Exercise 1 set 4

Wiktor Hamberger
308982

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Given is function $f(x,y) = C(x+y)exp\{-(x+y)\}$, where $x > 0, y > 0$.

1. Compute the value of C such that f(x, y) is the density of 2-dimensional r.v. (X, Y).

$$\int_0^\infty \int_0^\infty f(x,y)dxdy = 1$$

$$= \int_0^\infty \int_0^\infty C(x+y)e^{-(x+y)}dxdy = C\int_0^\infty \int_0^\infty (x+y)e^{-x}e{-}ydxdy =$$
$$= C\int_0^\infty e^{-y}\int_0^\infty xe^{-x}{+}ye^{-x}dxdy = C\int_0^\infty e^{-y}(\int_0^\infty xe^{-x}dx{+}\int_0^\infty ye^{-x}dx)dy =$$
$$= C\int_0^\infty e^{-y}(0-0+0-(-1)+y(0-(-1)))dy = C\int_0^\infty e^{-y}(1+y)dy =$$
$$= C(\int_0^\infty e^{-y}dy{+}\int_0^\infty e^{-y}ydy) = C(0{-}(-1){+}0{-}0{-}0{-}(-1)) = 2C = 1 \implies C = \frac{1}{2}$$

2. Check if variables X, Y are independent.
Variables X, Y are independent if $\forall x, y \in \mathbf{R}\ f(x,y) = f_1(x) * f_2(y)$

$$f_1(x) = \int_0^\infty \frac{1}{2}(x+y)e^{-x}e^{-y}dy = \frac{1}{2}e^{-x}\int_0^\infty (x+y)e^{-y}dy = \frac{1}{2}e^{-x}(x+1)$$

$$f_2(y) = \int_0^\infty \frac{1}{2}(x+y)e^{-x}e^{-y}dx = \frac{1}{2}e^{-y}(y+1)$$

For $x, y = 2$, $f_1(2) = \frac{3}{2e^2}$, $f_2(2) = \frac{3}{2e^2}$, $f(2,2) = \frac{2}{e^4} \implies$ they are not independent.
Also f(x, y) is not defined for $x, y \leq 0$, so it's another proof that X and Y are not independent.

3. Find moments m_{10}, m_{01}.

$$m_{pq} = \int_{\mathbf{R}}\int_{\mathbf{R}} x^p y^q f(x,y)dydx$$

$$m_{10} = \int_0^\infty \int_0^\infty x^1 y^0 \frac{1}{2}(x+y)e^{-x}e^{-y}dydx = \int_0^\infty \frac{1}{2}xe^{-x}(x\int_0^\infty e^{-y}dy + \int_0^\infty ye^{-y}dy)dx =$$
$$= \int_0^\infty \frac{1}{2}xe^{-x}(x+1)dx = \frac{1}{2}(\int_0^\infty x^2e^{-x}dx + \int_0^\infty xe^{-x}dx) = \frac{1}{2}[(-e^{-x}x^2)\Big|_0^\infty + 3\int_0^\infty xe^{-x}dx] =$$
$$= \frac{1}{2}(0 + 3 * 1) = 1.5$$

m_{10} *and* m_{01} are symmetric so $m_{10} = m_{01} = 1.5$

p_i | $\bar{u}$ u u

$E(Y) = 10 + 8 + 5 + 12{,}5 = 35{,}5$

<u>Liota 4.</u>

(2) $f(x,y) = Cxy + x + y$ for $\begin{matrix} 0 \le x \le 3 \\ 1 \le y \le 2 \end{matrix}$ - density

conditions: 1. $f(x,y) \geq 0$ for $(x,y) \in R^2$

2. $\int_R \int_R f(x,y)\, dy\, dx = 1$

2. $\int_0^3 \int_1^2 Cxy + x + y \; dy\, dx = 1$

$$\int_0^3 \int_1^2 Cxy + x + y \; dy\, dx = \int_0^3 \left[\int_1^2 Cxy \; dy + \int_1^2 x \; dy + \int_1^2 y \; dy \right] dx =$$

$$= \int_0^3 \left[\frac{Cx}{2} y^2 \Big|_1^2 + xy \Big|_1^2 + \frac{y^2}{2} \Big|_1^2 \right] = \int_0^3 \left[2Cx - \frac{Cx}{2} + x + 2 - \frac{1}{2} \right] dx =$$

$$= \int_0^3 \left(\frac{3}{2} Cx + x + \frac{3}{2} \right) dx = \frac{3}{2} \int_0^3 Cx \; dx + \int_0^3 x \; dx + \frac{3}{2} \cdot \int_0^3 1 \; dx =$$

$$= \frac{3}{2} \cdot \frac{C}{2} x^2 \Big|_0^3 + \frac{x^2}{2} \Big|_0^3 + \frac{3}{2} \cdot x \Big|_0^3 = \frac{3}{2} \cdot \frac{9}{2} C + \frac{9}{2} + \frac{9}{2} =$$

$$= \frac{27}{4} C + 9 \quad \Rightarrow \quad \frac{27}{4} C + 9 = 1 \quad \Rightarrow \quad 1. \; f(x,y) \geq 0$$

$$\frac{27}{4} C = -8$$

$$f(x,y) = -\frac{32}{27} xy + x + y$$

$$27C = -32$$

$$C = -\frac{32}{27}$$

$$\begin{cases} x = 3 \\ y = 2 \end{cases} \Rightarrow f(x,y) = -\frac{32}{9} \cdot 2 + 5 =$$

$$= -\frac{64}{9} + 5 = -7\frac{1}{9} + 5 =$$

$$= -2\frac{1}{9} < 0, \text{ so}$$

there is no such C value

Question 3

Andrzej Tkaczyk, 309181

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1. Calculate marginal density

f_1(x)=$\int_0^1(-xy+x)\ dy = \int_0^1 x(1-y)\ dy = x\int_0^1 1-y\ dy = x * \left[y - \frac{y^2}{2}\right]_0^1 = x * ((1-\frac{1}{2}+c)-(0-\frac{0}{2}+c)) = \frac{x}{2}$

f$_2$(y)=$\int_0^2(-xy+x)\ dx = \int_0^2 x(1-y)\ dx = (1-y)\int_0^2 x\ dx = (1-y) * \left[\frac{x^2}{2}\right]_0^2 = (1-y) * ((\frac{4}{2}+c)-(\frac{0}{2}+c)) = 2-2y$

2. Check, that $f(x,y) = f_1(x)$ * $f_2(y)$

$f_1(x)$ * $f_2(y)$= $\frac{x}{2} * (2-2y) = x - xy =$$f(x,y)$

3. Answer

$f(x,y) = f_1(x)$ * $f_2(y)$ so X and Y are independent.

Ex. 4

$$P(1 \leq X \leq 3,\ 0 \leq Y \leq 0{,}5) = \int_1^2 \int_0^{\frac{1}{2}} -xy + x \, dy \, dx = \int_1^2 -x \left(\int_0^{\frac{1}{2}} y - 1 \, dy \right) dx =$$

$$= \int_1^2 -\frac{1}{8}x + \frac{1}{2}x \, dx = \int_1^2 \frac{3}{8} x \, dx = \frac{9}{16}$$

Problem no 5

Suppose that $X \sim U[0;1]$, and let $Y = X^n$. Prove that $f_Y(y) = \frac{y^{1/n-1}}{n}$, where $0 \leqslant y \leqslant 1$.

Lets take cdf of var. Y:

$$F_Y(t) = P(Y < t) = P(X^n < t) = P(X < t^{1/n}) = F_X(t^{1/n})$$

Lets take the derivative of both sides:

$$f_Y(y) = \frac{y^{1/n-1}}{n} f_X(y^{1/n}) = \frac{y^{1/n-1}}{n}$$

since $f_X(x) = 1$. As for the range of Y, we know that $X \in [0,1]$, and since the result of operation X^n will remain in the same interval for any X, Y will remain in the same range.

RPiS

Kacper Puchalski 310271

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1 Task 3.11

$$E(X) = \sum x_i p_i = 40 \cdot \frac{40}{142} + 32 \cdot \frac{32}{142} + 20 \cdot \frac{20}{142} + 50 \cdot \frac{50}{142} \approx 38,9 \tag{1}$$

$$E(Y) = \sum y_i p_i = 40 \cdot \frac{1}{4} + 32 \cdot \frac{1}{4} + 20 \cdot \frac{1}{4} + 50 \cdot \frac{1}{4} = 35,5 \tag{2}$$

2 Task 4.6

F is CDF, f is density

$$\begin{aligned} F_Y(y) = P(Y < y) = P(X^2 < y) = P(-\sqrt{y} < X < \sqrt{y}) = \\ = P(X < \sqrt{y}) - P(X < -\sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned} \tag{3}$$

$$f_Y(y) = F_Y'(y) = (F_X(\sqrt{y}) - F_X(\sqrt{y}))' = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}} \tag{4}$$

7. Let $X \sim [-1, 1]$ find density of $Y = |X|$

$f_y(y) = ?$

$F_y(y) = P(Y < y) = P(|X| < y) = P(y > X > -y) =$

$= P(X < y) - P(X < -y) = F_x(y) - F_x(-y)$

We know that $f_y(y) = (F_y(y))'$ and $X \sim [-1, 1] \Rightarrow f_x(x) = \frac{1}{2}$

$f_y(y) = (F_y(y))' = (F_x(y) - F_x(-y))' = (F_x(y))' - (F_x(-y))' =$

$= f_x(y) + f_x(-y) = \frac{1}{2} + \frac{1}{2} = 1$

~~Y=x~~ $Y \in [0, 1]$ and $f_y(y) = 1 = \frac{1}{1-0} \Rightarrow Y \sim [0; 1]$

Task 8

April 2, 2020

THEOREM 1 *If X is continuous r.v. then $Y = F_x(X) \sim U[0;1]$.*

PROOF 1 $Y = F_x(X) \therefore y \in [0;1]$.

$$F_Y(y) = P(Y \leq y) = P(F_x(X) \leq y) = P(F_x^{-1}(F_x(X)) \leq F_x^{-1}(y)) = F_x(F_x^{-1}(y)) = y$$

$$f_y(y) = (F_y(y))' = y' = 1$$

$$y \in [0;1] \wedge f_y(y) = 1 \implies Y \sim U[0;1]$$

9. $f(x) = x e^{-x}, \quad x \geq 0$

$Y = X^2$

FROM PROBLEM 6 WE KNOW THAT:

$$f_Y(y) = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}$$

$$f_Y(y) = \frac{\sqrt{y}\, e^{-\sqrt{y}} + (-\sqrt{y}\, e^{\sqrt{y}})}{2\sqrt{y}} = \frac{1}{2} \cdot \left(e^{-\sqrt{y}} - e^{\sqrt{y}}\right)$$

Paweł Głowacki (308023)

Probability and Statistics

Task 10

Smolec, 2 kwietnia 2020

10. Let X $\sim$ U[a;b]. Find value of variance V(X)

Symbol X $\sim$ U[a,b] means that random variable X has uniform distribution on the interval [a,b]. In other words: $f_X(x) = \frac{1}{b-a}$, where $x \in [a, b]$.

$V(X) = E(X^2) - E(X)^2$

$$E(X) = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{b-a}(\frac{x^2}{2})\Big|_a^b = \frac{1}{b-a}(\frac{b^2-a^2}{2}) = \frac{b+a}{2}$$

$$E(X^2) = \int_a^b x^2 \cdot \frac{1}{b-a} dx = \frac{1}{b-a}(\frac{x^3}{3})\Big|_a^b = \frac{1}{b-a}(\frac{b^3-a^3}{3}) = \frac{a^2+ab+b^2}{3}$$

$V(X) = E(X^2) - E(X)^2 = \frac{a^2+ab+b^2}{3} - (\frac{b+a}{2})^2 = \frac{4a^2+4ab+4b^2}{12} - \frac{3a^2+6ab+3b^2}{12} = \frac{a^2-2ab++b^2}{12} = \frac{(a-b)^2}{12}$

Probability & Statistic
Problem set 4

Małgorzata Maciejewska

2 kwietnia 2020

1 Task 11

Random variable X has (standard) Cauchy distribution $f_X(x) = \frac{1}{\pi(1+x^2)}$. We prove that $Y = \frac{1}{X}$ has also (standard) Cauchy distribution.

Proof

Consider cdf of random variable Y.

$$F_Y(t) = P(Y < t) = P(\frac{1}{X} < t) = P(X > \frac{1}{t}) = \int_{1/t}^{\infty} \frac{1}{\pi(1+x^2)} dx = \frac{1}{\pi} arctan(x)|_{1/t}^{\infty} = \frac{1}{2} - \frac{arctan(1/t)}{\pi}$$

From cdf we get density of random variable Y.

$$f_Y(t) = (F_Y(t))' = (\frac{1}{2} - \frac{arctan(1/t)}{\pi})' = \frac{1}{\pi} \cdot \frac{1}{(1/t)^2 + 1} \cdot \frac{-1}{t^2} = \frac{1}{\pi(1+x^2)}$$

Density of random variables X and Y are equal, so Y has also Cauchy distribution. □