Chapter 6 Solutions

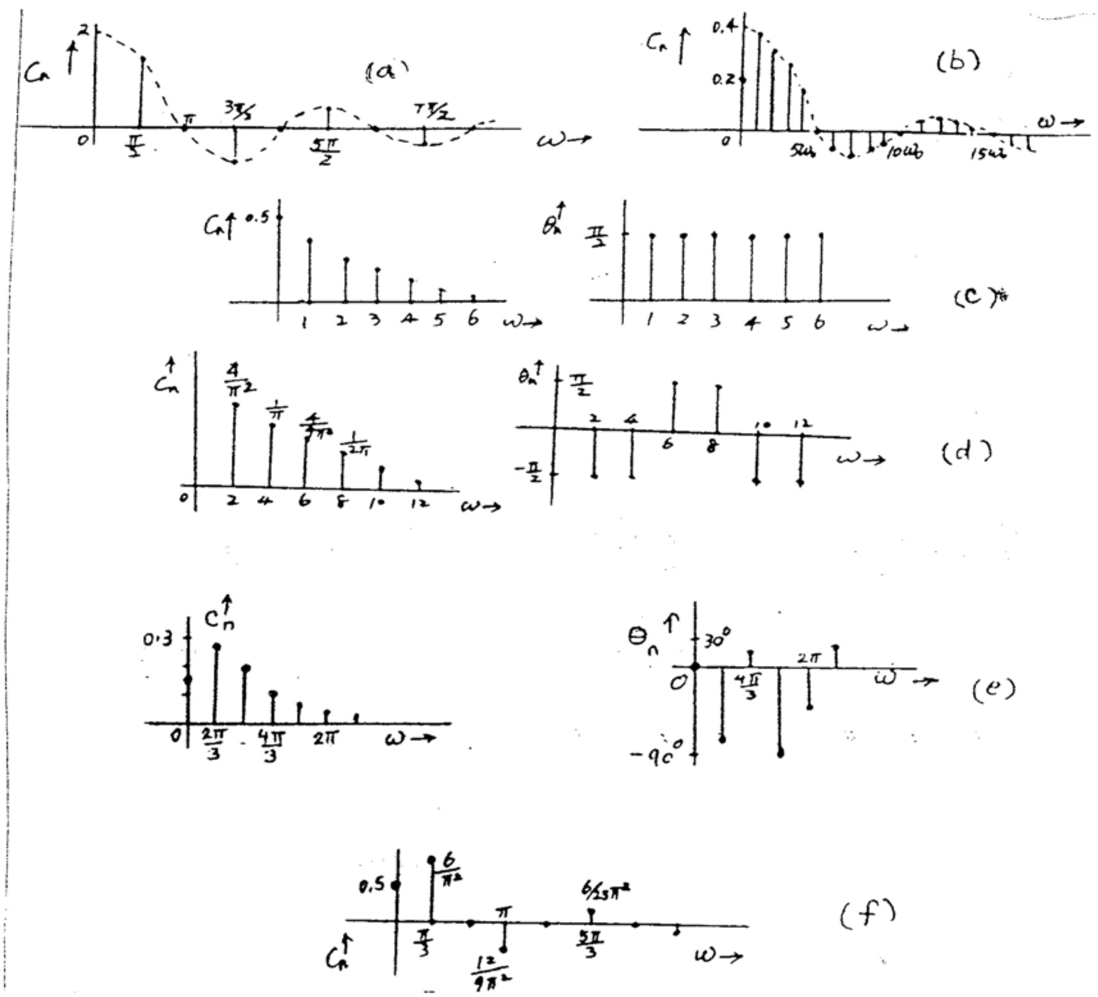

Figure S6.1-1

6.1-1. (a) $T_0 = 4$, $\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$. Because of even symmetry, all sine terms are zero.

$$
\begin{aligned}
x(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}t\right) \\
a_0 &= 0 \text{ (by inspection)} \\
a_n &= \frac{4}{4}\left[\int_0^1 \cos\left(\frac{n\pi}{2}t\right)\,dt - \int_1^2 \cos\left(\frac{n\pi}{2}t\right)\,dt\right] = \frac{4}{n\pi}\sin\frac{n\pi}{2}
\end{aligned}
$$

Therefore, the Fourier series for $x(t)$ is

$$
x(t) = \frac{4}{\pi}\left(\cos\frac{\pi t}{2} - \frac{1}{3}\cos\frac{3\pi t}{2} + \frac{1}{5}\cos\frac{5\pi t}{2} - \frac{1}{7}\cos\frac{7\pi t}{2} + \cdots\right)
$$

Here $b_n = 0$, and we allow C_n to take negative values. Figure S6.1-1a shows the plot of C_n.

(b) $T_0 = 10\pi$, $\omega_0 = \frac{2\pi}{T_0} = \frac{1}{5}$. Because of even symmetry, all the sine terms are zero.

$$
\begin{aligned}
x(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n}{5}t\right) + b_n \sin\left(\frac{n}{5}t\right) \\
a_0 &= \frac{1}{5} \quad \text{(by inspection)} \\
a_n &= \frac{2}{10\pi}\int_{-\pi}^{\pi} \cos\left(\frac{n}{5}t\right)\,dt = \frac{1}{5\pi}\left(\frac{5}{n}\right)\sin\left(\frac{n}{5}t\right)\Big|_{-\pi}^{\pi} = \frac{2}{\pi n}\sin\left(\frac{n\pi}{5}\right) \\
b_n &= \frac{2}{10\pi}\int_{-\pi}^{\pi} \sin\left(\frac{n}{5}t\right)\,dt = 0 \qquad \text{(integrand is an odd function of } t\text{)}
\end{aligned}
$$

Here $b_n = 0$, and we allow C_n to take negative values. Note that $C_n = a_n$ for $n = 0, 1, 2, 3, \cdots$. Figure S6.1-1b shows the plot of C_n.

(c) $T_0 = 2\pi$, $\omega_0 = 1$.

$$
x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \quad \text{with} \quad a_0 = 0.5 \quad \text{(by inspection)}
$$

$$
a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{t}{2\pi}\cos nt\,dt = 0, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{t}{2\pi}\sin nt\,dt = -\frac{1}{\pi n}
$$

and

$$
\begin{aligned}
x(t) &= 0.5 - \frac{1}{\pi}\left(\sin t + \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t + \frac{1}{4}\sin 4t + \cdots\right) \\
&= 0.5 + \frac{1}{\pi}\left[\cos\left(t + \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(2t + \frac{\pi}{2}\right) + \frac{1}{3}\cos\left(3t + \frac{\pi}{2}\right) + \cdots\right]
\end{aligned}
$$

The reason for vanishing of the cosines terms is that when 0.5 (the dc component) is subtracted from $x(t)$, the remaining function has odd symmetry. Hence, the Fourier series would contain dc and sine terms only. Figure S6.1-1c shows the plot of C_n and θ_n.

(d) $T_0 = \pi$, $\omega_0 = 2$ and $x(t) = \frac{4}{\pi}t$. $a_0 = 0$ (by inspection). $a_n = 0$ ($n >$

0) because of odd symmetry.

$$b_n = \frac{4}{\pi}\int_0^{\pi/4} \frac{4}{\pi} t \sin 2nt\, dt = \frac{2}{\pi n}\left(\frac{2}{\pi n}\sin\frac{\pi n}{2} - \cos\frac{\pi n}{2}\right)$$

$$\begin{aligned} x(t) &= \frac{4}{\pi^2}\sin 2t + \frac{1}{\pi}\sin 4t - \frac{4}{9\pi^2}\sin 6t - \frac{1}{2\pi}\sin 8t + \cdots \\ &= \frac{4}{\pi^2}\cos\left(2t - \frac{\pi}{2}\right) + \frac{1}{\pi}\cos\left(4t - \frac{\pi}{2}\right) + \frac{4}{9\pi^2}\cos\left(6t + \frac{\pi}{2}\right) + \frac{1}{\pi}\cos\left(8t + \frac{\pi}{2}\right) + \cdots \end{aligned}$$

Figure S6.1-1d shows the plot of C_n and θ_n.

(e) $T_0 = 3$, $\omega_0 = 2\pi/3$.

$$a_0 = \frac{1}{3}\int_0^1 t\, dt = \frac{1}{6}$$

$$a_n = \frac{2}{3}\int_0^1 t\cos\frac{2n\pi}{3} t dt = \frac{3}{2\pi^2 n^2}[\cos\frac{2\pi n}{3} + \frac{2\pi n}{3}\sin\frac{2\pi n}{3} - 1]$$

$$b_n = \frac{2}{3}\int_0^1 t\sin\frac{2n\pi}{3} t dt = \frac{3}{2\pi^2 n^2}[\sin\frac{2\pi n}{3} - \frac{2\pi n}{3}\cos\frac{2\pi n}{3}]$$

Therefore $C_0 = \frac{1}{6}$ and

$$C_n = \frac{3}{2\pi^2 n^2}\left[\sqrt{2 + \frac{4\pi^2 n^2}{9} - 2\cos\frac{2\pi n}{3} - \frac{4\pi n}{3}\sin\frac{2\pi n}{3}}\right]$$

and

$$\theta_n = \tan^{-1}\left(\frac{\frac{2\pi n}{3}\cos\frac{2\pi n}{3} - \sin\frac{2\pi n}{3}}{\cos\frac{2\pi n}{3} + \frac{2\pi n}{3}\sin\frac{2\pi n}{3} - 1}\right)$$

(f) $T_0 = 6$, $\omega_0 = \pi/3$, $a_0 = 5$ (by inspection). Even symmetry; $b_n = 0$.

$$\begin{aligned} a_n &= \frac{4}{6}\int_0^3 x(t)\cos\frac{n\pi}{3}\, dt \\ &= \frac{2}{3}\left[\int_0^1 \cos\frac{n\pi}{3}\, dt + \int_1^2 (2-t)\cos\frac{n\pi}{3} t\, dt\right] \\ &= \frac{6}{\pi^2 n^2}\left[\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right] \end{aligned}$$

$$x(t) = 0.5 + \frac{6}{\pi^2}\left(\cos\frac{\pi}{3}t - \frac{2}{9}\cos\pi t + \frac{1}{25}\cos\frac{5\pi}{3}t + \frac{1}{49}\cos\frac{7\pi}{3}t + \cdots\right)$$

Observe that even harmonics vanish. The reason is that if the dc (0.5) is subtracted from $x(t)$, the resulting function has half-wave symmetry. (See Prob. 6.1-5). Figure S6.1-1f shows the plot of C_n.

6.1-2. (a) Here $T_0 = \pi$, and $\omega_0 = \frac{2\pi}{T_0} = 2$. Therefore

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2nt + b_n \sin 2nt$$

To compute the coefficients, we shall use the interval π to 0 for integration. Thus

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{0} e^{t/2}\,dt = 0.504 \\
a_n &= \frac{2}{\pi}\int_{-\pi}^{0} e^{t/2}\cos 2nt\,dt = 0.504\left(\frac{2}{1+16n^2}\right) \\
b_n &= \frac{2}{\pi}\int_{-\pi}^{0} e^{t/2}\sin 2nt\,dt = -0.504\left(\frac{8n}{1+16n^2}\right)
\end{aligned}$$

Therefore

$$\begin{aligned}
C_0 &= a_0 = 0.504 \\
C_n &= \sqrt{a_n^2+b_n^2} = 0.504\left(\frac{2}{\sqrt{1+16n^2}}\right) \\
\theta_n &= \tan^{-1}\left(\frac{-b_n}{a_n}\right) = \tan^{-1} 4n \\
x(t) &= 0.504 + 0.504\sum_{n=1}^{\infty}\frac{2}{\sqrt{1+16n^2}}\cos\left(2nt + tan^{-1}4n\right)
\end{aligned}$$

(b) This Fourier series is identical to that in Eq. (6.15a) with t replaced by $-t$.

(c) If $x(t) = C_0 + \sum C_n\cos(n\omega_0 t + \theta_n)$, then

$$x(-t) = C_0 + \sum C_n\cos(-n\omega_0 t + \theta_n) = C_0 + \sum C_n\cos(n\omega_0 t - \theta_n)$$

Thus, time inversion of a signal merely changes the sign of the phase θ_n. Everything else remains unchanged. Comparison of the above results in part (a) with those in Example 6.1 confirms this conclusion.

6.1-3. (a) Here $T_0 = \pi/2$, and $\omega_0 = \frac{2\pi}{T_0} = 4$. Therefore

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos 4nt + b_n\sin 4nt$$

where

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_{0}^{\pi/2} e^{-t}\,dt = 0.504 \\
a_n &= \frac{4}{\pi}\int_{0}^{\pi/2} e^{-t}\cos 4nt\,dt = 0.504\left(\frac{2}{1+16n^2}\right)
\end{aligned}$$

and

$$b_n = \frac{4}{\pi}\int_{0}^{\pi/2} e^{-t}\sin 4nt\,dt = 0.504\left(\frac{8n}{1+16n^2}\right)$$

Therefore $C_0 = a_0 = 0.504$, $C_n = \sqrt{a_n^2+b_n^2} = 0.504\left(\frac{2}{\sqrt{1+16n^2}}\right)$, $\theta_n = -\tan^{-1} 4n$

(b) This Fourier series is identical to that in Eq. (6.15a) with t replaced by $2t$.

(c) If $x(t) = C_0 + \sum C_n \cos(n\omega_0 t + \theta_n)$, then

$$x(at) = C_0 + \sum C_n \cos(n(a\omega_0)t + \theta_n)$$

Thus, time scaling by a factor a merely scales the fundamental frequency by the same factor a. Everything else remains unchanged. If we time compress (or time expand) a periodic signal by a factor a, its fundamental frequency increases by the same factor a (or decreases by the same factor a). Comparison of the results in part **(a)** with those in Example 6.1 confirms this conclusion. This result applies equally well

6.1-4. (a) Here $T_0 = 2$, and $\omega_0 = \frac{2\pi}{T_0} = \pi$. Also $x(t)$ is an even function of t. Therefore

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi t$$

where, by inspection $a_0 = 0$ and from Eq. (6.18b)

$$a_n = \frac{4}{2}\int_0^1 A(-2t+1)\cos n\pi t\,dt = -\frac{4}{\pi^2 n^2}\,(\cos n\pi t - 1)|_0^1 = \begin{cases} 0 & n \text{ even} \\ \frac{8A}{n^2\pi^2} & n \text{ odd} \end{cases}$$

Therefore

$$x(t) = \frac{8A}{\pi^2}\left[\cos \pi t + \frac{1}{9}\cos 3\pi t + \frac{1}{25}\cos 5\pi t + \frac{1}{49}\cos 7\pi t + \cdots\right]$$

(b) This Fourier series is identical to that in Eq. (6.16) with t replaced by $t + 0.5$.

(c) If $x(t) = C_0 + \sum C_n \cos(n\omega_0 t + \theta_n)$, then

$$x(t+T) = C_0 + \sum C_n \cos[n\omega_0(t+T) + \theta_n] = C_0 + \sum C_n \cos[n\omega_0 t + (\theta_n + n\omega_0 T)]$$

Thus, time shifting by T merely increases the phase of the nth harmonic by $n\omega_0 T$.

6.1-5. (a) For half wave symmetry

$$x(t) = -x\left(t \pm \frac{T_0}{2}\right)$$

and

$$a_n = \frac{2}{T_0}\int_0^{T_0} x(t)\cos n\omega_0 t\,dt = \frac{2}{T_0}\int_0^{T_0/2} x(t)\cos n\omega_0 t\,dt + \int_{T_0/2}^{T_0} x(t)\cos n\omega_0 t\,dt$$

Let $\tau = t - T_0/2$ in the second integral. This gives

$$\begin{aligned} a_n &= \frac{2}{T_0}\left[\int_0^{T_0/2} x(t)\cos n\omega_0 t\,dt + \int_0^{T_0/2} x\left(\tau + \frac{T_0}{2}\right)\cos n\omega_0\left(\tau + \frac{T_0}{2}\right)d\tau\right] \\ &= \frac{2}{T_0}\left[\int_0^{T_0/2} x(t)\cos n\omega_0 t\,dt + \int_0^{T_0/2} -x(\tau)[-\cos n\omega_0 \tau]\,d\tau\right] \\ &= \frac{4}{T_0}\left[\int_0^{T_0/2} x(t)\cos n\omega_0 t\,dt\right] \end{aligned}$$

In a similar way we can show that

$$b_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\sin n\omega_0 t\, dt$$

(b) (i) $T_0 = 8$, $\omega_0 = \frac{\pi}{4}$, $a_0 = 0$ (by inspection). Half wave symmetry. Hence

$$\begin{aligned}
a_n &= \frac{4}{8}\left[\int_0^4 x(t)\cos\frac{n\pi}{4}t\, dt\right] = \frac{1}{2}\left[\int_0^2 \frac{t}{2}\cos\frac{n\pi}{4}t\, dt\right] \\
&= \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} + \frac{n\pi}{2}\sin\frac{n\pi}{2} - 1\right) \quad (n \text{ odd}) \\
&= \frac{4}{n^2\pi^2}\left(\frac{n\pi}{2}\sin\frac{n\pi}{2} - 1\right) \quad (n \text{ odd})
\end{aligned}$$

Therefore

$$a_n = \begin{cases} \frac{4}{n^2\pi^2}\left(\frac{n\pi}{2} - 1\right) & n = 1, 5, 9, 13, \cdots \\ -\frac{4}{n^2\pi^2}\left(\frac{n\pi}{2} + 1\right) & n = 3, 7, 11, 15, \cdots \end{cases}$$

Similarly

$$b_n = \frac{1}{2}\int_0^2 \frac{t}{2}\sin\frac{n\pi}{4}t\, dt = \frac{4}{n^2\pi^2}\left(\sin\frac{n\pi}{2} - \frac{n\pi}{2}\cos\frac{n\pi}{2}\right) = \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \quad (n \text{ odd})$$

and

$$x(t) = \sum_{n=1,3,5,\cdots}^{\infty} a_n\cos\frac{n\pi}{4}t + b_n\sin\frac{n\pi}{4}t$$

(ii) $T_0 = 2\pi$, $\omega_0 = 1$, $a_0 = 0$ (by inspection). Half wave symmetry. Hence

$$x(t) = \sum_{n=1,3,5,\cdots}^{\infty} a_n\cos nt + b_n\sin nt$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} e^{-t/10}\cos nt\, dt \\
&= \frac{2}{\pi}\left[\frac{e^{-t/10}}{n^2 + 0.01}(-0.1\cos nt + n\sin nt)\right]_0^{\pi} \quad (n \text{ odd}) \\
&= \frac{2}{\pi}\left[\frac{e^{-\pi/10}}{n^2 + 0.01}(0.1) - \frac{1}{n^2 + 0.01}(-0.1)\right] \\
&= \frac{2}{10\pi(n^2 + 0.01)}(e^{-\pi/10} - 1) = \frac{0.0465}{n^2 + 0.01}
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} e^{-t/10}\sin nt\, dt \\
&= \frac{2}{\pi}\left[\frac{e^{-t/10}}{n^2 + 0.01}(-0.1\sin nt - n\cos nt)\right]_0^{\pi} \quad (n \text{ odd}) \\
&= \frac{2n}{(n^2 + 0.01)}(e^{-\pi/10} - 1) = \frac{1.461n}{n^2 + 0.01}
\end{aligned}$$

6.1-6. (a) Here, we need only cosine terms and $\omega_0 = \frac{\pi}{2}$. Hence, we must construct a pulse such that it is an even function of t, has a value t over the interval $0 \le t \le 1$, and repeats every 4 seconds as shown in Fig. S6.1-6a. We selected the pulse width $W = 2$ seconds. But it can be anywhere from 2 to 4, and still satisfy these conditions. Each value of W results in different series. Yet all of them converge to t over 0 to 1, and satisfy the other requirements. Clearly, there are infinite number of Fourier series that will satisfy the given requirements. The present choice yields

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}\right)t$$

By inspection, we find $a_0 = 1/4$. Because of symmetry $b_n = 0$ and

$$a_n = \frac{4}{4}\int_0^1 t\cos\frac{n\pi}{2}t\,dt = \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right) - 1\right]$$

(b) Here, we need only sine terms and $\omega_0 = 2$. Hence, we must construct a pulse with odd symmetry, which has a value t over the interval $0 \le t \le 1$, and repeats every π seconds as shown in Fig. S6.1-6b. As in the case (a), the pulse width can be anywhere from 1 to π. For the present case

$$x(t) = \sum_{n=1}^{\infty} b_n \sin 2nt$$

Because of odd symmetry, $a_n = 0$ and

$$b_n = \frac{4}{\pi}\int_0^1 t\sin 2nt\,dt = \frac{1}{\pi n^2}(\sin 2n - 2n\cos 2n)$$

(c) Here, we need both sine and cosine terms and $\omega_0 = \frac{\pi}{2}$. Hence, we must construct a pulse such that it has no symmetry of any kind, has a value t over the interval $0 \le t \le 1$, and repeats every 4 seconds as shown in Fig. S6.1-6c. As usual, the pulse width can be have any value in the range 1 to 4.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{2}\right)t + b_n \sin\left(\frac{n\pi}{2}\right)t$$

By inspection, $a_0 = 1/8$ and

$$\begin{aligned} a_n &= \frac{2}{4}\int_0^1 t\cos\frac{n\pi}{2}t\,dt = \frac{2}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right) - 1\right] \\ b_n &= \frac{2}{4}\int_0^1 t\sin\frac{n\pi}{2}t\,dt = \frac{2}{n^2\pi^2}\left[\sin\left(\frac{n\pi}{2}\right) - \frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right)\right] \end{aligned}$$

(d) Here, we need only cosine terms with $\omega_0 = 1$ and odd harmonics only. Hence, we must construct a pulse such that it is an even function of t, has a value t over the interval $0 \le t \le 1$, repeats every 2π seconds and has half-wave symmetry as shown in Fig. S6.1-6d. Observe that the first half cycle (from 0 to π) and the second half cycle (from π to 2π) are negatives of each other as required in half-wave symmetry. This will cause even harmonics to vanish. The pulse has an even and half-wave symmetry. This yields

$$x(t) = a_0 + \sum_{\substack{n=1 \\ n\ odd}}^{\infty} a_n \cos nt$$

By inspection, $a_0 = 0$. Because of even symmetry $b_n = 0$. Because of half-wave symmetry (see Prob. 6.1-5),

$$a_n = \frac{4}{2\pi}\left[\int_0^{\pi/2} t \cos nt\, dt - \int_{\pi/2}^{\pi} (t-\pi)\cos nt\, dt\right] = \frac{2}{\pi n^2}(\cos n\pi - 1) + \frac{2}{n}\sin\frac{n\pi}{2} \qquad n \text{ odd}$$

(e) Here, we need only sine terms with $\omega_0 = \pi$ and odd harmonics only. Hence, we must construct a pulse such that it is an odd function of t, has a value t over the interval $0 \leq t \leq 1$, repeats every 4 seconds and has half-wave symmetry as shown in Fig. S6.1-6e. Observe that the first half cycle (from 0 to 2) and the second half cycle (from 2 to 4) are negatives of each other as required in half-wave symmetry. This will cause even harmonics to vanish. The pulse has an odd and half-wave symmetry. This yields

$$x(t) = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} b_n \sin\frac{n\pi}{2}t$$

By inspection, $a_0 = 0$. Because of odd symmetry $a_n = 0$. Because of half-wave symmetry (see Prob. 6.1-5),

$$b_n = \frac{4}{4}\int_0^1 t \sin\frac{n\pi}{2}t\, dt + \int_1^2 (-t+2)\sin\frac{n\pi}{2}t\, dt = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad n \text{ odd}$$

(f) Here, we need both sine and cosine terms with $\omega_0 = 1$ and odd harmonics only. Hence, we must construct a pulse such that it has half-wave symmetry, but neither odd nor even symmetry, has a value t over the interval $0 \leq t \leq 1$, and repeats every 2π seconds as shown in Fig. S6.1-6f. Observe that the first half cycle from 0 to π) and the second half cycle (from π to 2π) are negatives of each other as required in half-wave symmetry. By inspection, $a_0 = 0$. This yields

$$x(t) = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} a_n \cos nt + b_n \sin nt$$

Because of half-wave symmetry (see Prob. 6.1-5),

$$a_n = \frac{4}{2\pi}\int_0^1 t \cos nt\, dt = \frac{2}{\pi n^2}(\cos n + n \sin n - 1)$$

$$b_n = \frac{4}{2\pi}\int_0^1 t \sin nt\, dt = \frac{2}{\pi n^2}(\sin n - n \cos n) \qquad n \text{ odd}$$

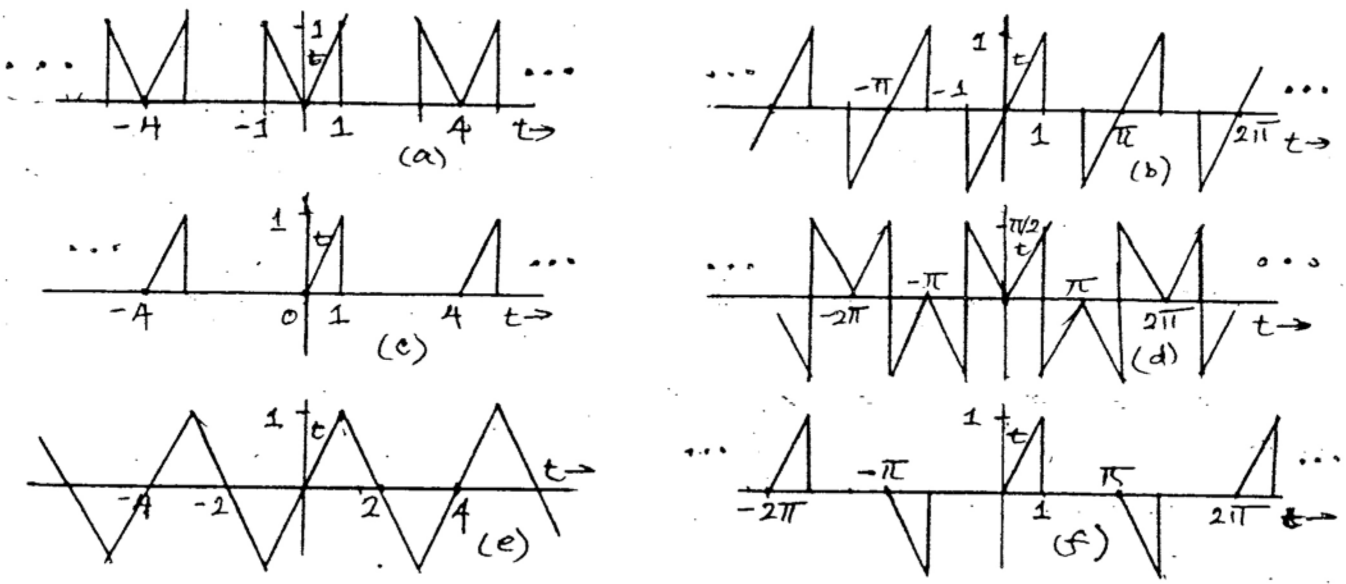

Figure S6.1-6

6.1-7.

	a	b	c	d	e	f	g	h	i
periodic ?	yes	yes	no	yes	no	yes	yes	yes	yes
ω_0	1	1		π		$\frac{1}{70}$	$\frac{3}{4}$	1	2
period	2π	2π		2		140π	$\frac{8\pi}{3}$	2π	π

6.3-1. (a) $T_0 = 4, \omega_0 = \pi/2$. Also $D_0 = 0$ (by inspection).

$$D_n = \frac{1}{2\pi}\int_{-1}^{1} e^{-j(n\pi/2)t}\,dt - \int_{1}^{3} e^{-j(n\pi/2)t}\,dt = \frac{2}{\pi n}\sin\frac{n\pi}{2} \qquad |n| \geq 1$$

(b) $T_0 = 10\pi$, $\omega_0 = 2\pi/10\pi = 1/5$

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\frac{n}{5}t},$$

where $$D_n = \frac{1}{10\pi}\int_{\pi}^{\pi} e^{-j\frac{n}{5}t}\,dt = \frac{j}{2\pi n}\left(-2j\sin\frac{n\pi}{5}\right) = \frac{1}{\pi n}\sin\left(\frac{n\pi}{5}\right)$$

(c)

$$x(t) = D_0 + \sum_{n=-\infty}^{\infty} D_n e^{jnt}, \quad \text{where, by inspection} \qquad D_0 = 0.5$$

$$D_n = \frac{1}{2\pi}\int_0^{2\pi} \frac{t}{2\pi} e^{-jnt}\,dt = \frac{j}{2\pi n},$$

so that $$|D_n| = \frac{1}{2\pi n}, \quad \text{and} \quad \angle D_n = \begin{cases} \frac{\pi}{2} & n > 0 \\ \frac{-\pi}{2} & n < 0 \end{cases}$$

(d) $T_0 = \pi$, $\omega_0 = 2$ and $D_n = 0$

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j2nt},$$

$$\text{where} \qquad D_n = \frac{1}{\pi}\int_{-\pi/4}^{\pi/4} \frac{4t}{\pi} e^{-j2nt}\, dt = \frac{-j}{\pi n}\left(\frac{2}{\pi n}\sin\frac{\pi n}{2} - \cos\frac{\pi n}{2}\right)$$

(e) $T_0 = 3, \omega_0 = \frac{2\pi}{3}$.

$$x(t) = \sum_{n=\infty}^{\infty} D_n\, e^{j\frac{2\pi n}{3}t},$$

$$\text{where} \qquad D_n = \frac{1}{3}\int_0^1 t\, e^{-j\frac{2\pi n}{3}t}\, dt = \frac{3}{4\pi^2 n^2}\left[e^{-j\frac{2\pi n}{3}}\left(\frac{j2\pi n}{3} + 1\right) - 1\right]$$

Therefore

$$|D_n| = \frac{3}{4\pi^2 n^2}\left[\sqrt{2 + \frac{4\pi^2 n^2}{9} - 2\cos\frac{2\pi n}{3} - \frac{4\pi n}{3}\sin\frac{2\pi n}{3}}\right]$$

$$\text{and } \angle D_n = \tan^{-1}\left(\frac{\frac{2\pi n}{3}\cos\frac{2\pi n}{3} - \sin\frac{2\pi n}{3}}{\cos\frac{2\pi n}{3} + \frac{2\pi n}{3}\sin\frac{2\pi n}{3} - 1}\right)$$

(f) $T_0 = 6$, $\omega_0 = \pi/3$ $D_0 = 0.5$

$$x(t) = 0.5 + \sum_{n=-\infty}^{\infty} D_n e^{\frac{j\pi n t}{3}}$$

$$\begin{aligned} D_n &= \frac{1}{6}\left[\int_{-2}^{-1}(t+2)e^{-\frac{j\pi n t}{3}}\, dt + \int_{-1}^{1} e^{-\frac{j\pi n t}{3}}\, dt + \int_{1}^{2}(-t+2)e^{-\frac{j\pi n t}{3}}\, dt\right] \\ &= \frac{3}{\pi^2 n^2}\left(\cos\frac{n\pi}{3} - \cos\frac{2\pi n}{3}\right) \end{aligned}$$

6.3-2. Note that the signal $x(t)$ is defined as

$$x(t) = \begin{cases} \frac{1}{A}t & 0 \le t < A \\ 1 & A \le t < \pi \\ 0 & \pi \le t < 2\pi \\ x(t+2\pi) & \text{otherwise} \end{cases}$$

The exponential Fourier series coefficients are determined by

$$D_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt$$

Since $T_0 = 2\pi$, $\omega_0 = \frac{2\pi}{T_0} = 1$. For $n = 0$,

$$D_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$$

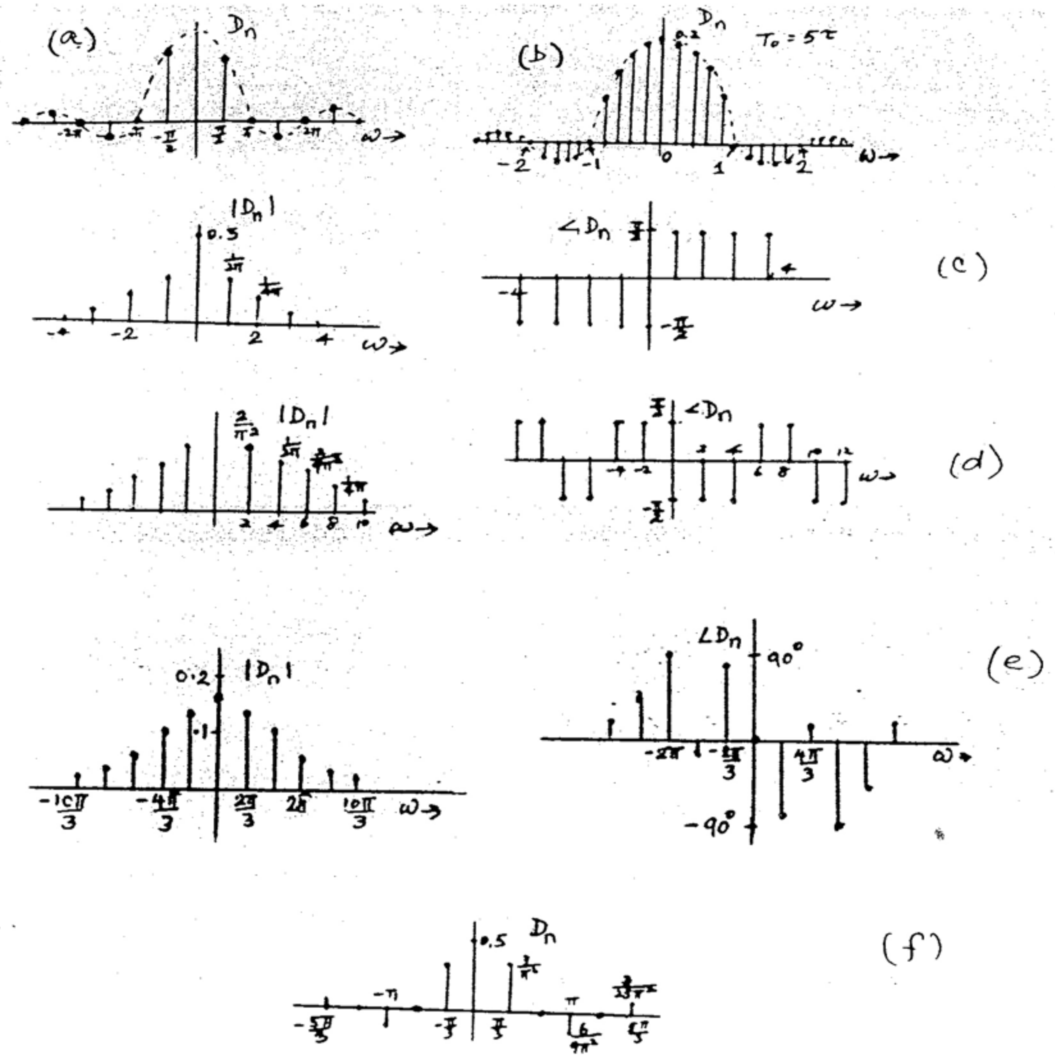

Figure S6.3-1

$$
\begin{aligned}
&= \frac{1}{2\pi}\left(\int_0^A \frac{t}{A}dt + \int_A^\pi dt\right) \\
&= \frac{1}{2\pi}\left(\left.\frac{t^2}{2A}\right|_{t=0}^{A} \left. t\right|_{t=A}^{\pi}\right) \\
&= \frac{1}{2\pi}\left(\frac{A}{2} + \pi - A\right) \\
&= \frac{2\pi - A}{4\pi}
\end{aligned}
$$

For $n \neq 0$,

$$
\begin{aligned}
D_n &= \frac{1}{T_0}\int_{T_0} e^{-jn\omega_0 t}x(t)dt \\
&= \frac{1}{2\pi}\left(\int_0^A \frac{t}{A}e^{-jn\omega_0 t}dt + \int_A^\pi e^{-jn\omega_0 t}dt\right)
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{2\pi}\left(\frac{te^{-jnt}}{-jAn}\bigg|_{t=0}^{A} - \int_0^A \frac{e^{-jnt}}{jAn}dt + \frac{e^{-jnt}}{-jn}\bigg|_{t=A}^{\pi}\right) \\
&= \frac{1}{2\pi}\left(\frac{e^{-jnA}}{-jn} - \frac{e^{-jnt}}{-An^2}\bigg|_{t=0}^{A} + \frac{e^{-jn\pi}-e^{-jnA}}{-jn}\right) \\
&= \frac{1}{2\pi}\left(\frac{je^{-jn\pi}}{n} + \frac{e^{-jnA}-1}{An^2}\right) \\
&= \frac{1}{2\pi n}\left(\frac{e^{-jnA}-1}{An} + je^{-jn\pi}\right)
\end{aligned}$$

Thus,

$$D_n = \begin{cases} \frac{2\pi - A}{4\pi} & n = 0 \\ \frac{1}{2\pi n}\left(\frac{e^{-jnA}-1}{nA} + je^{-jn\pi}\right) & \text{otherwise} \end{cases}$$

6.3-3. (a)

$$x(t) = 3\cos t + \sin\left(5t - \frac{\pi}{6}\right) - 2\cos\left(8t - \frac{\pi}{3}\right)$$

For a compact trigonometric form, all terms must have cosine form and amplitudes must be positive. For this reason, we rewrite $x(t)$ as

$$\begin{aligned}
x(t) &= 3\cos t + \cos\left(5t - \frac{\pi}{6} - \frac{\pi}{2}\right) + 2\cos\left(8t - \frac{\pi}{3} - \pi\right) \\
&= 3\cos t + \cos\left(5t - \frac{2\pi}{3}\right) + 2\cos\left(8t - \frac{4\pi}{3}\right)
\end{aligned}$$

In the preceding expression, we could have expressed the term $2\cos\left(8t - \frac{4\pi}{3}\right)$ as $2\cos\left(8t + \frac{2\pi}{3}\right)$. Figure S6.3-3a shows amplitude and phase spectra.

(b) By inspection of the trigonometric spectra in Fig. S6.3-3a, we plot the exponential spectra as shown in Fig. S6.3-3b.

(c) By inspection of exponential spectra in Fig. S6.3-3a, we obtain

$$\begin{aligned}
x(t) &= \frac{3}{2}(e^{jt} + e^{-jt}) + \frac{1}{2}\left[e^{j(5t-\frac{2\pi}{3})} + e^{-j(5t-\frac{2\pi}{3})}\right] + \left[e^{j(8t-\frac{4\pi}{3})} + e^{-j(8t-\frac{4\pi}{3})}\right] \\
&= \frac{3}{2}e^{jt} + \left(\frac{1}{2}e^{-j\frac{2\pi}{3}}\right)e^{j5t} + \left(e^{-j\frac{4\pi}{3}}\right)e^{j8t} + \frac{3}{2}e^{-jt} + \left(\frac{1}{2}e^{j\frac{2\pi}{3}}\right)e^{-j5t} + \left(e^{j\frac{4\pi}{3}}\right)e^{-j8t}
\end{aligned}$$

(d) By inspection of the first line in part (c), we can immediately write $x(t)$ in the trigonometric form as

$$\begin{aligned}
x(t) &= 3\cos t + \cos\left(5t - \frac{2\pi}{3}\right) + 2\cos\left(8t - \frac{4\pi}{3}\right) \\
&= 3\cos t + \sin\left(5t - \frac{\pi}{6}\right) - 2\cos\left(8t - \frac{\pi}{3}\right)
\end{aligned}$$

6.3-4. (a) In compact trigonometric form, all terms are of cosine form and amplitudes are positive. We can express $x(t)$ as

$$x(t) = 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \cos\left(3t - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(5t + \frac{\pi}{3} - \pi\right)$$

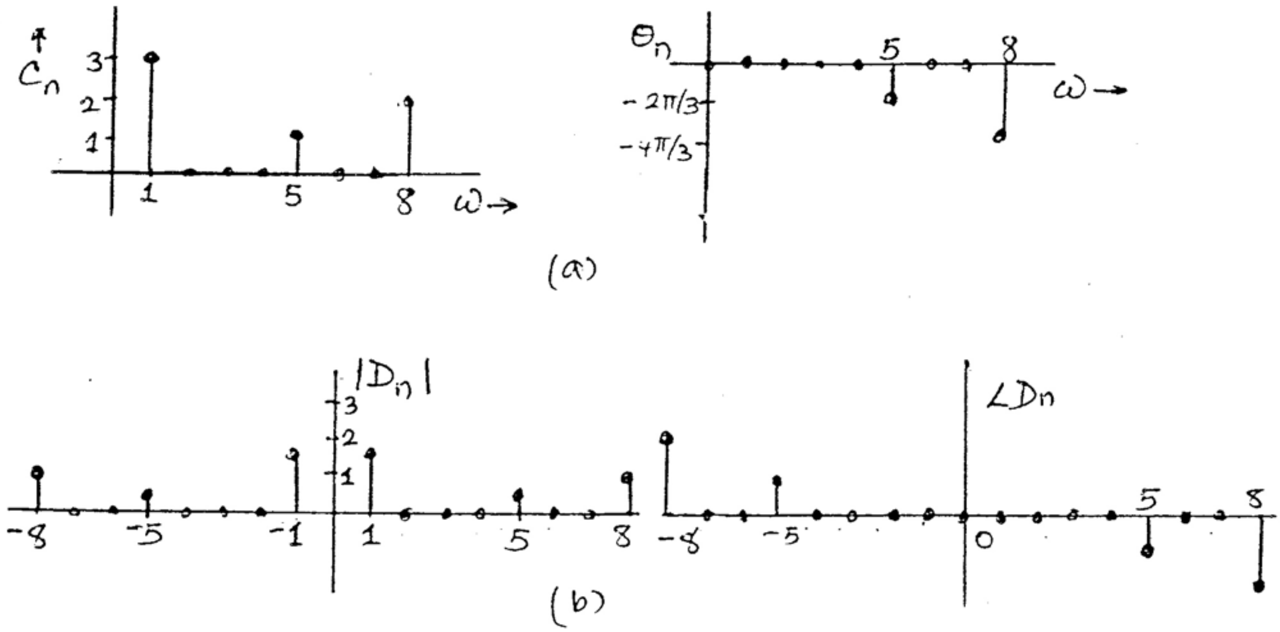

Figure S6.3-3

$$= \quad 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \cos\left(3t - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(5t - \frac{2\pi}{3}\right)$$

From this expression we sketch the trigonometric Fourier spectra as shown in Fig. S6.3-4a.

(b) By inspection of trigonometric spectra, we sketch the exponential Fourier spectra shown in Fig. S6.3-4b.

(c) From these exponential spectra, we can now write the exponential Fourier series as

$$x(t) = 3+e^{j(2t-\frac{\pi}{6})}+e^{-j(2t-\frac{\pi}{6})}+\frac{1}{2}e^{j(3t-\frac{\pi}{2})}+\frac{1}{2}e^{-j(3t-\frac{\pi}{3})}+\frac{1}{4}e^{j(5t-\frac{2\pi}{3})}+\frac{1}{4}e^{-j(5t-\frac{2\pi}{3})}$$

(d) By inspection of the first line in part (c), we can immediately write $x(t)$ in the trigonometric form as

$$\begin{aligned} x(t) &= 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \cos\left(3t - \frac{\pi}{2}\right) + \frac{1}{2}\cos\left(5t - \frac{2\pi}{3}\right) \\ &= 3 + 2\cos\left(2t - \frac{\pi}{6}\right) + \sin 3t - \frac{1}{2}\cos\left(5t + \frac{\pi}{3}\right) \end{aligned}$$

6.3-5. (a) The exponential Fourier series can be expressed with coefficients in Polar form as

$$x(t) = (2\sqrt{2}e^{j\pi/4})e^{-j3t} + 2e^{j\pi/2}e^{-jt} + 3 + 2e^{-j\pi/2}e^{jt} + (2\sqrt{2}e^{-j\pi/4})e^{j3t}$$

From this expression the exponential Spectra are sketched as shown in Figure S6.3-5a.

(b) By inspection of the exponential spectra in Figure S6.3-5a, we sketch the trigonometric spectra as shown in Figure S6.3-5b. From these spectra, we can write the

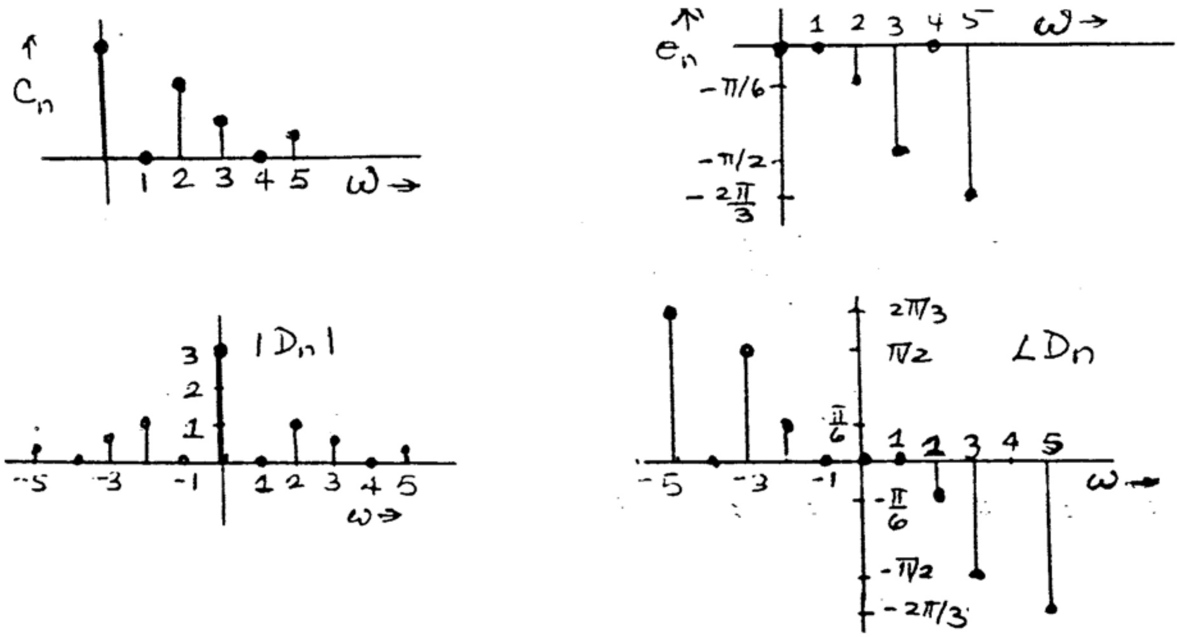

Figure S6.3-4

compact trigonometric Fourier series as

$$x(t) = 3 + 4\cos\left(t - \frac{\pi}{2}\right) + 4\sqrt{2}\cos\left(3t - \frac{\pi}{4}\right)$$

(c) Since, the trigonometric series in part (b) is obtained from the exponential series in part (a), the two series are equivalent.

(d) The lowest frequency in the spectrum is 0 and the highest frequency is 3. Therefore the bandwidth is 3 rad/s or $\frac{3}{2\pi}$ Hz.

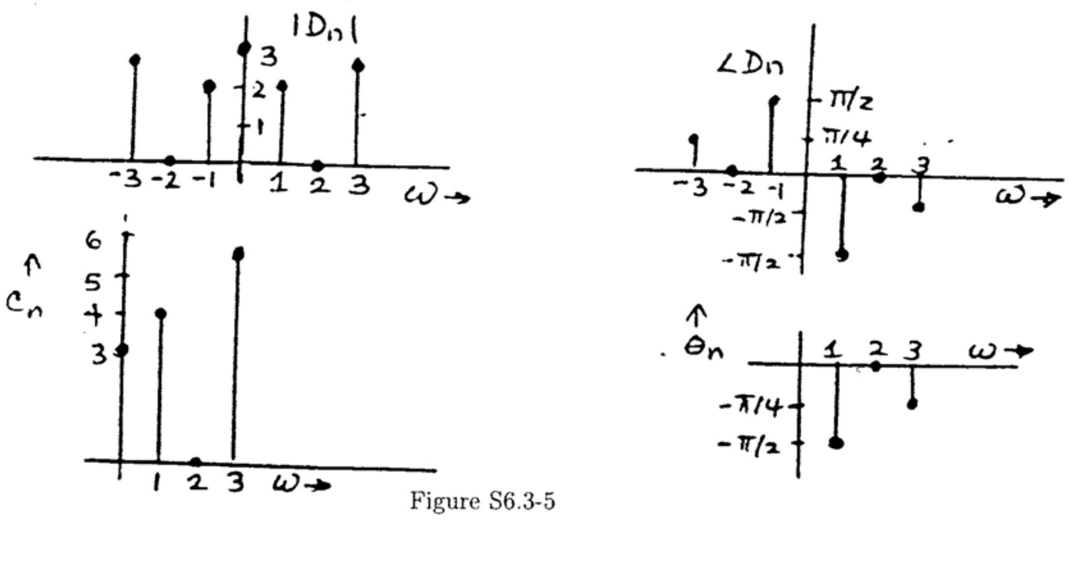

Figure S6.3-5

6.3-6. (a)

$$\begin{aligned} x(t) &= 2 + 2\cos(2t - \pi) + \cos(3t - \frac{\pi}{2}) \\ &= 2 - 2\cos 2t + \sin 3t \end{aligned}$$

(b) The exponential spectra are shown in Figure S6.3-6.

(c) By inspection of exponential spectra

$$\begin{aligned} x(t) &= 2 + \left[e^{(2t-\pi)} + e^{-j(2t-\pi)}\right] + \frac{1}{2}\left[e^{j(3t-\frac{\pi}{2})} + e^{-j(3t-\frac{\pi}{2})}\right] \\ &= 2 + 2\cos(2t - \pi) + \cos\left(3t - \frac{\pi}{2}\right) \end{aligned}$$

(d) Observe that the two expressions (trigonometric and exponential Fourier series) are equivalent.

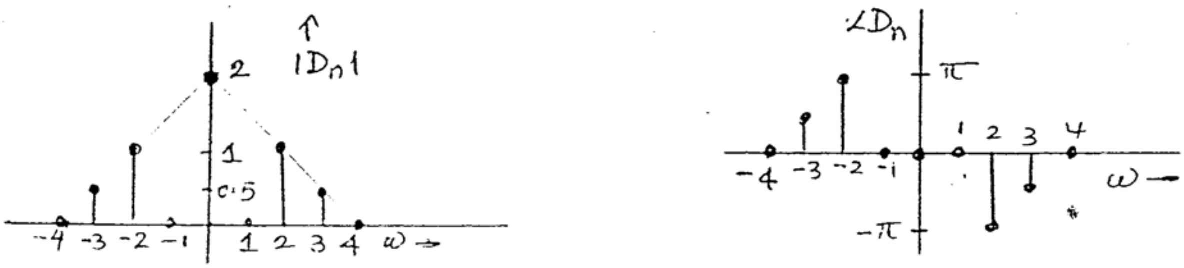

Figure S6.3-6

6.3-7. (a) The exponential Fourier series, as found by inspection of Figure P6.3-6, is

$$x(t) = 2 + 2e^{j(t+\frac{2\pi}{3})} + 2e^{-j(t+\frac{2\pi}{3})} + e^{j(2t+\frac{\pi}{3})} + e^{-j(2t+\frac{\pi}{3})}$$

(b) To find the corresponding trigonometric series, we consider only the positive frequency components, then double the exponential amplitudes (except for dc, which is kept the same), and maintain the same phase values to obtain the trigonometric spectrum, Figure S6.3-7.

(c) By inspection of the trigonometric spectra

$$x(t) = 2 + 4\cos\left(t + \frac{2\pi}{3}\right) + 2\cos\left(2t + \frac{\pi}{3}\right)$$

(d)

$$\begin{aligned} x(t) &= 2 + 4\cos\left(t + \frac{2\pi}{3}\right) + 2\cos\left(2t + \frac{\pi}{3}\right) \\ &= 2 + 2e^{j(t+\frac{2\pi}{3})} + 2e^{-j(t+\frac{2\pi}{3})} + e^{j(2t+\frac{\pi}{3})} + e^{-j(2t+\frac{\pi}{3})} \end{aligned}$$

6.3-8. (a) The period is $T_0 = 8$ and $\omega_0 = \pi/4$. Also $D_0 = 0$ (by inspection), and

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\frac{\pi}{4}t}$$

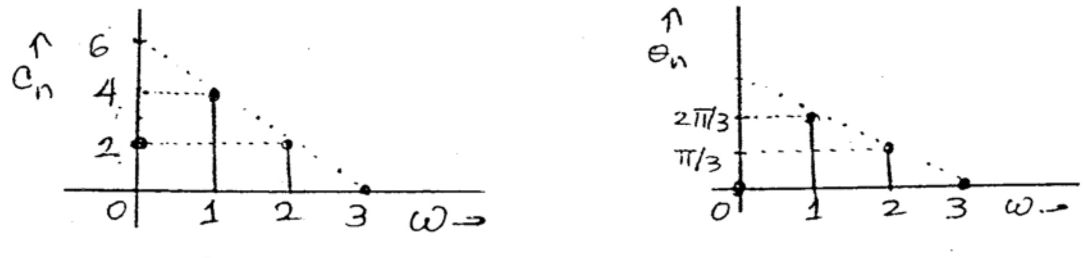

Figure S6.3-7

$$D_n = \frac{1}{8}\left[\int_{-4}^{0}\left(\frac{t}{2}+1\right)e^{-j2n(\pi/4)t}\,dt + \int_{0}^{4}\left(-\frac{t}{2}+1\right)e^{-j2n(\pi/4)t}\,dt\right] =$$

This yields

$$D_n = \begin{cases} \frac{4}{\pi^2 n^2} & n = \pm1, \pm3, \pm5, \pm7, \cdots \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$x(t) = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} \frac{4}{\pi^2 n^2} e^{jn\frac{\pi}{4}t}$$

(b) Observe that $\hat{x}(t)$ is the same as $x(t)$ in Figure P6.3-8a delayed by 2 seconds. Therefore

$$\hat{x}(t) = x(t-2) = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} D_n e^{jn\frac{\pi}{4}(t-2)} = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} D_n e^{-jn\pi/2} e^{jn\frac{\pi}{4}t}$$

Therefore

$$\hat{x}(t) = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} \hat{D}_n e^{jn\frac{\pi}{4}t}$$

where

$$\hat{D}_n = D_n e^{j\frac{n\pi}{2}} = \frac{4}{\pi^2 n^2} e^{-j\frac{n\pi}{2}}$$

(c) Observe that $\tilde{x}(t)$ is the same as $x(t)$ in Figure P6.2-8a time-compressed by a factor 2. Therefore

$$\tilde{x}(t) = x(2t) = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} D_n e^{jn\frac{\pi}{4}(2t)} = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} D_n e^{jn\frac{\pi}{2}t}$$

Therefore

$$\tilde{x}(t) = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} \tilde{D}_n e^{jn\frac{\pi}{2}t}$$

where

$$\tilde{D}_n = D_n = \frac{4}{\pi^2 n^2}$$

6.3-9. (a)

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \\ \hat{x}(t) &= x(t-T) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 (t-T)} = \sum_{n=-\infty}^{\infty} (D_n e^{-jn\omega_0 T}) e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \hat{D}_n e^{jn\omega_0 t} \end{aligned}$$

$$\hat{D}_n = D_n e^{-jn\omega_0 t} \quad \text{so that} \quad |\hat{D}_n| = |D_n|, \quad \text{and} \quad \angle\hat{D}_n = \angle D_n - jn\omega_0 T$$

(b)

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \\ \hat{x}(t) &= x(at) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 (at)} \end{aligned}$$

6.3-10. (a) From Exercise E6.1a

$$x(t) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi t \qquad -1 \le t \le 1$$

The power of $x(t)$ is

$$P_x = \frac{1}{2}\int_{-1}^{1} t^4\, dt = \frac{1}{5}$$

Moreover, from Parseval's theorem Eq. (6.40)

$$P_x = C_0^2 + \sum_{1}^{\infty} \frac{C_n^2}{2} = \left(\frac{1}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{\pi^2 n^2}\right)^2 = \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{9} + \frac{8}{90} = \frac{1}{5}$$

(b) If the N-term Fourier series is denoted by $w(t)$, then

$$w(t) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{N-1} \frac{(-1)^n}{n^2} \cos n\pi t \qquad -1 \le t \le 1$$

The power P_x is required to be $99\% P_x = 0.198$. Therefore

$$P_x = \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{N-1}\frac{1}{n^4} = 0.198$$

For $N = 1$, $P_x = 0.1111$; for $N = 2$, $P_x = 0.19323$, For $N = 3$, $P_x = 0.19837$, which is greater than 0.198. Thus, $N = 3$.

6.3-11. (a) From Exercise E6.1b

$$x(t) = \frac{2A}{\pi}(-1)^{n+1}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi t \qquad -\pi \le t \le \pi$$

The power of $x(t)$ is

$$P_x = \frac{1}{2}\int_{-1}^{1}(At)^2\,dt = \frac{A^2}{3}$$

Moreover, from Parseval's theorem [Eq. (6.40)]

$$P_x = C_0^2 + \sum_1^{\infty}\frac{C_n^2}{2} = \frac{1}{2}\sum_1^{\infty}\frac{4A^2}{\pi^2 n^2} = \frac{2A^2}{\pi^2}\sum_1^{\infty}\frac{1}{n^2} = \frac{A^2}{3}$$

(b) If the N-term Fourier series is denoted by $w(t)$, then

$$w(t) = \frac{2A}{\pi}(-1)^{n+1}\sum_{n=1}^{N}\frac{1}{n}\sin n\pi t \qquad\qquad -\pi \le t \le \pi$$

The power P_w is required to be no less than $0.90\frac{A^2}{3} = 0.3A^2$. Therefore

$$P_w = \frac{1}{2}\sum_1^{N}\frac{4A^2}{\pi^2 n^2} \ge 0.3A^2$$

For $N = 1$, $P_w = 0.2026A^2$; for $N = 2$, $P_w = 0.2533A^2$, for $N = 5$, $P_w = 0.29658A^2$, for $N = 6$, $P_w = 0.30222A^2$, which is greater than $0.3A^2$. Thus, $N = 6$.

6.3-12. The power of a rectified sine wave is the same as that of a sine wave, that is, 1/2. Thus $P_x = 0.5$. Let the $2N+1$ term truncated Fourier series be denoted by $\hat{x}(t)$. The power $P_{\hat{x}}$ is required to be no less than $0.9975P_x = 0.49875$. Using the Fourier series coefficients in Exercise E6.5, we have

$$P_{\hat{x}} = \sum_{n=-N}^{N}|D_n|^2 = \frac{4}{\pi^2}\sum_{n=-N}^{N}\frac{1}{(1-4n^2)^2} \ge 0.49875$$

Direct calculations using the above equation gives $P_{\hat{x}} = 4/\pi^2 = 0.4053$ for $N = 0$ (only dc), $P_{\hat{x}} = 0.49535$ for $N = 1$ (3 terms), and $P_{\hat{x}} = 0.49895$ for $N = 2$ (5 terms). Thus, a 5-term Fourier series yields a signal whose power is 99.79% of the power of the rectified sine wave. The power of the error in the approximation of $x(t)$ by $\hat{x}(t)$ is only 0.21% of the signal power P_x.

6.4-1. Period $T_0 = \pi$, and $\omega_0 = 2$, and

$$H(j\omega) = \frac{j\omega}{(-\omega^2+3)+j2\omega}, \qquad \text{and from Eq. (6.30b)} \qquad D_n = \frac{0.504}{1+j4n}$$

Therefore, $$y(t) = \sum_{n=-\infty}^{\infty} D_n H(jn\omega_0)e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty}\frac{j1.08n}{(1+j4n)(-\omega^2+3+j2\omega)}e^{j2nt}$$

6.4-2. (a)

$$\begin{aligned}\cos 5t\sin 3t &= \frac{1}{2}[\sin 8t - \sin 2t]\\ &= \frac{1}{4j}\left[e^{j8t} - e^{-j8t} - e^{j2t} + e^{-j2t}\right]\end{aligned}$$

$$= \frac{1}{4j}\left[e^{j(8t-\frac{\pi}{2})} - e^{-j(8t-\frac{\pi}{2})} - e^{j(2t+\frac{\pi}{2})} + e^{-j(2t+\frac{\pi}{2})}\right]$$

This is the desired exponential Fourier series.

(b) There are four spectral components at $\omega = \pm 8$ and ± 2. The phases are either $\frac{\pi}{2}$ or $-\frac{\pi}{2}$, as shown in the spectrum in Figure S6.4-2b.

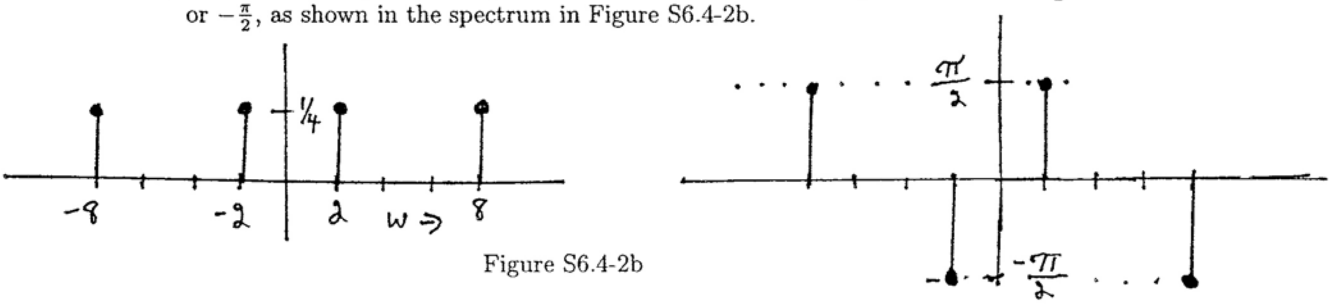

Figure S6.4-2b

(c) Since none of the spectral components of $x(t)$ appear in the pass-band of the filter, the output is $y(t) = 0$.

6.4-3.

$$D_n = \int_0^1 e^{-t}e^{-jn\omega_o t}dt = \frac{(e-1)(1-j2\pi n)}{e(1+4\pi^2n^2)}$$

The transfer function of the R-C circuit is

$$H(j\omega) = \frac{1}{1+(\frac{1}{j\omega})} = \frac{j\omega}{j\omega+1}$$

The input $x(t)$ can be expressed as a Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{(e-1)(1-j2\pi n)}{e(1+4\pi^2n^2)}e^{j2\pi nt}$$

Hence the output $y(t)$ is given by

$$\begin{aligned}
y(t) &= \sum_{n=-\infty}^{\infty} D_n H(j2\pi n)e^{j2\pi nt} \\
&= \sum_{n=-\infty}^{\infty} \frac{(e-1)(1-j2\pi n)(j2\pi n)}{e(1+4\pi^2n^2)(j2\pi n+1)}e^{j2\pi nt} \\
&= \sum_{n=-\infty}^{\infty} \frac{2\pi n(e-1)(2\pi n+j)}{e(1+4\pi^2n^2)^2}e^{j2\pi nt}
\end{aligned}$$

6.5-1. Equating the derivative (with respect to c) yields

$$2c|\mathbf{y}|^2 = 2\mathbf{x}.\mathbf{y}$$

which yields the desired result.

6.5-2. (a)

$$e(t) = x(t) - cx(t)$$

Also

$$\int_{t_1}^{t_2} x(t)[x(t) - cx(t)]\,dt = \int_{t_1}^{t_2} x(t)x(t)\,dt - c\int_{t_1}^{t_2} x^2(t)\,dt$$

But

$$c = \frac{\int_{t_1}^{t_2} x(t)x(t)\,dt}{\int_{t_1}^{t_2} x^2(t)\,dt}$$

Substitution of c in the earlier equation yields

$$\int_{t_1}^{t_2} x(t)[x(t) - cx(t)]\,dt = 0$$

Therefore $x(t)$ and $[x(t) - cx(t)]$ are orthogonal.

(b) We can readily see result from Figure 6.17. The error vector $\mathbf{e}$ is orthogonal to vector $\mathbf{x}$.

(c)

$$e(t) = \begin{cases} 1 - \frac{4}{\pi}\sin t & 0 \le t \le \pi \\ -1 - \frac{4}{\pi}\sin t & \pi \le t \le 2\pi \end{cases}$$

$$\begin{aligned}\int_0^{2\pi} e^2(t)\,dt &= \int_0^{\pi}\left(1 - \frac{4}{\pi}\sin t\right)^2 dt - \int_{\pi}^{2\pi}\left(1 + \frac{4}{\pi}\sin t\right)^2 dt \\ &= -\frac{8}{\pi}\left[\int_0^{\pi}\sin t\,dt + \int_{\pi}^{2\pi}\sin t\,dt\right] = -\frac{8}{\pi}\int_0^{2\pi}\sin t\,dt = 0\end{aligned}$$

6.5-3. (a) If $x(t)$ and $y(t)$ are orthogonal, then we showed [see Eq. (6.67)] the energy of $x(t) + y(t)$ is $E_x + E_y$. We now find the energy of $x(t) - y(t)$:

$$\begin{aligned}\int_{-\infty}^{\infty} |x(t) - y(t)|^2\,dt &= \int_{-\infty}^{\infty} |x(t)|^2\,dt + \int_{-\infty}^{\infty} |y(t)|^2\,dt \\ &\quad - \int_{-\infty}^{\infty} x(t)y^*(t)\,dt - \int_{-\infty}^{\infty} x^*(t)y(t)\,dt \\ &= \int_{-\infty}^{\infty} |x(t)|^2\,dt + \int_{-\infty}^{\infty} |y(t)|^2\,dt\end{aligned}$$

The last result follows from the fact that because of orthogonality, the two integrals of the cross products $x(t)y^*(t)$ and $x^*(t)y(t)$ are zero [see Eq. (6.80)]. Thus the energy of $x(t) + y(t)$ is equal to that of $x(t) - y(t)$ if $x(t)$ and $y(t)$ are orthogonal.

(b) Using similar argument, we can show that the energy of $c_1x(t) + c_2y(t)$ is equal to that of $c_1x(t) - c_2y(t)$ if $x(t)$ and $y(t)$ are orthogonal. This energy is given by $|c_1|^2E_x + |c_2|^2E_y$.

(c) If $z(t) = x(t) \pm y(t)$, then

$$\begin{aligned}\int_{-\infty}^{\infty} |x(t) \pm y(t)|^2\,dt &= \int_{-\infty}^{\infty} |x(t)|^2\,dt + \int_{-\infty}^{\infty} |y(t)|^2\,dt \\ &\quad \pm \int_{-\infty}^{\infty} x(t)y^*(t)\,dt \pm \int_{-\infty}^{\infty} x^*(t)y(t)\,dt\end{aligned}$$

$$= \quad E_x + E_y \pm (E_{xy} + E_{yx})$$

6.5-4. (a) In this case $E_x = \int_0^1 dt = 1$, and

$$c = \frac{1}{E_x}\int_0^1 x(t)y(t)\,dt = \frac{1}{1}\int_0^1 t\,dt = 0.5$$

(b) Thus, $x(t) \approx 0.5y(t)$, and the error $e(t) = t - 0.5$ over $(0 \le t \le 1)$, and zero outside this interval. Also E_x and E_e (the energy of the error) are

$$E_x = \int_0^1 x^2(t)\,dt = \int_0^1 t^2\,dt = 1/3 \quad \text{and} \quad E_e = \int_0^1 (t-0.5)^2\,dt = 1/12$$

The error $(t - 0.5)$ is orthogonal to $y(t)$ because

$$\int_0^1 (t-0.5)(1)\,dt = 0$$

By inspection of $y(t)$, we obtain $E_y = 1$. Note that $E_x = c^2 E_y + E_e$. To explain these results in terms of vector concepts we observe from Figure 6.17 that the error vector $\mathbf{e}$ is orthogonal to the component $c\mathbf{x}$. Because of this orthogonality, the length-square of $\mathbf{x}$ [energy of $x(t)$] is equal to the sum of the square of the lengths of $c\mathbf{y}$ and $\mathbf{e}$ [sum of the energies of $cy(t)$ and $e(t)$].

6.5-5. In this case $E_x = \int_0^1 x^2(t)\,dt = \int_0^1 t^2\,dt = 1/3$, and

$$c = \frac{1}{E_x}\int_0^1 y(t)x(t)\,dt = 3\int_0^1 t\,dt = 1.5$$

Thus, $y(t) \approx 1.5x(t)$, and the error $e(t) = y(t) - 1.5x(t) = 1 - 1.5t$ over $(0 \le t \le 1)$, and zero outside this interval. Also E_e (the energy of the error) is $E_e = \int_0^1 (1-1.5t)^2\,dt = 1/4$.

6.5-6.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi nt + b_n \sin 2\pi nt \qquad \left(\omega_0 = \frac{2\pi}{1}\right)$$

$$a_0 = 1\int_0^1 x(t)\,dt = \int_0^1 t\,dt = \frac{1}{2}$$

$$a_n = 2\int_0^1 t\cos 2\pi nt\,dt = 0 \quad n \ge 1 \quad (n\,\text{integer})$$

$$b_n = 2\int_0^1 t\sin 2\pi nt\,dt = \frac{-1}{\pi n}$$

Hence

$$\begin{aligned} x(t) &= \frac{1}{2} - \frac{1}{\pi}\left(\sin 2\pi t + \frac{1}{2}\sin 4\pi t + \frac{1}{3}\sin 6\pi t + \cdots\right) \\ &= \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2\pi nt \end{aligned}$$

From Eq. (6.77)

$$\epsilon_k = 1\int_0^1 x^2(t)\,dt - \left[\left(\frac{1}{2}\right)^2 + \frac{1}{2}\left[\left(\frac{1}{\pi}\right)^2 + \left(\frac{1}{2\pi}\right)^2 + \cdots + \left(\frac{1}{(k-1)\pi}\right)^2\right]\right]$$

(Note that $K_j^2 = 1/2$ for $j = 1, 2, \ldots$ and $K_0^2 = 1$)

$$\begin{aligned}
\epsilon_1 &= \int_0^1 t^2\,dt - \frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \\
\epsilon_2 &= \frac{1}{3} - \frac{1}{4} - \frac{1}{2\pi^2} = 0.03267 \\
\epsilon_3 &= \frac{1}{3} - \frac{1}{4} - \frac{1}{2\pi^2} - \frac{1}{8\pi^2} = 0.02 \\
\epsilon_4 &= \frac{1}{3} - \frac{1}{4} - \frac{1}{2\pi^2} - \frac{1}{8\pi^2} - \frac{1}{18\pi^2} = 0.014378
\end{aligned}$$

6.5-7. (a) Figure S6.5-7a shows $x_1(t)$ that is a periodic extension of $x(t)$ to yield a series with $\omega_0 = 2\pi$ and only sine terms. This requires $T_0 = 2\pi/2\omega = 1$ and odd symmetry. From inspection, the dc component is 0.5. If we subtract dc (0.5) from $x_1(t)$, the remaining signal $x_1(t) - 0.5$ has odd symmetry (only sine terms). Therefore

$$\begin{aligned}
x_1(t) &= 0.5 + \sum_{n=1}^{\infty} b_n \sin 2\pi n t \\
b_n &= 2\int_0^1 t\,\sin 2\pi n t\,dt = -\frac{1}{\pi n} \\
x_1(t) &= \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin 2\pi n t
\end{aligned}$$

(b) $\omega_0 = \pi$ and $T_0 = 2\pi/\pi = 2$. For sine terms only, we need odd symmetry. Figure S6.5-7b shows a suitable function $x_2(t)$. It has no dc.

$$\begin{aligned}
x_2(t) &= \sum_{n=1}^{\infty} b_n \sin n\pi t \\
b_n &= \frac{4}{2}\int_0^1 t \sin n\pi t\,dt = (-1)^{n+1}\frac{2}{n\pi}
\end{aligned}$$

(c) $\omega_0 = \pi$, $T_0 = 2\pi/\omega_0 = 2$. For cosine terms only, we need an even function $x_3(t)$ as shown in Figure S6.5-7c. By inspection dc is 0.5. Therefore

$$\begin{aligned}
x_3(t) &= \frac{1}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi t \\
a_n &= \frac{4}{2}\int_0^1 t \cos n\pi t\,dt = -\frac{4}{\pi^2 n^2} \qquad n = 1,\, 3,\, 5,\, \ldots
\end{aligned}$$

6.5-8. (a) The signal $g(t)$ is the same as the signal $x(t)$ in Example 6.12 (Figure 6.23)

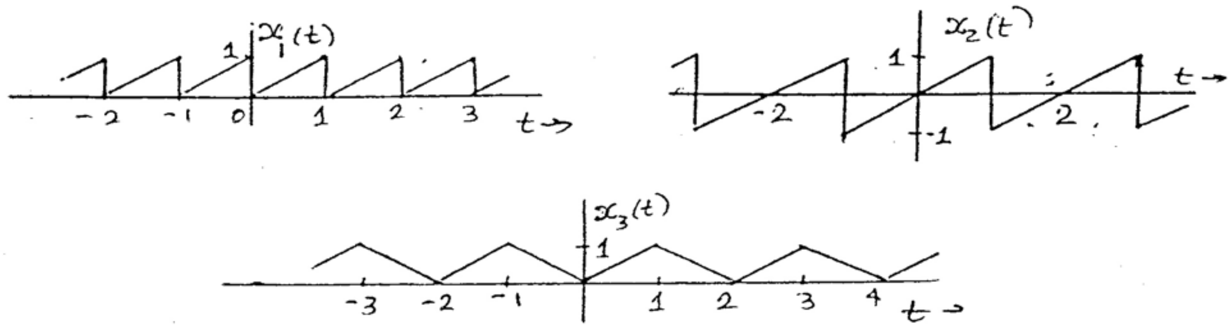

Figure S6.5-7

time-expanded by a factor π. Therefore from Eq. (6.90), we have

$$g(t) = x\left(\frac{t}{\pi}\right) = -\frac{3}{2}\left(\frac{t}{\pi}\right) + \frac{7}{8}\left[\frac{5}{2}\left(\frac{t}{\pi}\right)^3 - \frac{3}{2}\left(\frac{t}{\pi}\right)\right] + \cdots \qquad (1)$$

For the representation in Eq. (6.90) for $x(t)$ in Figure 6.23

$$\int_{-1}^{1} x^2(t)\,dt = 2\pi \qquad \text{and}$$

Therefore from Eq. (6.77)

$$\begin{aligned}\epsilon_1 &= \int x^2(t)\,dt - \frac{1}{3}c_1^2 = 2 - \frac{3}{2} = 0.5\\ \epsilon_2 &= \int x^2(t)\,dt - \frac{1}{3}c_1^2 - \frac{1}{7}c_3^2 = 0.28125\end{aligned}$$

Since $g(t)$ is the same as $x(t)$ time-expanded by a factor π, all energies are increased by the same factor (π). Therefore

$$\begin{aligned}\epsilon_1 &= 0.5\pi\\ \epsilon_2 &= 0.28125\pi\end{aligned}$$

6.5-9.

$$x(t) = c_0x_0(t) + c_1x_1(t) + \cdots + c_7x_7(t)$$

The energy E_n of $x_n(t)$, for all $n = 1, 2, 3, \ldots, 8$ is given by

$$E_n = \int_0^1 x_n^2(t)\,dt = 1$$

Hence

$$c_0 = \int_0^1 x(t)x_0(t)\,dt = \frac{1}{2}$$

$$\begin{aligned}
c_1 &= \int_0^1 x(t)x_1(t)\,dt = -\frac{1}{4} \\
c_2 &= c_4 = c_5 = c_6 = 0 \\
c_3 &= \int_0^1 x(t)x_3(t)\,dt = -\frac{1}{8} \\
c_7 &= \int_0^1 x(t)x_7(t)\,dt = -\frac{1}{16}
\end{aligned}$$

Hence

$$x(t) \simeq \frac{1}{2}x_0(t) - \frac{1}{4}x_1(t) - \frac{1}{8}x_3(t) - \frac{1}{16}x_7(t)$$

Also

$$\int_0^1 x^2(t)\,dt = \frac{1}{3} \quad \text{and} \quad E_n = 1$$

If $E_e(N)$ is the energy of the error signal in the approximation using first N terms, then From Eq. (6.77)

$$\begin{aligned}
E_e(1) &= \frac{1}{3} - c_0^2 = \frac{1}{12} = 0.0833 \\
E_e(2) &= \frac{1}{3} - c_0^2 - c_1^2 = \frac{1}{48} = 0.0204 \\
E_e(3) &= \frac{1}{3} - c_0^2 - c_1^2 - c_3^2 = \frac{1}{192} = 0.0052 \\
E_e(4) &= \frac{1}{3} - c_0^2 - c_1^2 - c_3^2 - c_7^2 = \frac{1}{768} = 0.001302
\end{aligned}$$

the corresponding trigonometric Fourier series found in Prob. 6.5-6 are 0.0833, 0.03267, 0.02, 0.014378. Clearly, the Walsh Fourier series gives smaller error than the corresponding trigonometric Fourier series for the same number of terms in the approximation.

6.M-1. (a) To determine a suitable set of $N = 10$ frequencies ω_n, we first determine ten points logarithmically spaced from 1 to 100.

```
>> f = logspace(0,2,10)
f =    1.0000    1.6681    2.7826    4.6416    7.7426
      12.9155   21.5443   35.9381   59.9484 100.0000
```

The problem with these points is that they are not all rational, and the resulting signal $m(t)$ is thus aperiodic. Truncating to the four decimal places shown makes the frequencies rational, but the resulting period T_0 is excessively long. An approximately logarithmic sequence that results in smaller T_0 is generated by rounding the logarithmic frequencies to the nearest tenths of a hertz.

```
>> f = round(10*logspace(0,2,10))/10
f =    1.0000    1.7000    2.8000    4.6000    7.7000
      12.9000   21.5000   35.9000   59.9000 100.0000
```

With these frequencies, the signal $m(t) = \sum_{n=1}^{N} \cos(\omega_n t + \theta_n)$ has period $T_0 = 10$. Thus, one reasonable choice of frequencies is

$\omega_n = 2\pi[1, 1.7, 2.8, 4.6, 7.7, 12.9, 21.5, 35.9, 59.9, 100]$ for which $m(t)$ has period $T_0 = 10$.

MATLAB is used to plot $m(t)$ when all θ_n are set to zero.

```
>> m = inline('sum(cos(omega*t+theta*ones(size(t))))',...
       'theta','t','omega');
>> omega = 2*pi*f'; theta = zeros(size(omega));
>> t = (-5:.01:5); plot(t,m(theta,t,omega),'k');
>> xlabel('t [sec]'); ylabel('m(t) [volts]');
```

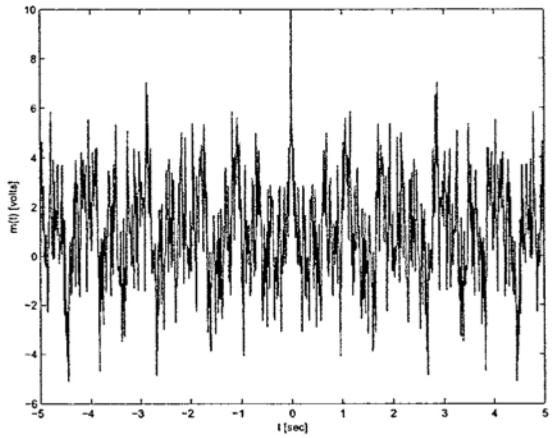

Figure S6.M-1a: Signal $m(t)$ with log-spaced ω_n and $\theta_n = 0$.

As expected, this worst-case version of $m(t)$ has a maximum amplitude of 10, which is also the number of sinusoids comprising the signal.

(b) MATLAB is used to try and find an optimal set of phases θ_n that minimizes the maximum amplitude of $m(t)$. The procedure followed is the same as that presented in MATLAB Session 6. To proceed, the code from part 6.M-1a needs to be first executed.

```
>> maxmagm = inline('max(abs(sum(cos(omega*t+theta*ones(size(t))))))',...
       'theta','t','omega');
>> t = [-5:.001 :5];
>> rand('state',0); theta_init = 2*pi*rand(N,1);
>> theta_opt = fminsearch(maxmagm,theta_init,[],t,omega);
>> mmag = max(abs(m(theta_opt,t,omega)))
mmag = 6.7711
```

The result of 6.7711 shows a reasonable reduction in maximum amplitude from the worst-case value of 10. Notice, a finely-spaced time vector `t` is required for the function `fminsearch` to determine a reliable result.

To make sure the result is good and not just a local minimum, the sequence is run again with a different initial guess for the phases.

```
>> theta_init = 2*pi*rand(N,1);
>> theta_opt = fminsearch(maxmagm,theta_init,[],t,omega);
>> mmag = max(abs(m(theta_opt,t,omega)))
mmag = 6.6854
```

Although the second result coincides well with the first, it is not exactly the same. To be safe, the sequence is therefore run several times, and the best solution is preserved.

```
>> mmag_opt = mmag; mmag = [mmag,zeros(1,9)];
>> for trial = 2:10;
>>     theta_init = 2*pi*rand(N,1);
>>     theta = fminsearch(maxmagm,theta_init,[],t,omega);
>>     mmag(trial) = max(abs(m(theta,t,omega)))
>>     if (mmag(trial)<mmag_opt),
>>         theta_opt = theta'; mmag_opt = mmag(trial);
>>     end
>> end
>> mmag, theta_opt
mmag =        6.6854  6.7069  6.6568  6.6421  6.7301
              6.4846  6.5906  6.5294  6.5292  6.5888
theta_opt = 2.7839  5.7162  5.1883  4.2464  5.1741
              4.0850  2.2366  1.9871  1.6831  3.7462
```

Thus, a good (but unlikely globally best) choice of phases is

$$\theta_n = 2\pi[2.7839, 5.7162, 5.1883, 4.2464, 5.1741, 4.0850, 2.2366, 1.9871, 1.6831, 3.7462].$$

In this case, the maximum value of $m(t)$ is 6.4846, as shown in Figure 6.M-1b.

```
>> plot(t,m(theta_opt,t,omega),'k');
>> xlabel('t [sec]'); ylabel('m(t) [volts]');
```

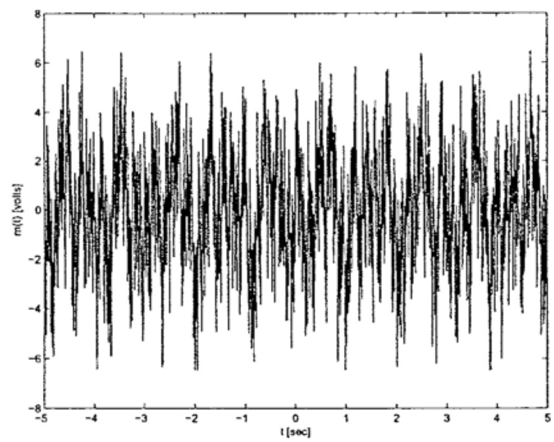

Figure S6.M-1b: Signal $m(t)$ with log-spaced ω_n and optimized phases.

(c) For environments with $1/f$ noise, it is appropriate to have lower frequency components have greater strength than higher frequency components. One simple possibility is to adjust the magnitude of each sinusoidal component to match the noise power at that frequency. In this way, the signal-to-noise ratio is kept constant for any frequency bin of the signal.

$$m(t) = \sum_{n=1}^{N} \frac{k}{\sqrt{\omega_n}} \cos(\omega_n t + \theta_n).$$

The constant k is selected to achieve the final desired signal power for the entire signal $m(t)$.