Chapter 2 Solutions

2.2-1. The characteristic polynomial is $\lambda^2+5\lambda+6$. The characteristic equation is $\lambda^2+5\lambda+6 = 0$. Also $\lambda^2+5\lambda+6 = (\lambda+2)(\lambda+3)$. Therefore the characteristic roots are $\lambda_1 = -2$ and $\lambda_2 = -3$. The characteristic modes are e^{-2t} and e^{-3t}. Therefore

$$y_0(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

and

$$\dot{y}_0(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$$

Setting $t = 0$, and substituting initial conditions $y_0(0) = 2$, $\dot{y}_0(0) = -1$ in this equation yields

$$\left.\begin{array}{r} c_1 + c_2 = 2 \\ -2c_1 - 3c_2 = -1 \end{array}\right\} \Longrightarrow \begin{array}{l} c_1 = 5 \\ c_2 = -3 \end{array}$$

Therefore

$$y_0(t) = 5e^{-2t} - 3e^{-3t}$$

2.2-2. The characteristic polynomial is $\lambda^2+4\lambda+4$. The characteristic equation is $\lambda^2+4\lambda+4 = 0$. Also $\lambda^2+4\lambda+4 = (\lambda+2)^2$, so that the characteristic roots are -2 and -2 (repeated twice). The characteristic modes are e^{-2t} and te^{-2t}. Therefore

$$y_0(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

and

$$\dot{y}_0(t) = -2c_1 e^{-2t} - 2c_2 t e^{-2t} + c_2 e^{-2t}$$

Setting $t = 0$ and substituting initial conditions yields

$$\left.\begin{array}{r} 3 = c_1 \\ -4 = -2c_1 + c_2 \end{array}\right\} \Longrightarrow \begin{array}{l} c_1 = 3 \\ c_2 = 2 \end{array}$$

Therefore

$$y_0(t) = (3 + 2t)e^{-2t}$$

2.2-3. The characteristic polynomial is $\lambda(\lambda+1) = \lambda^2 + \lambda$. The characteristic equation is $\lambda(\lambda+1) = 0$. The characteristic roots are 0 and -1. The characteristic modes are 1 and e^{-t}. Therefore

$$y_0(t) = c_1 + c_2 e^{-t}$$

and

$$\dot{y}_0(t) = -c_2 e^{-t}$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{array}{r} 1 = c_1 + c_2 \\ 1 = -c_2 \end{array}\right\} \Longrightarrow \begin{array}{l} c_1 = 2 \\ c_2 = -1 \end{array}$$

Therefore

$$y_0(t) = 2 - e^{-t}$$

2.2-4. The characteristic polynomial is λ^2+9. The characteristic equation is $\lambda^2+9=0$ or $(\lambda+j3)(\lambda-j3)=0$. The characteristic roots are $\pm j3$. The characteristic modes are e^{j3t} and e^{-j3t}. Therefore

$$y_0(t) = c\cos(3t+\theta)$$

and

$$\dot{y}_0(t) = -3c\sin(3t+\theta)$$

Setting $t=0$, and substituting initial conditions yields

$$\left.\begin{array}{r} 0 = c\cos\theta \\ 6 = -3c\sin\theta \end{array}\right\} \Longrightarrow \left.\begin{array}{r} c\cos\theta = 0 \\ c\sin\theta = -2 \end{array}\right\} \Longrightarrow \begin{array}{l} c = 2 \\ \theta = -\pi/2 \end{array}$$

Therefore

$$y_0(t) = 2\cos(3t-\frac{\pi}{2}) = 2\sin 3t$$

2.2-5. The characteristic polynomial is $\lambda^2+4\lambda+13$. The characteristic equation is $\lambda^2+4\lambda+13=0$ or $(\lambda+2-j3)(\lambda+2+j3)=0$. The characteristic roots are $-2\pm j3$. The characteristic modes are $c_1e^{(-2+j3)t}$ and $c_2e^{(-2-j3)t}$. Therefore

$$y_0(t) = ce^{-2t}\cos(3t+\theta)$$

and

$$\dot{y}_0(t) = -2ce^{-2t}\cos(3t+\theta) - 3ce^{-2t}\sin(3t+\theta)$$

Setting $t=0$, and substituting initial conditions yields

$$\left.\begin{array}{r} 5 = c\cos\theta \\ 15.98 = -2c\cos\theta - 3c\sin\theta \end{array}\right\} \Longrightarrow \left.\begin{array}{r} c\cos\theta = 5 \\ c\sin\theta = -8.66 \end{array}\right\} \Longrightarrow \begin{array}{l} c = 10 \\ \theta = -\pi/3 \end{array}$$

Therefore

$$y_0(t) = 10e^{-2t}\cos(3t-\frac{\pi}{3})$$

2.2-6. The characteristic polynomial is $\lambda^2(\lambda+1)$ or $\lambda^3+\lambda^2$. The characteristic equation is $\lambda^2(\lambda+1)=0$. The characteristic roots are 0, 0 and -1 (0 is repeated twice). Therefore

$$y_0(t) = c_1 + c_2t + c_3e^{-t}$$

and

$$\begin{aligned} \dot{y}_0(t) &= c_2 - c_3e^{-t} \\ \ddot{y}_0(t) &= c_3e^{-t} \end{aligned}$$

Setting $t=0$, and substituting initial conditions yields

$$\left.\begin{array}{r} 4 = c_1 + c_3 \\ 3 = c_2 - c_3 \\ -1 = c_3 \end{array}\right\} \Longrightarrow \begin{array}{l} c_1 = 5 \\ c_2 = 2 \\ c_3 = -1 \end{array}$$

Therefore

$$y_0(t) = 5 + 2t - e^{-t}$$

2.2-7. The characteristic polynomial is $(\lambda+1)(\lambda^2+5\lambda+6)$. The characteristic equation is $(\lambda+1)(\lambda^2+5\lambda+6)=0$ or $(\lambda+1)(\lambda+2)(\lambda+3)=0$. The characteristic roots are -1, -2 and -3. The characteristic modes are e^{-t}, e^{-2t} and e^{-3t}. Therefore

$$y_0(t) = c_1e^{-t} + c_2e^{-2t} + c_3e^{-3t}$$

and

$$\dot{y}_0(t) = -c_1e^{-t} - 2c_2e^{-2t} - 3c_3e^{-3t}$$

$$\ddot{y}_0(t) \quad = \quad c_1 e^{-t} + 4c_2 e^{-2t} + 9c_3 e^{-3t}$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{array}{r} 2 = c_1 + c_2 + c_3 \\ -1 = -c_1 - 2c_2 - 3c_3 \\ 5 = c_1 + 4c_2 + 9c_3 \end{array}\right\} \Longrightarrow \quad \begin{array}{l} c_1 = 6 \\ c_2 = -7 \\ c_3 = 3 \end{array}$$

Therefore

$$y_0(t) = 6e^{-t} - 7e^{-2t} + 3e^{-3t}$$

2.2-8. The zero-input response for a LTIC system is given as $y_0(t) = 2e^{-t} + 3$. Since two modes are visible, the system must have, at least, the characteristic roots $\lambda_1 = 0$ and $\lambda_2 = -1$.

(a) No, it is not possible for the system's characteristic equation to be $\lambda + 1 = 0$ since the required mode at $\lambda = 0$ is missing.

(b) Yes, it is possible for the system's characteristic equation to be $\sqrt{3}(\lambda^2 + \lambda) = 0$ since this equation has the two required roots $\lambda_1 = 0$ and $\lambda_2 = -1$.

(c) Yes, it is possible for the system's characteristic equation to be $\lambda(\lambda+1)^2 = 0$. This equation supports a general zero-input response of $y_0(t) = c_1 + c_2 e^{-t} + c_3 t e^{-t}$. By letting $c_1 = 3$, $c_2 = 2$, and $c_3 = 0$, the observed zero-input response is possible.

2.3-1. The characteristic equation is $\lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3) = 0$. The characteristic modes are e^{-t} and e^{-3t}. Therefore

$$y_n(t) = c_1 e^{-t} + c_2 e^{-3t}$$
$$\dot{y}_n(t) = -c_1 e^{-t} - 3c_2 e^{-3t}$$

Setting $t = 0$, and substituting $y(0) = 0$, $\dot{y}(0) = 1$, we obtain

$$\left.\begin{array}{r} 0 = c_1 + c_2 \\ 1 = -c_1 - 3c_2 \end{array}\right\} \Longrightarrow \quad \begin{array}{l} {}_1 = \frac{1}{2} \\ c_2 = -\frac{1}{2} \end{array}$$

Therefore

$$y_n(t) = \frac{1}{2}(e^{-t} - e^{-3t})$$

$$h(t) = [P(D)y_n(t)]u(t) = [(D+5)y_n(t)]u(t) = [\dot{y}_n(t) + 5y_n(t)]u(t) = (2e^{-t} - e^{-3t})u(t)$$

2.3-2. The characteristic equation is $\lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3) = 0$. and

$$y_n(t) = c_1 e^{-2t} + c_2 e^{-3t}$$
$$\dot{y}_n(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$$

Setting $t = 0$, and substituting $y(0) = 0$, $\dot{y}(0) = 1$, we obtain

$$\left.\begin{array}{r} 0 = c_1 + c_2 \\ 1 = -2c_1 - 3c_2 \end{array}\right\} \Longrightarrow \quad \begin{array}{l} c_1 = 1 \\ c_2 = -1 \end{array}$$

Therefore

$$y_n(t) = e^{-2t} - e^{-3t}$$

and

$$[P(D)y_n(t)]u(t) = [\ddot{y}_n(t) + 7\dot{y}_n(t) + 11y_n(t)]u(t) = (e^{-2t} + e^{-3t})u(t)$$

Hence

$$h(t) = b_n\delta(t) + [P(D)y_n(t)]u(t) = \delta(t) + (e^{-2t} + e^{-3t})u(t)$$

2.3-3. The characteristic equation is $\lambda + 1 = 0$ and

$$y_n(t) = ce^{-t}$$

In this case the initial condition is $y_n^{n-1}(0) = y_n(0) = 1$. Setting $t = 0$, and using $y_n(0) = 1$, we obtain $c = 1$, and

$$y_n(t) = e^{-t}$$
$$P(D)y_n(t) = [-\dot{y}_n(t) + y_n(t)]u(t) = 2e^{-t}u(t)$$

Hence $$h(t) = b_n\delta(t) + [P(D)y_n(t)]u(t) = -\delta(t) + 2e^{-t}u(t)$$

2.3-4. The characteristic equation is $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$. Therefore

$$y_n(t) = (c_1 + c_2 t)e^{-3t}$$
$$\dot{y}_n(t) = [-3(c_1 + c_2 t) + c_2]e^{-3t}$$

Setting $t = 0$, and substituting $y_n(0) = 1$, $\dot{y}_n(0) = 1$, we obtain

$$\left.\begin{array}{l} 0 = c_1 \\ 1 = -3c_1 + c_2 \end{array}\right\} \Longrightarrow \begin{array}{l} c_1 = 0 \\ c_2 = 1 \end{array}$$

and $$y_n(t) = te^{-3t}$$

Hence

$$h(t) = [P(D)y_n(t)]u(t) = [2\dot{y}_n(t) + 9y_n(t)]u(t) = (2 + 3t)e^{-3t}u(t)$$

2.4-1.

$$\begin{aligned} A_c &= \int_{-\infty}^{\infty} c(t)\,dt \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)g(t-\tau)\,d\tau\right] dt \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\,d\tau\right] g(t-\tau)\,dt \\ &= A_x \int_{-\infty}^{\infty} g(t-\tau)\,dt \\ &= A_x A_g \end{aligned}$$

This property can be readily verified from Examples 2.7 and 2.8. For Example 2.6, we note that

$$\int_{-\infty}^{\infty} e^{-at}\,dt = \frac{1}{a}$$

Use of this result yields $A_x = 1$, $A_h = 0.5$, and $A_y = 1 - 0.5 = 0.5 = A_x A_h$ For example 2.8, $A_x = 2$, $A_g = 1.5$, and

$$\begin{aligned} A_c &= \int_{-1}^{1} -\frac{1}{6}(t+1)^2\,dt + \int_{1}^{2} \frac{2}{3}t\,dt + \int_{2}^{4} -\frac{1}{6}(t^2 - 2t - 8)\,dt \\ &= \frac{4}{9} + 1 + \frac{14}{9} = 3 = A_x A_g \end{aligned}$$

2.4-2.

$$\begin{aligned} x(at) * g(at) &= \int_{-\infty}^{\infty} x(a\tau)g[a(t-\tau)]\,d\tau \\ &= \frac{1}{a}\int_{-\infty}^{\infty} x(w)g(at-w)\,dw \\ &= \frac{1}{a}c(at) \qquad a \geq 0 \end{aligned}$$

When $a < 0$, the limits of integration become from ∞ to $-\infty$, which is equivalent to the limits from $-\infty$ to ∞ with a negative sign. Hence, $x(at) * g(at) = |\frac{1}{a}|c(at)$.

2.4-3. Let $x(t) * g(t) = c(t)$. Using the time scaling property in Prob. 2.4-2 with $a = -1$, we have $x(-t) * g(-t) = c(-t)$. Now, if $x(t)$ and $g(t)$ are both even functions of t, then $x(t) = x(-t)$ and $g(-t) = g(t)$. Clearly $c(t) = c(-t)$. Using a parallel argument, we can show that if both functions are odd, $c(t) = c(-t)$, indicating that $c(t)$ is even. But if one is odd and the other is even, $c(t) = -c(-t)$, indicating that $c(t)$ is odd.

2.4-4.

$$\begin{aligned} e^{-at}u(t) * e^{-bt}u(t) &= \int_0^t e^{-a\tau}e^{-b(t-\tau)}\,d\tau = e^{-bt}\int_0^t e^{(b-a)\tau}\,d\tau \\ &= \frac{e^{-bt}}{b-a}e^{(b-a)\tau}\Big|_0^t = \frac{e^{-bt}}{b-a}[e^{(b-a)t}-1] = \frac{e^{-at}-e^{-bt}}{a-b} \end{aligned}$$

Because both functions are causal, their convolution is zero for $t < 0$. Therefore

$$e^{-at}u(t) * e^{-bt}u(t) = \left(\frac{e^{-at}-e^{-bt}}{a-b}\right)u(t)$$

2.4-5. (i)

$$\begin{aligned} u(t) * u(t) &= \int_0^t u(\tau)u(t-\tau)\,d\tau = \int_0^t d\tau = \tau\Big|_0^t = t \quad \text{for } t \geq 0 \\ &= 0 \qquad \text{for } t < 0 \end{aligned}$$

Therefore

$$u(t) * u(t) = tu(t)$$

(ii) Because both functions are causal

$$\begin{aligned} e^{-at}u(t) * e^{-at}u(t) = \int_0^t e^{-a\tau}e^{-a(t-\tau)}\,d\tau &= e^{-at}\int_0^t d\tau \\ &= te^{-at} \qquad t \geq 0 \end{aligned}$$

and

$$e^{-at}u(t) * e^{-at}u(t) = te^{-at}u(t)$$

(iii) Because both functions are causal

$$tu(t) * u(t) = \int_0^t \tau u(\tau)u(\tau - t)\,d\tau$$

The range of integration is $0 \leq \tau \leq t$. Therefore $\tau > 0$ and $\tau - t > 0$ so that

$u(\tau) = u(\tau - t) = 1$ and

$$tu(t) * u(t) = \int_0^t \tau \, d\tau = \frac{t^2}{2} \qquad t \geq 0$$

and

$$tu(t) * u(t) = \frac{1}{2} t^2 u(t)$$

2.4-6. (i)

$$\sin tu(t) * u(t) = \left(\int_0^t \sin \tau \, u(\tau) u(t - \tau) \, d\tau \right) u(t)$$

Because τ and $t - \tau$ are both nonnegative (when $0 \leq \tau \leq t$), $u(\tau) = u(t - \tau) = 1$, and

$$\sin t \, u(t) * u(t) = \left(\int_0^t \sin \tau \, d\tau \right) u(t) = (1 - \cos t) u(t)$$

(ii) Similarly

$$\cos t \, u(t) * u(t) = \left(\int_0^t \cos \tau \, d\tau \right) u(t) = \sin t \, u(t)$$

2.4-7. In this problem, we use Table 2.1 to find the desired convolution.

(a) $y(t) = h(t) * x(t) = e^{-t} u(t) * u(t) = (1 - e^{-t}) u(t)$

(b) $y(t) = h(t) * x(t) = e^{-t} u(t) * e^{-t} u(t) = te^{-t} u(t)$

(c) $y(t) = e^{-t} u(t) * e^{-2t} u(t) = (e^{-t} - e^{-2t}) u(t)$

(d) $y(t) = \sin 3tu(t) * e^{-t} u(t)$
Here we use pair 12 (Table 2.1) with $\alpha = 0$, $\beta = 3$, $\theta = -90°$ and $\lambda = -1$. This yields

$$\phi = \tan^{-1} \left[\frac{-3}{-1} \right] = -108.4°$$

and

$$\begin{aligned} \sin 3t \, u(t) * e^{-t} u(t) &= \frac{(\cos 18.4°) e^{-t} - \cos(3t + 18.4°)}{\sqrt{10}} u(t) \\ &= \frac{0.9486 e^{-t} - \cos(3t + 18.4°)}{\sqrt{10}} u(t) \end{aligned}$$

2.4-8. (a)

$$\begin{aligned} y(t) = (2e^{-3t} - e^{-2t}) u(t) * u(t) &= 2e^{-3t} u(t) * u(t) - e^{-2t} u(t) * u(t) \\ &= \left[\frac{2(1 - e^{-3t})}{3} - \frac{1 - e^{-2t}}{2} \right] u(t) \\ &= \left(\frac{1}{6} - \frac{2}{3} e^{-3t} + \frac{1}{2} e^{-2t} \right) u(t) \end{aligned}$$

(b)

$$\begin{aligned} (2e^{-3t} - e^{-2t}) u(t) * e^{-t} u(t) &= 2e^{-3t} u(t) * e^{-t} u(t) - e^{-2t} u(t) * e^{-t} u(t) \\ &= \left[\frac{2(e^{-t} - e^{-3t})}{2} - \frac{e^{-t} - e^{-2t}}{1} \right] u(t) \\ &= (e^{-2t} - e^{-3t}) u(t) \end{aligned}$$

(c)

$$\begin{aligned} y(t) = (2e^{-3t} - e^{-2t})u(t) * e^{-2t}u(t) &= 2e^{-3t}u(t) * e^{-2t}u(t) - e^{-2t}u(t) * e^{-2t}u(t) \\ &= \left[\frac{2(e^{-2t} - e^{-3t})}{1} - te^{-2t}\right] u(t) \\ &= [(2-t)e^{-2t} - 2e^{-3t}]u(t) \end{aligned}$$

2.4-9.

$$\begin{aligned} y(t) = (1-2t)e^{-2t}u(t) * u(t) &= e^{-2t}u(t) * u(t) - 2te^{-2t}u(t) * u(t) \\ &= \left[\left(\frac{1-e^{-2t}}{2}\right) - \left(\frac{1}{2} - \frac{1}{2}e^{-2t} - te^{-2t}\right)\right] u(t) \\ &= te^{-2t}u(t) \end{aligned}$$

2.4-10. (a) For $y(t) = 4e^{-2t}\cos 3t\, u(t) * u(t)$, We use pair 12 with $\alpha = 2$, $\beta = 3$, $\theta = 0$, $\lambda = 0$. Therefore

$$\phi = \tan^{-1}\left[\frac{-3}{2}\right] = -56.31^\circ$$

and

$$\begin{aligned} y(t) &= 4\left[\frac{\cos(56.31^\circ) - e^{-2t}\cos(3t + 56.31^\circ)}{\sqrt{4+9}}\right] u(t) \\ &= \frac{4}{\sqrt{13}}\left[0.555 - e^{-2t}\cos(3t + 56.31^\circ)\right] u(t) \end{aligned}$$

(b) For $y(t) = 4e^{-2t}\cos 3tu(t) * e^{-t}u(t)$, we use pair 12 with $\alpha = 2$, $\beta = 3$, $\theta = 0$, and $\lambda = -1$. Therefore

$$\phi = \tan^{-1}\left[\frac{-3}{1}\right] = -71.56^\circ$$

and

$$\begin{aligned} y(t) &= 4\left[\frac{\cos(71.56^\circ)e^{-t} - e^{-2t}\cos(3t + 71.56^\circ)}{\sqrt{10}}\right] u(t) \\ &= \frac{4}{\sqrt{10}}\left[0.316e^{-t} - e^{-2t}\cos(3t + 71.56^\circ)\right] u(t) \\ &= 4\left[e^{-t} - \frac{1}{\sqrt{10}}e^{-2t}\cos(3t + 71.56^\circ)\right] u(t) \end{aligned}$$

2.4-11. (a) $y(t) = e^{-t}u(t) * e^{-2t}u(t) = (e^{-t} - e^{-2t})u(t)$

(b) $e^{-2(t-3)}u(t) = e^6 e^{-2t}u(t)$, and $y(t) = e^6\left[e^{-t}u(t) * e^{-2t}u(t)\right] = e^6(e^{-t} - e^{-2t})u(t)$

(c) $e^{-2t}u(t-3) = e^{-6}e^{-2(t-3)}u(t-3)$. Now from the result in part (a) and the shift property of the convolution [Eq. (2.34)]: $y(t) = e^{-6}\left[e^{-(t-3)}u(t) - e^{-2(t-3)}\right]u(t-3)$

(d) $x(t) = u(t) - u(t-1)$. Now $y_1(t)$, the system response to $u(t)$ is given by

$$y_1(t) = e^{-t}u(t) * u(t) = (1 - e^{-t})u(t)$$

The system response to $u(t-1)$ is $y_1(t-1)$ because of time-invariance property. Therefore the response $y(t)$ to $x(t) = u(t) - u(t-1)$ is given by

$$y(t) = y_1(t) - y_1(t-1) = (1 - e^{-t})u(t) - [1 - e^{-(t-1)}]u(t-1)$$

The response is shown in Figure S2.4-11d.

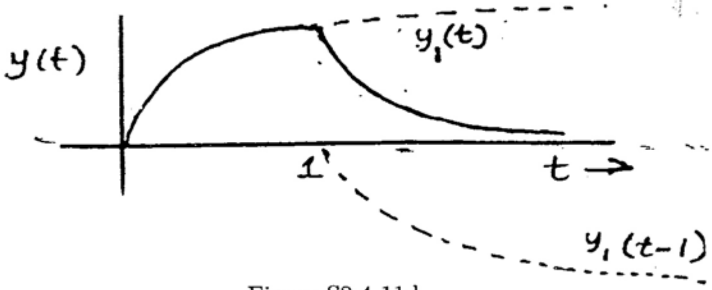


Figure S2.4-11d

2.4-12. (a)

$$
\begin{aligned}
y(t) &= [-\delta(t) + 2e^{-t}u(t)] * e^{t}u(-t) \\
&= -\delta(t) * e^{t}u(-t) + 2e^{-t}u(t) * e^{t}u(-t) \\
&= -e^{t}u(-t) + [e^{-t}u(t) + e^{t}u(-t)] \\
&= e^{-t}u(t)
\end{aligned}
$$

(b) Refer to Figure S2.4-12b.

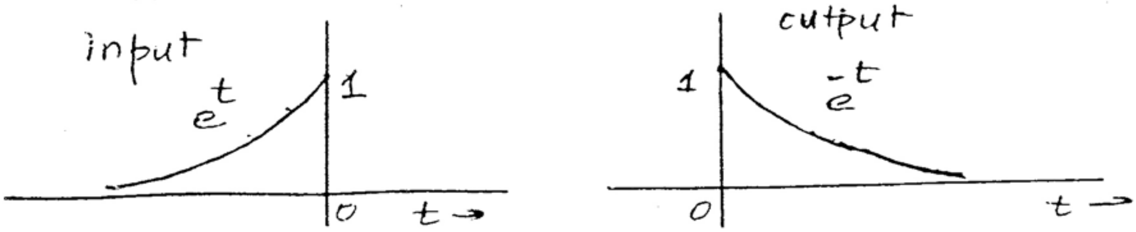
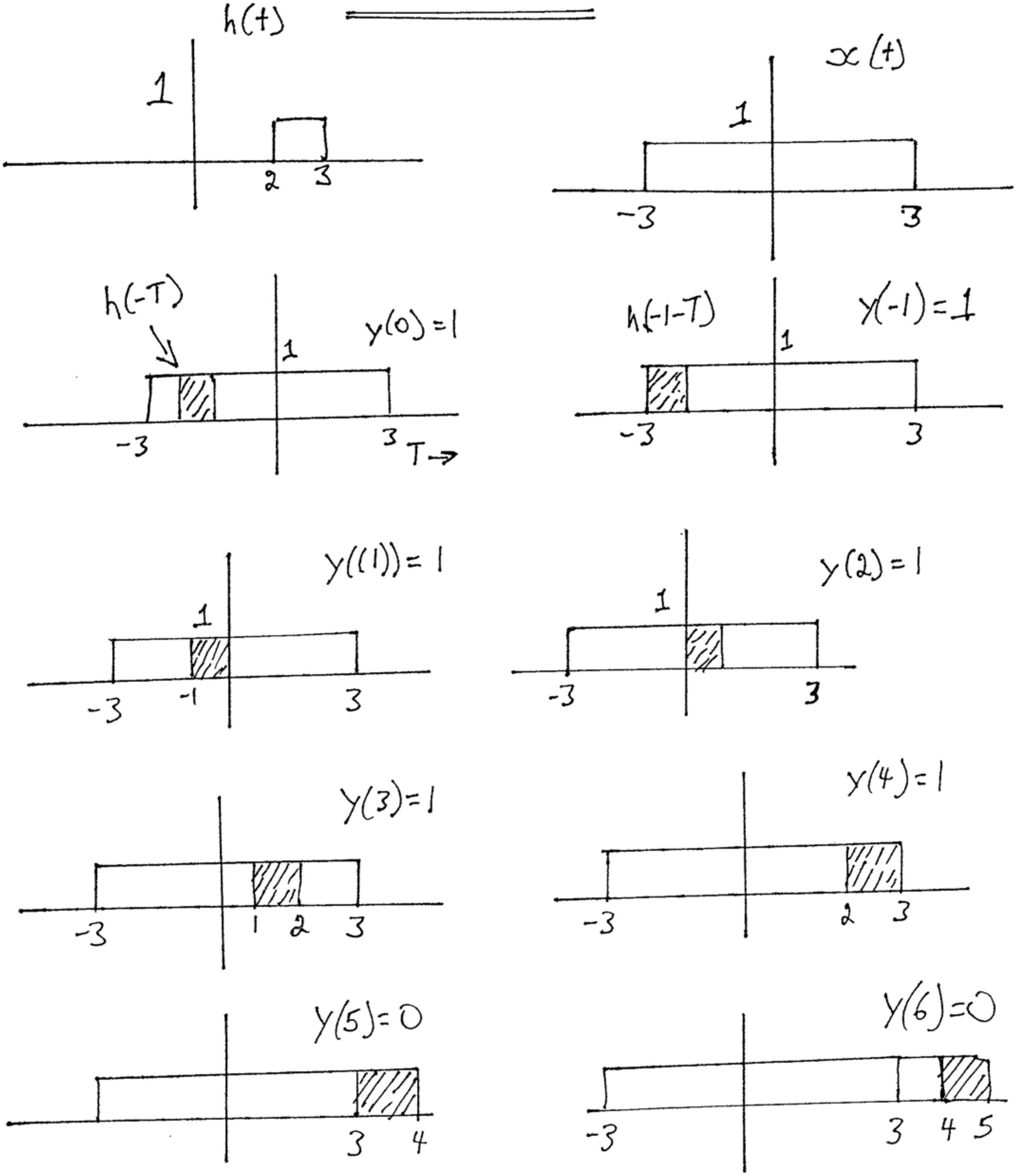


Figure S2.4-12b

2.4-13. Refer to Figure S2.4-13.

2.4-14. The output has the term $e^{-3t}u(t)$ that is not in the input. Hence, the input should have the term $e^{-3t}u(t)$. There is also a possibility of an impulse term in the input

FIGURE S2.4-13



that will result in a term of the form $e^{-2t}u(t)$ in the output. Let us try

$$x(t) = a\delta(t) + be^{-3t}u(t)$$

This yields the output

$$\begin{aligned} y(t) &= 2e^{-2t}u(t)\left[a\delta(t) + be^{-3t}u(t)\right] \\ &= 2e^{-2t}u(t) + 2b\left[e^{-2t} - e^{-3t}\right]u(t) \\ &= (2a+2b)e^{-2t} - 2be^{-3t}u(t) \end{aligned}$$

Matching the coefficients of similar terms yields

$$\left.\begin{aligned} 2a+2b &= 4 \\ -2b &= 6 \end{aligned}\right\} \Longrightarrow \begin{aligned} a &= 5 \\ b &= -3 \end{aligned}$$

Hence $\quad y(t) = 5\delta(t) - 3e^{-3t}u(t).$

2.4-15.

$$\frac{1}{t^2+1} * u(t) = \int_{-\infty}^{\infty} \frac{1}{\tau^2+1}u(t-\tau)\,d\tau$$

Because $u(t-\tau) = 1$ for $\tau < t$ and is 0 for $\tau > t$, we need integrate only up to $\tau = t$.

$$\frac{1}{t^2+1} * u(t) = \int_{-\infty}^{t} \frac{1}{\tau^2+1}\,d\tau = \tan^{-1}\tau\Big|_{-\infty}^{t} = \tan^{-1}t + \frac{\pi}{2}$$

Figure S2.4-15 shows $\frac{1}{t^2+1}$ and $c(t)$ (the result of the convolution)

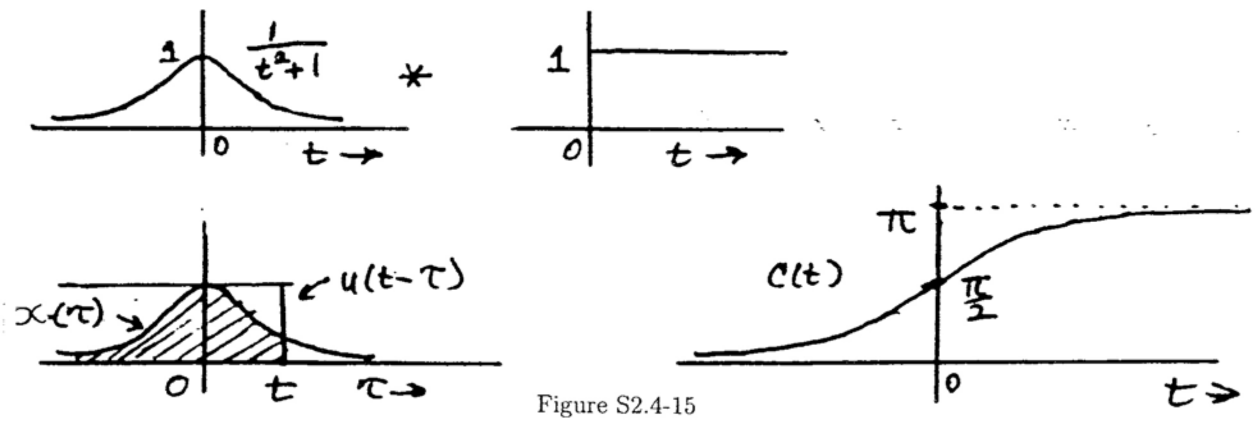


Figure S2.4-15

2.4-16. For $t < 2\pi$ (see Figure S2.4-16)

$$c(t) = x(t) * g(t) = \int_0^t \sin\tau\,d\tau = 1 - \cos t \qquad 0 \le t \le 2\pi$$

For $t \ge 2\pi$, the area of one cycle is zero and

$$x(t) * g(t) = 0 \qquad t \ge 2\pi \quad \text{and} \quad t < 0$$

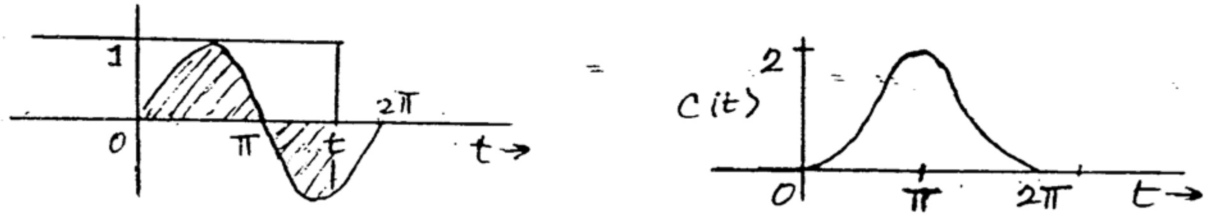


Figure S2.4-16

2.4-17. For $0 \leq t \leq 2\pi$ (see Figure S2.4-17a)

$$x(t) * g(t) = \int_0^t \sin\tau \, d\tau = 1 - \cos t \qquad 0 \leq t \leq 2\pi$$

For $2\pi \leq t \leq 4\pi$ (Figure S2.4-17b)

$$x(t) * g(t) = \int_{t-2\pi}^{2\pi} \sin\tau \, d\tau = \cos t - 1 \qquad 2\pi \leq t \leq 4\pi$$

For $t > 4\pi$ (also for $t < 0$), $\quad x(t) * g(t) = 0$. Figure S2.4-17c shows $c(t)$.

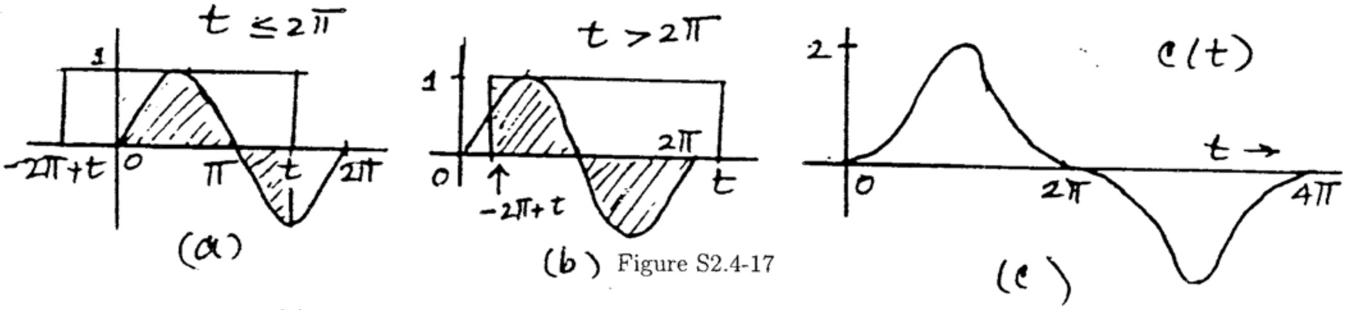


Figure S2.4-17

2.4-18. (a)

$$c(t) = \int_{2+t}^{2.5+t} AB \, d\tau = \tfrac{AB}{2} \qquad 0 \leq t \leq 0.5$$
$$c(t) = \int_{2+t}^{3} AB \, d\tau = AB(1-t) \qquad 0.5 \leq t \leq 1$$
$$c(t) = \int_{2}^{2.5+t} AB \, d\tau = AB(t+0.5) \qquad -0.5 \leq t \leq 0$$
$$c(t) = 0 \quad t \geq 1 \quad \text{or} \quad t \leq -0.5$$

(b)

$$c(t) = \int_{1.5+t}^{2.5} AB \, d\tau = AB(1-t) \qquad 0 \leq t \leq 1$$
$$c(t) = \int_{1.5}^{2.5+t} AB \, d\tau = AB(t+1) \qquad -1 \leq t \leq 0$$
$$c(t) = 0 \qquad \text{for} \quad |t| \geq 1$$

(c)

$$c(t) = \int_{-1+t}^{2+t} d\tau = 3 \qquad t > -1$$
$$c(t) = \int_{-2}^{2+t} d\tau = t + 4 \qquad -1 \geq t \geq -4$$
$$c(t) = 0 \qquad t \leq -4$$

(d)

$$
\begin{aligned}
c(t) &= \int_t^{3+t} e^{-\tau}\, d\tau = e^{-t}(1-e^{-3}) = 0.95e^{-t} \qquad t \geq 0 \\
&= \int_0^{3+t} e^{-\tau}\, d\tau = 1 - e^{-(3+t)} = 1 - 0.0498e^{-t} \qquad 0 \geq t \geq -3 \\
&= 0 \qquad t \leq -3
\end{aligned}
$$

(e)

$$c(t) = \int_{-\infty}^{-1+t} \frac{1}{\tau^2+1}\, d\tau = \tan^{-1}(t-1) + \frac{\pi}{2} \quad t \leq 1$$

$$c(t) = \int_{-\infty}^{0} \frac{1}{\tau^2+1}\, d\tau = \tan^{-1}\tau \Big|_{-\infty}^{0} = \frac{\pi}{2} \quad t \geq 1$$

(f)

$$c(t) = \int_0^t e^{-\tau}\, d\tau = 1 - e^{-t} \qquad 0 \leq t \leq 3$$

$$c(t) = \int_{t-3}^t e^{-\tau}\, d\tau = e^{-(t-3)} - e^{-t} \qquad t \geq 3$$

$$c(t) = 0 \qquad t \leq 0$$

(g) This problem is more conveniently solved by inverting $x_1(t)$ rather than $x_2(t)$

$$c(t) = \int_t^{t+1} (\tau - t)\, d\tau = \frac{1}{2} \qquad t \geq 0$$

$$c(t) = \int_0^{t+1} (\tau - t)\, d\tau = \frac{1}{2}(1 - t^2) \qquad 0 \geq t \geq -1$$

$$c(t) = 0 \qquad \text{for} \quad t \geq 0$$

(h) $x_1(t) = e^t, \quad x_2(t) = e^{-2t}, \quad x_1(\tau) = e^\tau, \quad x_2(t-\tau) = e^{-2(t-\tau)}.$

$$
\begin{aligned}
c(t) &= \int_{-1+t}^{0} e^\tau e^{-2(t-\tau)}\, d\tau = e^{-2t} \int_{-1+t}^{0} e^{3\tau}\, d\tau = \frac{1}{3}[e^{-2t} - e^{t-3}] \qquad 0 \leq t \leq 1 \\
c(t) &= \int_{-1+t}^{t} e^\tau e^{-2(t-\tau)}\, d\tau = e^{-2t} \int_{-1+t}^{t} e^{3\tau}\, d\tau = \frac{1}{3}[e^{t} - e^{t-3}] \qquad 0 \geq t \geq -1 \\
c(t) &= \int_{-2}^{t} e^\tau e^{-2(t-\tau)}\, d\tau = e^{-2t} \int_{-2}^{t} e^{3\tau}\, d\tau = \frac{1}{3}[e^{t} - e^{-2(t+3)}] \qquad -1 \geq t \geq -2 \\
c(t) &= 0 \qquad t \leq -2
\end{aligned}
$$

2.4-19.

$$\dot{x}(t) = \delta(t) - \delta(t-2)$$

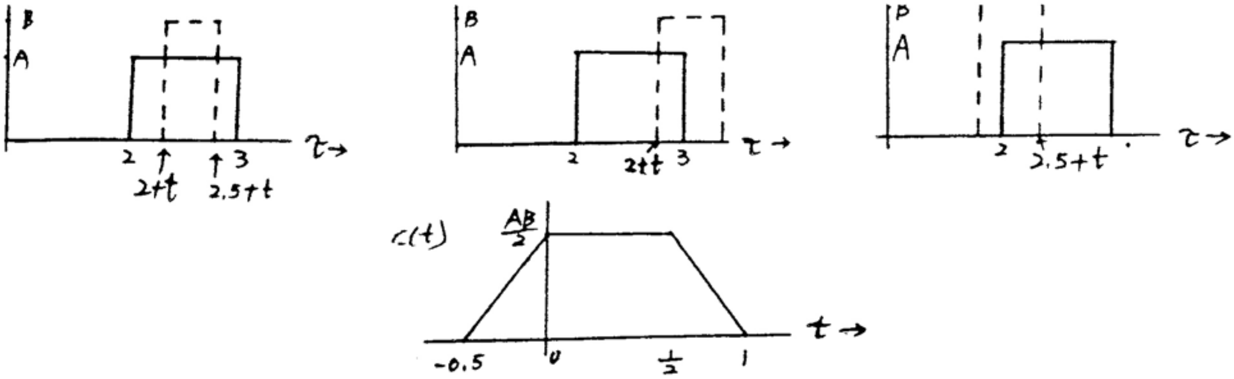
By inspection, we find

$$\int_0^t W(\tau) d\tau = \Delta\left(\frac{t-1}{2}\right)$$

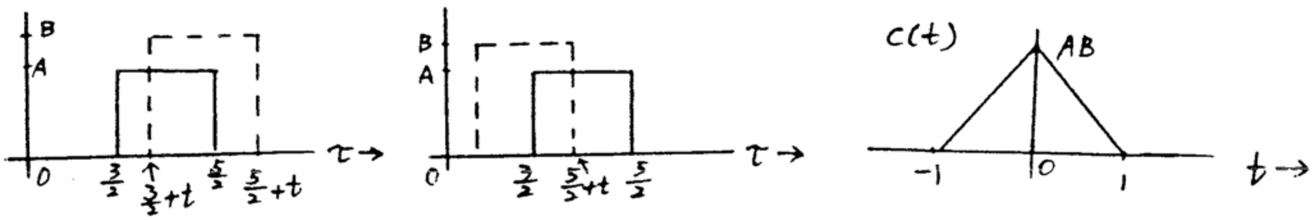
Therefore

$$
\begin{aligned}
x(t) * W(t) &= [\delta(t) - \delta(t-2)] * \Delta\left(\frac{t-1}{2}\right) \\
&= \Delta\left(\frac{t-1}{2}\right) - \Delta\left(\frac{t-3}{2}\right)
\end{aligned}
$$

64,d (Fig. S2.4-18)

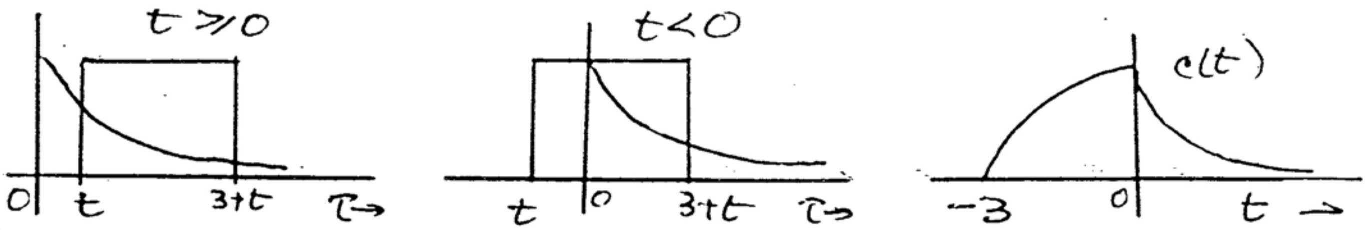


(a)

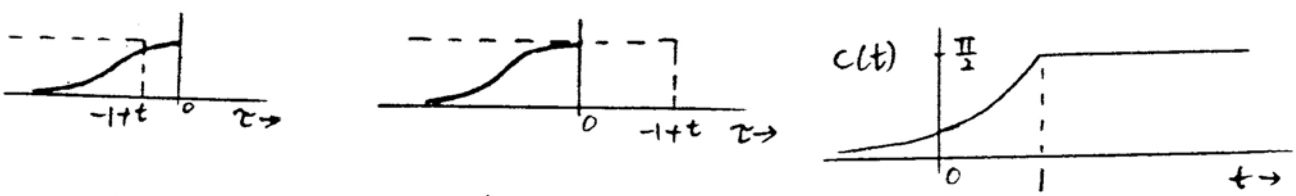


(b)

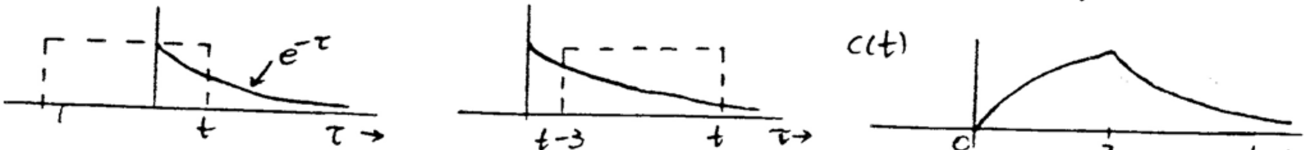
(c)



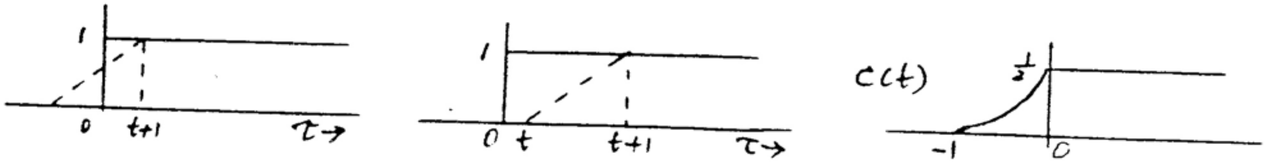
(d)



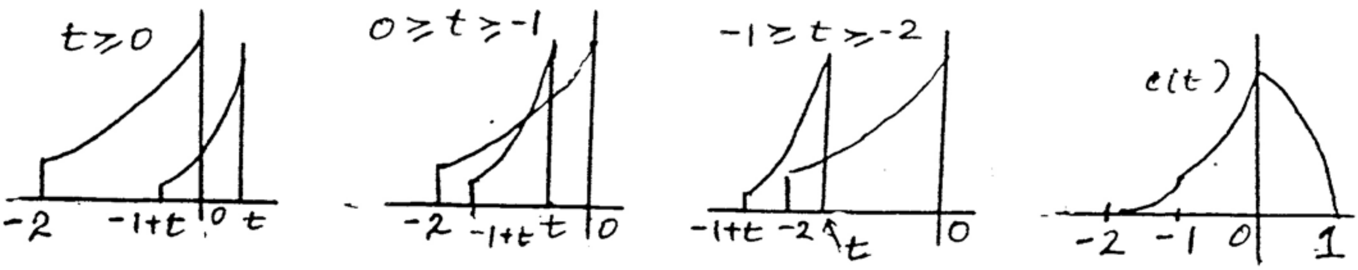
(e)



(f)



(g)



(h)

Figure S2.4-19 shows $x(t) * W(t)$

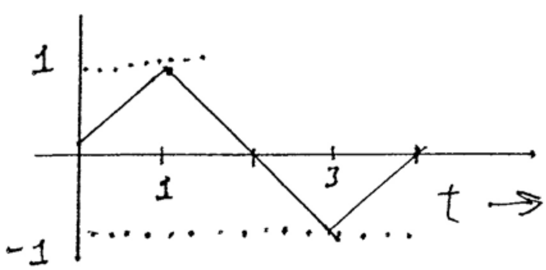


Figure S2.4-19

2.4-20. The unit impulse response of an ideal delay of T seconds is $h(t) = \delta(t - T)$. Using Eq. (2.48), we obtain

$$H(s) = \int_{-\infty}^{\infty} \delta(t - T)e^{-st}dt = e^{-sT}$$

For an input $x(t) = e^{st}$, the output of the delay is $y(t) = s^{s(t-T)}$. Hence, according to Eq. (2.49)

$$H(s) = \frac{e^{s(t-T)}}{e^{st}} = e^{-sT}$$

2.4-21. $y(t) = x(t) * h(t)$. For $t < -1$, $y(t) = 0$;

For $-1 \le t < 0$, $y(t) = \int_{-1}^{t}(\tau+1)d\tau = \tau^2/2 + \tau\big|_{\tau=-1}^{t} = t^2/2 + t + 1/2$.

For $t \ge 0$, $y(t) = \int_{-1}^{0}(\tau+1)d\tau = 1/2$.

Thus,

$$y(t) = \begin{cases} 0 & t < -1 \\ t^2/2 + t + 1/2 & -1 \le t < 0 \\ 1/2 & t \ge 0 \end{cases}.$$

2.4-22. Using the graph of the system response, $h(t) = (-t/2+1)(u(t) - u(t-2))$. $y(1) = \int_{-\infty}^{\infty} h_{\text{total}}(\tau)x(1-\tau)d\tau$. Since $x(t)$ is causal, the upper limit of the integral is one. Furthermore, since $h(t)$ is causal, the total response $h_{\text{total}}(t) = h(t) * h(t)$ is also causal, which makes the lower limit of the integral zero. Over $[0,1]$, $x(t) = u(t) = 1$. Thus, $y(1) = \int_0^1 h_{\text{total}}(\tau)d\tau$. To compute $y(1)$, it is only necessary to know $h_{\text{total}}(t)$ up to $t = 1$.

Over $(0 \le t < 2)$, $h_{\text{total}}(t) = \int_0^t(-\tau/2+1)(-(t-\tau)/2+1)d\tau = \int_0^t(-\tau/2+1)(\tau/2+1-t/2)d\tau = \int_0^t\left(-\tau^2/4 + \tau(1-1+t/2)/2 + (1-t/2)\right)d\tau = -\frac{t^3}{12} + \frac{t^3}{8} + (1-t/2)t = \frac{t^3}{24} - \frac{t^2}{2} + t$.

Thus,

$$y(1) = \int_0^1 (\tau^3/24 - \tau^2/2 + \tau)d\tau = \frac{\tau^4}{96} - \frac{\tau^3}{6} + \frac{\tau^2}{2}\bigg|_{\tau=0}^{1} = \frac{1}{96} - \frac{1}{6} + \frac{1}{2} = \frac{11}{32} = 0.34375.$$

2.4-23. (a) Using KVL, $x(t) = v_L(t) + y(t)$. Also, $i_C(t) = C\frac{dy}{dt}$ and $v_L(t) = L\frac{di_L}{dt} = L\frac{di_C}{dt} = LC\frac{d^2y}{dt^2}$. Combining yields

$$\frac{d^2y}{dt^2} + \frac{1}{LC}y(t) = \frac{1}{LC}x(t).$$

(b) The characteristic equation is

$$\lambda^2 + \frac{1}{LC} = 0.$$

The characteristic roots are

$$\lambda_{1,2} = \frac{\pm j}{\sqrt{LC}}.$$

(c) The form of the zero-input response is $y_0(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}$. Using $\lambda_1 = -\lambda_2$, $i_c(0) = 0 = C\frac{dy}{dt}\Big|_{t=0} = C(c_1\lambda_1 + c_2\lambda_2) = C(c_1\lambda_1 - c_2\lambda_1) = C\lambda_1(c_1 - c_2)$. Thus, $c_1 = c_2$. Also, $v_c(0) = 1 = y(0) = c_1 + c_2$. Combining yields $2c_1 = 2c_2 = 1$ or $c_1 = c_2 = 0.5$. The zero-input response is thus $y_0(t) = 0.5(e^{jt/\sqrt{LC}} + e^{-jt/\sqrt{LC}})$. Using Euler's identity yields

$$y_0(t) = \cos(t/\sqrt{LC}).$$

(d) MATLAB is used to plot $y_0(t)$ for a short time after $t \ge 0$.

```
>> t = linspace(0,6*pi,201); y_0 = cos(t); plot(t,y_0,'k');
>> axis tight; xlabel('t(LC)^{-1/2}'); ylabel('y_0(t)');
```

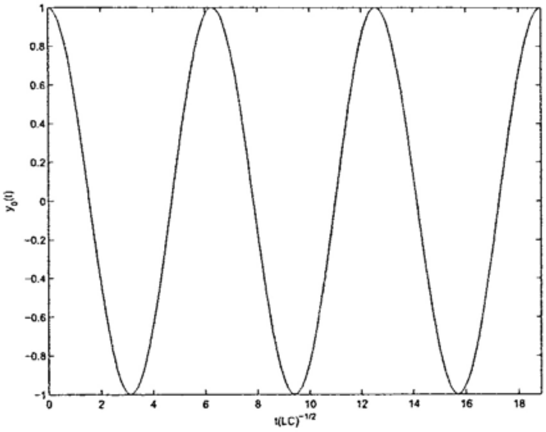


Figure S2.4-23d: Plot of $y_0(t)$

Since $y_0(t)$ is a non-decaying sinusoid, it continues forever; the initial conditions never die out.

(e) Since $L = C = 1$, $\lambda_{1,2} = \pm \jmath$. Let $\tilde{y}_0(t) = \tilde{c}_1 e^{\jmath t} + \tilde{c}_2 e^{-\jmath t}$. Using $\tilde{y}_0(0) = 0 = \tilde{c}_1 + \tilde{c}_2$, we know $\tilde{c}_1 = -\tilde{c}_2$. Combining with $\tilde{y}_0^{(1)}(0) = 1 = \jmath\tilde{c}_1 - \jmath\tilde{c}_2$, we know $2\jmath\tilde{c}_1 = 1$ or $\tilde{c}_1 = -\jmath 0.5$. Thus, $\tilde{c}_2 = \jmath 0.5$ and $\tilde{y}_0(t) = \frac{e^{\jmath t} - e^{-\jmath t}}{2\jmath} = \sin(t)$. From this, the system response is determined to be $h(t) = \frac{1}{LC}\sin(t)u(t) = \sin(t)u(t)$.

Next, the zero-state response is computed as $x(t) * h(t) = \int_0^t \sin\tau e^{-(t-\tau)} d\tau = \left(e^{-t}\int_0^t \mathrm{Imag}\left(e^{\jmath\tau}e^{\tau}\right) d\tau\right) u(t) = \left(\mathrm{Imag}\left(e^{-t}\int_0^t e^{\tau(1+\jmath)} d\tau\right)\right) u(t) = \left(\mathrm{Imag}\left(e^{-t}\left.\frac{e^{\tau(1+\jmath)}}{1+\jmath}\right|_{\tau=0}^{t}\right)\right) u(t) = \left(\mathrm{Imag}\left(e^{-t}\frac{e^{t(1+\jmath)}-1}{1+\jmath}\right)\right) u(t) = \left(\mathrm{Imag}\left(\frac{e^{\jmath t}-e^{-t}}{1+\jmath}\right)\right) u(t) = \left(\mathrm{Imag}\left(0.5e^{\jmath t} - \jmath 0.5 e^{\jmath t} - 0.5(1-\jmath)e^{-t}\right)\right) u(t) = \left(0.5\sin(t) - 0.5\cos(t) + 0.5e^{-t}\right) u(t)$.

Summing the zero-state response and the zero-input response calculated in 2.4-23c yields the total response, $y(t) = x(t) * h(t) + y_0(t) = \left(0.5\sin(t) - 0.5\cos(t) + 0.5e^{-t} + \cos(t)\right) u(t)$. Thus,

$$y(t) = \left(0.5\sin(t) + 0.5\cos(t) + 0.5e^{-t}\right) u(t).$$

2.4-24. (a) MATLAB is used to sketch $h_1(t)$ and $h_2(t)$.

```
>> h1 = inline('(1-t).*((t>=0)-(t>=1))');
>> h2 = inline('t.*((t>=-2)-(t>=2))');
>> t = linspace(-2.5,2.5,501);
>> subplot(211),plot(t,h1(t),'k');
>> axis([-2.5 2.5 -2.5 2.5]); xlabel('t'); ylabel('h_1(t)');
>> subplot(212),plot(t,h2(t),'k');
>> axis([-2.5 2.5 -2.5 2.5]); xlabel('t'); ylabel('h_2(t)');
```

(b) For a parallel connection, $h_p(t) = h_1(t) + h_2(t)$. MATLAB is used to plot $h_p(t)$.

```
>> h1 = inline('(1-t).*((t>=0)-(t>=1))');
>> h2 = inline('t.*((t>=-2)-(t>=2))');
>> t = linspace(-2.5,2.5,501); hp = h1(t)+h2(t);
```

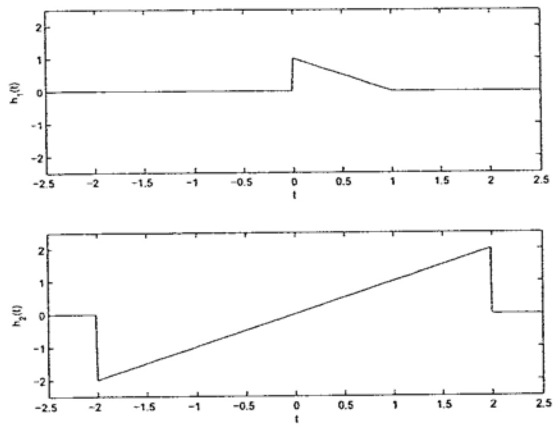


Figure S2.4-24a: Plots of $h_1(t)$ and $h_2(t)$.

```
>> plot(t,hp,'k');
>> axis([-2.5 2.5 -2.5 2.5]); xlabel('t'); ylabel('h_p(t)');
```

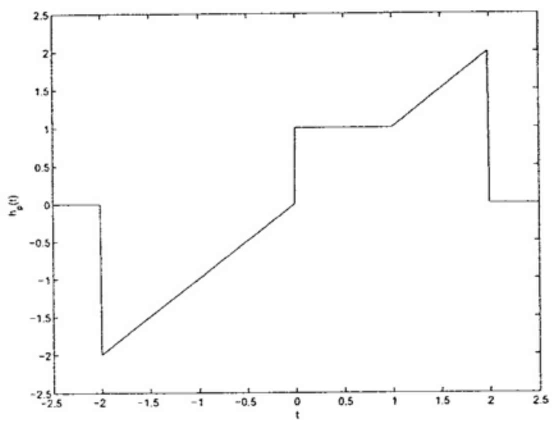


Figure S2.4-24b: Plot of $h_p(t)$.

(c) For a series connection, $h_s(t) = h_1(t) * h_2(t)$.
For $(t < -2)$, $h_s(t) = 0$.
For $(-2 \leq t < -1)$, $h_s(t) = \int_0^{t+2}(1-\tau)(t-\tau)d\tau = \int_0^{t+2}\left(t - \tau(t+1) + \tau^2\right)d\tau = t\tau - (t+1)\tau^2/2 + \tau^3/3\big|_{\tau=0}^{t+2} = t(t+2) - (t+1)(t+2)^2/2 + (t+2)^3/3 = -t^3/6 + t^2/2 + 2t + 2/3$.
For $(-1 \leq t < 2)$, $h_s(t) = \int_0^1(1-\tau)(t-\tau)d\tau = \int_0^1\left(t - \tau(t+1) + \tau^2\right)d\tau = t\tau - (t+1)\tau^2/2 + \tau^3/3\big|_{\tau=0}^{1} = t - (t+1)/2 + 1/3 = t/2 - 1/6$.
For $(2 \leq t < 3)$, $h_s(t) = \int_{t-2}^1(1-\tau)(t-\tau)d\tau = \int_{t-2}^1\left(t - \tau(t+1) + \tau^2\right)d\tau = t\tau - (t+1)\tau^2/2 + \tau^3/3\big|_{\tau=t-2}^{1} = t/2 - 1/6 - \left(t(t-2) - (t+1)(t-2)^2/2 + (t-2)^3/3\right) = t/2 - 1/6 - \left(-t^3/6 + t^2/2 + 2t - 14/3\right) = t^3/6 - t^2/2 - 3t/2 + 9/2$.
For $(t > 3)$, $h_s(t) = 0$.

Combining all pieces yields

$$h_s(t) = \begin{cases} -t^3/6 + t^2/2 + 2t + 2/3 & -2 \le t < -1 \\ t/2 - 1/6 & -1 \le t < 2 \\ t^3/6 - t^2/2 - 3t/2 + 9/2 & 2 \le t < 3 \\ 0 & \text{otherwise} \end{cases}.$$

MATLAB is used to plot $h_s(t)$.

```
>> t = linspace(-2.5,3.5,501);
>> hs = (-t.^3/6+t.^2/2+2*t+2/3).*((t>=-2)&(t<-1));
>> hs = hs+(t/2-1/6).*((t>=-1)&(t<2));
>> hs = hs+(t.^3/6-t.^2/2-3*t/2+9/2).*((t>=2)&(t<3));
>> plot(t,hs,'k'); xlabel('t'); ylabel('h_s(t)');
```

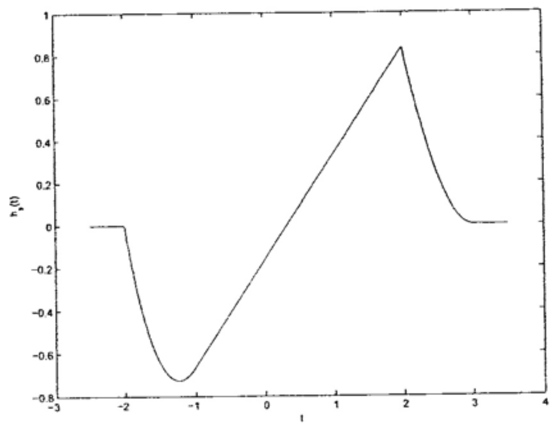


Figure S2.4-24c: Plot of $h_s(t)$.

2.4-25. (a) Using KVL, $x(t) = RC\dot{y}(t) + y(t)$ or $\dot{y}(t) + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$. The characteristic root is $\lambda = \frac{-1}{RC}$.

The zero-input response has form $y_0(t) = c_1 e^{-t/(RC)}$. Using the IC, $y_0(0) = 2 = c_1$. Thus, $y_0(t) = 2e^{-t/(RC)}$.

The zero-state response is $x(t) * h(t)$, where $h(t) = b_0\delta(t) + [P(D)\tilde{y}_0(t)]u(t)$. For this first-order system, $\tilde{y}_0(t) = \tilde{c}_1 e^{-t/(RC)}$ and $\tilde{y}_0(0) = 1 = \tilde{c}_1$. Using $\tilde{y}_0(t) = e^{-t/(RC)}$, $b_0 = 0$, and $P(D) = \frac{1}{RC}$, the impulse response is $h(t) = \frac{1}{RC}e^{-t/(RC)}u(t)$. Thus, the zero-state response is $\left(\int_0^t \frac{1}{RC}e^{-\tau/(RC)}d\tau\right)u(t) = \left(-e^{-\tau/(RC)}\big|_{\tau=0}^{t}\right)u(t) = \left(1 - e^{-t/(RC)}\right)u(t)$.

For $t \ge 0$, the total response is the sum of the zero-input response and the zero state response,

$$y(t) = \left(1 + e^{-t/(RC)}\right)u(t).$$

(b) From 2.4-25a, we know the zero-input response is $y_0(t) = y_0(0)e^{-t/(RC)}$. Since the system is time-invariant, the unit step response from 2.4-25a is shifted by one to provide the response to $x(t) = u(t-1)$. Thus, the zero-state response is $\left(1 - e^{-(t-1)/(RC)}\right)u(t-1)$. Summing the two parts together and evaluating at

$t = 2$ yields $y(2) = 1/2 = y_0(0)e^{-2/(RC)} + (1 - e^{-1/(RC)})$. Solving for $y_0(0)$ yields

$$y_0(0) = e^{1/(RC)} - 0.5e^{2/(RC)}.$$

2.4-26. Notice, $x(2t)$ is a compressed version of $x(t)$. The convolution $y(t) = x(t) * x(2t)$ has several distinct regions.

For $t < 0$ and $t \geq 3/2$, $y(t) = 0$.

For $0 \leq t < 1/2$, $y(t) = \int_0^t 2\tau(t-\tau)d\tau = t\tau^2 - 2\tau^3/3\big|_{\tau=0}^{t} = t^3/3$.

For $1/2 \leq t < 1$, $y(t) = \int_0^{1/2} 2\tau(t-\tau)d\tau = t\tau^2 - 2\tau^3/3\big|_{\tau=0}^{1/2} = t/4 - 1/12$.

For $1 \leq t < 3/2$, $y(t) = \int_{t-1}^{1/2} 2\tau(t-\tau)d\tau = t\tau^2 - 2\tau^3/3\big|_{\tau=t-1}^{1/2} = t/4 - 1/12 - (t^3 - 2t^2 + t - 2t^3/3 + 2t^2 - 2t + 2/3) = -t^3/3 + 5t/4 - 3/4$.

Thus,

$$y(t) = \begin{cases} t^3/3 & 0 \leq t < 1/2 \\ t/4 - 1/12 & 1/2 \leq t < 1 \\ -t^3/3 + 5t/4 - 3/4 & 1 \leq t < 3/2 \\ 0 & \text{otherwise} \end{cases}.$$

MATLAB is used to plot $y(t)$.

```
>> t = linspace(-1/2,2,251);
>> y = (t.^3/3).*((t>=0)&(t<1/2));
>> y = y+(t/4-1/12).*((t>=1/2)&(t<1));
>> y = y+(-t.^3/3+5*t/4-3/4).*((t>=1)&(t<3/2));
>> plot(t,y,'k'); xlabel('t'); ylabel('y(t)');
```

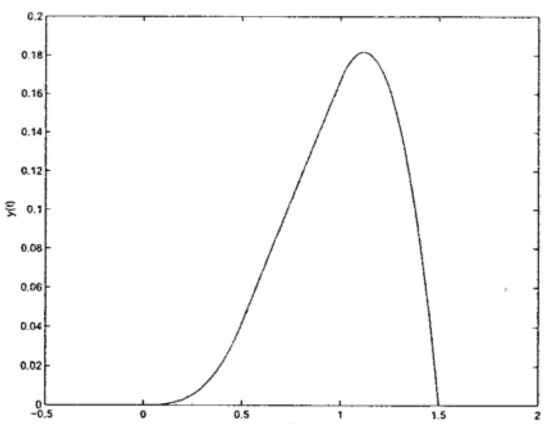


Figure S2.4-26: Plot of $y(t) = x(t) * x(2t)$.

2.4-27. Notice, $v_R(t) = v_{L_1}(t) = v_{L_2}(t) = v(t)$.

(a) KCL at the top node gives $x(t) = y(t) + i_{L_1}(t) + i_R$. Since $v(t) = L_2\dot{y}(t)$, we know $i_R(t) = v(t)/R = \frac{L_2}{R}\dot{y}(t)$. Thus, $x(t) = y(t) + \frac{L_2}{R}\dot{y}(t) + i_{L_1}(t)$. Differentiating this expression yields $\dot{x}(t) = \dot{y}(t) + \frac{L_2}{R}\ddot{y}(t) + i_{L_1}^{(1)}(t)$. However, $i_{L_1}^{(1)}(t) = v(t)/L_1 =$

$\frac{L_2}{L_1}\dot{y}(t)$. Thus, $\dot{x}(t) = \dot{y}(t) + \frac{L_2}{R}\ddot{y}(t) + \frac{L_2}{L_1}\dot{y}(t)$ or

$$\ddot{y}(t) + \left(\frac{R}{L_1} + \frac{R}{L_2}\right)\dot{y}(t) = \frac{R}{L_2}\dot{x}(t).$$

(b) The characteristic equation is $\lambda^2 + \left(\frac{R}{L_1} + \frac{R}{L_2}\right)\lambda = 0$ which yields characteristic roots of $\lambda_1 = 0$ and $\lambda_2 = -\left(\frac{R}{L_1} + \frac{R}{L_2}\right)$.

(c) The zero-input response has form $y_0(t) = c_1 + c_2 e^{\lambda_2 t}$. Each inductor has an initial current of one amp each. Thus, $y_0(0) = 1 = c_1 + c_2$. The initial resistor current is $i_R(0) = -i_{L_1}(0) - i_{L_2}(0) = -2$ and the initial resistor voltage is $v(0) = i_R(0)R = -2R$. Thus, $\dot{y}_0(0) = -\frac{2R}{L_2} = \lambda_2 c_2$. Solving yields $c_2 = \frac{2L_1}{L_1+L_2}$ and $c_1 = 1 - c_2 = \frac{L_2-L_1}{L_1+L_2}$. Thus,

$$y_0(t) = \frac{L_2 - L_1}{L_1 + L_2} + \frac{2L_1}{L_1 + L_2}e^{-tR/L_1 - tR/L_2}.$$

2.4-28. Since the system step response is $s(t) = e^{-t}u(t) - e^{-2t}u(t)$, the system impulse response is $h(t) = \frac{d}{dt}s(t) = -e^{-t}u(t) + \delta(t) + 2e^{-2t}u(t) - \delta(t) = (2e^{-2t} - e^{-t})u(t)$. The input $x(t) = \delta(t-\pi) - \cos(\sqrt{3})u(t)$ is just a sum of a shifted delta function and a scaled step function. Since the system is LTI, the output is quickly computed using just $h(t)$ and $s(t)$. That is,

$$y(t) = h(t-\pi) - \cos(\sqrt{3})s(t) = (2e^{-2(t-\pi)} - e^{-(t-\pi)})u(t-\pi) - \cos(\sqrt{3})(e^{-t} - e^{-2t})u(t).$$

2.4-29. Since $x(t)$ is $(T = 2)$-periodic, the convolution $y(t) = x(t) * h(t)$ is also $(T = 2)$-periodic. Thus, it is sufficient to evaluate $y(t)$ over any interval of length two.

For $0 \leq t < 1/2$, $y(t) = \int_0^t \tau d\tau + \int_{t+1}^{3/2} \tau d\tau = \left.\frac{\tau^2}{2}\right|_{\tau=0}^{t} + \left.\frac{\tau^2}{2}\right|_{\tau=t+1}^{3/2} = t^2/2 + 9/8 - (t^2/2 + t + 1/2) = -t + 5/8$.

For $1/2 \leq t < 1$, $y(t) = \int_0^t \tau d\tau = t^2/2$.

For $1 \leq t < 3/2$, $y(t) = \int_{t-1}^t \tau d\tau = \left.\frac{\tau^2}{2}\right|_{\tau=t-1}^{t} = t^2/2 - (t^2/2 - t + 1/2) = t - 1/2$.

For $3/2 \leq t < 2$, $y(t) = \int_{t-1}^{3/2} \tau d\tau = \left.\frac{\tau^2}{2}\right|_{\tau=t-1}^{3/2} = 9/8 - (t^2/2 - t + 1/2) = -t^2/2 + t + 5/8$.

Combining,

$$y(t) = \begin{cases} -t + 5/8 & 0 \leq t < 1/2 \\ t^2/2 & 1/2 \leq t < 1 \\ t - 1/2 & 1 \leq t < 3/2 \\ -t^2/2 + t + 5/8 & 3/2 \leq t < 2 \\ y(t+2) & \forall t \end{cases}.$$

MATLAB is used to plot $y(t)$ over $(-3 \leq t \leq 3)$. This interval includes three periods of the $(T = 2)$-periodic function $y(t)$.

```
>> t = linspace(-3,3,601); tm = mod(t,2);
>> y = (-tm+5/8).*((tm>=0)&(tm<1/2));
>> y = y+(tm.^2/2).*((tm>=1/2)&(tm<1));
>> y = y+(tm-1/2).*((tm>=1)&(tm<3/2));
>> y = y+(-tm.^2/2+tm+5/8).*((tm>=3/2)&(tm<2));
>> plot(t,y,'k'); xlabel('t'); ylabel('y(t)');
```

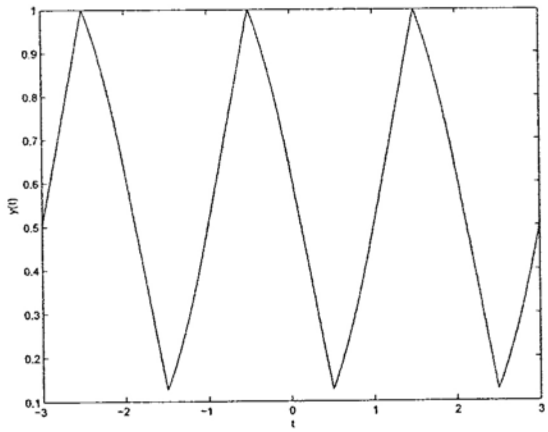


Figure S2.4-29: Plot of $y(t) = x(t) * h(t)$ over $(-3 \leq t \leq 3)$.

2.4-30. (a) Using KVL, $x(t) = i(t)R + v_{C_1}(t) + y(t) = RC_2\dot{y}(t) + v_{C_1}(t) + y(t)$. Differentiating yields $\dot{x}(t) = RC_2\ddot{y}(t) + \dot{v}_{C_1}(t) + \dot{y}(t) = RC_2\ddot{y}(t) + \frac{1}{C_1}i(t) + \dot{y}(t) = RC_2\ddot{y}(t) + \frac{C_2}{C_1}\dot{y}(t) + \dot{y}(t)$. Thus,

$$\ddot{y}(t) + \left(\frac{1}{RC_1} + \frac{1}{RC_2}\right)\dot{y}(t) = \frac{1}{RC_2}\dot{x}(t).$$

(b) Since $R = 1$, $C_1 = 1$, and $C_2 = 2$, the differential equation becomes $\ddot{y}(t) + 3/2\dot{y}(t) = 1/2\dot{x}(t)$.
The characteristic equation is $\lambda^2 + 3/2\lambda = 0$, and the characteristic roots are $\lambda_1 = 0$ and $\lambda_2 = -3/2$. Thus, the form of the zero-input response is $y_0(t) = c_1 + c_2e^{-3t/2}$. Using the first IC, $y(0) = 1 = c_1 + c_2$. The initial voltage across the resistor is $v_R(0) = -3$ which yields $i_R(0) = -3/R = -3$. Also, $i_R(0) = -3 = i_{C_2}(0) = C_2\dot{y}(0) = 2\dot{y}(0)$. Thus, $\dot{y}(0) = -3/2 = -3c_2/2$. Solving yields $c_2 = 1$ and $c_1 = 0$. Thus,

$$y_0(t) = e^{-3t/2}.$$

The zero-state response is $x(t) * h(t)$, where $h(t) = b_0\delta(t) + [P(D)\tilde{y}_0(t)]u(t)$. For this second-order system, $\tilde{y}_0(t) = \tilde{c}_1 + \tilde{c}_2e^{-3t/2}$, $\tilde{y}_0(0) = 0 = \tilde{c}_1 + \tilde{c}_2$ and $\tilde{y}_0^{(1)}(0) = 1 = -3\tilde{c}_2/2$. Thus, $\tilde{c}_2 = -2/3$ and $\tilde{c}_1 = 2/3$. Using $b_0 = 0$, and $P(D) = 0.5D$, the impulse response is $h(t) = 0.5D(\tilde{y}_0(t))u(t) = 0.5D(2/3 - 2/3e^{-3t/2})u(t) = 0.5(-2/3(-3/2)e^{-3t/2})u(t) = 0.5e^{-3t/2}u(t)$. Using $x(t) = 4te^{-3t/2}u(t)$, the zero-state response is $\left(\int_0^t (4\tau e^{-3\tau/2})(0.5e^{-3(t-\tau)/2})d\tau\right)u(t) = \left(2e^{-3t/2}\int_0^t \tau d\tau\right)u(t) = \left(2e^{-3t/2}t^2/2\right)u(t)$. Thus,

$$x(t) * h(t) = t^2e^{-3t/2}u(t).$$

Since the input is driving a natural mode, resonance is expected; thus, the t^2 term seems sensible.
For $(t \geq 0)$, the total response is the sum of the zero-input response and the zero-state response.

$$y(t) = y_0(t) + x(t) * h(t) = \left(e^{-3t/2} + t^2e^{-3t/2}\right)u(t).$$

2.4-31. Since $h(t)$ is only provided for over $(0 \leq t < 0.5)$, it is not possible to determine with certainty whether or not the system is causal or stable. However, when looking at $h(t)$ the waveform appears to have a DC offset. This apparent DC offset can be very troubling if $h(t)$ is truly an impulse response function. If a DC offset is present, the system is neither causal nor stable. Imagine, a non-causal, unstable heart! Something is probably wrong.

One simple explanation is that a blood-filled heart always has some ventricular pressure. Unless removed, this relaxed-state pressure would likely appear as a DC offset to any measurements. It would likely be most appropriate to subtract this offset when trying to measure the impulse response function.

Another problem is that the impulse response function is most appropriate in the study of linear, time-invariant systems. It is quite unlikely that the heart is either linear or time-invariant. Even if the impulse response could be reliably measured at a particular time, it might not provide much useful information.

2.4-32. (a) $x(t) * x(-t) = \int_{-\infty}^{\infty} x(\tau)x(-(t-\tau))d\tau = \int_{-\infty}^{\infty} x(\tau)x(\tau - t)d\tau = r_{xx}(t)$.

(b) Since $r_{xx}(t)$ is an even function, we only need to compute $r_{xx}(t)$ for either $t \geq 0$ or $t \leq 0$. In either case, the autocorrelation function is computed by convolving the original signal with its reflection.
For $t < -2$, $r_{xx}(t) = 0$.
For $-2 \leq t < -1$, $r_{xx}(t) = \int_0^{t+2} \tau d\tau = 0.5\tau^2\Big|_{\tau=0}^{t+2} = t^2/2 + 2t + 2$.
For $-1 \leq t < 0$, $r_{xx}(t) = \int_0^{t+1} \tau(\tau - t)d\tau + \int_{t+1}^{1} \tau d\tau + \int_1^{t+2} d\tau = \left(\frac{\tau^3}{3} - t\frac{\tau^2}{2}\Big|_{\tau=0}^{t+1}\right) + \frac{\tau^2}{2}\Big|_{\tau=t+1}^{1} + \tau\Big|_{\tau=1}^{t+2} = \frac{t^3+3t^2+3t+1}{3} - \frac{t^3+2t^2+t}{2} + \frac{1}{2} - \frac{t^2+2t+1}{2} + (t+2-1) = -t^3/6 - t^2/2 + t/2 + 4/3$.
Combining and using $r_{xx}t = r_{xx}-t$ yields

$$r_{xx}(t) = \begin{cases} t^2/2 + 2t + 2 & -2 \leq t < -1 \\ -t^3/6 - t^2/2 + t/2 + 4/3 & -1 \leq t < 0 \\ t^3/6 - t^2/2 - t/2 + 4/3 & 0 \leq t < 1 \\ t^2/2 - 2t + 2 & 1 \leq t < 2 \\ 0 & \text{otherwise} \end{cases}$$

MATLAB is used to plot the result.

```
>> t = linspace(-2.5,2.5,501);
>> rxx = (t.^2/2+2*t+2).*((t>=-2)&(t<-1));
>> rxx = rxx+(-t.^3/6-t.^2/2+t/2+4/3).*((t>=-1)&(t<0));
>> rxx = rxx+(t.^3/6-t.^2/2-t/2+4/3).*((t>=0)&(t<1));
>> rxx = rxx+(t.^2/2-2*t+2).*((t>=1)&(t<2));
>> plot(t,rxx,'k'); xlabel('t'); ylabel('r_{xx}(t)');
```

2.4-33. (a) KCL at the negative terminal of the op-amp yields $\frac{x(t)-0}{R} + C\dot{y}(t) = 0$. Thus,

$$\dot{y}(t) = -\frac{1}{RC}x(t).$$

(b) The zero-state response is $y(t) = x(t)*h(t)$, where $h(t) = b_0\delta(t) + [P(D)\tilde{y}_0(t)]u(t)$. This is a first order system with $\lambda = 0$, thus $\tilde{y}_0(t) = \tilde{c}_1 e^{\lambda t} = \tilde{c}_1$. Since $\tilde{y}_0(0) = 1 = \tilde{c}_1$, $b_0 = 0$, and $P(D) = -\frac{1}{RC}$, the impulse response is $h(t) = -\frac{1}{RC}u(t)$.

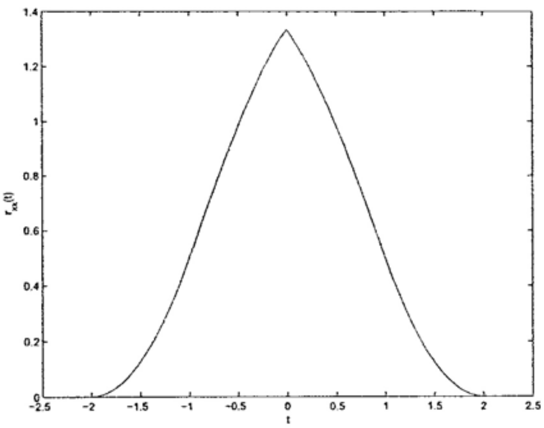


Figure S2.4-32b: Plot of $r_{xx}(t)$.

Thus,

$$y(t) = \left(\int_0^t -\frac{1}{RC}d\tau\right)u(t) = -\frac{t}{RC}u(t).$$

Notice, $|y(t)|$ ramps toward infinity as time increases. Intuitively, this makes sense; a DC input to an integrator should output an unbounded ramp function.

2.4-34. The system response to $u(t)$ is $g(t)$ and the response to step $u(t-\tau)$ is $g(t-\tau)$. The input $x(t)$ is made up of step components. The step component at τ has a height $\triangle f$ which can be expressed as

$$\triangle f = \frac{\triangle f}{\triangle \tau}\triangle\tau = \dot{x}(\tau)\triangle\tau$$

The step component at $n\triangle\tau$ has a height $\dot{x}(n\triangle\tau)\triangle\tau$ and it can be expressed as $[\dot{x}(n\triangle\tau)\triangle\tau]u(t-n\triangle\tau)$. Its response $\triangle y(t)$ is

$$\triangle y(t) = [\dot{x}(n\triangle\tau)\triangle\tau]g(t-n\triangle\tau)$$

The total response due to all components is

$$\begin{aligned} y(t) &= \lim_{\triangle\tau\to 0}\sum_{n=-\infty}^{\infty}\dot{x}(n\triangle\tau)g(t-n\triangle\tau)\triangle\tau \\ &= \int_{-\infty}^{\infty}\dot{x}(\tau)g(t-\tau)\,d\tau = \dot{x}(\tau)*g(\tau) \end{aligned}$$

2.4-35. Consider the input $x(t) = e^{j\omega_o t}$. Letting $s = j\omega_o$ in Eq. (2.47), the system response is found as

$$y(t) = H(j\omega_o)e^{j\omega_o t}$$

Using Eq. (2.40), the system response to input $\hat{x}(t) = \cos\omega_o t = Re[e^{j\omega_o t}]$ is $\hat{y}(t)$, where

$$\begin{aligned} \hat{y}(t) &= Re[H(j\omega_o)e^{j\omega_o t}] \\ &= Re\left\{|H(j\omega_o)|e^{j[\omega_o t + \angle H(j\omega_o)]}\right\} \\ &= |H(j\omega_o)|\cos[\omega_o t + \angle H(j\omega_o)] \end{aligned}$$

Where $H(j\omega)$ is $H(s)|_{s=j\omega}$ in Eq. (2.48). Hence

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau$$

2.4-36. An element of length $\triangle\tau$ at point $n\triangle\tau$ has a charge (Figure S2.4-36). A point x is at a distance $x - n\triangle\tau$ from this charge. The electric field at point x due to the charge $Q(n\triangle\tau)\triangle\tau$ is

$$\triangle E = \frac{Q(n\triangle\tau)\triangle\tau}{4\pi\epsilon(x - n\triangle\tau)^2}$$

The total field due to the charge along the entire length is

$$\begin{aligned} E(x) &= \lim_{\triangle\tau\to 0} \sum_{n=-\infty}^{\infty} \frac{Q(n\triangle\tau)\triangle\tau}{4\pi\epsilon(x - n\triangle\tau)^2} \\ &= \int_{-\infty}^{\infty} \frac{Q(\tau)}{4\pi\epsilon(x-\tau)^2}\,d\tau = Q(x) * \frac{1}{4\pi\epsilon x} \end{aligned}$$

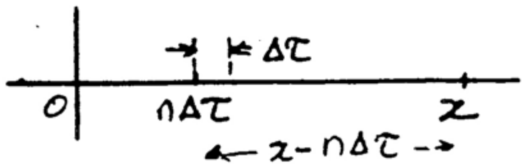


Figure S2.4-36

2.4-37. (a) KCL at the negative terminal of the op-amp yields $\frac{x(t)-0}{R_{in}} + \frac{y(t)-0}{R_f} + i_c(t) = 0$. Also, $i_c(t) = C\dot{y}(t)$. Thus, $\frac{x(t)}{R_{in}} + \frac{y(t)}{R_f} + C\dot{y}(t) = 0$ or

$$\dot{y}(t) + \frac{1}{CR_f}y(t) = \frac{-1}{CR_{in}}x(t).$$

The characteristic equation is $\lambda + \frac{1}{CR_f} = 0$, and the characteristic root is

$$\lambda = \frac{-1}{CR_f}.$$

(b) The zero-state response is $y(t) = x(t)*h(t)$, where $h(t) = b_0\delta(t) + [P(D)\tilde{y}_0(t)]u(t)$. This is a first order system with $\lambda = \frac{-1}{CR_f}$, thus $\tilde{y}_0(t) = \tilde{c}_1 e^{\lambda t}$. Since $\tilde{y}_0(0) = 1 = \tilde{c}_1$, $b_0 = 0$, and $P(D) = \frac{-1}{CR_{in}}$, the impulse response is $h(t) = \frac{-1}{CR_{in}}e^{-t/(CR_f)}u(t)$. Thus, $y(t) = \left(\int_0^t \frac{-1}{CR_{in}}e^{-\tau/(CR_f)}d\tau\right)u(t) = \left(\frac{R_f}{R_{in}}e^{-\tau/(CR_f)}\Big|_{\tau=0}^{t}\right)u(t) = \frac{R_f}{R_{in}}(e^{-t/(CR_f)} - 1)u(t)$.

$$y(t) = \frac{R_f}{R_{in}}(e^{-t/(CR_f)} - 1)u(t).$$

Notice, $y(t)$ approaches $\frac{-R_f}{R_{in}}$ as time increases; unlike a true integrator, the "lossy" integrator provides a bounded output in response to a DC input.

(c) For this system, the characteristic root is only affected by C and R_f. Using 10% resistors, the resistor R_f is generally expected to lie in the range $(0.9R_f, 1.1R_f)$. Using 25% capacitors, the capacitor C is generally expected to lie in the range $(.25C, 1.25C)$. Since $\lambda = \frac{-1}{CR_f}$, the characteristic root is expected to lie in the range $(\lambda/[(0.9)(0.75)], \lambda/[(1.1)(1.25)])$. Thus,

The characteristic root is expected within the interval $(1.48\lambda, 0.73\lambda)$.

2.4-38. Identify the output of the first op-amp as $v(t)$.

(a) KCL at the negative terminal of the first op-amp yields $\frac{x(t)}{R_1} + C_1\dot{v}(t) = 0$ or $\frac{1}{R_1C_1}x(t) = -\dot{v}(t)$. KCL at the negative terminal of the second op-amp yields $\frac{v(t)}{R_2} + \frac{y(t)}{R_3} + C_2\dot{y}(t) = 0$ or $v(t) = -\frac{R_2}{R_3}y(t) - R_2C_2\dot{y}(t)$. Substituting this expression for $v(t)$ into the first expression yields $\frac{1}{R_1C_1}x(t) = \frac{R_2}{R_3}\dot{y}(t) + R_2C_2\ddot{y}(t)$. Thus,

$$\ddot{y}(t) + \frac{1}{R_3C_2}\dot{y}(t) = \frac{1}{R_1R_2C_1C_2}x(t).$$

The characteristic equation is $\lambda^2 + \frac{1}{R_3C_2}\lambda = 0$ and the characteristic roots are

$$\lambda_1 = 0 \text{ and } \lambda_2 = -\frac{1}{R_3C_2}.$$

Substituting $C_1 = C_2 = 10\mu\text{F}$, $R_1 = R_2 = 100k\Omega$, and $R_3 = 50k\Omega$ yields

$$\ddot{y}(t) + 2\dot{y}(t) = x(t), \lambda_1 = 0, \text{ and } \lambda_2 = -2.$$

Since one root lies on the ω-axis, the circuit is not BIBO stable. In particular, a DC input results in an unbounded output.

(b) The zero-input response has form $y_0(t) = c_1 + c_2e^{-2t}$. Each op-amp has an initial output of one volt. Thus, $y_0(0) = 1 = c_1 + c_2$. KCL at the negative terminal of the second op-amp yields $\frac{1}{R_2} + \frac{1}{R_3} + C_2\dot{y}_0(0) = 0$ or $\dot{y}_0(0) = -\frac{1}{R_2C_2} - \frac{1}{R_3C_2} = -1-2 = -3$. Thus, $\dot{y}_0(0) = -3 = -2c_2$. Thus, $c_2 = 3/2$ and $c_1 = 1-3/2 = -1/2$ and

$$y_0(t) = -1/2 + 3/2e^{-2t}.$$

(c) The zero-state response is $y(t) = x(t)*h(t)$, where $h(t) = b_0\delta(t) + [P(D)\tilde{y}_0(t)]u(t)$. This is a second order system with $\lambda_1 = 0$ and $\lambda_2 = -2$, so $\tilde{y}_0(t) = \tilde{c}_1 + \tilde{c}_2e^{-2t}$. Solving $\tilde{y}_0(0) = 0 = \tilde{c}_1 + \tilde{c}_2$ and $\tilde{y}_0^{(1)}(t) = 1 = -2\tilde{c}_2$ yields $\tilde{c}_2 = -1/2$ and $\tilde{c}_1 = 1/2$. Since $b_0 = 0$ and $P(D) = 1$, $h(t) = \left(1/2 - e^{-2t}/2\right)u(t)$.

Next, $y(t) = x(t) * h(t) = \left(\int_0^t (1/2 - e^{-2\tau}/2)d\tau\right)u(t) = \left(\tau/2 + e^{-2\tau}/4\big|_{\tau=0}^{t}\right)u(t) = \left(t/2 + e^{-2t}/4 - 1/4\right)u(t)$.

$$y(t) = \left(t/2 + e^{-2t}/4 - 1/4\right)u(t).$$

As expected, the DC nature of the unit step input results in an unbounded output.

(d) For this system, $\lambda_1 = 0$ is not affected by the components and λ_2 is only affected by C_2 and R_3. Using 10% resistors, the resistor R_3 is generally expected to lie in the range $(0.9R_3, 1.1R_3)$. Using 25% capacitors, the capacitor C_2 is generally expected to lie in the range $(.25C_2, 1.25C_2)$. Since $\lambda_2 = -\frac{1}{R_3C_2}$, the characteristic root is expected to lie in the range $(\lambda_2/[(0.9)(0.75)], \lambda_2/[(1.1)(1.25)])$. Thus,

$$\lambda_1 \text{ is unaffected and } \lambda_2 \text{ is expected to lie within } (-2.9630, -1.4545).$$

2.4-39.

2.4-40. (a) Yes, the system is causal since $h(t) = 0$ for $(t < 0)$.

(b) To compute the zero-state response $y_1(t)$, the convolution of two rectangular pulses is required: a pulse of amplitude $\jmath$ and width two and a pulse of amplitude one and a width of one. The convolution involves several regions.
For $t < 0$, $y_1(t) = 0$.
For $0 \leq t < 1$, $y_1(t) = \int_0^t \jmath dt = \jmath t$.
For $1 \leq t < 2$, $y_1(t) = \int_{t-1}^t \jmath dt = \jmath(t - (t-1)) = \jmath$.
For $2 \leq t < 3$, $y_1(t) = \int_{t-1}^2 \jmath dt = \jmath(2 - (t-1)) = \jmath(3-t)$.
For $t \geq 0$, $y_1(t) = 0$.
Thus,

$$y_1(t) = \begin{cases} \jmath t & 0 \leq t < 1 \\ \jmath & 1 \leq t < 2 \\ \jmath(3-t) & 2 \leq t < 3 \\ 0 & \text{otherwise} \end{cases}.$$

(c) To compute $y_2(t)$, first note that $x_2(t) = 2x_1(t-1) - x_1(t-2)$. Using the system properties of linearity and time-invariance, the output $y_2(t)$ is given by

$$y_2(t) = 2y_1(t-1) - y_1(t-2).$$

2.5-1.

$$\lambda^2 + 7\lambda + 12 = (\lambda+3)(\lambda+4)$$

The natural response is

$$y_n(t) = K_1e^{-3t} + K_2e^{-4t}$$

(a) For $x(t) = u(t) = e^{0t}u(t)$, $\quad y_\phi(t) = H(0) = \frac{P(0)}{Q(0)} = \frac{1}{6}$

$$\begin{aligned} y(t) &= K_1e^{-3t} + K_2e^{-4t} + \frac{1}{6} \\ \dot{y}(t) &= -3K_1e^{-3t} - 4K_2e^{-4t} \end{aligned}$$

Setting $t = 0$ and substituting initial conditions, we obtain

$$\left.\begin{aligned} 0 &= K_1 + K_2 + \tfrac{1}{6} \\ 1 &= -3K_1 - 4K_2 \end{aligned}\right\} \Longrightarrow \qquad \begin{aligned} K_1 &= -\tfrac{2}{3} \\ K_2 &= \tfrac{1}{2} \end{aligned}$$

and

$$y(t) = -\frac{2}{3}e^{-3t} + \frac{1}{2}e^{-4t} + \frac{1}{6} \qquad t \geq 0$$

(b) $x(t) = e^{-t}u(t)$, $\quad y_\phi(t) = H(-1) = \frac{P(-1)}{Q(-1)} = \frac{1}{6}$

$$\begin{aligned} y(t) &= K_1e^{-3t} + K_2e^{-4t} + \frac{1}{6}e^{-t} \\ \dot{y}(t) &= -3K_1e^{-3t} - 4K_2e^{-4t} - \frac{1}{6}e^{-t} \end{aligned}$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{array}{l} 0 = K_1 + K_2 + \frac{1}{6} \\ 1 = -3K_1 - 4K_2 - \frac{1}{6} \end{array}\right\} \Longrightarrow \qquad \begin{array}{l} K_1 = -\frac{1}{2} \\ K_2 = -\frac{2}{3} \end{array}$$

and

$$y(t) = \frac{1}{2}e^{-3t} - \frac{2}{3}e^{-4t} + \frac{1}{6}e^{-t} \qquad t \geq 0$$

(c) $\quad x(t) = e^{-2t}u(t), \quad y_\phi(t) = H(-2) = 0$

$$\begin{aligned} y(t) &= K_1e^{-3t} + K_2e^{-4t} \\ \dot{y}(t) &= -3K_1e^{-3t} - 4K_2e^{-4t} \end{aligned}$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{array}{l} 0 = K_1 + K_2 \\ 1 = -3K_1 - 4K_2 \end{array}\right\} \Longrightarrow \qquad \begin{array}{l} K_1 = 1 \\ K_2 = -1 \end{array}$$

and

$$y(t) = e^{-3t} - e^{-4t} \qquad t \geq 0$$

2.5-2. $\quad \lambda^2 + 6\lambda + 25 = (\lambda + 3 - j4)(\lambda + 3 + j4)$ characteristic roots are $-3 \pm j4$

$$y_n(t) = Ke^{-3t}\cos(4t + \theta)$$

For $x(t) = u(t), \quad y_\phi(t) = H(0) = \frac{3}{25}$ so that

$$\begin{aligned} y(t) &= Ke^{-3t}\cos(4t + \theta) + \frac{3}{25} \\ \dot{y}(t) &= -3Ke^{-3t}\cos(4t + \theta) - 4Ke^{-3t}\cos(4t + \theta) \end{aligned}$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{array}{l} 0 = K\cos\theta + \frac{3}{25} \\ 2 = -3K\cos\theta - 4K\sin\theta \end{array}\right\} \Longrightarrow \quad \left.\begin{array}{l} K\cos\theta = \frac{-3}{25} \\ K\sin\theta = \frac{-41}{100} \end{array}\right\} \Longrightarrow \quad \begin{array}{l} K = 0.427 \\ \theta = -106.3 \end{array}$$

and

$$y(t) = 0.427e^{-3t}\cos(4t - 106.3^\circ) + \frac{3}{25} \qquad t \geq 0$$

2.5-3. Characteristic polynomial is $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2$. The roots are -2 repeated twice.

$$y_n(t) = (K_1 + K_2t)e^{-2t}$$

(a) $\quad$ For $x(t) = e^{-3t}u(t), \quad y_\phi(t) = H(-3) = -2e^{-3t}$

$$\begin{aligned} y(t) &= (K_1 + K_2t)e^{-2t} - 2e^{-3t} \\ \dot{y}(t) &= -2(K_1 + K_2t)e^{-2t} + K_2e^{-2t} + 6e^{-3t} \end{aligned}$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{array}{l} \frac{9}{4} = K_1 - 2 \\ 5 = -2K_1 + K_2 + 6 \end{array}\right\} \Longrightarrow \qquad \begin{array}{l} K_1 = \frac{17}{4} \\ K_2 = \frac{15}{2} \end{array}$$

and

$$y(t) = (\frac{17}{4} + \frac{15}{2}t)e^{-2t} - 2e^{-3t} \qquad t \geq 0$$

(b) $\quad x(t) = e^{-t}u(t), \quad y_\phi(t) = H(-1)e^{-t} = 0$

$$\begin{aligned} y(t) &= (K_1 + K_2 t)e^{-2t} \\ \dot{y}(t) &= -2(K_1 + K_2 t)e^{-2t} + K_2 e^{-2t} \end{aligned}$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{array}{l} \frac{9}{4} = K_1 \\ 5 = -2K_1 + K_2 \end{array}\right\} \Longrightarrow \quad \begin{array}{l} K_1 = \frac{9}{4} \\ K_2 = \frac{19}{2} \end{array}$$

and

$$y(t) = (\frac{9}{4} + \frac{19}{2}t)e^{-2t} \qquad t \geq 0$$

2.5-4. Because $\quad (\lambda^2 + 2\lambda) = \lambda(\lambda + 2)$, the characteristic roots are 0 and -2.

$$y_n(t) = K_1 + K_2 e^{-2t}$$

In this case $x(t) = u(t)$. The input itself is a characteristic mode. Therefore

$$y_\phi(t) = \beta t$$

But $y_\phi(t)$ satisfied the system equation

$$(D^2 + 2D)y_\phi(t) = (D + 1)y(t) = \ddot{y}_\phi(t) + 2\dot{y}_\phi(t) = \dot{x}(t) + x(t)$$

Substituting $x(t) = u(t)$ and $y_\phi(t) = \beta t$, we obtain

$$0 + 2\beta = 0 + 1 \quad \Longrightarrow \quad \beta = \frac{1}{2}$$

Therefore $y_\phi(t) = \frac{1}{2}t$.

$$\begin{aligned} y(t) &= K_1 + K_2 e^{-2t} + \frac{1}{2}t \\ \dot{y}(t) &= -2K_2 e^{-2t} + \frac{1}{2} \end{aligned}$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{array}{l} 2 = K_1 + K_2 \\ 1 = -2K_2 + \frac{1}{2} \end{array}\right\} \Longrightarrow \quad \begin{array}{l} K_1 = \frac{9}{4} \\ K_2 = -\frac{1}{4} \end{array}$$

and

$$y(t) = \frac{9}{4} - \frac{1}{4}e^{-2t} + \frac{1}{2}t \qquad t \geq 0$$

2.5-5. The natural response $\quad y_n(t)$ is found in Prob. 2.5-1:

$$y_n(t) = K_1 e^{-3t} + K_2 e^{-4t}$$

The input $x(t) = e^{-3t}$ is a characteristic mode. Therefore

$$y_\phi(t) = \beta t e^{-3t}$$

Also $y_\phi(t)$ satisfies the system equation:

$$(D^2 + 7D + 12)y_\phi(t) = (D + 2)x(t)$$

or $$\ddot{y}_\phi(t) + 7\dot{y}_\phi(t) + 12y_\phi(t) = \dot{x}(t) + 2x(t)$$

Substituting $x(t) = e^{-3t}$ and $y_\phi(t) = \beta t e^{-3t}$ in this equation yields

$$(9\beta t - 6\beta)e^{-3t} + 7(-3\beta t + \beta)e^{-3t} + 12\beta t e^{-3t} = -3e^{-3t} + 2e^{-3t}$$

or $$\beta e^{-3t} = -e^{-3t} \quad \Longrightarrow \quad \beta = -1$$

Therefore

$$\begin{aligned} y(t) &= K_1 e^{-3t} + K_2 e^{-4t} - te^{-3t} \\ \dot{y}(t) &= -3K_1 e^{-3t} - 4K_2 e^{-4t} + 3te^{-3t} - e^{-3t} \end{aligned}$$

Setting $t = 0$, and substituting initial conditions yields

$$\left.\begin{aligned} 0 &= K_1 + K_2 \\ 1 &= -3K_1 - 4K_2 - 1 \end{aligned}\right\} \Longrightarrow \qquad \begin{aligned} K_1 &= 2 \\ K_2 &= -2 \end{aligned}$$

and

$$\begin{aligned} y(t) &= 2e^{-3t} - 2e^{-4t} - te^{-3t} \qquad t \geq 0 \\ &= (2-t)e^{-3t} - 2e^{-4t} \qquad t \geq 0 \end{aligned}$$

2.6-1. (a) $\lambda^2 + 8\lambda + 12 = (\lambda+2)(\lambda+6)$
Both roots are in LHP. The system is BIBO stable and also asymptotically stable.

(b) $\lambda(\lambda^2 + 3\lambda + 2) = \lambda(\lambda+1)(\lambda+2)$
Roots are 0, -1, -2. One root on imaginary axis and none in RHP. The system is BIBO ustable and marginally stable.

(c) $\lambda^2(\lambda^2+2) = \lambda^2(\lambda + j\sqrt{2})(\lambda - j\sqrt{2})$
Roots are 0 (repeated twice) and $\pm j\sqrt{2}$. Multiple roots on imaginary axis. The system is BIBO unstable and asymptotically unstable.

(d) $(\lambda+1)(\lambda^2 - 6\lambda + 5) = (\lambda+1)(\lambda-1)(\lambda-5)$
Roots are -1, 1 and 5. Two roots in RHP. The system is BIBO unstable and asymptotically unstable.

2.6-2. (a) $(\lambda+1)(\lambda^2+2\lambda+5)^2 = (\lambda+1)(\lambda+1-j2)^2(\lambda+1+j2)^2$
Roots -1, $-1 \pm j2$ (repeated twice) are all in LHP. The system is BIBO stable and asymptotically stable.

(b) $(\lambda+1)(\lambda^2+9) = (\lambda+1)(\lambda+j3)(\lambda-j3)$
Roots are -1, $\pm j3$. Two (simple) roots on imaginary axis, none in RHP. The system is BIBO unstable and marginally stable.

(c) $(\lambda+1)(\lambda^2+9)^2 = (\lambda+1)(\lambda+j3)^2(\lambda-j3)^2$
Roots are -1 and $\pm j3$ repeated twice. Multiple roots on imaginary axis. The system is BIBO unstable and asymptotically unstable.

(d) $(\lambda^2+1)(\lambda^2+4)(\lambda^2+9) = (\lambda+j1)(\lambda-j1)(\lambda+j2)(\lambda-j2)(\lambda+j3)(\lambda-j3)$
The roots are $\pm j1$, $\pm j2$ and $\pm j3$. All roots are simple and on imaginary axis. None in RHP. The system is BIBO unstable and marginally stable.

2.6-3. (a) Because $u(t) = e^{0t}u(t)$, the characteristic root is 0.

(b) The root lies on the imaginary axis, and the system is marginally stable.

(c) $\int_0^\infty h(t)\,dt = \infty$
The system is BIBO unstable.

(d) The integral of $\delta(t)$ is $u(t)$. The system response to $\delta(t)$ is $u(t)$. Clearly, the system is an ideal integrator.

2.6-4. Assume that a system exists that violates Eq. (2.64) and yet produces a bounded output for every bounded input. The response at $t = t_1$ is

$$y(t_1) = \int_0^\infty h(\tau)x(t_1 - \tau)\,d\tau$$

Consider a bounded input $x(t)$ such that at some instant t_1

$$x(t_1 - \tau) = \begin{cases} 1 & \text{if } h(\tau) > 0 \\ -1 & \text{if } h(\tau) < 0 \end{cases}$$

In this case

$$h(\tau)x(t_1 - \tau) = |h(\tau)|$$

and

$$y(t_1) = \int_0^\infty |h(\tau)|\,d\tau = \infty$$

This violates the assumption.

2.6-5. (a) For this convolution, there are several regions. For $(t < -2)$ and $(t \geq 4)$, $y(t) = 0$.
For $(-2 \leq t < 0)$, $y(t) = \int_0^{t+2} \tau d\tau = (t+2)^2/2 = t^2/2 + 2t + 2$.
For $(-2 \leq t < 0)$, $y(t) = \int_0^2 \tau d\tau = 2$.
For $(-2 \leq t < 0)$, $y(t) = \int_{t-2}^2 \tau d\tau = 2^2/2 - (t-2)^2/2 = 2 - t^2/2 - 2t - 2 = -t^2/2 + 2t$.
Combining yields

$$y(t) = \begin{cases} t^2/2 + 2t + 2 & -2 \leq t < 0 \\ 2 & 0 \leq t < 2 \\ -t^2/2 + 2t & 2 \leq t < 4 \\ 0 & \text{otherwise} \end{cases}.$$

MATLAB is used to plot the result.

```
>> t = linspace(-3,5,401);
>> y = (t.^2/2+2*t+2).*((t>=-2)&(t<0));
>> y = y+2.*((t>=0)&(t<2));
>> y = y+(-t.^2/2+2*t).*((t>=2)&(t<4));
>> plot(t,y,'k'); axis([-3 5 -.5 2.5]);
>> xlabel('t'); ylabel('y(t)');
```

(b) Yes, the system is stable since $\int h(t) = 4 < \infty$.
No, the system is not causal since $h(t) \neq 0$ for all $t < 0$.

2.6-6. Yes, the system is stable since $h(t)$ is absolutely integrable. That is, $\int_{-\infty}^{\infty} h(t)dt = \int_0^1 1dt = 1 < \infty$.
Yes, the system is causal since $h(t) = 0$ for $t < 0$.s

2.6-7. Expanding

$$h(t) = \sum_{i=0}^{\infty} (0.5)^i \delta(t - i)$$

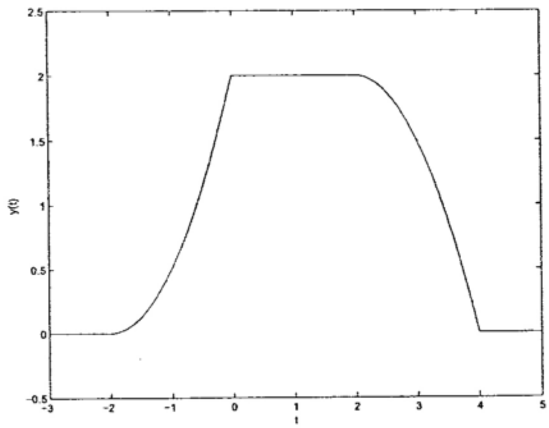


Figure S2.6-5a: Plot of $y(t) = x(t) * h(t)$.

yields

$$h(t) = (\delta(t) + 0.5\delta(t-1) + 0.25\delta(t-2) + 0.125\delta(t-3) + \cdots)$$

(a) Yes, the system is causal since $h(t) = 0$ for $t < 0$.

(b) Yes, the system is stable since the impulse response is absolutely integrable. That is, $\int_{-\infty}^{\infty} \sum_{i=0}^{\infty} (0.5)^i \delta(t-i)dt = \sum_{i=0}^{\infty} (0.5)^i \int_{-\infty}^{\infty} \delta(t-i) = \sum_{i=0}^{\infty} (0.5)^i = \frac{1-0}{1-0.5} = 2 < \infty$.

2.7-1. (a) The time-constant (rise-time) of the system is $T_h = 10^{-5}$. The rate of pulse communication $< \frac{1}{T_h} = 10^5$ pulses/sec. The channel cannot transmit million pulses/second.

(b) The bandwidth of the channel is

$$B = \frac{1}{T_h} = 10^5 \,\text{Hz}$$

The channel can transmit audio signal of bandwidth 15 kHz readily.

2.7-2.

$$T_h = \frac{1}{B} = \frac{1}{10^4} = 10^{-4} = 0.1 \,\text{ms}$$

The received pulse width $= (0.5 + 0.1) = 0.6$ ms. Each pulse takes up 0.6 ms interval. The maximum pulse rate (to avoid interference between successive pulses) is

$$\frac{1}{0.6 \times 10^{-3}} \simeq 1667 \,\text{pulses/sec}$$

2.7-3. Using Eqs. (2.67) and (2.68)

(a)

$$T_r = T_h = -\frac{1}{\lambda} = 10^{-4}$$

(b) The bandwidth $f_c = 1/T_h = 1/T_r = 10^4$.

(c) The pulse transmission rate is $f_c = 10^4$ pulses/sec.

2.7-4. (a) For a causal system with finite duration $h(t)$, the rise time is exactly equal to the time when the signal is last non-zero. That is,

$$T_r = 4 \text{ seconds.}$$

(b) The impulse response function $h(t)$ is consistent with a channel that has the following three characteristics: 1) a channel with delay from input to output (for example, signal propagation delay), 2) a channel with low-pass character (pulse dispersion that results in a $\delta(t)$ input spreading into a square pulse), and 3) a channel with two signal paths (for example, a primary signal path and an echo path).

For systems with predominantly low-pass character, digital information can be transmitted without significant interference at a rate of $\mathcal{F}_c = \frac{1}{T_r} = 1/4$. However, this estimate is too conservative for the present system. Notice that $h(t) = 0$ for $0 \leq t < 1$, corresponding to a transmission delay in the primary signal path. The remaining portion of $h(t)$ has a width of three, so it is therefore practical to transmit at rates of $\mathcal{F}_c = 1/3$. By clever interleaving of data, it is possible to transmit at rates of $\mathcal{F}_c = 1/2$. Consider transmitting the binary sequence $\{b_0, b_1, b_2, b_3, \ldots\}$ using a $(t = 1)$-spaced delta train weighted by the pulse sequence $\{b_0, b_1, 0, 0, b_2, b_3, 0, 0, \ldots\}$. The output is the series of non-overlapping unit-duration pulses given by $\{b_0, b_1, b_0, b_1, b_2, b_3, b_2, b_3, \ldots\}$. The effective transmission rate is 0.5 bits per unit time.

(c) The resulting convolution $y(t) = x(t) * h(t)$ has many regions.

For $t < 1$, $y(t) = 0$.

For $1 \leq t < 2$, $y(t) = \int_1^t (-1)dt = 1 - t$.

For $2 \leq t < 3$, $y(t) = \int_1^2 (-1)dt = -1$.

For $3 \leq t < 4$, $y(t) = \int_{t-2}^2 (-1)dt + \int_3^t (-1)dt = (t-2) - 2 + 3 - t = -1$.

For $4 \leq t < 5$, $y(t) = \int_3^4 (-1)dt = -1$.

For $5 \leq t < 6$, $y(t) = \int_{t-2}^4 (-1)dt = (t-2) - 4 = t - 6$.

For $6 \leq t$, $y(t) = 0$.

Thus,

$$y(t) = \begin{cases} 1-t & 1 \leq t < 2 \\ -1 & 2 \leq t < 5 \\ t-6 & 5 \leq t < 6 \\ 0 & \text{otherwise} \end{cases}.$$

MATLAB is used to plot the result.

```
>> t = [0:.01:10]; y = zeros(size(t));
>> y = y + (1-t).*((t>=1)&(t<2));
>> y = y + (-1).*((t>=2)&(t<5));
>> y = y + (t-6).*((t>=5)&(t<6));
>> plot(t,y,'k'); axis([0 10 -1.2 .2]);
>> xlabel('t'); ylabel('y(t)');
```

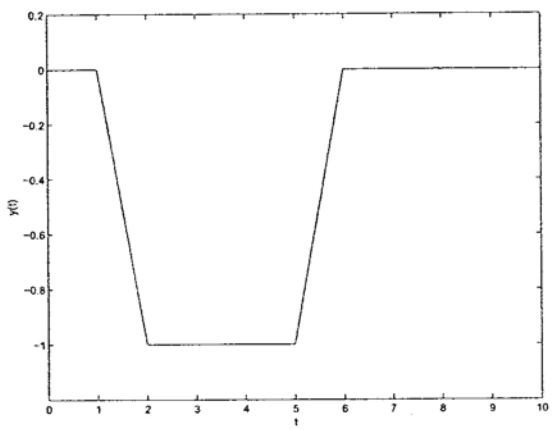


Figure S2.7-4c: Plot of $y(t) = x(t) * h(t)$.