Zad 1

We know that

$P(A \cap B) = \frac{1}{4}$

$P(A^c) = \frac{1}{3}$

$P(B) = \frac{1}{2}$

$P(A \cup B) = ?$

$$\begin{cases} P(\Omega) = 1 \\ P(A^c) = \frac{1}{3} \end{cases} \Rightarrow P(A) = \frac{2}{3}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{2}{3} + \frac{1}{2} - \frac{1}{4} = \frac{11}{12}$$

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Task 2 Solution, Author: Szymon Kiczak 306389

home > szymon > rozne > rpis32.py

```python
from datetime import date, timedelta

beg = date( 1601, 1, 1 )
end = date( 2000, 12, 31 )

dal = date( 1601, 1, 2 )
oneday = dal - beg

dayscount = [ 0, 0, 0, 0, 0, 0, 0 ]

currentdate = beg
d = 0
while( currentdate != end ):
    if( currentdate.day == 13 ):
        dayscount[ d ] += 1
    d = ( d + 1 ) % 7
    currentdate += oneday

print( dayscount )
print( "How many times in 400-year cycle 13. day of the month was:" )
print( "Monday: " + str( dayscount[0] ) )
print( "Tuesday: " + str( dayscount[1] ) )
print( "Wednesday: " + str( dayscount[2] ) )
print( "Thursday: " + str( dayscount[3] ) )
print( "Friday: " + str( dayscount[4] ) )
print( "Saturday: " + str( dayscount[5] ) )
print( "Sunday: " + str( dayscount[6] ) )
```

Figure 1: Code of my program

```
[685, 685, 687, 684, 688, 684, 687]
How many times in 400-year cycle 13. day of the month was:
Monday: 685
Tuesday: 685
Wednesday: 687
Thursday: 684
Friday: 688
Saturday: 684
Sunday: 687
```

Figure 2: Output of my program

Probability class

Task 3.

R.v. X has binomial distribution $B(n_1, p)$ and r.v Y $B(n_2, p)$ distribution. X, Y are independent. Prove that $Z = X + Y$ has $B(n_1 + n_2, p)$ distribution).

26 marca 2020

We are given two random variables X and Y and their distributions:

$X \sim B(n_1, p)$ and $Y \sim B(n_2, p)$.

We can write them down by finding their probability mass functions (probability mass function is a function that gives the probability that a discrete random variable is exactly equal to some value).

$P(X = k) = \binom{n_1}{k} p^k (1-p)^{n_1 - k}$

$P(Y = k) = \binom{n_2}{k} p^k (1-p)^{n_2 - k}$

Now let's introduce a new random variable Z which equals the sum of X and Y so $Z = X + Y$. Let's calculate the probability mass function for the r.v Z. We can write it in that way:

$P(X + Y = k) = P(Z = k) = \sum\limits_{i=0}^{k} P(X = i, Y = k - i) =$
$\sum\limits_{i=0}^{k} P(X = i) \cdot P(Y = k - i)$

The sum of X and Y must equal k, so if the probability of X is i, the probability of Y has to be $k - i$. Then we are using the fact that X and Y are independent. Knowing that, we can proceed to calculating the probability function.

$$P(Z = K) = \sum_{i=0}^{k} \binom{n_1}{i} p^i (1-p)^{n_1-i} \cdot \binom{n_2}{k-i} p^{k-i} (1-p)^{n_2-k+i} =$$

$$= p^k (1-p)^{n_1+n_2-k} \cdot \sum_{i=0}^{k} \binom{n_1}{i} \binom{n_2}{k-i} = p^k (1-p)^{n_1+n_2-k} \binom{n_1+n_2}{k}.$$

In the last transformation, we use the equation:

$$\sum_{k=-m}^{n} \binom{r}{m+k} \binom{s}{n-k} = \binom{r+s}{m+n}$$

The last expression equals $B(n_1 + n_2, p)$, and this is what we wanted to prove. $\triangle$

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Probability and Statistics

Task 4

Smolec, 26 marca 2020

4. Independent r.vs. X,Y have Poisson distribution with parameters λ_1 i λ_2. Prove that r.v. $Z = X + Y$ has Poisson distribution with parameter $\lambda_1 + \lambda_2$.

Poisson distribution: $P(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$

We know, that X and Y are independent variables.

$P_1(X = k) = e^{-\lambda_1} \cdot \frac{\lambda_1^k}{k!}$
$P_2(Y = k) = e^{-\lambda_2} \cdot \frac{\lambda_2^k}{k!}$

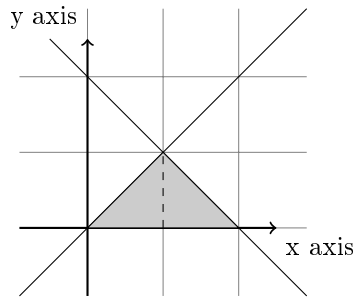
$Z = X + Y$

$$P(Z = X+Y = k) = \sum_{i=0}^{k} P_1(X = i) \cdot P_2(Y = k-i) = \sum_{i=0}^{k} e^{-\lambda_1} \frac{\lambda_1^i}{i!} \cdot e^{-\lambda_2} \frac{\lambda_2^{(k-i)}}{(k-i)!} = e^{-(\lambda_1+\lambda_2)} \cdot \sum_{i=0}^{k} \frac{\lambda_1^i \lambda_2^{(k-i)}}{i!(k-i)!} =$$

$$= e^{-(\lambda_1+\lambda_2)} \cdot \frac{\sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \lambda_1^i \lambda_2^{(k-i)}}{k!} = e^{-(\lambda_1+\lambda_2)} \cdot \frac{\sum_{i=0}^{k} \binom{k}{i} \lambda_1^i \lambda_2^{(k-i)}}{k!} = e^{-(\lambda_1+\lambda_2)} \cdot \frac{(\lambda_1 + \lambda_2)^k}{k!}$$

Problem no 5

Density of random v. (X, Y) has a form $f(x,y) = 3xy$ on a area bounded by $y = 0,\ y = x,\ y = 2 - x$. Find marginal densities $f_1(x), f_2(y)$
The area bounded by these lines:



To calculate $f_1(x)$ we will divide that area into two parts, that are separated on a picture by a dashed line.

$$f_1(x) = \int_{\mathbb{R}} 3xy\mathrm{d}y = \begin{cases} \int_0^x 3xy\mathrm{d}y & x \in [0,1] \\ \int_0^{2-x} 3xy\mathrm{d}y & x \in [1,2] \end{cases} = \begin{cases} \frac{3xy^2}{2}\Big|_0^x & x \in [0,1] \\ \frac{3xy^2}{2}\Big|_0^{2-x} & x \in [1,2] \end{cases} = \begin{cases} \frac{3x^3}{2} & x \in [0,1] \\ \frac{3x(2-x)^2}{2} & x \in [1,2] \end{cases}$$

As for $f_2(y)$:

$$f_2(y) = \int_{\mathbb{R}} 3xy\mathrm{d}x = \int_y^{2-x} 3xy\mathrm{d}x = \frac{3x^2y}{2}\Big|_y^{2-y} = \frac{3(2-y)^2y}{2} - \frac{3y^3}{2} = \frac{3y}{2}(4-4y+y^2-y^2) = 6y(1-y)$$

The bounds for integration are from these reverse function

$$y = x \iff x = y$$

$$y = 2 - x \iff x = 2 - y$$

(6) $f_2(y) = 6y(1-y) \leftarrow Y$

from definition:

$$E(Y) = \int_0^1 y \cdot f_2(y)\, dy = \int_0^1 y \cdot 6y(1-y)\, dy = 6 \cdot \int_0^1 y^2 - y^3\, dy =$$

$$= 6 \cdot \left(\frac{y^3}{3} \Big|_0^1 - \frac{y^4}{4} \Big|_0^1 \right) = 6 \cdot \left(\frac{1}{3} - \frac{1}{4} \right) = 6 \cdot \frac{1}{12} = \frac{1}{2}$$

X, Y independent $\Leftrightarrow \forall x, y \in \mathbb{R} \quad f(x,y) = f_1(x) \cdot f_2(y)$

for $\begin{cases} x = 1 \\ y = 1 \end{cases}$ we have $f_1(1) \cdot f_2(1) = \frac{3}{2} \cdot 0 = 0 \neq 3 = f(x,y)$

They are NOT independent.

7. Density function $f(k)$ is an answer to question "how many possible ways you can get 3 successes and $k-3$ failures?"

So $f(k) = p\binom{k-1}{2}p^2(1-p)^{k-3} = \underline{\underline{p^3\binom{k-1}{2}(1-p)^{k-3}}}$, because

p - probability of success in k-th attempt

$\binom{k-1}{2}$ - from $k-1$ attempts we choose 2 with success

p^2 - chance of success for those 2 attempts

$(1-p)^{k-3}$ - probability of failures for the rest $k-3$ attempts.

Bernoulli

Now, let's find expected value X:

$$EX = \sum_{k=0}^{\infty} k f(k) = \sum_{k\geq 3} k\, p^3\binom{k-1}{2}(1-p)^{k-3} = p^3 \sum_{k\geq 3} k \frac{(k-1)!}{2!\,(k-3)!}\cdot(1-p)^{k-3} =$$

$$= \frac{p^3}{2}\sum_{k\geq 3} k(k-1)(k-2)(1-p)^{k-3} = \frac{p^3}{2}\left(\sum_{k\geq 3}(1-p)^k\right)^{(3)} =$$

third deriv.

sum of the convergent series ($p<1$)

$$= \frac{p^3}{2}\left(\frac{(1-p)^3}{1-(1-p)}\right)^{(3)} = \frac{p^3}{2}\left(\frac{(1-p)^3}{p}\right)^{(3)} =$$

$$= \frac{p^3}{2}\cdot\frac{6}{p^4} = \underline{\underline{\frac{3}{p}}}$$ - expected value

Ex. 9

$$f(x) = \begin{cases} \frac{1}{4}, & x \in [-2,2] \\ 0, & \text{otherwise} \end{cases}$$

a) $P(a < X < b) = \int_a^b f(x)\,dx = \int_a^b \frac{1}{4}\,dx, \quad a, b \in [-2,2]$

$$F_X(t) = P(X < t) = \int_{-2}^{t} \frac{1}{4}\,dx, \quad t \in [-2,2]$$

$Y = |X|$

$$F_Y(t) = P(Y < t) = P(-t < X < t) = \int_{-t}^{t} \frac{1}{4}\,dx = \int_0^t \frac{1}{4}\,dx + \int_{-t}^{0} \frac{1}{4}\,dx =$$

$$= 2\int_0^t \frac{1}{4}\,dx, \quad t \in [0,2]$$

density function: $f_Y(t) = \begin{cases} \frac{1}{2}, & t \in [0,2] \\ 0, & \text{otherwise} \end{cases}$

b) $Y = X^3$

$$f(x) = \begin{cases} \frac{1}{2}, & x \in [-2,2] \\ 0, & \text{otherwise} \end{cases}$$

$$F_Y(t) = P(Y < t) = P(X^3 < t) = P(X < \sqrt[3]{t}) = \int_{-1}^{\sqrt[3]{t}} \frac{1}{2}\,dx = \frac{1}{2}(\sqrt[3]{t} + 1), \quad t \in [-1,1]$$

$$\int_{-1}^{t} f_Y(x)\,dx = \frac{1}{2}(\sqrt[3]{t} + 1)$$

density function: $f_Y(t) = \frac{1}{6\sqrt[3]{t^2}}, \quad t \in [-1,1]$

$Z = X^2$

$$F_Z(t) = P(Z < t) = P(X^2 < t) = P(-\sqrt{t} < X < \sqrt{t}) = \int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{2}\,dx = \sqrt{t}, \quad t \in [0,1]$$

$$\int_0^t f_Z(x)\,dx = \sqrt{t} \qquad f_Z(t) = \begin{cases} \frac{1}{2\sqrt{t}}, & t \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

$i=1$

(10) $X \sim Geom(p) \Rightarrow P(X=x) = p(1-p)^{x-1}$; We know that $V(X) = E(X^2) + [E(X)]^2$

1°) $E(X) = \sum_{x\geq 1} x P(X=x) = \sum_{x\geq 1} x \cdot p(1-p)^{x-1} = p \cdot \sum_{x\geq 1} x(1-p)^{x-1} = p\left(\sum_{x\geq 1} (1-p)^x\right)' = p \cdot \left(-\sum_{x\geq 1}(1-p)^x\right)' =$

$= p \cdot \left(-\frac{1-p}{1-(1-p)}\right)' = p \cdot \left(-\frac{1-p}{p}\right)' = p \cdot \left(-\frac{1}{p}+1\right)' = \frac{1}{p}$

$E(X^2) = \sum_{x\geq 1} x^2 P(X=x) = \sum_{x\geq 1} x^2 \cdot p \cdot (1-p)^{x-1} = p \cdot \sum_{x\geq 1} x^2 (1-p)^{x-1} = p\left(\sum_{x\geq 1} -x(1-p)^x\right)' = p\left(-\sum_{x\geq 1} x(1-p)^x \frac{p}{1-p} \cdot \frac{1-p}{p}\right)' =$

$= p\left(-\underbrace{\sum_{x\geq 1} x\, p(1-p)^{x-1}}_{E(X)} \cdot \frac{1-p}{p}\right)' = p \cdot \left(-\frac{1}{p} \cdot \frac{1-p}{p}\right)' = p\left(-\frac{1}{p^2} + \frac{1}{p}\right)' = p \cdot \left(2\frac{1}{p^3} - \frac{1}{p^2}\right) =$

$= 2\frac{1}{p^2} - \frac{p}{p^2} = \frac{2-p}{p^2}$

$V(X) = E(X^2) + [E(X)]^2 = \frac{1}{p^2} + \frac{2-p}{p^2} = \frac{1-p}{p^2}$ □

RPiS

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March 26, 2020

1 Task 3.11

$$E(X) = \sum x_i p_i = 40 \cdot \frac{40}{142} + 32 \cdot \frac{32}{142} + 20 \cdot \frac{20}{142} + 50 \cdot \frac{50}{142} \approx 38,9 \tag{1}$$

$$E(Y) = \sum y_i p_i = 40 \cdot \frac{1}{4} + 32 \cdot \frac{1}{4} + 20 \cdot \frac{1}{4} + 50 \cdot \frac{1}{4} = 35,5 \tag{2}$$