**INSTRUCTOR'S SOLUTIONS MANUAL for**

# LINEAR SYSTEMS AND SIGNALS

**Second Edition**

**B.P. Lathi**
California State University – Sacramento

**Roger A. Green**
North Dakota State University

New York Oxford
OXFORD UNIVERSITY PRESS
2005

Oxford University Press

Oxford New York
Auckland Bangkok Buenos Aires Cape Town Chennai
Dar es Salaam Delhi Hong Kong Istanbul Karachi Kolkata
Kuala Lumpur Madrid Melbourne Mexico City Mumbai Nairobi
São Paulo Shanghai Taipei Tokyo Toronto



Published by Oxford University Press, Inc.
198 Madison Avenue, New York, New York 10016
www.oup.com



**Library of Congress Cataloging-in-Publication Data**

Printing number: 9 8 7 6 5 4 3 2 1

Printed in the United States of America
on acid-free paper

# Contents

# Chapter B Solutions

B.1. Given $w = re^{\jmath\theta} = r(\cos(\theta) + \jmath\sin(\theta)) = x + \jmath y$,

$$w^* = (x + \jmath y)^* = x - \jmath y = r(\cos(\theta) - \jmath\sin(\theta)) = re^{-\jmath\theta}.$$

B.2. (a) For $1 + \jmath$, $r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \arctan\left(\frac{1}{1}\right) = \pi/4 = 0.584$. Thus,

$$1 + \jmath = \sqrt{2}e^{\jmath\pi/4} = 1.414e^{\jmath 0.584}.$$

(b) For $-4 + \jmath 3$, $r = \sqrt{(-4)^2 + 3^2} = 5$ and $\theta = \arctan\left(\frac{3}{-4}\right) = -0.643 + \pi = 2.498$. Thus,

$$-4 + \jmath 3 = 5e^{\jmath 2.498}.$$

(c) Using the results from B.2a and B.2b,

$$(1 + \jmath)(-4 + \jmath 3) = (\sqrt{2}e^{\jmath\pi/4})(5e^{\jmath 2.498}) = 7.071e^{\jmath 3.283}.$$

(d) $e^{j\pi/4} + 2e^{-j\pi/4} = \frac{1+\jmath}{\sqrt{2}} + \frac{2-\jmath 2}{\sqrt{2}} = \frac{3-\jmath}{\sqrt{2}}$. Thus, $r = \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2} = \sqrt{5} = 2.236$ and $\theta = \arctan\left(\frac{-1}{3}\right) = -0.322$, which yields

$$e^{j\pi/4} + 2e^{-j\pi/4} = 2.236e^{-\jmath 0.322}.$$

(e) $e^{\jmath} + 1 = \cos(1) + \jmath\sin(1) + 1$. Thus, $r = \sqrt{(\cos(1) + 1)^2 + (\sin(1))^2} = 1.755$ and $\theta = \arctan\left(\frac{\sin(1)}{\cos(1)+1}\right) = 0.500$, which yields

$$e^{\jmath} + 1 = 1.755e^{\jmath/2}.$$

(f) Using the results from B.2a and B.2b,

$$\frac{1 + \jmath}{-4 + \jmath 3} = \frac{\sqrt{2}e^{\jmath\pi/4}}{5e^{\jmath 2.498}} = 0.283e^{-\jmath 1.713}.$$

B.3. (a) Using Euler's identity,

$$3e^{\jmath\pi/4} = 3\cos(\pi/4) + \jmath 3\sin(\pi/4) = 2.121 + \jmath 2.121.$$

(b) Using Euler's identity,

$$\frac{1}{e^{\jmath}} = e^{-\jmath} = \cos(-1) + \jmath\sin(-1) = 0.540 - \jmath 0.841.$$

(c) Expanding,

$$(1+\jmath)(-4+\jmath 3) = (-4-3) + \jmath(-4+3) = -7 - \jmath.$$

(d) Using Euler's identity,

$$e^{j\pi/4} + 2e^{-j\pi/4} = \frac{1+\jmath}{\sqrt{2}} + \frac{2-\jmath 2}{\sqrt{2}} = \frac{3}{\sqrt{2}} + \jmath\frac{-1}{\sqrt{2}}.$$

(e) Using Euler's identity,

$$e^{\jmath} + 1 = \cos(1) + \jmath\sin(1) + 1 = (\cos(1)+1) + \jmath\sin(1).$$

(f) Start by expressing the denominator in standard polar form, $\frac{1}{2^{\jmath}} = \frac{1}{e^{\jmath\ln(2)}} = e^{-\jmath\ln(2)}$. Using Euler's identity,

$$\frac{1}{2^{\jmath}} = \cos(\ln(2)) - \jmath\sin(\ln(2)) = 0.769 - \jmath 0.639.$$

B.4. For each proof, substitute the Cartesian form for $w$.

(a)

$$\frac{w+w^*}{2} = \frac{x+\jmath y + x - \jmath y}{2} = x = \mathrm{Re}\,(x+\jmath y) = \mathrm{Re}\,(w)\,.$$

(b)

$$\frac{w-w^*}{2\jmath} = \frac{x+\jmath y - x + \jmath y}{2\jmath} = y = \mathrm{Im}\,(x+\jmath y) = \mathrm{Im}\,(w)\,.$$

B.5. Using the results from B.4,

(a) $\mathrm{Re}\,(e^{w}) = \mathrm{Re}\,(e^{x-\jmath y}) = \frac{e^{x}e^{-\jmath y}+e^{x}e^{\jmath y}}{2} = e^{x}\frac{e^{-\jmath y}+e^{\jmath y}}{2}$. Using Euler's identity yields

$$\mathrm{Re}\,(e^{w}) = e^{x}\cos(y).$$

(b) $\mathrm{Im}\,(e^{w}) = \mathrm{Im}\,(e^{x-\jmath y}) = \frac{e^{x}e^{-\jmath y}-e^{x}e^{\jmath y}}{2\jmath} = e^{x}\frac{e^{-\jmath y}-e^{\jmath y}}{2\jmath}$. Using Euler's identity yields

$$\mathrm{Re}\,(e^{w}) = -e^{x}\sin(y).$$

B.6. For arbitrary complex constants $w_1$ and $w_2$,

(a) $\mathrm{Re}\,(\jmath w_1) = \mathrm{Re}\,(\jmath(x_1+\jmath y_1)) = \mathrm{Re}\,(-y_1+\jmath x_1) = -y_1$. Also, $-\mathrm{Im}\,(w_1) = -\mathrm{Im}\,(x_1+\jmath y_1) = -y_1$. Thus,

$$\text{True.} \qquad \mathrm{Re}\,(\jmath w_1) = -\mathrm{Im}\,(w_1)\,.$$

(b) $\mathrm{Im}\,(\jmath w_1) = \mathrm{Im}\,(\jmath(x_1+\jmath y_1)) = \mathrm{Im}\,(-y_1+\jmath x_1) = x_1$. Also, $\mathrm{Re}\,(w_1) = x_1$. Clearly,

$$\text{True.} \qquad \mathrm{Im}\,(\jmath w_1) = \mathrm{Re}\,(w_1)\,.$$

(c) $\text{Re}(w_1)+\text{Re}(w_2) = x_1+x_2$. Also, $\text{Re}(w_1+w_2) = \text{Re}(x_1+jy_1+x_2+jy_2) = x_1+x_2$. Thus,

$$\text{True.} \qquad \text{Re}(w_1) + \text{Re}(w_2) = \text{Re}(w_1 + w_2).$$

(d) $\text{Im}(w_1)+\text{Im}(w_2) = y_1+y_2$. Also, $\text{Im}(w_1+w_2) = \text{Im}(x_1+jy_1+x_2+jy_2) = y_1+y_2$. Thus,

$$\text{True.} \qquad \text{Im}(w_1) + \text{Im}(w_2) = \text{Im}(w_1 + w_2).$$

(e) $\text{Re}(w_1)\text{Re}(w_2) = x_1x_2$. Also, $\text{Re}(w_1w_2) = \text{Re}((x_1 + jy_1)(x_2 + jy_2)) = \text{Re}(x_1x_2 - y_1y_2 + j(x_1y_2 + x_2y_1)) = x_1x_2 - y_1y_2)$. In general $x_1x_2 \neq x_1x_2 - y_1y_2$, so

$$\text{False.} \qquad \text{Re}(w_1)\text{Re}(w_2) \neq \text{Re}(w_1w_2).$$

(f) $\text{Im}(w_1)/\text{Im}(w_2) = y_1/y_2$. Also, $\text{Im}(w_1/w_2) = \text{Im}\left(\frac{x_1+jy_1}{x_2+jy_2}\right) = \text{Im}\left(\frac{x_1+jy_1}{x_2+jy_2}\frac{x_2-jy_2}{x_2-jy_2}\right) = \text{Im}\left(\frac{x_1x_2-y_1y_2+j(x_1y_2+x_2y_1)}{x_2^2+y_2^2}\right) = \frac{x_1y_2+x_2y_1}{x_2^2+y_2^2}$. In general $y_1/y_2 \neq \frac{x_1y_2+x_2y_1}{x_2^2+y_2^2}$, so

$$\text{False.} \qquad \text{Im}(w_1)/\text{Im}(w_2) \neq \text{Im}(w_1/w_2).$$

B.7. First, express $w_1$ in both rectangular and polar coordinates. By inspection, $w_1 = x_1 + jy_1 = 3 + j4$. Next, $r_1 = \sqrt{3^2+4^2} = 5$ and $\theta_1 = \arctan\left(\frac{4}{3}\right) = 0.927$ so $w_1 = r_1e^{j\theta_1} = 5e^{j0.927}$.

Second, express $w_2$ in both rectangular and polar coordinates. By inspection, $w_2 = r_2e^{j\theta_2} = 2e^{j\pi/4} = 2e^{j0.785}$. Next, $x_2 = r_2\cos(\theta_2) = 2\cos(\pi/4) = \sqrt{2} = 1.414$ and $y_2 = r_2\sin(\theta_2) = 2\sin(\pi/4) = \sqrt{2} = 1.414$. Thus, $w_2 = x_2 + jy_2 = 1.414 + j1.414$.

(a) From above,

$$w_1 = r_1e^{j\theta_1} = 5e^{j0.927}.$$

(b) From above,

$$w_2 = x_2 + jy_2 = 1.414 + j1.414.$$

(c)

$$|w_1|^2 = r_1^2 = 5^2 = 25.$$

Similarly,

$$|w_2|^2 = r_2^2 = 4.$$

(d)

$$w_1 + w_2 = (x_1 + x_2) + j(y_1 + y_2) = (3 + 1.414) + j(4 + 1.414) = 4.414 + j5.414.$$

(e) $w_1 - w_2 = (x_1+x_2) - j(y_1+y_2) = (3-1.414)+j(4-1.414) = 1.586+j2.586$. Converting to polar form, $r = \sqrt{(1.586)^2 + (2.586)^2} = 3.033$ and $\theta = \arctan\left(\frac{2.586}{1.586}\right) = 1.021$. Thus,

$$w_1 - w_2 = re^{j\theta} = 3.033e^{j1.021}.$$

(f) $w_1w_2 = r_1e^{j\theta_1}r_2e^{j\theta_2} = 10e^{j1.713}$. Converting to Cartesian form, $x = 10\cos(1.713) = -1.414$ and $y = 10\sin(1.713) = 9.899$. Thus,

$$w_1w_2 = x + jy = -1.414 + j9.899.$$

(g)

$$\frac{w_1}{w_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)} = 2.5 e^{j0.142}.$$

B.8. First, express $w_1$ in both rectangular and polar coordinates. For rectangular form, $w_1 = (3+j4)^2 = 9-16+j(12+12) = -7+j24$. For polar form, $r_1 = \sqrt{(-7)^2 + 24^2} = 25$ and $\theta_1 = \arctan\left(\frac{24}{-7}\right) = -1.287 + \pi = 1.855$. Thus, $w_1 = r_1 e^{j\theta_1} = 25 e^{j1.855}$.

Second, express $w_2$ in both rectangular and polar coordinates. Since $j = e^{j\pi/2}$ and $e^{-j40\pi} = 1$, rectangular form is $w_2 = x_2 + jy_2 = j2.5$. For polar form, $w_2 = r_2 e^{j\theta_2} = 2.5 e^{j\pi/2} = 2.5 e^{j1.571}$.

(a) From above,

$$w_1 = r_1 e^{j\theta_1} = 25 e^{j1.855}.$$

(b) From above,

$$w_2 = x_2 + jy_2 = j2.5.$$

(c)

$$|w_1|^2 = r_1^2 = 25^2 = 625.$$

Similarly,

$$|w_2|^2 = r_2^2 = 2.5^2 = 6.25.$$

(d)

$$w_1 + w_2 = (x_1 + x_2) + j(y_1 + y_2) = (-7 + 0) + j(24 + 2.5) = -7 + j26.5.$$

(e) $w_1 - w_2 = (x_1 + x_2) - j(y_1 + y_2) = (-7 - 0) + j(24 - 2.5) = -7 + j21.5$. Converting to polar form, $r = \sqrt{(-7)^2 + (21.5)^2} = 22.611$ and $\theta = \arctan\left(\frac{21.5}{-7}\right) = -1.256 + \pi = 1.886$. Thus,

$$w_1 - w_2 = re^{j\theta} = 22.611 e^{j1.886}.$$

(f) $w_1 w_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = 62.5 e^{j3.425}$. Converting to Cartesian form, $x = 62.5\cos(3.425) = -60$ and $y = 62.5\sin(3.425) = -17.5$. Thus,

$$w_1 w_2 = x + jy = -60 + j - 17.5.$$

(g)

$$\frac{w_1}{w_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)} = 10 e^{j0.284}.$$

B.9. First, express $w_1$ in both rectangular and polar coordinates. By inspection, $w_1 = x_1 + jy_1 = e^{\pi/4} + j = 2.193 + j$. Next, $r_1 = \sqrt{2.193^2 + 1^2} = 2.410$ and $\theta_1 = \arctan\left(\frac{1}{2.193}\right) = 0.428$ so $w_1 = r_1 e^{j\theta_1} = 2.410 e^{j0.428}$.

Second, express $w_2$ in both rectangular and polar coordinates. Using Euler's identity, $w_2 = \cos(j) = \frac{e^{jj} + e^{-jj}}{2} = \frac{e^{-1} + e^{1}}{2} = \cosh(1) = 1.543$. Thus, $w_2 = x_2 + jy_2 = 1.543$. Polar form is $w_2 = r_2 e^{j\theta_2} = 1.543 e^{j0}$.

(a) From above,

$$w_1 = r_1 e^{j\theta_1} = 2.410 e^{j0.428}.$$

(b) From above,

$$w_2 = x_2 + \jmath y_2 = 1.543.$$

(c)

$$|w_1|^2 = r_1^2 = 5.810.$$

Similarly,

$$|w_2|^2 = r_2^2 = 2.381.$$

(d)

$$w_1 + w_2 = (x_1 + x_2) + \jmath(y_1 + y_2) = (2.193 + 1.543) + \jmath(1 + 0) = 3.736 + \jmath.$$

(e) $w_1 - w_2 = (x_1 + x_2) - \jmath(y_1 + y_2) = (2.193 - 1.543) + \jmath(1 - 0) = 0.650 + \jmath$. Converting to polar form, $r = \sqrt{(0.650)^2 + (1)^2} = 1.193$ and $\theta = \arctan\left(\frac{1}{0.650}\right) = 0.994$. Thus,

$$w_1 - w_2 = re^{\jmath\theta} = 1.193e^{\jmath 0.994}.$$

(f) $w_1 w_2 = r_1 e^{\jmath\theta_1} r_2 e^{\jmath\theta_2} = 3.720e^{\jmath 0.428}$. Converting to Cartesian form, $x = 3.720\cos(0.428) = 3.384$ and $y = 3.720\sin(0.428) = 1.543$. Thus,

$$w_1 w_2 = x + \jmath y = 3.384 + \jmath 1.543.$$

(g)

$$\frac{w_1}{w_2} = \frac{r_1 e^{\jmath\theta_1}}{r_2 e^{\jmath\theta_2}} = \frac{r_1}{r_2} e^{\jmath(\theta_1 - \theta_2)} = 1.562e^{\jmath 0.428}.$$

B.10. (a) Note, we can rewrite $2.5\cos(3t) - 1.5\sin(3t + \pi/3) = c\cos(3t + \phi)$ as $\mathrm{Re}\left(2.5e^{\jmath 3t} + \jmath 1.5e^{\jmath(3t+\pi/3)}\right) = \mathrm{Re}\left(ce^{\jmath(3t+\phi)}\right)$. Working with the left-hand side, $\mathrm{Re}\left(2.5e^{\jmath 3t} + \jmath 1.5e^{\jmath(3t+\pi/3)}\right) = \mathrm{Re}\left(e^{\jmath 3t}(2.5 + 1.5e^{\jmath(\pi/3+\pi/2)})\right)$. The unknown constants $c$ and $\phi$ are determined by comparing the left- and right-hand sides.

$$c = |2.5 + 1.5e^{\jmath(\pi/3+\pi/2)}| = \sqrt{(2.5 + 1.5\cos(5\pi/6))^2 + (1.5\sin(5\pi/6))^2} = 1.416$$

and

$$\phi = \angle\left(2.5 + 1.5e^{\jmath(\pi/3+\pi/2)}\right) = \arctan\left(\frac{1.5\sin(5\pi/6)}{2.5 + 1.5\cos(5\pi/6)}\right) = 0.558.$$

(b) Note, $\cos(\theta \pm \phi) = \mathrm{Re}\left(e^{\jmath(\theta\pm\phi)}\right) = \mathrm{Re}\left((\cos(\theta) + \jmath\sin(\theta))(\cos(\phi) \pm \jmath\sin(\phi))\right) = \mathrm{Re}\left((\cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi)) + \jmath(\sin(\theta)\cos(\phi) \pm \cos(\theta)\sin(\phi))\right) = (\cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi))$. Thus,

$$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi).$$

(c) Noting that $\sin(\alpha x) = \frac{e^{\jmath\alpha x} - e^{-\jmath\alpha x}}{2\jmath}$, first solve the indefinite integral $\int e^{wx}\sin(\alpha x)dx = \int e^{wx}\frac{e^{\jmath\alpha x} - e^{-\jmath\alpha x}}{2\jmath}dx = \int \frac{e^{x(w+\jmath\alpha)} - e^{x(w-\jmath\alpha)}}{2\jmath} = \frac{1}{2\jmath(w+\jmath\alpha)}e^{x(w+\jmath\alpha)} - \frac{1}{2\jmath(w-\jmath\alpha)}e^{x(w-\jmath\alpha)}$. Substituting the limits of integration yields

$$\int_a^b e^{wx}\sin(\alpha x)dx =$$

$$\frac{1}{2j(w+j\alpha)}\left(e^{b(w+j\alpha)}-e^{a(w+j\alpha)}\right)-\frac{1}{2j(w-j\alpha)}\left(e^{b(w-j\alpha)}-e^{a(w-j\alpha)}\right).$$

B.11. Solutions to this problem are based on Euler's identity.

(a)

$$\begin{aligned}\cosh(w) &= \cosh(x+jy)=\frac{e^{x+jy}+e^{-x-jy}}{2}\\ &= 0.5\left((\cos(y)+j\sin(y))e^{x}+(\cos(y)-j\sin(y))e^{-x}\right)\\ &= 0.5\left(\cos(y)(e^{x}+e^{-x})+j\sin(y)(e^{x}-e^{-x})\right)\\ &= \cos(y)\cosh(x)+j\sin(y)\sinh(x)\end{aligned}$$

Thus,

$$\cosh(w)=\cosh(x+jy)=\cosh(x)\cos(y)+j\sinh(x)\sin(y).$$

(b)

$$\begin{aligned}\sinh(w) &= \sinh(x+jy)=\frac{e^{x+jy}-e^{-x-jy}}{2}\\ &= 0.5\left((\cos(y)+j\sin(y))e^{x}-(\cos(y)-j\sin(y))e^{-x}\right)\\ &= 0.5\left(\cos(y)(e^{x}-e^{-x})+j\sin(y)(e^{x}+e^{-x})\right)\\ &= \cos(y)\sinh(x)+j\sin(y)\cosh(x)\end{aligned}$$

Thus,

$$\sinh(w)=\sinh(x+jy)=\sinh(x)\cos(y)+j\cosh(x)\sin(y).$$

B.12. (a) $(w)^4=-1=e^{jj(\pi+2\pi k)}\Rightarrow w=(e^{j(\pi+2\pi k)})^{1/4}$. Thus,

$$w=e^{j\pi(1/4+k/2)}\quad\text{for}\quad k=[0,1,2,3].$$

```
>> k = [0:3]; w = exp(j*pi*(1/4+k/2)); t = linspace(0,2*pi,200);
>> h = plot(real(w),imag(w),'kx',cos(t),sin(t),'k:'); axis equal;
>> xlabel('Real'); ylabel('Imag'); grid;
>> set(h(1),'markersize',10,'linewidth',2);
```

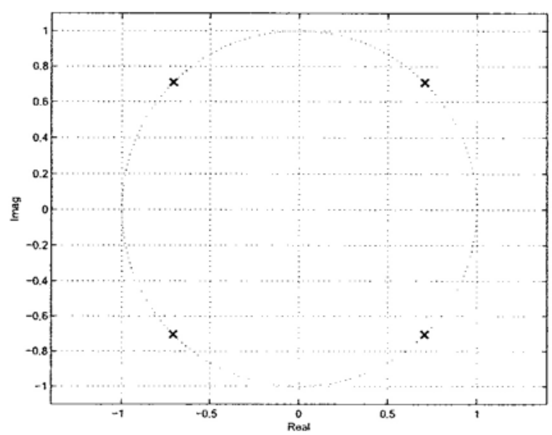


Figure SB.12a: Solutions to $(w)^4=-1$.

(b) Notice,

$$(w-(1+j2))^5 = \frac{32}{\sqrt{2}}(1+j) = 32e^{j(\pi/4+2\pi k)}.$$

This implies that

$$w-(1+j2) = \left(32e^{j(\pi/4+2\pi k)}\right)^{1/5} = 2e^{j(\pi/20+2\pi k/5)}.$$

Thus,

$$w = (1+j2)+2e^{j(\pi/20+2\pi k/5)} \quad \text{for} \quad k=[0,1,2,3,4].$$

```
>> k = [0:4]; w = (1+j*2)+2*exp(j*(pi/20+2*pi*k/5));
>> t = linspace(0,2*pi,200);
>> h = plot(real(w),imag(w),'kx',1+2*cos(t),2+2*sin(t),'k:');
>> axis equal; xlabel('Real'); ylabel('Imag'); grid;
>> set(h(1),'markersize',10,'linewidth',2);
```

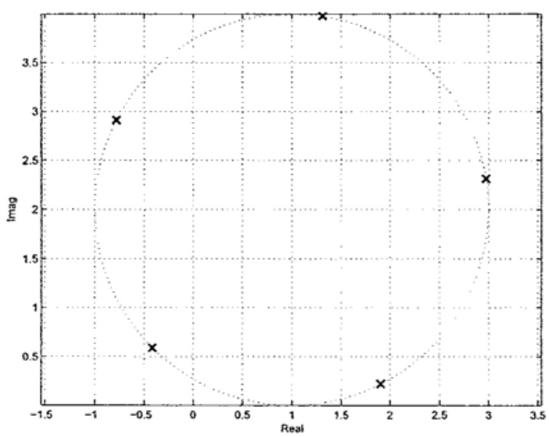


Figure SB.12b: Solutions to $(w-(1+j2))^5 = \frac{32}{\sqrt{2}}(1+j)$.

(c) The solution set of $|\ w-2j\ |=3$ describes a circle. To see this, note that $|\ w-2j\ |^2=(w-2j)(w-2j)^* = (x+j(y-2))(x+j(2-y)) = x^2+(y-2)^2 = 3^2 = 9$. The circle has center $(0,2)$ and radius $r=3$.

```
>> theta = linspace(0,2*pi,201); x = 3*cos(theta); y = 2+3*sin(theta);
>> plot(x,y,'k-'); axis equal; grid; xlabel('Real'); ylabel('Imag');
```

(d) Graph $w(t) = (1+t)e^{jt}$ for $(-10 \le t \le 10)$.

```
>> t = [-10:.01:10]; w = (1+t).*exp(j*t);
>> im10 = find(t==-10); im = find(t<0);i0 = find(t==0);
>> ip = find(t>0); ip10 = find(t==10);
>> plot(real(w(im10)),imag(w(im10)),'vk',real(w(im)),imag(w(im)),'k-',...
>> real(w(i0)),imag(w(i0)),'ok',real(w(ip)),imag(w(ip)),'k:',...
>> real(w(ip10)),imag(w(ip10)),'k^'); axis equal; xlabel('Real');
>> ylabel('Imag'); legend('t=-10','t<0','t=0','t>0','t=10',0)
```

B.13. Since four distinct solutions are indicated, we know $n=4$. The solutions to $w^n = w_2 = r_2e^{j\theta_2}$ lie on a circle of radius $r_2^{1/n}$. The solutions to $(w-w_1)^n = w_2$ lie on the same circle shifted by $w_1$. To find $w_1$, drop perpendicular lines from the circle center to

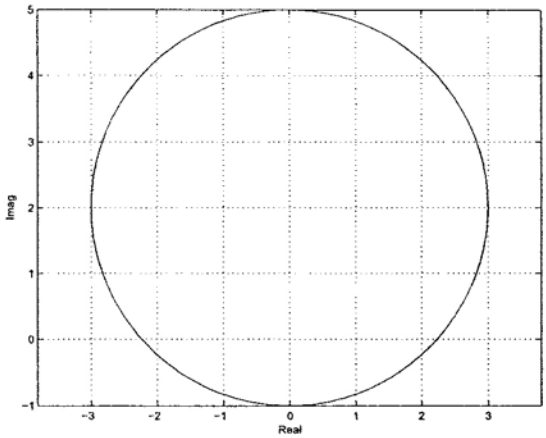


Figure SB.12c: Graph of $| w - 2j |= 3$.

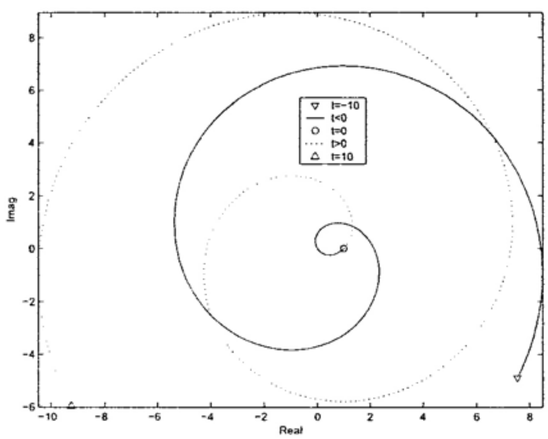


Figure SB.12d: Graph of $w(t) = (1 + t)e^{jt}$ for $(-10 \leq t \leq 10)$.

the real and imaginary axes, respectively. As shown, two similar triangles are formed. The circle center is $w_1 = A + jA$. Furthermore, we know that $A + B = \sqrt{3} + 1$ and $A - B = \sqrt{3} - 1$. Clearly, $A = \sqrt{3}$ and $B = 1$. Thus, $w_1 = \sqrt{3} + j\sqrt{3}$. The value of $w_2$ is now easily found by substitution: $w_2 = (\sqrt{3}+1-(\sqrt{3}+j\sqrt{3}))^4 = (1+j\sqrt{3})^4 = 16e^{j2\pi/3}$. Thus,

$$n = 4, w_1 = \sqrt{3} + j\sqrt{3}, \text{ and } w_2 = 16e^{j2\pi/3}.$$

B.14. We can write $(j - w)^{1.5} = (j - w)^{3/2} = \sqrt{8}e^{j\pi/4}$. Squaring both sides yields $(j - w)^3 = 8e^{j(\pi/2+2\pi k)}$. Taking the third root of each side yields $(j - w) = 2e^{j(\pi/6+2\pi k/3)}$. Rearranging yields three distinct solutions

$$w = j - 2e^{j(\pi/6+2\pi k/3)} \quad \text{for} \quad k = [0, 1, 2].$$

```
>> k = [0:2]; w = j-2*exp(j*(pi/6+2*pi*k/3)); t = linspace(0,2*pi,200);
>> h = plot(real(w),imag(w),'kx',...
      real(j-2*exp(j*t)),imag(j-2*exp(j*t)),'k-');
>> xlabel('Real'); ylabel('Imag'); grid;
```

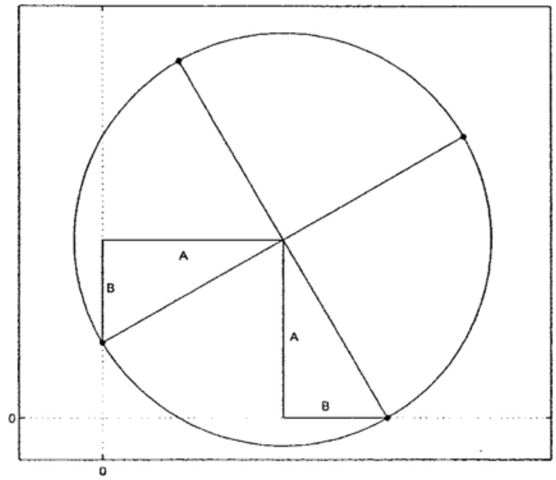


Figure SB.13: Distinct solutions to $(w - w_1)^n = w_2$.

```
>> axis equal; axis([-2.5 2.5 -1.5 3.5]);
>> set(h(1),'markersize',10,'linewidth',2);
```

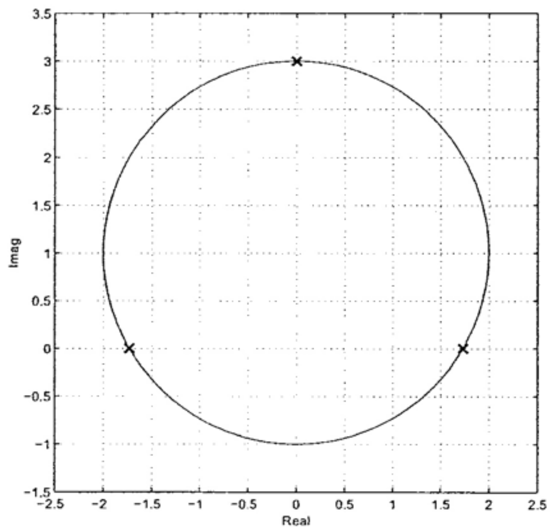


Figure SB.14: Distinct solutions to $(\jmath - w)^{1.5} = 2 + \jmath 2$.

B.15. $w = \sqrt{\jmath} = \left(e^{\jmath(\pi/2+2\pi k)}\right)^{1/2} = e^{\jmath(\pi/4+\pi k)}$. Thus, there are two distinct solutions

$$w = e^{\jmath(\pi/4+\pi k)} \quad \text{for} \quad k = [0,1].$$

That is, $w = \pm(1+\jmath)/\sqrt{2}$.

B.16. $\ln(-e) = \ln(e^{1+\jmath(\pi+2\pi k)}) = 1 + \jmath(\pi + 2\pi k)$. Since $k$ can be any integer, there are an infinite number of solutions

$$\ln(-e) = 1 + \jmath(\pi + 2\pi k) \quad \text{for integer } k.$$

MATLAB and other calculating devices generally give only the $k = 0$ solution $1 + \jmath\pi$.

B.17. $\log_{10}(-1) = \log_{10} e^{\jmath(\pi+2\pi k)} = \jmath(\pi + 2\pi k)\log_{10}(e)$. Since $k$ can be any integer, there are an infinite number of solutions

$$\log_{10}(-1) = 0 + \jmath(\pi + 2\pi k)\log_{10}(e) \quad \text{for integer } k.$$

MATLAB and other calculating devices generally give only the $k = 0$ solution $\jmath\pi\log_{10}(e)$.

B.18. (a) $\ln\left(\frac{1}{1+\jmath}\right) = \ln\left(\frac{1}{\sqrt{2}e^{\jmath\pi/4}}\right) = \ln\left((\sqrt{2})^{-1}e^{\jmath(-\pi/4+2\pi k)}\right) = -\ln(\sqrt{2}) + \jmath(-\pi/4+2\pi k)$. Since $k$ can be any integer, there are an infinite number of solutions

$$\ln\left(\frac{1}{1+\jmath}\right) = -\ln(\sqrt{2}) + \jmath(-\pi/4 + 2\pi k) \quad \text{for integer } k.$$

MATLAB and other calculating devices generally give only the $k = 0$ solution.

(b) $\cos(1+\jmath) = 0.5\left(e^{\jmath(1+\jmath)} + e^{-\jmath(1+\jmath)}\right) = 0.5\left(e^{-1}(\cos(1) + \jmath\sin(1)) + e^{1}(\cos(1) - \jmath\sin(1))\right) = \cos(1)\cosh(1) - \jmath\sin(1)\sinh(1)$. That is

$$\cos(1+\jmath) = \cos(1)\cosh(1) - \jmath\sin(1)\sinh(1).$$

(c) $(1 - \jmath)^{\jmath} = \left(\sqrt{2}e^{-\jmath\pi/4}\right)^{\jmath} = \left(e^{\ln(\sqrt{2})}e^{-\jmath\pi/4}\right)^{\jmath} = e^{\jmath\ln(\sqrt{2})}e^{\pi/4} = e^{\pi/4}\left(\cos(\ln(\sqrt{2})) + \jmath\sin(\ln(\sqrt{2}))\right)$. Thus,

$$(1-\jmath)^{\jmath} = e^{\pi/4}\cos(\ln(\sqrt{2})) + \jmath e^{\pi/4}\sin(\ln(\sqrt{2})).$$

B.19. Letting $w = \jmath y$, $\cos(w) = \cos(\jmath y) = 0.5\left(e^{\jmath\jmath y} + e^{-\jmath\jmath y}\right) = 0.5\left(e^{-y} + e^{y}\right) = 2$. Multiplying both sides by $2e^{y}$ yields $1 + (e^{y})^2 - 4e^{y} = (e^{y})^2 - 4e^{y} + 1 = 0$. This is a quadratic equation in $e^{y}$. Applying the quadratic formula yields $e^{y} = \frac{4\pm\sqrt{16-4}}{2} = 2\pm\sqrt{3}$. Solving for $y$ gives $y = \ln\left(2 \pm \sqrt{3}\right)$. Thus,

$$w = \jmath y = \jmath\ln\left(2 \pm \sqrt{3}\right) = \pm\jmath 1.3170.$$

B.20. The general form is $x(t) = e^{-at}\cos(\omega t)$. At $t = 0$, $e^{-at} = 1$. Thus, a fifty percent decrease in two seconds requires $0.5 = e^{-a2}$, or $a = 0.5\ln(2)$. To oscillate three times per second requires $\omega = 6\pi$.

```
>> w = 3*2*pi; a = 0.5*log(2);
>> t = [-2:.01:2]; x = exp(-a*t).*cos(w*t);
>> plot(t,x,'k-'); xlabel('t'); ylabel('x(t)');
```

B.21. (a) $x_1(t) = \mathrm{Re}(2e^{(-1+\jmath 2\pi)t}) = 2e^{-t}\cos(2\pi t)$. This is 1Hz cosine wave that exponentially decays by a factor of $1 - e^{-1} = 0.632$ every second. A signal peak is near $t = 0$, where the signal has an amplitude of 2. See Figure SB.22a.

(b) $x_2(t) = \mathrm{Im}(3 - e^{(1-\jmath 2\pi)t}) = e^{t}\sin(2\pi t)$. This is a 1Hz sine wave that exponentially grows by a factor of $e^1 = 2.718$ every second. A signal peak is near $t = 1/4$, where the signal has an amplitude of 1.284. See Figure SB.22b.

(c) $x_3(t) = 3 - \mathrm{Im}(e^{(1-\jmath 2\pi)t}) = 3 + e^{t}\sin(2\pi t)$. This is a 1Hz sine wave that exponentially grows by a factor of $e^1 = 2.718$ every second and has an offset of 3.

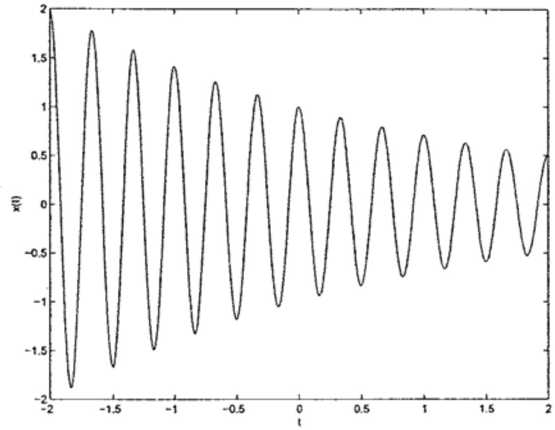


Figure SB.20: Plot of $x(t) = e^{-at}\cos(\omega t)$ for $\omega = 3(2\pi)$ and $a = 0.5\ln(2)$.

A signal peak is near $t = 1/4$, where the signal has an amplitude of 4.284. See Figure SB.22c.

B.22. (a)
```
>> t = [0:.001:3]; x_1 = 2*exp(-t).*cos(2*pi*t);
>> plot(t,x_1,'k-',t,2*exp(-t),'k:',t,-2*exp(-t),'k:');
>> xlabel('t'); ylabel('x_1(t)');
```

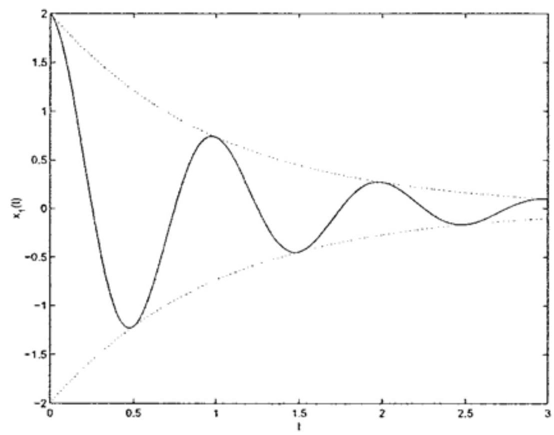


Figure SB.22a: Plot of $x_1(t) = 2e^{-t}\cos(2\pi t)$.

(b)
```
>> t = [0:.001:3]; x_2 = exp(t).*sin(2*pi*t);
>> plot(t,x_2,'k-',t,exp(t),'k:',t,-exp(t),'k:');
>> xlabel('t'); ylabel('x_2(t)');
```

(c)
```
>> t = [0:.001:3]; x_3 = 3+exp(t).*sin(2*pi*t);
>> plot(t,x_3,'k-',t,3+exp(t),'k:',t,3-exp(t),'k:');
>> xlabel('t'); ylabel('x_3(t)');
```

B.23. Since $\cos(t)$ oscillates at $\frac{1}{2\pi}$Hz, $t$ should cover at least $2\pi$ seconds to span one period. Since $\sin(20t)$ has a period of $\frac{2\pi}{20} = 0.314$ seconds, the step size of $t$ should be less than 0.0314 to ensure at least ten samples per period of this fastest component.

```
>> t = [0:.01:8]; x = cos(t).*sin(20*t);
```

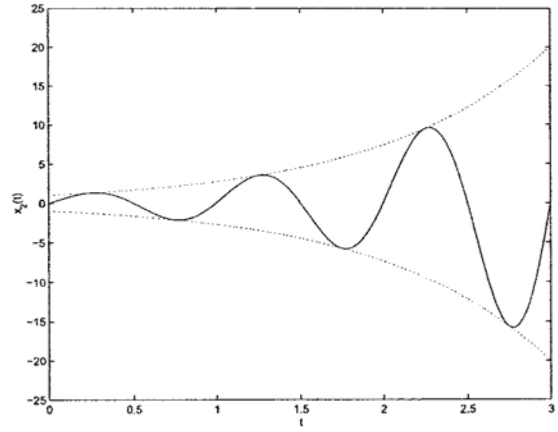


Figure SB.22b: Plot of $x_2(t) = e^t \sin(2\pi t)$.

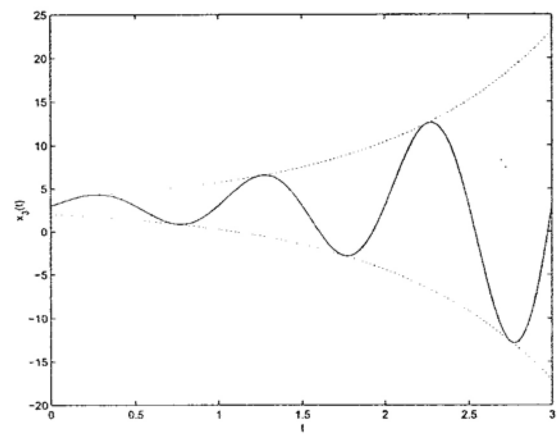


Figure SB.22c: Plot of $x_3(t) = 3 + e^t \sin(2\pi t)$.

```
>> plot(t,x,'k-'); xlabel('t'); ylabel('x(t)');
```

B.24. The highest frequency is 10Hz, so the step size of $t$ should be 0.01 or less to provide ten samples per period of the fastest component. The lowest frequency is 1Hz, so $t$ should span at least one second to cover one period of the slowest component.

```
>> t = [0:.005:2]; kt = (1:10)'*t; x = sum(cos(2*pi*kt));
>> plot(t,x,'k'); xlabel('t'); ylabel('x(t)');
```

B.25. There are many approximations possible for the sound of a bell. In the most simple case, we can model a bell as a decaying exponential. A small, light bell will have a high pitch and not sustain a sound for long. Thus, we might choose a base oscillation of 1kHz. A reasonably quick decay rate is obtained if the envelop decreases by 90% every second, or $e^{\ln(0.1)t}$. Thus, our bell model is $x(t) = e^{\ln(0.1)t} \cos(2\pi 1000t)$. The result, however, is somewhat "flat". Adding harmonics, such as $\cos(2\pi 2000t)$, adds richness to the sound. Furthermore, some low frequency modulation, perhaps as a result of a hand initially ringing the bell, improves the sound. For example,

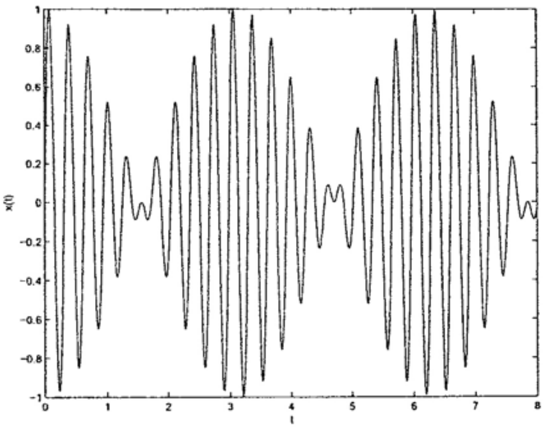


Figure SB.23: Plot of $x(t) = \cos(t)\sin(20t)$.

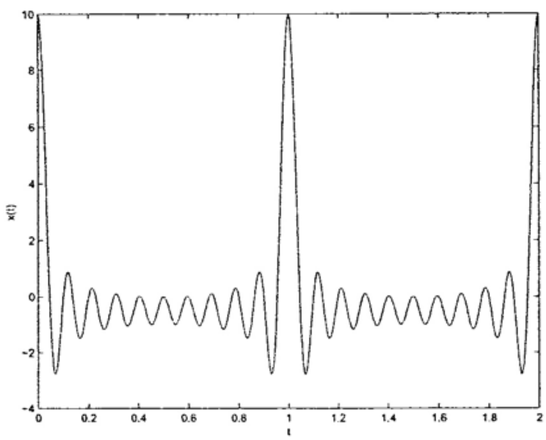


Figure SB.24: Plot of $x(t) = \sum_{k=1}^{10} \cos(2\pi kt)$.

$y(t) = e^{\ln(0.1)t}\cos(2\pi 3t)\left(\cos(2\pi 1000t) + 0.1\cos(2\pi 2000t)\right)$ sounds more natural than $x(t)$. The possibilities are endless.

```
>> t = [0:1/8000:3.5]; a = log(0.1); x = exp(a*t).*(cos(2*pi*1000*t));
>> y = exp(a*t).*(cos(2*pi*3*t).*(cos(2*pi*1000*t)+0.1*cos(2*pi*2000*t)));
>> sound([x,y],8000);
```

If a large, heavy bell is desired, the frequency and decay rates need to be reduced. For example, $z(t) = e^{\ln(0.5)t}\cos(2\pi 3t)\left(\cos(2\pi 100t) + 0.1\cos(2\pi 200t)\right)$.

```
>> t = [0:1/8000:5]; a = log(0.5);
>> z = exp(a*t).*(cos(2*pi*3*t).*(cos(2*pi*200*t)+0.1*cos(2*pi*400*t)));
>> sound(z,8000);
```

B.26. (a) To express $e^{-x^2}$ as a Taylor series, recall that a Taylor series of $e^u$ about zero is given by $e^u = \sum_{i=0}^{\infty} \frac{u^i}{i!}$. Substituting $-x^2$ for $u$ yields

$$e^{-x^2} = \sum_{i=0}^{\infty} \frac{\left(-x^2\right)^i}{i!}.$$

(b) Integrating yields $\int e^{-x^2}dx = \int \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{i!}dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \int x^{2i}dx$ or

$$\int e^{-x^2}dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{x^{2i+1}}{2i+1}.$$

(c) Since the lower limit of the definite integral is zero, it does not make any contribution. Thus $\int_0^1 e^{-x^2}dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{x^{2i+1}}{2i+1}\Big|_{x=1} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(2i+1)}$. First, MATLAB is used to compute the first 10 terms of the sum.

```
>> i = 0:9; terms = (-1).^i./(gamma(i+1).*(2*i+1));
```

Next, one to ten term truncations are obtained using the MATLAB's cumulative sum command.

```
>> cumsum(terms)
```

The results are

$$(1.0000, 0.6667, 0.7667, 0.7429, 0.7475, 0.7467, 0.7468, 0.7468, 0.7468, 0.7468).$$

At a seven-term truncation, the result appears to converge to four digits.

B.27. (a) To express $e^{-x^3}$ as a Taylor series, recall that a Taylor series of $e^u$ about zero is given by $e^u = \sum_{i=0}^{\infty} \frac{u^i}{i!}$. Substituting $-x^3$ for $u$ yields

$$e^{-x^3} = \sum_{i=0}^{\infty} \frac{\left(-x^3\right)^i}{i!}.$$

(b) Integrating yields $\int e^{-x^3}dx = \int \sum_{i=0}^{\infty} \frac{(-1)^i x^{3i}}{i!}dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \int x^{3i}dx$ or

$$\int e^{-x^3}dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{x^{3i+1}}{3i+1}.$$

(c) Since the lower limit of the definite integral is zero, it does not make any contribution. Thus $\int_0^1 e^{-x^3}dx = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{x^{3i+1}}{3i+1}\Big|_{x=1} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(3i+1)}$. First, MATLAB is used to compute the first 10 terms of the sum.

```
>> i = 0:9; terms = (-1).^i./(gamma(i+1).*(3*i+1));
```

Next, one to ten term truncations are obtained using the MATLAB's cumulative sum command.

```
>> cumsum(terms)
```

The results are

$$(1.0000, 0.7500, 0.8214, 0.8048, 0.8080, 0.8074, 0.8075, 0.8075, 0.8075, 0.8075).$$

At a seven-term truncation, the result appears to converge to four digits.

B.28. (a) To express $\cos(x^2) = 0.5\left(e^{jx^2} + e^{-jx^2}\right)$ as a Taylor series, recall that a Taylor series of $e^u$ about zero is given by $e^u = \sum_{i=0}^{\infty} \frac{u^i}{i!}$. Substituting $\pm jx^2$ for $u$ yields

$$\cos(x^2) = \sum_{i=0}^{\infty} 0.5\left(\frac{\left(jx^2\right)^i}{i!} + \frac{\left(-jx^2\right)^i}{i!}\right).$$

(b) Integrating yields $\int \cos(x^2)dx = \int \sum_{i=0}^{\infty} 0.5\left(\frac{(\jmath x^2)^i}{i!} + \frac{(-\jmath x^2)^i}{i!}\right) dx = \sum_{i=0}^{\infty} 0.5\frac{(\jmath)^i(1+(-1)^i)}{i!} \int (x^2)^i\, dx$ or

$$\int \cos(x^2)dx = \sum_{i=0}^{\infty} 0.5\frac{(\jmath)^i(1+(-1)^i)}{i!}\frac{x^{2i+1}}{2i+1}.$$

(c) Since the lower limit of the definite integral is zero, it does not make any contribution. Thus $\int_0^1 \cos(x^2)dx = \sum_{i=0}^{\infty} 0.5\frac{(\jmath)^i(1+(-1)^i)}{i!}\frac{x^{2i+1}}{2i+1}\Big|_{x=1} = \sum_{i=0}^{\infty} 0.5\frac{(\jmath)^i(1+(-1)^i)}{i!(2i+1)}$. First, MATLAB is used to compute the first 10 terms of the sum.

```
>> i = 0:9; terms = 0.5*(j).^i.*(1+(-1).^i)./(gamma(i+1).*(2*i+1));
```

Next, one to ten term truncations are obtained using the MATLAB's cumulative sum command.

```
>> cumsum(terms)
```

The results are

$$(1.0000, 1.0000, 0.9000, 0.9000, 0.9046, 0.9046, 0.9045, 0.9045, 0.9045, 0.9045).$$

At a seven-term truncation, the result appears to converge to four digits.

B.29. (a) Using synthetic division, express $f_1(x) = \frac{1}{2-x^2} = \frac{1}{2} + \frac{1}{4}x^2 + \frac{1}{8}x^4 + \frac{1}{16}x^6 + \cdots$. Thus,

$$f_1(x) = \sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{i+1} x^{2i}.$$

(b) Rewrite as $f_2(x) = (0.5)^x = e^{-\ln(2)x}$. Recall that a Taylor series of $e^u$ about zero is given by $e^u = \sum_{i=0}^{\infty}\frac{u^i}{i!}$. Substituting $-\ln(2)x$ for $u$ yields

$$f_2(x) = \sum_{i=0}^{\infty}\frac{(-\ln(2)x)^i}{i!}.$$

B.30. (a) Begin by choosing a point on the unit circle, $w = e^{\jmath\Omega}$. Multiplying $w$ by itself yields $ww = w^2 = e^{\jmath 2\Omega}$. Taking this result and again multiplying by $w$ yields $ww^2 = w^3 = e^{\jmath 3\Omega}$. At step $n$, the result is $w^n = e^{\jmath n\Omega}$. From Euler's identity, we know $w^n = e^{\jmath n\Omega} = \cos(n\Omega) + \jmath\sin(n\Omega)$. The process does indeed provide the desired quadrature sinusoids: the real part provides the cosine term and the imaginary part yields the sine term.

(b) To produce a periodic signal, $\Omega$ needs to be a rational multiple of $2\pi$. Most simply, choose $\Omega = 2\pi/N$, where $N$ is the number of points computed per oscillation of each sinusoid. For reasonable quality sinusoids, $N$ should be some moderately large integer, say 10 or 20. Although the quality of the sinusoids increases as $N$ is increased, the required processing speed also increases with $N$. Thus, $N$ represents a compromise between signal quality and processor speed. Taking $N = 20$, for example, yields $w = e^{\jmath\pi/10}$. In this case, only $\frac{1}{100000N} = 500e-9$ seconds (500ns) are available to process each sample. This is feasible with current processor technologies.

(c) Although not required by the procedure, a vector $x[n]$ is maintained so that the signal outputs can be plotted.

```
>> N = 20; w = exp(j*2*pi/N); w_n = w;
>> I = 40; x = zeros(1,I); x(1) = w_n;
>> for i = 1:I; x(i) = w_n; w_n = w_n*w; end
>> plot([1:I],real(x),'k-',[1:I],imag(x),'k:');
>> xlabel('n'); ylabel('Amplitude');
```

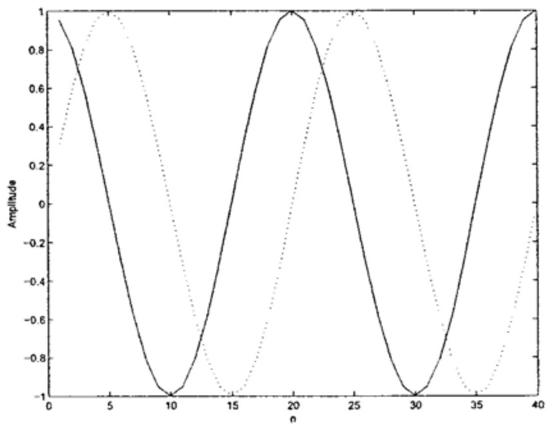


Figure SB.30c: Simulation to generate quadrature sinusoids.

(d) To work, this procedure requires several assumptions. First, we assume $N$ is chosen large enough to provide good-quality sinusoids yet provide ample time during each step to compute the next value. For periodicity, the frequency $\Omega$ needs to be a rational multiple of $2\pi$. Each of these assumptions can generally be met. However, there are at least two limitations that may affect the suitability of this procedure:

i. Since digital processors represent numbers with a finite number of bits, there is often an error associated with representing $w$. Instead of $w$, the computer stores $w + \Delta$. Due to the iterative nature of the procedure, the error grows with time. Generally, if $|w + \Delta| > 1$ then the signals will exponentially grow and if $|w + \delta| < 1$ the signals will exponentially decay. This limitation can prevent the procedure from working correctly over an indefinite time period.

ii. For the output signals to be truly periodic, the processor must take exactly the same amount of time between steps. This is impossible; timing errors are always present. Additionally, if the desired output frequency is not divisible by the processor clock speed, the resulting signals with either not be truly periodic, have slight frequency errors, or both.

B.31. First, the system of equations is written in matrix form.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = Ax = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}.$$

$|A| = 0 + 3 - 1 - (2 - 3 - 0) = 3.$

(a) $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 3 \\ -3 & -1 & 0 \end{vmatrix} = 0 - 9 - 3 - (-6 - 3 - 0) = -3$. Thus,

$$x_1 = \frac{-3}{|\boldsymbol{A}|} = \frac{-3}{3} = -1.$$

The same result is obtained in MATLAB by

```
>> A = [1 1 1;1 2 3;1 -1 0];
>> x_1 = det([[1;3;-3],A(:,2:3)])/det(A)
x_1 = -1
```

(b) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & -3 & 0 \end{vmatrix} = 0 + 3 - 3 - (3 - 9 - 0) = 6$. Thus,

$$x_2 = \frac{6}{|\boldsymbol{A}|} = \frac{6}{3} = 2.$$

The same result is obtained in MATLAB by

```
>> A = [1 1 1;1 2 3;1 -1 0];
>> x_2 = det([A(:,1),[1;3;-3],A(:,3)])/det(A)
x_2 = 2
```

(c) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{vmatrix} = -6 - 1 + 3 - (2 - 3 - 3) = 0$. Thus,

$$x_3 = \frac{0}{|\boldsymbol{A}|} = \frac{0}{3} = 0.$$

The same result is obtained in MATLAB by

```
>> A = [1 1 1;1 2 3;1 -1 0];
>> x_3 = det([A(:,1:2),[1;3;-3]])/det(A)
x_3 = 0
```

B.32. (a) A matrix representation is

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{A}\boldsymbol{x} = \boldsymbol{y} = \begin{bmatrix} c \\ f \end{bmatrix}.$$

(b) By inspection, $x_1 = 3$ and $x_2 = -2$ can be obtained by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Thus, $a = 1, b = 0, c = 3, d = 0, e = 1$, and $f = -2$ is one possible set of constants. These constants are not unique. Any linear combination of the rows yields the same solution set. For example, $a = 2, b = 0, c = 6, d = 1, e = 1$, and $f = 1$ also works.

To ensure unique values of $x_1$ and $x_2$, the matrix $\boldsymbol{A}$ must be full rank.

(c) For no solutions to exist, the matrix $\boldsymbol{A}$ must be rank deficient, and $[\boldsymbol{A}, \boldsymbol{y}]$ must increase the rank of $\boldsymbol{A}$ by one. For example,

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The rank of $\boldsymbol{A}$ is one and the rank of $[\boldsymbol{A}, \boldsymbol{y}]$ is two. Thus, there are no solutions. MATLAB verifies the desired ranks are obtained.

```
>> A = [1 1;2 2]; y = [1;1];
>> [rank(A), rank([A,y])]
ans =
     1     2
```

(d) For an infinite number of solutions to exist, the matrix $\boldsymbol{A}$ must be rank deficient, and $[\boldsymbol{A}, \boldsymbol{y}]$ must not increase the rank of $\boldsymbol{A}$. For example,

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The rank of $\boldsymbol{A}$ is one and the rank of $[\boldsymbol{A}, \boldsymbol{y}]$ is also one. Thus, there are an infinite number of solutions. MATLAB verifies the desired ranks are obtained.

```
>> A = [1 1;2 2]; y = [1;2];
>> [rank(A), rank([A,y])]
ans =
     1     1
```

B.33. The system of equations is first written in matrix form.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ -2 \end{bmatrix}.$$

Next, the result is obtained using MATLAB.

```
>> A = [1 1 1 1;1 1 1 -1;1 1 -1 -1;1 -1 -1 -1];
>> x = A\[4;2;0;-2]
x =  1
     1
     1
     1
```

That is, $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, and $x_4 = 1$.

B.34. First, the system of equations is written in matrix form.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -2 & 3 & 0 \\ 1 & 0 & -1 & 7 \\ 0 & -2 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

The result is obtained using MATLAB.

```
>> A = [1 1 1 1;1 -2 3 0;1 0 -1 7;0 -2 3 -4];
>> x = A\[1;2;3;4]
```

```
x = -30.0000
      8.0000
     16.0000
      7.0000
```

That is, $x_1 = -30$, $x_2 = 8$, $x_3 = 16$, and $x_4 = 7$.

B.35. (a) $H_1(s) = \frac{s^2+5s+6}{s^3+s^2+s+1} = \frac{s^2+5s+6}{(s-j)(s+j)(s+1)} = \frac{k_1}{s-j} + \frac{k_2}{s+j} + \frac{k_3}{s+1}$. Using the method of residues, $k_1 = \left.\frac{s^2+5s+6}{(s+j)(s+1)}\right|_{s=j} = \frac{5(1+j)}{2j(1+j)} = -2.5j$. Since the system is real, $k_2 = k_1^* = 2.5j$. Lastly, $k_3 = \left.\frac{s^2+5s+6}{s^2+1}\right|_{s=-1} = 1$. Thus,

$$H_1(s) = \frac{1}{s+1} + \frac{-2.5j}{s-j} + \frac{2.5j}{s+j} = \frac{1}{s+1} + \frac{5}{s^2+1}.$$

(b) $H_2(s) = \frac{1}{H_1(s)} = \frac{s^3+s^2+s+1}{s^2+5s+6} = s - 4 + \frac{15s+25}{(s+2)(s+3)} = s - 4 + \frac{k_1}{s+2} + \frac{k_2}{s+3}$. Using the method of residues, $k_1 = \left.\frac{15s+25}{s+3}\right|_{s=-2} = -5$ and $k_2 = \left.\frac{15s+25}{s+2}\right|_{s=-3} = 20$. Thus,

$$H_2(s) = s - 4 + \frac{-5}{s+2} + \frac{20}{s+3}.$$

(c) $H_3(s) = \frac{1}{(s+1)^2(s^2+1)} = \frac{1}{(s+1)^2(s+j)(s-j)} = \frac{k_1}{s-j} + \frac{k_2}{s+j} + \frac{\tilde{a}_0}{(s+1)^2} + \frac{\tilde{a}_1}{(s+1)}$. Using the method of residues, $k_1 = \left.\frac{1}{(s+1)^2(s+j)}\right|_{s=j} = \frac{1}{(1+j2-1)(j2)} = -0.25$. Since the corresponding roots are complex conjugates, $k_2 = k_1^* = -0.25$. $\tilde{a}_0 = \left.\frac{1}{s^2+1}\right|_{s=-1} = 0.5$ and $\tilde{a}_1 = \frac{d}{ds}(s^2+1)^{-1}\big|_{s=-1} = -(s^2+1)^{-2}(2s)\big|_{s=-1} = \frac{-2}{-4} = 0.5$. Thus,

$$H_3(s) = \frac{-0.25}{s-j} + \frac{-0.25}{s+j} + \frac{0.5}{(s+1)^2} + \frac{0.5}{(s+1)}.$$

(d) $H_4(s) = \frac{s^2+5s+6}{3s^2+2s+1} = \frac{1}{3} + \frac{13s/3+17/3}{3s^2+2s+1} = \frac{13s/9+17/9}{s^2+2s/3+1/3}$. In some cases, this form is sufficient. A complete partial fraction expansion, however, requires the denominator roots $s = \frac{-2/3\pm\sqrt{4/9-4/3}}{2} = \frac{-1\pm j\sqrt{2}}{3} = -0.3333 \pm 0.4714j$. Thus, $H_4(s) = \frac{1}{3} + \frac{k_1}{s-(-1-j\sqrt{2})/3} + \frac{k_2}{s-(-1+j\sqrt{2})/3}$. Using the method of residues, $k_1 = \left.\frac{13s/9+17/9}{s-(-1+j\sqrt{2})/3}\right|_{s=(-1-j\sqrt{2})/3} = 0.7222 + 1.4928j$. Since the system is real, $k_2 = k_1^* = 0.7222 - 1.4928j$. Thus,

$$H_4(s) = \frac{1}{3} + \frac{0.7222+1.4928j}{s+0.3333+0.4714j} + \frac{0.7222-1.4928j}{s+0.3333-0.4714j}.$$

B.36. The MATLAB `residue` command computes the partial fraction expansion of a rational function by providing three quantities: the residues, the poles, and the direct terms.

(a)
```
>> [r,p,k] = residue([1 5 6],[1 1 1 1])
r =  1.0000
     0.0000 - 2.5000i
     0.0000 + 2.5000i
p = -1.0000
    -0.0000 + 1.0000i
```

```
   -0.0000 - 1.0000i
k = []
```

Thus,

$$H_1(s) = \frac{1}{s+1} + \frac{-2.5j}{s-j} + \frac{2.5j}{s+j} = \frac{1}{s+1} + \frac{5}{s^2+1}.$$

(b)
```
>> [r,p,k] = residue([1 1 1 1],[1 5 6])
r = 20.0000
   -5.0000
p = -3.0000
   -2.0000
k =  1    -4
```

Thus,

$$H_2(s) = \frac{20}{s+3} + \frac{-5}{s+2} + s - 4.$$

(c)
```
>> [r,p,k] = residue(1,poly([-1,-1,j,-j]))
r =  0.5000
     0.5000
    -0.2500 - 0.0000i
    -0.2500 + 0.0000i
p = -1.0000
    -1.0000
     0.0000 + 1.0000i
     0.0000 - 1.0000i
k =  []
```

Thus,

$$H_3(s) = \frac{0.5}{(s+1)} + \frac{0.5}{(s+1)^2} + \frac{-0.25}{s-j} + \frac{-0.25}{s+j}.$$

(d)
```
>> [r,p,k] = residue([1 5 6],[3 2 1])
r =  0.7222 - 1.4928i
     0.7222 + 1.4928i
p = -0.3333 + 0.4714i
    -0.3333 - 0.4714i
k =  0.3333
```

Thus,

$$H_4(s) = \frac{1}{3} + \frac{0.7222 - 1.4928j}{s + 0.3333 - 0.4714j} + \frac{0.7222 + 1.4928j}{s + 0.3333 + 0.4714j}.$$

B.37. First, express both sides of the expression with a common denominator $F(s) = \frac{s}{(s+1)^3} = \frac{a_0}{(s+1)^3} + \frac{a_1}{(s+1)^2} + \frac{a_2}{(s+1)} = \frac{a_0+a_1(s+1)+a_2(s+1)^2}{(s+1)^3} = \frac{a_2s^2+(a_1+2a_2)s+(a_0+a_1+a_2)}{(s+1)^3}$. Equating the coefficients of $s^2$ yields $a_2 = 0$. Thus $(a_1 + 2a_2) = a_1 = 1$. Finally $a_0 + a_1 + a_2 = a_0 + 1 + 0 = 0$ implies that $a_0 = -1$.

$$a_0 = -1, a_1 = 1, \text{ and } a_2 = 0.$$

B.38. Many solutions are possible to this problem, but the procedure is the same in each case. Consider a fictitious phone number 555-5555. Then, $H_N(s) = \frac{5s^2+5s+5+5s^{-1}}{5s^2+5s+5}\frac{s}{s} = \frac{5s^3+5s^2+5s+5}{5s^3+5s^2+5s+0}$. The partial fraction expansion of $H_N(s)$ is obtained using the MATLAB residue command.

```
>> [r,p,k] = residue([5 5 5 5],[5 5 5 0])
```

```
r = -0.5000 + 0.2887i
    -0.5000 - 0.2887i
     1.0000
p = -0.5000 + 0.8660i
    -0.5000 - 0.8660i
     0
k =  1
```

Thus,

$$H_N(s) = 1 + \frac{-0.5000 + 0.2887\jmath}{s + 0.5000 - 0.8660\jmath} + \frac{-0.5000 - 0.2887\jmath}{s + 0.5000 + 0.8660\jmath} + \frac{1}{s}.$$

B.39. (a)

```
>> omega = linspace(-pi,pi,201);
>> fr = cos(omega); fi = 0.1*sin(2*omega);
>> plot(fr,fi,'k-'); xlabel('Re(f)'); ylabel('Im(f)');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
```

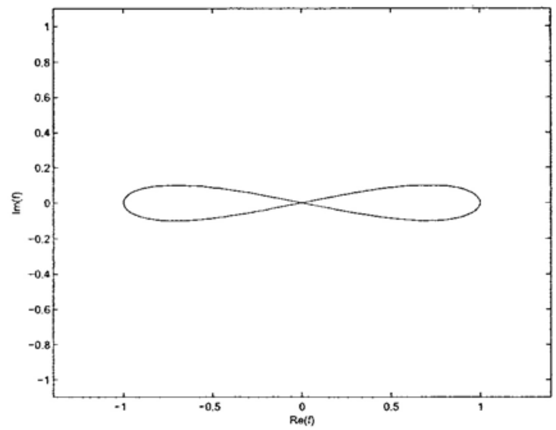


Figure SB.39a: Lissajous figure resembling horizontal propeller.

(b) Multiplying $w$ by $e^{\jmath\theta}$ adds $\theta$ to the angle of $w$ and thereby rotates $w$ by $\theta$. Also, $we^{\jmath\theta} = (x+\jmath y)(\cos(\theta)+\jmath\sin(\theta)) = (x\cos(\theta)-y\sin(\theta))+\jmath(x\sin(\theta)+y\cos(\theta))$. Furthermore, $\mathbf{R}\mathbf{w} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{bmatrix}$. Thus, $\mathbf{R}\mathbf{w}$ and $we^{\jmath\theta}$ are equivalent, and $\mathbf{R}\mathbf{w}$ rotates $w$ by $\theta$.

(c)

```
>> theta = 10*pi/180; R = [cos(theta) -sin(theta);sin(theta) cos(theta)];
>> f = [fr;fi]; f = R*f;
>> plot(f(1,:),f(2,:),'k-'); xlabel('Re(Rf)'); ylabel('Im(Rf)');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
```

(d) If $Rf$ rotates $f$ by $\theta$, then $RRf = R(Rf)$ rotates $f$ by $2\theta$. Similarly, $RRRf$ rotates $f$ by $3\theta$. In general, $(R^N)f$ rotates $f$ by $N\theta$.

(e) As suggested in B.39b, multiplying $f(\omega)$ by the function $e^{\jmath\theta}$ simply rotates $f$ by $\theta$. For example, the previous plot is also obtained by

```
>> f = fr + j*fi; f = f*exp(j*theta);
>> plot(real(f),imag(f),'k-');
>> xlabel('Re(fe^{j\theta})'); ylabel('Im(fe^{j\theta})');
>> axis([-1.1 1.1 -1.1 1.1]); axis equal;
```

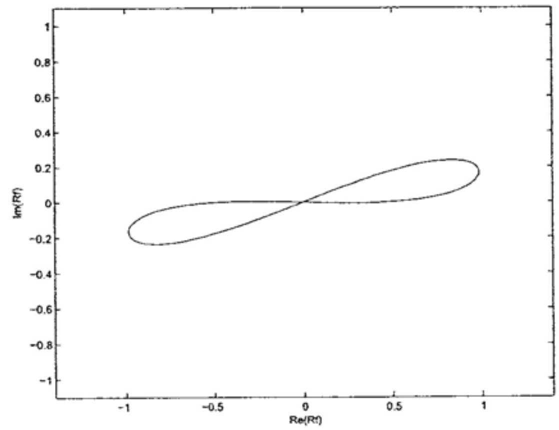

Figure SB.39c: Lissajous figure rotated 10 degrees CCW.

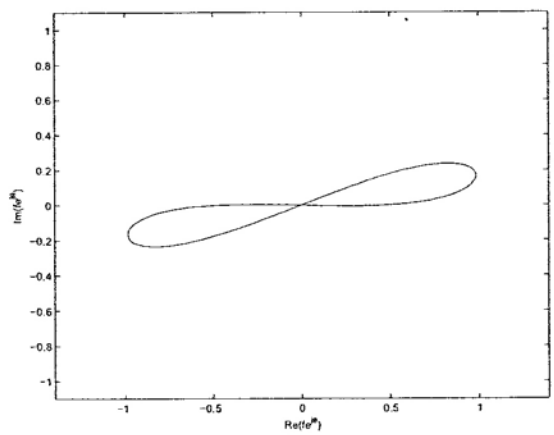

Figure SB.39e: Lissajous figure rotated by alternate method.