Let $\dot{x} = F(t, x_0) = \begin{bmatrix} F_1(t, x_0) \\ \vdots \\ F_n(t, x_0) \end{bmatrix}$, $x \in \mathbb{R}^n$

$\dot{x}_i = F_i(t, x_0)$ $\qquad F_i : \mathbb{R} \to \mathbb{R}$

Let φ_i s be basis functions from $\mathbb{R} \to \mathbb{R}$, spanning a hilbert space $\mathcal{H}$.
and g_is be observables from $\mathbb{R} \to \mathbb{R}$.

For some $g \in \mathcal{H}$, $g(t) = \sum_{k=1}^{\infty} a_k \varphi_k(t)$

where $a_k = \langle g, \varphi_k \rangle = \int_0^T g(\tau) \overline{\varphi_k(\tau)} \, d\tau$

If $g \circ F_i \in \mathcal{H}$,

$g \circ F_i(t, x_0) = \sum_{j=1}^{\infty} b_j \varphi_j(t)$,

$$b_j = \int_0^T g \circ F_i(\tau, x_0) \overline{\varphi_j(\tau)} \, d\tau$$

$$b_j = \left\langle \left(\sum_{k=1}^{\infty} a_k \varphi_k(t) \right) \circ F_i(t, x_0), \varphi_j(t) \right\rangle$$

$$= \sum_{k=1}^{\infty} a_k \langle \varphi_k \circ F_i(t, x_0), \varphi_j(t) \rangle$$

$$g \circ F_i(t, x_0) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_k \langle \varphi_k \circ F_i(t, x_0), \varphi_j(t) \rangle \cdot \varphi_j(t)$$

switch

$$= \sum_{k=1}^{\infty} a_k \underbrace{\sum_{j=1}^{\infty} \langle \varphi_k \circ F_i(t, x_0), \varphi_j(t) \rangle \varphi_j(t)}_{\varphi_k \circ F_i(t, x_0)}$$

$$g \circ F_i(t, x_0) = \sum_{K=1}^{\infty} a_K \varphi_K \circ F_i(t, x_0)$$

$$= \sum_{K=1}^{\infty} \langle g, \varphi_K \rangle \varphi_K \circ F_i(t, x_0)$$

$$= \sum_{K=1}^{\infty} \varphi_K \circ F_i(t, x_0) \langle \bar{\varphi}_K(t), \underset{\uparrow \text{real}}{g}(t) \rangle$$

$$= \sum_{K=1}^{\infty} \varphi_K [F_i(t, x_0)] \int_0^T \bar{\varphi}_K(\tau) g(\tau) d\tau$$

switch

$$= \int_0^T \sum_{K=1}^{\infty} \varphi_K [F_i(t, x_0)] \bar{\varphi}_K(\tau) \cdot g(\tau) d\tau$$

$$\mathcal{K}_F(t, \tau) = \sum_{K=1}^{\infty} \varphi_K [F_i(t, x_0)] \bar{\varphi}_K(\tau)$$

$$\Rightarrow g[F_i(t, x_0)] = \int_0^T \mathcal{K}_{F_i}(t, \tau) \cdot g(\tau) d\tau$$

Let $t^1, t^2, \ldots, t^I, \ldots \in \mathbb{R}$ be points in $[0, T]$

$$K_F = \{ \mathcal{K}_F(t^I, \tau^J) \} \Delta\tau$$

$$K_F = \begin{bmatrix} \mathcal{K}_F(t^1, \tau^1) & \mathcal{K}_F(t^1, \tau^2) & \cdots \\ \mathcal{K}_F(t^2, \tau^1) & \mathcal{K}_F(t^2, \tau^2) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \Delta\tau$$

Kernel Matrix

Components evaluated at each grid point $\mathcal{K}_{F_i}(t^I, \tau^J)$

$$\vec{g} = \begin{bmatrix} g(\tau^1) \\ g(\tau^2) \\ \vdots \end{bmatrix}, \quad \vec{g}_{F_i} = \begin{bmatrix} g(F_i(t^1, x_0)) \\ g(F_i(t^2, x_0)) \\ \vdots \end{bmatrix}$$

$$g[F_i(t, x_0)] = \int \mathcal{K}_{F_i}(t, \tau) \cdot g(\tau)\, d\tau$$

becomes $\vec{g}_{F_i} = K_{F_i} \vec{g}$ (Infinite Dimensional)

State Transition Equation

This holds for any function $g \in \mathcal{H}$.

Since $\dot{x}_i(t) = F_i(t, x_0)$,

$$g(\dot{x}_i(t)) = K_{F_i} \cdot g(t)$$