Graphs

Unweighted Graphs

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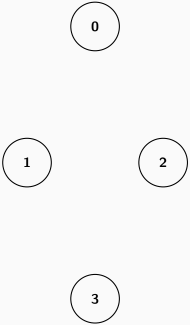
Today we're going to cover

- Graph basics
- Graph representation (recap)
- Depth-first search
- Connected components
- Breadth-first search
- Shortest paths in unweighted graphs

What is a graph?

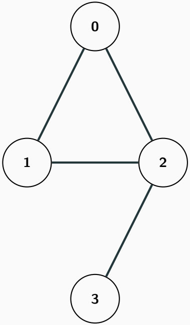
What is a graph?

- Vertices
    - Road intersections
    - Computers
    - Floors in a house
    - Objects



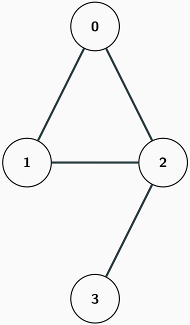
What is a graph?

- Vertices
  - Road intersections
  - Computers
  - Floors in a house
  - Objects
- Edges
  - Roads
  - Ethernet cables
  - Stairs or elevators
  - Relation between objects



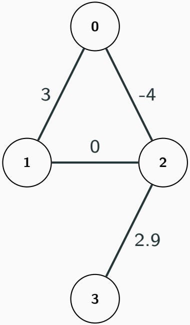
Types of edges

- Unweighted



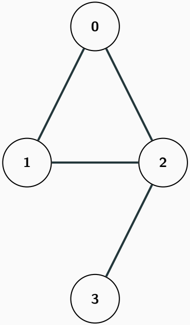
Types of edges

- Unweighted or Weighted



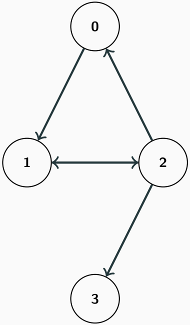
Types of edges

- Unweighted or Weighted
- Undirected



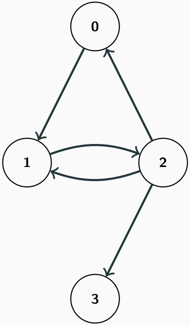
Types of edges

- Unweighted or Weighted
- Undirected or Directed

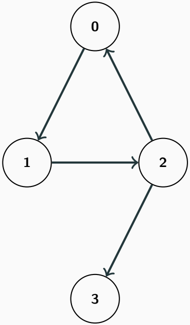


Types of edges

- Unweighted or Weighted
- Undirected or Directed

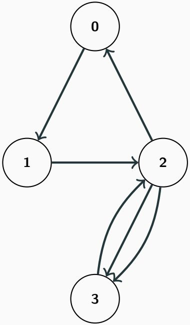


Multigraphs



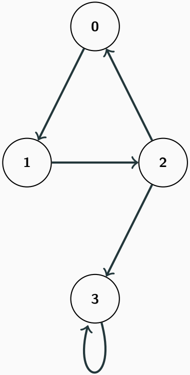
Multigraphs

- Multiple edges



Multigraphs

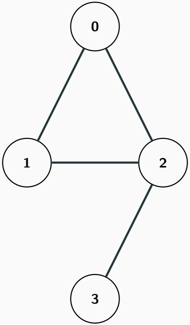
- Multiple edges
- Self-loops



Adjacency list

```
0: 1, 2
1: 0, 2
2: 0, 1, 3
3: 2
```

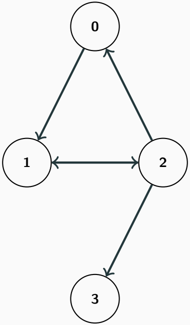
```
vector<int> adj[4];
adj[0].push_back(1);
adj[0].push_back(2);
adj[1].push_back(0);
adj[1].push_back(2);
adj[2].push_back(0);
adj[2].push_back(1);
adj[2].push_back(3);
adj[3].push_back(2);
```



Adjacency list (directed)

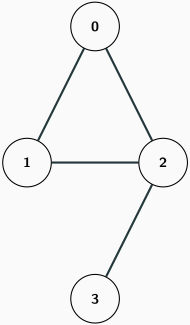
```
0: 1
1: 2
2: 0, 1, 3
3:
```

```
vector<int> adj[4];
adj[0].push_back(1);
adj[1].push_back(2);
adj[2].push_back(0);
adj[2].push_back(1);
adj[2].push_back(3);
```



Vertex properties (undirected graph)

- Degree of a vertex
  - Number of adjacent edges
  - Number of adjacent vertices



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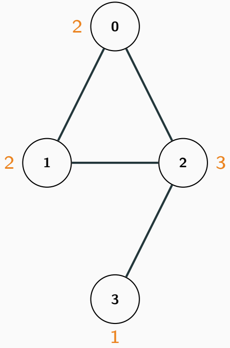
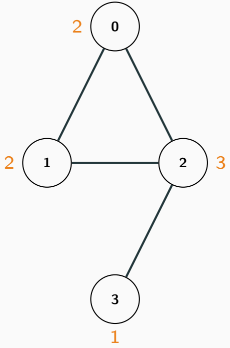


Vertex properties (undirected graph)

- Degree of a vertex
  - Number of adjacent edges
  - Number of adjacent vertices
- Handshaking lemma

$$\sum_{v \in V} \deg(v) = 2|E|$$

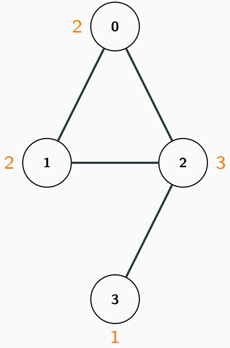


Vertex properties (undirected graph)

- Degree of a vertex
    - Number of adjacent edges
    - Number of adjacent vertices
- Handshaking lemma

$$\sum_{v \in V} \deg(v) = 2|E|$$

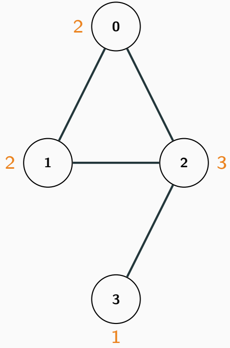
$$2 + 2 + 3 + 1 = 2 \times 4$$



Vertex properties (undirected graph)

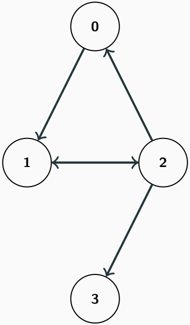
```
0: 1, 2
1: 0, 2
2: 0, 1, 3
3: 2

adj[0].size() // 2
adj[1].size() // 2
adj[2].size() // 3
adj[3].size() // 1
```



Vertex properties (directed graph)

- Outdegree of a vertex
    - Number of outgoing edges



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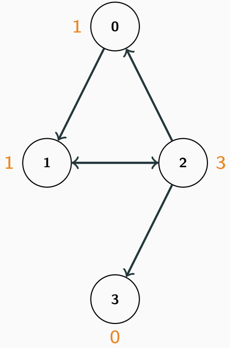
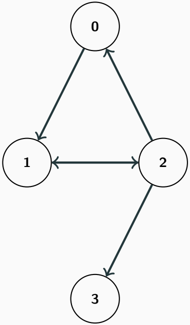


Vertex properties (directed graph)

- Outdegree of a vertex
  - Number of outgoing edges
- Indegree of a vertex
  - Number of incoming edges



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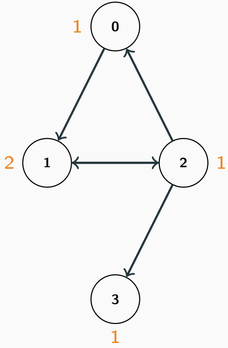

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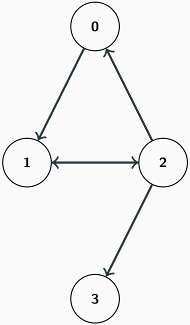

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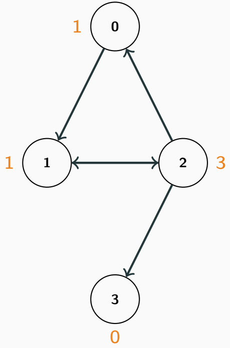
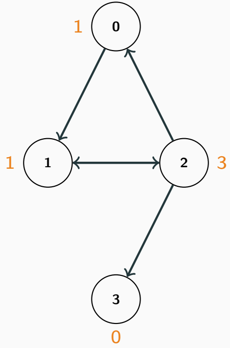


Adjacency list (directed)

```
0: 1
1: 2
2: 0, 1, 3
3:

adj[0].size() // 1
adj[1].size() // 1
adj[2].size() // 3
adj[3].size() // 0
```



Paths

- Path / Walk / Trail:

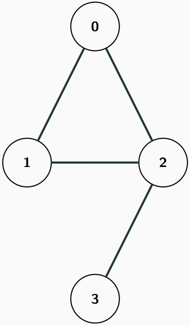
$$e_1 e_2 \dots e_k$$

such that

$$e_i \in E$$

$$e_i = e_j \Rightarrow i = j$$

$$\mathrm{to}(e_i) = \mathrm{from}(e_{i+1})$$



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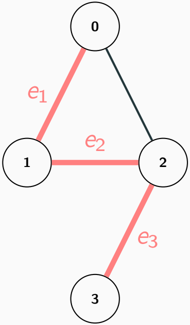

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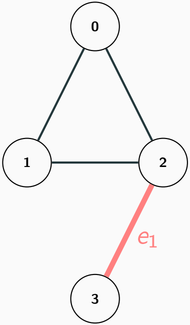

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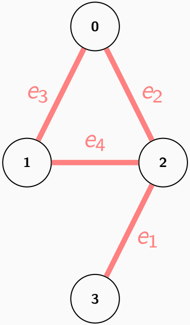

Cycles

- Cycle / Circuit / Tour:

$$e_1 e_2 \dots e_k$$

such that

$$e_i \in E$$

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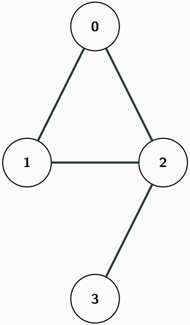

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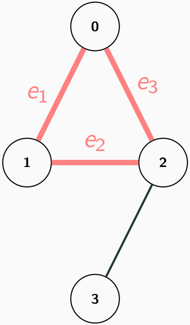

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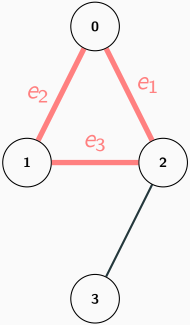

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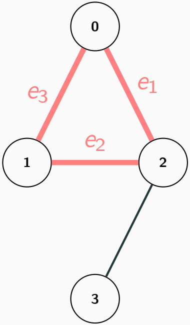


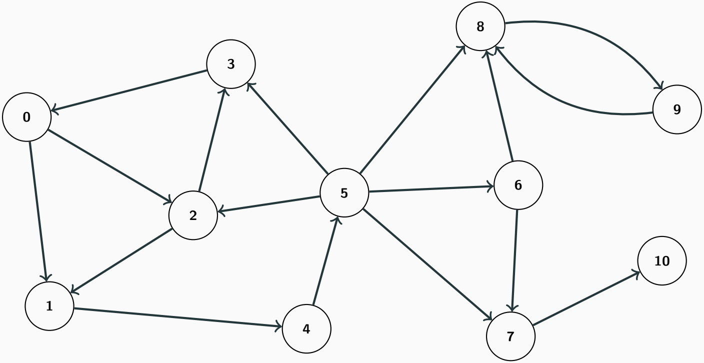
Depth-first search

- Given a graph (either directed or undirected) and two vertices u and v, does there exist a path from u to v?
- Depth-first search is an algorithm for finding such a path, if one exists

- It traverses the graph in depth-first order, starting from the initial vertex u
- We don't actually have to specify a v, since we can just let it visit all reachable vertices from u (and still same time complexity)

- But what is the time complexity?
- Each vertex is visited once, and each edge is traversed once
- $O(n + m)$

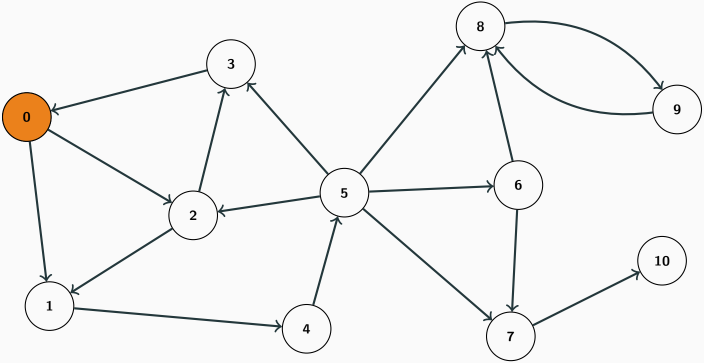
Depth-first search



Stack: |

	0	1	2	3	4	5	6	7	8	9	10
marked	0	0	0	0	0	0	0	0	0	0	0

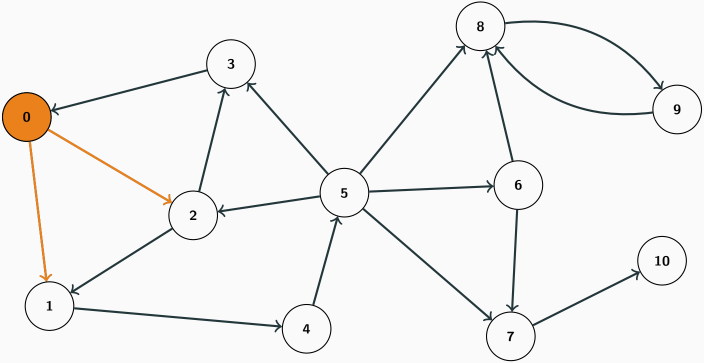
Depth-first search



Stack: 0 |

	0	1	2	3	4	5	6	7	8	9	10
marked	1	0	0	0	0	0	0	0	0	0	0

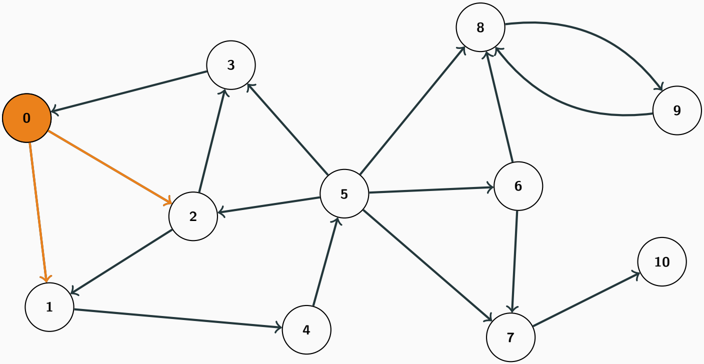
Depth-first search



Stack: 0 |

	0	1	2	3	4	5	6	7	8	9	10
marked	1	0	0	0	0	0	0	0	0	0	0

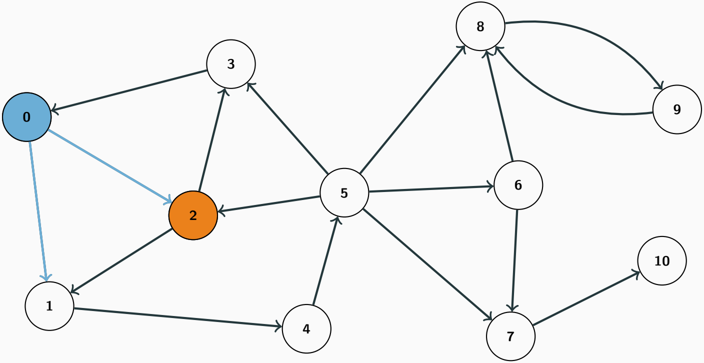
Depth-first search



Stack: 0 | 2 1

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	0	0	0	0	0	0	0	0

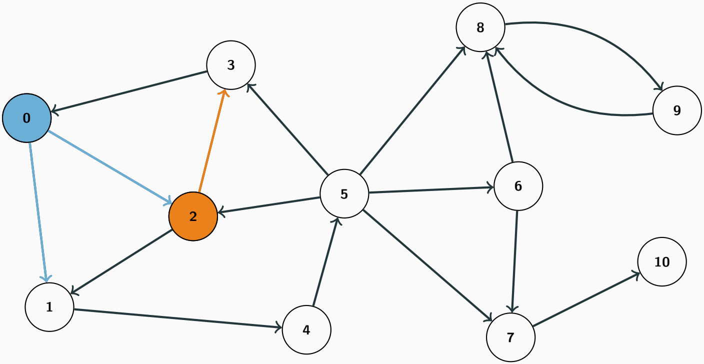
Depth-first search



Stack: 2 | 1

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	0	0	0	0	0	0	0	0

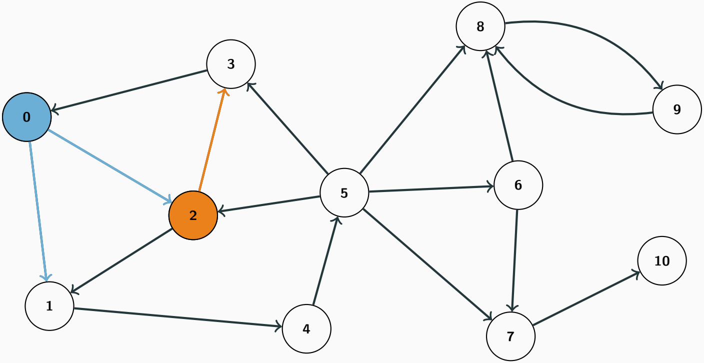
Depth-first search



Stack: 2 | 1

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	0	0	0	0	0	0	0	0

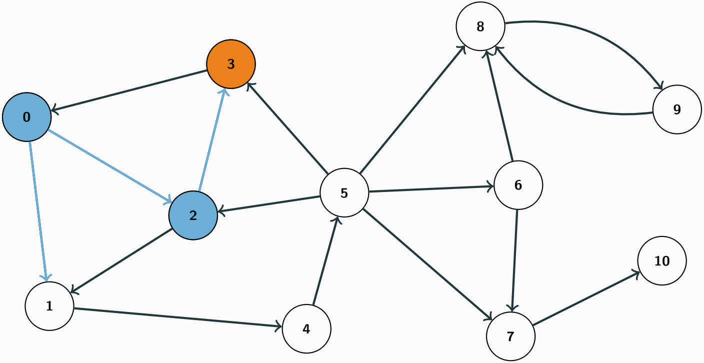
Depth-first search



Stack: 2 | 3 1

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	0	0	0	0	0	0	0

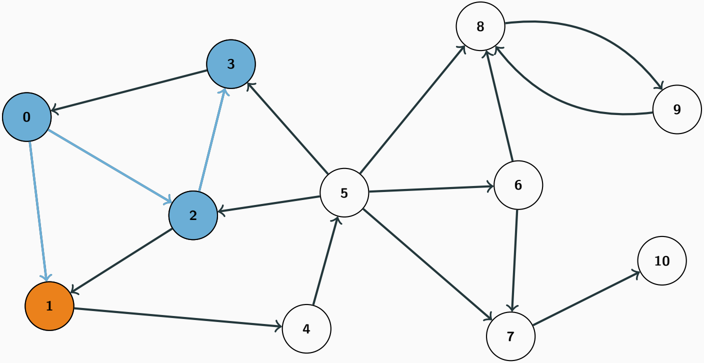
Depth-first search



Stack: 3 | 1

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	0	0	0	0	0	0	0

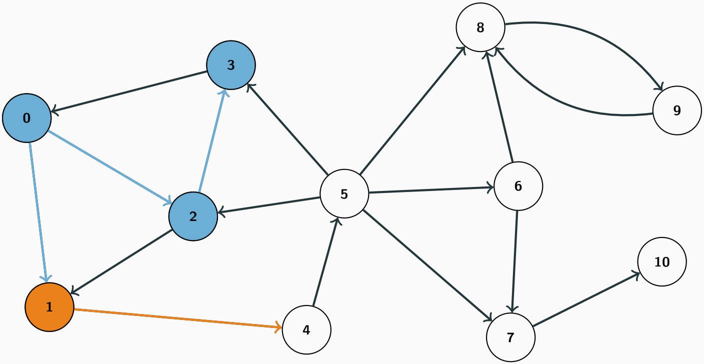
Depth-first search



Stack: 1 |

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	0	0	0	0	0	0	0

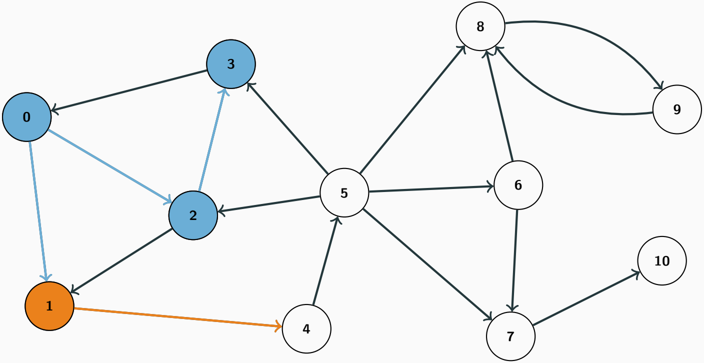
Depth-first search



Stack: 1 |

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	0	0	0	0	0	0	0

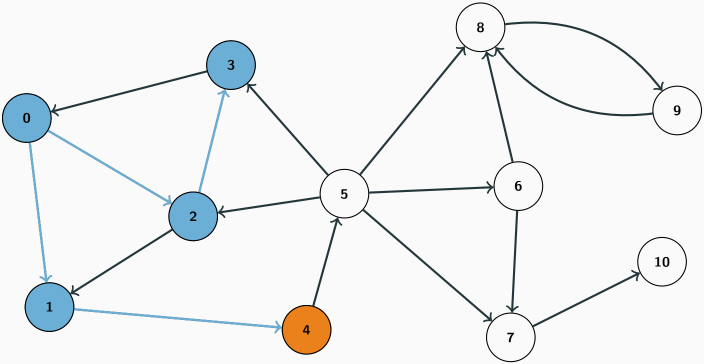
Depth-first search



Stack: 1 | 4

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	0	0	0	0	0	0

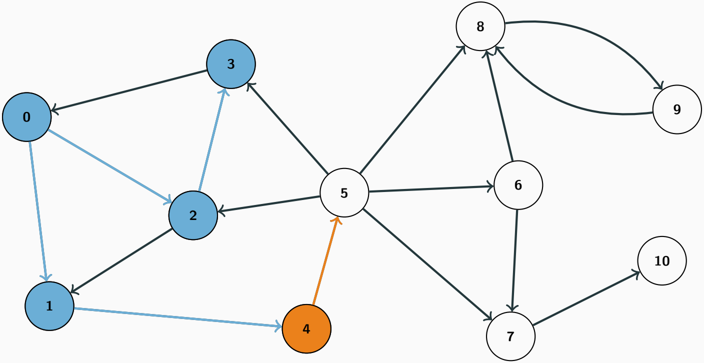
Depth-first search



Stack: 4 |

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	0	0	0	0	0	0

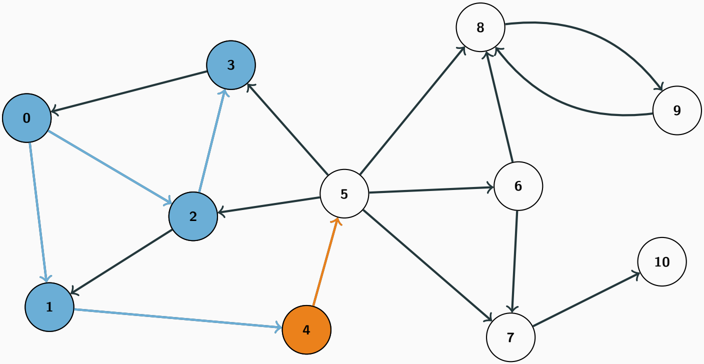
Depth-first search



Stack: 4 |

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	0	0	0	0	0	0

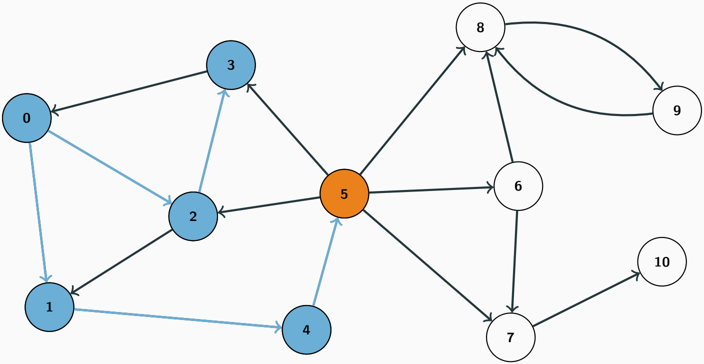
Depth-first search



Stack: 4 | 5

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	0	0	0	0	0

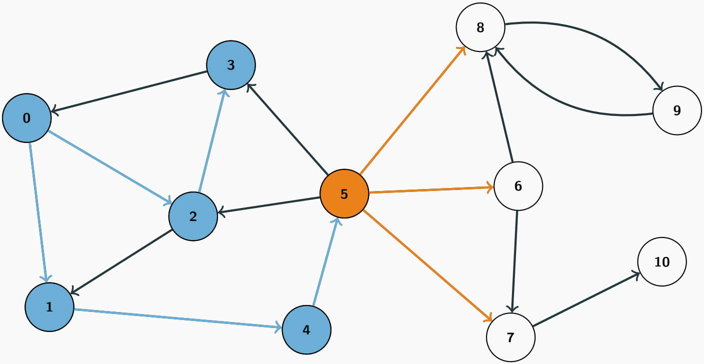
Depth-first search



Stack: 5 |

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	0	0	0	0	0

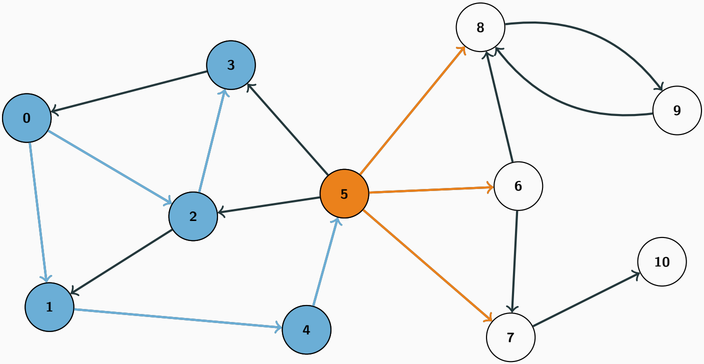
Depth-first search



Stack: 5 |

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	0	0	0	0	0

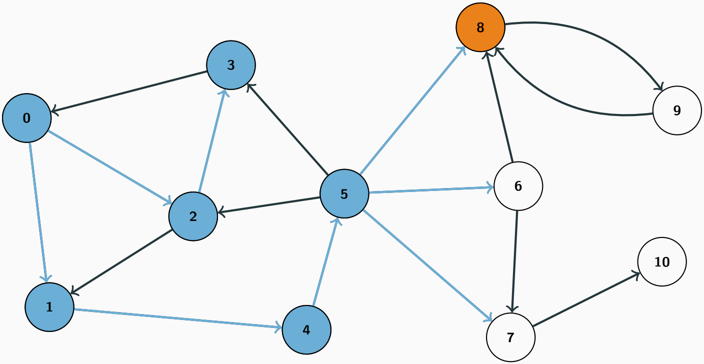
Depth-first search




```
Stack:   5 |  8 6 7
```

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	0	0

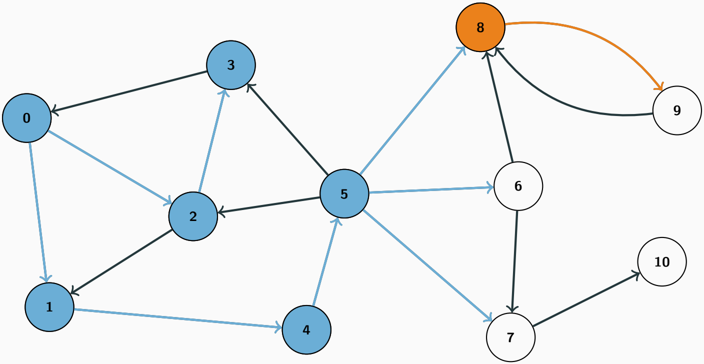
Depth-first search



Stack: 8 | 6 7

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	0	0

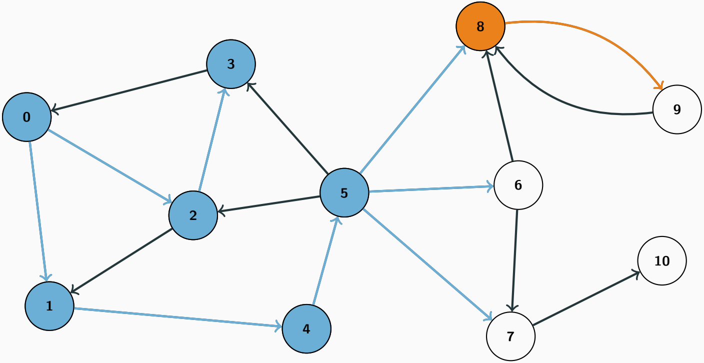
Depth-first search



Stack: 8 | 6 7

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	0	0

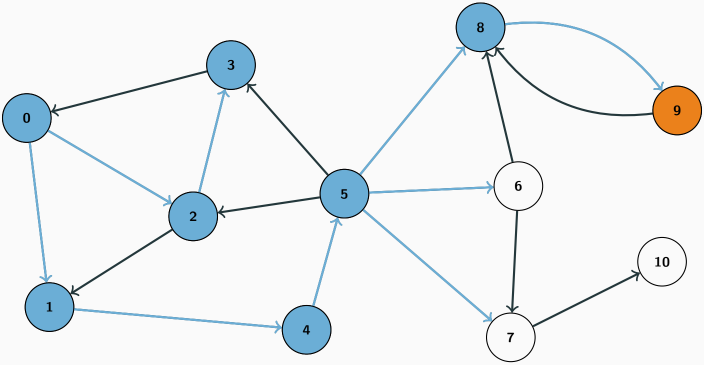
Depth-first search



Stack: 8 | 9 6 7

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	1	0

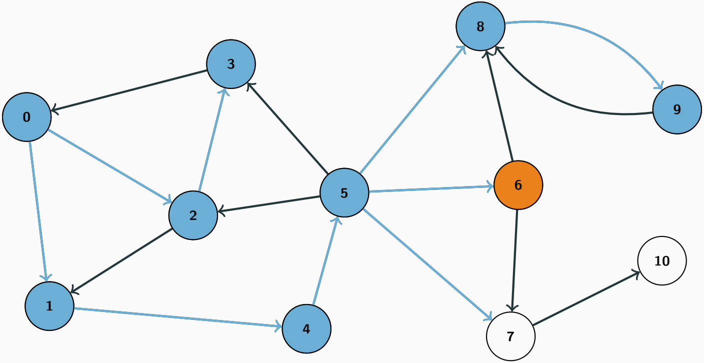
Depth-first search



Stack: 9 | 6 7

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	1	0

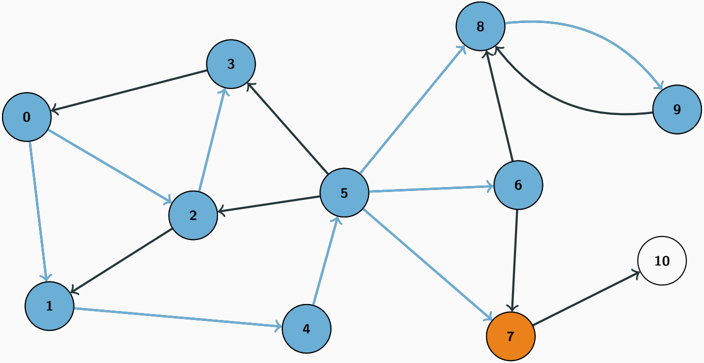
Depth-first search



Stack: 6 | 7

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	1	0

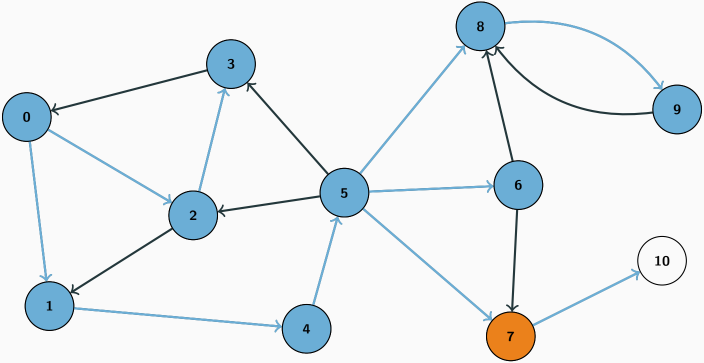
Depth-first search



Stack: 7 |

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	1	0

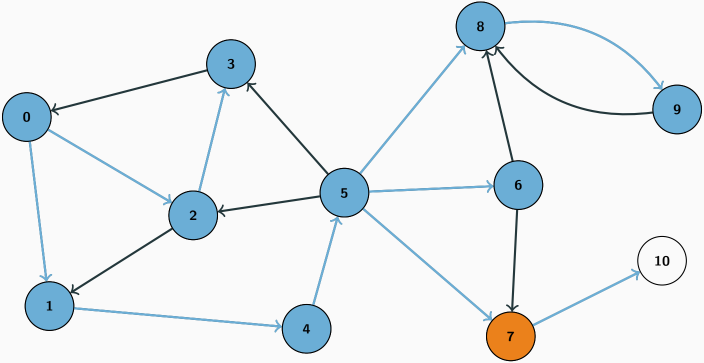
Depth-first search



Stack: 7 |

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	1	0

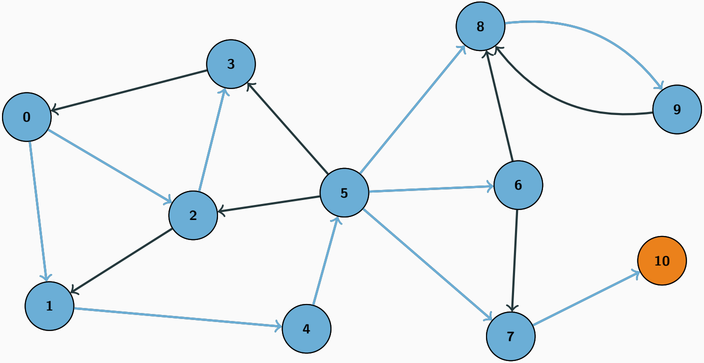
Depth-first search




```
Stack:  7 |  10
```

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	1	1

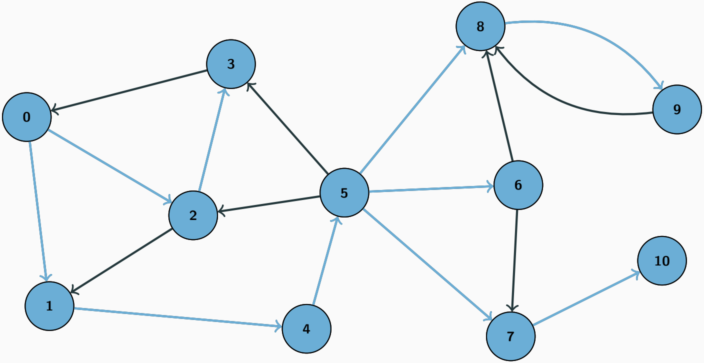
Depth-first search



Stack: 10 |

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	1	1

Depth-first search



Stack: |

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	1	1

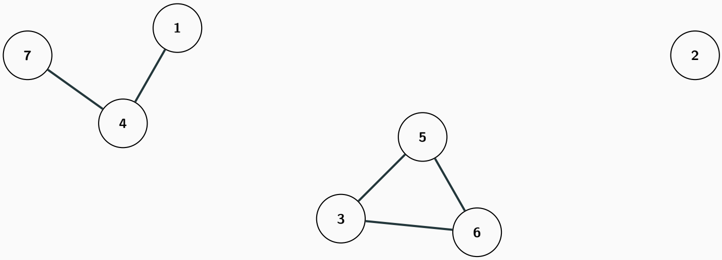
Depth-first search

```
vector<int> adj[1000];
vector<bool> visited(1000, false);

void dfs(int u) {
    if (visited[u]) {
        return;
    }

    visited[u] = true;

    for (int i = 0; i < adj[u].size(); i++) {
        int v = adj[u][i];
        dfs(v);
    }
}
```

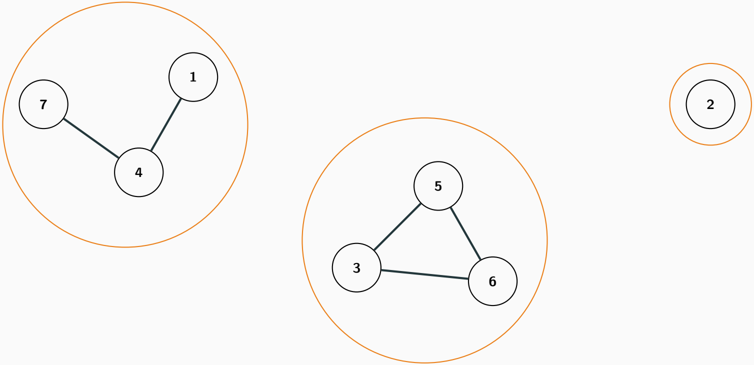
Connected components

- An *undirected graph* can be partitioned into connected components
- A connected component is a maximal subset of the vertices such that each pair of vertices is reachable from each other
- We've already seen this in a couple of problems, but we've been using Union-Find to keep track of the components

Connected components



Connected components



Connected components

- Also possible to find these components using depth-first search
- Pick some vertex we don't know anything about, and do a depth-first search from that vertex
- All vertices reachable from that starting vertex are in the same component
- Repeat this process until you have all the components
- Time complexity is $O(n + m)$

Connected components

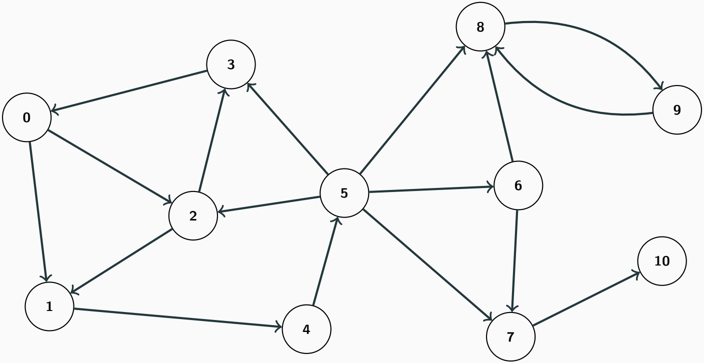
```
vector<int> adj[1000];
vector<int> component(1000, -1);

void find_component(int cur_comp, int u) {
    if (component[u] != -1) {
        return;
    }

    component[u] = cur_comp;

    for (int i = 0; i < adj[u].size(); i++) {
        int v = adj[u][i];
        find_component(cur_comp, v);
    }
}

int components = 0;
for (int u = 0; u < n; u++) {
    if (component[u] == -1) {
        find_component(components, u);
        components++;
    }
}
```

Depth-first search tree

- When we do a depth-first search from a certain vertex, the edges that we go over form a tree
- When we go from a vertex to another vertex that we haven't visited before, the edge that we take is called a *forward edge*
- When we go from a vertex to another vertex that we've already visited before, the edge that we take is called a *backward edge*
- To be more specific: the forward edges form a tree

- *see example*

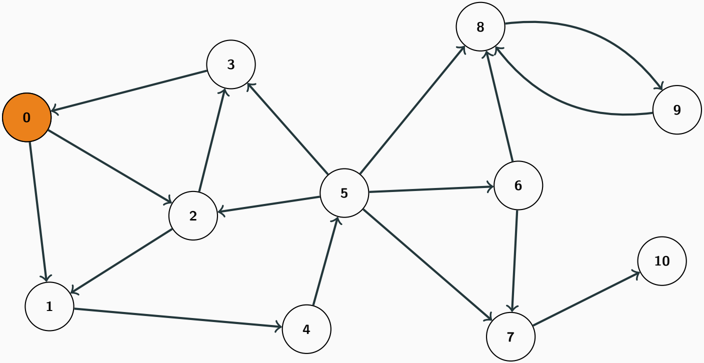
Breadth-first search



Queue:

	0	1	2	3	4	5	6	7	8	9	10
marked	0	0	0	0	0	0	0	0	0	0	0

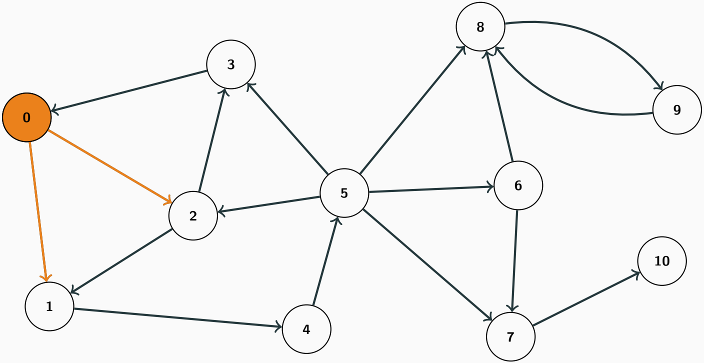
Breadth-first search



Queue: 0

	0	1	2	3	4	5	6	7	8	9	10
marked	1	0	0	0	0	0	0	0	0	0	0

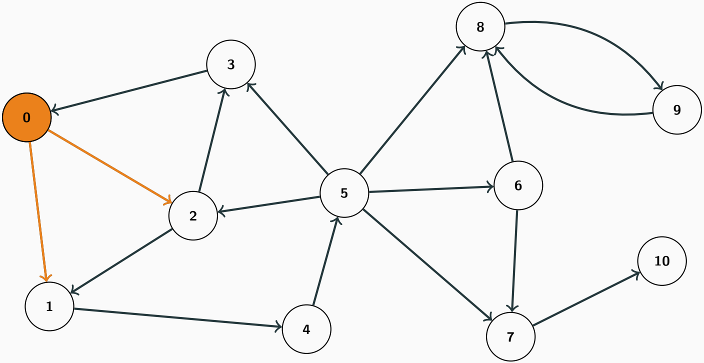
Breadth-first search



Queue: 0

	0	1	2	3	4	5	6	7	8	9	10
marked	1	0	0	0	0	0	0	0	0	0	0

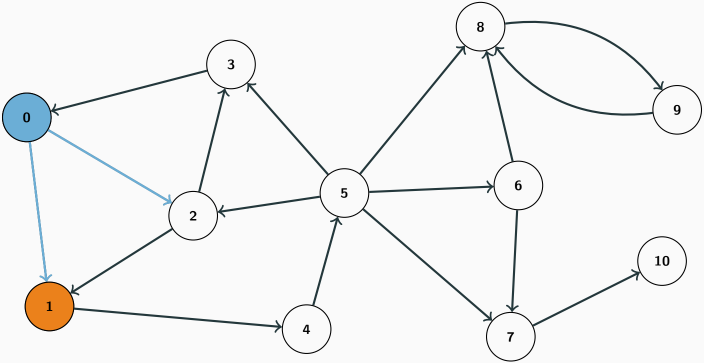
Breadth-first search



Queue: 0 1 2

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	0	0	0	0	0	0	0	0

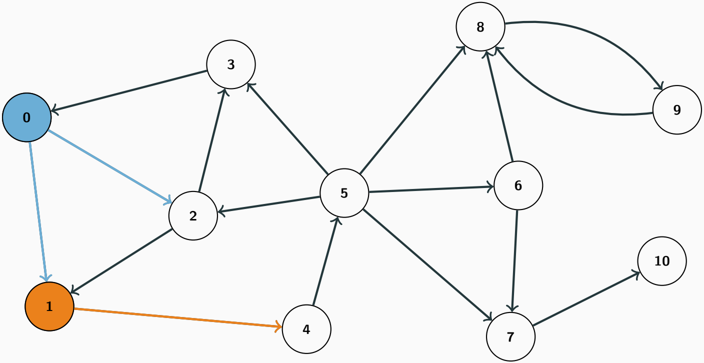
Breadth-first search



Queue: 1 2

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	0	0	0	0	0	0	0	0

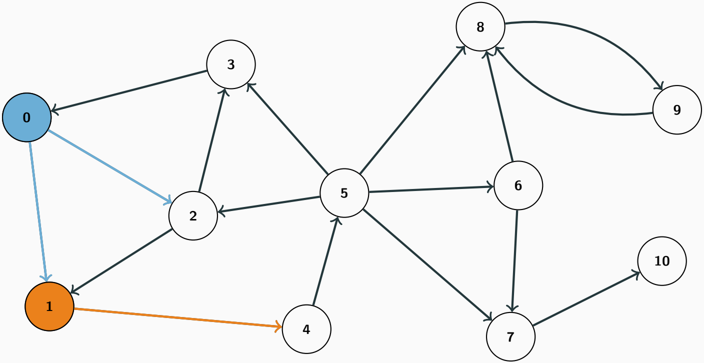
Breadth-first search



Queue: 1 2

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	0	0	0	0	0	0	0	0

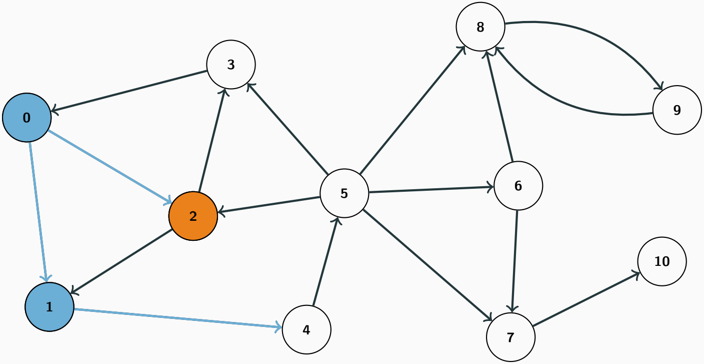
Breadth-first search



Queue: 1 2 4

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	0	1	0	0	0	0	0	0

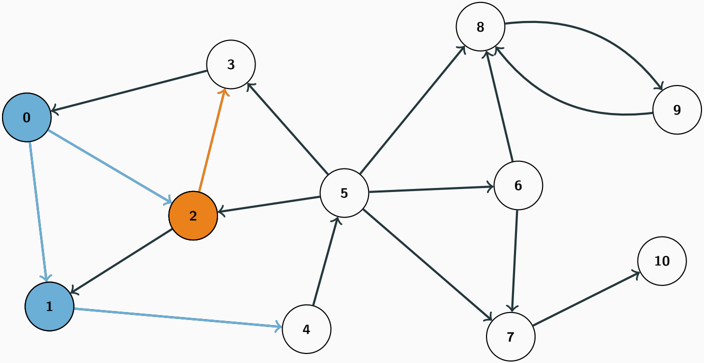
Breadth-first search



Queue: 2 4

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	0	1	0	0	0	0	0	0

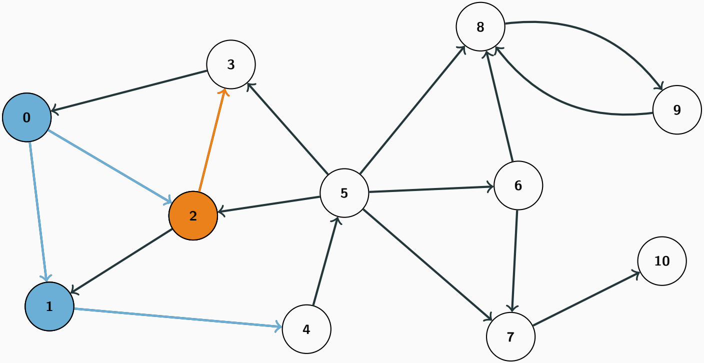
Breadth-first search



Queue: 2 4

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	0	1	0	0	0	0	0	0

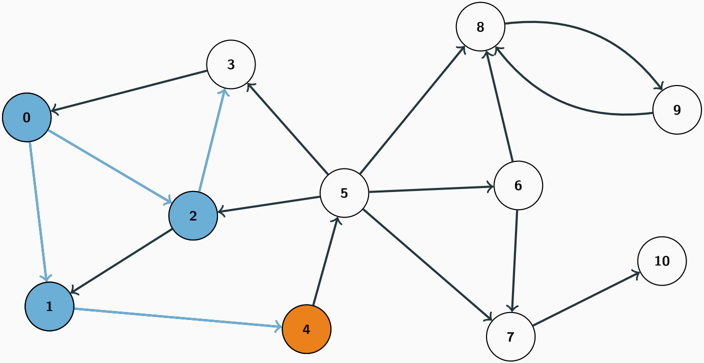
Breadth-first search



Queue: 2 4 3

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	0	0	0	0	0	0

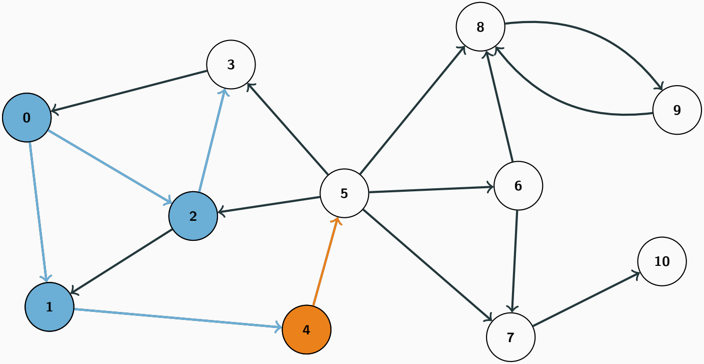
Breadth-first search



Queue: 4 3

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	0	0	0	0	0	0

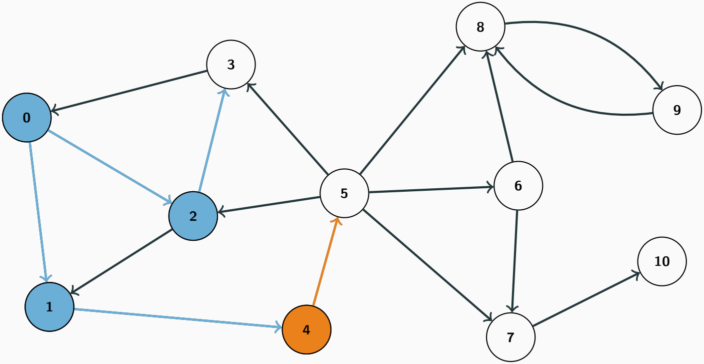
Breadth-first search



Queue: 4 3

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	0	0	0	0	0	0

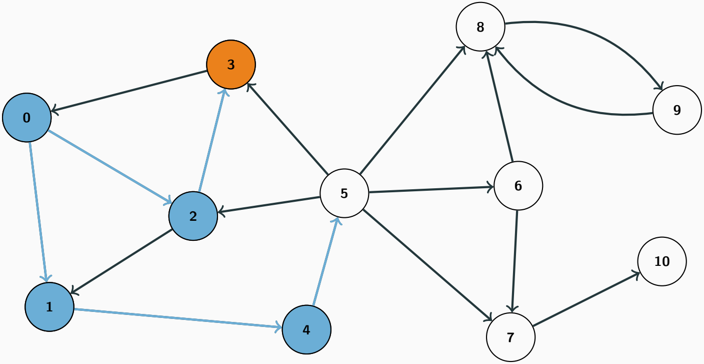
Breadth-first search



Queue: 4 3 5

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	0	0	0	0	0

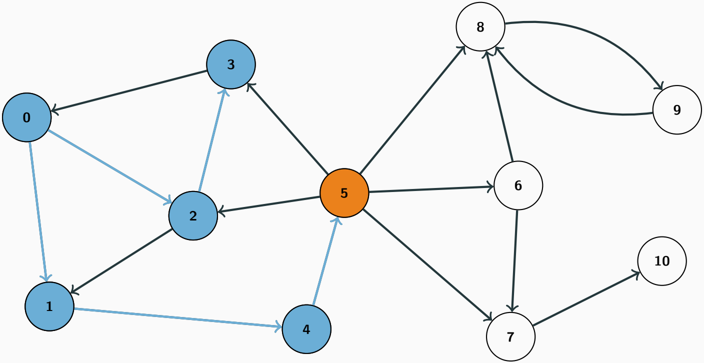
Breadth-first search



Queue: 3 5

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	0	0	0	0	0

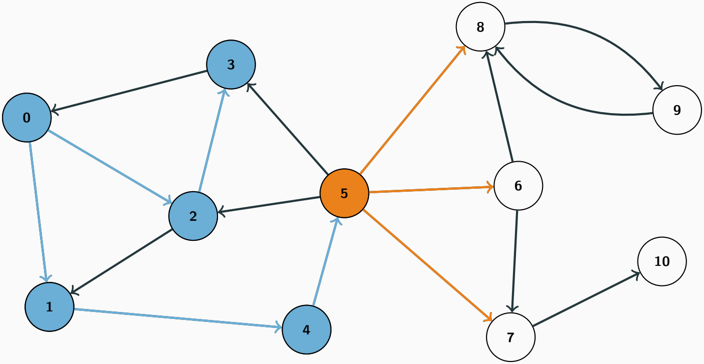
Breadth-first search



Queue: 5

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	0	0	0	0	0

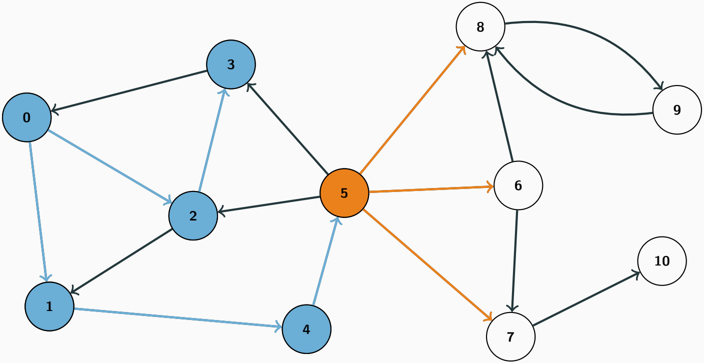
Breadth-first search



Queue: 5

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	0	0	0	0	0

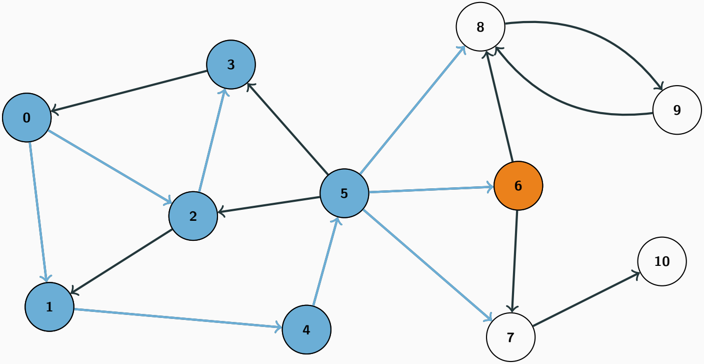
Breadth-first search



Queue: 5 6 7 8

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	0	0

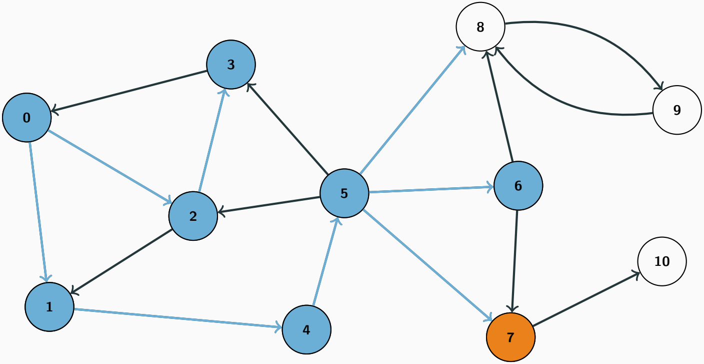
Breadth-first search



Queue: 6 7 8

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	0	0

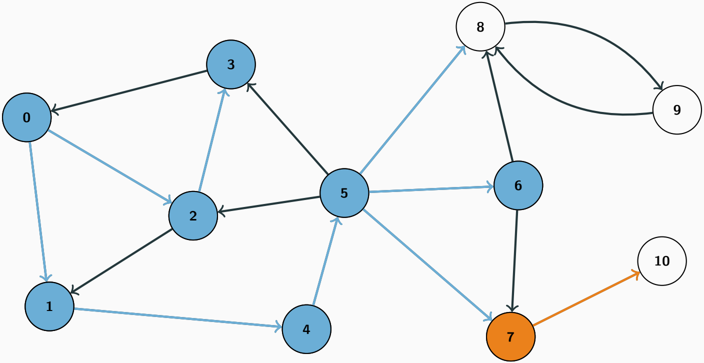
Breadth-first search



Queue: 7 8

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	0	0

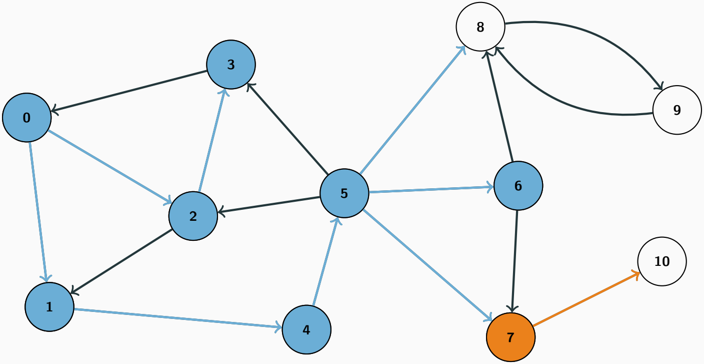
Breadth-first search



Queue: 7 8

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	0	0

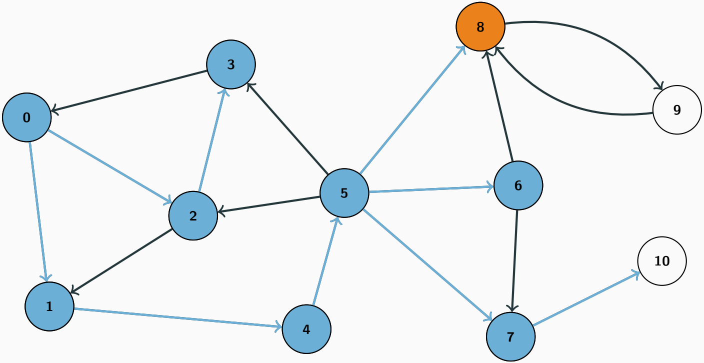
Breadth-first search



Queue: 7 8 10

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	0	1

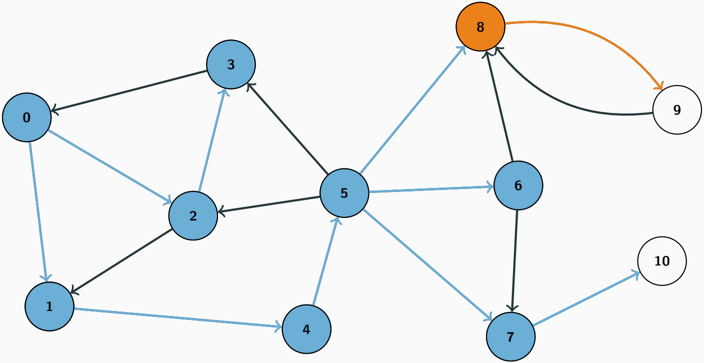
Breadth-first search



Queue: 8 10

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	0	1

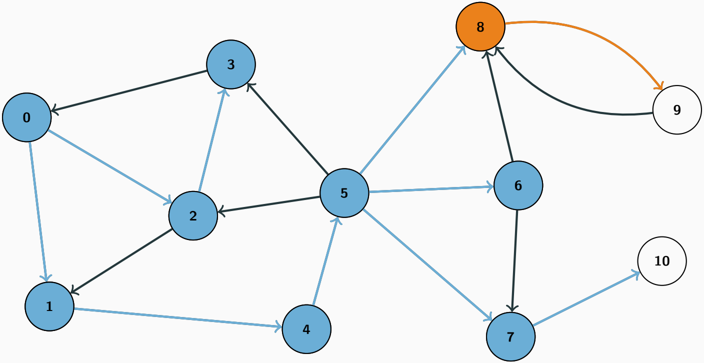
Breadth-first search



Queue: 8 10

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	0	1

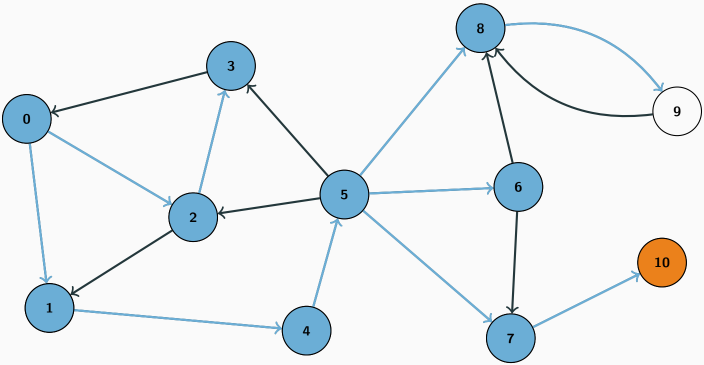
Breadth-first search



Queue: 8 10 9

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	1	1

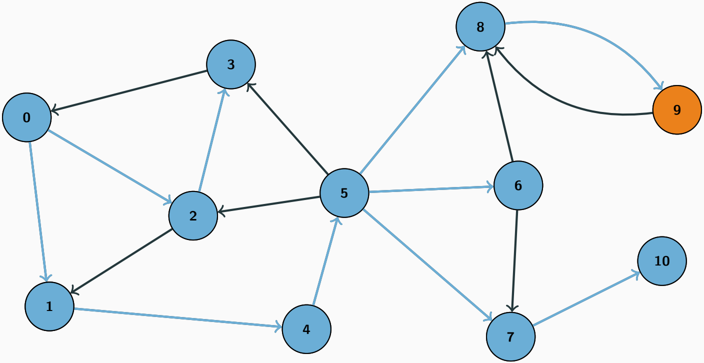
Breadth-first search



Queue: 10 9

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	1	1

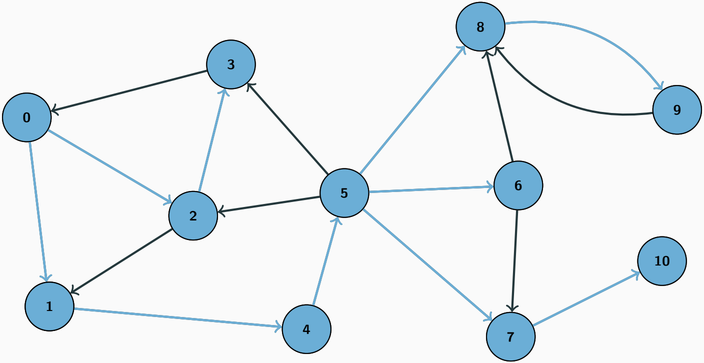
Breadth-first search



Queue: 9

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	1	1

Breadth-first search



Queue:

	0	1	2	3	4	5	6	7	8	9	10
marked	1	1	1	1	1	1	1	1	1	1	1

Breadth-first search

```
vector<int> adj[1000];
vector<bool> visited(1000, false);

queue<int> Q;
Q.push(start);
visited[start] = true;

while (!Q.empty()) {
    int u = Q.front(); Q.pop();

    for (int i = 0; i < adj[u].size(); i++) {
        int v = adj[u][i];
        if (!visited[v]) {
            Q.push(v);
            visited[v] = true;
        }
    }
}
```

Shortest path in unweighted graphs

- We have an unweighted graph, and want to find the shortest path from A to B
- That is, we want to find a path from A to B with the minimum number of edges

- Breadth-first search goes through the vertices in increasing order of distance from the start vertex
- Just do a single breadth-first search from A, until we find B

- Or let the search continue through the whole graph, and then we have the shortest paths from A to all other vertices
- Shortest path from A to all other vertices: $O(n + m)$

Shortest path in unweighted graphs

```
vector<int> adj[1000];
vector<int> dist(1000, -1);

queue<int> Q;
Q.push(A);
dist[A] = 0;

while (!Q.empty()) {
    int u = Q.front(); Q.pop();

    for (int i = 0; i < adj[u].size(); i++) {
        int v = adj[u][i];
        if (dist[v] == -1) {
            Q.push(v);
            dist[v] = 1 + dist[u];
        }
    }
}

printf("%d\n", dist[B]);
```