EXERCICIOS SOBRE LIMITES:

Calcule os seguintes limites.

1) $\lim_{x\to 2} \frac{x^2-7x+10}{x^2-4}$

Solução:

$$\lim_{x\to 2} \frac{x^2 - 7x + 10}{x^2 - 10} = \lim_{x\to 2} \frac{(x-2)(x-5)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-5}{x+2} = -\frac{3}{4}$$

2) $\lim_{x\to -1} \frac{x^2+x-2}{x^2-1}$

Solução:

$$\begin{aligned}\lim_{x\to -1} \frac{x^2+x-2}{x^2-1} &= \lim_{x\to -1} \frac{(x+2)(x-1)}{(x-1)(x+1)} \\ &= \text{does not exist}\end{aligned}$$

3) $\lim_{x\to 5} \frac{x^2+2x-35}{x^2-10x+25}$

Solução:

$$\begin{aligned}\lim_{x\to 5} \frac{x^2+2x-35}{x^2-10x+25} &= \lim_{x\to 5} \frac{(x+7)(x-5)}{(x-5)^2} \\ &= \lim_{x\to 5} \frac{x+7}{x-5} \\ &= \text{does not exist}\end{aligned}$$

4) $\lim_{x\to 25} \frac{5-\sqrt{x}}{25-x}$

Solução:

$$\begin{aligned}\lim_{x\to 25} \frac{5-\sqrt{x}}{25-x} &= \lim_{x\to 25} \frac{5-\sqrt{x}}{25-x}\frac{5+\sqrt{x}}{5+\sqrt{x}} \\ &= \lim_{x\to 25} \frac{25-x}{(25-x)(5+\sqrt{x})} \\ &= \lim_{x\to 25} \frac{1}{5+\sqrt{x}} \\ &= \frac{1}{10}\end{aligned}$$

5) $\lim_{x\to 0} \frac{(x+3)^3-27}{x}$

Solução:

$$\begin{aligned}\lim_{x\to 0} \frac{(x+3)^3-27}{x} &= \lim_{x\to 0} \frac{(x^3+9x^2+27x+27)-27}{x} \\ &= \lim_{x\to 0} \frac{x^3+9x^2+27x}{x} \\ &= \lim_{x\to 0} \frac{x(x^2+9x+27)}{x} \\ &= \lim_{x\to 0} x^2+9x+27 \\ &= 27\end{aligned}$$

6) $\lim_{x\to 0} \frac{x^2}{\sqrt{x^2+12}-\sqrt{12}}$

Solução:

$$\begin{aligned}\lim_{x\to 0} \frac{x^2}{\sqrt{x^2+12}-\sqrt{12}} &= \lim_{x\to 0} \frac{x^2}{\sqrt{x^2+12}-\sqrt{12}}\frac{\sqrt{x^2+12}+\sqrt{12}}{\sqrt{x^2+12}+\sqrt{12}} \\ &= \lim_{x\to 0} \frac{x^2(\sqrt{x^2+12}+\sqrt{12})}{(x^2+12)-12} \\ &= \lim_{x\to 0} \frac{x^2(\sqrt{x^2+12}+\sqrt{12})}{x^2} \\ &= \lim_{x\to 0} \sqrt{x^2+12}+\sqrt{12} \\ &= 2\sqrt{12}\end{aligned}$$

7)Calcule: $\lim_{x\to 0} \frac{3}{x}\left(\frac{1}{5+x} - \frac{1}{5-x}\right)$

Solução:

$$\begin{aligned}
\lim_{x\to 0} \frac{3}{x}\left(\frac{1}{5+x} - \frac{1}{5-x}\right) &= \lim_{x\to 0} \frac{3}{x}\left(\frac{(5-x)-(5+x)}{(5+x)(5-x)}\right) \\
&= \lim_{x\to 0} \frac{3}{x}\left(\frac{-2x}{25-x^2}\right) \\
&= \lim_{x\to 0} \frac{-6}{25-x^2} \\
&= \frac{-6}{25}
\end{aligned}$$

8) Calcule: $\lim_{x\to 4} \frac{(x-4)^3}{|4-x|}$

Solução:

9) Calcule: $\lim_{x\to 0} \frac{x\sin(x)}{|x|}$

Solução:

$$\text{desde que } -1 \le \frac{x}{|x|} \le 1, \quad \text{temos:} \ -\text{sen}(x) \le \frac{x.\text{sen}(x)}{|x|} \le \text{sen}(x)$$

Pelo Teorema do Confronto (Sandwich) como $\lim_{x\to\infty} (\pm \text{sen} x) = 0$, então

$$\lim_{x\to 0} \frac{x.\text{sen}(x)}{|x|} = 0$$

10)Calcule $\lim_{x\to\infty} \frac{100}{x^2+5}$

Solução:

$$\lim_{x\to\infty} \frac{100}{x^2+5} = \frac{100}{\infty}$$

The numerator is always 100 and the denominator $x^2 + 5$ approaches ∞ as x approaches ∞, so that the resulting fraction approaches 0.

11) Calcule $\lim_{x\to -\infty} \frac{7}{x^3-20}$.

Solução:

$$\lim_{x\to -\infty} \frac{7}{x^3-20} = \frac{7}{-\infty} = 0$$

The numerator is always 7 and the denominator $x^3 - 7$ approaches $-\infty$ as x approaches $-\infty$, so that the resulting fraction approaches 0.

12) Calcule: $\lim_{x\to\infty} x^5 - x^2 + x - 10$

Solução:

Note that the expression $x^5 - x^2$ leads to the indeterminate form $\infty - \infty$. Circumvent this by appropriate factoring:

$$\lim_{x\to\infty} \{\, x^2(x^3-1) + (x-10) \,\}.$$

As x approaches ∞, each of the three expressions x^2, $(x^3 - 1)$, and $(x - 10)$ approaches ∞. Temos, então:

$$\text{“}\infty.\infty+\infty\text{”}$$
$$\text{“}\infty + \infty\text{”}$$
$$= \infty$$

Thus, the limit does not exist. Note that an alternate solution follows by first factoring out x^5, the highest power of x . Try it.

13) Calcule $\lim_{x\to -\infty} \frac{x+7}{3x+5}$

Solução:

$$\lim_{x\to -\infty} \frac{x+7}{3x+5} = \frac{-\infty}{-\infty}$$

Dividindo o numerador e denominador da expressão por x, teremos

$$\lim_{x\to -\infty} \frac{x+7}{3x+5} = \lim_{x\to\infty} \frac{\frac{x}{x}+\frac{7}{x}}{\frac{3x}{x}+\frac{5}{x}} = \lim_{x\to\infty} \frac{1+\frac{7}{x}}{3+\frac{5}{x}}$$ e quando $x \to \infty$, $\frac{7}{x}$ e $\frac{5}{x}$ tendem a zero

Desta forma, $\lim_{x\to -\infty} \frac{x+7}{3x+5}$ tende a $\frac{1}{3}$

14) Calcule $\lim_{x\to\infty} \frac{x^2-3x+7}{x^3+10x-4}$

Solução:

Note that the expression $x^2 - 3x$ leads to the indeterminate form $\infty - \infty$ as *x* se approaches ∞. Circumvent this by dividing each of the terms in the original problem by x^3, the highest power of *x* in the problem .

$$\lim_{x\to\infty} \left\{ \frac{x^2-3x+7}{x^3+10x-4} \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \right\}$$

$$\lim_{x\to\infty} \frac{\frac{x^2}{x^3}-\frac{3x}{x^3}+\frac{7}{x^3}}{\frac{x^3}{x^3}+\frac{10x}{x^3}-\frac{4}{x^3}}$$

$$\lim_{x\to\infty} \frac{\frac{1}{x}-\frac{3}{x^2}+\frac{7}{x^3}}{1+\frac{10}{x^2}-\frac{4}{x^3}}$$

When $x \to \infty$ the $\lim_{x\to\infty} \frac{x^2-3x+7}{x^3+10x-4}$ approaches 0

$$\frac{0-0+0}{1+0-0}$$

$= 0$

15) Calcule $\lim_{x\to\infty} x - \sqrt{x^2+7}$

Solução:

$$\lim_{x\to\infty} x - \sqrt{x^2+7} = \lim_{x\to\infty} \frac{(x-\sqrt{x^2+7})(x+\sqrt{x^2+7})}{x+\sqrt{x^2+7}} = \lim_{x\to\infty} \frac{x^2-x^2-7}{x+\sqrt{x^2+7}} =$$

$\lim_{x\to\infty} \frac{-7}{x+\sqrt{x^2+7}}$ e quando $x \to \infty$, $\lim_{x\to\infty} x - \sqrt{x^2+7} = \lim_{x\to\infty} \frac{-7}{x+\sqrt{x^2+7}}$ tende a zero

16) (Circumvent this indeterminate form by using the conjugate of the expression

Calcule $\lim_{x\to\infty} \frac{\sin(x)}{x}$

SOLUTION : First note that

$$-1 \le \sin x \le +1$$

because of the well-known properties of the sine function. Since we are computing the limit as *x* goes to infinity, it is reasonable to assume that $x > 0$. Thus,

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

Since

$$\lim_{x\to\infty} \frac{-1}{x} = 0 = \lim_{x\to\infty} \frac{1}{x},$$

it follows from the Squeeze Principle that

$$\lim_{x\to\infty}\frac{\sin x}{x}=0$$

17) Calcule $\lim_{x\to\infty}\frac{2-\cos x}{x+3}$

SOLUTION :

First note that

$$-1\le\cos x\le+1$$

because of the well-known properties of the cosine function. Now multiply by -1, reversing the inequalities and getting

$$+1\ge-\cos x\ge-1$$

or

$$-1\le-\cos x\le+1.$$

Next, add 2 to each component to get

$$1\le 2-\cos x\le 3.$$

Since we are computing the limit as x goes to infinity, it is reasonable to assume that $x+3>0$. Thus,

$$\frac{1}{x+3}\le\frac{2-\cos x}{x+3}\le\frac{3}{x+3}.$$

Since

$$\lim_{x\to\infty}\frac{1}{x+3}=0=\lim_{x\to\infty}\frac{3}{x+3},$$

it follows from the Squeeze Principle that

$$\lim_{x\to\infty}\frac{2-\cos x}{x+3}=0.$$

18) Calcule $\lim_{x\to\infty}\frac{2-\cos x}{3-2x}$

SOLUTION : First note that

$$-1\le\cos(2x)\le+1$$

because of the well-known properties of the cosine function, and therefore

$$0\le\cos^2(2x)\le+1.$$

Since we are computing the limit as x goes to infinity, it is reasonable to assume that $3-2x<0$. Now divide each component by $3-2x$, reversing the inequalities and getting

$$\frac{0}{3-2x}\ge\frac{\cos^2(2x)}{3-2x}\ge\frac{1}{3-2x},$$

or

$$\frac{1}{3-2x}\le\frac{\cos^2(2x)}{3-2x}\le 0.$$

Since

$$\lim_{x\to\infty}\frac{1}{3-2x}=0=\lim_{x\to\infty}0,$$

it follows from the Squeeze Principle that

$$\lim_{x\to\infty}\frac{\cos^2(2x)}{3-2x}=0$$

19) Calcule $\lim_{x\to 0^-}x^3\cos\left(\frac{2}{x}\right)$

SOLUTION : Note that $\lim\limits_{x\to 0^-} x^3 \cos\left(\frac{2}{x}\right)$ DOES NOT EXIST since values of $\cos\left(\frac{x}{2}\right)$ oscillate between -1 and +1 as x approaches 0 from the left. However, this does NOT necessarily mean that $\lim\limits_{x\to 0^-} x^3 \cos\left(\frac{2}{x}\right)$ does not exist ! ? Indeed, $x^3 < 0$ and

$$-1 \le \cos\left(\frac{2}{x}\right) \le +1$$

for $x < 0$. Multiply each component by x^3, reversing the inequalities and getting

$$-x^3 \ge x^3 \cos\left(\frac{2}{x}\right) \ge x^3$$

or

$$x^3 \le x^3 \cos\left(\frac{2}{x}\right) \le -x^3 .$$

Since

$$\lim_{x\to 0^-} x^3 = 0 = \lim_{x\to 0^-} \{-x^3\} ,$$

it follows from the Squeeze Principle that

$$\lim_{x\to 0^-} x^3 \cos\left(\frac{2}{x}\right) = 0 .$$

20) Calcule

$$\lim_{x\to\infty} \frac{x^2(2+\sin^2 x)}{x+100} .$$

SOLUTION : First note that

$$-1 \le \sin x \le +1 ,$$

so that

$$0 \le \sin^2 x \le 1$$

and

$$2 \le 2 + \sin^2 x \le 3 .$$

Since we are computing the limit as x goes to infinity, it is reasonable to assume that $x+100 > 0$. Thus, dividing by $x+100$ and multiplying by x^2, we get

$$\frac{2}{x+100} \le \frac{2+\sin^2 x}{x+100} \le \frac{3}{x+100}$$

and

$$\frac{2x^2}{x+100} \le \frac{x^2(2+\sin^2 x)}{x+100} \le \frac{3x^2}{x+100} .$$

Then

$$\lim_{x\to\infty} \frac{2x^2}{x+100} = \lim_{x\to\infty} \frac{\frac{2x^2}{x}}{\frac{x+100}{x}} = \lim_{x\to\infty} \frac{2x}{1+\frac{100}{x}} \text{ quando x tende a } \infty$$

$$\lim_{x\to\infty} \frac{x^2(2+\sin^2 x)}{x+100} \quad \text{é igual a} \quad \frac{\infty}{1+0} = \infty$$

Similarly,

$$\lim_{x\to\infty} \frac{3x^2}{x+100} = \infty .$$

Thus, it follows from the Squeeze Principle that

$$\lim_{x\to\infty} \frac{x^2(2+\sin^2 x)}{x+100} = \infty \text{(does not exist).}$$

21) Calcule

$$\lim_{x \to -\infty} \frac{5x^2 - \sin(3x)}{x^2 + 10}.$$

SOLUTION : First note that

$$-1 \le \sin(3x) \le +1,$$

so that

$$-1 \le -\sin(3x) \le +1,$$

$$5x^2 - 1 \le 5x^2 - \sin(3x) \le 5x^2 + 1,$$

and

$$\frac{5x^2 - 1}{x^2 + 10} \le \frac{5x^2 - \sin(3x)}{x^2 + 10} \le \frac{5x^2 + 1}{x^2 + 10}.$$

Then

$$\lim_{x \to -\infty} \frac{5x^2 - 1}{x^2 + 10}$$

$$\lim_{x \to -\infty} \frac{5x^2 - 1}{x^2 + 10} \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$\lim_{x \to -\infty} \frac{5 - \frac{1}{x^2}}{1 + \frac{10}{x^2}}$$

$$\frac{5 - 0}{1 + 0}$$

$$= 5 .$$

Similarly,

$$\lim_{x \to \infty} \frac{5x^2 + 1}{x^2 + 10} = 5$$

Thus, it follows from the Squeeze Principle that

$$\lim_{x \to -\infty} \frac{5x^2 - \sin(3x)}{x^2 + 10} = 5 .$$

22) Calcule

$$\lim_{x \to -\infty} \frac{x^2(\sin x + \cos^3 x)}{(x^2 + 1)(x - 3)}$$

SOLUTION : First note that

$$-1 \le \sin x \le +1$$

and

$$-1 \le \cos x \le +1,$$

so that

$$-1 \le \cos^3 x \le +1$$

and

$$-2 \le \sin x + \cos^3 x \le +2.$$

Since we are computing the limit as x goes to negative infinity, it is reasonable to assume that x-3 < 0. Thus, dividing by x-3, we get

$$\frac{-2}{x - 3} \ge \frac{\sin x + \cos^3 x}{x - 3} \ge \frac{2}{x - 3}$$

or

$$\frac{2}{x - 3} \le \frac{\sin x + \cos^3 x}{x - 3} \le \frac{-2}{x - 3}.$$

Now divide by $x^2 + 1$ and multiply by x^2 , getting

$$\frac{2x^2}{(x^2+1)(x-3)} \le \frac{x^2(\sin x + \cos^3 x)}{(x^2+1)(x-3)} \le \frac{-2x^2}{(x^2+1)(x-3)}.$$

Then

$$\lim_{x \to -\infty} \frac{2x^2}{(x^2+1)(x-3)}$$

$$\lim_{x \to -\infty} \frac{2x^2}{x^3-3x^2+x-3}$$

$$\lim_{x \to -\infty} \frac{\frac{2}{x}}{1-\frac{3}{x}+\frac{1}{x^2}-\frac{3}{x^3}} = \frac{0}{1-0+0-0} = 0$$

Similarly,

$$\lim_{x \to -\infty} \frac{-2x^2}{(x^2+1)(x-3)} = 0$$

It follows from the Squeeze Principle that

$$\lim_{x \to -\infty} \frac{x^2(\sin x + \cos^3 x)}{(x^2+1)(x-3)} = 0$$

LIMITES – CONSTINUIDADE

1) Determine se a seguinte função é contínua em x=1 .

$$f(x) = \begin{cases} 3x-5, & \text{if } x \neq 1 \\ 2\ \ , & \text{if } x = 1 \end{cases}$$

SOLUTION :: Function f is defined at x = 1 since

i.) $f(1) = 2$.

The limit

$$\lim_{x \to 1} f(x) = \lim_{x \to 1}(3x-5)$$

$= 3(1) - 5$

$= -2$,

i.e.,

ii.) $\lim_{x \to 1} f(x) = -2$.

But

iii.) $\lim_{x \to 1} f(x) \neq f(1)$,

so condition iii.) is not satisfied and function f is NOT continuous at $x = 1$.

2) Determine se a seguinte função é contínua em $x = -2$.

$$f(x) = \begin{cases} x^2+2x, & \text{if } x \le -2 \\ x^3-6x, & \text{if } x > -2 \end{cases}$$

SOLUTION : Function f is defined at x=-2 since

$f(-2) = (-2)^2 + 2(-2) = 4-4 = 0$.

The left-hand limit

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-}(x^2+2x)$$

$= (-2)^2 + 2(-2)$

$= 4 - 4$

$= 0$.

The right-hand limit

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+}(x^3-6x)$$

$= (-2)^3 - 6(-2)$

$= -8 + 12$

$= 4$.

Since the left- and right-hand limits are not equal, ,

ii.) $\lim_{x\to -2} f(x)$ does not exist,

and condition ii.) is not satisfied. Thus, function *f* is NOT continuous at $x = -2$.

3) Determine se a seguinte função é contínua em $x = 0$.

$$f(x) = \begin{cases} \dfrac{x-6}{x-3}, & \text{if } x < 0 \\ 2, & \text{if } x = 0 \\ \sqrt{4+x^2}, & \text{if } x > 0 \end{cases}$$

SOLUTION : Function *f* is defined at $x = 0$ since

i.) $f(0) = 2$.

The left-hand limit

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x-6}{x-3}$$

$$= \frac{-6}{-3}$$

$$= 2 .$$

The right-hand limit

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{4+x^2}$$

$$= \sqrt{4+(0)^2}$$

$$= \sqrt{4}$$

$$= 2 .$$

Thus, $\lim_{x\to 0} f(x)$ exists with

ii.) $\lim_{x\to 0} f(x) = 2$.

Since

iii.) $\lim_{x\to 0} f(x) = 2 = f(0)$,

all three conditions are satisfied, and *f* is continuous at $x=0$.

4) Determine se a função $h(x) = \dfrac{x^2+1}{x^3+1}$ é contínua at $x = -1$.

SOLUTION : Function *h* is not defined at $x = -1$ since it leads to division by zero. Thus, h (-1) does not exist, condition i.) is violated, and function *h* is NOT continuous at $x = -1$.

5) Check the following function for continuity at $x = 3$ and $x = -3$.

$$f(x) = \begin{cases} \dfrac{x^3-27}{x^2-9}, & \text{if } x \neq 3 \\ \dfrac{9}{2}, & \text{if } x = 3 \end{cases}$$

SOLUTION : First, check for continuity at $x=3$. Function *f* is defined at $x=3$ since

$$f(3) = \frac{9}{2} .$$

The limit

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^3-27}{x^2-9} = "\frac{0}{0}"$$

(Circumvent this indeterminate form by factoring the numerator and the denominator.)

$$= \lim_{x\to 3} \frac{x^3-3^3}{x^2-3^2}$$

(Recall that $A^2 - B^2 = (A - B)(A + B)$ and $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$.)

$$= \lim_{x\to 3} \frac{(x-3)(x^2+3x+9)}{(x-3)(x+3)}$$

(Divide out a factor of (x - 3) .)

$$= \lim_{x \to 3} \frac{x^2 + 3x + 9}{x + 3}$$

$$\frac{(3)^2 + 3(3) + 9}{(3) + 3}$$

$$= \frac{9}{2}$$
,

i.e.,

ii.) $\lim_{x \to 3} f(x) = \frac{9}{2}$.

Since,

iii.) $\lim_{x \to 3} f(x) = \frac{9}{2} = f(3)$,

all three conditions are satisfied, and *f* is continuous at x=3 . Now, check for continuity at x = -3 . Function *f* is not defined at x = -3 because of division by zero. Thus,

i.) $f(-3)$

does not exist, condition i.) is violated, and *f* is NOT continuous at x = -3 .

6) Para que valores de x a função é $f(x) = \frac{x^2 + 3x + 5}{x^2 + 3x - 4}$ contínua ?

SOLUTION 6 : Functions $y = x^2 + 3x + 5$ and $y = x^2 + 3x - 4$ are continuous for all values of x since both are polynomials. Thus, the quotient of these two functions, $f(x) = \frac{x^2 + 3x + 5}{x^2 + 3x - 4}$, is continuous for all values of x where the denominator, $y = x^2 + 3x - 4 = (x - 1)(x + 4)$, does NOT equal zero. Since $(x - 1)(x + 4) = 0$ for $x = 1$ and $x = -4$, function *f* is continuous for all values of x EXCEPT $x = 1$ and $x = -4$.

7) Para que valores de x a função é $g(x) = \left(\sin(x^{20} + 5)\right)^{1/3}$ contínua ?

SOLUTION: First describe function g using functional composition. Let $f(x) = x^{1/3}$, h(x) = sin(x), and $k(x) = x^{20} + 5$. Function k is continuous for all values of x since it is a polynomial, and functions *f* and h are well-known to be continuous for all values of x . Thus, the functional compositions

$$h(k(x)) = \sin(k(x)) = \sin(x^{20} + 5)$$

and

$$f(h(k(x))) = (h(k(x)))^{1/3} = (\sin(x^{20} + 5))^{1/3}$$

are continuous for all values of x . Since

$$g(x) = (\sin(x^{20} + 5))^{1/3} = f(h(k(x)))$$
,

function g is continuous for all values of x .

8) Para que valores de x a função é $f(x) = \sqrt{x^2 - 2x}$ contínua ?

SOLUTION : First describe function *f* using functional composition. Let $g(x) = x^2 - 2x$ and $h(x) = \sqrt{x}$. Function g is continuous for all values of x since it is a polynomial, and function h is well-known to be continuous for $x \geq 0$. Since $g(x) = x^2 - 2x = x(x-2)$, it follows easily that $g(x) \leq 0$ for $x \leq 0$ and $x \geq 2$. Thus, the functional composition

$$h(g(x)) = \sqrt{g(x)} = \sqrt{x^2 - 2x}$$

is continuous for $x \leq 0$ and $x \geq 2$ and. Since

$$f(x) = \sqrt{x^2 - 2x} = h(g(x))$$
,

function *f* is continuous for $x \leq 0$ and $x \geq 2$ and.

9) Para que valores de x a função é $f(x) = \ln\left(\frac{x - 1}{x + 2}\right)$ contínua ?

SOLUTION : First describe function *f* using functional composition. Let

$g(x) = \dfrac{x-1}{x+2}$ and h(x) = ln(x). Since *g* is the quotient of polynomials $y = x - 1$ and $y = x + 2$, function *g* is continuous for all values of *x* EXCEPT where $x+2 = 0$, i.e., EXCEPT for $x = -2$. Function *h* is well-known to be continuous for $x > 0$. Since $g(x) = \dfrac{x-1}{x+2}$, it follows easily that $g(x) > 0$ for $x < -2$ and $x > 1$. Thus, the functional composition

$$h(g(x)) = \ln(g(x)) = \ln\left(\frac{x-1}{x+2}\right)$$

is continuous for $x < -2$ and $x > 1$. Since

$$f(x) = \ln\left(\frac{x-1}{x+2}\right) = h(g(x)),$$

function *f* is continuous for $x < -2$ and $x > 1$.

10) Para que valores de *x* a função é $f(x) = \dfrac{e^{\sin x}}{4 - \sqrt{x^2 - 9}}$ contínua ?

SOLUTION 10 : First describe function *f* using functional composition. Let $g(x) = \sin(x)$ and $h(x) = e^x$, both of which are well-known to be continuous for all values of *x*. Thus, the numerator $y = e^{\sin(x)} = h(g(x))$ is continuous (the functional

composition of continuous functions) for all values of *x*. Now consider the denominator $y = 4 - \sqrt{x^2 - 9}$. Let $g(x) = 4$, $h(x) = x^2 - 9$, and $k(x) = \sqrt{x}$. Functions *g* and *h* Are continuous for all values of *x* since both are polynomials, and it is well-known that function *k* is continuous for $x \geq 0$. Since $h(x) = x^2 - 9 = (x-3)(x+3) = 0$ when $x = 3$ or $x = -3$, it follows easily that $h(x) \geq 0$ for $x \geq 3$ and $x \leq -3$ for and, so that $y = 4 - \sqrt{x^2 - 9} = k(h(x))$ is continuous (the functional composition of continuous functions) for $x \geq 3$ and $x \leq -3$ and. Thus, the denominator $y = 4 - \sqrt{x^2 - 9}$ is continuous (the difference of continuous functions) for $x \geq 3$ and $x \leq -3$ and. There is one other important consideration. We must insure that the DENOMINATOR IS NEVER ZERO. If

$$y = 4 - \sqrt{x^2 - 9} = 0$$

then

$$4 = \sqrt{x^2 - 9}.$$

Squaring both sides, we get

$$16 = x^2 - 9$$

so that

$$x^2 = 25$$

when

$$x = 5 \text{ or } x = -5 .$$

Thus, the denominator is zero if $x = 5$ or $x = -5$. Summarizing, the quotient of these continuous functions, $f(x) = \dfrac{e^{\sin x}}{4 - \sqrt{x^2 - 9}}$, is continuous for $x \geq 3$ and $x \leq -3$ and, but NOT for $x = 5$ and $x = -5$.

Para que valores de *x* é a seguinte função contínua ?

$$f(x) = \begin{cases} \dfrac{x-1}{\sqrt{x}-1}, & \text{if } x > 1 \\ 5 - 3x, & \text{if } -2 \leq x \leq 1 \\ \dfrac{6}{x-4}, & \text{if } x < -2 \end{cases}$$

SOLUTION : Consider separately the three component functions which determine *f*.

Function $y = \frac{x - 1}{\sqrt{x} - 1}$ is continuous for $x > 1$ since it is the quotient of continuous functions and the denominator is never zero. Function $y = 5 - 3x$ is continuous for $-2 \le x \le 1$ since it is a polynomial. Function $y = \frac{6}{x - 4}$ is continuous for $x < -2$ since it is the quotient of continuous functions and the denominator is never zero. Now check for continuity of f where the three components are joined together, i.e., check for continuity at $x = 1$ and $x = -2$. For $x = 1$ function f is defined since

i.) $f(1) = 5 - 3(1) = 2$.

The right-hand limit

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x - 1}{\sqrt{x} - 1} = \frac{0}{0}$$

(Circumvent this indeterminate form one of two ways. Either factor the numerator as the difference of squares, or multiply by the conjugate of the denominator over itself.)

$$= \lim_{x \to 1^+} \frac{(\sqrt{x})^2 - (1)^2}{\sqrt{x} - 1}$$

$$= \lim_{x \to 1^+} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{\sqrt{x} - 1}$$

$$= \lim_{x \to 1^+} (\sqrt{x} + 1)$$

$$= (\sqrt{1} + 1)$$

$$= 2 .$$

The left-hand limit

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (5 - 3x)$$

$$= 5 - 3(1)$$

$$= 2 .$$

Thus,

ii.) $\lim_{x \to 1} f(x) = 2$.

Since

iii.) $\lim_{x \to 1} f(x) = 2 = f(1)$,

all three conditions are satisfied, and function f is continuous at $x = 1$. Now check for continuity at $x = -2$. Function f is defined at $x = -2$ since

i.) $f(-2) = 5 - 3(-2) = 11$.

The right-hand limit

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (5 - 3x)$$

$$= 5 - 3(-2)$$

$$= 11 .$$

The left-hand limit

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \frac{6}{x - 4} =$$

$$= \frac{6}{(-2) - 4}$$

$$= -1 .$$

Since the left- and right-hand limits are different,

ii.) $\lim_{x \to -2^-} f(x)$ does NOT exist,

condition ii.) is violated, and function f is NOT continuous at $x=-2$. Summarizing, function f is continuous for all values of x EXCEPT $x = -2$.

12. Determine todos os valores da constante A para que a seguinte função seja contínua para todos os valores de x .

$$f(x) = \begin{cases} A^2x - A , & \text{if } x \ge 3 \\ 4 , & \text{if } x < 3 \end{cases}$$

SOLUTION : First, consider separately the two components which determine function f .

Function $y = A^2 x - A$ is continuous for $x \geq 3$ for any value of A since it is a polynomial. Function $y = 4$ is continuous for $x < 3$ since it is a polynomial. Now determine A so that function f is continuous at x=3 . Function f must be defined at $x = 3$, so

i.) $f(3)= A^2 (3) - A = 3 A^2 - A$.

The right-hand limit

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (A^2 x - A)$$
$$= A^2 (3) - A$$
$$= 3 A^2 - A .$$

The left-hand limit

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 4 = 4 .$$

For the limit to exist, the right- and left-hand limits must exist and be equal. Thus,

ii.) $\lim_{x \to 3} f(x) = 3A^2 - A = 4$,

so that

$$= 3 A^2 - A - 4 = 0 .$$

Factoring, we get

$$(3A - 4)(A + 1) = 0$$

for

$$A = 4/3 \text{ or } A = -1 .$$

For either choice of A ,

iii.) $\lim_{x \to 3} f(x) = 4 = f(3)$,

all three conditions are satisfied, and f is continuous at $x = 3$. Therefore, function f is continuous for all values of x if $A = 4/3$ or $A = -1$.

13. Determine todos os va;ores das constantes A e B para que a função seja contínua para todos os valores de x .

$$f(x) = \begin{cases} Ax - B , & \text{if } x \leq -1 \\ 2x^2 + 3Ax + B , & \text{if } -1 < x \leq 1 \\ 4 , & \text{if } x > 1 \end{cases}$$

SOLUTION : First, consider separately the three components which determine function f . Function $y = Ax - B$ is continuous for $x \leq -1$ for any values of A and B since it is a polynomial. Function $y = 2x^2 + 3Ax + B$ is continuous for $-1 \leq x \leq 1$ for any values of A and B since it is a polynomial. Function $y = 4$ is continuous for $x > 1$ since it is a polynomial. Now determine A and B so that function f is continuous at x=-1 and x=1 . First, consider continuity at $x = -1$. Function f must be defined at $x = -1$, so

i.) $f(-1)= A(-1) - B = - A - B$.

The left-hand limit

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (Ax - B) =$$
$$= A (-1) - B$$
$$= - A - B .$$

The right-hand limit

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (2x^2 + 3Ax + B) =$$
$$= 2(-1)^2 + 3A(-1) + B$$
$$= 2 - 3A + B .$$

For the limit to exist, the right- and left-hand limits must exist and be equal. Thus,

ii.) $\lim_{x \to -1} f(x) = - A - B = 2 - 3A + B$,

so that

$$2A - 2B = 2 ,$$

or

(Equation 1)

$$A - B = 1 .$$

Now consider continuity at x=1 . Function f must be defined at x=1 , so

i.) $f(1)= 2(1)^2 + 3A(1) + B = 2 + 3A + B$.

The left-hand limit

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (2x^2 + 3Ax + B) =$$

$$= 2(1)^2 + 3A(1) + B$$

$$= 2 + 3A + B .$$

The right-hand limit

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} 4 =$$

$$= 4 .$$

For the limit to exist, the right- and left-hand limits must exist and be equal. Thus,

$$\text{ii.) } \lim_{x \to 1} f(x) = 2 + 3A + B = 4 ,$$

or

(Equation 2)

$$3A + B = 2 .$$

Now solve Equations 1 and 2 simultaneously. Thus,

$$A - B = 1 \text{ and } 3A + B = 2$$

are equivalent to

$$A = B + 1 \text{ and } 3A + B = 2 .$$

Use the first equation to substitute into the second, getting

$$3 (B + 1) + B = 2 ,$$

$$3 B + 3 + B = 2 ,$$

and

$$4 B = -1 .$$

Thus,

$$B = \frac{-1}{4}$$

and

$$A = B + 1 = \frac{-1}{4} + 1 = \frac{3}{4} .$$

For this choice of A and B it can easily be shown that

$$\text{iii.) } \lim_{x \to 1} f(x) = 4 = f(1)$$

and

$$\text{iii.) } \lim_{x \to -1} f(x) = \frac{-1}{2} = f(-1) ,$$

so that all three conditions are satisfied at both x=1 and x=-1 , and function f is continuous at both x=1 and x=-1 . Therefore, function f is continuous for all values of x if

$A = 3/4$ and $B = -1/4$ and.