Chapter 7 Solutions

7.1-1.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} x(t)\cos\omega t\,dt - j\int_{-\infty}^{\infty} x(t)\sin\omega t\,dt$$

If $x(t)$ is an even function of t, $x(t)\sin\omega t$ is an odd function of t, and the second integral vanishes. Moreover, $x(t)\cos\omega t$ is an even function of t, and the first integral is twice the integral over the interval 0 to ∞. Thus when $x(t)$ is even

$$X(\omega) = 2\int_{0}^{\infty} x(t)\cos\omega t\,dt \tag{1}$$

Similar argument shows that when $x(t)$ is odd

$$X(\omega) = -2j\int_{0}^{\infty} x(t)\sin\omega t\,dt \tag{2}$$

If $x(t)$ is also real (in addition to being even), the integral (1) is real. Moreover from (1)

$$X(-\omega) = 2\int_{0}^{\infty} x(t)\cos\omega t\,dt = X(\omega)$$

Hence $X(\omega)$ is real and even function of ω. Similar arguments can be used to prove the rest of the properties.

7.1-2.

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|e^{j\angle X(\omega)}e^{j\omega t}\,d\omega \\ &= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} |X(\omega)|\cos[\omega t + \angle X(\omega)]\,d\omega + j\int_{-\infty}^{\infty} |X(\omega)|\sin[\omega t + \angle X(\omega)]\,d\omega\right] \end{aligned}$$

Since $|X(\omega)|$ is an even function and $\angle X(\omega)$ is an odd function of ω, the integrand in the second integral is an odd function of ω, and therefore vanishes. Moreover the integrand in the first integral is an even function of ω, and therefore

$$x(t) = \frac{1}{\pi}\int_{0}^{\infty} |X(\omega)|\cos[\omega t + \angle X(\omega)]\,d\omega$$

7.1-3. (a) Because $x(t) = x_o(t) + x_e(t)$ and $e^{-j\omega t} = \cos\,\omega t + j\sin\,\omega t$

$$X(\omega) = \int_{-\infty}^{\infty} [x_o(t) + x_e(t)]e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} [x_o(t) + x_e(t)] \cos \omega t dt - j \int_{-\infty}^{\infty} [x_o(t) + x_e(t)] \sin \omega t dt$$

Because $x_e(t) \cos \omega t$ and $x_o(t) \sin \omega t$ are even functions and $x_o(t) \cos \omega t$ and $x_e(t) \sin \omega t$ are odd functions of t, these integrals [properties in Eqs. (B.43), p. 38] reduce to

$$X(\omega) = 2 \int_0^{\infty} x_e(t) \cos \omega t \, dt - 2j \int_0^{\infty} x_o(t) \sin \omega t \, dt \qquad (1)$$

Also, from the results of Prob. 7.1-1, we have

$$\mathcal{F}\{x_e(t)\} = 2 \int_0^{\infty} x_e(t) \cos \omega t \, dt \quad \text{and} \quad \mathcal{F}\{x_0(t)\} = -2j \int_0^{\infty} x_o(t) \sin \omega t \, dt \quad (2)$$

From Eqs. (1) and (2), the desired result follows.

(b) We can express $u(t)$ in terms of its even and odd components as follows

$$\begin{aligned} u(t) &= \frac{1}{2}[u(t) + u(-t)] + \frac{1}{2}[u(t) - u(-t)] \\ &= \underbrace{\frac{1}{2}}_{x_e(t)} + \underbrace{\frac{1}{2}\text{sgn}(t)}_{x_o(t)} \end{aligned}$$

and

$$X_e(\omega) = \pi \delta(\omega) \qquad \text{and} \qquad X_o(\omega) = \frac{1}{j\omega}$$

Clearly, $X_e(\omega)$ is the real part and $X_o(\omega)$ is the odd part of $X(\omega)$.
We follow the same procedure for $x(t) = e^{-at}u(t)$.

$$e^{-at}u(t) = \underbrace{\frac{1}{2}[e^{-at}u(t) + e^{-at}u(-t)]}_{x_e(t)} + \underbrace{\frac{1}{2}[e^{-at}u(t) - e^{-at}u(-t)]}_{x_o(t)}$$

Also

$$X_e(\omega) = \frac{1}{2}\left[\frac{1}{j\omega + a} - \frac{1}{j\omega - a}\right] = \frac{2a}{\omega^2 + a^2}$$

and

$$X_o(\omega) = \frac{1}{2}\left[\frac{1}{j\omega + a} + \frac{1}{j\omega - a}\right] = \frac{2j\omega}{\omega^2 + a^2}$$

Clearly, $X_e(\omega)$ is the real part and $X_o(\omega)$ is the odd part of $X(\omega)$.

7.1-4. (a)

$$X(\omega) = \int_0^T e^{-at} e^{-j\omega t} \, dt = \int_0^T e^{-(j\omega + a)t} \, dt = \frac{1 - e^{-(j\omega + a)T}}{j\omega + a}$$

(b)

$$X(\omega) = \int_0^T e^{at} e^{-j\omega t} \, dt = \int_0^T e^{-(j\omega - a)} \, dt = \frac{1 - e^{-(j\omega - a)T}}{j\omega - a}$$

7.1-5. (a)

$$X(\omega) = \int_0^1 4e^{-j\omega t}\,dt + \int_1^2 2e^{-j\omega t}\,dt = \frac{4 - 2e^{-j\omega} - 2e^{-j2\omega}}{j\omega}$$

(b)

$$X(\omega) = \int_{-\tau}^{0} -\frac{t}{\tau}e^{-j\omega t}\,dt + \int_0^{\tau} \frac{t}{\tau}e^{-j\omega t}\,dt = \frac{2}{\tau\omega^2}[\cos\omega\tau + \omega\tau\sin\omega\tau - 1]$$

This result could also be derived by observing that $x(t)$ is an even function. Therefore from the result in Prob. 7.1-1

$$X(\omega) = \frac{2}{\tau}\int_0^{\tau} t\cos\omega t\,dt = \frac{2}{\tau\omega^2}[\cos\omega\tau + \omega\tau\sin\omega\tau - 1]$$

7.1-6. (a)

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} \omega^2 e^{j\omega t}\,d\omega = \frac{1}{2\pi}\,\frac{e^{j\omega t}}{(jt)^3}\left[-\omega^2t^2 - 2j\omega t + 2\right]\Big|_{-\omega_0}^{\omega_0} \\ &= \frac{(\omega_0^2t^2 - 2)\sin\omega_0 t + 2\omega_0 t\cos\omega_0 t}{\pi t^3} \end{aligned}$$

(b) The derivation can be simplified by observing that $X(\omega)$ can be expressed as a sum of two gate functions $X_1(\omega)$ and $X_2(\omega)$ as shown in Figure S7.1-6. Therefore

$$x(t) = \frac{1}{2\pi}\int_{-2}^{2}[X_1(\omega)+X_2(\omega)]e^{j\omega t}\,d\omega = \frac{1}{2\pi}\left\{\int_{-2}^{2} e^{j\omega t}\,d\omega + \int_{-1}^{1} e^{j\omega t}\,d\omega\right\} = \frac{\sin 2t + \sin t}{\pi t}$$

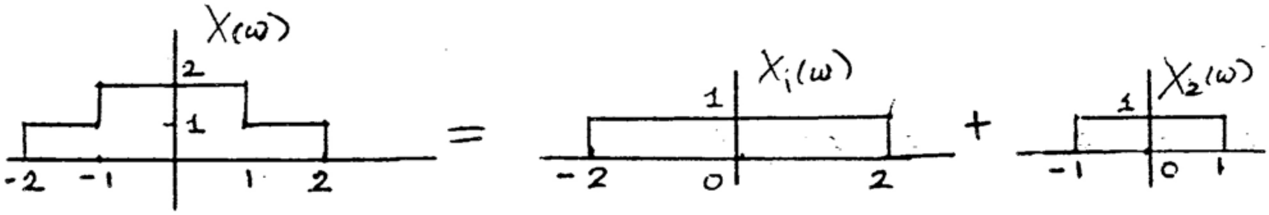

Figure S7.1-6

7.1-7. (a)

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos\omega\, e^{j\omega t}\,d\omega \\ &= \frac{e^{j\omega t}}{2\pi(1-t^2)}\{jt\cos\omega + \sin\omega\}_{-\pi/2}^{\pi/2} \\ &= \frac{1}{\pi(1-t^2)}\cos\left(\frac{\pi t}{2}\right) \end{aligned}$$

(b)

$$x(t) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} X(\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\left[\int_{-\pi/2}^{\pi/2} X(\omega)\cos\omega t\,d\omega + j\int_{-\pi/2}^{\pi/2} X(\omega)\sin\omega t\,d\omega\right]$$

Because $X(\omega)$ is even function, the second integral on the right-hand side vanishes. Also the integrand of the first term is an even function. Therefore

$$\begin{aligned} x(t) &= \frac{1}{\pi}\int_0^{\pi/2} \frac{\omega}{\omega_0}\cos t\omega\, d\omega = \frac{1}{\pi\omega_0}\left[\frac{\cos t\omega + t\omega\sin t\omega}{t^2}\right]_0^{\omega_0} \\ &= \frac{1}{\pi\omega_0 t^2}[\cos\omega_0 t + \omega_0 t \sin\omega_0 t - 1] \end{aligned}$$

7.1-8.

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ \text{Hence}\quad X(0) &= \int_{-\infty}^{\infty} x(t)dt \\ \text{Also}\quad x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \\ \text{and}\quad x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)d\omega \end{aligned}$$

Because $\text{sinc}(t) \leftrightarrow \pi\text{rect}(\frac{\omega}{2})$

$$\pi\text{rect}(0) = \pi = \int_{-\infty}^{\infty} \text{sinc}(t)dt$$

Also $\text{sinc}^2(t) \leftrightarrow \pi\Delta(\frac{\omega}{4})$

$$\pi\Delta(0) = \pi = \int_{-\infty}^{\infty} \text{sinc}^2(t)dt$$

7.2-1. Figure S7.2-1 shows the plots of various functions. The function in part (a) is a gate function centered at the origin and of width 2. The function in part (b) can be expressed as $\Delta\left(\frac{\omega}{100/3}\right)$. This is a triangle pulse centered at the origin and of width 100/3. The function in part (c) is a gate function $\text{rect}(\frac{t}{8})$ delayed by 10. In other words it is a gate pulse centered at $t = 10$ and of width 8. The function in part (d) is a sinc pulse centered at the origin and the first zero occurring at $\frac{\pi\omega}{5} = \pi$, that is at $\omega = 5$. The function in part (e) is a sinc pulse $\text{sinc}(\frac{\omega}{5})$ delayed by 10π. For the sinc pulse $\text{sinc}(\frac{\omega}{5})$, the first zero occurs at $\frac{\omega}{5} = \pi$, that is at $\omega = 5\pi$. Therefore the function is a sinc pulse centered at $\omega = 10\pi$ and its zeros spaced at intervals of 5π as shown in the figure S7.2-1e. The function in part (f) is a product of a gate pulse (centered at the origin) of width 10π and a sinc pulse (also centered at the origin) with zeros spaced at intervals of 5π. This results in the sinc pulse truncated beyond the interval $\pm 5\pi$ ($|t| \geq 5\pi$) as shown in Fig. f.

7.2-2.

$$\begin{aligned} X(\omega) &= \int_{4.5}^{5.5} e^{-j\omega t}\,dt = -\frac{1}{j\omega}e^{-j\omega t}\Big|_{4.5}^{5.5} = \frac{1}{j\omega}[e^{-j4.5\omega} - e^{-j5.5\omega}] \\ &= \frac{e^{-j5\omega}}{j\omega}[e^{j\omega/2} - e^{-j\omega/2}] = \frac{e^{-j5\omega}}{j\omega}\left[2j\sin\frac{\omega}{2}\right] \\ &= \text{sinc}\left(\frac{\omega}{2}\right)e^{-j5\omega} \end{aligned}$$

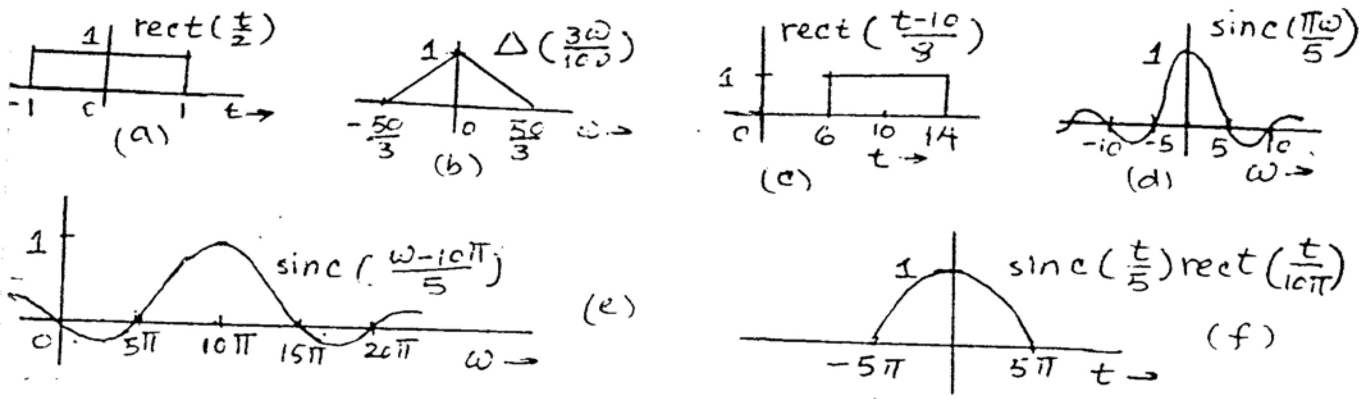

Figure S7.2-1

7.2-3.

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{10-\pi}^{10+\pi} e^{j\omega t}\,d\omega = \left.\frac{e^{j\omega t}}{2\pi(j\omega)}\right|_{10-\pi}^{10+\pi} = \frac{1}{j2\pi\omega}\left[e^{j(10+\pi)t} - e^{j(10-\pi)t}\right] \\ &= \frac{e^{j10t}}{j2\pi\omega}[2j\sin\pi t] = \text{sinc}(\pi t)e^{j10t} \end{aligned}$$

7.2-4. (a)

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{-j\omega t_0}e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega(t-t_0)}\,d\omega \\ &= \left.\frac{1}{(2\pi)j(t-t_0)}e^{j\omega(t-t_0)}\right|_{-\omega_0}^{\omega_0} = \frac{\sin\omega_0(t-t_0)}{\pi(t-t_0)} = \frac{\omega_0}{\pi}\text{sinc}[\omega_0(t-t_0)] \end{aligned}$$

(b)

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\left[\int_{-\omega_0}^{0} je^{j\omega t}\,d\omega + \int_{0}^{\omega_0} -je^{j\omega t}\,d\omega\right] \\ &= \left.\frac{1}{2\pi t}e^{j\omega t}\right|_{-\omega_0}^{0} - \left.\frac{1}{2\pi t}e^{j\omega t}\right|_{0}^{\omega_0} = \frac{1-\cos\omega_0 t}{\pi t} \end{aligned}$$

7.2-5. (a) When $a > 0$, we cannot find the Fourier transform of $e^{at}u(t)$ by setting $s = j\omega$ in the Laplace transform of $e^{at}u(t)$ because the ROC is $Re s > a$, which does not include the $j\omega$-axis.

(b) The Laplace transform of $x(t)$ is

$$X(s) = \int_0^T e^{at}e^{-st}dt = \int_0^T e^{-(s-a)t}dt = \frac{1}{s-a}\left[1 - e^{-(s-a)T}\right]$$

Interestingly, because $x(t)$ has a finite width, the ROC of its $X(s)$ is the entire s-plane, which includes $j\omega$-axis. Hence, the Fourier transform

$$X(\omega) = X(s)\,|_{s=j\omega} = \frac{1}{s-a}\left[1 - e^{-(j\omega-a)T}\right]$$

To verify this, we find the Fourier transform of $x(t)$

$$X(\omega) = \int_0^T e^{at} e^{-j\omega t} dt = \int_0^T e^{-(j\omega - a)t} dt = \frac{1}{j\omega - a}\left[1 - e^{-(j\omega - a)T}\right]$$

Which agrees with $X(j\omega)$

7.3-1. (a)

$$\underbrace{u(t)}_{x(t)} \Longleftrightarrow \underbrace{\pi\delta(\omega) + \frac{1}{j\omega}}_{X(\omega)}$$

Application of duality property yields

$$\underbrace{\pi\delta(t) + \frac{1}{jt}}_{X(t)} \Longleftrightarrow \underbrace{2\pi u(-\omega)}_{2\pi x(-\omega)}$$

or

$$\frac{1}{2}\left[\delta(t) + \frac{1}{j\pi t}\right] \Longleftrightarrow u(-\omega)$$

Application of Eq. (4.35) yields

$$\frac{1}{2}\left[\delta(-t) - \frac{1}{j\pi t}\right] \Longleftrightarrow u(\omega)$$

But $\delta(t)$ is an even function, that is $\delta(-t) = \delta(t)$, and

$$\frac{1}{2}[\delta(t) + \frac{j}{\pi t}] \Longleftrightarrow u(\omega)$$

(b)

$$\underbrace{\cos \omega_0 t}_{x(t)} \Longleftrightarrow \underbrace{\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]}_{X(\omega)}$$

Application of duality property yields

$$\underbrace{\pi[\delta(t + \omega_0) + \delta(t - \omega_0)]}_{X(t)} \Longleftrightarrow \underbrace{2\pi \cos(-\omega_0\omega)}_{2\pi x(-\omega)} = 2\pi \cos(\omega_0\omega)$$

Setting $\omega_0 = T$ yields

$$\delta(t + T) + \delta(t - T) \Longleftrightarrow 2\cos T\omega$$

(c)

$$\underbrace{\sin \omega_0 t}_{x(t)} \Longleftrightarrow \underbrace{j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]}_{X(\omega)}$$

Application of duality property yields

$$\underbrace{j\pi[\delta(t + \omega_0) - \delta(t - \omega_0)]}_{X(t)} \Longleftrightarrow \underbrace{2\pi \sin(-\omega_0\omega)}_{2\pi x(-\omega)} = -2\pi \sin(\omega_0\omega)$$

Setting $\omega_0 = T$ yields

$$\delta(t+T) - \delta(t-T) \Longleftrightarrow 2j \sin T\omega$$

7.3-2. Refer to the solution of Prob. 1.2-3 for description of these signals.

(a)

$$x_1(t) = x(t+1) + x(1-t)$$

$$x(t+1) \Longleftrightarrow X(\omega)e^{j\omega}$$

Time inversion of $x(t+1)$ yields $x(-t+1)$, Hence

$$x(1-t) \Longleftrightarrow X(-\omega)e^{-j\omega}$$

$$\text{Hence} \qquad x_1(t) \Longleftrightarrow X(\omega)e^{j\omega} + X(-\omega)e^{-j\omega}$$

(b)

$$x_2(t) = x\left(\frac{t+1}{2}\right) + x\left(\frac{1-t}{2}\right)$$

$$x\left(t+\frac{1}{2}\right) \Longleftrightarrow X(\omega)e^{\frac{j\omega}{2}}$$

$$\text{and} \qquad x\left(\frac{t+1}{2}\right) = x\left(\frac{t}{2}+\frac{1}{2}\right) \Longleftrightarrow 2X(2\omega)e^{j\omega}$$

$$\text{Therefore} \qquad x\left(\frac{1-t}{2}\right) \Longleftrightarrow 2X(-2\omega)e^{-j\omega}$$

$$\text{Hence} \qquad x_2(t) \Longleftrightarrow 2\left[X(2\omega)e^{j\omega} + X(-2\omega)e^{-j\omega}\right]$$

(c)

$$x_3(t) = x\left(\frac{t+2}{4}\right) + x\left(\frac{2-t}{4}\right) + x\left(\frac{t}{2}\right) + x\left(\frac{-t}{2}\right)$$

$$x\left(\frac{t+2}{4}\right) \Longleftrightarrow 4X(4\omega)e^{j2\omega}$$

$$x\left(\frac{2-t}{4}\right) \Longleftrightarrow 4X(-4\omega)e^{-j2\omega}$$

$$x\left(\frac{t}{2}\right) \Longleftrightarrow 2X(2\omega) \qquad \text{and} \qquad x\left(-\frac{t}{2}\right) \Longleftrightarrow 2X(-2\omega)$$

$$\text{Hence} \qquad x_3(t) \Longleftrightarrow 4\left[X(4\omega)e^{j2\omega} + X(-4\omega)e^{-j2\omega} + 2X(2\omega) + 2X(-2\omega)\right]$$

(d)

$$x_4(t) = \frac{4}{3}x\left(\frac{t+2}{2}\right) + \frac{4}{3}x\left(\frac{2-t}{2}\right) - \frac{1}{3}x\left(\frac{t+2}{4}\right) - \frac{1}{3}x\left(\frac{2-t}{4}\right)$$

$$x\left(\frac{t+2}{2}\right) \Longleftrightarrow 2X(2\omega)e^{j2\omega} \qquad \text{and} \qquad x\left(\frac{2-t}{2}\right) \Longleftrightarrow 2X(-2\omega)e^{-j2\omega}$$

$$x\left(\frac{t+2}{4}\right) \Longleftrightarrow 4X(4\omega)e^{j2\omega} \qquad \text{and} \qquad x\left(\frac{2-t}{4}\right) \Longleftrightarrow 4X(-4\omega)e^{-j2\omega}$$

Hence $\quad x_4(t) \Longleftrightarrow \frac{8}{3}\left[X(2\omega)e^{j2\omega} + X(-2\omega)e^{-j2\omega}\right] - \frac{4}{3}\left[X(4\omega)e^{j2\omega} + X(-4\omega)e^{-j2\omega}\right]$

(e)

$$x_5(t)x(t+0.5) + x(0.5-t) + x(t+1.5) + x(1.5-t)$$

Hence $\quad x_5(t) \Longleftrightarrow X(\omega)e^{\frac{j\omega}{2}} + X(-\omega)e^{\frac{-j\omega}{2}} + X(\omega)e^{1.5j\omega} + X(-\omega)e^{-1.5j\omega}$

In all these expressions, we substitute

$$X(\omega) = \frac{1}{\omega^2}\left[ej\omega - j\omega e^{j\omega} - 1\right]$$

7.3-3. (a)

$$x(t) = \text{rect}\left(\frac{t+T/2}{T}\right) - \text{rect}\left(\frac{t-T/2}{T}\right)$$

$$\begin{aligned}
\text{rect}\left(\frac{t}{T}\right) &\Longleftrightarrow T\text{sinc}\left(\frac{\omega T}{2}\right) \\
\text{rect}\left(\frac{t \pm T/2}{T}\right) &\Longleftrightarrow T\text{sinc}\left(\frac{\omega T}{2}\right)e^{\pm j\omega T/2}
\end{aligned}$$

$$\begin{aligned}
X(\omega) &= T\text{sinc}\left(\frac{\omega T}{2}\right)\left[e^{j\omega T/2} - e^{-j\omega T/2}\right] \\
&= 2jT\text{sinc}\left(\frac{\omega T}{2}\right)\sin\frac{\omega T}{2} \\
&= \frac{j4}{\omega}\sin^2\left(\frac{\omega T}{2}\right)
\end{aligned}$$

(b) From Figure S7.3-3b we verify that

$$x(t) = \sin t\, u(t) + \sin(t-\pi)u(t-\pi)$$

Note that $\sin(t-\pi)u(t-\pi)$ is $\sin\, t\,u(t)$ delayed by π. Now, $\sin t\, u(t) \Longleftrightarrow \frac{\pi}{2j}[\delta(\omega-1) - \delta(\omega+1)] + \frac{1}{1-\omega^2}$ and

$$\sin(t-\pi)u(t-\pi) \Longleftrightarrow \left\{\frac{\pi}{2j}[\delta(\omega-1) - \delta(\omega+1)] + \frac{1}{1-\omega^2}\right\}e^{-j\pi\omega}$$

Therefore

$$X(\omega) = \left\{\frac{\pi}{2j}[\delta(\omega-1) - \delta(\omega+1)] + \frac{1}{1-\omega^2}\right\}(1+e^{-j\pi\omega})$$

Recall that $g(x)\delta(x-x_0) = g(x_0)\delta(x-x_0)$. Therefore $\delta(\omega \pm 1)(1+e^{-j\pi\omega}) = 0$, and

$$X(\omega) = \frac{1}{1-\omega^2}(1+e^{-j\pi\omega})$$

(c) From Figure S7.3-3c we verify that

$$x(t) = \cos t\left[u(t) - u\left(t - \frac{\pi}{2}\right)\right] = \cos t\, u(t) - \cos t\, u\left(t - \frac{\pi}{2}\right)$$

But $\sin(t - \frac{\pi}{2}) = -\cos t$. Therefore

$$\begin{aligned} x(t) &= \cos t\, u(t) + \sin\left(t - \frac{\pi}{2}\right) u\left(t - \frac{\pi}{2}\right) \\ X(\omega) &= \frac{\pi}{2}[\delta(\omega - 1) + \delta(\omega + 1)] + \frac{j\omega}{1 - \omega^2} + \left\{\frac{\pi}{2j}[\delta(\omega - 1) - \delta(\omega + 1)] + \frac{1}{1 - \omega^2}\right\} e^{-j\pi\omega/2} \end{aligned}$$

Also because $g(x)\delta(x - x_0) = g(x_0)\delta(x - x_0)$,

$$\delta(\omega \pm 1)e^{-j\pi\omega/2} = \delta(\omega \pm 1)e^{\pm j\pi/2} = \pm j\delta(\omega \pm 1)$$

Therefore

$$X(\omega) = \frac{j\omega}{1 - \omega^2} + \frac{e^{-j\pi\omega/2}}{1 - \omega^2} = \frac{1}{1 - \omega^2}[j\omega + e^{-j\pi\omega/2}]$$

(d)

$$\begin{aligned} x(t) &= e^{-at}[u(t) - u(t - T)] = e^{-at}u(t) - e^{-at}u(t - T) \\ &= e^{-at}u(t) - e^{-aT}e^{-a(t-T)}u(t - T) \\ X(\omega) &= \frac{1}{j\omega + a} - \frac{e^{-aT}}{j\omega + a}e^{-j\omega T} = \frac{1}{j\omega + a}[1 - e^{-(a+j\omega)T}] \end{aligned}$$

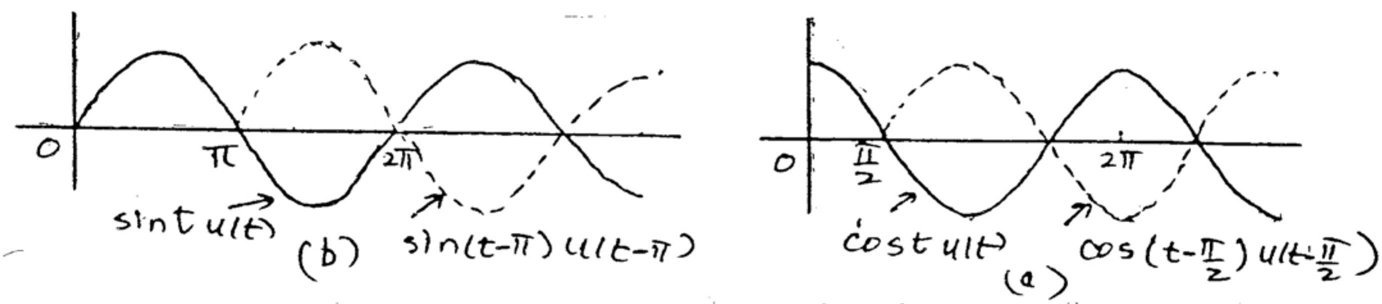

Figure S7.3-3

7.3-4. From time-shifting property

$$x(t \pm T) \Longleftrightarrow X(\omega)e^{\pm j\omega T}$$

Therefore

$$x(t + T) + x(t - T) \Longleftrightarrow X(\omega)e^{j\omega T} + X(\omega)e^{-j\omega T} = 2X(\omega)\cos\omega T$$

We can use this result to derive transforms of signals in Figure P7.3-4.

(a) Here $x(t)$ is a gate pulse as shown in Figure S7.3-4a.

$$x(t) = \text{rect}\left(\frac{t}{2}\right) \Longleftrightarrow 2\,\text{sinc}(\omega)$$

Also $T = 3$. The signal in Figure S7.3-4a is $x(t+3) + x(t-3)$, and

$$x(t+3) + x(t-3) \Longleftrightarrow 4\,\text{sinc}(\omega)\cos 3\omega$$

(b) Here $x(t)$ is a triangular pulse shown in Figure S7.3-4b. From the Table 4.1 (pair 19)

$$x(t) = \Delta\left(\frac{t}{2}\right) \Longleftrightarrow \text{sinc}^2\left(\frac{\omega}{2}\right)$$

Also $T = 3$. The signal in Figure P7.3-4b is $x(t+3) + x(t-3)$, and

$$x(t+3) + x(t-3) \Longleftrightarrow 2\,\text{sinc}^2\left(\frac{\omega}{2}\right)\cos 3\omega$$

Figure S7.3-4

7.3-5. Frequency-shifting property states that

$$x(t)e^{\pm j\omega_0 t} \Longleftrightarrow X(\omega \mp \omega_0)$$

Therefore

$$x(t)\sin\omega_0 t = \frac{1}{2j}[x(t)e^{j\omega_0 t} + x(t)e^{-j\omega_0 t}] = \frac{1}{2j}[X(\omega - \omega_0) + X(\omega - \omega_0)]$$

Time-shifting property states that

$$x(t \pm T) \Longleftrightarrow X(\omega)e^{\pm j\omega T}$$

Therefore

$$x(t+T) - x(t-T) \Longleftrightarrow X(\omega)e^{j\omega T} - X(\omega)e^{-j\omega T} = 2jX(\omega)\sin\omega T$$

and

$$\frac{1}{2j}[x(t+T) - x(t-T)] \Longleftrightarrow X(\omega)\sin T\omega$$

The signal in Figure P7.3-5 is $x(t+3) - x(t-3)$ where

$$x(t) = \text{rect}\left(\frac{t}{2}\right) \Longleftrightarrow 2\text{sinc}(\omega)$$

Therefore

$$x(t+3) - x(t-3) \Longleftrightarrow 2j[2\text{sinc}(\omega)\sin 3\omega] = 4j\,\text{sinc}(\omega)\sin 3\omega$$

7.3-6. (a) The signal $x(t)$ in this case is a triangle pulse $\Delta(\frac{t}{2\pi})$ (Figure S7.3-6) multiplied by $\cos 10t$.

$$x(t) = \Delta\left(\frac{t}{2\pi}\right)\cos\,10t$$

Also from Table 4.1 (pair 19) $\Delta(\frac{t}{2\pi}) \Longleftrightarrow \pi\,\text{sinc}^2(\frac{\pi\omega}{2})$ From the modulation property (4.41), it follows that

$$x(t) = \Delta\left(\frac{t}{2\pi}\right)\cos 10t \Longleftrightarrow \frac{\pi}{2}\left\{\text{sinc}^2\left[\frac{\pi(\omega-10)}{2}\right] + \text{sinc}^2\left[\frac{\pi(\omega+10)}{2}\right]\right\}$$

The Fourier transform in this case is a real function and we need only the amplitude spectrum in this case as shown in Figure S7.3-6a.

(b) The signal $x(t)$ here is the same as the signal in (a) delayed by 2π. From time shifting property, its Fourier transform is the same as in part (a) multiplied by $e^{-j\omega(2\pi)}$. Therefore

$$X(\omega) = \frac{\pi}{2}\left\{\text{sinc}^2\left[\frac{\pi(\omega-10)}{2}\right] + \text{sinc}^2\left[\frac{\pi(\omega+10)}{2}\right]\right\}e^{-j2\pi\omega}$$

The Fourier transform in this case is the same as that in part (a) multiplied by $e^{-j2\pi\omega}$. This multiplying factor represents a linear phase spectrum $-2\pi\omega$. Thus we have an amplitude spectrum [same as in part (a)] as well as a linear phase spectrum $\angle X(\omega) = -2\pi\omega$ as shown in Figure S7.3-6b. the amplitude spectrum in this case as shown in Figure S7.3-6b.
Note: In the above solution, we first multiplied the triangle pulse $\Delta(\frac{t}{2\pi})$ by $\cos 10t$ and then delayed the result by 2π. This means the signal in (b) is expressed as $\Delta(\frac{t-2\pi}{2\pi})\cos 10(t-2\pi)$.
We could have interchanged the operation in this particular case, that is, the triangle pulse $\Delta(\frac{t}{2\pi})$ is first delayed by 2π and then the result is multiplied by $\cos 10t$. In this alternate procedure, the signal in (b) is expressed as $\Delta(\frac{t-2\pi}{2\pi})\cos 10t$.
This interchange of operation is permissible here only because the sinusoid $\cos 10t$ executes integral number of cycles in the interval 2π. Because of this both the expressions are equivalent since $\cos 10(t-2\pi) = \cos 10t$.

(c) In this case the signal is identical to that in (b), except that the basic pulse is $\text{rect}(\frac{t}{2\pi})$ instead of a triangle pulse $\Delta(\frac{t}{2\pi})$. Now

$$\text{rect}\left(\frac{t}{2\pi}\right) \Longleftrightarrow 2\pi\,\text{sinc}(\pi\omega)$$

Using the same argument as for part (b), we obtain

$$X(\omega) = \pi\{\text{sinc}[\pi(\omega+10)] + \text{sinc}[\pi(\omega-10)]\}e^{-j2\pi\omega}$$

7.3-7. (a)

$$X(\omega) = \text{rect}\left(\frac{\omega-4}{2}\right) + \text{rect}\left(\frac{\omega+4}{2}\right)$$

Also

$$\frac{1}{\pi}\text{sinc}(t) \Longleftrightarrow \text{rect}\left(\frac{\omega}{2}\right)$$

Therefore

$$x(t) = \frac{2}{\pi}\text{sinc}(t)\cos 4t$$

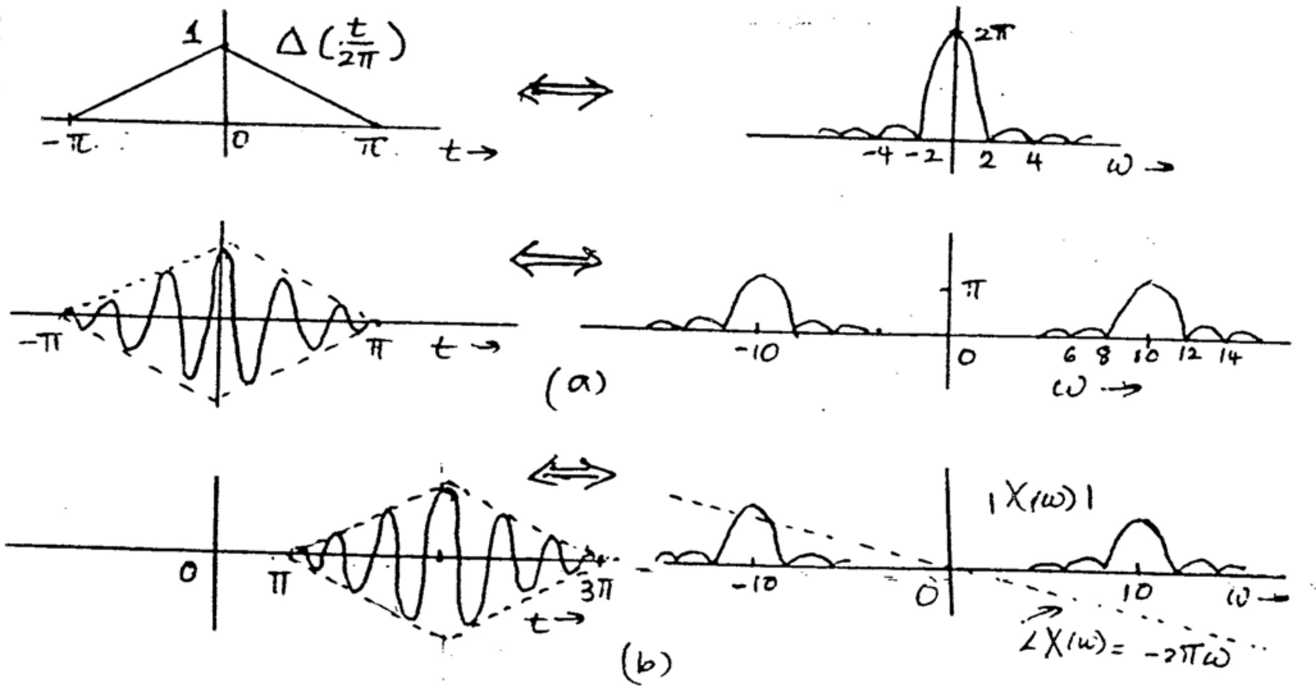

Figure S7.3-6

(b)

$$X(\omega) = \triangle\left(\frac{\omega+4}{4}\right) + \triangle\left(\frac{\omega-4}{4}\right)$$

Also

$$\frac{1}{\pi}\text{sinc}^2(t) \Longleftrightarrow \triangle\left(\frac{\omega}{4}\right)$$

Therefore

$$x(t) = \frac{2}{\pi}\text{sinc}^2(t)\cos 4t$$

7.3-8. (a)

$$e^{\lambda t}u(t) \Longleftrightarrow \frac{1}{j\omega - \lambda} \quad \text{and} \quad u(t) \Longleftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

If $x(t) = e^{\lambda t}u(t) * u(t)$, then

$$\begin{aligned}
X(\omega) &= \left(\frac{1}{j\omega-\lambda}\right)\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) \\
&= \frac{\pi\delta(\omega)}{j\omega-\lambda} + \left[\frac{1}{j\omega(j\omega-\lambda)}\right] \\
&= -\frac{\pi}{\lambda}\delta(\omega) + \left[\frac{-\frac{1}{\lambda}}{j\omega} + \frac{\frac{1}{\lambda}}{j\omega-\lambda}\right] \quad \text{because } g(x)\delta(x) = g(0)\delta(x) \\
&= \frac{1}{\lambda}\left[\frac{1}{j\omega-\lambda} - \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)\right]
\end{aligned}$$

Taking the inverse transform of this equation yields

$$x(t) = \frac{1}{\lambda}(e^{\lambda t} - 1)u(t)$$

(b)

$$e^{\lambda_1 t}u(t) \Longleftrightarrow \frac{1}{j\omega - \lambda_1} \quad \text{and} \quad e^{\lambda_2 t}u(t) \Longleftrightarrow \frac{1}{j\omega - \lambda_2}$$

If $x(t) = e^{\lambda_1 t}u(t) * e^{\lambda_2 t}u(t)$, then

$$X(\omega) = \frac{1}{(j\omega - \lambda_1)(j\omega - \lambda_2)} = \frac{\frac{1}{\lambda_1 - \lambda_2}}{j\omega - \lambda_1} - \frac{\frac{1}{\lambda_1 - \lambda_2}}{j\omega - \lambda_2}$$

Therefore

$$x(t) = \frac{1}{\lambda_1 - \lambda_2}(e^{\lambda_1 t} - e^{\lambda_2 t})u(t)$$

(c)

$$e^{\lambda_1 t}u(t) \Longleftrightarrow \frac{1}{j\omega - \lambda_1} \quad \text{and} \quad e^{\lambda_2 t}u(-t) \Longleftrightarrow -\frac{1}{j\omega - \lambda_2}$$

If $x(t) = e^{\lambda_1 t}u(t) * e^{\lambda_2 t}u(-t)$, then

$$X(\omega) = \frac{-1}{(j\omega - \lambda_1)(j\omega - \lambda_2)} = \frac{\frac{1}{\lambda_2 - \lambda_1}}{j\omega - \lambda_1} - \frac{\frac{1}{\lambda_2 - \lambda_1}}{j\omega - \lambda_2}$$

Therefore

$$x(t) = \frac{1}{\lambda_2 - \lambda_1}[e^{\lambda_1 t}u(t) + e^{\lambda_2 t}u(-t)]$$

Note that because $\lambda_2 > 0$, the inverse transform of $\frac{-1}{j\omega - \lambda_2}$ is $e^{\lambda_2 t}u(-t)$ and not $-e^{\lambda_2 t}u(t)$. The Fourier transform of the latter does not exist because $\lambda_2 > 0$.

(d)

$$e^{\lambda_1 t}u(-t) \Longleftrightarrow -\frac{1}{j\omega - \lambda_1} \quad \text{and} \quad e^{\lambda_2 t}u(-t) \Longleftrightarrow -\frac{1}{j\omega - \lambda_2}$$

If $x(t) = e^{\lambda_1 t}u(-t) * e^{\lambda_2 t}u(-t)$, then

$$X(\omega) = \frac{1}{(j\omega - \lambda_1)(j\omega - \lambda_2)} = \frac{\frac{-1}{\lambda_2 - \lambda_1}}{j\omega - \lambda_1} - \frac{\frac{-1}{\lambda_2 - \lambda_1}}{j\omega - \lambda_2}$$

Therefore

$$x(t) = \frac{1}{\lambda_2 - \lambda_1}(e^{\lambda_1 t} - e^{\lambda_2 t})u(-t)$$

The remarks at the end of part (c) apply here also.

7.3-9. From the frequency convolution property, we obtain

$$x^2(t) \Longleftrightarrow \frac{1}{2\pi}X(\omega) * X(\omega)$$

Because of the width property of the convolution, the width of $X(\omega) * X(\omega)$ is twice the width of $X(\omega)$. Repeated application of this argument shows that the bandwidth of $x^n(t)$ is nB Hz (n times the bandwidth of $x(t)$).

7.3-10. (a)

$$X(\omega) = \int_{-T}^{0} e^{-j\omega t}\,dt - \int_{0}^{T} e^{-j\omega t}\,dt = -\frac{2}{j\omega}[1 - \cos\omega T] = \frac{j4}{\omega}\sin^2\left(\frac{\omega T}{2}\right)$$

(b)

$$x(t) = \text{rect}\left(\frac{t+T/2}{T}\right) - \text{rect}\left(\frac{t-T/2}{T}\right)$$

$$\begin{aligned}
\text{rect}\left(\frac{t}{T}\right) &\iff T\text{sinc}\left(\frac{\omega T}{2}\right) \\
\text{rect}\left(\frac{t \pm T/2}{T}\right) &\iff T\text{sinc}\left(\frac{\omega T}{2}\right) e^{\pm j\omega T/2}
\end{aligned}$$

$$\begin{aligned}
X(\omega) &= T\text{sinc}\left(\frac{\omega T}{2}\right)[e^{j\omega T/2} - e^{-j\omega T/2}] \\
&= 2jT\text{sinc}\left(\frac{\omega T}{2}\right)\sin\frac{\omega T}{2} \\
&= \frac{j4}{\omega}\sin^2\left(\frac{\omega T}{2}\right)
\end{aligned}$$

(c)

$$\frac{df}{dt} = \delta(t+T) - 2\delta(t) + \delta(t-T)$$

The Fourier transform of this equation yields

$$j\omega X(\omega) = e^{j\omega T} - 2 + e^{-j\omega T} = -2[1 - \cos\omega T] = -4\sin^2\left(\frac{\omega T}{2}\right)$$

Therefore

$$X(\omega) = \frac{j4}{\omega}\sin^2\left(\frac{\omega T}{2}\right)$$

7.3-11. (a)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \quad \text{and} \quad \frac{dF}{d\omega} = \frac{d}{d\omega}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Changing the order of differentiation and integration yields

$$\frac{dF}{d\omega} = \int_{-\infty}^{\infty} \frac{d}{d\omega}(x(t)e^{-j\omega t}) = \int_{-\infty}^{\infty} [-jtx(t)]e^{-j\omega t}\,dt$$

Therefore

$$-jtx(t) \iff \frac{dF}{d\omega}$$

(b)

$$
\begin{aligned}
e^{-at}u(t) &\iff \frac{1}{j\omega + a} \\
-jte^{-at}u(t) &\iff \frac{d}{d\omega}\left(\frac{1}{j\omega + a}\right) = \frac{-j}{(j\omega + a)^2}
\end{aligned}
$$

$$
te^{-at}u(t) \iff \frac{1}{(j\omega + a)^2}
$$

7.4-1.

$$
H(\omega) = \frac{1}{j\omega + 1}
$$

(a)

$$
\begin{aligned}
X(\omega) &= \frac{1}{j\omega + 2} \\
Y(\omega) &= \frac{1}{(j\omega + 1)(j\omega + 2)} = \frac{1}{j\omega + 1} - \frac{1}{j\omega + 2} \\
y(t) &= (e^{-t} - e^{-2t})u(t)
\end{aligned}
$$

(b)

$$
\begin{aligned}
X(\omega) &= \frac{1}{j\omega + 1} \\
Y(\omega) &= \frac{1}{(j\omega + 1)^2} \\
y(t) &= te^{-at}u(t)
\end{aligned}
$$

(c)

$$
\begin{aligned}
X(\omega) &= -\frac{1}{j\omega - 1} \\
Y(\omega) &= \frac{-1}{(j\omega + 1)(j\omega - 1)} = \frac{1/2}{j\omega + 1} - \frac{1/2}{j\omega - 1} \\
y(t) &= \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{t}u(-t)
\end{aligned}
$$

(d)

$$
\begin{aligned}
X(\omega) &= \pi\delta(\omega) + \frac{1}{j\omega} \\
Y(\omega) &= \frac{1}{j\omega + 1}\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] \\
&= \pi\delta(\omega) + \frac{1}{j\omega(j\omega + 1)} \qquad [\text{because} g(x)\delta(x) = g(0)\delta(x)] \\
&= \pi\delta(\omega) + \frac{1}{j\omega} - \frac{1}{j\omega + 1} \\
y(t) &= (1 - e^{-t})u(t)
\end{aligned}
$$

7.4-2. (a)

$$X(\omega) = \frac{1}{j\omega + 1} \qquad \text{and} \qquad H(\omega) = \frac{-1}{j\omega - 2}$$

and

$$Y(\omega) = \frac{-1}{(j\omega - 2)(j\omega + 1)} = \frac{1}{3}\left[\frac{1}{j\omega + 1} - \frac{1}{j\omega - 2}\right]$$

Therefore

$$y(t) = \frac{1}{3}\left[e^{-t}u(t) + e^{2t}u(-t)\right]$$

(b)

$$X(\omega) = \frac{-1}{j\omega - 1} \qquad \text{and} \qquad H(\omega) = \frac{-1}{j\omega - 2}$$

and

$$Y(\omega) = \frac{1}{(j\omega - 1)(j\omega - 2)} = \frac{-1}{j\omega - 1} - \frac{-1}{j\omega - 2}$$

Therefore

$$y(t) = [e^{t} - e^{2t}]u(-t)]$$

7.4-3.

$$X_1(\omega) = \text{sinc}\left(\tfrac{\omega}{20000}\right) \qquad \text{and} \quad X_2(\omega) = 1$$

Figure S7.4-3 shows $X_1(\omega)$, $X_2(\omega)$, $H_1(\omega)$ and $H_2(\omega)$. Now

$$\begin{aligned} Y_1(\omega) &= X_1(\omega)H_1(\omega) \\ Y_2(\omega) &= X_2(\omega)H_2(\omega) \end{aligned}$$

The spectra $Y_1(\omega)$ and $Y_2(\omega)$ are also shown in Figure S7.4-3. Because $y(t) = y_1(t)y_2(t)$, the frequency convolution property yields $Y(\omega) = Y_1(\omega) * Y_2(\omega)$. From the width property of convolution, it follows that the bandwidth of $Y(\omega)$ is the sum of bandwidths of $Y_1(\omega)$ and $Y_2(\omega)$. Because the bandwidths of $Y_1(\omega)$ and $Y_2(\omega)$ are 10 kHz, 5 kHz, respectively, the bandwidth of $Y(\omega)$ is 15 kHz.

7.4-4.

$$H(\omega) = 10^{-3}\,\text{sinc}\left(\tfrac{\omega}{2000}\right) \qquad \text{and} \quad P(\omega) = 0.5 \times 10^{-6}\,\text{sinc}^2\left(\tfrac{\omega}{4\times 10^6}\right)$$

The two spectra are sketched in Figure S7.4-4. It is clear that $H(\omega)$ is much narrower than $P(\omega)$, and we may consider $P(\omega)$ to be nearly constant of value $P(0) = 10^{-6}/2$ over the entire band of $H(\omega)$. Hence,

$$Y(\omega) = P(\omega)H(\omega) \approx P(0)H(\omega) = 0.5 \times 10^{-6}H(\omega) \quad \Longrightarrow \quad y(t) = 0.5 \times 10^{-6}h(t)$$

Recall that $h(t)$ is the unit impulse response of the system. Hence, the output $y(t)$ is equal to the system response to an input $0.5 \times 10^{-6}\delta(\omega) = A\delta(\omega)$.

7.4-5.

$$H(\omega) = 10^{-3}\,\text{sinc}\left(\tfrac{\omega}{2000}\right) \qquad \text{and} \quad P(\omega) = 0.5\,\text{sinc}^2\left(\tfrac{\omega}{4}\right)$$

The two spectra are sketched in Figure S7.4-5. It is clear that $P(\omega)$ is much narrower than $H(\omega)$, and we may consider $H(\omega)$ to be nearly constant of value $H(0) = 10^{-3}$

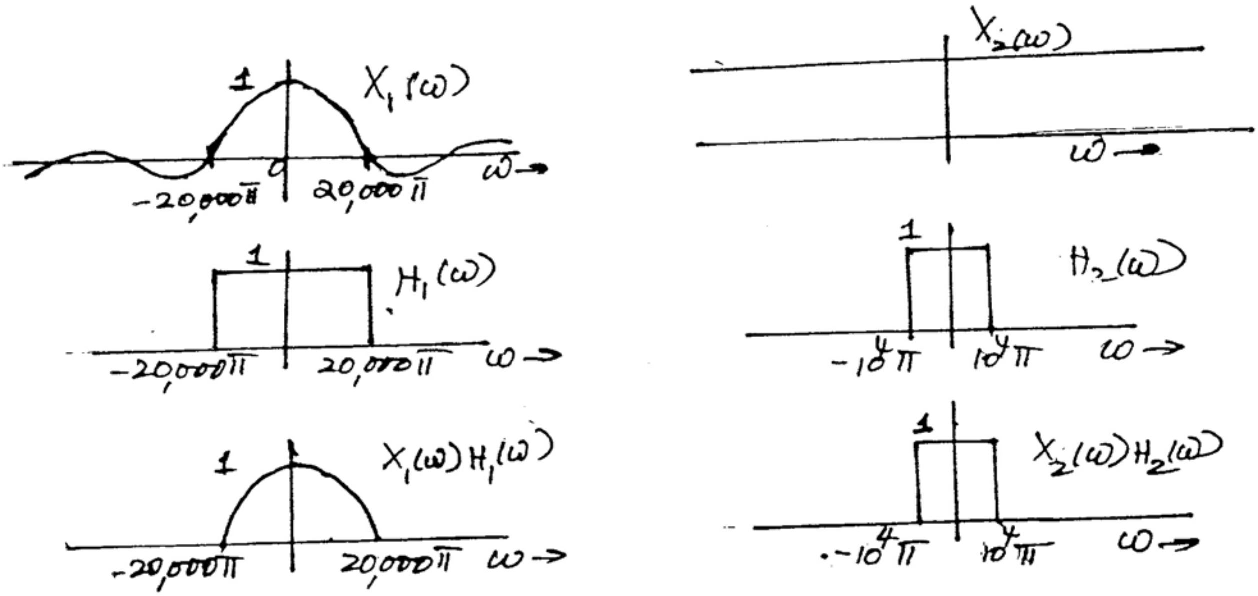

Figure S7.4-3

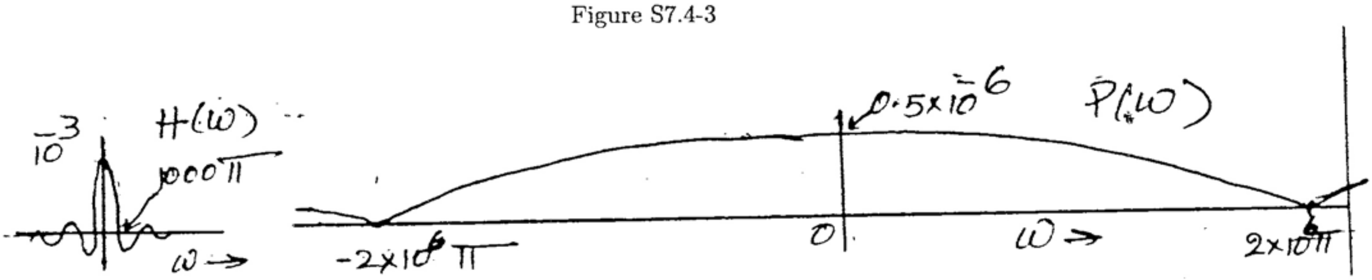

Figure S7.4-4

over the entire band of $P(\omega)$. Hence,

$$Y(\omega) = P(\omega)H(\omega) \approx P(\omega)H(0) = 10^{-3}P(\omega) \quad \Longrightarrow \quad y(t) = 10^{-3}p(t)$$

Note that the dc gain of the system is $k = H(0) = 10^{-3}$. Hence, the output is nearly $kP(t)$.

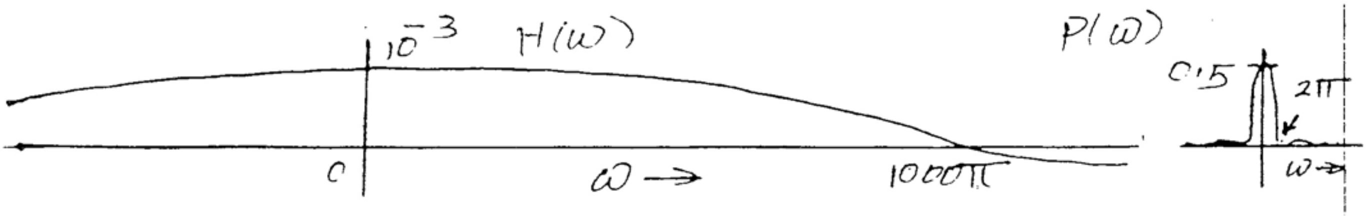

Figure S7.4-5

7.4-6. Every signal can be expressed as a sum of even and odd components (see Sec. 1.5-2). Hence, $h(t)$ can be expressed as a sum of its even and odd components a

$$h(t) = h_e(t) + h_o(t)$$

where $h_e(t) = \frac{1}{2}[h(t)u(t) + h(-t)u(-t)]$ and $h_o(t) = \frac{1}{2}[h(t)u(t) - h(-t)u(-t)]$. From these equations, we make an important observation that

$$h_e(t) = h_o(t)\,\mathrm{sgn}(t) \qquad \text{and} \qquad h_o(t) = h_e(t)\,\mathrm{sgn}(t) \tag{1}$$

provided that $h(t)$ has no impulse at the origin. This result applies only if $h(t)$ is causal. The graphical proof of this result may be seen in Figure 1.24.
Moreover, we have proved in Prob. 7.1-1 that the Fourier transform of a real and even signal is a real and even function of ω, and the Fourier transform of a real and odd signal is an imaginary odd function of ω. Therefore, if $X(\omega) = R(\omega) + jX(\omega)$, then

$$h_e(t) \Longleftrightarrow R(\omega) \qquad \text{and} \qquad h_o(t) \Longleftrightarrow jX(\omega) \tag{2}$$

Applying the convolution property to Eq. (1), we obtain

$$R(\omega) = \frac{1}{2\pi} jX(\omega) * \frac{2}{j\omega} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{X(y)}{\omega - y}\,dy$$

and

$$jX(\omega) = \frac{1}{2\pi} R(\omega) * \frac{2}{j\omega} = \frac{1}{j\pi}\int_{-\infty}^{\infty} \frac{R(y)}{\omega - y}\,dy$$

or

$$X(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{R(y)}{\omega - y}\,dy$$

7.5-1.

$$H(\omega) = e^{-k\omega^2} e^{-j\omega t_0}$$

Using pair 22 (Table 4.1) and time-shifting property, we get

$$h(t) = \frac{1}{\sqrt{4\pi k}} e^{-(t-t_0)^2/4k}$$

This is noncausal. Hence the filter is unrealizable. Also

$$\int_{-\infty}^{\infty} \frac{|\ln|H(\omega)||}{\omega^2 + 1}\,d\omega = \int_{-\infty}^{\infty} \frac{k\omega^2}{\omega^2 + 1}\,d\omega = \infty$$

Hence the filter is noncausal and therefore unrealizable. Since $h(t)$ is a Gaussian function delayed by t_0, it looks as shown in the adjacent figure. Choosing $t_0 = 3\sqrt{2k}$, $h(0) = e^{-4.5} = 0.011$ or 1.1% of its peak value. Hence $t_0 = 3\sqrt{2k}$ is a reasonable choice to make the filter approximately realizable.

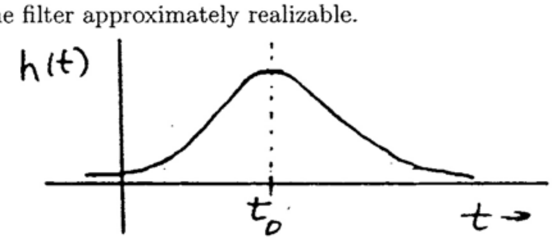

Figure S7.5-1

7.5-2.

$$H(\omega) = \frac{2 \times 10^5}{\omega^2 + 10^{10}} e^{-j\omega t_0}$$

From pair 3, Table 4.1 and time-shifting property, we get

$$h(t) = e^{-10^5|t-t_0|}$$

The impulse response is noncausal, and the filter is unrealizable.

The exponential delays to 1.8% at 4 times constants. Hence $t_0 = 4/a = 4 \times 10^{-5} = 40\mu s$ is a reasonable choice to make this filter approximately realizable.

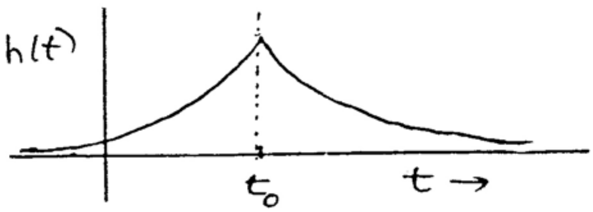

Figure S7.5-2

7.5-3. The unit impulse response is the inverse Fourier transform of $H(\omega)$. Hence, we have

$$h(t) = \text{(a)}\ 0.5\,\text{rect}\left(\frac{t}{2 \times 10^{-6}}\right) \quad \text{(b)}\ \text{sinc}^2(10,000\pi t) \quad \text{(c)}\ 1$$

All the three systems are noncausal (and, therefore, unrealizable) because all the three impulse responses start before $t = 0$.
For (a), the impulse response is a rectangular pulse starting at $t = -10^{-6}$. Hence, delaying the $h(t)$ by 1 μsecond will make it realizable. This will not change anything in the system behavior except the time delay of 1 μsecond in the system response.
For (b), the impulse response is a sinc square pulse, which extends all the way to $-\infty$. Clearly, this system cannot be made realizable with a finite time delay. The delay has to be infinite. However, because the sinc square pulse decays rapidly (see Figure 4.24d), we may truncate it at $t = 10^{-4}$, and then delay the resulting $h(t)$ by 10^{-4}. This makes the filter approximately realizable by allowing a time delay of 100 μseconds in the system response.
For (c), the impulse response is 1, which never decays. Consequently, this filter cannot be realized with any amount of delay.

7.6-1.

$$E_x = \int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-t^2/\sigma^2}\,dt$$

Letting $\frac{t}{\sigma} = \frac{x}{\sqrt{2}}$ and consequently $dt = \frac{\sigma}{\sqrt{2}} dx$

$$E_x = \frac{1}{2\pi\sigma^2} \frac{\sigma}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \frac{\sqrt{2\pi}}{2\sqrt{2}\pi\sigma} = \frac{1}{2\sigma\sqrt{\pi}}$$

Also from pair 22 (Table 4.1)

$$X(\omega) = e^{-\sigma^2\omega^2/2}$$

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\sigma^2\omega^2}\,d\omega$$

Letting $\sigma\omega = \frac{x}{\sqrt{2}}$ and consequently $d\omega = \frac{1}{\sigma\sqrt{2}}dx$

$$E_x = \frac{1}{2\pi}\frac{1}{\sigma\sqrt{2}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \frac{\sqrt{2\pi}}{2\pi\sigma\sqrt{2}} = \frac{1}{2\sigma\sqrt{\pi}}$$

7.6-2. Consider a signal

$$x(t) = \text{sinc}(kt) \quad \text{and} \quad X(\omega) = \frac{\pi}{k}\text{rect}(\frac{\omega}{2k})$$

$$\begin{aligned} E_x = \int_{-\infty}^{\infty} \text{sinc}^2(kt)\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\pi^2}{k^2}\left[\text{rect}\left(\frac{\omega}{2k}\right)\right]^2 d\omega \\ &= \frac{\pi}{2k^2}\int_{-k}^{k} d\omega = \frac{\pi}{k} \end{aligned}$$

7.6-3. If $x^2(t) \Longleftrightarrow A(\omega)$, then the output $Y(\omega) = A(\omega)H(\omega)$, where $H(\omega)$ is the lowpass filter transfer function (Figure S7.6-3). Because this filter band $\Delta f \to 0$, we may express it as an impulse function of area $4\pi\Delta f$. Thus,

$$H(\omega) \approx [4\pi\Delta f]\delta(\omega) \quad \text{and} \quad Y(\omega) \approx [4\pi A(\omega)\Delta f]\delta(\omega) = [4\pi A(0)\Delta f]\delta(\omega)$$

Here we used the property $g(x)\delta(x) = g(0)\delta(x)$ [Eq. (1.23a)]. This yields

$$y(t) = 2A(0)\Delta f$$

Next, because $x^2(t) \Longleftrightarrow A(\omega)$, we have

$$A(\omega) = \int_{-\infty}^{\infty} x^2(t)e^{-j\omega t}\,dt \quad \text{so that} \quad A(0) = \int_{-\infty}^{\infty} x^2(t)\,dt = E_x$$

Hence, $y(t) = 2E_x\Delta f$.

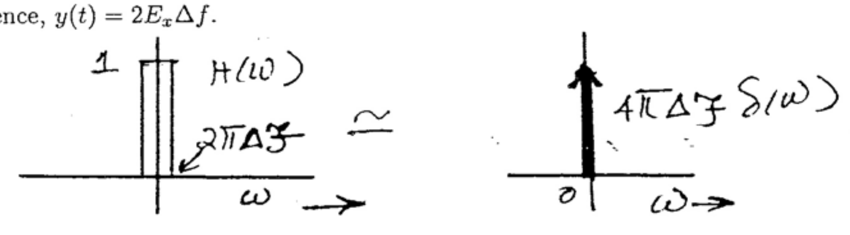

Figure S7.6-3

7.6-4. Recall that

$$x_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega)e^{j\omega t}\,d\omega \quad \text{and} \quad \int_{-\infty}^{\infty} x_1(t)e^{j\omega t}\,dt = X_1(-\omega)$$

Therefore

$$\int_{-\infty}^{\infty} x_1(t)x_2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} x_1(t)[\int_{-\infty}^{\infty} X_2(\omega)e^{j\omega t}\,d\omega]\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(\omega)[\int_{-\infty}^{\infty} x_1(t)e^{j\omega t}\,dt]\,d\omega = \frac{1}{2\pi}\int X_1(-\omega)X_2(\omega)\,d\omega$$

Interchanging the roles of $x_1(t)$ and $x_2(t)$ in the above development, we can show that

$$\int_{-\infty}^{\infty} x_1(t)x_2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)X_2(-\omega)\,d\omega$$

7.6-5. In the generalized Parseval's theorem in Prob. 7.6-4, if we identify $g_1(t) = \text{sinc}(Wt - m\pi)$ and $g_2(t) = \text{sinc}(Wt - n\pi)$, then

$$G_1(\omega) = \frac{\pi}{W}\,\text{rect}\left(\frac{\omega}{2W}\right)e^{\frac{jm\pi\omega}{W}}, \quad \text{and} \quad G_2(\omega) = \frac{\pi}{W}\,\text{rect}\left(\frac{\omega}{2W}\right)e^{\frac{jn\pi\omega}{W}}$$

Therefore

$$\int_{-\infty}^{\infty} g_1(t)g_2(t)\,dt = \frac{1}{2\pi}\left(\frac{\pi}{W}\right)^2\int_{-\infty}^{\infty}\left[\text{rect}\left(\frac{\omega}{2W}\right)\right]^2 e^{\frac{j(n-m)\pi\omega}{W}}\,d\omega$$

But $\text{rect}\left(\frac{\omega}{2W}\right) = 1$ for $|\omega| \le W$, and is 0 otherwise. Hence

$$\int_{-\infty}^{\infty} g_1(t)g_2(t)\,dt = \frac{\pi}{2W^2}\int_{-W}^{W} e^{\frac{j(n-m)\pi\omega}{W}}\,d\omega = \begin{cases} 0 & n \ne m \\ \frac{\pi}{W} & n = m \end{cases}$$

In evaluating the integral, we used the fact that $e^{\pm j2\pi k} = 1$ when k is an integer.

7.6-6. Application of duality property [Eq. (4.31)] to pair 3 (Table 4.1) yields

$$\frac{2a}{t^2 + a^2} \Longleftrightarrow 2\pi e^{-a|\omega|}$$

The signal energy is given by

$$E_x = \frac{1}{\pi}\int_0^{\infty} |2\pi e^{-a\omega}|^2\,d\omega = 4\pi\int_0^{\infty} e^{-2a\omega}\,d\omega = \frac{2\pi}{a}$$

The energy contained within the band (0 to W) is

$$E_W = 4\pi\int_0^{W} e^{-2a\omega}\,d\omega = \frac{2\pi}{a}[1 - e^{-2aW}]$$

If $E_W = 0.99E_x$, then

$$e^{-2aW} = 0.01 \quad \Longrightarrow W = \frac{2.3025}{a}\,\text{rad/s} = \frac{0.366}{a}\,\text{Hz}$$

7.7-1. (i) For $m(t) = \cos 1000t$

$$\begin{aligned}\varphi_{DSB-SC}(t) &= m(t)\cos 10,000t = \cos 1000t\cos 10,000t \\ &= \frac{1}{2}[\underbrace{\cos 9000t}_{\text{LSB}} + \underbrace{\cos 11,000t}_{\text{USB}}]\end{aligned}$$

(ii) For $m(t) = 2\cos 1000t + \cos 2000t$

$$\varphi_{DSB-SC}(t) = m(t)\cos 10,000t = [2\cos 1000t + \cos 2000t]\cos 10,000t$$

$$\begin{aligned} &= \cos 9000t + \cos 11,000t + \frac{1}{2}[\cos 8000t + \cos 12,000t] \\ &= \underbrace{[\cos 9000t + \frac{1}{2}\cos 8000t]}_{\text{LSB}} + \underbrace{[\cos 11,000t + \frac{1}{2}\cos 12,000t]}_{\text{USB}} \end{aligned}$$

(iii) For $m(t) = \cos 1000t \cos 3000t$

$$\begin{aligned} \varphi_{DSB-SC}(t) &= m(t)\cos 10,000t = \frac{1}{2}[\cos 2000t + \cos 4000t]\cos 10,000t \\ &= \frac{1}{2}[\cos 8000t + \cos 12,000t] + \frac{1}{2}[\cos 6000t + \cos 14,000t] \\ &= \frac{1}{2}\underbrace{[\cos 8000t + \cos 6000t]}_{\text{LSB}} + \frac{1}{2}\underbrace{[\cos 12,000t + \cos 14,000t]}_{\text{USB}} \end{aligned}$$

This information is summarized in a table below. Figure S7.7-1 shows various spectra.

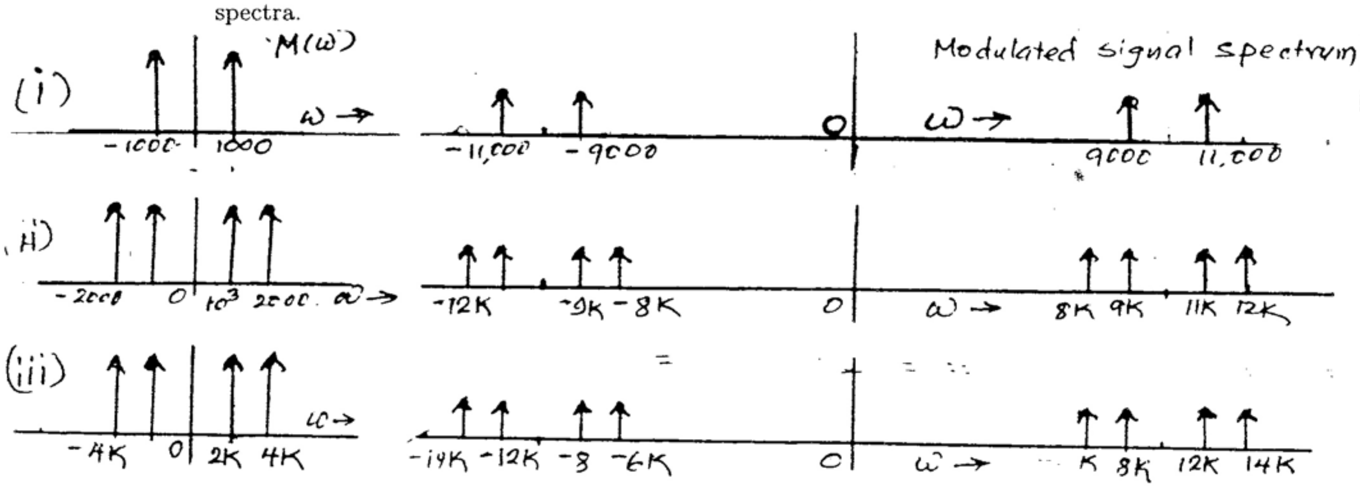

Figure S7.7-1

case	Baseband frequency	DSB frequency	LSB frequency	USB frequency
i	1000	9000 and 11,000	9000	11,000
ii	1000	9000 and 11,000	9000	11,000
	2000	8000 and 12,000	8000	12,000
iii	2000	8000 and 12,000	8000	12,000
	4000	6000 and 14,000	6000	14,000

7.7-2. (a) The signal at point b is

$$\begin{aligned} x_a(t) &= m(t)\cos^3 \omega_c t \\ &= m(t)\left[\frac{3}{4}\cos \omega_c t + \frac{1}{4}\cos 3\omega_c t\right] \end{aligned}$$

The term $\frac{3}{4}m(t)\cos \omega_c t$ is the desired modulated signal, whose spectrum is centered at $\pm\omega_c$. The remaining term $\frac{1}{4}m(t)\cos 3\omega_c t$ is the unwanted term, which represents the modulated signal with carrier frequency $3\omega_c$ with spectrum centered at $\pm 3\omega_c$ as shown in Figure S7.7-2. The bandpass filter centered at $\pm\omega_c$ allows to pass the desired term $\frac{3}{4}m(t)\cos\omega_c t$, but suppresses the unwanted term $\frac{1}{4}m(t)\cos 3\omega_c t$. Hence, this system works as desired with the output $\frac{3}{4}m(t)\cos\omega_c t$.

(b) Figure S7.7-2 shows the spectra at points b and c.

(c) The minimum usable value of ω_c is $2\pi B$ in order to avoid spectral folding at dc.

(d)

$$\begin{aligned} m(t)\cos^2\omega_c t &= \frac{m(t)}{2}[1+\cos 2\omega_c t] \\ &= \frac{1}{2}m(t) + \frac{1}{2}m(t)\cos 2\omega_c t \end{aligned}$$

The signal at point b consists of the baseband signal $\frac{1}{2}m(t)$ and a modulated signal $\frac{1}{2}m(t)\cos 2\omega_c t$, which has a carrier frequency $2\omega_c t$, not the desired value $\omega_c t$. Both the components will be suppressed by the filter, whose center center frequency is ω_c. Hence, this system will not do the desired job.

(e) The reader may verify that the identity for $\cos n\omega_c t$ contains a term $\cos\omega_c t$ when n is odd. This is not true when n is even. Hence, the system works for a carrier $\cos^n\omega_c t$ only when n is odd.

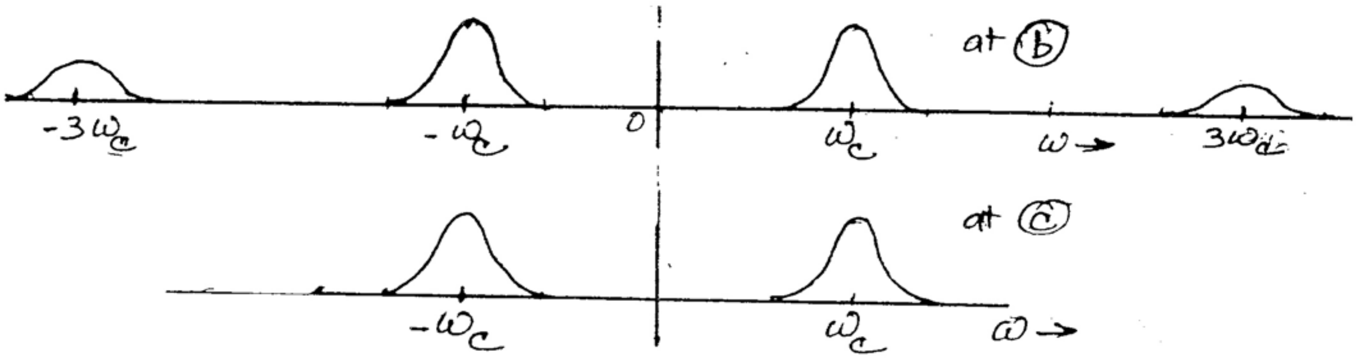

Figure S7.7-2

7.7-3. This signal is identical to that in Figure 3.8a with period T_0 (instead of 2π). We find the Fourier series for this signal as

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left[\cos\,\omega_c t - \frac{1}{3}\cos\,3\omega_c t + \frac{1}{5}\cos\,5\omega_c t + \cdots\right]$$

Hence, $y(t)$, the output of the multiplier is

$$y(t) = m(t)x(t) = m(t)\left[\frac{1}{2} + \frac{2}{\pi}\left(\cos\,\omega_c t - \frac{1}{3}\cos\,3\omega_c t + \frac{1}{5}\cos\,5\omega_c t + \cdots\right)\right]$$

The bandpass filter suppresses the signals $m(t)$ and $m(t)\cos\,n\omega_c t$ for all $n \neq 1$. Hence, the bandpass filter output is

$$km(t)\cos\omega_c t = \frac{2}{\pi}m(t)\cos\omega_c t$$

7.7-4. (a) Figure S7.7-4 shows the signals at points a, b, and c.

(b) From the spectrum at point c, it is clear that the channel bandwidth must be at least 30,000 rad/s (from 5000 to 35,000 rad/s.).

(c) Figure S7.7-4 shows the receiver to recover $m_1(t)$ and $m_2(t)$ from the received modulated signal.

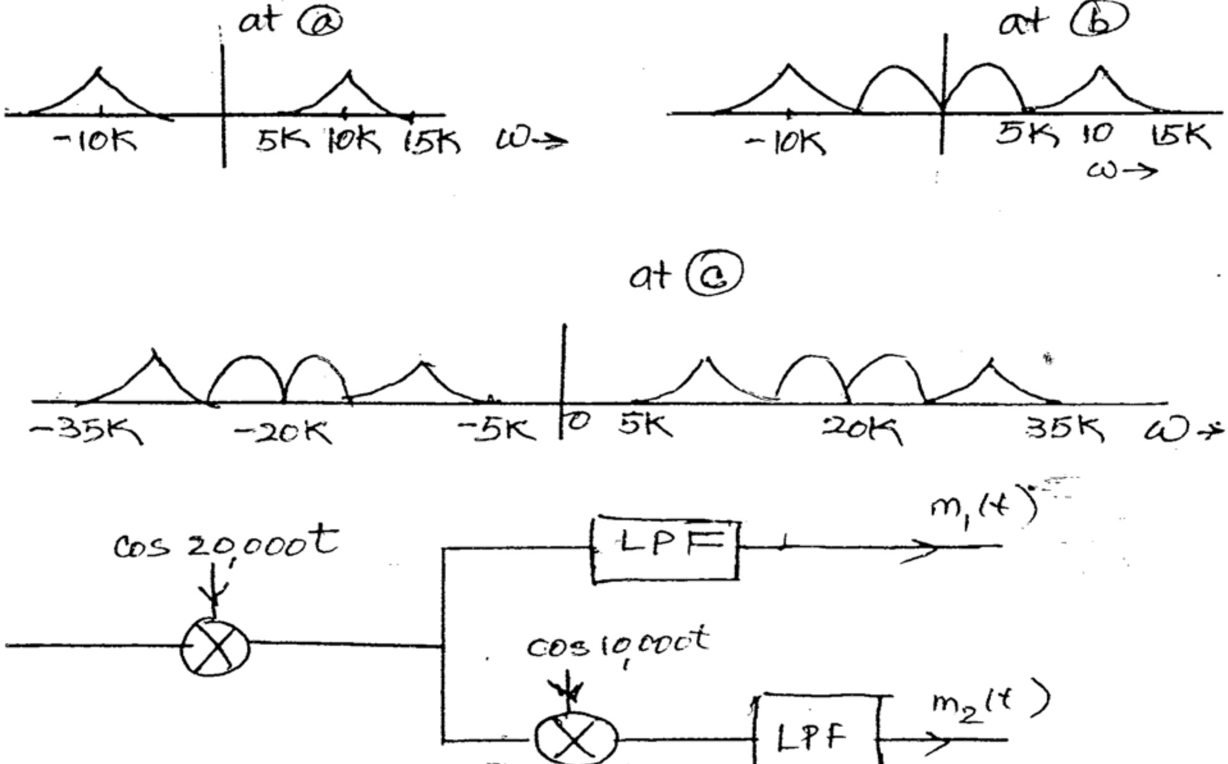

Figure S7.7-4

7.7-5. (a) Figure S7.7-5 shows the output signal spectrum $Y(\omega)$.

(b) Observe that $Y(\omega)$ is the same as $M(\omega)$ with the frequency spectrum inverted, that is, the high frequencies are shifted to lower frequencies and vice versa. Thus, the scrambler in Figure P7.7-5 inverts the frequency spectrum. To get back the original spectrum $M(\omega)$, we need to invert the spectrum $Y(\omega)$ once again. This can be done by passing the scrambled signal $y(t)$ through the same scrambler.

7.7-6. $x_a(t) = [A + m(t)]\cos\omega_c t$. Hence,

$$\begin{aligned} x_b(t) &= [A+m(t)]\cos^2\omega_c t \\ &= \frac{1}{2}[A+m(t)] + \frac{1}{2}[A+m(t)]\cos 2\omega_c t \end{aligned}$$

The first term is a lowpass signal because its spectrum is centered at $\omega = 0$. The lowpass filter allows this term to pass, but suppresses the second term, whose spectrum

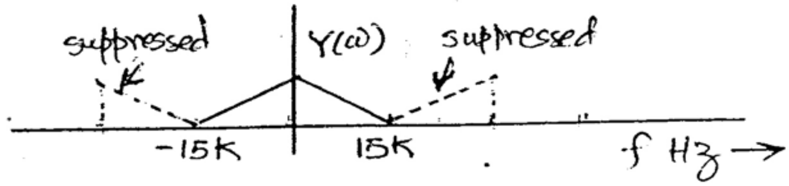

Figure S7.7-5

is centered at $\pm 2\omega_c$. Hence the output of the lowpass filter is

$$y(t) = A + m(t)$$

When this signal is passed through a dc block, the dc term A is suppressed yielding the output $m(t)$. This shows that the system can demodulate AM signal regardless of the value of A. This is a synchronous or coherent demodulation.

7.7-7. (a) $\mu = 0.5 = \frac{m_p}{A} = \frac{10}{A} \quad \Rightarrow A = 20$
(b) $\mu = 1.0 = \frac{m_p}{A} = \frac{10}{A} \quad \Rightarrow A = 10$
(c) $\mu = 2.0 = \frac{m_p}{A} = \frac{10}{A} \quad \Rightarrow A = 5$
(d) $\mu = \infty = \frac{m_p}{A} = \frac{10}{A} \quad \Rightarrow A = 0$

This means that $\mu = \infty$ represents the DSB-SC case. Figure S7.7-7 shows various waveforms.

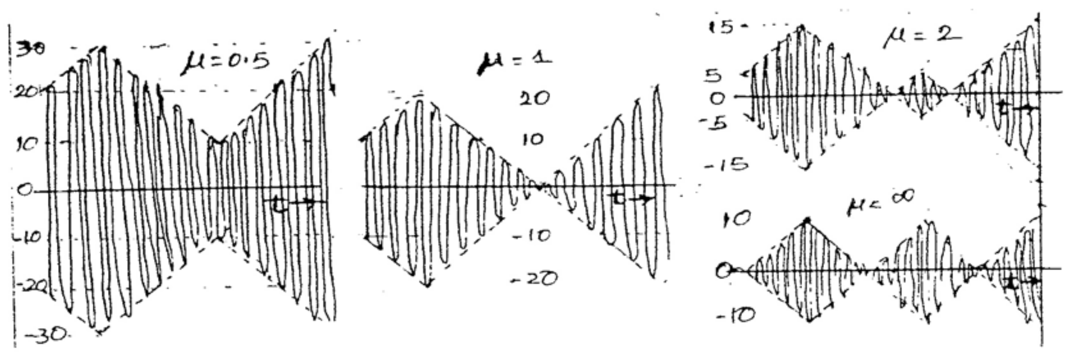

Figure S7.7-7

7.M-1. The signal $x(t) = e^{-at}u(t)$ has Fourier Transform given by $X(\omega) = \frac{1}{j\omega + a}$ and energy $E_x = \frac{1}{2a}$. Using this information, MATLAB program MS7P2 is modified.

```
function [W,E_W] = MS7P2mod1(a,beta,tol)
% MS7P2mod1.m
% Function M-file computes essential bandwidth W for exp(-at)u(t).
% INPUTS:   a = decay parameter of x(t)
%           beta = fraction of signal energy desired in W
%           tol = tolerance of relative energy error
% OUTPUTS:  W = essential bandwidth [rad/s]
```

```
%           E_W = Energy contained in bandwidth W
W = 0; step = a;         % Initial guess and step values
X_squared = inline('1./(omega.^2+a.^2)','omega','a');
E = beta/(2*a);          % Desired energy in W
relerr = (E-0)/E;        % Initial relative error is 100 percent
while(abs(relerr) > tol),
    if (relerr>0),       % W too small
        W=W+step;        % Increase W by step
    elseif (relerr<0),   % W too large
        step = step/2; W = W-step; % Decrease step size and then W.
    end
    E_W = 1/(2*pi)*quad(X_squared,-W,W,[],[],a);
    relerr = (E - E_W)/E;
end
```

(a) Setting $a = 1$ and using 95% signal energy results in

```
>> [W,E_W]=MS7P2mod1(1,.95,1e-9)
W = 12.7062
E_W = 0.4750
```

Thus,

$$W_1 = 12.7062.$$

From the text example, the essential bandwidth corresponding to 95% signal energy is derived as $W = 12.706a$ radians per second. For $a = 1$, this corresponds nicely with the computed value of $W_1 = 12.7062$.

(b) Setting $a = 2$ and using 90% signal energy results in

```
>> [W,E_W]=MS7P2mod1(2,.90,1e-9)
W = 12.6275
E_W = 0.2250
```

Thus,

$$W_2 = 12.6275.$$

(c) Setting $a = 3$ and using 75% signal energy results in

```
>> [W,E_W]=MS7P2mod1(3,.75,1e-9)
W = 7.2426
E_W = 0.1250
```

Thus,

$$W_3 = 7.2426.$$

7.M-2. To solve this problem, program MS7P2 is modified to solve for the pulse width to achieve a desired essential bandwidth, rather than solving for the essential bandwidth that corresponds to a desired pulse.

```
function [tau,E_W] =  MS7P2mod2(W,beta,tol)
% MS7P2mod2.m
% Function M-file computes essential bandwidth W for square pulse.
% INPUTS:   W = essential bandwidth [rad/s]
%           beta = fraction of signal energy desired in W
%           tol = tolerance of relative energy error
% OUTPUTS:  tau = pulse width
```

```
%           E_W = Energy contained in bandwidth W
tau = 1; step = 1;       % Initial guess and step values
X_squared = inline('(tau*MS7P1(omega*tau/2)).^2','omega','tau');
E_W = 1/(2*pi)*quad(X_squared,-W,W,[],[],tau);
E = beta*tau;            % Desired energy in W
relerr = (E - E_W)/E;
while(abs(relerr) > tol),
    if (relerr>0),       % tau too small
        tau=tau+step;    % Increase tau by step
    elseif (relerr<0),   % tau too large
        step = step/2;
        tau = tau-step;  % Decrease step size and then tau.
    end
    E_W = 1/(2*pi)*quad(X_squared,-W,W,[],[],tau);
    E = beta*tau; % Desired energy in W
    relerr = (E - E_W)/E;
end
```

(a) Set $W = 2\pi 5$ and select 95% signal energy.

```
>> [tau,E_W] = MS7P2mod2(2*pi*5,.95,1e-9)
tau = 0.4146
E_W = 0.3939
```

Thus,

$$\tau_1 = 0.4146.$$

(b) Set $W = 2\pi 10$ and select 90% signal energy.

```
>> [tau,E_W] = MS7P2mod2(2*pi*10,.90,1e-9)
tau = 0.0849
E_W = 0.0764
```

Thus,

$$\tau_2 = 0.0849.$$

(c) Set $W = 2\pi 20$ and select 75% signal energy.

```
>> [tau,E_W] = MS7P2mod2(2*pi*20,.75,1e-9)
tau = 0.0236
E_W = 0.0177
```

Thus,

$$\tau_3 = 0.0236.$$

7.M-3. To solve this problem, program MS7P2 is modified to solve for the decay parameter a to achieve a desired essential bandwidth, rather than solving for the essential bandwidth that corresponds to a desired decay parameter.

```
function [a,E_W] =  MS7P2mod3(W,beta,tol)
% MS7P2mod3.m
% Function M-file computes decay parameter a needed to
%                     achieve a given essential bandwidth.
% INPUTS:    W = essential bandwidth [rad/s]
%            beta = fraction of signal energy desired in W
%            tol = tolerance of relative energy error
```

```
% OUTPUTS:   a = decay parameter
%            E_W = Energy contained in bandwidth W
a = 1; step = 1;          % Initial guess and step values
X_squared = inline('1./(omega.^2+a.^2)','omega','a');
E_W = 1/(2*pi)*quad(X_squared,-W,W,[],[],a);
E = beta/(2*a); % Desired energy in W
relerr = (E - E_W)/E;
while(abs(relerr) > tol),
    if (relerr<0),        % a too small
        a=a+step;         % Increase tau by step
    elseif (relerr>0),    % a too large
        step = step/2;
        a = a-step;       % Decrease step size and then tau.
    end
    E_W = 1/(2*pi)*quad(X_squared,-W,W,[],[],a);
    E = beta/(2*a); % Desired energy in W
    relerr = (E - E_W)/E;
end
```

(a) Set $W = 2\pi 5$ and select 95% signal energy.

```
>> [a,E_W] = MS7P2mod3(2*pi*5,.95,1e-9)
a = 2.4725
E_W = 0.1921
```

Thus,

$$a_1 = 2.4725.$$

(b) Set $W = 2\pi 10$ and select 90% signal energy.

```
>> [a,E_W] = MS7P2mod3(2*pi*10,.90,1e-9)
a = 9.9524
E_W = 0.0452
```

Thus,

$$a_2 = 9.9524.$$

(c) Set $W = 2\pi 20$ and select 75% signal energy.

```
>> [a,E_W] = MS7P2mod3(2*pi*20,.75,1e-9)
a = 52.0499
E_W = 0.0072
```

Thus,

$$a_3 = 52.0499.$$

7.M-4. Call the desired unit-amplitude, unit duration triangle function $x(t)$. First, notice that $x(t)$ can be constructed by convolving two rectangular pulses, each of width $\tau = 0.5$ and height $A = \sqrt{2}$. The energy of $x(t)$ is $E_x = 2\int_{t=0}^{0.5}(2t)^2dt = 1/3$. Furthermore, using the convolution-in-time property and spectrum of a rectangular pulse, we know that $X(\omega) = \left(\frac{\sqrt{2}}{2}\text{sinc}(\omega/4)\right)^2$.

Next, program MS7P2 is modified to solve for the essential bandwidths of this signal for various signal energies.

```
function [W,E_W] = MS7P2mod4(beta,tol)
% MS7P2mod4.m
% Function M-file computes essential bandwidth W for a
%                  unit-amplitude, unit duration triangle function.
% INPUTS:   beta = fraction of signal energy desired in W
%           tol = tolerance of relative energy error
% OUTPUTS:  W = essential bandwidth [rad/s]
%           E_W = Energy contained in bandwidth W
W = 0; step = 1;        % Initial guess and step values
X_squared = inline('(sqrt(2)/2*MS7P1(omega/4)).^4','omega');
E = beta/3;             % Desired energy in W
relerr = (E-0)/E;       % Initial relative error is 100 percent
while(abs(relerr) > tol),
    if (relerr>0),      % W too small
        W=W+step;       % Increase W by step
    elseif (relerr<0),  % W too large
        step = step/2; W = W-step; % Decrease step size and then W.
    end
    E_W = 1/(2*pi)*quad(X_squared,-W,W);
    relerr = (E - E_W)/E;
end
```

Use 95% signal energy to compute the essential bandwidth:

```
>> [W,E_W] = MS7P2mod4(.95,1e-9)
W = 6.2877
E_W = 0.3167
```

Use 90% signal energy to compute the essential bandwidth:

```
>> [W,E_W] = MS7P2mod4(.9,1e-9)
W = 5.3350
E_W = 0.3000
```

Use 75% signal energy to compute the essential bandwidth:

```
>> [W,E_W] = MS7P2mod4(.75,1e-9)
W = 3.7872
E_W = 0.2500
```

Thus, the essential bandwidths are

$$W_{0.95} = 6.2877\text{rad/s},\ W_{0.90} = 5.3350\text{rad/s},\ W_{0.75} = 3.7872\text{rad/s}.$$

7.M-5. Following the example in MATLAB Session 7, the first 10 Fourier series coefficients of a 1/3 duty-cycle square wave are

$$D_n = \frac{\tau}{T_0}\text{sinc}\left(\frac{n\pi\tau}{T_0}\right).$$

(a) Setting $T_0 = 2\pi$ and $\tau = 2\pi/3$ yields

$$D_n = \frac{1}{3}\text{sinc}\left(\frac{n\pi}{3}\right).$$

MATLAB is used to evaluate and plot the first ten coefficients.

```
>> tau = 2*pi/3; T_0 = 2*pi; n = [0:10];
>> D_n = tau/T_0*MS7P1(n*pi*tau/T_0);
>> stem(n,D_n,'k'); xlabel('n'); ylabel('D_n');
>> axis([-0.5 10.5 -0.2 0.55]);
```

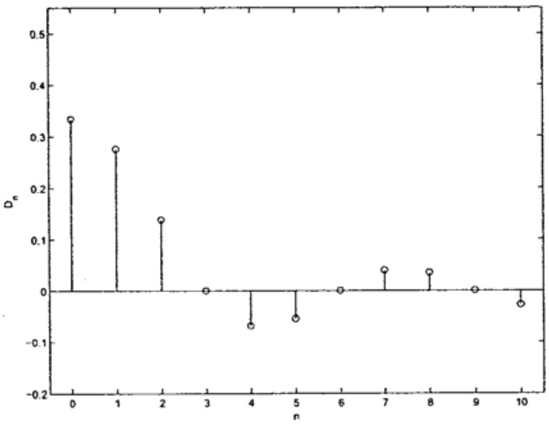

Figure S7.M-5a: Fourier series coefficients D_n for $x(t)$.

(b) Setting $T_0 = \pi$ and $\tau = \pi/3$ yields

$$D_n = \frac{1}{3}\text{sinc}\left(\frac{n\pi}{3}\right).$$

Notice, the coefficients D_n depend only on the duty-cycle of the signal, not the period. Since the duty cycle is fixed, the coefficients D_n are identical to those determined in 7.M-5a. Refer to solution 7.M-5a for the MATLAB code and plot.

7.M-6.

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} e^{-t^2}e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} e^{-(t^2+j\omega t+(j\omega/2)^2-(j\omega/2)^2)}dt \\ &= e^{(j\omega/2)^2}\int_{-\infty}^{\infty} e^{-(t+j\omega/2)^2}dt \end{aligned}$$

Substituting $t'/\sqrt{2} = t$ and $dt'/\sqrt{2} = dt$ yields

$$X(\omega) = \frac{e^{-\omega^2/4}}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-(t'+j\omega^2/\sqrt{2})/2}dt'.$$

However, $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-(t-a)^2}{2}}dt = 1$ for any a, so $\int_{-\infty}^{\infty} e^{\frac{-(t'+j\omega^2)^2}{2}}dt' = \sqrt{2\pi}$. Thus,

$$X(\omega) = \sqrt{\pi}e^{-\omega^2/4}.$$

MATLAB is used to plot $x(t)$ and $X(\omega)$.

```
>> t = linspace(-10,10,1001); x = exp(-t.^2);
>> omega = linspace(-10,10,1001); X = sqrt(pi)*exp(-omega.^2/4);
```

```
>> subplot(211); plot(t,x,'k');
>> xlabel('t'); ylabel('x(t)');
>> subplot(212); plot(t,X,'k');
>> xlabel('\omega'); ylabel('X(\omega)');
```

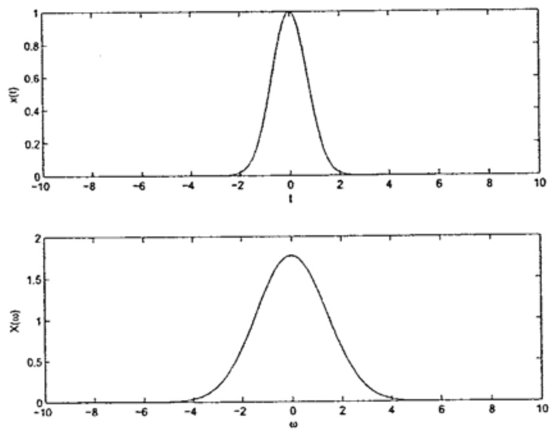

Figure S7.M-6: $x(t) = e^{-t^2}$ and $X(\omega) = \sqrt{\pi}e^{-\omega^2/4}$.

Figure S7.M-6 confirms that $X(\omega)$ is just a scaled and stretched version of $x(t)$. This is something remarkable; the Fourier Transform of a Gaussian pulse is itself a Gaussian pulse!