Task 1

Continuous random variable (X, Y) has density function f(x, y). Prove that: E(X + Y) = E(X) + E(Y).

Proof.

$$E(X+Y) = \int_{\mathbb{R}}\int_{\mathbb{R}}(x+y)f(x,y)dydx = \int_{\mathbb{R}}\int_{\mathbb{R}}xf(x,y)dydx + \int_{\mathbb{R}}\int_{\mathbb{R}}yf(x,y)dxdy =$$

$$= \int_{\mathbb{R}}\left(x\int_{\mathbb{R}}f(x,y)dy\right)dx + \int_{\mathbb{R}}\left(y\int_{\mathbb{R}}f(x,y)dx\right)dy =$$

$$= \int_{\mathbb{R}}xf_1(x)dx + \int_{\mathbb{R}}yf_2(y)dy = E(X) + E(Y)$$

$\square$

L5.Z2

we know that $E(aX+b) \overset{①}{=} aE(X)+b$ $\quad a, b \in \mathbb{R}$

$$V(X) = E((X-E(X))^2)$$

$$V(Y) = E[(aX+b-E(aX+b))^2] \overset{①}{=}$$

$$= E[(aX+b-aE(X)-b)^2] = E[(a(X-E(X))^2] =$$

$$= E[a^2(X-E(X)^2] = a^2 E((X-E(X))^2) = \underbrace{a^2 V(X)}$$

:)

Probability and Statistics set nr 5

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08.04.2020

Task nr 3

Contents:
Random variable X has a normal distribution, i.e. $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $x \in \mathbb{R}$. (In short: $X \sim N(0,1)$). Find distribution (density $f_Y(y) \equiv g(y)$) of variable $Y = X^2$.
Solution:
We'll use cumulative distribution:

$$F_Y(t) = P(Y < t) = P(X^2 < t) = P(|X| < \sqrt{t}) =$$
$$= P(X < \sqrt{t}) - P(X < -\sqrt{t}) = F_X(\sqrt{t}) - F_X(-\sqrt{t})$$
$$F_Y(t) = F_X(\sqrt{t} - F_X(-\sqrt{t})) \implies f_Y(t) = (F_X(\sqrt{t}) - F_X(-\sqrt{t}))'$$
$$f_Y(t) = \frac{f_X(\sqrt{t})}{2\sqrt{t}} + \frac{f_X(-\sqrt{t})}{2\sqrt{t}} = \frac{f_X(\sqrt{t}) + f_X(-\sqrt{t})}{2\sqrt{t}} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{t}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{t}{2}}}{2\sqrt{t}} = \frac{e^{-\frac{t}{2}}}{\sqrt{2t\pi}}$$
$$f_Y(y) = \frac{e^{-\frac{y}{2}}}{\sqrt{2y\pi}}, y > 0$$

Task 5.4

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt, \ p > 0$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t} dt$$

We use substitution

$$t = \frac{x^2}{2}$$
$$dt = x \ dx$$

$$\int_0^\infty \left(\frac{x^2}{2}\right)^{-1/2} e^{-x^2/2} x \ dx = \int_0^\infty \frac{\sqrt{2}}{x} e^{-x^2/2} x \ dx = \sqrt{2} \int_0^\infty e^{-x^2/2} dx =$$
$$= \frac{\sqrt{2}}{2} \int_{-\infty}^\infty e^{-x^2/2} dx \text{ (because integrated function is even)}$$

From task 1.6 we know that

$$\left(\int_{-\infty}^\infty e^{-x^2/2} dx\right)^2 = 2\pi, \text{ since the value of integral is positive}$$

$$\int_{-\infty}^\infty e^{-x^2/2} dx = \sqrt{2\pi}$$

So

$$\Gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}} \int_{-\infty}^\infty e^{-x^2/2} dx = \frac{1}{\sqrt{2}} \cdot \sqrt{2\pi} = \sqrt{\pi}$$

Question 5

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April 8, 2020

From question 4 we know that:

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

We can use it now:
$f_Y(y) = \frac{e^{\frac{-y}{2}}}{\sqrt{2\pi y}} = \frac{1}{\sqrt{\pi} * \sqrt{2} * \sqrt{y}} * exp(-\frac{1}{2}y) = \frac{1}{\Gamma(\frac{1}{2}) * \sqrt{2}} * y^{\frac{-1}{2}} * exp(-\frac{1}{2}y) = \frac{(\frac{1}{2})^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} * y^{\frac{1}{2}-1} * exp(-\frac{1}{2}y)$

$Y \sim Gamma(\frac{1}{2}, \frac{1}{2})$

6. We know, that: $X \sim N(0,1)$, $Y = \sigma X + \mu$, $\sigma > 0$, $\mu \in \mathbb{R}$

$X \sim N(0,1)$, $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}$.

$$F_Y(t) = P(Y < t) = P(\sigma X + \mu < t) = P\left(X < \frac{t-\mu}{\sigma}\right) = F_X\left(\frac{t-\mu}{\sigma}\right)$$

$$f_Y(t) = f_X\left(\frac{t-\mu}{\sigma}\right) \cdot \frac{1}{\sigma} = \frac{1}{\sigma} \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\left(\frac{t-\mu}{\sigma}\right)^2}{2}\right\} =$$

$$= \underline{\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{(t-\mu)^2}{2\sigma^2}\right\}}.$$

So from the note No. 5 we know that $Y \sim N(\mu, \sigma^2)$.

RPIS

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March 2020

Exercise 7

$$F_{xy}(s,t) = \int_{min(s,0)}^{min(s,2)} \int_{min(t,0)}^{min(t,1)} xydxdy =$$

$$= \int_{min(s,0)}^{min(s,2)} xdx \int_{min(t,0)}^{min(t,1)} ydy =$$

$$= \frac{min(s,2)^2 - min(s,0)^2}{2} * \frac{min(t,1)^2 - min(t,0)^2}{2}$$

• $(s,t) \in D$

$$F_{XY}(s,t) = \int_0^{2t}\int_0^{t} xy\,dx\,dy = \frac{4 \cdot t^2}{4} = t^2$$

• $(s,t) \in E$

$$F_{XY}(s,t) = 1$$

Zad 8/5

$(X,Y) \sim f(x,y) = xy \quad [0,2] \times [0,1]$

$Z = X+Y \qquad g(z) = ?$ ← density

Start with transformation

$(X,Y) \mapsto (Z,T)$ Let $Z = X+Y$, $T = Y$

We reverse the transformation:

$X = Z - T \qquad Y = T$

We compute the Jacobian of reverse transformation:

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

We calculate the density of (Z,T)

$$g(z,t) = f(x(z,t), y(z,t))\,|J| = (z-t)t$$

$g(z)$ is one of marginal distributions of (Z,T)

So we integrate $g(z,t)$ by the variable t.

To find the value of $g_1(z)$ $(z \in [0,3])$

the range of the variable t should be determined

$$\begin{cases} 0 < x < 2 \\ 0 < y < 1 \end{cases} \quad \begin{cases} 0 < z-t < 2 \\ 0 < t < 1 \end{cases} \quad \begin{cases} z-2 < t < z \\ 0 < t < 1 \end{cases}$$

Integration interval for the variable t is $[\max\{0, z-2\}, \min\{1, z\}]$

For $z \in [0,1]$ we have $t \in [0, z]$

for $z \in [1,2]$ we have $t \in [0, 1]$

and for $z \in [2,3]$ we have $t \in [z-2, 1]$

We calculate the indefinite integral

$$\int g(z,t)\,dt = \int (z-t)\,t\,dt = z\int t\,dt - \int t^2\,dt =$$

$$z\frac{t^2}{2} - \frac{t^3}{3}$$

Finally

$$g_1(z) = \begin{cases} z\frac{t^2}{2} - \frac{t^3}{3}\Big|_0^z = \frac{z^3}{6} & : z \in [0,1] \\ z\frac{t^2}{2} - \frac{t^3}{3}\Big|_0^1 = \frac{z}{2} - \frac{1}{3} & : z \in [1,2] \\ z\frac{t^2}{2} - \frac{t^3}{3}\Big|_{z-2}^1 = \frac{-z(z-2)^2}{2} + \frac{(z-2)^3}{3} + \frac{z}{2} - \frac{1}{3} & : z \in [2,3] \end{cases}$$

$$= \frac{-z^3 + 12z - 16}{6} + \frac{z}{2} - \frac{1}{3}$$

$$= -\frac{z^3}{6} + \frac{5}{2}z - 3$$

$$\int_0^3 g_1(z)\,dz = \frac{1}{24} + \frac{2^2}{4} - \frac{2}{3} - \frac{1}{4} + \frac{1}{3}$$

$$-\frac{81}{24} + \frac{5}{4}\cdot 9 - 9 + \frac{16}{24} - \frac{5}{4}\cdot 4 + 6 =$$

$$= -\frac{64}{24} + \frac{48}{4} - \frac{1}{3} - 9 + 1 = -3 + 12 - 9 + 1 = 1$$

Probability class

Task 9.

List 5

9 kwietnia 2020

Variable (X, Y) is of continuos type, variables X,Y are independent. Prove that $Cov(X, Y) = 0$.

1. Definition of covariance.
$Cov(X, Y) = E(X \cdot Y) - E(X) \cdot E(Y)$

2. From the definition of independence we have:
$E(X \cdot Y) = EX \cdot EY$

3.Proof:
$cov(X, Y) = E(X \cdot Y) - EX \cdot EY = EX \cdot EY - EX \cdot EY = 0$

4.On the integrals:
$E(X) = \int_{\mathbb{R}} x f_1(x) dx$
$E(Y) = \int_{\mathbb{R}} y f_2(y) dy$

$E(X) \cdot E(Y) = \int_{\mathbb{R}} \int_{\mathbb{R}} x f_1(x) y f_2(y) dx dy =$
$\int_{\mathbb{R}} \int_{\mathbb{R}} xy f(x, y) dx dy = E(X \cdot Y)$

$\square$