$\boxed{L6.Z1}$

$X \sim Geom(p)$ (is discrete) Prove that $M_X(t)$ is of the form $M_X(t) = \frac{pe^t}{1-qe^t}$, so geometric means

~~[illegible]~~

$p + q = 1$

that $p(x) = (1-p)^{x-1} p = q^x p$

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} p(x) = \sum_{x=1}^{\infty} e^{tx} q^{x-1} p = \sum_{x=1}^{\infty} e^{tx} q^x \frac{p}{q} =$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (e^t q)^x = \frac{p}{q} \Big[\underbrace{(e^t q) + (e^t q)^2 + \dots}_{a + aq + aq^2 + \dots = \frac{a}{1-q}} \Big] =$$

geometric sequence

$$= \frac{p}{\cancel{q}} \cdot \frac{e^t \cancel{q}}{1 - e^t q} = \frac{pe^t}{1 - qe^t}$$

From 1.: $M_x(t) = \frac{pe^t}{1-qe^t}$, $M_x'(t) = \frac{pe^t}{(1-qe^t)^2}$, $M_x'(0) = \frac{p}{(1-q)^2} = \frac{1}{p}$

We know: $E(X) = M_x'(0) = \frac{1}{p}$

$$V(X) = E[(X - E(X))^2] = E[X^2 - 2XE(X) + (E(X))^2] =$$

$$= E(X^2) - 2E(X)E(X) + E(X)^2 = E(X^2) - E(X)^2$$

$$E(X^2) = M_x''(0)$$

$$M_x''(t) = \frac{p \cdot e^t (qe^t + 1)}{(1 - qe^t)^3}$$

$$M_x''(0) = \frac{p(q+1)}{p^3} = \frac{2-p}{p^2}$$

$$V(X) = E(X^2) - E(X)^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

Task nr 3

Contents:
Let $X \sim N(\mu, \sigma^2)$. That means $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $x \in \mathbb{R}$. Prove (using definition $M_X(t) = E(e^{tX})$) that $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$
Solution:
We'll use the definition:

$$M_X(t) = E(e^{tX}) = \int_{\mathbb{R}} e^{tx} f_X(x) dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} e^{tx - \frac{(x-\mu)^2}{2\sigma^2}} dx =$$

$$= \left\{ u = \frac{x-\mu}{\sqrt{2}\sigma}, du = \frac{1}{\sqrt{2}\sigma} dx \right\} = \frac{\sqrt{2}\sigma}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} e^{(\sqrt{2}\sigma u + \mu)t - u^2} du =$$

$$= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{\sqrt{2}\sigma u t + \mu t - u^2} du = \frac{e^{\mu t}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-(u^2 - \sqrt{2}\sigma u t)} du =$$

$$= \frac{e^{\mu t}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-((u - \frac{\sqrt{2}}{2}\sigma t)^2 + \frac{1}{2}\sigma^2 t^2)} du = \frac{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-((u - \frac{\sqrt{2}}{2}\sigma t)^2)} du =$$

$$= \left\{ w = u - \frac{\sqrt{2}}{2}\sigma t, dw = du \right\} = \frac{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-w^2} dw =$$

$$= \frac{\sqrt{\pi} e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{\sqrt{\pi}} = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

4. $X_k \sim N(\mu, \sigma^2)$

$X_1, \dots, X_n$ independent

$\overline{X} = \frac{1}{n} \sum X_i$

We know that: if $Z = X + Y$ then $M_Z(t) = M_X(t) \cdot M_Y(t)$

~~$M_{\overline{X}}(t) = \prod M_{X_k}$~~ and if $C = aX + b$ then $M_C(t) = e^{tb} \cdot M_X(at)$

So $\overline{X} = \frac{1}{n} \sum X_i = \sum \frac{1}{n} X_i$ and $M_{X_k}(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$

$$M_{\overline{X}}(t) = \prod M_{X_k}\left(\frac{t}{n}\right) = \left(\exp\left(\mu \frac{t}{n} + \frac{\sigma^2 t^2}{2n^2}\right)\right)^n = \exp\left(n\mu \frac{t}{n} + \frac{n\sigma^2 t^2}{2n^2}\right) =$$

$$= \exp\left(\mu t + \frac{\sigma^2 t^2}{2n}\right) \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Task 5

Let $X \sim N(\mu, \sigma^2)$. Find distribution of $Y = \left(\frac{X-\mu}{\sigma}\right)^2$

$$F_Y(t) = P(Y \leq t) = P\left(\left(\frac{X-\mu}{\sigma}\right)^2 \leq t\right) = P\left(-\sqrt{t} \leq \frac{X-\mu}{\sigma} \leq \sqrt{t}\right) =$$

$$= P(X \leq \sqrt{t}\sigma + \mu) - P(X < -\sqrt{t}\sigma + \mu) =$$

$$= F_X(\sqrt{t}\sigma + \mu) - F_X(-\sqrt{t}\sigma + \mu)$$

After differentiation we have:

$$f_Y(t) = f_X(\sqrt{t}\sigma + \mu)\frac{\sigma}{2\sqrt{t}} + f_X(-\sqrt{t}\sigma + \mu) \cdot \frac{\sigma}{2\sqrt{t}}$$

We know that $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$, so:

$$f_Y(t) = \frac{\sigma}{2\sqrt{t}} \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(\sqrt{t}\sigma + \mu - \mu)^2}{2\sigma^2}\right) +$$

$$+ \frac{\sigma}{2\sqrt{t}} \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(-\sqrt{t}\sigma + \mu - \mu)^2}{2\sigma^2}\right) =$$

$$= \frac{1}{2\sqrt{2\pi t}}\left(\exp\left(\frac{-t\sigma^2}{2\sigma^2}\right) + \exp\left(\frac{-t\sigma^2}{2\sigma^2}\right)\right) =$$

$$= \underline{\underline{\frac{1}{\sqrt{t}} \cdot \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t}{2}\right)}}$$

Task 6

April 15, 2020

THEOREM 1 *Random variables $X_1, ..., X_n$ are independent and $X_k \sim N(\mu, \sigma^2)$. Find distribution of random variable $Z_n = \sum_{k=1}^{n} (\frac{X_k - \mu}{\sigma})^2$.*

PROOF 1 *Let $Y_k = (\frac{X_k - \mu}{\sigma})^2$. From task 5 we know that*

$$Y_k \sim Gamma(\frac{1}{2}, \frac{1}{2}) \wedge M_{Y_k}(t) = (1 - \frac{t}{\frac{1}{2}})^{-\frac{1}{2}} = (1 - 2t)^{-\frac{1}{2}}$$

$X_1, ..., X_n$ are independent thus $Y_1, ..., Y_n$ are also independent. We can then calculate

$$M_{Z_n}(t) = \prod_{k=1}^{n} M_{Y_k}(t) = \prod_{k=1}^{n} (1 - 2t)^{-\frac{1}{2}} = (1 - 2t)^{-\frac{n}{2}} \sim Gamma(\frac{1}{2}, \frac{n}{2}).$$

Question 7

Andrzej Tkaczyk, 309181

April 15, 2020

$X_k \sim Gamma(b, p_k) \implies \mathrm{M}_{X_k}(t) = (1 - \frac{t}{b})^{-p_k}$

$$M_Z(t) = \prod_{k=1}^{n} M_{X_k}(t) = \prod_{k=1}^{n} (1 - \frac{t}{b})^{-p_k} = (1 - \frac{t}{b})^{-\sum_{k=1}^{n} p_k} \implies Z \sim Gamma(b, \sum_{k=1}^{n} p_k)$$

Task 8

$$Z = \sum_{k=1}^{n} X_k$$

$$X_k \sim B(m_k, p), \quad k = 1, \dots, n$$

For $X \sim B(n, p)$ probability of p_k equals

$$p_k = \binom{n}{k} p^k (1-p)^{n-k}$$

$$M_X(t) = \sum_{k=0}^{n} e^{tk} p_k = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} =$$

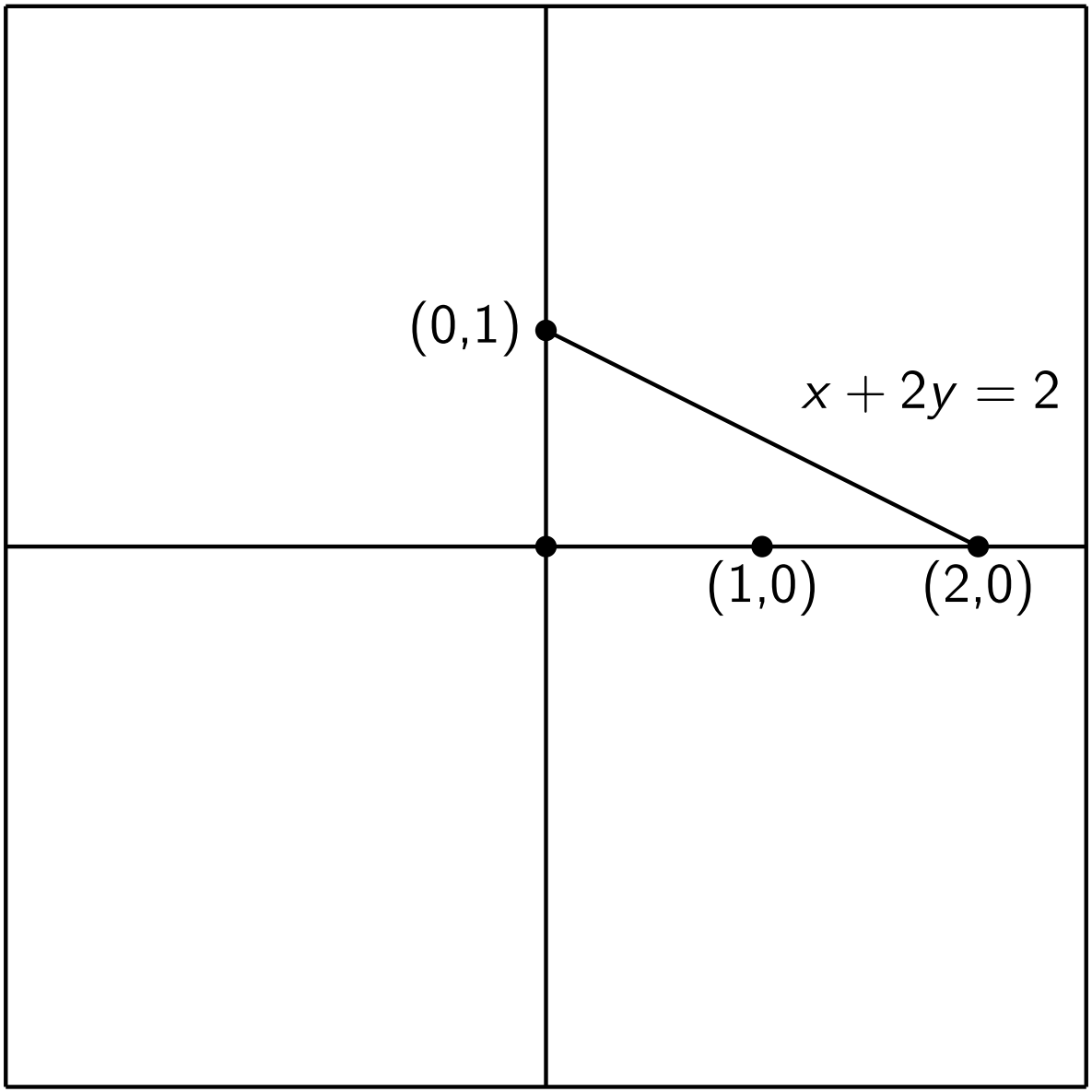
$$= \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k (1-p)^{n-k} = (pe^t + 1 - p)^n$$

$$M_Z(t) = \prod_{k=1}^{n} M_{X_k}(t) =$$ because X_k are independent

$$= \prod_{k=1}^{n} (pe^t + 1 - p)^{m_k} = (pe^t + 1 - p)^{\sum_{k=1}^{n} m_k}$$

$$Z \sim B\left(\sum_{k=1}^{n} m_k, p\right)$$

Exercise 9



Probability Density Function

$$f(x, y) = \frac{15}{2}x^2y, \text{within the given triangle}$$

Verifying the density function

$$\int_0^1 \int_0^{2-2y} \frac{15}{2} x^2 y \; dx \; dy$$

Verifying the density function

$$\int_0^1 \int_0^{2-2y} \frac{15}{2} x^2 y \; dx \; dy = \int_0^1 \frac{15}{2} \frac{(2-2y)^3}{3} y \; dy$$

Verifying the density function

$$\begin{aligned}\int_0^1 \int_0^{2-2y} \frac{15}{2} x^2 y \; dx \; dy &= \int_0^1 \frac{15}{2} \frac{(2-2y)^3}{3} y \; dy \\ &= \frac{120}{6} \int_0^1 (1-y)^3 y \; dy\end{aligned}$$

Verifying the density function

$$\begin{aligned}
\int_0^1 \int_0^{2-2y} \frac{15}{2} x^2 y \; dx \; dy &= \int_0^1 \frac{15}{2} \frac{(2-2y)^3}{3} y \; dy \\
&= \frac{120}{6} \int_0^1 (1-y)^3 y \; dy \\
&= 20 \int_0^1 (y - 3y^2 + 3y^3 - y^4) \; dy
\end{aligned}$$

Verifying the density function

$$\begin{aligned}
\int_0^1 \int_0^{2-2y} \frac{15}{2} x^2 y \; dx \; dy &= \int_0^1 \frac{15}{2} \frac{(2-2y)^3}{3} y \; dy \\
&= \frac{120}{6} \int_0^1 (1-y)^3 y \; dy \\
&= 20 \int_0^1 (y - 3y^2 + 3y^3 - y^4) \; dy \\
&= 20 \cdot \left(\frac{y^2}{2} - \frac{3y^3}{3} + \frac{3y^4}{4} - \frac{y^5}{5}\right)\Big|_{y=0}^{y=1}
\end{aligned}$$

Verifying the density function

$$
\begin{aligned}
\int_0^1 \int_0^{2-2y} \frac{15}{2} x^2 y \; dx \; dy &= \int_0^1 \frac{15}{2} \frac{(2-2y)^3}{3} y \; dy \\
&= \frac{120}{6} \int_0^1 (1-y)^3 y \; dy \\
&= 20 \int_0^1 (y - 3y^2 + 3y^3 - y^4) \; dy \\
&= 20 \cdot \left(\frac{y^2}{2} - \frac{3y^3}{3} + \frac{3y^4}{4} - \frac{y^5}{5}\right)\Big|_{y=0}^{y=1} \\
&= 20 \cdot \left(\frac{1}{2} - \frac{3}{3} + \frac{3}{4} - \frac{1}{5}\right)
\end{aligned}
$$

Verifying the density function

$$
\begin{aligned}
\int_0^1 \int_0^{2-2y} \frac{15}{2} x^2 y \; dx \; dy &= \int_0^1 \frac{15}{2} \frac{(2-2y)^3}{3} y \; dy \\
&= \frac{120}{6} \int_0^1 (1-y)^3 y \; dy \\
&= 20 \int_0^1 (y - 3y^2 + 3y^3 - y^4) \; dy \\
&= 20 \cdot \left(\frac{y^2}{2} - \frac{3y^3}{3} + \frac{3y^4}{4} - \frac{y^5}{5}\right)\Big|_{y=0}^{y=1} \\
&= 20 \cdot \left(\frac{1}{2} - \frac{3}{3} + \frac{3}{4} - \frac{1}{5}\right) \\
&= 20 \cdot \left(\frac{5}{4} - \frac{6}{5}\right)
\end{aligned}
$$

Verifying the density function

$$
\begin{aligned}
\int_0^1 \int_0^{2-2y} \frac{15}{2} x^2 y \; dx \; dy &= \int_0^1 \frac{15}{2} \frac{(2-2y)^3}{3} y \; dy \\
&= \frac{120}{6} \int_0^1 (1-y)^3 y \; dy \\
&= 20 \int_0^1 (y - 3y^2 + 3y^3 - y^4) \; dy \\
&= 20 \cdot \left(\frac{y^2}{2} - \frac{3y^3}{3} + \frac{3y^4}{4} - \frac{y^5}{5}\right)\Big|_{y=0}^{y=1} \\
&= 20 \cdot \left(\frac{1}{2} - \frac{3}{3} + \frac{3}{4} - \frac{1}{5}\right) \\
&= 20 \cdot \left(\frac{5}{4} - \frac{6}{5}\right) \\
&= 20 \cdot \frac{25-24}{20}
\end{aligned}
$$

Verifying the density function

$$
\begin{aligned}
\int_0^1 \int_0^{2-2y} \tfrac{15}{2} x^2 y \; dx \; dy &= \int_0^1 \tfrac{15}{2} \tfrac{(2-2y)^3}{3} y \; dy \\
&= \tfrac{120}{6} \int_0^1 (1-y)^3 y \; dy \\
&= 20 \int_0^1 (y - 3y^2 + 3y^3 - y^4) \; dy \\
&= 20 \cdot \left(\tfrac{y^2}{2} - \tfrac{3y^3}{3} + \tfrac{3y^4}{4} - \tfrac{y^5}{5}\right)\Big|_{y=0}^{y=1} \\
&= 20 \cdot \left(\tfrac{1}{2} - \tfrac{3}{3} + \tfrac{3}{4} - \tfrac{1}{5}\right) \\
&= 20 \cdot \left(\tfrac{5}{4} - \tfrac{6}{5}\right) \\
&= 20 \cdot \tfrac{25-24}{20} \\
&= 20 \cdot \tfrac{1}{20}
\end{aligned}
$$

Verifying the density function

$$
\begin{aligned}
\int_0^1 \int_0^{2-2y} \frac{15}{2} x^2 y \; dx \; dy &= \int_0^1 \frac{15}{2} \frac{(2-2y)^3}{3} y \; dy \\
&= \frac{120}{6} \int_0^1 (1-y)^3 y \; dy \\
&= 20 \int_0^1 (y - 3y^2 + 3y^3 - y^4) \; dy \\
&= 20 \cdot \left(\frac{y^2}{2} - \frac{3y^3}{3} + \frac{3y^4}{4} - \frac{y^5}{5}\right)\Big|_{y=0}^{y=1} \\
&= 20 \cdot \left(\frac{1}{2} - \frac{3}{3} + \frac{3}{4} - \frac{1}{5}\right) \\
&= 20 \cdot \left(\frac{5}{4} - \frac{6}{5}\right) \\
&= 20 \cdot \frac{25-24}{20} \\
&= 20 \cdot \frac{1}{20} \\
&= 1
\end{aligned}
$$

Transformation of the variables

- $T = \frac{X}{Y}$
- $Z = Y$

Transformation of the variables

- $T = \frac{X}{Y}$
- $Z = Y$

$$\Longrightarrow$$

- $X = TZ$
- $Y = Z$

Transformation of the variables

- $T = \frac{X}{Y}$
- $Z = Y$

$$\Longrightarrow$$

- $X = TZ$
- $Y = Z$

The jacobian

$$\frac{\partial(X,Y)}{\partial(T,Z)} = \begin{vmatrix} \frac{\partial X}{\partial T} & \frac{\partial X}{\partial Z} \\ \frac{\partial Y}{\partial T} & \frac{\partial Y}{\partial Z} \end{vmatrix} = \begin{vmatrix} Z & T \\ 0 & 1 \end{vmatrix} = Z$$

Transformation of the density function

$$
\begin{aligned}
g(z,t) = & \quad f(x(z,t), y(z,t)).|J| \\
= & \quad \tfrac{15}{2} t^2 z^2 \cdot z \cdot z \\
= & \quad \tfrac{15}{2} t^2 z^4
\end{aligned}
$$

- $0 \leq Y \leq 1$
- $0 \leq X \leq 2 - 2Y$

$$\Longrightarrow$$

- $0 \leq Z \leq 1$
- $0 \leq TZ \leq 2 - 2Y$

- $0 \leq Y \leq 1$
- $0 \leq X \leq 2 - 2Y$

$$\Rightarrow$$

- $0 \leq Z \leq 1$
- $0 \leq TZ \leq 2 - 2Y$

$$TZ \leq 2 - 2Z \Rightarrow Z(T+2) \leq 2 \Rightarrow Z \leq \frac{2}{T+2}$$

- $0 \leq Y \leq 1$
- $0 \leq X \leq 2 - 2Y$

$\Longrightarrow$

- $0 \leq Z \leq 1$
- $0 \leq TZ \leq 2 - 2Y$

$$TZ \leq 2 - 2Z \Rightarrow Z(T+2) \leq 2 \Rightarrow Z \leq \frac{2}{T+2}$$

- $0 \leq Z \leq 1$
- $0 \leq Z \leq \frac{2}{T+2}$

$\Longrightarrow$

- $0 \leq Z \leq \frac{2}{T+2}$

Marginal Distribution of $Z = \frac{X}{Y}$

$$\int_0^{\frac{2}{t+2}} \frac{15}{2} t^2 z^4 \, dz \, dt = \frac{48t^2}{(t+2)^5} \, dt, \quad z \in [0, \infty)$$

Frame Title

WolframAlpha computational intelligence.

int_0^infty int_0^(2/(t+2)) 15/2t^2z^4 dz dt

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Definite integral:

$$\int_0^\infty \int_0^{\frac{2}{t+2}} \frac{15\, t^2\, z^4}{2}\, dz\, dt = 1$$

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Aleksander Szymański, List 6

(10) $M_U(t) = E(e^{tU}) = 1 + t \cdot E(U) + \frac{t^2 E(X^2)}{2!} + \frac{t^3 E(X^3)}{3!} + \dots$

$M_U(t) = \frac{2}{2-3t}$

a)

$$\frac{d}{dt} M_U(t) = \frac{d}{dt} E(e^{tU}) = \frac{d}{dt}\left(1 + t \cdot E(U) + \frac{t^2 \cdot E(X^2)}{2!} + \frac{t^3 E(X^3)}{3!} + \dots\right) =$$

$$= 0 + E(U) + \frac{\cancel{2} \cdot t E(X^2)}{\cancel{2}!} + \frac{\cancel{3} \cdot t^2 \cdot E(X^3)}{\cancel{3} \cdot 2!} + \dots$$

$$\frac{d}{dt} M_U(0) = 0 + E(U) + 0 + \dots \Rightarrow \frac{d}{dt} M_U(0) = E(U)$$

$$\frac{d}{dt} M_U(t) = \left(\frac{2}{2-3t}\right)' = \frac{0 - 2 \cdot (-3)}{(2-3t)^2} = \frac{6}{4 - 6t + 9t^2}$$

$$E(U) = \frac{6}{4} = \frac{3}{2}$$

b) $V(U) = E(U^2) - (E(U))^2$.

$$(M_U(t))'' = \left(\frac{6}{(2-3t)^2}\right)' = \frac{0 - 6 \cdot 2 \cdot (2-3t) \cdot (-3)}{(2-3t)^4} =$$

$$= \frac{36}{(2-3t)^3}, \quad E(U^2) = (M_U(0))'' = \frac{36}{8} = \frac{9}{2}$$

$$V(U) = \frac{9}{2} - \left(\frac{3}{2}\right)^2 = \frac{9}{2} - \frac{9}{4} = \frac{9}{4}$$

c) $X = 0{,}9U$

$$M_X(t) = E(e^{t \cdot (0{,}9U)}) = E(e^{(0{,}9t)U}) = M_U(0{,}9t) = \frac{2}{2 - 3 \cdot 0{,}9t} = \frac{2}{2 - 2{,}7t}$$

Ex. 11

a) $M_Y(t) = M_X(2t) \cdot M_X(4t)$

Let X_1, X_2 be independent random variables, MGF of X_1 and X_2 is the same as X.

Then $Y = 2X_1 + 4X_2$

b) $Y = X + 2$

c) $M_Y(t) = 4 M_X(t)$

$M_Y(0) = 4 M_Y(0)$ MGF (0) must be equal to 1!

Either Y or X is not a random variable