4\times4}$
  $W_u = 0.1 \times I_{2\times2}$
  where, I is identity matrix of given dimension.

MPC Optimization Problem

The model predictive control problem at the sampling instant k is defined as a constrained optimization problem whereby the future manipulated input moves

$$U_{f,k} = \{u(k+j|k) : j = 0, 1, ...p-1\} \tag{5}$$

are determined by minimizing a cost function defined over prediction horizon p. Note that use of control horizon $q < p$ implies inclusion of the following constraints on the future manipulated inputs

$$u(k+j) = u(k+q-1), \quad j = q, q+1, ..., p-1 \tag{6}$$

Conversion of the MPC formulation to a quadratic programming optimization problem was discussed in detail in the class. The constraints arising from use of control horizon $q < p$ can be incorporated as additional equality constraints.

The MPC formulation as a quadratic programming problem can be written as

$$\arg\min_{U_{f,k}} J = \frac{1}{2}(U_{f,k})^T H(U_{f,k}) + (F_k)^T U_{f,k}$$

$$\text{s.t.}$$

$$\widetilde{A}U_{f,k} \leqslant \widetilde{b}$$

$$A_{eq}U_{f,k} = b_{eq}$$

where, construction of $\widetilde{A}, \widetilde{b}$ was discussed in the class. Construct A_{eq}, b_{eq} to incorporate the equality constraints arising from the use of control horizon which is less than prediction horizon.

NOTE: The MATLAB inbuilt command 'quadprog' can be used to solve the above quadratic programming problem. $x = quadprog(H, f, \widetilde{A}, \widetilde{b}, Aeq, beq)$. Please refer to MATLAB help to understand how to use the command.

Performance Indices

Compute the following quantities for the two cases (linear, nonlinear plant):

- **Sum Squared Error**
  $SSE_i = \sum_{k=1}^{N_s}(X_i(k) - R)^2$, for $i = 1, 2, 3, 4$.
  This is a measure of regulatory performance.

- **Sum Squared Manipulated variables**
  $SSMV_i = \sum_{k=1}^{N_s}(U_i(k) - U_s)^2$, for $i = 1, 2$.
  This is a measure of perturbations in the manipulated variables (a proxy for control effort).

Plots to generate

1. Plot controlled states of linear plant, Nonlinear plant, and setpoint for each state in the same plot for case 1 and case 2. You can generate four figures: one for each state. In each subplot, provide state trajectory obtained with linear and nonlinear plant, along with setpoint.

2. Plot each manipulated input given to linear plant and nonlinear plant in the same plot for case 1 and case 2. Once again can have two figures: one for each input.

MATLAB Code To Submit

- Submit all files in zipped folder. The submission should have a main MATLAB file, which on execution, will generate the various plots and returns value of performance indices. Keep the file name as $Assn\_02\_ROLLNO.m$ where ROLLNO is your roll number. The .mat file provided for continuous time linear perturbation model should be present in the same folder. [**Get in touch with TA for any issues.**]

- Make sure that the plots are labeled properly with axes names, titles, and legends.

Additional Learning Activity (For the curious; not to be submitted): In this assignment, we have considered the case of no disturbance. Now also implement MPC when the plant is influenced by disturbance (use some constant value for example). You will now find that MPC does not work well since its predictions don't match the actual plant states. Also, the set-points correspond to the steady state computed without considering disturbance. So, disturbance estimation and incorporation within MPC is important.

Learning is fun. Best of Luck!