RPiS

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April 30, 2020

1 Task 8.1

We know that

$$\beta_1 = \frac{\sum_{i=1}^{n}(x_i - \overline{x}) \cdot (y_i - \overline{y})}{\sum_{i=1}^{n}(x_i - \overline{x})^2} \tag{1}$$

$$\beta_0 = \overline{y} - \beta_1 \overline{x} \tag{2}$$

Where $\overline{x}$ is average value of X, same for Y. Using the formula, we can easily calculate the regression line of temperature (Y) with respect to altitude (X) $y = \beta_0 + \beta_1 x$

```
function calculate()
    numerator=0
    denominator=0
    for i in 1:16
        numerator +=(altitude[i]-average_x)*(temperature[i]-average_y)
        denominator+=square(altitude[i]-average_x)
    end
    b1=numerator/denominator
    tab=[b1, average_y-b1*average_x]
    return tab
end
```

We can also use Exels build in function REGLINP.
The answer is (up to 6th decimal point) $y = 9.211601 + 0.000395 \cdot x$

Problem Set 8, task 2. Find regression line of temperature with respect to latitude and longitude. (Z depends on X and Y). let's name it different, Y=Z, X=X1, Y=X2/

Y – temperature
X1– latitude
X2– longitude

We need to find function y = a + b1*X1 + b2*X2. It's 3 dimensional function, value (y) depends on 2 other variables X1 and X2.

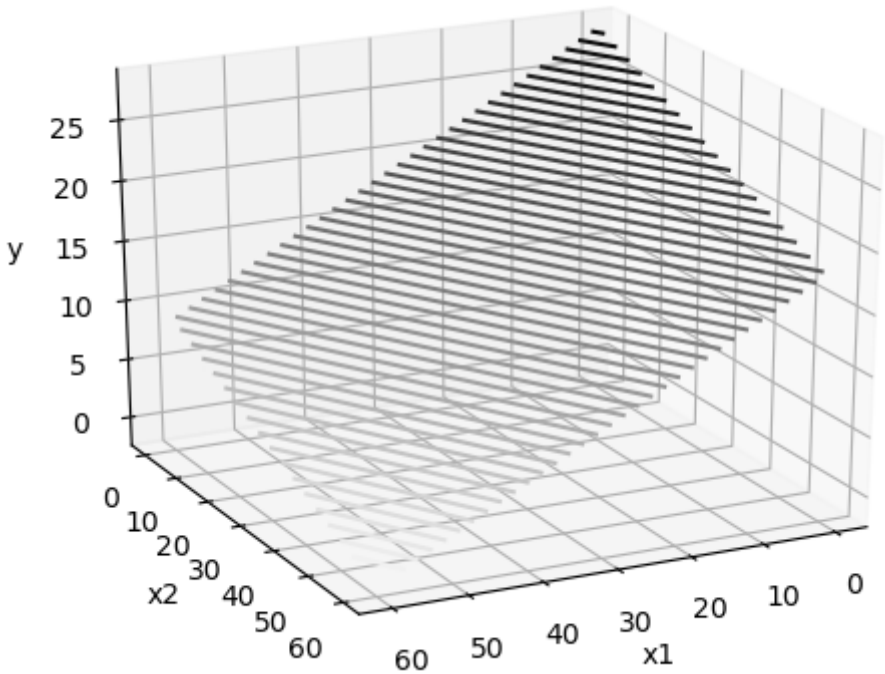
Y is linear combination of X1, X2. From note 8 we've got

$$\begin{array}{ccc} X & \beta & \approx & Y \\ \begin{bmatrix} 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22} \\ \vdots & \vdots & \vdots \\ 1 & x_{n1} & x_{n2} \end{bmatrix} & \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} & \approx & \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} . \end{array}$$

And we need to calculate vector β for example using Excel functions

$$\beta = \left(X^T X\right)^{-1} X^T Y .$$

X				Y	
1	51.11	17.02222222222		9.7	
1	50.66472222222	17.92694444444		8.4	
1	50.26416666667	19.02361111111		7.9	
1	50.06138888889	19.93833333333		8.7	
1	50.03361111111	22.00472222222		7.5	
1	50.87416666667	20.63333333333		7.8	
1	51.77666666667	19.45472222222		8.6	
1	52.40833333333	16.93416666667		8.4	
1	51.93972222222	15.505		8.8	
1	53.43805555556	14.54222222222		8	
1	54.3475	18.64527777778		6.7	
1	53.125	18.00111111111		8.4	
1	52.23222222222	21.00833333333		8.2	
1	53.13527777778	23.14555555556		6.8	
1	51.24805555556	22.57027777778		8.4	
1	53.77305555556	20.47611111111		7.2	
				B0	28.5301
				B1	-0.32809
				B2	-0.17775


y
25
20
15
10
5
0
0
10
20
30
40
50
60
x2
60
50
40
30
20
10
0
x1

L8Z3

Filip Knefel

April 30, 2020

1 Problem

Random variable X has discrete uniform distribution, i.e.

$$P(X = i) = \frac{1}{100}, i \in 1, 2, ..., 99, 100.$$

Let r. vs. Y and Z be defined by

$$Y = \begin{cases} 1, 2|X \vee 3|X \\ 0, \text{ elsewhere} \end{cases} \quad Z = \begin{cases} 1, 3|X, \\ 0, \text{ elsewhere} \end{cases}$$

Find correlation coefficient ρ of variables Y and Z.

1.1 Solution

We can obtain ρ from equasion

$$\rho = \frac{Cov(Y, Z)}{\sqrt{VY \cdot VZ}}$$

First we calculate p_0, p_1 for Y, Z.
For Y:

$$p_1 = P(Y = 1) = \lfloor \frac{100}{2} \rfloor + \lfloor \frac{100}{3} \rfloor - \lfloor \frac{100}{6} \rfloor = \frac{50 + 33 - 16}{100} = \frac{67}{100}$$

thus

$$p_0 = 1 - \frac{67}{100} = \frac{33}{100}.$$

For Z:

$$p_1 = P(Z = 1) = \lfloor \frac{100}{3} \rfloor = \frac{33}{100}.$$

$$p_0 = 1 - \frac{33}{100} = \frac{67}{100}$$

Now we calculate EY, EY^2 and EZ, EZ^2.

$$EY = EY^2 = 1 \cdot p_1 + 0 \cdot p_0 = p_1 = \frac{67}{100}$$

$$EZ = EZ^2 = 1 \cdot p_1 + 0 \cdot p_0 = p_1 = \frac{33}{100}$$

From which we can calculate VY, VZ:

$$VY = EY^2 - (EY)^2 = \frac{67}{100} - (\frac{67}{100})^2 = \frac{2211}{10000}$$

$$VZ = EZ^2 - (EZ)^2 = \frac{33}{100} - (\frac{33}{100})^2 = \frac{2211}{10000}$$

$\sqrt{VY \cdot VZ}$ is then equal to:

$$\sqrt{(\frac{2211}{10000})^2} = \frac{2211}{10000}$$

Now let's calculate Cov(Y, Z). For that we will need to know propabilities for (Y, Z):

$$p_{0,0} = P(Y = 0, Z = 0) = P(3 \nmid X \wedge 2 \nmid X) = \frac{33}{100}$$

$$p_{1,0} = P(Y = 1, Z = 0) = P(3 \nmid X \wedge 2|X) = \frac{34}{100}$$

$$p_{0,1} = P(Y = 0, Z = 1) = P(3 \nmid X \wedge 3|X) = 0$$

$$p_{1,1} = P(Y = 1, Z = 1) = P(3|X) = \frac{33}{100}$$

Having those we can calculate Cov(Y, Z) using following equation:

$$Cov(Y, Z) = \sum_{(y,z) \in S} f(y, z)(y - EY)(z - EZ) =$$

$$\frac{33}{100}(0 - \frac{67}{100})(0 - \frac{33}{100}) + \frac{34}{100}(1 - \frac{67}{100})(0 - \frac{33}{100}) + \frac{33}{100}(1 - \frac{67}{100})(1 - \frac{33}{100}) =$$

$$\frac{33}{100}\frac{67}{100}\frac{33}{100} - \frac{34}{100}\frac{33}{100}\frac{33}{100} + \frac{33}{100}\frac{33}{100}\frac{67}{100} =$$

$$(\frac{33}{100})^2(2 \cdot \frac{67}{100} - \frac{34}{100}) = (\frac{33}{100})^2$$

Now we finaly get

$$\rho = \frac{Cov(Y, Z)}{\sqrt{VY \cdot VZ}} = \frac{33^2 \cdot 10000}{100^2 \cdot 2211} = \frac{33^2}{33 \cdot 67} = \frac{33}{67}$$

Aleksander Szymański, 309785, list 8

(5)

$$f_{(2)}(x) = \left(F_{(2)}(x)\right)'$$

$$F_{(2)}(x) = P(X_{(2)} < x) = P(X_1 < x)P(X_2 < x)P(X_3 > x) + P(X_1 < x)P(X_3 < x)P(X_2 > x) +$$

$$+ P(X_2 < x)P(X_3 < x)P(X_1 > x) + P(X_1 < x)P(X_2 < x)P(X_3 < x) =$$

$$= F(x) \cdot F(x) \cdot (1 - F(x)) + F(x) \cdot F(x) \cdot (1 - F(x)) + F(x) \cdot F(x) \cdot (1 - F(x)) + F(x)F(x)F(x) =$$

$$= 3 \cdot F^2(x) \cdot (1 - F(x)) + F^3(x)$$

$$f_{(2)}(x) = \left(F_{(2)}(x)\right)' = 6F(x)(1 - F(x))f(x) + 3F^2(x)(-f(x)) + 3F^2(x) \cdot f(x) = 6F(x)(1 - F(x)) \cdot f(x)$$

5. X_1, X_2, X_3 – INDEPENDENT

(DF - $F(x)$, DENSITY - $f(x)$

$X_k \sim U[0, a]$, $k = 1, 2, 3$

$$Y_1 = \frac{X_1 + X_2 + X_3}{3}, \quad Y_2 \; X_{(2)}, \quad Y_3 = \frac{X_{(1)} + X_{(3)}}{2}$$

PROOF THAT $E(Y_1) = \frac{a}{2}$:

$$E(Y_1) = E\left(\frac{X_1 + X_2 + X_3}{3}\right) = \frac{E(X_1) + E(X_2) + E(X_3)}{3} = \frac{3 \cdot \frac{a}{2}}{3} = \frac{a}{2} \quad \square$$

PROOF THAT $E(Y_2) = \frac{a}{2}$:

FROM TASK 4 WE KNOW THAT

$$f_{(2)}(x) = 6F(x)(1 - F(x))f(x)$$

$X_k \sim U[0, a]$, so:

$$f(x) = \frac{1}{a}, \quad F(x) = \frac{x}{a}$$

$$f_2(x) = \frac{6x}{a}\left(1 - \frac{x}{a}\right)\frac{1}{a} = \frac{6x}{a^2}\left(1 - \frac{x}{a}\right)$$

$$E(Y_2) = \int_0^a f_2(x)\, x \, dx = \int_0^a \frac{6x^2}{a^2}\left(1 - \frac{x}{a}\right) dx =$$

$$= \frac{6}{a^2}\left(\int_0^a x^2 dx - \frac{1}{a}\int_0^a x^3 dx\right) = \frac{6}{a^2}\left(\frac{a^3}{3} - \frac{a^3}{4}\right) = 2a - \frac{3}{2}a = \frac{a}{2} \quad \square$$

PROOF THAT $E(Y_3) = \frac{a}{2}$

$$E(Y_3) = E\left(\frac{X_{(1)} + X_{(3)}}{2}\right), \quad Y_3 = \frac{3Y_1 - Y_2}{2}$$

$$E(Y_3) = E\left(\frac{3Y_1 - Y_2}{2}\right) = \frac{3E(Y_1) - E(Y_2)}{2} = \frac{2\frac{a}{2}}{2} = \frac{a}{2} \quad \square$$

$$E(Y_1) = E(Y_2) = E(Y_3) = \frac{a}{2}$$

Problem Set 8

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30.04.2020

1 Task 6

We know that $Y_1 = \frac{X_1+X_2+X_3}{3}$ and $Y_2 = X_2$. We have to proof that $V(Y_1) = \frac{a^2}{36}$ and $V(Y_2) = \frac{a^2}{20}$.

$$V(Y_1) = V(\frac{X_1 + X_2 + X_3}{3}) = \frac{1}{9}(V(X_1) + V(X_2) + V(X_3)) = \frac{a^2}{36},$$

because $V(X_k) = \frac{a^2}{12}$

$$V(Y_2) = E(Y_2^2) - (E(Y_2))^2 = \int_0^a \frac{6x^3(1-\frac{x}{a})}{a^2} - \frac{a^2}{4} = \frac{3a^2}{10} - \frac{a^2}{4} = \frac{a^2}{20}$$

⑦ $X, Y \sim N(0,1)$, XY - independent

X, Y - point, R, Θ - polar coordinates

prove that $g(r,\theta) = \frac{1}{2\pi} r \cdot \exp\{-\frac{r^2}{2}\}$, $0 < \theta < 2\pi$
$0 < r < \infty$

$$\left.\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\ f_Y(y) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \end{aligned}\right\} f_{XY}(x,y) = \frac{1}{2\pi} \cdot e^{-\frac{x^2+y^2}{2}}$$

(because independent)

We know that

$$\begin{cases} x = r \cdot \cos\theta \\ y = r \cdot \sin\theta \end{cases} \Rightarrow J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r \cdot \sin\theta \\ \sin\theta & r \cdot \cos\theta \end{vmatrix} =$$

$$= r \cdot \cos^2\theta + r \cdot \sin^2\theta = r$$

so $g(r,\theta) = f(r\cos\theta, r\sin\theta) \cdot r =$

$$= \frac{1}{2\pi} r \cdot \exp\{-\frac{r^2\cos^2\theta + r^2\sin^2\theta}{2}\} =$$

$$= \frac{1}{2\pi} r \cdot \exp\{-\frac{r^2}{2}\} \quad \square$$

Problem 8

Random Variable (X, Y) is the same like in previous exercise. Let

$$D = R^2 = X^2 + Y^2, \Theta = \tan^{-1} \tfrac{Y}{X}$$

From previous exercise we have $f_{X,Y}(x, y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$. Similarly to the previous problem we can substitute $X = \sqrt{D}\cos\Theta$, $Y = \sqrt{D}\sin\Theta$. (Because $R = \sqrt{D}$

$$J = \begin{vmatrix} \frac{\cos\Theta}{2\sqrt{D}} & -\sqrt{D}\sin\Theta \\ \frac{\sin\Theta}{2\sqrt{D}} & \sqrt{D}\cos\Theta \end{vmatrix} = \frac{\cos^2\Theta + \sin^2\Theta}{2} = \frac{1}{2}$$

$$f_{D,\Theta}(d, \theta) = \frac{1}{2}\frac{1}{2\pi}\exp -\frac{d(\sin^2\theta + \cos^2\theta)}{2} = \frac{1}{4\pi}e^{-\frac{d}{2}}$$

Now we will calculate distribution of variable D.

$$f_D(d) = \int_0^{2\pi} \frac{1}{4\pi}e^{-\frac{d}{2}}\ d\Theta = \frac{1}{4\pi}e^{-\frac{d}{2}}\int_0^{2\pi} 1\ d\Theta = \frac{1}{2}e^{-\frac{d}{2}}$$

Therefore distribution of D is exponential with parameter $\frac{1}{2}$. Now we will calculate marginal density of Θ:

$$\int_0^{\infty} \frac{1}{4\pi}e^{-\frac{d}{2}}\ dd = \frac{1}{4\pi}\int_0^{\infty} e^{-\frac{d}{2}}\ dd = \frac{1}{4\pi}\Big| - 2e^{-\frac{x}{2}}\Big|_0^{\infty} = \frac{1}{2\pi}$$

To check whether these variables are independent we can calculate:

$$\frac{1}{2\pi}\cdot\frac{1}{2}e^{-\frac{d}{2}} = \frac{1}{4\pi}e^{-\frac{d}{2}}$$

Which shows that the variables are independent.

$$f(x) = \frac{b^p}{\Gamma(p)} x^{p-1} e^{-bx} \quad , x \in (0, \infty)$$

b) $M_U(t) = M_X(t) \cdot M_Y(t) = \frac{b^p}{\Gamma(p)} \cdot \int_0^\infty e^{tx} x^{p-1} e^{-bx} dx \cdot$

$$\cdot \frac{b^q}{\Gamma(q)} \int_0^\infty e^{tx} x^{q-1} e^{-bx} dx = \left| \begin{array}{l} u = (b-t)x \\ du = (b-t)dx \end{array} \right| =$$

$$= \frac{b^p}{\Gamma(p)} \frac{1}{(b-t)^p} \int_0^\infty e^{-u} u^{p-1} du \cdot \frac{b^q}{\Gamma(q)} \frac{1}{(b-t)^q} \int_0^\infty e^{-u} u^{q-1} du =$$

$$= \left(\frac{b}{b-t}\right)^p \cdot \left(\frac{b}{b-t}\right)^q = \left(\frac{b}{b-t}\right)^{p+q}$$

#

c)

$$V = \frac{X}{X+Y}$$

$$U = X+Y$$

Solving for X and Y

$$X = UV$$

$$Y = U(1-V)$$

and Jacobian

$$J = \begin{vmatrix} u & v \\ -u & 1-v \end{vmatrix} = u(1-v) + uv = u$$

Because X and Y are independent random variables, the joint probability density function of X and Y is:

$$f_{X,Y}(x,y) = \frac{b^{p+q}}{\Gamma(p)\Gamma(q)} x^{p-1} y^{q-1} e^{-(x+y)b}$$

By the transformation technique the probability density function of V and U is:

$$f_{U,V}(u,v) = f_{X,Y}(uv, u(1-v)) \cdot |J| =$$

$$= \frac{b^{p+q}}{\Gamma(p)\Gamma(q)} (uv)^{p-1} (u(1-v))^{q-1} e^{-ub} |u|$$

for $0 < v < 1$, $u > 0$.

To calculate marginal distribution $f_V(v)$, we integrate with respect to u:

$$f_V(v) = \frac{b^{p+q}}{\Gamma(p)\Gamma(q)} \int_0^\infty (uv)^{p-1} (u(1-v))^{q-1} e^{-ub} |u| \, du =$$

$$= \frac{b^{p+q}}{\Gamma(p)\Gamma(q)} \int_0^\infty u^{p-1} v^{p-1} u^{q-1} e^{-ub} \cdot u(1-v)^{q-1} du =$$

$$= \frac{b^{p+q}}{\Gamma(p)\Gamma(q)} \int_0^\infty u^{p+q-1} e^{-ub} v^{p-1} (1-v)^{q-1} du = \qquad 0<v<1$$

$$= \left| \begin{array}{l} r = ub \\ \frac{dr}{b} = du \end{array} \right| =$$

$$= f_V(v) = \frac{b^{p+q}}{\Gamma(p)\Gamma(q)} \int_0^\infty \left(\frac{r}{b}\right)^{p+q-1} e^{-r} v^{p-1} (1-v)^{q-1} \frac{1}{b} dr$$

Using the definition of the gamma function for $\Gamma(p+q)$, this becomes:

$$f_V(v) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} v^{p-1} (1-v)^{q-1} = \qquad 0<v<1$$

$$= f_V(v) = \frac{1}{B(p,q)} v^{p-1} (1-v)^{q-1} \qquad \#$$

a) Now we check if $\text{Gamma}(b, p+q) \cdot \text{Beta}(p,q) = f_{U,V}(u,v)$

$$\frac{b^{p+q}}{\Gamma(p+q)} u^{p+q-1} e^{-bu} \cdot \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} v^{p-1} (1-v)^{q-1} =$$

$$= \frac{b^{p+q}}{\Gamma(p)\Gamma(q)} (uv)^{p-1} (u(1-v))^{q-1} e^{-ub} \qquad \#$$