Koopman Lifting Linearization

Hybrid Dynamical Systems: Multiple sets of dynamic equations switched depending on state of the system.

Find a global, unified representation of hybrid dynamical systems based on Koopman Lifting Linearization.

$$\text{Low Dimensional Space Non-linear} \xrightarrow[\text{Linearization}]{\text{Lifting}} \text{High-Dimensional Meta-space}$$

Koopman operator theory:

Non-linear autonomous system represented as linear dynamical system in infinite dimensional space. (Exact linearization)

$$K_f \varphi = \varphi \circ f$$

Math Review:

1) Inner Product:

V is vector space. For $x, y \in V$, $\langle x, y \rangle$ inner product has properties:

1) $\langle x, x \rangle \geq 0$, 0 iff $x = 0$
2) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

E.g. $V = \ell^2$, $\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$ Inf. Dim. vec. space

$V = C[0,1]$ with $\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}\,dx$ Func. space

2) Square-Integrable Functions:

Also called L^2 function.

$$f(x) \in L^2[-\infty, \infty] \Rightarrow \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$$

Finite

3) Orthonormal Functions:

$\varphi_i(x), \varphi_j(x)$ orthonormal if

$$\int_a^b \varphi_i(x)\, \varphi_j(x)\, w(x)\, dx = \delta_{ij}$$

Weight

4) Hilbert Space:

Complete inner product space (cauchy sequences converge within the space.

Let $\varphi_1(x), \varphi_2(x), \ldots$ be orthonormal basis in Hilbert Space $\mathcal{H}$. Then F.A.E

1) $\varphi_1, \varphi_2, \varphi_3, \ldots$ are complete.

2) $f = \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle \varphi_k \quad \forall f \in \mathcal{H}$

3) $\|f\|^2 = \sum_{k=1}^{\infty} |\langle f, \varphi_k \rangle|^2$

4) $\langle f, g \rangle = \sum_{K=1}^{\infty} \langle f, \varphi_K \rangle \overline{\langle g, \varphi_K \rangle}$

Proof of Existence of Koopman operator:

$g \circ F = K_F g$

Let $\mathcal{H}$ be hilbert space with L^2 functions, and

$$\langle f, g \rangle = \int_X f(x) \overline{g(x)} \, dx \qquad f, g \in \mathcal{H}$$

Let $\varphi_1, \varphi_2, \ldots, \varphi_K, \ldots$ span $\mathcal{H}$.

$g(x) = \sum_{K=1}^{\infty} a_K \varphi_K(x)$, where $a_K = \int_X g(x) \overline{\varphi_K(x)} dx$

Let $F : \mathbb{R}^n \to \mathbb{R}^n$, s.t $g \circ F = g[F(x)] \in \mathcal{H}$

So, $g \circ F(x) = \sum_{i=1}^{\infty} b_i \varphi_i(x)$, where

$$b_i = \langle g \circ F, \varphi_i \rangle$$
$$= \int_X g \circ F(x) \overline{\varphi_i(x)} dx$$

Now,

$$b_i = \left\langle \left(\sum_{K=1}^{\infty} a_K \varphi_K(x) \right) \circ F(x), \varphi_i(x) \right\rangle$$

Composition distributable, hence

$$b_i = \sum_{K=1}^{\infty} a_K \langle \varphi_K \circ F(x), \varphi_i(x) \rangle$$

Now, $g \circ F(x) =$

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_k \langle \varphi_k \circ F(x), \varphi_i(x) \rangle \varphi_i(x)$$

switch

$$= \sum_{k=1}^{\infty} a_k \underbrace{\sum_{i=1}^{\infty} \langle \varphi_k \circ F(x), \varphi_i(x) \rangle \varphi_i(x)}_{\varphi_k \circ F(x)}$$

$$= \sum_{k=1}^{\infty} a_k \varphi_k \circ F(x) = \sum_{k=1}^{\infty} \langle g, \varphi_k \rangle \varphi_k \circ F(x)$$

$$= \sum_{k=1}^{\infty} \varphi_k \circ F(x) \langle \bar{\varphi}_k(x), \underset{\uparrow \text{real}}{g(x)} \rangle$$

$$= \sum_{k=1}^{\infty} \varphi_k[F(x)] \int_X \bar{\varphi}_k(\xi) g(\xi) d\xi$$

switch

$$= \int_X \underbrace{\sum_{k=1}^{\infty} \varphi_k[F(x)] \bar{\varphi}_k(\xi)}_{\text{A Kernel}} \cdot g(\xi) d\xi$$

$$K_F(x, \xi) = \sum_{k=1}^{\infty} \varphi_k[F(x)] \bar{\varphi}_k(\xi)$$

$$\Rightarrow g[F(x)] = \int_X \underbrace{K_F}_{\text{Linear operator}}(x,\xi)\cdot g(\xi)\,d\xi$$

K_F : Koopman operator

$$g \circ F = K_F \cdot g$$

Let $x^1, x^2, \ldots, x^i, \ldots \in \mathbb{R}^n$ be gridpoints in $\mathbb{R}^n$

$$K_F = \{K_F(x^i, \xi^j)\}\Delta\xi$$

Kernel Matrix

$$K_F = \begin{bmatrix} K_F(x^1,\xi^1) & K_F(x^1,\xi^2) & \cdots \\ K_F(x^2,\xi^1) & K_F(x^2,\xi^2) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}\Delta\xi$$

Components evaluated at each grid point $K_F(x^i,\xi^j)$ $\xi^j \in \mathbb{R}^n$

$$\vec{g} = \begin{bmatrix} g(\xi^1) \\ g(\xi^2) \\ \vdots \end{bmatrix}, \quad \vec{g}_F = \begin{bmatrix} g(F(x^1)) \\ g(F(x^2)) \\ \vdots \end{bmatrix}$$

$$g[F(x)] = \int_X K_F(x,\xi)\cdot g(\xi)\,d\xi$$

becomes $\vec{g}_F = K_F\,\vec{g}$ (Infinite Dimensional)

State Transition Equation

$g_i(x_j)$

This holds for any f^n $g \in \mathcal{H}$.

$$[g_{1F}, g_{2F}, \cdots, g_{MF}, \cdots] = K_F \cdot [g_1, g_2, \cdots, g_M, \cdots]$$

Considering a discrete-time, non-linear dynamical system:

$$x_{t+1} = F(x_t), \quad y_t = g(x_t)$$

$$g_{1F}(x_t^i) = g_1[F(x_t^i)] = g_1(x_{t+1}^i)$$

x_t^i are all grid points at time t.

$$\underbrace{\begin{bmatrix} g_1(x_{t+1}^1) & g_2(x_{t+1}^1) & \cdots \\ g_1(x_{t+1}^2) & g_2(x_{t+1}^2) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}}_{Y} = K_F \cdot \underbrace{\begin{bmatrix} g_1(x_t^1) & g_2(x_t^1) & \cdots \\ g_1(x_t^2) & g_2(x_t^2) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}}_{X}$$

$$[g_1(x_t) \; g_2(x_t) \cdots] \xrightarrow[K_F]{} [g_1(x_{t+1}) \; g_2(x_{t+1}) \cdots]$$

Each row represents set of observables. Koopman operator transfers system from $t \to t+1$.

Limiting to N grid points and taking transpose,

$$\begin{bmatrix} g_1(x_{t+1}^0) & \cdots & g_1(x_{t+1}^N) \\ \vdots & & \\ g_M(x_{t+1}^0) & \cdots & g_M(x_{t+1}^N) \end{bmatrix} = \underbrace{\begin{bmatrix} g_1(x_t^0) & \cdots & g_1(x_t^N) \\ \vdots & & \\ g_M(x_t^0) & \cdots & g_M(x_t^N) \end{bmatrix}}_{X^T} \cdot K_F^T$$

Defining A s.t $X^T K_F^T = A X^T$, i.e

$$A = X^T K_F^T (X^T)^\dagger$$

Thus, State Transition eq^n is

$$\begin{bmatrix} g_1(x_{t+1}) \\ g_2(x_{t+1}) \\ \vdots \\ g_M(x_{t+1}) \end{bmatrix} = A \cdot \begin{bmatrix} g_1(x_t) \\ g_2(x_t) \\ \vdots \\ g_M(x_t) \end{bmatrix}$$

New state vector

Let $z_t \equiv \begin{bmatrix} g_1(x_t) \\ g_2(x_t) \\ \vdots \end{bmatrix}$ be new state vector in lifted space

$x_{t+1} = F(x_t)$ can now be represented as

$$z_{t+1} = A z_t$$

where $A = Q^T (R^\dagger)^T$

$$R = \begin{bmatrix} \langle g_1, g_1 \rangle & \langle g_1, g_2 \rangle & \cdots \\ \langle g_2, g_1 \rangle & \langle g_2, g_2 \rangle & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

$$Q = \begin{bmatrix} \sum_k \langle \varphi_k \circ F, g_1 \rangle \langle g_1, \varphi_k \rangle & \sum_k \langle \varphi_k \circ F, g_1 \rangle \langle g_2, \varphi_k \rangle & \cdots \\ \sum_k \langle \varphi_k \circ F, g_2 \rangle \langle g_1, \varphi_k \rangle & \sum_k \langle \varphi_k \circ F, g_2 \rangle \langle g_2, \varphi_k \rangle & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

If g_is are φ_is, then $R = I$,

$$Q_{ij} = \sum_K \langle \varphi_K \circ F, \varphi_i \rangle \underbrace{\langle \varphi_j, \varphi_K \rangle}_{\delta_{jK}}$$

$$= \langle \varphi_j \circ F, \varphi_i \rangle$$

$$A = Q^T (R^{\dagger})^T = Q^T$$

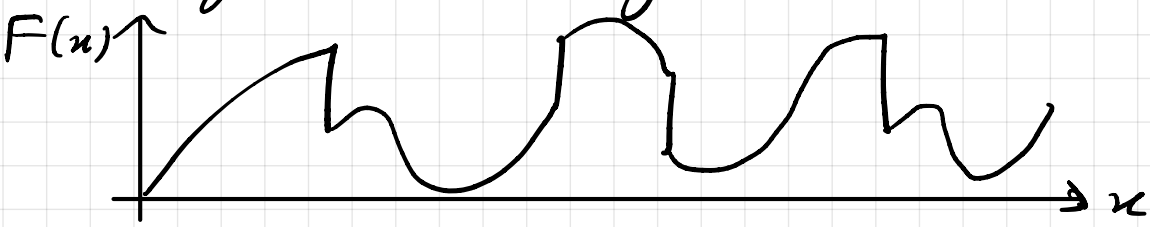
Time evolution: $\begin{bmatrix} \varphi_1(x_{t+1}) \\ \varphi_2(x_{t+1}) \\ \vdots \end{bmatrix} = \underset{(Q^T)}{A} \begin{bmatrix} \varphi_1(x_t) \\ \varphi_2(x_t) \\ \vdots \end{bmatrix}$

R.H.S is $\begin{bmatrix} \sum_i \langle \varphi_1 \circ F, \varphi_i \rangle \varphi_i(x_t) \\ \sum_i \langle \varphi_2 \circ F, \varphi_i \rangle \varphi_i(x_t) \\ \vdots \end{bmatrix} = \begin{bmatrix} \varphi_1 \circ F(x_t) \\ \varphi_2 \circ F(x_t) \\ \vdots \end{bmatrix}$

$$= \begin{bmatrix} \varphi_1(x_{t+1}) \\ \varphi_2(x_{t+1}) \\ \vdots \end{bmatrix}$$

Can be applied to Hybrid systems, where $F(x)$ varies dynamically, but conditions are

1) F is absolutely integrable.
2) Gaps are finite.
3) Finite discontinuities in finite interval.
4) No. of max, minima finite.



If K_F correctly constructed, applies to any observable function in the Hilbert space.

For example, we can use trigonometric functions or polynomial functions for observables, but same K_F kernel matrix.

Dynamic Mode Decomposition:

$A = X^T K_F^T (X^T)^\dagger$ was found using Koopman.

Instead, if we have the data of X, Y,

$$Y \cong AX, \text{ hence } A = YX^\dagger,$$

SVD done for $X^\dagger$.

Linear relationship is assumed in DMD.

If K_F exists, DMD is valid. i.e, if observables belong to a hilbert space, as well as $g \circ F$.

1) Effective finite set of observables needed.

2) Exact linearization applicable only to autonomous systems.

Physical systems are linear in structure, e.g. spring-mass-damper systems

$K^{-1}g(x)$ $\quad Kg(x) = Ag(x)$

$g(x) = x$

x, Ax, A^2x

$$K\begin{bmatrix} \langle g, \varphi_1 \rangle \\ \langle g, \varphi_2 \rangle \\ \vdots \end{bmatrix} \quad x = \sum a_i \varphi_i$$

$$x = \sum \langle x, g_i \rangle g_i \qquad \sum a_i g_i$$

$$x = [\langle x, \varphi_1 \rangle \quad \cdots \quad \langle x, \varphi_m \rangle] \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_m \end{bmatrix}$$