TASK 1. SZYMON KIŁZAK

GOAL: PROVE THAT DENSITY OF VARIABLE $(Y_1, \dots, Y_N)$ IS

$$f_{Y_1, \dots Y_N}(y_1, \dots, y_n) = \lambda^n \cdot e^{-\lambda y_n}$$

$X_1, \dots, X_N$ - INDEPENDENT AND HAVE DISTRIBUTION $Exp(\lambda)$, SO

$$f_{X_1, \dots, X_n}(x_1, \dots, x_n) = \lambda e^{-\lambda x_1} \cdot \lambda e^{-\lambda x_2} \cdot \dots \cdot \lambda e^{-\lambda x_n} =$$

$$= \lambda^n e^{-\lambda \sum_i^n x_i}$$

WE KNOW THAT

$Y_i = X_1 + \dots + X_i$ FOR $i = 1, \dots, n$, SO

$Y_1 = X_1$

$Y_2 = X_1 + X_2 \iff X_2 = Y_2 - X_1 = Y_2 - Y_1$

$Y_3 = X_1 + X_2 + X_3 \iff X_3 = Y_3 - X_1 - X_2 = Y_3 - Y_2$

$\vdots$

$Y_N = X_1 + \dots + X_n \iff X_N = Y_N - Y_{n-1}$

DENSITY OF VARIABLES $Y_1, \dots, Y_n$ EQUALS TO DENSITY OF $X_1, \dots, X_n$ MULTIPLIED BY DETERMINANT OF JACOBIAN:

$$f_{Y_1, \dots, Y_n}(y_1, \dots, y_n) = f_{X_1, \dots, X_n}(x_1, \dots, x_n) \cdot |J| = \lambda^n e^{-\lambda \sum_1^n x_i} \cdot |J|$$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \dots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \dots & \frac{\partial x_2}{\partial y_n} \\ \vdots & & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \dots & \frac{\partial x_n}{\partial y_n} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & 0 & & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{vmatrix} = 1$$

$Y_n = \sum_i^n X_i$, SO

$$f_{Y_1, \dots, Y_n}(y_1, \dots, y_n) = \lambda^n e^{-\lambda \sum_i^n x_i} \cdot 1 = \lambda^n e^{-\lambda y_n}$$

(2) $f_{Y_1 \dots Y_n}(y_1 \dots y_n) = \lambda^n \exp(-\lambda y_n)$

prove that $f_n(y_n) = \lambda^n \frac{y_n^{n-1}}{(n-1)!} \exp(-\lambda y_n)$, $y_n > 0$

$0 < y_1 < \dots < y_n$

$$\int_0^{y_n} \int_0^{y_{n-1}} \dots \int_0^{y_2} f(y_1 \dots y_n)\, dy_{n-1} \dots dy_1 = \lambda^n \exp(-\lambda y_n) \cdot$$

$$\cdot \int_0^{y_n} \int_0^{y_{n-1}} \dots \int_0^{y_2} 1\, dy_{n-1} \dots dy_1 = \lambda^n \exp(-\lambda y_n) \cdot \int_0^{y_n} \dots \int_0^{y_3} y_2\, dy_2\, dy_3 \dots =$$

$$= \lambda^n \exp(-\lambda y_n) \cdot \int_0^{y_n} \dots \int_0^{y_4} \frac{y_3^2}{2}\, dy_3 \dots dy_{n-1} =$$

$$= \lambda^n \exp(-\lambda y_n) \cdot \int_0^{y_n} \dots \int_0^{y_5} \frac{y_4^3}{2 \cdot 3}\, dy_4 \dots dy_{n-1} =$$

$$= \lambda^n \exp(-\lambda y_n) \cdot \frac{y_n^{n-1}}{(n-1)!} \quad \square$$

TASK 3

$f(x) = \frac{1}{2a}$, where $x \in [\theta - a, \theta + a]$

$L(x;\theta) = P(X_1 = x_1, \dots, X_n = x_n) = \prod_{k=1}^{n} f(x_k, \theta)$

We want L function to reach its maximum at some point $\hat{\theta}$.

$$L(x;\theta) = \prod_{k=1}^{n} f(x_k, \theta) =$$

$$= \prod_{k=1}^{n} \frac{1}{2a} \cdot \mathbb{1}_{[\theta - a, \theta + a]}(x_k) =$$

$$= \left(\frac{1}{2a}\right)^n \cdot \mathbb{1}_{[\theta - a, \theta + a]}\left(\min_k (x_k)\right) \cdot \mathbb{1}_{[\theta - a, \theta + a]}\left(\max_k (x_k)\right)$$

$$= \left(\frac{1}{2a}\right)^n \cdot \mathbb{1}_{[\theta - a, \infty)}\left(\min_k (x_k)\right) \cdot \mathbb{1}_{(-\infty, \theta + a]}\left(\max_k (x_k)\right)$$

Maximum value of $L(x, \theta)$ is $\left(\frac{1}{2a}\right)^n$ and it is reached when

$$\begin{cases} \theta - a \leq \min_k (x_k) \\ \theta + a \geq \max_k (x_k) \end{cases}$$

Therefore

$$\max_k (x_k) - a \leq \hat{\theta} \leq \min_k (x_k) + a$$

Ex. 4

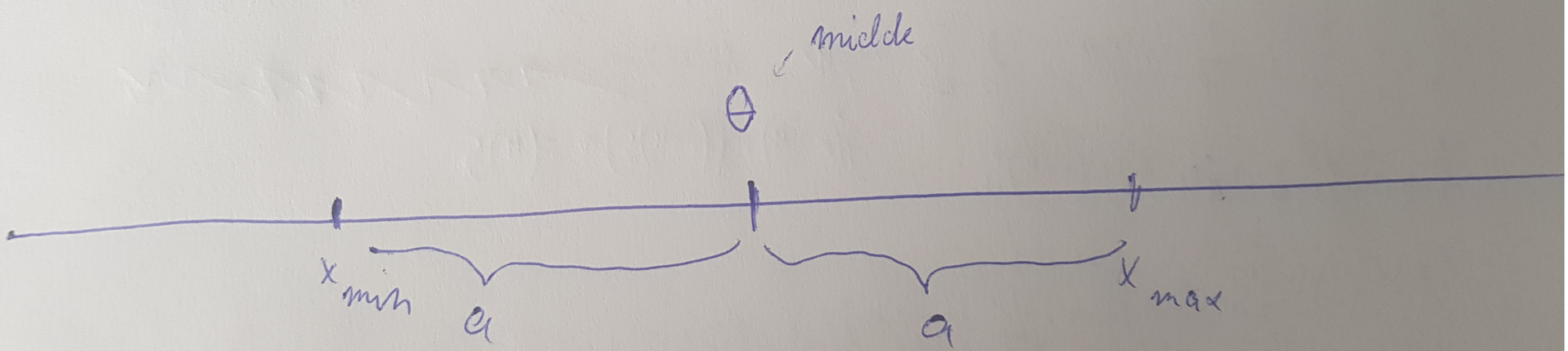
1) we do not choose a - same as task 3.; we choose any θ

2) we do choose a and θ:

We want a as small as possible, since $L(x, \theta) = \frac{1}{(2a)^n}$:

$$\theta = \frac{\min\{x_1, x_2, \dots, x_n\} + \max\{x_1, x_2, \dots, x_n\}}{2}$$

$$a = \frac{\max\{x_1, x_2, \dots, x_n\} - \min\{x_1, x_2, \dots, x_n\}}{2}$$



RPiS

Kacper Puchalski 310271

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1 Task 9.5

We know that $p = P(X_1 < X_2 > X_3 < X_4 > X_5)$. Let $Y = F_X(X)$. Then we can calculate

$$F_Y(y) = P(Y \leq y) = P(F_X(X) \leq y) = P(F_X^{-1}(F_X(X)) \leq F_X^{-1}(y)) = \\ = P(X \leq F_X^{-1}(y)) = F_x(F_X^{-1}(y)) = y$$

Therefore $Y \sim U[0,1]$ and $f_Y(y) = 1$ so p does not depend on density $f(x)$ of variables X_k.

$$p = P(X_1 < X_2 > X_3 < X_4 > X_5) =$$

$$= P(F(X_1) < F(X_2), F(X_2) > F(X_3), F(X_3) < F(X_4), F(X_4) > F(X_5)) =$$

$$= \int\int\int\int\int_{y_1<y_2, y_2>y_3, y_3<y_4, y_4>y_5} f_Y(y_1)f_Y(y_2)f_Y(y_3)f_Y(y_4)f_Y(y_5)dy_5dy_4dy_3dy_2dy_1 =$$

$$= \int\int\int\int\int_{y_1<y_2, y_2>y_3, y_3<y_4, y_4>y_5} 1dy_5dy_4dy_3dy_2dy_1 =$$

$$= \int_{-\infty}^{\infty}\int_{y_1}^{\infty}\int_{-\infty}^{y_2}\int_{y_3}^{\infty}\int_{-\infty}^{y_4} 1dy_5dy_4dy_3dy_2dy_1$$

Because $Y \sim U[0,1]$ we can change $\infty \to 1$ and $-\infty \to 0$

$$= \int_0^1\int_{y_1}^1\int_0^{y_2}\int_{y_3}^1\int_0^{y_4} 1dy_5dy_4dy_3dy_2dy_1 =$$

$$= \int_0^1\int_{y_1}^1\int_0^{y_2}\int_{y_3}^1 y_4dy_4dy_3dy_2dy_1 = \int_0^1\int_{y_1}^1\int_0^{y_2} \frac{1-y_3^2}{2}dy_3dy_2dy_1 = \cdots = 2/15$$

2.6 $X, Y, Z \sim U[0,1]$

$$P(X>YZ) = \int_0^1\int_0^1\int_{yz}^1 1 \, dx\, dy\, dz = \int_0^1\int_0^1 x\Big|_{yz}^1 dy\, dz = \int_0^1\int_0^1 (1-yz)\, dy\, dz =$$

$$= \int_0^1 \left(y - \frac{y^2 z}{2}\right)\Big|_{y=0}^1 dz = \int_0^1 \left(1 - \frac{z}{2}\right) dz = \left(z - \frac{z^2}{4}\right)\Big|_{z=0}^1 = 1 - \frac{1}{4} = \frac{3}{4}$$

L9Z7

Filip Knefel

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1 Problem

$X_1, X_2, X_3 \sim Exp(\lambda)$ and independent.
Find distribution of $(Y_1, Y_2, Y_3) = (X_1 + X_2, X_1 + X_3, X_2 + X_3)$.

2 Solution

$f(x_1, x_2, x_3) = \lambda^3 e^{-\lambda(x_1+x_2+x_3)}$ because variables are independent. In order to solve Jacobian we calculate:

$$\begin{cases} x_1 = y_1 - x_2 \\ x_1 = y_2 - x_3 \\ x_2 = y_3 - x_3 \end{cases} \implies \begin{cases} y_2 - x_3 = y_1 - (y_3 - x_3) \\ x_1 = y_2 - x_3 \\ x_2 = y_3 - x_3 \end{cases} \implies$$

$$\implies \begin{cases} \frac{y_3+y_2-y_1}{2} = x_3 \\ x_1 = y_2 - \frac{y_3+y_2-y_1}{2} \\ x_2 = y_3 - \frac{y_3+y_2-y_1}{2} \end{cases} \implies \begin{cases} x_1 = \frac{-y_3+y_2+y_1}{2} \\ x_2 = \frac{y_3-y_2+y_1}{2} \\ x_3 = \frac{y_3+y_2-y_1}{2} \end{cases}$$

Now we calculate Jacobian:

$$\begin{vmatrix} \frac{\delta x_1}{\delta y_1} & \frac{\delta x_1}{\delta y_2} & \frac{\delta x_1}{\delta y_3} \\ \frac{\delta x_2}{\delta y_1} & \frac{\delta x_2}{\delta y_2} & \frac{\delta x_2}{\delta y_3} \\ \frac{\delta x_3}{\delta y_1} & \frac{\delta x_3}{\delta y_2} & \frac{\delta x_3}{\delta y_3} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

And then we get

$$g(y_1, y_2, y_3) = |J| \cdot f(x_1(y_1, y_2, y_3), x_2(y_1, y_2, y_3), x_3(y_1, y_2, y_3)) = \frac{1}{2} \cdot \lambda^3 e^{-\lambda \frac{y_1+y_2+y_3}{2}}$$

Task nr 8

Contents:

$X_1, ..., X_n$ are independent and of identical continuous distribution random variables. At time j record occurs ($j \leqslant n$), if $X_j \geqslant X_i$ for $1 \leqslant i \leqslant j$.Let r.v. Z equals number of record values in sequence $\{X_k\}$. Prove that $E(Z) = \sum\limits_{j=1}^{n} \frac{1}{j}$

Solution:

At first, I want to say my interpretation of Z: it says, how many times, when creating a set and adding to it r.v. from X_1 to X_n, we will find a r.v., that is, in that particular moment, a new maximum of the set. Now, I want to make some observations. If we have n r.v. X, the probability, that $X_1 \leqslant X_2 \leqslant ... \leqslant X_n$, is equal to $\frac{1}{n!}$. Basically, every possible shuffle has $\frac{1}{n!}$ chance to occur. We can deduce that using knowledge from task nr 5. Now, for the same variables, the probability, that $X_n = max\{X_1, X_2, ..., X_n\}$, is equal to $\frac{1}{n!} \cdot (n-1)! = \frac{1}{n}$. We can justify it by saying, that if we only care about the placement of the last r.v., then the rest of them can be shuffled randomly, and we have $(n-1)!$ ways to shuffle $n-1$ elements. To say it in a different way, let's define a new variable: let $Y = max\{X_1, X_2, ..., X_k\}$. In other words, $Y \geqslant X_i$ for $1 \leqslant i \leqslant k$. Probability, that $Y = X_k$, is equal to $\frac{1}{k}$. Now, following my interpretation, $E(Z)$ is equal to sum of probabilities, that the last value is the maximum in sets, where m-th set has m elements, and these elements are X_i, for $1 \leqslant i \leqslant m$. In other words, if we have one r.v. X, then $E(Z) = 1$, if we have two r.v. X, then $E(Z) = 1 + \frac{1}{2}$. And, because we have n r.v. X, $E(Z) = \sum\limits_{j=1}^{n} \frac{1}{j}$

$\square$

L9.Z9 Find density of $Z = \frac{X}{Y}$

$f(x,y) = 1 \quad 0 < \leq x, y \leq 1$ (X, Y are independent it's constant 1 and boundaries are 0, 1)

let's define (Z, A) such as $A = Y$

$$Z = \frac{X}{Y} \qquad X = ZA$$
$$A = Y \qquad Y = A$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial A} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial A} \end{vmatrix} = \begin{vmatrix} A & Z \\ 0 & 1 \end{vmatrix} = A, \text{ so } f(Z, A) = f(\overset{\frac{x}{y}}{\overset{\shortparallel}{\frac{x}{y}}}, y) \cdot |J| = A$$

Now let's check how does A changes respect to Z

what do we know

$0 < X < 1 \qquad 0 < ZA < 1 \qquad 0 < A < \frac{1}{Z}$

$0 < Y < 1 \qquad 0 < A < 1 \qquad 0 < A < 1$

we have got two cases, basically $z \in [0, \infty)$, $z \in [0, \frac{1}{A}]$

- when $Z \in [0; 1]$ then $A \in [0, 1]$, and $F_Z(z) = \int_0^1 a \, da = \frac{1}{2}$
- when $Z \in (1; \infty)$ then $A \in [0; \frac{1}{Z}]$, and $F_Z(z) = \int_1^\infty a \, da = \frac{1}{2z^2}$

$$f_Z(z) = \begin{cases} \frac{1}{2}, & z \in [0, 1] \\ \frac{1}{2z^2}, & z \in (1, \infty) \end{cases}$$

Set 9 exercise 10

Wiktor Hamberger 308982

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We have to find probability that first significant digit of Z equals 1. From task 9. we know the density function of r.v. Z:

$$f_z(x) = \begin{cases} \frac{1}{2} & 0 \leq x \leq 1 \\ \frac{1}{2x^2} & 1 < x \end{cases}$$

We will need to integrals:

$$\int \frac{1}{2} dx = \frac{x}{2} + C$$

$$\int \frac{1}{2x^2} dx = -\frac{1}{2x} dx + C$$

Knowing that let's calculate:

$$\begin{gathered} P(\text{first significant digit of Z equals 1}) = \\ = P(1 \leq Z < 2) + P(10 \leq Z < 20) + P(100 \leq Z < 200) + \cdots + \\ + P(0,1 \leq Z < 0,2) + P(0,01 \leq Z < 0,02) + P(0,001 \leq Z < 0,002) + \cdots = \\ = \int_1^2 \frac{1}{2x^2} dx + \int_{10}^{20} \frac{1}{2x^2} dx + \int_{100}^{200} \frac{1}{2x^2} dx + \cdots + \\ + \int_{0,1}^{0,2} \frac{1}{2} dx + \int_{00,1}^{0,02} \frac{1}{2} dx + \int_{0,001}^{0,002} \frac{1}{2} dx + \cdots = \\ = \frac{1}{4} + \frac{1}{40} + \frac{1}{400} + \cdots + \frac{1}{20} + \frac{1}{200} + \frac{1}{2000} = \\ = \frac{1}{4} * \frac{1}{1 - \frac{1}{10}} + \frac{1}{20} * \frac{1}{1 - \frac{1}{10}} = \frac{1}{3} \end{gathered}$$