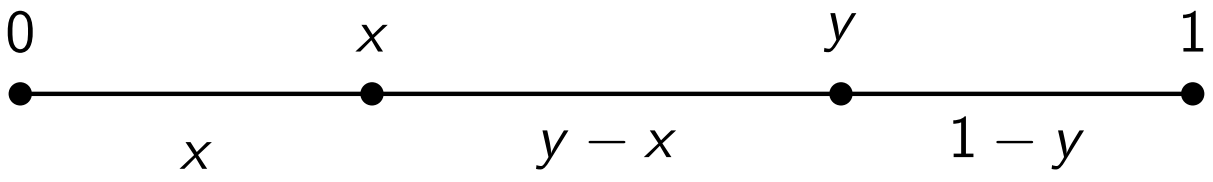
Exercise 1



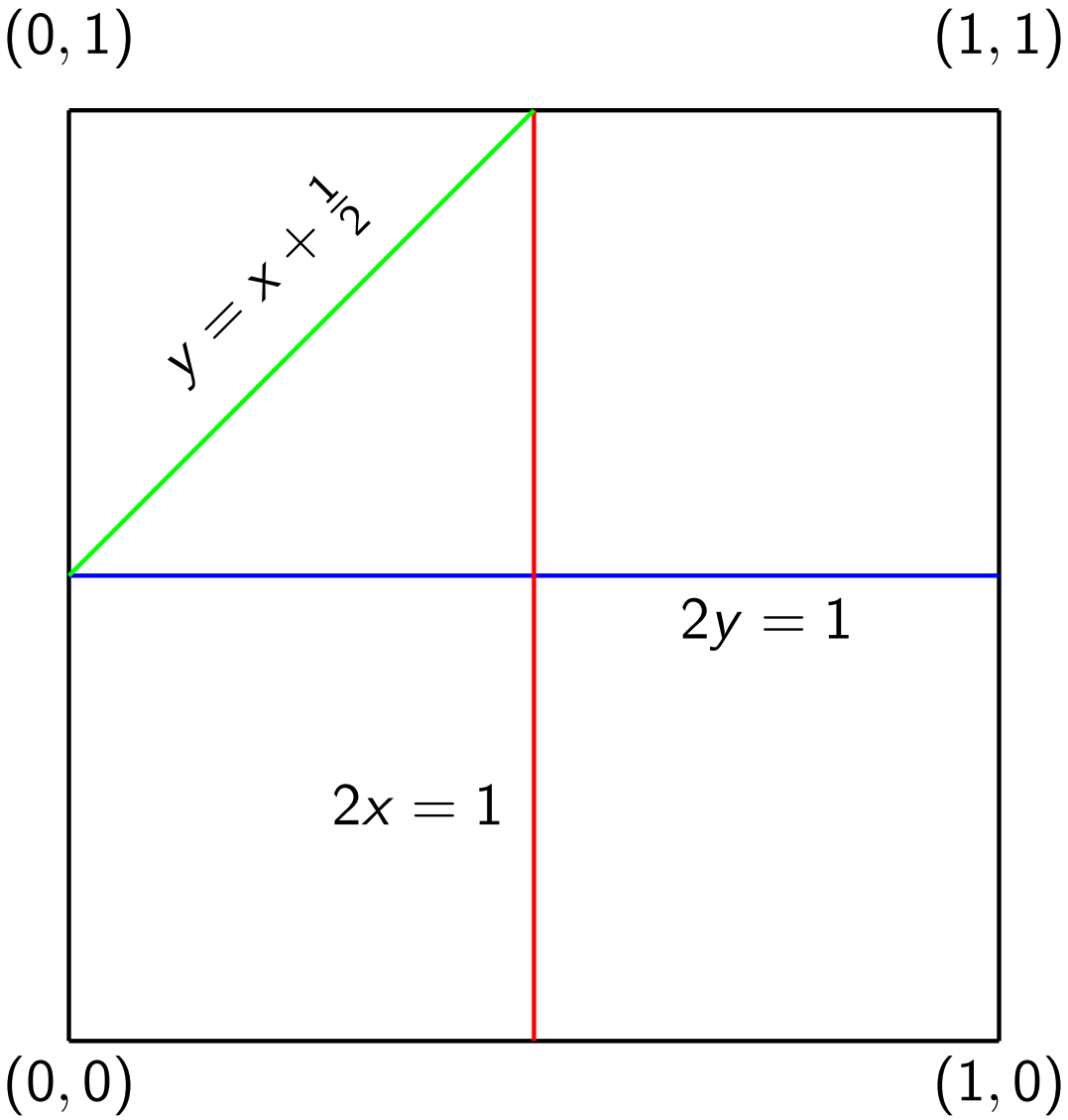
Let us assume that $x < y$. Therefore the three segments are x, $y - x$, $1 - y$. These segments create a triangle if and only if the sum of any two line segments are bigger than the third line segment.

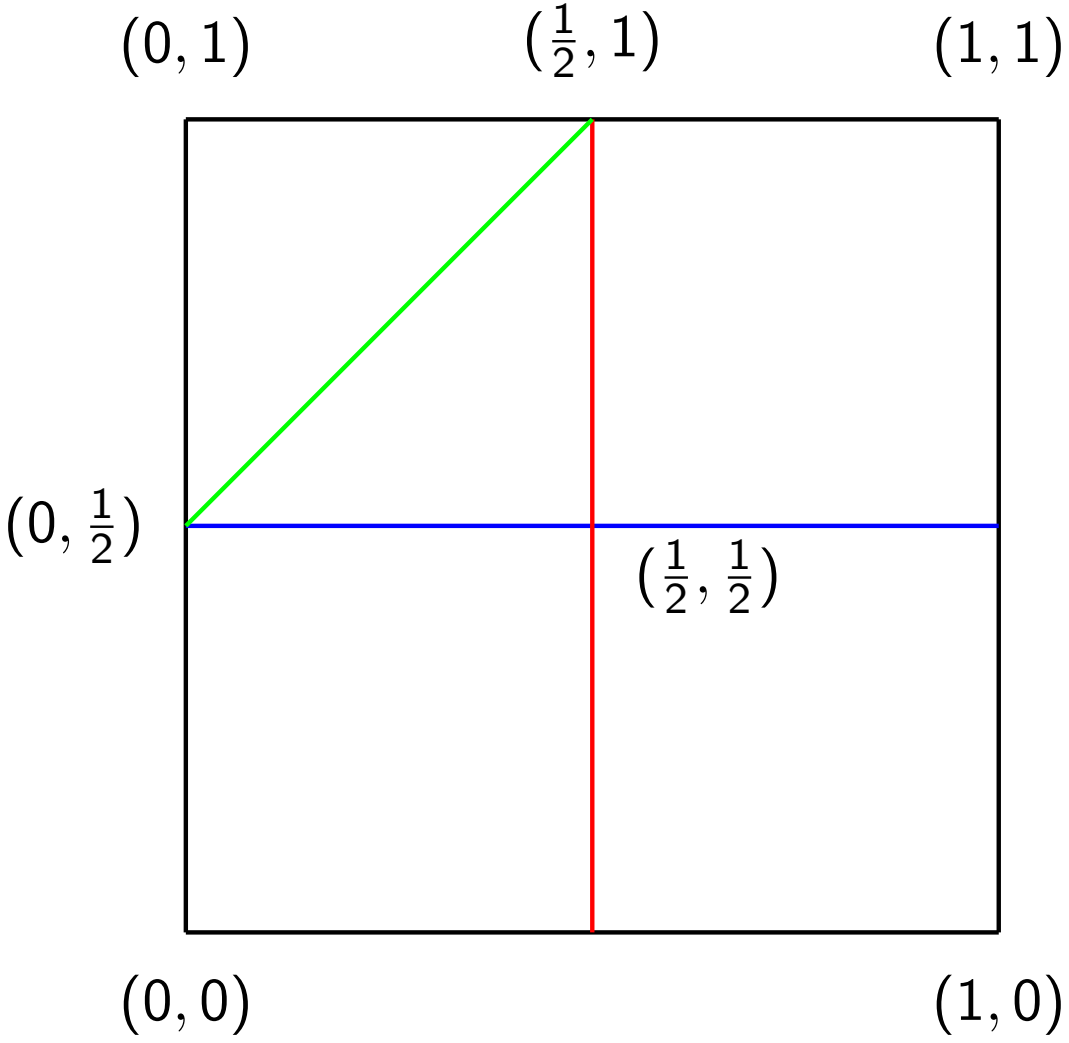
The inequalities

- $x + (y - x) > (1 - y)$
- $x + (1 - y) > (y - x)$
- $(y - x) + (1 - y) > x$

$\Longrightarrow$

- $2y > 1$
- $2x + 1 > 2y$
- $2x < 1$


$(0,1)$
$(1,1)$
$y = x + \frac{1}{2}$
$2y = 1$
$2x = 1$
$(0,0)$
$(1,0)$



Area of the Triangle

The area of the triangle is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

The bijection

The Function

Let x, y be two numbers from the set $[0, 1]$. Let us consider the function

$$f : x, y \longrightarrow (x, y)$$

where (x, y) is a point in the unit length square.

It is clear that the function is bijective.

Probability

- The area of the rectangle is 1.
- The area of the triangle when $y < x$ is also $\frac{1}{8}$.

Probability

The probability of the three segments creating a triangle is $\frac{1}{8} \times 2 = \frac{1}{4}$.

Exercise 2

Input

$f(x_1, x_2) = \frac{1}{\pi}$ where $0 < x_1^2 + x_2^2 < 1$

Solution

$$f_1(x_1) = \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \frac{1}{\pi} dx = \frac{2}{\pi}\sqrt{1 - x_1^2}$$

$$f_2(x_2) = \int_{-\sqrt{1-x_2^2}}^{\sqrt{1-x_2^2}} \frac{1}{\pi} dx = \frac{2}{\pi}\sqrt{1 - x_2^2}$$

3. $f(x_1, x_2) = \frac{1}{\pi}$ when $0 < x_1^2 + x_2^2 < 1$

$$\rho(x_1, x_2) = \frac{\mathrm{cov}(X_1, X_2)}{\sqrt{V(X_1)\,V(X_2)}} \leftarrow \text{this needs to be } 0$$

$$\mathrm{Cov}(X_1, X_2) = E(X_1, X_2) - E(X_1) \cdot E(X_2)$$

$$E(X_1) = \int_{-1}^{1} \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \frac{x_1}{\pi}\, dx_2\, dx_1 = \int_{-1}^{1} \frac{2x_1\sqrt{1-x_1^2}}{\pi} = 0$$

$$E(X_2) = \int_{-1}^{1} \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \frac{x_2}{\pi}\, dx_2\, dx_1 = \int_{-1}^{1} 0\, dx_1 = 0$$

$$E(X_1 X_2) = \int_{-1}^{1} \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \frac{x_1 x_2}{\pi}\, dx_2\, dx_1 = \int_{-1}^{1} 0\, dx_1 = 0$$

$\mathrm{Cov} = 0 \Rightarrow \rho(x_1, x_2) = 0$

$x_2 = \sqrt{1-x_1^2}$ not independent

Probability class

Task 9.

List 5

23 kwietnia 2020

Let $X_1 = Y_1 cosY_2$, $X_2 = Y_1 sinY_2$ $(0 < Y_1 < 1, 0 \leq Y_2 \leq 2\pi)$. Find density $g(y_1, y_2)$ of variable (Y_1, Y_2). Check if variables Y_1,Y_2 are independent. Let (X_1, X_2) be two-dimensional random variable with density function $f(x_1, x_2) = \frac{1}{\pi}$, when $0 < x_1^2 + x_2^2 < 1$.

To find the $g(y_1, y_2)$ of variable (Y_1, Y_2) we will transform the density function $f(x_1, x_2)$ using the formula:

$g(y_1, y_2) = f(w_1(y_1, y_2), w_2(y_1, y_2)) \cdot |J|$

where J is the Jacobian transformation matrix, and w a transformation function of r.v. X_1 and X_2.

In this case we already have the w functions given:

$X_1 = Y_1 cosY_2$
$X_2 = Y_1 sinY_2$

Let's write the Jacobian matrix $J(X_1, X_2|Y_1, Y_2)$ of this transformation and find its determinant.

$$|J| = \begin{vmatrix} cos(Y_2) & -Y_1(sinY2) \\ sin(Y_2) & Y_1 cos(Y_2) \end{vmatrix} = |Y_1 \cdot cos^2(Y_2) + Y_1 \cdot sin^2(Y_2)| = |Y_1| = Y_1 \quad (1)$$

Now we can find the value of $g(y_1, y_2)$:

$g(y_1, y_2) = \frac{1}{\pi} \cdot y_1 = \frac{y_1}{\pi}$

So this is the density function of variable (Y_1, Y_2), and now we are going to find the densities of Y_1 and Y_2 so we can check if they are independent.

$$g(y_1) = \int_0^{2\pi} g(y_1, y_2) dy_2 = \int_0^{2\pi} \frac{y_1}{\pi} = \left[\frac{y_1 y_2}{\pi}\right]_0^{2\pi} = 2y_1$$
$$g(y_2) = \int_0^{1} g(y_1, y_2) dy_1 = \int_0^{1} \frac{y_1}{\pi} = \left[\frac{y_1^2}{2\pi}\right]_0^{1} = \frac{1}{2\pi}$$

Random variables are independent if $f(x, y) = f(x) \cdot f(y)$. Let's check if random variables Y_1 and Y_2 are independent.

$$g(y_1, y_2) = \frac{y_1}{\pi} = \frac{1}{2\pi} \cdot 2y_1 = g(y_2) \cdot g(y_1)$$

That means random variables Y_1 and Y_2 are independent.

Szymon Kosakowski 309980

April 2020

List 7 Exercise 5

We are given n-dimensional variable $X = (X_1, \dots, X_n)^T$. Variable $Y = (Y_1, \dots, Y_n)^T$ is specified by:

$$Y_1 = \overline{X},\ Y_k = X_k - \overline{X}\ dla\ k = 2, \dots, n$$

. Find the Jakobian of transformation.

$$X_k = Y_k + Y_1,\ dla\ k = 2, \dots, n$$

$$nY_1 = \sum_{k=1}^{n} X_k \Rightarrow nY_1 = X_1 + X_2 + \dots + X_n = X_1 + (Y_2 + Y_1) + (Y_3 + Y_1) + \dots + (Y_n + Y_1) \Rightarrow X_1 = Y_1 - Y_2 - Y_3 - \dots - Y_n$$

Let's also calculate X_2:

$$nY_1 = X_1 + X_2 + (Y_3 + Y_1) + \dots + (Y_n + Y_1) \Rightarrow X_2 = nY_1 - X_1 - [(Y_3 + Y_1) + (Y_4 + Y_1) + \dots + (Y_n + Y_1)]$$

Inserting $X_1 = Y_1 - Y_2 - \dots - Y_n$

$$X_2 = Y_1 + Y_2$$

Here we can assume the formula for next variables:

$$X_3 = Y_1 + Y_3$$
$$X_4 = Y_1 + Y_4$$

,

$$\dots$$

,

$$X_n = Y_1 + Y_n$$

therefore we get Jakobian:

$$\begin{vmatrix} 1 & -1 & -1 & \dots & -1 \\ 1 & 1 & & & \\ 1 & & 1 & & \\ 1 & & & \ddots & \\ 1 & & & & 1 \end{vmatrix}$$

Which determinant we already calculated before on 1st list.

Probability and statistics

Set 7, Task 6

Kamil Fedzin

April 23, 2020

1 Task 6

Given are random variables $X_1, ..., X_n$. Prove that

$$\sum_{k=1}^{n}(X_k - \mu)^2 = \sum_{k=1}^{n}(X_k - \overline{X})^2 + n(\overline{X} - \mu)^2$$

2 Solution

Proof.

$$\sum_{k=1}^{n}(X_k-\mu)^2 = \sum_{k=1}^{n}(X_k-\overline{X}+\overline{X}-\mu)^2 = \sum_{k=1}^{n}\Big((X_k - \overline{X})^2 + (\overline{X} - \mu)^2 + 2(X_k - \overline{X})(\overline{X} - \mu)\Big) =$$

$$= \sum_{k=1}^{n}(X_k-\overline{X})^2+n(\overline{X}-\mu)^2+2(\overline{X}-\mu)\left(\sum_{k=1}^{n}(X_k - \overline{X})\right) = \sum_{k=1}^{n}(X_k-\overline{X})^2+n(\overline{X}-\mu)^2+2(\overline{X}-\mu)\cdot$$

$$\cdot\left(\sum_{k=1}^{n}X_k - \sum_{k=1}^{n}\overline{X}\right) = \sum_{k=1}^{n}(X_k - \overline{X})^2 + n(\overline{X} - \mu)^2 + 2(\overline{X} - \mu)\left(nY_1 - n\overline{X}\right) =$$

$$= \sum_{k=1}^{n}(X_k - \overline{X})^2 + n(\overline{X} - \mu)^2 + 2(\overline{X} - \mu)\cdot 0 = \sum_{k=1}^{n}(X_k - \overline{X})^2 + n(\overline{X} - \mu)^2$$

□

Set 7 Exercise 7

Wiktor Hamberger
308982

April 23, 2020

We have to find distribution of $M = \frac{n}{\sigma^2}(\overline{X}-\mu)^2 = (\frac{\sqrt{n}}{\sigma}\overline{X} - \frac{\sqrt{n}}{\sigma}\mu)^2$, assuming that independent random variables $X_k \sim N(\mu, \sigma^2)$

$$M_{X_k}(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$M_{\sum\limits_{k=1}^{n} X_k}(t) = e^{n\mu t + \frac{n}{2}\sigma^2 t^2}$$

$$M_{\frac{1}{n}\sum\limits_{k=1}^{n} X_k}(t) = M_{\overline{X}}(t) = e^{\mu t + \frac{1}{2n}\sigma^2 t^2}$$

$$M_{\frac{\sqrt{n}}{\sigma}\overline{X}}(t) = e^{\mu \frac{\sqrt{n}}{\sigma} t + \frac{1}{2n}\sigma^2 \frac{n}{\sigma^2} t^2} = e^{\mu \frac{\sqrt{n}}{\sigma} t + \frac{1}{2} t^2}$$

$$M_{\frac{\sqrt{n}}{\sigma}\overline{X} - \frac{\sqrt{n}}{\sigma}\mu}(t) = e^{\mu \frac{\sqrt{n}}{\sigma} t + \frac{1}{2} t^2} e^{-\frac{\sqrt{n}}{\sigma}\mu t} = e^{\frac{1}{2} t^2} \Rightarrow$$

$$\Rightarrow \frac{\sqrt{n}}{\sigma}\overline{X} - \frac{\sqrt{n}}{\sigma}\mu \sim N(0,1)$$

Let's $Z = \frac{\sqrt{n}}{\sigma}\overline{X} - \frac{\sqrt{n}}{\sigma}\mu$, then $Z \sim N(0,1) \Rightarrow Z^2 \sim \chi^2(1)$ and $Z^2 = M$, so $\underline{\underline{M \sim \chi^2(1)}}$.

Task nr 8

Contents:

Let r.vs. $\overline{\mathbf{X}} = \frac{1}{n}\sum\limits_{k=1}^{n} X_k$ and $S^2 = \frac{1}{n}\sum\limits_{k=1}^{n}(X_k - \overline{\mathbf{X}})^2$ be independent. Using equation (1) prove that $\frac{nS^2}{\delta^2} \equiv Gamma\left(\frac{1}{2}, \frac{n-1}{2}\right) \sim \chi^2(n-1)$

Solution:

We can see that $S^2 = \frac{1}{n}\sum\limits_{k=1}^{n}(X_k - \overline{\mathbf{X}})^2 \Rightarrow nS^2 = \sum\limits_{k=1}^{n}(X_k - \overline{\mathbf{X}})^2$, also, from previous tasks we know, that $\chi^2(n) = \sum\limits_{k=1}^{n}\left(\frac{X_k - \mu}{\delta}\right)^2, \chi^2(1) = n\left(\frac{\overline{\mathbf{X}} - \mu}{\delta}\right)^2$, that $M_{\chi^2(k)}(t) = (1-2t)^{-\frac{k}{2}}$, and that $M_{Gamma(\alpha,\beta)}(t) = (1-\frac{t}{\beta})^{-\alpha}$ Now, armed in all these facts, we can calculate:

$$\sum_{k=1}^{n}(X_k - \mu)^2 = \sum_{k=1}^{n}(X_k - \overline{\mathbf{X}})^2 + n(\overline{\mathbf{X}} - \mu)^2$$

$$\frac{1}{\delta^2}\sum_{k=1}^{n}(X_k - \mu)^2 = \frac{nS^2}{\delta^2} + \frac{n}{\delta^2}(\overline{\mathbf{X}} - \mu)^2$$

$$M_{\frac{1}{\delta^2}\sum\limits_{k=1}^{n}(X_k-\mu)^2}(t) = M_{\frac{nS^2}{\delta^2}}(t) M_{\frac{n}{\delta^2}(\overline{\mathbf{X}}-\mu)^2}(t)$$

$$M_{\frac{nS^2}{\delta^2}}(t) = \frac{M_{\frac{1}{\delta^2}\sum\limits_{k=1}^{n}(X_k-\mu)^2}(t)}{M_{\frac{n}{\delta^2}(\overline{\mathbf{X}}-\mu)^2}(t)} = \frac{M_{\chi^2(n)}(t)}{M_{\chi^2(1)}(t)} = \frac{(1-2t)^{-\frac{n}{2}}}{(1-2t)^{-\frac{1}{2}}} = (1-2t)^{-\frac{n-1}{2}}$$

$$\frac{nS^2}{\delta^2} = Gamma\left(\frac{1}{2}, \frac{n-1}{2}\right) \sim \chi^2(n-1)$$

$\square$

Question 9

Andrzej Tkaczyk, 309181

April 23, 2020

X_1, X_2 are independent and $X_1 \sim U[1,2], X_2 \sim U[1,2]$.

We know, that $E(X+Y) = EX+EY$ and $E(XY) = EX*EY$, if X and Y are independent.

$E(Y_1) = E(2X_1 + 2X_2) = E(2X_1) + E(2X_2) = 2 * E(X_1) + 2 * E(X_2) = 2 * \int_1^2 x\ dx + 2 * \int_1^2 x\ dx = 6.$

$E(Y_2) = E(X_1X_2) = E(X_1) * E(X_2) = \int_1^2 x\ dx * \int_1^2 x\ dx = \frac{3}{2} * \frac{3}{2} = \frac{9}{4}.$

$V(X+Y) = V(X) + V(Y)$, if X and Y are independent.

Proof from Wiki.

$V(Y_1) = V(X_1 + X_2) = V(X_1) + V(X_2) = E(4X_1^2) - E^2(2X_1) + E(4X_2^2) - E^2(2X_2) =$

$= 4 * \int_1^2 x^2\ dx - 3^2 + 4 * \int_1^2 x^2\ dx - 3^2 = 2 * 4 * \frac{7}{3} - 18 = \frac{56}{3} - \frac{54}{3} = \frac{2}{3}.$

$V(Y_2) = V(X_1X_2) = E(X_1^2X_2^2) - E^2(X_1X_2) = E(X_1^2) * E(X_2^2) - E^2(X_1) * E^2(X_2) =$

$= \int_1^2 x^2\ dx * \int_1^2 x^2\ dx - (\frac{3}{2})^2 * (\frac{3}{2})^2 = (\frac{7}{3})^2 - (\frac{3}{2})^4 = \frac{49}{9} - \frac{81}{16} = \frac{784}{144} - \frac{729}{144} = \frac{55}{144}.$

TASK 10

$$Y_1 = 2X_1 + 2X_2$$
$$Y_2 = X_1 X_2 \qquad \rho = ?$$

We know that $EY_1 = 6$, $EY_2 = \frac{9}{4}$, $V(Y_1) = \frac{2}{3}$
$EX_1 = \frac{3}{2}$, $EX_2 = \frac{3}{2}$, $V(Y_2) = \frac{55}{144}$

$$E(Y_1 Y_2) = E((2X_1 + 2X_2) \cdot X_1 X_2) =$$
$$2E(X_1^2 X_2 + X_1 X_2^2) = 2E(X_1^2)EX_2 + 2EX_1 EX_2^2 =$$
$$= 2 \cdot \frac{7}{3} \cdot \frac{3}{2} + 2 \cdot \frac{3}{2} \cdot \frac{7}{3} = 14$$

$$E(X_2^2) = E(X_1^2) = \int_1^2 x^2 \, dx = \frac{x^3}{3}\Big|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

$$\mathrm{cov}(Y_1, Y_2) = E(Y_1 Y_2) - EY_1 EY_2 =$$
$$14 - 6 \cdot \frac{9}{4} = \frac{28 - 27}{2} = 1/2$$

$$\rho = \frac{\mathrm{cov}(Y_1, Y_2)}{\sigma_1 \sigma_2} = \frac{\mathrm{cov}(Y_1, Y_2)}{\sqrt{V(Y_1)V(Y_2)}} = \frac{1/2}{\sqrt{\frac{2}{3} \cdot \frac{55}{144}}} = \frac{1/2}{\sqrt{\frac{110}{3}}} =$$
$$= \frac{6\sqrt{3}}{\sqrt{110}} = \frac{3\sqrt{330}}{55}$$

where $\sigma_1 = \sqrt{V(Y_1)}$, $\sigma_2 = \sqrt{V(Y_2)}$

Ex. 11

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 3X_1 + X_2 \\ -4X_1 + 2X_2 \end{bmatrix} = \overset{A}{\begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$X \sim N\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 38 & -5 \\ -5 & 4 \end{bmatrix} \right)$$

$$Y \sim N\left(\underbrace{A \begin{bmatrix} 1 \\ 4 \end{bmatrix}}_{M}, \underbrace{A \begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix} A^T}_{\Sigma} \right)$$

$$\begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

$$M = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 316 & -458 \\ -458 & 704 \end{bmatrix}$$

$$Y \sim N\left(\begin{bmatrix} 7 \\ 4 \end{bmatrix}, \begin{bmatrix} 316 & -458 \\ -458 & 704 \end{bmatrix} \right)$$

Ex. 12

$$\boxed{\begin{aligned} \Sigma_{ii} &= V[X_i] \\ \Sigma_{ij} &= Cov(X_i, X_j) \\ i \neq j \end{aligned}}$$

$$Y = N\left(A \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \underbrace{A \begin{bmatrix} 38 & -5 \\ -5 & 4 \end{bmatrix} A^T}_{\Sigma} \right)$$

$$\Sigma = \begin{bmatrix} 248 & 320 \\ 320 & 544 \end{bmatrix} = \begin{bmatrix} V[Y_1] & Cov(Y_1, Y_2) \\ Cov(Y_2, Y_1) & V[Y_2] \end{bmatrix}$$

$$\rho_{Y_1, Y_2} = \frac{Cov(Y_1, Y_2)}{\sqrt{V[Y_1] V[Y_2]}} = \frac{320}{\sqrt{248 \cdot 544}} = \frac{20\sqrt{527}}{527}$$