E11)

$X \sim Gamma(b, p)$, $Y \sim Gamma(b, q)$, $U = X + Y$, $V = \frac{X}{X+Y}$

$$\begin{cases} X = \frac{VY}{1-V} \\ Y = U - X \end{cases} \qquad \begin{cases} Y = \frac{U}{1-\frac{1}{1-V}} \\ X = \frac{VY}{1-V} \end{cases} \qquad \begin{matrix} Y = U(1-V) \\ X = VU \end{matrix} \qquad \begin{matrix} u \in (0,\infty) \\ v \in (0,1) \end{matrix}$$

$$J = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u$$

$$f_{X,Y}(x,y) = \frac{b^p}{\Gamma(p)} \cdot x^{p-1} \cdot e^{-bx} \cdot \frac{b^q}{\Gamma(q)} \cdot y^{q-1} \cdot e^{-by} =$$

$$= \frac{b^{p+q} \cdot e^{-b(x+y)}}{\Gamma(p)\Gamma(q)} \cdot x^{p-1} \cdot y^{q-1}$$

$$g_{V,U}(v,u) = \frac{b^{p+q} \cdot e^{-b(u)}}{\Gamma(p) \cdot \Gamma(q)} \cdot (vu)^{p-1} \cdot u^{q-1} \cdot (1-v)^{q-1} \cdot u =$$

$$= \frac{b^{p+q} \cdot e^{-bu}}{\Gamma(p) \cdot \Gamma(q)} \cdot v^{p-1} \cdot (1-v)^{q-1} \cdot u^{q+p-1}$$

b) $$g_U(u) = \int_0^1 \frac{b^{p+q} \cdot e^{-bu}}{\Gamma(p) \cdot \Gamma(q)} \cdot \frac{\Gamma(p+q)}{\Gamma(p+q)} \cdot v^{p-1} \cdot (1-v)^{q-1} \cdot u^{q+p-1} \, dv =$$

$$= \frac{b^{p+q} \cdot e^{-bu}}{\Gamma(p+q)} \cdot u^{p+q-1} \cdot \underbrace{\int_0^1 \frac{\Gamma(p+q)}{\Gamma(p) \cdot \Gamma(q)} \cdot v^{p-1} \cdot (1-v)^{q-1} \, dv}_{Beta(p,q)} =$$

$$= \frac{b^{p+q} \cdot e^{-bu}}{\Gamma(p+q)} \cdot u^{p+q-1}$$, vagyis $U = X + Y \sim Gamma(b, p+q)$

c) $g_v(v) = \int_0^\infty \frac{b^{p+q} \cdot e^{-bu}}{\Gamma(p)\cdot\Gamma(q)} \cdot \frac{\Gamma(p+q)}{\Gamma(p+q)} \cdot v^{p-1} \cdot (1-v)^{q-1} \cdot u^{p+q-1} \, du =$

$= \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \cdot v^{p-1} \cdot (1-v)^{q-1} \cdot \underbrace{\int_0^\infty \frac{b^{p+q} \cdot e^{-bu}}{\Gamma(p+q)} \cdot u^{p+q-1} \, du}_{\text{Gamma}(b,\, p+q)} =$

$= \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \cdot v^{p-1} \cdot (1-v)^{q-1}$

czyli $V \sim Beta(p, q$

d) $g_u(u) \cdot g_v(v) = g_{u,v}(u,v)$ czyli niezależne

12)

$f(a,b) = \sum_{i=1}^{n} (b + a x_i - y_i)^2$ ← szukamy minimum

z wykładu:

$a = \frac{\sum x_i y_i - n \bar{x} \bar{y}}{\sum x_i^2 - n \bar{x}^2}$ $\quad b = \bar{y} - a\bar{x}$

13)

$y = ax + bz + c$

$f(a,b,c) = \sum_{i=1}^{n} (a x_i + b z_i + c - y_i)^2$

$\beta = (X^T X)^{-1} X^T Y$

Z14)

$P(X=i)=\frac{1}{100} \qquad i\in\{1,2,\dots,100\}$

$Y=\begin{cases}1, & 2|X \vee 3|X\\ 0, & \text{wpp}\end{cases} \qquad Z=\begin{cases}1, & 3|X\\ 0, & \text{wpp}\end{cases}$

$\rho=\frac{Cov(Y,Z)}{\sqrt{V(Z)\cdot V(Y)}} \qquad V(X)=E(X^2)-E(X)^2$

$Cov(X,Y)=E(XY)-E(X)\cdot E(Y)$

$E(Y)=\sum_{i=1}^{100} y_i\cdot p_i=\frac{1}{100}\cdot\sum_{i=1}^{100} y_i=\frac{1}{100}\cdot 67=0,67$

$50+33-16$

$2| \quad 3| \quad 2\wedge 3|$

$E(Z)=\sum_{i=1}^{100} z_i\cdot p_i=\frac{1}{100}\cdot\sum_{i=1}^{100} z_i=0,33$

$3|$

$E(ZY)=0,33 \quad \Leftarrow 3|$

$E(Z^2)=0,33$

podzielność ta sama

$E(Y^2)=0,67$

czyli

$\rho=\frac{\frac{33}{100}-\frac{33\cdot 67}{10000}}{\sqrt{\left(\frac{67}{100}-\left(\frac{67}{100}\right)^2\right)\cdot\left(\frac{33}{100}-\left(\frac{33}{100}\right)^2\right)}}=\frac{\frac{1089}{10000}}{\sqrt{\frac{2211}{10000}\cdot\frac{2211}{10000}}}=\frac{1089}{2211}=\frac{33}{67}$ ✓