Chapter 10 Solutions

10.1-1. (a)

$$\ddot{y} + 10\dot{y} + 2y = x$$

Choose: $q_1 = y \quad and \quad q_2 = \dot{y} = \dot{q}_1 \Longrightarrow \dot{q}_2 = \ddot{y}$

hence:
$$\begin{aligned}\dot{q}_1 &= q_2\\ \dot{q}_2 &= -2q_1 - 10q_2 + x\end{aligned}$$

In matrix form we get:

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -10\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}x$$

(b)

$$\ddot{y} + 2e^{y}\dot{y} + \log y = x$$

Choose $q_1 = y \quad and \quad q_2 = \dot{y} = \dot{q}_1$

hence:
$$\begin{aligned}\dot{q}_1 &= q_2\\ \dot{q}_2 &= -2e^{q_1}q_2 - \log q_1 + x\end{aligned}$$

It is easy to see that this set is nonlinear.

(c)

$$\ddot{y} + \phi_1(y)\dot{y} + \phi_2(y)y = x$$

Choose $q_1 = y \quad and \quad q_2 = \dot{y}$.

hence:
$$\begin{aligned}\dot{q}_1 &= q_2\\ \dot{q}_2 &= -\phi_1(q_1)q_2 - \phi_2(q_1)q_1 + x\end{aligned}$$

Also in this case we are dealing with a nonlinear set, since $\phi_2(q_1)$ and $\phi_1(q_1)$ are not constants.

10.2-1. Writing the loop equations we get:

$$x = q_1 + 2i + 3i_2 \qquad \text{where} \qquad i_2 = \frac{x - q_1 - \dot{q}_2}{2} - q_2$$

$$\text{and} \qquad i = \frac{x - q_1 - \dot{q}_2}{2}$$

Also we have: $$\frac{1}{2}\dot{q}_1 = \frac{x - q_1 - \dot{q}_2}{2} - q_1$$

Therefore $$\dot{q}_1 = x - q_1 - \dot{q}_2 - 2q_1 = -3q_1 - \dot{q}_2 + x \tag{1}$$

We can also write:

$$\dot{q}_2 = 3i_2 = 3\left[\frac{x - q_1 - \dot{q}_2}{2} - q_2\right] = \frac{3}{2}x - \frac{3}{2}q_1 - \frac{3}{2}\dot{q}_2 - 3q_2$$

$$\text{Hence} \qquad \frac{5}{2}\dot{q}_2 = -\frac{3}{2}q_1 - 3q_2 + \frac{3}{2}x$$

$$\text{or} \qquad \dot{q}_2 = -\frac{3}{5}q_1 - \frac{6}{5}q_2 + \frac{3}{5}x \qquad (2)$$

Substituting equation (2) in equation (1) we obtain:

$$\dot{q}_1 = -3q_1 + x - \left[-\frac{3}{5}q_1 - \frac{6}{5}q_2 + \frac{3}{5}x\right] = -\frac{12}{5}q_1 + \frac{6}{5}q_2 + \frac{2}{5}x$$

Hence the state equations are:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -\frac{12}{5} & \frac{6}{5} \\ -\frac{3}{5} & -\frac{6}{5} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \frac{2}{5} \\ \frac{3}{5} \end{bmatrix} x(t)$$

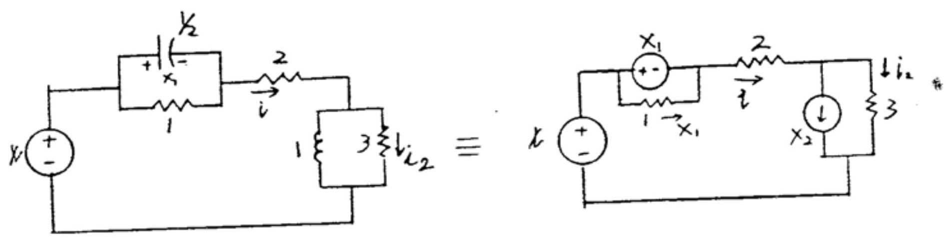

Figure S10.2-1

10.2-2. In the 1st loop, the current i_1 can be computed as:

$$x = \frac{1}{3}i_1 + q_1 \Longrightarrow i_1 = 3(x - q_1)$$

We also have: (using node equation)

$$\frac{1}{2}\dot{q}_1 = -2q_1 - q_2 - 3q_1 + 3x = -5q_1 - q_2 + 3x$$

$$\text{Hence} \qquad \dot{q}_1 = -10q_1 - 2q_2 + 6x \qquad (1)$$

Writing the equations in the rightmost loop we get:

$$q_1 = q_2 + \dot{q}_2 \quad \text{and} \quad \dot{q}_2 = q_1 - q_2 \qquad (2)$$

Hence from (1) and (2) the state equations are found as:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -10 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \end{bmatrix} x$$

The output equation is: $y = \dot{q}_2 = q_1 - q_2$

$$\text{or} \qquad y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

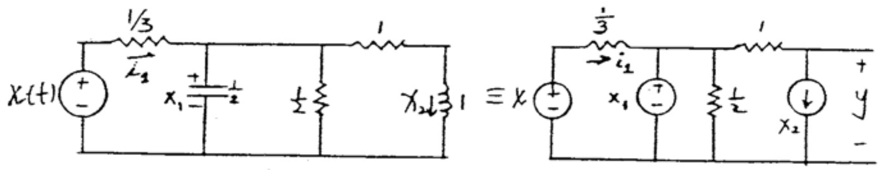

Figure S10.2-2

10.2-3. Let's choose the voltage across the capacitor and the current through the inductor as state variables q_1 and q_2, respectively. Writing the loop equations we get:

$$x_1 = q_1 + \frac{1}{5}[\dot{q}_1 - q_2]$$

Here we use the fact that: $\dot{q}_1 = i_1$ and $q_2 = i_2$.

$$x_2 = -\frac{1}{2}\dot{q}_2 - q_2 + \frac{1}{5}[\dot{q}_1 - q_2]$$

$$\text{And thus:} \qquad \begin{aligned} \dot{q}_1 &= -5q_1 + q_2 + 5x_1 \\ \dot{q}_2 &= -2q_1 - 2q_2 + 2x_1 - 2x_2 \end{aligned}$$

Hence the state equations are

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

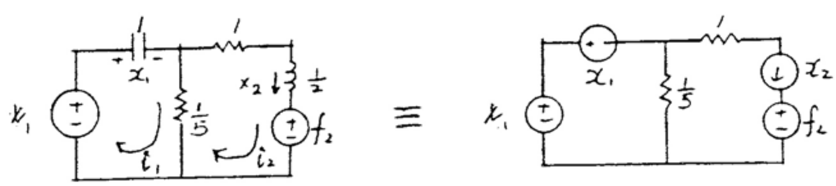

Figure S10.2-3

10.2-4. The loop equations yield:

$$\text{with} \qquad i_2 = \dot{q}_2 \qquad \text{and} \qquad i_1 = q_1 + i_2 = q_1 + \dot{q}_2$$

$$x = 2i_1 + q_1 + \dot{q}_1 = 2q_1 + 2\dot{q}_2 + q_1 + \dot{q}_1 = 3q_1 + \dot{q}_1 + 2\dot{q}_2 \qquad (1)$$

$$x = 2i_1 + \dot{q}_2 + q_2 = 2q_1 + 2\dot{q}_2 + \dot{q}_2 + q_2 = 2q_1 + q_2 + 3\dot{q}_2 \qquad (2)$$

The last equation gives:

$$\dot{q}_2 = -\frac{2}{3}q_1 - \frac{1}{3}q_2 + \frac{1}{3}x \qquad (3)$$

Substituting $\dot{q}_2$ in the equation (1) we get:

$$\dot{q}_1 = -\frac{5}{3}q_1 + \frac{2}{3}q_2 + \frac{5}{3}x \tag{4}$$

From (3) and (4) the state equations are obtained as:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} x(t)$$

And the output equations are: $y_1 = q_1$ and

$$y_2 = i_2 = \dot{q}_2 = -\frac{2}{3}q_1 - \frac{1}{3}q_2 + \frac{1}{3}x$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix} x(t)$$

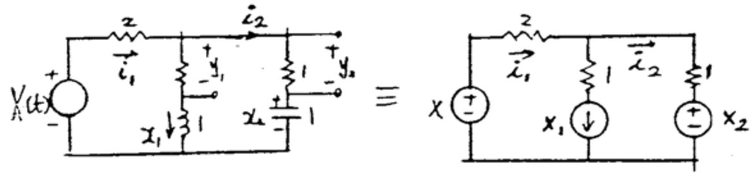

Figure S10.2-4

10.2-5. We have:

$$\begin{aligned} i &= q_1 + \dot{q}_1 \\ &= \frac{x - q_1}{2} + \frac{\dot{x} - \dot{q}_1}{2} \end{aligned}$$

Multiplying both sides of this equations by 2, we get:

$$\begin{aligned} 2q_1 + 2\dot{q}_1 &= x - q_1 + \dot{x} - \dot{q}_1 \\ \text{or} \qquad 3\dot{q}_1 &= -3q_1 + x + \dot{x} \\ \text{Hence} \qquad \dot{q}_1 &= -q_1 + \frac{x}{3} + \frac{\dot{x}}{3} \end{aligned}$$

Thus the only state equation is:

$$\dot{q}_1 = -q_1 + \frac{x}{3} + \frac{\dot{x}}{3}$$

The output equation is: $y = -q_1 + x$.
Note that although there are two capacitors, there is only one independent capacitor voltage, because the two capacitors form a loop with the voltage source. In such a case the state equation contains the terms x as well as $\dot{x}$. Similar situation exists when inductors along with current source(s) for a cut set.

10.2-6. Let us choose q_1, q_2 and q_3 as the outputs of the subsystem shown in the figure:

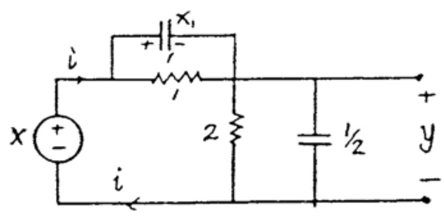

Figure S10.2-5

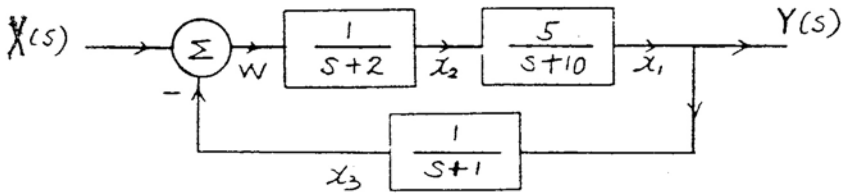

Figure S10.2-6

From the block diagram we obtain:

$$
\begin{aligned}
5q_2 &= \dot{q}_1 + 10q_1 \Longrightarrow \dot{q}_1 = -10q_1 + 5q_2 & (1)\\
q_1 &= \dot{q}_3 + q_3 \Longrightarrow \dot{q}_3 = q_1 - q_3 & (2)\\
w &= \dot{q}_2 + 2q_2 \Longrightarrow \dot{q}_2 = w - 2q_2 & (3)\\
\dot{q}_2 &= -2q_2 - q_3 + x & (4)
\end{aligned}
$$

From (1), (2) and (3) the state equations can be written as:

$$
\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} -10 & 5 & 0 \\ 0 & -2 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x
$$

And the output equation is:

$$
y = q_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}
$$

10.2-7. From Figure P10.2-7., it is easy to write the state equations as:

$$
\begin{aligned}
\dot{q}_1 &= \lambda_1 q_1 \\
\dot{q}_2 &= \lambda_2 q_2 + x_1 \\
\dot{q}_3 &= \lambda_3 q_3 + x_2 \\
\dot{q}_4 &= \lambda_4 q_4 + x_2
\end{aligned}
$$

or

$$
\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

The output equation is:

$$\begin{array}{l} y_1 = q_1 + q_2 \\ y_2 = q_2 + q_3 \end{array} \Longrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

10.2-8.

$$H(s) = \frac{3s + 10}{s^2 + 7s + 12}$$

Direct form II:
We can write the state and output equations straightforward from the transfer function $H(s)$. Thus we get:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x$$

$$y = \begin{bmatrix} 10 & 3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Transposed direct form II: In this case the block diagram can be drawn as shown in Figure S10.2-8a.

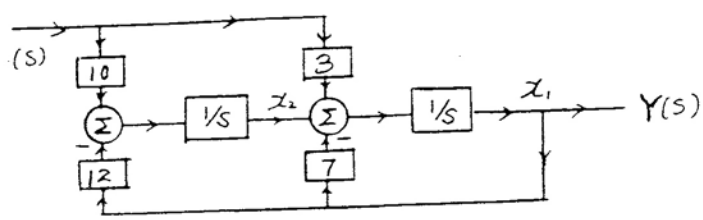

Figure S10.2-8a: transposed direct form II

hence:
$$\begin{aligned} \dot{q}_1 &= -7q_1 + q_2 + 3x \\ \dot{q}_2 &= -12q_1 + 10x \end{aligned}$$

or

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 10 \end{bmatrix} x$$

The output equation is:

$$y = q_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

The cascade form:

$$H(s) = \frac{3s+10}{s^2+7s+12} = \left(\frac{3s+10}{s+4}\right)\left(\frac{1}{s+3}\right)$$

Hence we can write:

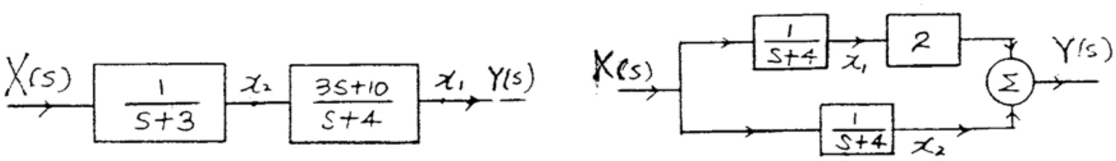

Figure S10.2-8b: cascade and parallel

$$\left.\begin{aligned} \dot{q}_1 + 4q_1 &= 3\dot{q}_2 + 10q_2 \\ \dot{q}_2 &= -3q_2 + x \end{aligned}\right\} \Longrightarrow \begin{aligned} \dot{q}_1 &= -4q_1 - 9q_2 + 10q_2 + 3x \\ \dot{q}_2 &= -3q_2 + x \end{aligned}$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} x$$

and

$$y = q_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Parallel form:

$$H(s) = \frac{2}{s+4} + \frac{1}{s+3}$$

$$\begin{aligned} \dot{q}_1 &= -4q_1 + x \\ \dot{q}_2 &= -3q_1 + x \end{aligned} \Longrightarrow \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x$$

And the output equation is:

$$y = 2q_1 + q_2 = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

10.2-9. (a)

$$H(s) = \frac{4s}{(s+1)(s+2)^2} = \frac{4s}{s^3+5s^2+8s+4}$$

Direct form II:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x$$

And

$$y = \begin{bmatrix} 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Transposed direct form II:

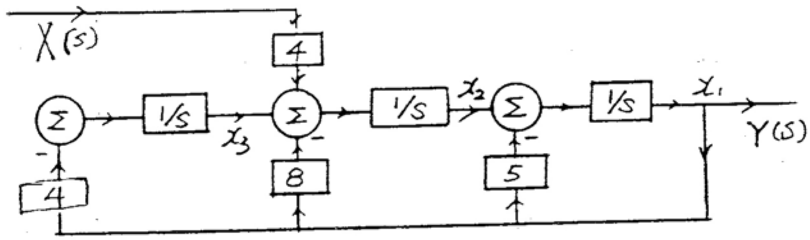

Figure S10.2-9a: transposed direct form II:

In this case:

$$\begin{aligned}\dot{q}_1 &= -5q_1 + q_2\\ \dot{q}_2 &= -8q_1 + q_3 + 4x\\ \dot{q}_3 &= -q_1\end{aligned}$$

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\end{bmatrix} = \begin{bmatrix}-5 & 1 & 0\\ -8 & 0 & 1\\ -4 & 0 & 0\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix} + \begin{bmatrix}0\\ 4\\ 0\end{bmatrix} x$$

And:

$$y = q_1 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix}$$

Cascade form:

$$H(s) = \left(\frac{1}{s+1}\right)\left(\frac{4s}{s+2}\right)\left(\frac{1}{s+2}\right)$$

From the block diagram we have:

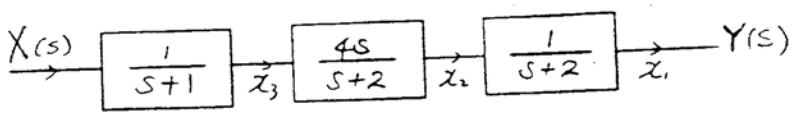

Figure S10.2-9a: cascade

$$\begin{aligned}\dot{q}_1 &= -2q_1 + q_2\\ \dot{q}_2 + 2q_2 &= 4\dot{q}_3\\ \dot{q}_3 &= -q_3 + x\end{aligned} \quad\Longrightarrow\quad \left\{\begin{aligned}\dot{q}_1 &= -2q_1 + q_2\\ \dot{q}_2 &= -4q_3 - 2q_2 + 4x\\ \dot{q}_3 &= -q_3 + x\end{aligned}\right.$$

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\end{bmatrix} = \begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & -4\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix} + \begin{bmatrix}0\\ 4\\ 1\end{bmatrix} x$$

And the output:

$$y = q_1 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix}$$

Parallel form:

$$H(s) = \frac{-4}{s+1} + \frac{4}{s+2} + \frac{8}{(s+2)^2}$$

We have:

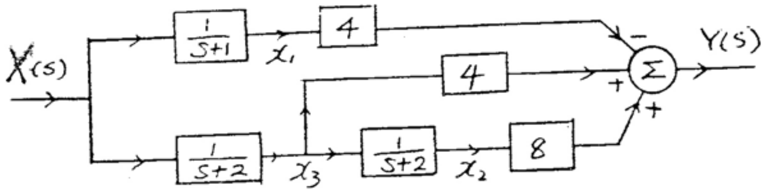

Figure S10.2-9a: parallel

$$\begin{aligned}\dot{q}_1 &= -q_1 + x \\ \dot{q}_2 &= -2q_2 + q_3 \\ \dot{q}_3 &= -2q_3 + x\end{aligned}$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x$$

And the output is:

$$y = -4q_1 + 8q_2 + 4q_3 = \begin{bmatrix} -4 & 8 & 4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

(b)

$$H(s) = \frac{s^3 + 7s^2 + 12s}{(s+1)^3(s+2)} = \frac{s^3 + 7s^2 + 12s}{s^4 + 5s^3 + 9s^2 + 7s + 2}$$

Direct form II:
Straightforward from $H(s)$, we have:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -7 & -9 & -5 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x$$

And the output is:

$$y = \begin{bmatrix} 0 & 12 & 7 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

Transposed direct form II:
We can write the state equation directly from $H(s)$ as in the direct form II.

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 0 & 0 \\ -9 & 0 & 1 & 0 \\ -7 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 7 \\ 12 \\ 0 \end{bmatrix} x$$

And

$$y = q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

Cascade form:

$$H(s) = \frac{s(s+3)(s+4)}{(s+2)(s+1)^3} = \left(\frac{1}{s+2}\right)\left(\frac{s}{s+1}\right)\left(\frac{s+3}{s+1}\right)\left(\frac{s+4}{s+1}\right)$$

Cascade form: From the block diagram we obtain:

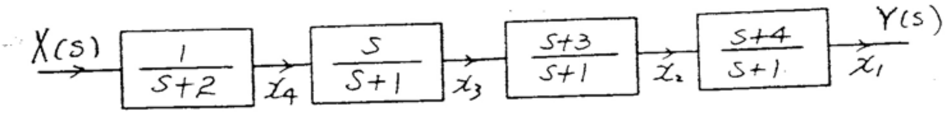

Figure S10.2-9b: cascade

$$\begin{array}{l} \dot{q}_1 + q_1 = \dot{q}_2 + 4q_2 \\ \dot{q}_2 + q_2 = \dot{q}_3 + 3q_3 \\ \dot{q}_3 = -q_3 + \dot{q}_4 \\ \dot{q}_4 = -2q_4 + x \end{array} \Longrightarrow \left\{ \begin{array}{l} \dot{q}_1 = -q_1 + 4q_2 - q_2 + 2q_3 - 2q_4 + x \\ \dot{q}_2 = -q_2 + 3q_3 - q_3 - 2q_4 + x \\ \dot{q}_3 = -q_3 - 2q_4 + x \\ \dot{q}_4 = -2q_4 + x \end{array} \right.$$

hence:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 2 & -2 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} x$$

And

$$y = q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

Parallel form: we can rewrite $H(s)$ as (after partial fraction expansion)

$$H(s) = \frac{6}{s+2} + \frac{11}{s+1} + \frac{7}{(s+1)^2} - \frac{6}{(s+1)^3}$$

$$\begin{aligned} \dot{q}_1 &= -2q_1 + x \\ \dot{q}_2 &= -q_2 + q_3 \\ \dot{q}_3 &= -q_3 + q_4 \\ \dot{q}_4 &= -q_4 + x \end{aligned}$$

From the block diagram, we have

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} x$$

And the output can be written as:

$$y = 6q_1 - 6q_2 + 7q_3 + 11q_4$$

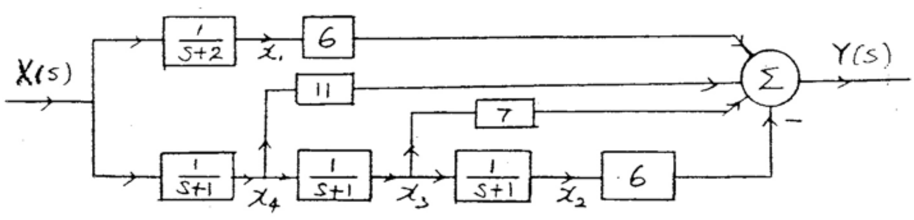

Figure S10.2-9b: parallel

or

$$y = \begin{bmatrix} 6 & -6 & 7 & 11 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

10.3-1.

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{B}\mathbf{x}$$

The solution of the state equation in the frequency domain is given by:

$$\mathbf{Q}(s) = \Phi(s)\mathbf{q}(0) + \Phi(s)\mathbf{B}\mathbf{X}(s)$$

but in this case $x(t) = 0 \Longrightarrow X(s) = 0$
hence: $\mathbf{Q}(s) = \Phi(s)\mathbf{q}(0)$ where $\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1}$

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} \qquad (s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix} \Longrightarrow \Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix} \frac{1}{s^2+3s+2}$$

$$\Phi(s) = \begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{2}{s^2+3s+2} \\ \frac{-1}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{2}{(s+1)(s+2)} \\ \frac{-1}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

And hence: $\mathbf{Q}(s) = \Phi(s)\mathbf{q}(0)$

$$\mathbf{Q}(s) = \begin{bmatrix} \frac{2(s+3)+2}{(s+1)(s+2)} \\ \frac{-2+s}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{2s+8}{(s+1)(s+2)} \\ \frac{s-2}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{6}{s+1} - \frac{4}{s+2} \\ \frac{-3}{s+1} + \frac{4}{s+2} \end{bmatrix}$$

And finally:

$$\mathbf{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \mathcal{L}^{-1}\left[\ \mathbf{Q}(s)\ \right] = \begin{bmatrix} (6e^{-t} - 4e^{-2t})u(t) \\ (-3e^{-t} + 4e^{-2t})u(t) \end{bmatrix}$$

10.3-2.

$$\mathbf{Q}(s) = \Phi(s)\mathbf{q}(0) + \Phi(s)\mathbf{B}\mathbf{X}(s) = \Phi(s)[\mathbf{q}(0) + \mathbf{B}\mathbf{X}(s)]$$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+5 & 6 \\ -1 & s \end{bmatrix} \quad \text{and} \quad \Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2+5s+6} \begin{bmatrix} s & -6 \\ 1 & s+5 \end{bmatrix}$$

$$\Phi(s) = \begin{bmatrix} \frac{s}{(s+3)(s+2)} & \frac{-6}{(s+3)(s+2)} \\ \frac{1}{(s+3)(s+2)} & \frac{s+5}{(s+3)(s+2)} \end{bmatrix}$$

And hence:

$$\begin{aligned} \mathbf{Q}(s) = \Phi(s)\left[\mathbf{q}(0) + \mathbf{B}\mathbf{X}(s)\right] &= \begin{bmatrix} \frac{s}{(s+3)(s+2)} & \frac{-6}{(s+3)(s+2)} \\ \frac{1}{(s+3)(s+2)} & \frac{s+5}{(s+3)(s+2)} \end{bmatrix} \begin{bmatrix} 5 + \frac{100}{s^2+10^4} \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{-34.02}{s+2} + \frac{39.03}{s+3} - \frac{10^{-2}s}{s^2+10^4} \\ \frac{17.01}{s+2} - \frac{10.01}{s+3} - \frac{0}{s^2+10^4} \end{bmatrix} \end{aligned}$$

hence: $\mathbf{q}(t) = \mathcal{L}^{-1}(\mathbf{Q}(s))$

$$\mathbf{Q}(s) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} -34.02e^{-2t} + 39.03e^{-3t} - 0.01\cos 100t \\ 17.01e^{-2t} - 10.01e^{-3t} \end{bmatrix}$$

10.3-3.

$$\mathbf{Q}(s) = \Phi(s)[\mathbf{q}(0) + \mathbf{B}\mathbf{X}(s)]$$

$$\begin{aligned} (s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+2 & 0 \\ -1 & s+1 \end{bmatrix} \text{ and } \Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} &= \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+1 & 0 \\ 1 & s+2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{s+1} \end{bmatrix} \end{aligned}$$

Also $x(t) = u(t) \Longrightarrow X(s) = \frac{1}{s}$

Hence: $\mathbf{B}\mathbf{X}(s) = \begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix}$ And $\mathbf{q}(0) + \mathbf{B}\mathbf{X}(s) = \begin{bmatrix} \frac{1}{s} \\ -1 \end{bmatrix}$

And thus:

$$\begin{aligned} \mathbf{Q}(s) = \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} \frac{1}{s} \\ -1 \end{bmatrix} &= \begin{bmatrix} \frac{1}{s(s+2)} \\ \frac{1}{s(s+1)(s+2)} - \frac{1}{s+1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2s} - \frac{1}{2(s+2)} \\ \frac{1}{2s} - \frac{2}{s+1} - \frac{1}{2(s+2)} \end{bmatrix} \end{aligned}$$

Hence:

$$\mathbf{q}(t) = \mathcal{L}^{-1}(\mathbf{Q}(s)) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} (\frac{1}{2} - \frac{1}{2}e^{-2t})u(t) \\ (\frac{1}{2} - 2e^{-t} + \frac{1}{2}e^{-2t})u(t) \end{bmatrix}$$

10.3-4.

$$\mathbf{Q}(s) = \Phi(s)[\mathbf{q}(0) + \mathbf{B}\mathbf{X}(s)]$$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix} \text{ and } \Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix}$$

$$\Phi(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$\text{and} \qquad x(t) = \begin{bmatrix} u(t) \\ \delta(t) \end{bmatrix} \Longrightarrow \mathbf{X}(s) = \begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix}$$

$$\mathbf{BX}(s) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{s+1}{s} \\ 1 \end{bmatrix}$$

$$\text{and:} \qquad \mathbf{q}(0) + \mathbf{BX}(s) = \begin{bmatrix} \frac{s+1}{s} + 1 \\ 3 \end{bmatrix} \begin{bmatrix} \frac{2s+1}{s} + 1 \\ 3 \end{bmatrix}$$

$$\begin{aligned} \mathbf{Q}(s) = \Phi(s)[\mathbf{q}(0) + \mathbf{BX}(s)] &= \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} \frac{2s+1}{s} \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{(2s+1)(s+2)+3s}{s(s+1)(s+2)} \\ \frac{3}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} + \frac{4}{s+1} - \frac{3}{s+2} \\ \frac{3}{s+2} \end{bmatrix} \end{aligned}$$

And hence:

$$\mathbf{q}(t) = \mathcal{L}^{-1}(\mathbf{Q}(s)) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} (1 + 4e^{-t} - 3e^{-2t})u(t) \\ 3e^{-2t}u(t) \end{bmatrix}$$

10.3-5.

$$\mathbf{Y}(s) = \mathbf{CQ}(s) + \mathbf{DX}(s) = \mathbf{C}\Phi(s)\mathbf{q}(0) + [\mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D}]\mathbf{X}(s)]$$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix} \text{ and } \Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}$$

$$\Phi(s) = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix} \qquad \text{and} \qquad \mathbf{BX}(s) = \begin{bmatrix} \frac{1}{s} \\ 1 \end{bmatrix}$$

Since $\mathbf{D} = 0 \Longrightarrow Y(s) = \mathbf{C}\Phi(s)[\mathbf{q}(0) + \mathbf{BX}(s)]$

$$\text{So} \qquad \mathbf{q}(0) + \mathbf{BX}(s) = \begin{bmatrix} 2 + \frac{1}{s} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2s+1}{s} \\ 0 \end{bmatrix}$$

$$\text{and} \qquad \Phi(s)[\mathbf{q}(0) + \mathbf{BX}(s)] = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} \frac{2s+1}{s} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2s+1}{(s+1)(s+2)} \\ \frac{-2(2s+1)}{s(s+1)(s+2)} \end{bmatrix}$$

$$\mathbf{Y}(s) = \mathbf{C}\Phi(s)[\mathbf{q}(0) + \mathbf{BX}(s)] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2s+1}{(s+1)(s+2)} \\ \frac{-2(2s+1)}{s(s+1)(s+2)} \end{bmatrix}$$

$$\mathbf{Y}(s) = \frac{-4s-2}{s(s+1)(s+2)} = \frac{-1}{s} - 2 \cdot \frac{1}{s+1} + \frac{3}{s+2}$$

$$y(t) = \mathcal{L}^{-1}[y(s)] = (-1 - 2e^{-t} + 3e^{-2t})u(t)$$

10.3-6.

$$\mathbf{y}(s) \quad = \quad \mathbf{CQ}(s) + \mathbf{DX}(s) = \mathbf{C}\Phi(s)\mathbf{q}(0) + [\mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D}]\mathbf{X}(\mathrm{s})$$

$$= \mathbf{C}\{\Phi(s)[\mathbf{q}(0) + \mathbf{BX}(s)]\} + \mathbf{DX}(s)$$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+1 & -1 \\ 1 & s+1 \end{bmatrix} \text{ and } \Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2+2s+2}\begin{bmatrix} s+1 & 1 \\ -1 & s+1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+1}{s^2+2s+2} & \frac{1}{s^2+2s+2} \\ \frac{-1}{s^2+2s+2} & \frac{s+1}{s^2+2s+2} \end{bmatrix}$$

$$\mathbf{BX}(s) = \begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix} \qquad \text{and} \qquad \mathbf{q}(0) + \mathbf{BX}(s) = \begin{bmatrix} 2 \\ \frac{s+1}{s} \end{bmatrix}$$

Hence $\quad \Phi(s)[\mathbf{q}(0) + \mathbf{BX}(s)] = \begin{bmatrix} \frac{s+1}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ \frac{-1}{(s+1)^2+1} & \frac{s+1}{(s+1)^2+1} \end{bmatrix}\begin{bmatrix} 2 \\ \frac{s+1}{s} \end{bmatrix}$

$$= \begin{bmatrix} \frac{2(s+1)}{(s+1)^2+1} + \frac{s+1}{s[(s+1)^2+1]} \\ \frac{-2}{(s+1)^2+1} + \frac{(s+1)^2}{s[(s+1)^2+1]} \end{bmatrix} = \begin{bmatrix} \frac{2s^2+3s+1}{s[(s+1)^2+1]} \\ \frac{s^2+1}{s[(s+1)^2+1]} \end{bmatrix}$$

$$\mathbf{C}\Phi(s)[\mathbf{q}(0) + \mathbf{BX}(s)] = \begin{bmatrix} 1 & 1 \end{bmatrix}\Phi(s)[\mathbf{q}(0) + \mathbf{BX}(s)] = \left[\frac{2s^2+3s+1+s^2+1}{s\{(s+1)^2+1\}}\right]$$

Also: $\quad \mathbf{DX}(s) = \frac{1}{s}$

Hence $\quad \mathbf{Y}(s) = \mathbf{C}\Phi(s)[\mathbf{q}(0) + \mathbf{BX}(s)] + \mathbf{DX}(s) = \frac{3s^2+3s+2}{s\{(s+1)^2+1\}} + \frac{1}{s} = \frac{4s^2+5s+4}{s\{(s+1)^2+1\}}$

$$Y(s) = \frac{4s^2+5s+4}{s(s^2+2s+2)} = \frac{\mathbf{C}}{s} + \frac{\mathbf{A}s+\mathbf{B}}{s^2+2s+2}$$

Using partial fractions and clearing fractions we get:

$$Y(s) = \frac{2}{s} + \frac{2s+1}{(s+1)^2+1^2} = \frac{2}{s} + 2\frac{(s+1)}{(s+1)^2+1^2} - \frac{1}{(s+1)^2+1^2}$$

$$\text{and} \qquad y(t) = \mathcal{L}^{-1}[Y(s)] = (2 + 2e^{-t}\cos t - e^{-t}\sin t)u(t)$$

10.3-7.

$$H(s) = \left(\frac{1}{s+3}\right)\left(\frac{3s+10}{s+4}\right) = \frac{3s+10}{s^2+7s+12}$$

This is the same transfer function as in Prob. 10.2-8, where the cascade form state equations were found to be

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}x$$

$$\text{And} \qquad y = q_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

In this case

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s+4 & -1 \\ 0 & s+3 \end{bmatrix} \text{ and } \Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{(s+3)(s+4)}\begin{bmatrix} s+3 & 1 \\ 0 & s+4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+4} & \frac{1}{(s+3)(s+4)} \\ 0 & \frac{1}{s+3} \end{bmatrix}$$

Also in our case:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{D} = 0$$

Hence $$\Phi(s)\mathbf{B} = \begin{bmatrix} \frac{1}{s+4} & \frac{1}{(s+3)(s+4)} \\ 0 & \frac{1}{s+3} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3(s+3)+1}{(s+3)(s+4)} \\ \frac{1}{s+3} \end{bmatrix} = \begin{bmatrix} \frac{3s+10}{(s+3)(s+4)} \\ \frac{1}{s+3} \end{bmatrix}$$

And $$\mathbf{C}\phi(s)\mathbf{B} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{3s+10}{(s+3)(s+4)} \\ \frac{1}{s+3} \end{bmatrix} = \frac{3s+10}{(s+3)(s+4)}$$

Hence: $$\mathbf{C}\Phi(s)\mathbf{B} = \frac{3s+10}{s^2+7s+12} = H(s)$$

10.3-8.

$$H(s) = \mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D}$$

in Prob. 10.3-5 we have found $\Phi(s)$. And

$$\Phi(s)\mathbf{B} = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{s}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} \end{bmatrix}$$

Hence

$$\mathbf{C}\Phi(s)\mathbf{B} = \begin{bmatrix} 1 & 0 \end{bmatrix} \Phi(s)\mathbf{B} = \frac{-2}{(s+1)(s+2)} \qquad \text{and since} \qquad \mathbf{D} = 0$$

$$H(s) = \mathbf{C}\Phi(s)\mathbf{B} = \frac{-2}{s^2+3s+2}$$

10.3-9. From Prob. 10.3-6,

$$\Phi(s)\mathbf{B} = \begin{bmatrix} \frac{s+1}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ \frac{-1}{(s+1)^2+1} & \frac{s+1}{(s+1)^2+1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{(s+1)^2+1} \\ \frac{s+1}{(s+1)^2+1} \end{bmatrix}$$

And:

$$\mathbf{C}\Phi(s)\mathbf{B} = \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{(s+1)^2+1} \\ \frac{s+1}{(s+1)^2+1} \end{bmatrix} = \frac{s+1+1}{(s+1)^2+1} = \frac{s+2}{(s+1)^2+1}$$

And

$$H(s) = \mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D} = \frac{s+2}{(s+1)^2+1} + 1 = \frac{s^2+3s+4}{s^2+2s+2}$$

10.3-10. In this case:

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix} \quad \text{and} \quad \Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} \;=\; \frac{1}{(s+1)^2} \begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix}$$

And:

$$\Phi(s)\mathbf{B} = \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{s+2}{(s+1)^2} \\ \frac{s}{(s+1)^2} & \frac{-1}{(s+1)^2} \end{bmatrix}$$

$$\mathbf{C}\Phi(s)\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{s+2}{(s+1)^2} \\ \frac{s}{(s+1)^2} & \frac{-1}{(s+1)^2} \end{bmatrix}$$

$$\text{and} \quad H(s) = \mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D} = \begin{bmatrix} \frac{2s+1}{(s+1)^2} & \frac{s}{(s+1)^2} \\ \frac{4+s}{(s+1)^2} & \frac{4s+7}{(s+1)^2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

10.3-11. In the time domain, the solution $\mathbf{q}(t)$ is given by:

$$\mathbf{q}(t) = e^{\mathbf{A}t}\mathbf{q}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{x}(\tau)\, d\tau$$

$$\mathbf{q}(t) = e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t)$$

where:

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \mathcal{L}^{-1}(\Phi(s))$$

From Prob. 10.3-1 we have found:

$$\Phi(s) = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{2}{(s+1)(s+2)} \\ \frac{-1}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{2}{s+1} - \frac{2}{s+2} \\ \frac{-1}{s+1} + \frac{1}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}(\Phi(s)) = \begin{bmatrix} 2e^{-t} - e^{-2t} & 2e^{-t} - 2e^{-2t} \\ -e^{-t} + e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$e^{\mathbf{A}t}\mathbf{q}(0) = \begin{bmatrix} 4e^{-t} - 2e^{-2t} + 2e^{-t} - 2e^{-2t} \\ -2e^{-t} + 2e^{-2t} - e^{-t} + 2e^{-2t} \end{bmatrix} = \begin{bmatrix} 6e^{-t} - 4e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix}$$

$$\text{Also:} \qquad \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \times 0 = 0$$

$$\text{hence:} \qquad \mathbf{q}(t) = \begin{bmatrix} (6e^{-t} - 4e^{-2t})u(t) \\ (-3e^{-t} + 4e^{-2t})u(t) \end{bmatrix}$$

which is the same thing as in Prob. 10.3-1.

10.3-12. From Prob. 10.3-2,

$$\Phi(s) = \begin{bmatrix} \frac{s}{(s+2)(s+3)} & \frac{-6}{(s+2)(s+3)} \\ \frac{1}{(s+2)(s+3)} & \frac{s+5}{(s+2)(s+3)} \end{bmatrix} = \begin{bmatrix} \frac{-2}{s+2} + \frac{3}{s+3} & \frac{-6}{s+2} + \frac{6}{s+3} \\ \frac{1}{s+2} - \frac{1}{s+3} & \frac{3}{s+2} - \frac{2}{s+3} \end{bmatrix}$$

$$\text{Hence:} \qquad e^{\mathbf{A}t} = \mathcal{L}^{-1}(\Phi(s)) = \begin{bmatrix} -2e^{-2t} + 3e^{-3t} & -6e^{-2t} + 6e^{-3t} \\ e^{-2t} - e^{-3t} & 3e^{-2t} - 2e^{-3t} \end{bmatrix}$$

And:

$$e^{\mathbf{A}t}\mathbf{q}(0) = \begin{bmatrix} -10e^{-2t} + 15e^{-3t} - 24e^{-2t} + 24e^{-3t} \\ 5e^{-2t} - 5e^{-3t} + 12e^{-2t} - 8e^{-3t} \end{bmatrix} = \begin{bmatrix} -34e^{-2t} + 39e^{-3t} \\ 17e^{-2t} - 13e^{-3t} \end{bmatrix}$$

$$\text{Also:} \qquad \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin 100t = \begin{bmatrix} \sin 100t \\ 0 \end{bmatrix}$$

$$\begin{aligned} \text{And} \qquad e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) &= \begin{bmatrix} -2e^{-2t} * \sin 100t + 3e^{-3t} * \sin 100t \\ e^{-2t} * \sin 100t - e^{-3t} * \sin 100t \end{bmatrix} \\ &= \begin{bmatrix} -\frac{2e^{-2t}}{100} + \frac{2\cos 100t}{100} + \frac{3e^{-3t}}{100} - \frac{3\cos 100t}{100} \\ +\frac{e^{-2t}}{100} - \frac{\cos 100t}{100} - \frac{e^{-3t}}{100} + \frac{\cos 100t}{100} \end{bmatrix} \\ &= \begin{bmatrix} -0.02e^{-2t} + 0.03e^{-3t} - 0.01\cos 100t \\ 0.01e^{-2t} - 0.01e^{-3t} \end{bmatrix} \end{aligned}$$

Hence:

$$\mathbf{q}(t) = e^{\mathbf{A}t}[\mathbf{q}(0)] + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} -34.02e^{-2t} + 39e^{-3t} + 0.01\cos 100t \\ 17.01e^{-2t} - 10.0e^{-3t} \end{bmatrix}$$

Hence

$$\mathbf{q}(t) = e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x} = \begin{bmatrix} -34.02e^{-2t} + 39.03e^{-3t} + 0.01\cos 100t \\ 17.01e^{-2t} - 10.01e^{-3t} \end{bmatrix}$$

This is the same result as in Prob. 10.3-2.

10.3-13. From Prob. 10.3-3,

$$\Phi(s) = \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} \end{bmatrix}$$

$$\text{Hence:} \qquad e^{\mathbf{A}t} = \mathcal{L}^{-1}(\Phi(s)) = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix}$$

$$\text{And:} \qquad e^{\mathbf{A}t}\mathbf{q}(0) = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -e^{-t} \end{bmatrix}$$

$$\text{Also:} \qquad \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) = \begin{bmatrix} u(t) \\ 0 \end{bmatrix}$$

$$\text{And:} \qquad e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} e^{-2t} * u(t) \\ e^{-t} * u(t) - e^{-2t} * u(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1 - e^{-2t})u(t) \\ (1 - e^{-t}) - \frac{1}{2}(1 - e^{-2t}) \end{bmatrix}$$

And hence:

$$\begin{aligned} \mathbf{q}(t) = e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) &= \begin{bmatrix} 0 \\ -e^{-t} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - \frac{1}{2}e^{-2t} \\ \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} - \frac{1}{2}e^{-2t} \\ \frac{1}{2} + \frac{1}{2}e^{-2t} - 2e^{-t} \end{bmatrix} \end{aligned}$$

10.3-14. From Prob. 10.3-4,

$$\Phi(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+2} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$\text{Hence:} \qquad e^{\mathbf{A}t} = \mathcal{L}^{-1}(\Phi(s)) = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

$$e^{\mathbf{A}t}\mathbf{q}(0) = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ 2e^{-2t} \end{bmatrix}$$

$$\mathbf{B}\mathbf{x}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u(t) \\ \delta(t) \end{bmatrix} = \begin{bmatrix} u(t) + \delta(t) \\ \delta(t) \end{bmatrix}$$

$$\text{And} \qquad e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} e^{-t} * u(t) + e^{-t} * \delta(t) + e^{-t} * \delta(t) - e^{-2t} * \delta(t) \\ e^{-2t} * \delta(t) \end{bmatrix}$$

$$e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} (1 - e^{-t}) + e^{-t} + e^{-t} - e^{-2t} \\ e^{-2t} \end{bmatrix} \begin{bmatrix} 1 + e^{-t} - e^{-2t} \\ e^{-2t} \end{bmatrix}$$

$$\begin{aligned} \text{And hence:} \quad \mathbf{q}(t) = e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) \quad &= \begin{bmatrix} 3e^{-t} - 2e^{-2t} + 1 + e^{-t} - e^{-2t} \\ 2e^{-2t} + e^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} 1 + 4e^{-t} - 3e^{-2t} \\ 3e^{-2t} \end{bmatrix} \end{aligned}$$

10.3-15. From Prob. 10.3-5,

$$\Phi(s) = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{-1}{s+1} + \frac{2}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{2}{s+1} - \frac{1}{s+2} \end{bmatrix}$$

$$\text{And } y(t) \text{ is given by:} \qquad y(t) = \mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t}\mathbf{B} * x(t)] + \mathbf{D}\mathbf{x}(t)$$

$$\text{where:} \qquad e^{\mathbf{A}t} = L^{-1}(\Phi(s)) = \begin{bmatrix} -e^{-t} + 2e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix}$$

$$\text{And:} \qquad e^{\mathbf{A}t}\mathbf{q}(0) = e^{\mathbf{A}t} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2e^{-t} + 4e^{-2t} \\ -2e^{-t} + 4e^{-2t} \end{bmatrix}$$

$$e^{\mathbf{A}t}\mathbf{B} = e^{\mathbf{A}t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -e^{-t} + 2e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\begin{aligned} e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} -e^{-t} + 2e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{bmatrix} * u(t) \quad &= \begin{bmatrix} -e^{-t} * u(t) + e^{-2t} * u(t) \\ -2e^{-t} * u(t) + 2e^{-2t} * u(t) \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} - e^{-2t} \\ -1 + 2e^{-t} - e^{-2t} \end{bmatrix} \end{aligned}$$

$$\text{Since} \qquad \mathbf{D} = 0 \Longrightarrow y(t) = \mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t)]$$

$$\begin{aligned}
\text{And:} \qquad e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) &= \begin{bmatrix} -2e^{-t} + 4e^{-2t} \\ -4e^{-t} + 4e^{-2t} \end{bmatrix} + \begin{bmatrix} e^{-t} - e^{-2t} \\ -1 + 2e^{-t} - e^{-2t} \end{bmatrix} \\
&= \begin{bmatrix} -e^{-t} + 3e^{-2t} \\ -1 - 2e^{-t} + 3e^{-2t} \end{bmatrix}
\end{aligned}$$

And hence:

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} + \begin{bmatrix} -e^{-t} + 3e^{-2t} \\ -1 - 2e^{-t} + 3e^{-2t} \end{bmatrix} = (-1 - 2e^{-t} + 3e^{-2t})u(t)$$

10.3-16.

$$y(t) = \mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t)] + \mathbf{D}\mathbf{x}(t)$$

From Prob. 10.3-6 we have obtained:

$$\Phi(s) = \begin{bmatrix} \frac{s+1}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ \frac{-1}{(s+1)^2+1} & \frac{s+1}{(s+1)^2+1} \end{bmatrix}$$

$$\text{Hence:} \qquad e^{\mathbf{A}t} = L^{-1}(\Phi(s)) = \begin{bmatrix} e^{-t}\cos t & e^{-t}\sin t \\ -e^{-t}\sin t & e^{-t}\cos t \end{bmatrix}$$

$$e^{\mathbf{A}t}\mathbf{q}(0) = e^{\mathbf{A}t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{-t}\cos t + e^{-t}\sin t \\ -2e^{-t}\sin t + e^{-t}\cos t \end{bmatrix}$$

$$\text{And:} \qquad e^{\mathbf{A}t}\mathbf{B} = e^{\mathbf{A}t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t}\sin t \\ e^{-t}\cos t \end{bmatrix}$$

$$\text{And:} \qquad e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} e^{-t}\sin t * u(t) \\ e^{-t}\cos t * u(t) \end{bmatrix} = \begin{bmatrix} \frac{\cos(\frac{\pi}{2}-\phi)}{\sqrt{2}} - \frac{e^{-t}}{\sqrt{2}}\cos(t - \frac{\pi}{2} - \phi) \\ \frac{\cos(-\phi)}{\sqrt{2}} - \frac{e^{-t}}{\sqrt{2}}\cos(t - \phi) \end{bmatrix}$$

where: $\phi = \tan^{-1}\frac{-1}{1} = -\frac{\pi}{4}$.

$$\text{And hence:} \qquad e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} \frac{1}{2} + \frac{3}{2}e^{-t}\cos t + \frac{1}{2}e^{-t}\sin t \\ \frac{1}{2} + \frac{1}{2}e^{-t}\cos t - \frac{3}{2}e^{-t}\sin t \end{bmatrix}$$

And

$$\begin{aligned}
y(t) &= \mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t)] + \mathbf{D}\mathbf{x}(t) \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} [e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t)] + u(t) \\
&= [1 + 2e^{-t}\cos t - e^{-t}\sin t + 1]u(t) = [2 + 2e^{-t}\cos t - e^{-t}\sin t]u(t)
\end{aligned}$$

10.3-17.

$$H(s) = \frac{3s + 10}{s^2 + 7s + 12}$$

From Eq. (10.65) we have:

$$\mathbf{h}(t) = \mathbf{C}\phi(t)\mathbf{B} + \mathbf{D}\delta(t) \qquad \text{where} \qquad \phi(t) = e^{\mathbf{A}t}$$

From Prob. 10.3-7 we obtained $\Phi(s)$ as:

$$\Phi(s) = \begin{bmatrix} \frac{1}{s+4} & \frac{1}{(s+3)(s+4)} \\ 0 & \frac{1}{s+1} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{D} = 0$$

$$\text{hence:} \qquad e^{\mathbf{A}t} = L^{-1}(\Phi(s)) = \begin{bmatrix} e^{-4t} & e^{-3t} - e^{-4t} \\ 0 & e^{-3t} \end{bmatrix}$$

$$\text{And:} \qquad \phi(t)\mathbf{B} = \begin{bmatrix} 3e^{-4t} + e^{-3t} - e^{-4t} \\ e^{-3t} \end{bmatrix} = \begin{bmatrix} e^{-3t} + 2e^{-4t} \\ e^{-3t} \end{bmatrix}$$

$$\begin{aligned} \text{Since} \qquad \mathbf{D} = 0, \quad h(t) = \mathbf{C}\phi(t)\mathbf{B} &= [1 \quad 0]\phi(t)\mathbf{B} \\ &= (e^{-3t} + 2e^{-4t})u(t) \end{aligned}$$

10.3-18. From Prob. 10.3-6,

$$\Phi(s) = \begin{bmatrix} \frac{s+1}{(s+1)^2+1} & \frac{1}{(s+1)^2+1} \\ \frac{-1}{(s+1)^2+1} & \frac{s+1}{(s+1)^2+1} \end{bmatrix}$$

$$\text{hence:} \qquad \phi(t) = L^{-1}(\Phi(s)) = \begin{bmatrix} e^{-t}\cos t & e^{-t}\sin t \\ -e^{-t}\sin t & e^{-t}\cos t \end{bmatrix}$$

$$\phi(t)\mathbf{B} = \phi(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t}\sin t \\ e^{-t}\cos t \end{bmatrix}$$

$$\text{And:} \qquad \mathbf{C}\phi(t)\mathbf{B} = \begin{bmatrix} 1 & 1 \end{bmatrix}\phi(t)\mathbf{B} = (e^{-t}\sin t + e^{-t}\cos t)$$

And

$$h(t) = \mathbf{C}\phi(t)\mathbf{B} + \delta(t) = \delta(t) + (e^{-t}\sin t + e^{-t}\cos t)u(t)$$

10.3-19. From Prob. 10.3-10,

$$\phi(s) = \begin{bmatrix} \frac{2s+1}{(s+1)^2} & \frac{s}{(s+1)^2} \\ \frac{4+s}{(s+1)^2} & \frac{4s+7}{(s+1)^2} \\ \frac{s+2}{s+1} & \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{(s+1)^2} & \frac{1}{s+1} - \frac{1}{(s+1)^2} \\ \frac{1}{s+1} + \frac{3}{(s+1)^2} & \frac{4}{s+1} + \frac{3}{(s+1)^2} \\ 1 + \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

And hence: the unit inputs response $\mathbf{h}(t)$ is given by:

$$\mathbf{h}(t) = \mathcal{L}^{-1}\{\mathbf{H}(s)\} = \begin{bmatrix} 2e^{-t} - te^{-t} & e^{-t} - te^{-t} \\ e^{-t} + 3te^{-t} & 4e^{-t} + 3te^{-t} \\ \delta(t) + e^{-t} & e^{-t} \end{bmatrix}$$

10.4-1.

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}x$$

$$\mathbf{w} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \mathbf{Pq}$$

The new state equation of the system is given by:

$$\dot{\mathbf{w}} = \mathbf{PAP}^{-1}\mathbf{w} + \mathbf{P}\dot{\mathbf{B}} = \hat{\mathbf{A}}\mathbf{w} + \hat{\mathbf{B}}x$$

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \Longrightarrow \mathbf{PA} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$$

$$\mathbf{PAP}^{-1} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix}$$

And: $$\mathbf{PB} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}x$$

Hence $$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}x$$

Eigenvalues in the original system:
The eigenvalues are the roots of the characteristic equation, thus: in the original system:

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -1 \\ 1 & s+1 \end{vmatrix} = (s+1)s + 1 = s^2 + s + 1 = 0$$

The roots are given by: $$s_{1,2} = \frac{-1 \pm j\sqrt{3}}{2}$$

In the transformed system, the characteristic equation is given by:

$$|s\mathbf{I} - \hat{\mathbf{A}}| = \left|\begin{bmatrix} s+2 & -1 \\ 3 & s-1 \end{bmatrix}\right| = (s+2)(s-1) + 3 = s^2 - s + 2s - 2 + 3 = s^2 + s + 1$$

And the eigenvalues are given by:

$$s_{1,2} = \frac{-1 \pm j\sqrt{3}}{2}$$

which are the same as in the original system.

10.4-2.

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}x(t)$$

(a) The characteristic equation is given by:

$$|s\mathbf{I} - \mathbf{A}| = 0 = \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = s(s+3) - 2 = s^2 + 3s + 2 = (s+1)(s+2) = 0$$

$\lambda_1 = -1$ and $\lambda_2 = -2$ are the eigenvalues. And

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$\mathbf{w} = \mathbf{Pq}$ and $\dot{\mathbf{w}} = \mathbf{PAP}^{-1}\mathbf{w} + \mathbf{PB}x = \Lambda\mathbf{w} + \hat{\mathbf{B}}x$
Hence we have to find $\mathbf{P}$ such that: $\mathbf{PAP}^{-1} = \Lambda$ or $\Lambda\mathbf{P} = \mathbf{PA}$

$$\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\left.\begin{array}{l} -p_{11} = 2p_{12} \Longrightarrow p_{11} = 2p_{12} \\ p_{12} = 3p_{12} - p_{11} \\ p_{21} = p_{22} \\ p_{22} = 3p_{22} - p_{21} \end{array}\right\} \Longrightarrow \begin{array}{l} p_{12} = 3p_{12} - 2p_{12} \\ \text{If we choose} \quad p_{11} = 2 \quad \text{then} \quad p_{12} = 1 \\ \text{And if} \quad p_{21} = 1 \quad \text{then} \quad p_{22} = 1 \end{array}$$

$$\text{Therefore} \qquad \mathbf{P} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\text{and hence} \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 2q_1 + q_2 \\ q_1 + q_2 \end{bmatrix}$$

(b) $\mathbf{y} = \mathbf{Cq} + \mathbf{D}x$ where $\mathbf{D} = 0 \Longrightarrow \mathbf{y} = \mathbf{Cq}$.
we have $\mathbf{w} = \mathbf{Pq} \Longrightarrow \mathbf{P}^{-1}\mathbf{w} = \mathbf{q} \Longrightarrow \mathbf{y} = \mathbf{CP}^{-1}\mathbf{w}$. hence:

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad \mathbf{CP}^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 5 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_2 \\ 5w_2 - 3w_1 \end{bmatrix}$$

10.4-3.

$$\dot{\mathbf{q}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \mathbf{q} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x$$

The characteristic equation is given by:

$$\begin{aligned} |s\mathbf{I} - \mathbf{A}| &= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 2 & s+3 \end{vmatrix} \\ &= s\{(s)(s+3) + 2\} = s(s^2 + 3s + 2) = s(s+1)(s+2) = 0 \end{aligned}$$

Hence the eigenvalues are: $\lambda_1 = 0$, $\lambda_2 = -1$ and $\lambda_3 = -2$.

$$\text{And} \qquad \Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

In the transformed system we have: $\mathbf{w} = \mathbf{Pq}$ and $\dot{\mathbf{w}} = \mathbf{PAP}^{-1}\mathbf{w} + \mathbf{PB}x$
We have to find $\mathbf{P}$ such that: $\mathbf{PAP}^{-1} = \Lambda$ or $\Lambda\mathbf{P} = \mathbf{PA}$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}$$

$$\Longrightarrow \left\{ \begin{array}{lll} p_{21} = 0 \qquad p_{31} = 0 & \text{if} & p_{11} = 2 \text{ then } p_{13} = 1 \text{ and } p_{12} = 3 \\ p_{11} = p_{13} & & \\ p_{12} = 3p_{13} & & \\ p_{22} = 2p_{23} - p_{21} & \text{if} & p_{23} = 1, \text{ then } p_{22} = 2 \text{ and } p_{23} = 1 \\ p_{23} = 3p_{23} - p_{22} & \text{if} & p_{32} = 1 \text{ then } p_{33} = 1 \\ 2p_{32} = 2p_{33} - p_{31} & & \\ 2p_{33} = 3p_{33} - p_{32} \Longrightarrow p_{33} = p_{32} & & \end{array} \right.$$

$$\mathbf{w} = \mathbf{P}\mathbf{q} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

10.4-4.

$$y(t) = \mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t)]$$

where: $$e^{\mathbf{A}t} = \mathcal{L}^{-1}(\phi(s))$$

$$(\phi(s))^{-1} = [s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+3 & 0 \\ 0 & 0 & s+2 \end{bmatrix}$$

$$\phi(t) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s+3} & 0 \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix} \text{ and } e^{\mathbf{A}t} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

And: $$e^{\mathbf{A}t}\mathbf{q}(0) = e^{\mathbf{A}t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ 2e^{-3t} \\ e^{-2t} \end{bmatrix}$$

$$e^{\mathbf{A}t}\mathbf{B} = e^{\mathbf{A}t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ e^{-3t} \\ e^{-2t} \end{bmatrix}$$

and: $$e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = e^{\mathbf{A}t} * \mathbf{B}u(t) = \begin{bmatrix} e^{-t} * u(t) \\ e^{-3t} * u(t) \\ e^{-2t} * u(t) \end{bmatrix} = \begin{bmatrix} (1 - e^{-t})u(t) \\ \frac{1}{3}(1 - e^{-3t})u(t) \\ \frac{1}{2}(1 - e^{-2t})u(t) \end{bmatrix}$$

Hence: $$e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t) = \begin{bmatrix} e^{-t} + 1 - e^{-t} \\ 2e^{-3t} + \frac{1}{3} - \frac{1}{3}e^{-3t} \\ e^{-2t} + \frac{1}{2} - \frac{1}{2}e^{-2t} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{3} + \frac{5}{3}e^{-3t} \\ \frac{1}{2} + \frac{1}{2}e^{-2t} \end{bmatrix}$$

And finally: $$y(t) = \mathbf{C}[e^{\mathbf{A}t}\mathbf{q}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{x}(t)] \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix}$$

$$y(t) = \left(1 + 1 + 5e^{-3t} + \frac{1}{2} + \frac{1}{2}e^{-2t}\right) = \left(\frac{5}{2} + \frac{1}{2}e^{-2t} + 5e^{-3t}\right)$$

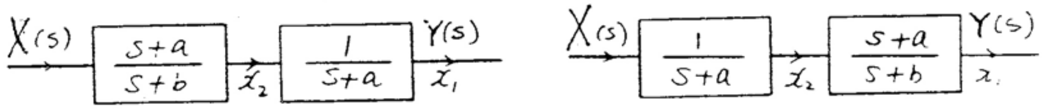

Figure S10.5-1: a and b

10.5-1. (a) state equations:

$$\begin{aligned} \dot{q}_2 + bq_2 = (a-b)x \quad &\Longrightarrow \dot{q}_2 = -bq_2 + (a-b)x \\ \dot{q}_1 + aq_1 = q_2 + x \quad &\Longrightarrow \dot{q}_1 = -aq_1 + q_2 + x \end{aligned}$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -a & 1 \\ 0 & -b \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 1 \\ (a-b) \end{bmatrix} x$$

the output is: $$y = q_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

The characteristic equation is

$$|s\mathbf{I} - \mathbf{A}| = 0 = \begin{vmatrix} s+a & -1 \\ 0 & s+b \end{vmatrix} = (s+a)(s+b) = 0$$

$\lambda_1 = -a$ and $\lambda_2 = -b$ are the eigenvalues.

$$\Lambda = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}$$

we also have: $\mathbf{w} = \mathbf{P}\mathbf{q}$ and $\dot{\mathbf{w}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}\mathbf{w} + \mathbf{P}\mathbf{B}x$.
We are looking for $\mathbf{P}$ such that: $\mathbf{P}\mathbf{A}\mathbf{P}^{-1} = \Lambda$ or $\Lambda\mathbf{P} = \mathbf{P}\mathbf{A}$

$$\begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} -a & 1 \\ 0 & -b \end{bmatrix}$$

$$\Longrightarrow \begin{cases} -ap_{11} = -ap_{11} \\ -bp_{21} = -ap_{21} \Longrightarrow p_{21} = 0 \\ -ap_{12} = p_{11} - bp_{12} = 0 \\ -bp_{22} = p_{21} - bp_{22} \Longrightarrow p_{21} = 0 \end{cases}$$

If $p_{11} = (b-a)$ then $p_{12} = 1, p_{21} = 0$ and p_{22} can be anything; let's take $p_{22} = 1$

And thus: $$\mathbf{w} = \mathbf{P}\mathbf{q} = \begin{bmatrix} b-a & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Observability: the output in terms of $\mathbf{w}$ is: $y = \mathbf{C}\mathbf{q} = \mathbf{C}\mathbf{P}^{-1}\mathbf{w} = \hat{\mathbf{C}}\mathbf{w}$.

where: $$\mathbf{P}^{-1} = \frac{1}{b-a}\begin{bmatrix} 1 & -1 \\ 0 & b-a \end{bmatrix} = \begin{bmatrix} \frac{1}{b-a} & \frac{-1}{b-a} \\ 0 & 1 \end{bmatrix}$$

hence: $$\hat{\mathbf{C}} = \mathbf{C}\mathbf{P}^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{b-a} & \frac{-1}{b-a} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{b-a} & \frac{-1}{b-a} \end{bmatrix}$$

We notice that in $\hat{\mathbf{C}}$, there is no column with all elements zeros, hence we conclude that the system is observable.
Controllability: In the new system (diagonalized form):

$$\hat{\mathbf{B}} = \mathbf{P}\mathbf{B} = \begin{bmatrix} b-a & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ a-b \end{bmatrix} = \begin{bmatrix} 0 \\ a-b \end{bmatrix}$$

the 1st row in $\hat{\mathbf{B}}$ is zero. We affirm that this system is not controllable.

(b) State equations:

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -b & 0 \\ 0 & -a \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}x$$

and: $$y = q_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

The matrix A is already in the diagonal form:

$$\mathbf{P} = \mathbf{A} = \begin{bmatrix} -b & 0 \\ 0 & -a \end{bmatrix} \Longrightarrow \mathbf{P}^{-1} = \frac{1}{ab} = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix} = \begin{bmatrix} -\frac{1}{b} & 0 \\ 0 & -\frac{1}{a} \end{bmatrix}$$

In the transformed system: $\dot{\mathbf{w}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}\mathbf{w} + \mathbf{P}\mathbf{B}x = \mathbf{A}\mathbf{w} + \hat{\mathbf{B}}x$.

Observability:

$$\hat{\mathbf{C}} = \mathbf{CP}^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{b} & 0 \\ 0 & -\frac{1}{a} \end{bmatrix} = \begin{bmatrix} -\frac{1}{b} & 0 \end{bmatrix}$$

the second column in $\hat{\mathbf{C}}$ vanishes. This system is not observable.
Controllability:

$$\hat{\mathbf{B}} = \mathbf{PB} = \begin{bmatrix} -b & 0 \\ 0 & -a \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} -b & -a \end{bmatrix}$$

in $\hat{\mathbf{B}}$, there is no row with all elements zeros; hence this system is controllable.

10.6-1. (a) Time-domain method: the output $y[n]$ is given by:

$$y[n] = \mathbf{CA}^n\mathbf{q}[0] + \mathbf{CA}^{n-1}u[n-1] * \mathbf{B}x[n] + \mathbf{D}x[n]$$

The characteristic equation of $\mathbf{A}$ is:

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda - 2 & 0 \\ -1 & \lambda - 1 \end{vmatrix} = (\lambda - 1)(\lambda - 2) = 0$$

$\lambda_1 = 1$ and $\lambda_2 = 2$ are the eigenvalues of $\mathbf{A}$. Also:

$$\mathbf{A}^n = \beta_0\mathbf{I} + \beta_1\mathbf{A} \qquad \text{where:} \qquad \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2^n \end{bmatrix}$$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2^n \end{bmatrix} = \begin{bmatrix} 2 - 2^n \\ -1 + 2^n \end{bmatrix}$$

hence:

$$\mathbf{A}^n = \begin{bmatrix} \beta_0 & 0 \\ 0 & \beta_0 \end{bmatrix} + \begin{bmatrix} 2\beta_1 & 0 \\ \beta_1 & \beta_1 \end{bmatrix} = \begin{bmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{bmatrix}$$

Hence: $\qquad \mathbf{CA}^n = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{A}^n = \begin{bmatrix} 2^n - 1 & 1 \end{bmatrix}$

And:

$$y_x[n] = \mathbf{CA}^n\mathbf{q}(0) = \mathbf{CA}^n \begin{bmatrix} 2 \\ 1 \end{bmatrix} = (2^{n+1} - 1)u[n]$$

The zero-state component is given by:

$$y_x[n] = \mathbf{CA}^{n-1}u[n-1] * \mathbf{B}x[n] + \mathbf{D}x[n]$$

But $\quad \mathbf{CA}^n u[n] * \mathbf{B}x[n] = \begin{bmatrix} 2^n - 1 & 1 \end{bmatrix} u[n] * \begin{bmatrix} 0 \\ u[n] \end{bmatrix} = (n+1)u[n]$

Hence

$$y_x[n] = nu[n-1] + \mathbf{D}x[n] = nu[n-1] + u[n] = (n+1)u[n]$$

$$\text{and} \qquad y[n] = y_x[n] + y_x[n] = [2^{n+1} + n]u[n]$$

(b) Frequency-domain method: in this case:

$$\mathbf{Y}(z) = \mathbf{C}(\mathbf{I} - z^{-1}\mathbf{A})^{-1}\mathbf{q}[0] + [\mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]X[z]$$

$$(\mathbf{I}-z^{-1}\mathbf{A})^{-1} = \begin{bmatrix} 1-2z^{-1} & 0 \\ -z^{-1} & 1-z^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} 1-\frac{2}{z} & 0 \\ -\frac{1}{z} & 1-\frac{1}{z} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{z-2}{z} & 0 \\ -\frac{1}{z} & \frac{z-1}{z} \end{bmatrix}^{-1}$$

$$= \frac{z^2}{(z-1)(z-2)} \begin{bmatrix} \frac{z-1}{z} & 0 \\ \frac{1}{z} & \frac{z-2}{z} \end{bmatrix} = \begin{bmatrix} \frac{z}{z-2} & 0 \\ \frac{z}{(z-1)(z-2)} & \frac{z}{z-1} \end{bmatrix}$$

Also:

$$(z\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} z-2 & 0 \\ -1 & z-1 \end{bmatrix}^{-1} = \frac{1}{(z-1)(z-2)} \begin{bmatrix} z-1 & 0 \\ 1 & z-2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{z-2} & 0 \\ \frac{1}{(z-1)(z-2)} & \frac{1}{z-1} \end{bmatrix}$$

$$\text{and} \qquad \mathbf{C}(\mathbf{I}-z^{-1}\mathbf{A})^{-1} = \begin{bmatrix} \frac{z}{(z-1)(z-2)} & \frac{z}{z-1} \end{bmatrix}$$

$$\mathbf{C}(\mathbf{I}-z^{-1}\mathbf{A})^{-1}\mathbf{q}(0) = \begin{bmatrix} \frac{2z}{(z-1)(z-2)} + \frac{z}{z-1} \end{bmatrix} = \frac{z^2}{(z-1)(z-2)}$$

Also

$$\mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} \frac{1}{(z-1)(z-2)} & \frac{1}{z-1} \end{bmatrix} \quad \text{and} \quad \mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \frac{1}{z-1}$$

$$\text{Hence:} \qquad \mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D} = \frac{1}{z-1} + \mathbf{D} = \frac{1}{z-1} + 1 = \frac{z}{z-1}$$

$$x[n] = u[n] \qquad \text{and} \qquad X(z) = \frac{z}{z-1}$$

$$\text{And hence:} \qquad (\mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D})X(z) = \left[\frac{z}{z-1}\right]^2 = \frac{z^2}{(z-1)^2}$$

$$\mathbf{Y}(z) = \mathbf{C}(\mathbf{I}-z^{-1}\mathbf{A})^{-1}\mathbf{q}(0) + [\mathbf{C}(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}]X(z) = \frac{z^2}{(z-1)(z-2)} + \frac{z^2}{(z-1)^2}$$

$$\frac{\mathbf{Y}(z)}{z} = \frac{1}{z-2} + \frac{z}{(z-1)^2} = \frac{2}{z-2} + \frac{1}{(z-1)^2}$$

$$\mathbf{Y}(z) = \frac{2z}{z-2} + \frac{z}{(z-1)^2}$$

$$\text{and} \qquad y[n] = z^{-1}[\mathbf{Y}(z)] = [2^n+1]u[n] + (n+1)u[n] = [2^{n+1}+n]u[n]$$

10.6-2.

$$y[n] = \frac{E+0.32}{E^2+E+0.16}x[n]$$

(a) In this case:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z+0.32}{z^2+z+0.16}$$

$$= \frac{z+0.32}{(z+0.2)(z+0.8)} = \frac{0.2}{z+0.2} + \frac{0.8}{z+0.8}$$

(b) State and output equations for the direct form II: using the output of each delay as a state variable we get:

$$\begin{aligned} q_1[n+1] &= q_2[n] \\ q_2[n+1] &= -0.16q_1[n] - q_2[n] + x[n] \end{aligned}$$

Direct form II

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n]$$

output equation:

$$y[n] = 0.32q_1[n] + q_[n] = \begin{bmatrix} 0.32 & 1 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}$$

State equations for the transposed direct form II:

$$\begin{aligned} q_1[n+1] &= -q_1[n] + q_2[n] + x[n] \\ q_2[n+1] &= -0.16q_1[n] + 0.32x[n] \end{aligned}$$

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -0.16 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0.32 \end{bmatrix} x[n]$$

The output equation is:

$$y[n] = q_1[n] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}$$

State equations for the cascade realization:

$$\begin{aligned} q_1[n+1] &= -0.8q_1[n] + q_2[n] \\ q_2[n+1] &= -0.2q_2[n] + x[n] \end{aligned}$$

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -0.8 & 1 \\ 0 & -0.2 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n]$$

The output equation is:

$$y[n] = 0.32q_1[n] - 0.8q_1[n] + q_2[n] = \begin{bmatrix} -0.48 & 1 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}$$

State equations for the parallel realization:

$$\begin{aligned} q_1[n+1] &= -0.2q_1[n] + x[n] \\ q_2[n+1] &= -0.8q_2[n] + x[n] \end{aligned}$$

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.8 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x[n]$$

The output equation is:

$$y[n] = 0.2q_1[n] + 0.8q_2[n] = \begin{bmatrix} 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}$$

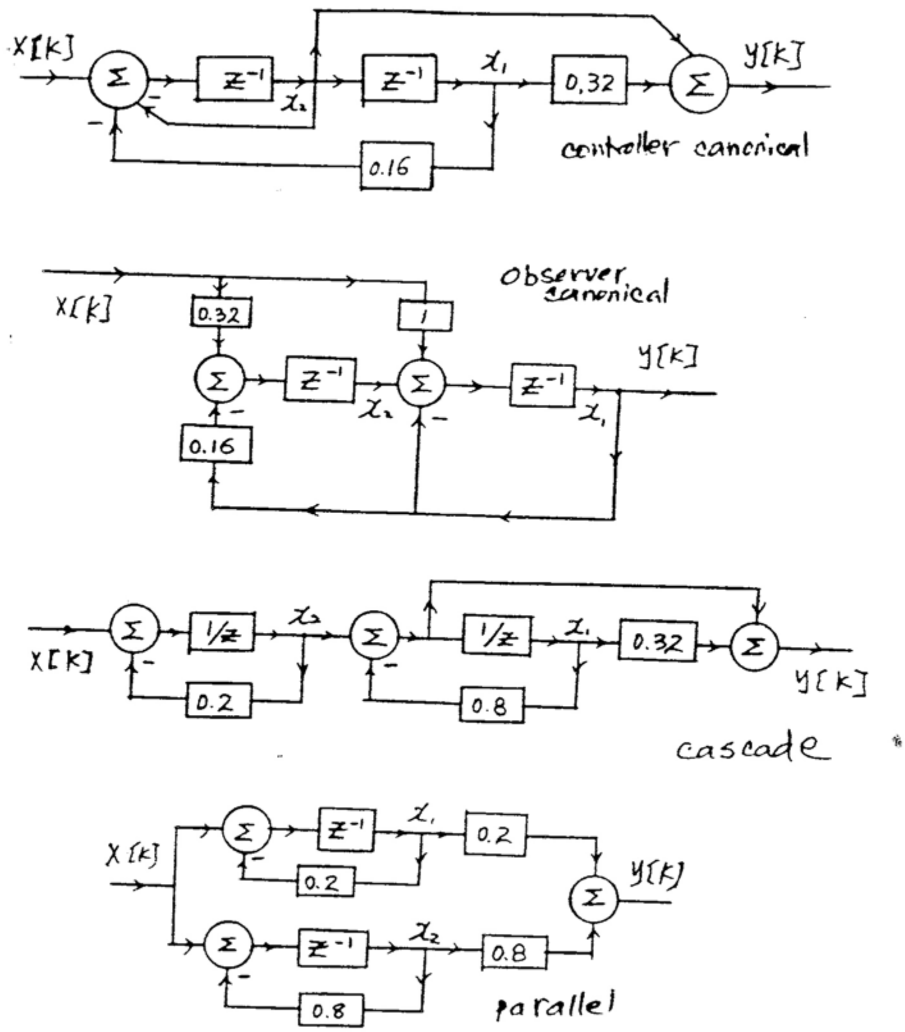

Figure S10.6-2:

10.6-3.

$$y[n] = \frac{E(2E+1)}{E^2+E-6}x[n]$$

(a)

$$\begin{aligned}
\frac{Y(z)}{X(z)} = H(z) &= \frac{z(2z+1)}{z^2+z-6} = \frac{2z^2+z}{z^2+z-6} \\
&= \frac{2z^2+z}{(z-2)(z+3)} = \left(\frac{z}{z-2}\right)\left(\frac{2z+1}{z+3}\right) \\
&= \frac{z}{z-2} + \frac{z}{z+3}
\end{aligned}$$

(b) State and output equations for the direct form II:

$$\begin{aligned} q_1[n+1] &= q_2[n] \\ q_2[n+1] &= 6q_1[n] - q_2[n] + x[n] \end{aligned}$$

and

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n]$$

The output equation is:

$$\begin{aligned} y[n] &= q_2[n] + 2[6q_1[n] - q_2[n] + x[n]] \\ &= 12q_1[n] - 2q_2[n] + 2x[n] \end{aligned}$$

$$\text{Hence} \qquad y[n] = \begin{bmatrix} 12 & -2 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + 2x[n]$$

State equations for the transposed direct form II:

$$\begin{aligned} q_1[n+1] &= -q_1[n] + q_2[n] + x[n] \\ q_2[n+1] &= 6q_1[n] \end{aligned}$$

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x[n]$$

The output equation is:

$$y[n] = q_1[n] + 2x[n] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + 2x[n]$$

State equations for the cascade realization:

$$\begin{aligned} q_1[n+1] &= -0.3q_1[n] + 2q_2[n] + x[n] \\ q_2[n+1] &= 2q_2[n] + x[n] \end{aligned}$$

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x[n]$$

The output equation is:

$$\begin{aligned} y[n] &= q_1[n] - 6q_1[n] + 4q_2[n] + 2x[n] \\ &= -5q_1[n] + 4q_2[n] + 2x[n] = \begin{bmatrix} -5 & 4 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + 2x[n] \end{aligned}$$

State equations for the parallel realization:

$$\begin{aligned} q_1[n+1] &= 2q_1[n] + x[n] \\ q_2[n+1] &= -3q_2[n] + x[n] \end{aligned}$$

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x[n]$$

The output equation is:

$$y[n] = 2q_1[n] + x[n] + x[n] - 3q_2[n]$$

$$y[n] = \begin{bmatrix} 2 & -3 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + 2x[n]$$

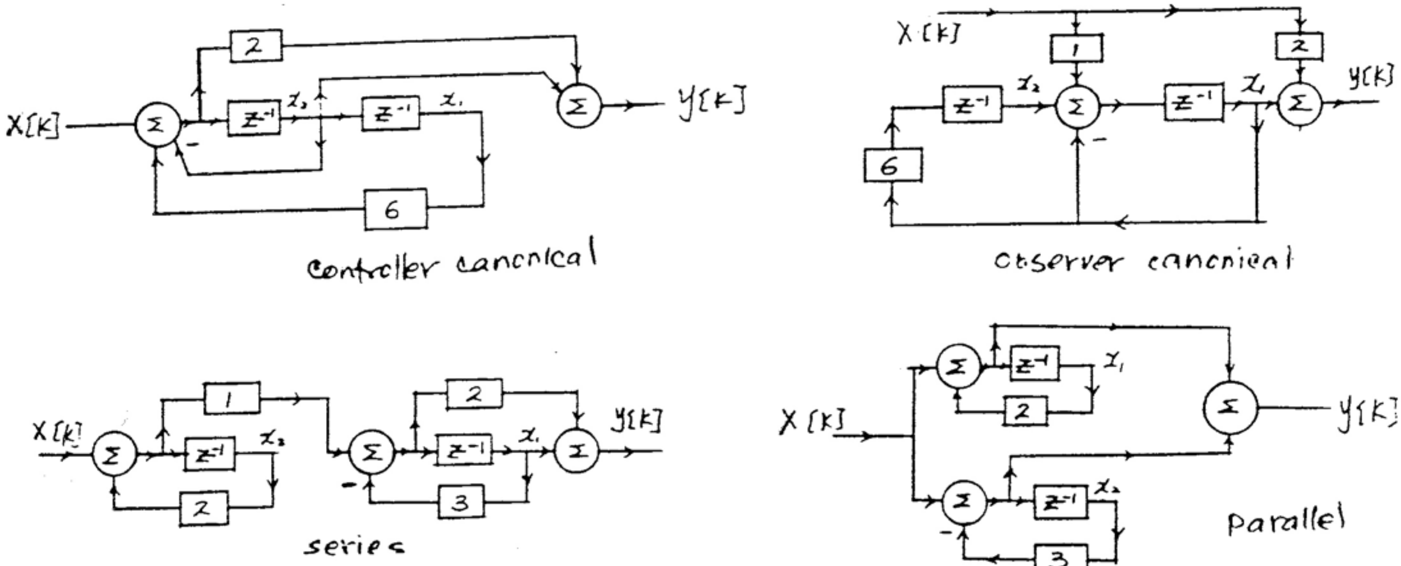

Figure S10.6-3

10.M-1. Figure 10.M-1 is used to help determine the state and output equations.

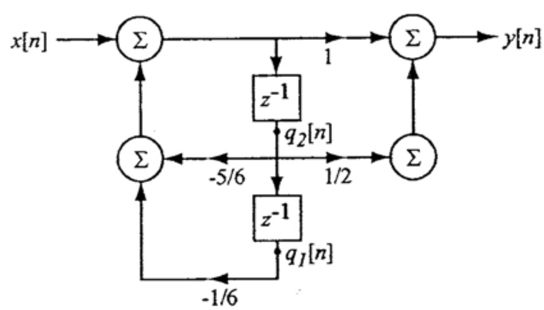

Figure S10.M-1: Direct form II realization of
$y[n] + \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] + \frac{1}{2}x[n-1]$.

Directly from the diagram, note that $q_1[n+1] = q_2[n] + 0x[n]$ and $q_2[n+1] = -\frac{5}{6}q_2[n] - \frac{1}{6}q_1[n] + x[n]$. Taken together, these yield the state equation

$$\mathbf{Q}[n+1] = \begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{6} & -\frac{5}{6} \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x[n] = \mathbf{A}\mathbf{Q}[n] + \mathbf{B}x[n]$$

The diagram is also used to write the output equation: $y[n] = \frac{1}{2}q_2[n] - \frac{5}{6}q_2[n] - \frac{1}{6}q_1[n] + x[n]$. Simplifying yields the output equation:

$$y[n] = \begin{bmatrix} -\frac{1}{6} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + 1x[n] = \mathbf{C}\mathbf{Q}[n] + \mathbf{D}x[n].$$

10.M-2. Figure 10.M-2 is used to help determine the state and output equations.

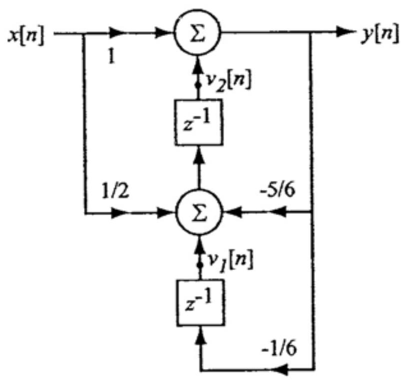

Figure S10.M-2: Transposed direct form II realization of
$y[n] + \frac{5}{6}y[n-1]\frac{1}{6}y[n-2] = x[n] + \frac{1}{2}x[n-1]$.

Directly from the diagram, note that $y[n] = v_2[n] + x[n]$. In standard form, the output equation is thus

$$y[n] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix} + 1x[n] = \mathbf{C}\mathbf{V}[n] + \mathbf{D}x[n].$$

Also using the diagram, note that $v_2[n+1] = v_1[n] + -\frac{5}{6}y[n] + \frac{1}{2}x[n]$ and $v_1[n+1] = -\frac{1}{6}y[n]$. Substituting $y[n] = v_2[n] + x[n]$ into each yields $v_2[n+1] = v_1[n] - \frac{5}{6}(v_2[n] + x[n]) + \frac{1}{2}x[n]$ and $v_1[n+1] = -\frac{1}{6}(v_2[n] + x[n])$. Simplifying to standard form, the state equations are represented in matrix form by

$$\mathbf{V}[n+1] = \begin{bmatrix} v_1[n+1] \\ v_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{6} \\ 1 & -\frac{5}{6} \end{bmatrix} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix} + \begin{bmatrix} -\frac{1}{6} \\ -\frac{1}{3} \end{bmatrix} x[n] = \mathbf{A}\mathbf{V}[n] + \mathbf{B}x[n].$$