Chapter 9 Solutions

9.1-1.

$$\begin{aligned}
x[n] &= 4\cos 2.4\pi n + 2\sin 3.2\pi n \\
&= 4\cos 0.4\pi n + 2\sin 1.2\pi n \\
&= 2[e^{j0.4\pi n} + e^{-j0.4\pi n}] + \frac{1}{j}[e^{j1.2\pi n} - e^{-j1.2\pi n}] \\
&= 2e^{j0.4\pi n} + 2e^{-j0.4\pi n} + e^{j(1.2\pi n - \pi/2)} + e^{-j(1.2\pi n - \pi/2)}
\end{aligned}$$

The fundamental $\Omega_0 = 0.4\pi$ and $N_0 = \frac{2\pi}{\Omega_0} = 5$. Note also that.

$$e^{-j0.4\pi n} = e^{j1.6\pi n} \quad \text{and} \quad e^{-j1.2\pi n} = e^{j0.8\pi n}$$

Therefore

$$x[n] = 2e^{j0.4\pi n} + 2e^{j1.6\pi n} + e^{j(1.2\pi n - \pi/2)} + e^{j(0.8\pi n + \pi/2)}$$

We have first, second, third and fourth harmonics with coefficients

$$\begin{aligned}
&D_1 = D_2 = 2 \quad D_3 = -j \quad D_4 = j \\
&|D_1| = |D_2| = 2 \quad |D_3| = |D_4| = 1 \\
&\angle D_1 = \angle D_2 = 0 \quad \angle D_3 = -\frac{\pi}{2} \quad \text{and} \quad \angle D_4 = \frac{\pi}{2}
\end{aligned}$$

The spectrum is shown in Figure S9.1-1.

Figure S9.1-1

9.1-2.

$$\begin{aligned}
x[n] = \cos 2.2\pi n \cos 3.3\pi n &= \frac{1}{2}[\cos 5.5\pi n + \cos 1.1\pi n] \\
&= \frac{1}{2}[\cos 1.5\pi n + \cos 1.1\pi n]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}[e^{j1.5\pi n} + e^{-j1.5\pi n} + e^{j1.1\pi n} + e^{-j1.1\pi n}] \\
&= \frac{1}{2}[e^{j1.5\pi n} + e^{j0.5\pi n} + e^{j1.1\pi n} + e^{j0.9\pi n}]
\end{aligned}
$$

The fundamental frequency $\Omega_0 = 0.1$ and $N_0 = \frac{2\pi}{\Omega_0} = 20$. There are only 5th, 9th, 11th and 15th harmonics with coefficients

$$
D_5 = D_9 = D_{11} = D_{15} = \frac{1}{2}
$$

All the form coefficients are real (phases zero). The spectrum is shown in Figure S9.1-2.

Figure S9.1-2

9.1-3.

$$
\begin{aligned}
x[n] &= 2\cos 3.2\pi(n-3) = 2\cos(3.2\pi n - 9.6\pi) = 2\cos(1.2\pi n - 1.6\pi) \\
&= e^{j(1.2\pi n - 1.6\pi)} + e^{-j(1.2\pi n - 1.6\pi)} \\
&= e^{j(1.2\pi n - 1.6\pi)} + e^{j(0.8\pi n + 1.6\pi)}
\end{aligned}
$$

The fundamental frequency $\Omega_0 = 0.4\pi$ and $N_0 = \frac{2\pi}{\Omega_0} = 5$. Only 2nd, and 3rd harmonics are present.

$$
|D_2| = |D_3| = 1 \quad \angle D_2 = 9.6\pi = 1.6\pi \quad \angle D_3 = -9.6\pi = -1.6\pi
$$

The spectrum is shown in Figure S9.1-3.

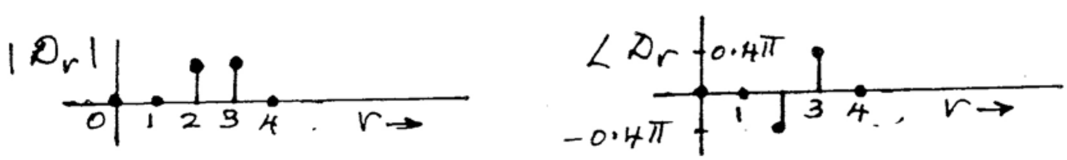

Figure S9.1-3

9.1-4. To compute coefficients D_r, we use Eq. (9.13) where summation is performed over any interval N_0. We choose this interval to be $-N_0/2$, $(N_0/2)-1$ (for even N_0). Therefore

$$
D_r = \frac{1}{N_0} \sum_{n=-N_0/2}^{(N_0/2)-1} x[n] e^{-jr\Omega_0 n}
$$

In the present case $N_0 = 6$, $\Omega_0 = \frac{2\pi}{N_0} = \frac{\pi}{3}$, and

$$D_r = \frac{1}{6}\sum_{n=-3}^{2} x[n]e^{-jr\frac{\pi}{3}n}$$

We have $x[0] = 3$, $x[\pm1] = 2$, $x[\pm2] = 1$, and $x[\pm3] = 0$. Therefore

$$\begin{aligned} D_r &= \frac{1}{6}[3 + 2(e^{j\frac{\pi}{3}r} + e^{-j\frac{\pi}{3}r}) + (e^{j\frac{2\pi}{3}r} + e^{-j\frac{2\pi}{3}r})] \\ &= \frac{1}{6}[3 + 4\cos(\frac{\pi}{3}r) + 2\cos(\frac{2\pi}{3}r)] \end{aligned}$$

$$D_0 = \frac{3}{2} \quad D_1 = \frac{2}{3} \quad D_2 = 0 \quad D_3 = \frac{1}{6} \quad D_4 = 0 \quad D_5 = \frac{2}{3}$$

9.1-5. In this case $N_0 = 12$ and $\Omega_0 = \frac{\pi}{6}$.

$$\begin{aligned} &x[0] = 0 \quad x[1] = 1 \quad x[-1] = -1 \quad x[2] = 2 \quad x[-2] = -2 \\ &x[3] = 3 \quad x[-3] = -3 \quad x[\pm4] = x[\pm5] = x[\pm6] = 0 \end{aligned}$$

Therefore

$$\begin{aligned} D_r &= \frac{1}{12}\sum_{n=-6}^{5} x[n]e^{-jr\frac{\pi}{6}n} \\ &= \frac{1}{12}[e^{-j\frac{\pi}{6}r} - e^{j\frac{\pi}{6}r} + 2(e^{-j\frac{2\pi}{6}r} - e^{j\frac{2\pi}{6}r}) + 3(e^{-j\frac{3\pi}{6}r} - e^{j\frac{3\pi}{6}r})] \\ &= \frac{-j}{12}[2\sin(\frac{\pi}{6}r) + 4\sin(\frac{\pi}{3}r) + 6\sin(\frac{\pi}{2}r)] \end{aligned}$$

9.1-6. Here, the period is N_0 and $\Omega_0 = 2\pi/N_0$. Using Eq. (9.9), we obtain

$$\mathcal{D}_r = \frac{1}{N_0}\sum_{n=0}^{N_0-1} a^n e^{-jr\Omega_0 n} = \frac{1}{N_0}\sum_{n=0}^{N_0-1}(ae^{-jr\Omega_0})^n$$

This is a geometric progression, whose sum is found from Sec. B.7-4 as

$$\mathcal{D}_r = \frac{1}{N_0}\frac{a^{N_0}e^{-jr\Omega_0 N_0} - 1}{ae^{-jr\Omega_0} - 1} = \frac{a^{N_0} - 1}{N_0(ae^{-jr\Omega_0} - 1)} \quad \text{because} \quad e^{-jr\Omega_0 N_0} = e^{-jr2\pi} = 1$$

Therefore

$$\begin{aligned} \frac{a^{N_0}}{N_0(ae^{-jr\Omega_0} - 1)} &= \frac{a^{N_0}}{N_0(a\cos r\Omega_0 - ja\sin\Omega_0 - 1)} \\ &= \underbrace{\frac{a^{N_0}}{N_0(\sqrt{a^2 - 2a\cos r\Omega_0 + 1}}}_{|\mathcal{D}_r|} \angle\underbrace{\{-\tan^{-1}\frac{-a\sin r\Omega_0}{a\cos r\Omega_0 - 1}\}}_{\angle\mathcal{D}_r} \end{aligned}$$

9.1-7. Because $|x[n]|^2 = x[n]x^*[n]$, using Eq. (9.8), we obtain

$$P_x = \frac{1}{N_0}\sum_{n=0}^{N_0-1}\left|\sum_{r=0}^{N_0-1} D_r e^{jr\Omega_0 n}\right|^2 = \frac{1}{N_0 - 1}\sum_{n=0}^{N_0-1}\left[\sum_{r=0}^{N_0-1} D_r e^{j\Omega_0 n}\sum_{m=0}^{N_0-1} D_m^* e^{-jm\Omega_0 n}\right]$$

Interchanging the order of summation yields

$$P_x = \frac{1}{N_0} \sum_{r=0}^{N_0-1} \sum_{m=0}^{N_0-1} D_r D_m^* \left[\sum_{n=0}^{N_0-1} e^{j(r-m)\Omega_0 n} \right]$$

From Eq. (5.43), in Appendix 5.1, the sum inside the parenthesis is N_0 when $r = m$, and is zero otherwise. Hence

$$P_x = \frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2 = \sum_{r=0}^{N_0-1} |D_r|^2$$

9.1-8. (a) Yes, the sum of aperiodic discrete-time sequences can be periodic. For example, consider two signals $x_1[n] = \sin(n)u[n]$ and $x_2[n]\sin(n)u[-n]$. The sum of these two aperiodic signals is the periodic function $x_1[n] + x_2[n] = sin(n)$.

(b) No, it is not possible for a sum of periodic discrete-time sequences to be aperiodic. Consider arbitrary periodic signals $x_1[n]$ and $x_2[n]$ with periods N_1 and N_2, respectively. Let $y[n] = x_1[n] + x_2[n]$. Notice that $y[n+N_1N_2] = x_1[n+N_1N_2] + x_2[n+N_1N_2]$. By periodicity, $x_1[n+kN_1] = x_1[n]$ and $x_2[n+kN_2] = x_2[n]$ for any k. Thus, $y[n+N_1N_2] = x_1[n]+x_2[n] = y[n]$. That is, the sum of two periodic signals must also be periodic.

9.2-1.

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega n}\,d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\Omega)|e^{j\angle X(\Omega)}e^{j\Omega n}\,d\Omega \\ &= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} |X(\Omega)|\cos[\Omega n + \angle X(\Omega)]\,d\Omega + j\int_{-\infty}^{\infty} |X(\Omega)|\sin[\Omega n + \angle X(\Omega)]\,d\Omega\right] \end{aligned}$$

Since $|X(\Omega)|$ is an even function and $\angle X(\Omega)$ is an odd function of Ω, the integrand in the second integral is an odd function of Ω, and therefore vanishes. Moreover the integrand in the first integral is an even function of Ω, and therefore

$$x[n] = \frac{1}{\pi}\int_{0}^{\infty} |X(\Omega)|\cos[\Omega n + \angle X(\Omega)]\,d\Omega$$

9.2-2. (a) Because $x[n] = x_o[n] + x_e[n]$ and $e^{-j\Omega n} = \cos\,\Omega n - j\sin\,\Omega n$

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} \{x_o[n] + x_e[n]\}e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} \{x_o[n] + x_e[n]\}\cos\Omega n - j\sum_{n=-\infty}^{\infty} \{x_o[n] + x_e[n]\}\sin\Omega n \end{aligned}$$

Because $x_e[n]\cos\Omega n$ and $x_o[n]\sin\Omega n$ are even functions and $x_o[n]\cos\Omega n$ and $x_e[n]\sin\Omega n$ are odd functions of n, these sums reduce to

$$X(\Omega) = 2\sum_{n=0}^{\infty} x_e[n]\cos\Omega n - 2j\sum_{n=0}^{\infty} x_o[n]\sin\Omega n \qquad (1)$$

Also, using the parallel development, we obtain the results similar to those of

Prob. 7.1-1, by which we have

$$\text{DTFT}\{x_e[n]\} = 2\sum_{n=0}^{\infty} x_e[n]\cos\Omega n \quad \text{and} \quad \text{DTFT}\{x_0[n]\} = -2j\sum_{n=0}^{\infty} x_o[n]\sin\Omega n \tag{2}$$

From Eqs. (1) and (2), the desired result follows.

(b) We shall prove the result for a general exponential $x[n] = a^n u[n]$. From Table 9.1, we obtain

$$X(\Omega) = \frac{1}{1-ae^{-j\Omega}} = \frac{1-a\cos\Omega}{1+a^2-2a\cos\Omega} + j\frac{-a\sin\Omega}{1+a^2-2a\cos\Omega}$$

The even and odd components of $x[n] = a^n u[n]$ are

$$x_e[n] = 0.5(a^n u[n] + a^{-n}u[-n]), \qquad \text{and} \qquad x_o[n] = 0.5(a^n u[n] - a^{-n}u[-n])$$

W e know that $a^n u[n] \Longleftrightarrow 1/(1-ae^{j\Omega})$. Moreover

$$a^{-n}u[-n] = \left(\frac{1}{a}\right)^n u[-(n+1)] + \delta[n]$$

Hence

$$a^{-n}u[-n] \Longleftrightarrow \frac{1}{\left(\frac{1}{a}\right)^n e^{-j\Omega} - 1} + 1 = \frac{1}{1-ae^{j\Omega}}$$

and

$$\text{DTFT}\{x_e[n]\} = 0.5\left(\frac{1}{1-ae^{-j\Omega}} + \frac{1}{1-ae^{\Omega}}\right) = \frac{1-a\cos\Omega}{1+a^2-2a\cos\Omega} = \text{Re}\,X(\Omega)$$

and

$$\text{DTFT}\{x_o[n]\} = 0.5\left(\frac{1}{1-ae^{-j\Omega}} - \frac{1}{1-ae^{\Omega}}\right) = \frac{-ja\sin\Omega}{1+a^2-2a\cos\Omega} = j\text{Im}\,X(\Omega)$$

9.2-3. (a)

$$X(\Omega) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\Omega n} = 1$$

(b)

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} \delta[n-k]e^{-j\Omega n} = e^{-j\Omega k} \\ |X(\Omega)| &= 1 \qquad \angle X(\Omega) = -\Omega k \end{aligned}$$

(c)

$$\begin{aligned} X(\Omega) &= \sum_{n=1}^{\infty} \gamma^n e^{-j\Omega n} = \sum_{n=1}^{\infty} (\gamma e^{-j\Omega})^n = \frac{(\gamma e^{-j\Omega})^\infty - (\gamma e^{-j\Omega})}{\gamma e^{-j\Omega} - 1} \\ &= \frac{0-\gamma e^{-j\Omega}}{\gamma e^{-j\Omega}-1} = \frac{\gamma}{e^{j\Omega}-\gamma} = \frac{\gamma}{(\cos\Omega - \gamma) + j\sin\Omega} \end{aligned}$$

$$|X(\Omega)| = \frac{\gamma}{\sqrt{(1+\gamma^2) - 2\gamma\cos\Omega}} \qquad \angle X(\omega) = -\tan^{-1}\left(\frac{\sin\Omega}{\cos\Omega - \gamma}\right)$$

Observe that

$$X(\Omega) = \frac{\gamma}{e^{j\Omega} - \gamma} = \frac{\gamma e^{-j\Omega}}{1 - \gamma e^{-j\Omega}}$$

Comparison of this equation with Eq. (9.34a) shows that $X(\Omega)$ in the present case is $\gamma e^{-j\Omega}$ times the $X(\Omega)$ for $\gamma^n u[n]$. Clearly, the amplitude spectrum in this case is γ times that in Figure 9.4b. Moreover, the angle spectrum in the present case is equal to $-\Omega$ plus that in Figure 9.4c. This is shown in Figure S9.2-3a.

(d)

$$X(\Omega) = \sum_{n=-1}^{\infty} (\gamma e^{-j\Omega})^n = \frac{(\gamma e^{-j\Omega})^{\infty} - (\gamma e^{-j\Omega})^{-1}}{\gamma e^{-j\Omega} - 1} = \frac{e^{j2\Omega}}{\gamma(e^{j\Omega} - \gamma)}$$

$$|X(\Omega)| = \frac{1}{\gamma\sqrt{1+\gamma^2 - 2\gamma\cos\Omega}} \qquad \angle X(\Omega) = 2\Omega - \tan^{-1}\left(\frac{\sin\Omega}{\cos\Omega - \gamma}\right)$$

Observe that

$$X(\Omega) = \frac{e^{j2\Omega}}{\gamma(e^{j\Omega} - \gamma)} = \frac{e^{j\Omega}/\gamma}{1 - \gamma e^{-j\Omega}}$$

Comparison of this equation with Eq. (9.34a) shows that $X(\Omega)$ in the present case is $\frac{1}{\gamma}e^{-j\Omega}$ times the $X(\Omega)$ for $\gamma^n u[n]$. Clearly, the amplitude spectrum in this case is $1/\gamma$ times that in Figure 9.4b. Moreover, the angle spectrum in the present case is equal to Ω plus that in Figure 9.4c. This is shown in Figure S9.2-3b.

(e)

$$\begin{aligned}
x[n] &= (-\gamma)^n u[n] \\
X(\Omega) &= \sum_{n=0}^{\infty} (-\gamma)^n e^{-j\Omega n} \\
&= \sum_{n=0}^{\infty} (-\gamma e^{-j\Omega})^n \\
&= \frac{1}{1 + \gamma e^{-j\Omega}} \\
&= \frac{1}{\sqrt{1+\gamma^2 + 2\gamma\cos\Omega}} \quad e^{-j\tan^{-1}\left[\frac{-\gamma\sin\Omega}{1+\gamma\cos\Omega}\right]}
\end{aligned}$$

(f)

$$\begin{aligned}
x[n] = \gamma^{|n|} &= \gamma^n u[n] + \gamma^{-n} u[-(n+1)] \\
&= \gamma^n u[n] + \left(\frac{1}{\gamma}\right)^n u[-(n+1)]
\end{aligned}$$

DTFT of both those components are found in the text Eq. (9.34) and Eq. (9.36). Hence

$$X(\Omega) = \frac{1}{1 - \gamma e^{-j\Omega}} + \frac{1}{\frac{1}{\gamma}e^{-j\Omega} - 1} = \frac{e^{-j\Omega}(1-\gamma^2)}{(1 - \gamma e^{-j\Omega})(e^{-j\Omega} - \gamma)}$$

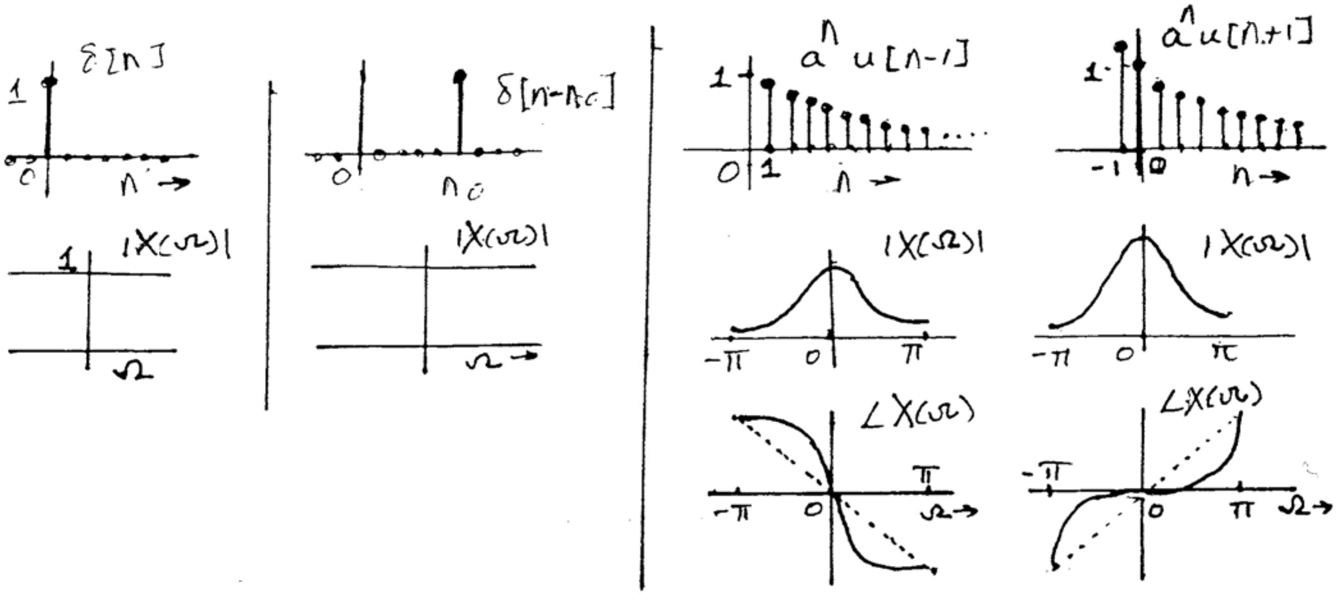

Figure S9.2-3

9.2-4. (a)

$$
\begin{aligned}
X(\Omega) &= e^{jk\Omega} \\
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{jk\Omega}e^{jn\Omega}d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(n+k)\Omega}d\Omega \\
&= \frac{1}{2\pi j}e^{j(n+k)\Omega}\Big|_{-\pi}^{\pi} \\
&= \text{sinc}[(n+k)\pi] \\
&= \delta[n+k]
\end{aligned}
$$

This follows from the fact that both n and k are integers and $\sin[(n+k)\pi] = 0$ for all $n \neq -k$. For $n = -k$, $\text{sinc}[(n+k)\pi] = 1$. Hence, the result.

(b)

$$X(\Omega) = \cos k\Omega = \frac{1}{2}[e^{jk\Omega} + e^{-jk\Omega}]$$

Hence, use of arguments in part (a) yields

$$x[n] = \frac{1}{2}(\delta[n+k] + \delta[n-k])$$

(c)

$$X(\Omega) = \cos^2(\frac{\Omega}{2}) = \frac{1}{2}(1+\cos\Omega) = \frac{1}{2} + \frac{1}{2}\cos\Omega$$

The IDTFT of $\frac{1}{2}$, denoted by $x_1[n]$, is given by

$$x_1[n] = \frac{1}{4\pi}\int_{-\pi}^{\pi} e^{j\Omega n}d\Omega = \frac{1}{2jn}e^{j\Omega n}\Big|_{-\pi}^{\pi} = \frac{1}{2}\text{sinc}(\pi n) = \frac{1}{2}\delta[n]$$

The DTFT if $x_2[n] = \frac{1}{2}\cos\Omega$ is found in part (b). Hence

$$x[n] = \frac{1}{2}\delta[n] + \frac{1}{4}\left(\delta[n+1] + \delta[n-1]\right)$$

(d)

$$\begin{aligned}
X(\Omega) &= \Delta\left(\frac{\Omega}{2\Omega_c}\right) = \begin{cases} 1 + \frac{\Omega}{\Omega_c} & \Omega < 0 \\ 1 - \frac{\Omega}{\Omega_c} & \Omega > 0 \end{cases} \\
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{0}\left(1 + \frac{\Omega}{\Omega_c}\right)e^{j\Omega n}d\Omega + \frac{1}{2\pi}\int_{0}^{\pi}\left(1 - \frac{\Omega}{\Omega_c}\right)e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\left[\int_{-\Omega_c}^{\Omega_c} e^{j\Omega n}d\Omega + \frac{1}{\Omega_c}\int_{-\Omega_c}^{0}\Omega e^{j\Omega n}d\Omega - \frac{1}{\Omega_c}\int_{0}^{-\Omega_c}\Omega e^{j\Omega n}d\Omega\right]
\end{aligned}$$

The detailed derivation of these integrals yields

$$x[n] = \frac{1}{2\pi\Omega_c n^2}[1 - 2\cos\Omega_c n] = \frac{4}{2\pi\Omega_c n^2}\sin^2\left(\frac{\Omega_c n}{2}\right) = \frac{\Omega_c}{2\pi}\text{sinc}^2\left(\frac{\Omega_c}{2}\right)$$

(e)

$$X(\Omega) = 2\pi\delta(\Omega - \Omega_0)$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\Omega - \Omega_0)e^{j\Omega n}d\Omega$$

Use of the sampling property of the impulse in Eq. (1.24) yields

$$x[n] = e^{j\Omega_0 n}$$

(f)

$$\begin{aligned}
X(\Omega) &= \pi\left[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\right] \\
x[n] &= \frac{\pi}{2\pi}\int_{-\pi}^{\pi}\left[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\right]e^{j\Omega n}d\Omega \\
&= \frac{1}{2}\left[e^{j\Omega_0 n} + e^{-j\Omega_0 n}\right] = \cos\Omega_0 n
\end{aligned}$$

9.2-5.

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi/4}^{3\pi/4} e^{j\Omega n}\,d\Omega = \left.\frac{e^{j\Omega n}}{2\pi jn}\right|_{-\pi/4}^{3\pi/4} = \frac{1}{j2\pi n}\left[e^{j(3\pi/4)n} - e^{-j(\pi/4)n}\right] \\
&= \frac{e^{j\pi n/4}}{j2\pi n}[2j\sin\pi n/2] = 0.5\,\text{sinc}(\pi n/2)e^{j\ pin/4}
\end{aligned}$$

9.2-6. (a) and (b)

$$X(\Omega) = \sum_{n=0}^{N_0} a^n e^{-j\Omega n} = \sum_{n=0}^{N_0}\left(ae^{-j\Omega}\right)^n = \frac{a^{N_0+1}e^{-j(N_0+1)\Omega} - 1}{ae^{-j\Omega} - 1}$$

The result applies to both parts (a) and (b). The only difference being in part (a),

$a < 1$ and in part (b), $a > 1$.

9.2-7. (a)

$$\begin{aligned} X(\Omega) &= 2\sum_{n=0}^{6} e^{-jn\Omega} + \sum_{n=7}^{12} e^{-jn\Omega} \\ &= 2\frac{e^{-j7\Omega}-1}{e^{-j\Omega}-1} + \frac{e^{-j13\Omega}-e^{-j7\Omega}}{e^{-j\Omega}-1} \\ &= \frac{e^{-j7\Omega}+e^{-j13\Omega}-2}{e^{-j\Omega}-1} \end{aligned}$$

(b)

$$\begin{aligned} X(\Omega) &= \sum_{n=-1}^{-(N_0-1)} \frac{-n}{N_0-1}e^{-jn\Omega} + \sum_{n=0}^{N_0-1} \frac{n}{N_0-1}e^{-jn\Omega} \\ &= \sum_{m=1}^{N_0-1} \frac{m}{N_0-1}e^{-jm\Omega} + \sum_{n=0}^{N_0-1} \frac{n}{N_0-1}e^{-jn\Omega} \\ &= \sum_{m=0}^{N_0-1} \frac{m}{N_0-1}e^{-jm\Omega} + \sum_{n=0}^{N_0-1} \frac{n}{N_0-1}e^{-jn\Omega} \\ &= \frac{1}{N_0-1}\left\{\frac{e^{j\Omega}+[(N_0-1)(e^{j\Omega}-1)-1]e^{jN_0\Omega}}{(e^{j\Omega}-1)^2}\right. \\ &\qquad \left. + \frac{e^{-j\Omega}+[(N_0-1)(e^{-j\Omega}-1)-1]e^{-jN_0\Omega}}{(e^{-j\Omega}-1)^2}\right\} \end{aligned}$$

9.2-8. (a)

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\Omega_0}^{\Omega_0} \Omega^2 e^{j\Omega n}\, d\Omega \\ &= \frac{1}{2\pi}\frac{e^{j\Omega n}}{(jn)^3}\left[-\Omega^2 n^2 - 2j\Omega n + 2\right]\Big|_{-\Omega_0}^{\Omega_0} \\ &= \frac{(\Omega_0^2 n^2 - 2)\sin\Omega_0 n + 2\Omega_0 n\cos\Omega_0 n}{\pi n^3} \end{aligned}$$

(b) The derivation can be simplified by observing that $X(\Omega)$ can be expressed as a sum of two gate functions $X_1(\Omega)$ and $X_2(\Omega)$ as shown in Figure S9.2-8. Therefore

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-2}^{2} [X_1(\Omega)+X_2(\Omega)]e^{j\Omega n}\, d\Omega \\ &= \frac{1}{2\pi}\left\{\int_{-2}^{2} e^{j\Omega n}\, d\Omega + \int_{-1}^{1} e^{j\Omega n}\, d\Omega\right\} \\ &= \frac{\sin 2n + \sin n}{\pi n} \end{aligned}$$

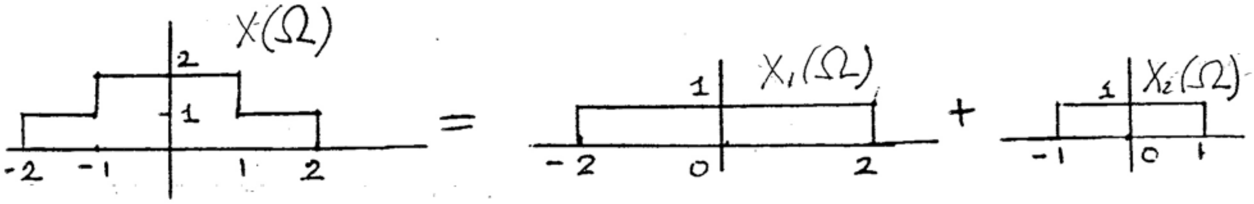

Figure S9.2-8b

9.2-9. (a)

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos\Omega\, e^{j\Omega n}\, d\Omega \\
&= \frac{e^{j\Omega n}}{2\pi(1-n^2)}\left\{jn\cos\Omega + \sin\Omega\right\}_{-\pi/2}^{\pi/2} \\
&= \frac{1}{\pi(1-n^2)}\cos\left(\frac{\pi n}{2}\right)
\end{aligned}
$$

(b)

$$
x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}\, d\Omega = \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} X(\Omega)\cos\Omega n\, d\Omega + j\int_{-\pi}^{\pi} X(\Omega)\sin\Omega n\, d\Omega\right]
$$

Because $X(\Omega)$ is even function, $X(\Omega)\,\sin\Omega n$ is an odd function of Ω. Hence, the second integral on the right-hand side vanishes. Also the integrand of the first term is an even function. Letting $\Omega_0 = \pi/4$, we obtain

$$
\begin{aligned}
x[n] &= \frac{1}{\pi}\int_{0}^{\Omega_0} \frac{\Omega}{\Omega_0}\cos n\Omega\, d\Omega = \frac{1}{\pi\Omega_0}\left[\frac{\cos n\Omega + n\Omega\sin n\Omega}{n^2}\right]\Bigg|_0^{\Omega_0} \\
&= \frac{1}{\pi\Omega_0 n^2}[\cos\Omega_0 n + \Omega_0 n\sin\Omega_0 n - 1] \\
&= \frac{4}{\pi^2 n^2}\left[\cos\frac{\pi n}{4} + \frac{\pi n}{4}\sin\frac{\pi n}{4} - 1\right]
\end{aligned}
$$

9.2-10. (a)

$$
\begin{aligned}
X(\Omega) = \sum_{n=-3}^{3} x[n]e^{-j\Omega n} &= 3 + 2(e^{-j\Omega} + e^{j\Omega}) + (e^{-j2\Omega} + e^{j2\Omega}) \\
&= 3 + 4\cos\Omega + 2\cos 2\Omega
\end{aligned}
$$

(b)

$$
\begin{aligned}
X(\Omega) = \sum_{n=0}^{6} x[n]e^{-jn\Omega} &= e^{-j\Omega} + 2e^{-j2\Omega} + 3e^{-j3\Omega} + 2e^{-j4\Omega} + e^{-j5\Omega} \\
&= e^{-j3\Omega}[(e^{j2\Omega} + e^{-j2\Omega}) + 2(e^{j\Omega} + e^{-j\Omega}) + 3] \\
&= e^{-j3\Omega}[3 + 4\cos\Omega + 2\cos 2\Omega]
\end{aligned}
$$

(c)

$$X(\Omega) = \sum_{n=-3}^{3} x[n]e^{-jn\Omega} = 3e^{-j\Omega} - 3e^{j\Omega} + 6e^{-j2\Omega} - 6e^{-j2\Omega} + 9e^{-j3\Omega} - 9e^{j3\Omega}$$
$$= 6j[\sin\Omega + 2\sin 2\Omega + 3\sin 3\Omega]$$

(d)

$$X(\Omega) = \sum_{n=-2}^{2} x[n]e^{-jn\Omega} = 2e^{-j\Omega} + 2e^{j\Omega} + 4e^{-j2\Omega} + 4e^{-j2\Omega}$$
$$= 4\cos\Omega + 8\cos 2\Omega$$

9.2-11. (a)

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\Omega_0}^{\Omega_0} e^{-j\Omega n_0} e^{j\Omega n}\, d\Omega = \frac{1}{2\pi}\int_{-\Omega_0}^{\Omega_0} e^{j\Omega(n-n_0)}\, d\Omega \\ &= \frac{1}{(2\pi)j(n-n_0)} e^{j\Omega(n-n_0)}\Big|_{-\Omega_0}^{\Omega_0} = \frac{\sin\Omega_0(n-n_0)}{\pi(n-n_0)} = \frac{\Omega_0}{\pi}\text{sinc}[\Omega_0(n-n_0)] \end{aligned}$$

(b)

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\left[\int_{-\Omega_0}^{0} je^{j\Omega n}\, d\Omega + \int_{0}^{\Omega_0} -je^{j\Omega n}\, d\Omega\right] \\ &= \frac{1}{2\pi n} e^{j\Omega n}\Big|_{-\Omega_0}^{0} - \frac{1}{2\pi n} e^{j\Omega n}\Big|_{0}^{\Omega_0} = \frac{1-\cos\Omega_0 n}{\pi n} \end{aligned}$$

9.2-12. (a) We shall show that

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-Lk] = \begin{cases} x\left[\frac{n}{L}\right] & n = 0, \pm L, \pm 2L, \cdots \\ 0 & \text{otherwise} \end{cases}$$

When $n \neq mL$ where m is an integer, then for any integer values of k, $n - Lk$ cannot be zero and $\delta[n - Lk] = 0$ for all k and the sum on the left-hand side is zero for all $n \neq mL$ (m integer). When $n = mL$ (m integer), then $\delta[mL - Lk] = 1$ for $k = m$ and is zero for all $k \neq m$. Hence, the sum on the left-hand side has only one term $x[m]$ when $k = m$. Therefore

$$\begin{aligned} \sum_{k=-\infty}^{\infty} x[k]\delta[n-Lk] &= \begin{cases} x\left[\frac{m}{L}\right] & n = 0, \pm 1, \pm 2, \cdots \\ 0 & \text{otherwise} \end{cases} \\ &= x_e[n] \end{aligned}$$

(b)

$$X_e(\Omega) = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-Lk]\right)e^{-j\Omega n}$$

Interchanging the order of the summation

$$\begin{aligned} X_e(\Omega) &= \sum_{n=-\infty}^{\infty} x[k] \left(\sum_{k=-\infty}^{\infty} \delta[n-Lk] e^{-j\Omega n} \right) \\ &= \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega L k} = X(L\Omega) \end{aligned}$$

(c) $z[n]$ is the signal $x[n] = 1$ expanded by factor $L = 3$. Hence, from the above result and pair 11, we obtain

$$Z(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(3\Omega - 2\pi k) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi k}{3})$$

9.2-13. (a) We shall consider spectra within the band $|\Omega| \leq \pi$ only.

Pair 8: $\frac{\Omega_c}{\pi} \text{sinc}\,(\Omega_c n) \Longleftrightarrow \text{rect}\left(\frac{\Omega}{2\Omega_c}\right)$

This is identical to pair 18 in Table 7.1 with ω_c replaced by Ω_c and t replaced by n.

Pair 9: In the same way, we see that pair 9 is identical to pair 20 in Table 7.1.

Pair 11: is identical to pair 7 in Table 7.1.

Pair 12: is identical to pair 8 in Table 7.1.

Pair 13: is identical to pair 9 in Table 7.1.

Pair 14: is identical to pair 10 in Table 7.1.

(b) This method cannot be used for pairs 2, 3, 4, 5, 6, 7, 10, 15 and 16 because in all these cases $X(\Omega)$ is not bandlimited.

9.2-14. (a) Not valid because not 2π-periodic.

(b) Valid because it is a constant.

(c) Valid because it is $\frac{2\pi}{10}$-periodic, hence also 2π-periodic.

(d) Not valid because it is 20π-periodic and not 2π-periodic.

(e) Not valid because it is not 2π-periodic.

9.3-1. (a)

$$\begin{aligned} x[n] &= u[n] - u[n-9] \\ X(\Omega) &= \left[\frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi\delta(\Omega)\right] - \left[\frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi\delta(\Omega)\right] e^{-j9\Omega} \end{aligned}$$

Observe that $\delta(\Omega)e^{-j9\Omega} = \delta(\Omega)$ because $e^{-j9\Omega} = 1$ at $\Omega = 0$. Therefore

$$\begin{aligned} X(\Omega) &= \frac{e^{j\Omega}}{e^{j\Omega}-1}\left[1 - e^{-j9\Omega}\right] \\ &= \frac{e^{j\Omega}e^{-j4.5\Omega}\left[e^{j4.5\Omega} - e^{-j4.5\Omega}\right]}{e^{j\Omega/2}\left[e^{j\Omega/2} - e^{-j\Omega/2}\right]} \\ &= \frac{\sin 4.5\Omega}{\sin 0.5\Omega} e^{-j4\Omega} \end{aligned}$$

(b)

$$a^{n-m}u[n-m]$$

because

$$a^n u[n] \Longleftrightarrow \frac{e^{j\Omega}}{e^{j\Omega}-\gamma}$$

we obtain

$$a^{n-m}u[n-m] \Longleftrightarrow \frac{e^{j\Omega}e^{-jm\Omega}}{e^{j\Omega}-\gamma} = \frac{e^{j(1-m)\Omega}}{e^{j\Omega}-\gamma}$$

(c)

$$x[n] = a^{n-3}(u[n]-u[n-10]) = a^{-3}a^n u[n] - a^7 a^{n-10}u[n-10]$$

Hence

$$\begin{aligned} X(\Omega) &= a^{-3}\frac{e^{j\Omega}}{e^{j\Omega}-a} - a^7\frac{e^{j\Omega}}{e^{j\Omega}-a}e^{-j10\Omega} \\ &= \frac{e^{j\Omega}\left(a^{-3}-a^7e^{-10\Omega}\right)}{e^{j\Omega}-a} \end{aligned}$$

(d)

$$x[n] = a^{n-m}u[n] = a^{-m}a^n u[n]$$

Hence

$$X(\Omega) = \frac{a^{-m}e^{j\Omega}}{e^{j\Omega}-a}$$

(e)

$$x[n] = a^n u[n-m] = a^m a^{n-m}u[n-m]$$

Hence

$$X(\Omega) = \frac{a^m e^{j\Omega}e^{-jm\Omega}}{e^{j\Omega}-a} = a^m\frac{e^{j(1-m)\Omega}}{e^{j\Omega}-a}$$

(f)

$$x[n] = (n-m)a^{n-m}u[n-m]$$

Apply time-shift property to pair 5 to obtain

$$X(\Omega) = \frac{\gamma e^{j\Omega}e^{-jm\Omega}}{(e^{j\Omega}-\gamma)^2} = \frac{\gamma e^{j(1-m)\Omega}}{(e^{j\Omega}-\gamma)^2}$$

(g)

$$x[n] = (n-m)a^n u[n] = na^n u[n] - ma^n u[n]$$

Hence

$$\begin{aligned} X(\Omega) &= \frac{\gamma e^{j\Omega}}{(e^{j\Omega}-\gamma)^2} - \frac{me^{j\Omega}}{(e^{j\Omega}-\gamma)^2} \\ &= \frac{e^{j\Omega}(\gamma - me^{j\Omega}+m\gamma)}{(e^{j\Omega}-\gamma)^2} \end{aligned}$$

(h)

$$x[n] = na^{n-m}u[n-m] = (n-m)a^{n-m}u[n-m] + ma^{n-m}u[n-m]$$

Hence

$$\begin{aligned} X(\Omega) &= \frac{\gamma e^{j\Omega}}{(e^{j\Omega}-\gamma)^2}e^{-jm\Omega} + m\frac{e^{j\Omega}}{e^{j\Omega}-\gamma}e^{-jm\Omega} \\ &= \frac{\gamma + m(e^{j\Omega}-\gamma)}{(e^{j\Omega}-\gamma)^2}e^{j(1-m)\Omega} \end{aligned}$$

9.3-2.

$$x_1[n] = x[n-4] + x[-n-4] - 4\delta[n]$$

Note that at $n = 0$, both $x[n-4]$ and $x[-n-4]$ have value 4. This duplication is corrected by the term $-4\delta[n]$. Thus

$$X_1(\Omega) = X(\Omega)e^{-j4\Omega} + X(-\Omega)e^{j4\Omega} - 4$$

$$\begin{aligned} x_2[n] &= x[n] + x[-n] \\ X_2(\Omega) &= X(\Omega) + X(-\Omega) \\ x_3[n] &= x[n-2] + x[-n-2] \\ X_3(\Omega) &= X(\Omega)e^{-j2\Omega} + X(-\Omega)e^{j2\Omega} \\ x_4[n] &= x[n-2] + x[-n-2] + x[n-7] + x[-n-7] \\ X_4(\Omega) &= X(\Omega)e^{-j2\Omega} + X(-\Omega)e^{j2\Omega} + X(\Omega)e^{-j7\Omega} + X(-\Omega)e^{j7\Omega} \end{aligned}$$

In all these expression, we substitute

$$X(\Omega) = \frac{4e^{j6\Omega} - 5e^{j5\Omega} + e^{j\Omega}}{(e^{j\Omega}-1)^2}$$

9.3-3. Let

$$Z(\Omega) = X(\Omega) \circledast Y(\Omega)$$

then

$$z[n] = 2\pi x[n]y[n]$$

$$x[n] = Z^{-1}\left[\sum_{k=0}^{4} a_k e^{-jk\Omega}\right] = a_0\delta[n] + a_1\delta[n-1] + a_2\delta[n-2] + a_3\delta[n-3] + a_4\delta[n-4]$$

and from pair 7 (Table 9.1)

$$y[n] = Z^{-1}\frac{\sin(5\Omega/2)}{\sin(\Omega/2)}e^{-j2\Omega} = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$$

Both $x[n]$ and $y[n]$ are nonzero over $0 \le n \le 4$ and are zero outside this range clearly.

$$x[n]y[n] = x[n] \qquad \text{and} \qquad z[n] = 2\pi x[n]$$

$$Z(\Omega) = X(\Omega) \circledast Y(\Omega) = 2\pi X(\Omega)$$

9.3-4. (a) Apply modulation property [Eq. (9.55)] to pair 2 to obtain

$$\begin{aligned} a^n \cos\Omega_0 n u[n] \quad &\Longleftrightarrow \quad \frac{1}{2}\left[\frac{e^{j(\Omega-\Omega_0)}}{e^{j(\Omega-\Omega_0)}-a}+\frac{e^{j(\Omega+\Omega_0)}}{e^{j(\Omega+\Omega_0)}-a}\right] \\ &= \quad \left[\frac{e^{j\Omega}-a\cos\Omega_0}{e^{j2\Omega}-2ae^{j\Omega}\cos\Omega_0+a^2}\right]e^{j\Omega} \end{aligned}$$

(b)

$$x[n] = n^2 a^n u[n]$$

Apply 'multiplication by n' property [Eq. (9.50)] to pair 2 to obtain

$$na^n u[n] \Longleftrightarrow j\frac{d}{d\Omega}\left[\frac{e^{j\Omega}}{e^{j\Omega}-a}\right] = \frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2}$$

Apply the same property again to this result to obtain

$$n^2 a^n u[n] \Longleftrightarrow j\frac{d}{d\Omega}\left[\frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2}\right] = \frac{ae^{j\Omega}(e^{j\Omega}+a)}{(e^{j\Omega}-a)^3}$$

(c)

$$x[n] = (n-k)a^{2n}u[n-m] = a^{2m}(n-m)a^{2(n-m)}u[n-m]+a^{2m}(m-k)a^{2(n-m)}u[n-m]$$

Application of 'multiplication by n' property to $na^n u[n]$ yields

$$na^{2n}u[n] \Longleftrightarrow \frac{a^2 e^{j\Omega}}{(e^{j\Omega}-a^2)^2}$$

Application of time-shift property now yields

$$\begin{aligned} X(\Omega) \quad &= \quad a^{2m}\frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2}e^{-jm\Omega}+a^{2m}(m-k)\frac{e^{j\Omega}}{e^{j\Omega}-a^2}e^{-jm\Omega} \\ &= \quad \frac{a^{2m}e^{-j(m-1)\Omega}}{e^{j\Omega}-a^2}\left[\frac{a}{e^{j\Omega}-a^2}+m-k\right] \end{aligned}$$

9.3-5. We shall consider spectra within the band $|\Omega| \leq \pi$ only
Pair#11

$$1 = u[n] + u[-(n+1)]$$

But

$$u[n] \Longleftrightarrow \frac{e^{j\Omega}}{e^{j\Omega}-1}+\pi\delta(\Omega) \quad |\Omega| \leq \pi$$

Hence

$$u[-n] \Longleftrightarrow \frac{e^{-j\Omega}}{e^{-j\Omega}-1}+\pi\delta(\Omega)$$

and

$$u[-(n+1)] = u[-n]-\delta[n] \Longleftrightarrow \frac{e^{-j\Omega}}{e^{-j\Omega}-1}+\pi\delta(\Omega)-1$$

and

$$1 \Longleftrightarrow \frac{e^{j\Omega}}{e^{j\Omega}-1}+\pi\delta(\Omega)+\frac{e^{-j\Omega}}{e^{-j\Omega}-1}+\pi\delta(\Omega)-1 = 2\pi\delta(\Omega)$$

Pair#12
Apply frequency-shift property [Eq. (9.54)] to the result for pair 11 to obtain

$$e^{j\Omega_0} \Longleftrightarrow 2\pi\delta(\Omega - \Omega_0) \qquad \Omega_0 \leq \pi$$

Pair#13
Apply modulation property [Eq. (9.55)] to the result for pair 11 to obtain

$$\cos\Omega_0 n \Longleftrightarrow \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)] \qquad \Omega_0 \leq \pi$$

Pair#14
Apply modulation property [Eq. (9.57)] with $\theta = -\pi/2$, to the result for pair 11 to obtain

$$\sin\Omega_0 n \Longleftrightarrow j\pi[\delta(\Omega + \Omega_0) - \delta(\Omega - \Omega_0)]$$

Pair#15

Apply modulation property [Eq. (9.55)] to the result for pair 10 to obtain

$$\begin{aligned}
\cos\Omega_0 n u[n] \quad &\Longleftrightarrow \quad \frac{1}{2}\left[\frac{e^{j(\Omega-\Omega_0)}}{e^{j(\Omega-\Omega_0)} - 1} + \frac{e^{j(\Omega+\Omega_0)}}{e^{j(\Omega+\Omega_0)} - 1}\right] + \frac{\pi}{2}[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)] \\
&= \quad \frac{e^{j2\Omega} - e^{j\Omega}\cos\Omega_0}{e^{j2\Omega} - 2e^{j\Omega}\cos\Omega_0 + 1} + \frac{\pi}{2}[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]
\end{aligned}$$

Pair# 16
Apply modulation property [Eq. (9.57)] with $\theta = -\pi/2$ to pair 10 to obtain

$$\begin{aligned}
\sin\Omega_0 n u[n] \quad &\Longleftrightarrow \quad \frac{j}{2}\left[\frac{e^{j(\Omega+\Omega_0)}}{e^{j(\Omega+\Omega_0)} - 1} - \frac{e^{j(\Omega-\Omega_0)}}{e^{j(\Omega-\Omega_0)} - 1}\right] + \frac{j\pi}{2}[\delta(\Omega + \Omega_0) + \delta(\Omega - \Omega_0)] \\
&= \quad \frac{e^{j2\Omega}\sin\Omega_0}{e^{j2\Omega} - 2e^{j\Omega}\cos\Omega_0 + 1} + \frac{\pi}{2j}[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]
\end{aligned}$$

9.3-6.

$$\begin{aligned}
x[n+k] \quad &\Longleftrightarrow \quad X(\Omega)e^{jk\Omega} \\
x[n-k] \quad &\Longleftrightarrow \quad X(\Omega)e^{-jk\Omega}
\end{aligned}$$

Hence

$$x[n+k] + x[n-k] \Longleftrightarrow X(\Omega)\left[e^{jk\Omega} + e^{-jk\Omega}\right] = 2X(\Omega)\cos k\Omega$$

(a) Let $z[n] = u[n+2] - u[n-3]$ (see Figure S9.3-6a)
Then $x[n] = z[n-4] + z[n+4]$
Moreover $z[n]$ is $u[n] - u[n-5]$ left-shifted by 2 units. Hence use of pair 7 and time-shifting property yields

$$Z(\Omega) = \frac{\sin 2.5\Omega}{\sin 0.5\Omega}$$

and

$$X(\Omega) = 2\frac{\sin 3\Omega}{\sin 0.5\Omega}\cos 4\Omega$$

(b) Figure S9.3-6b shows the signal $\Delta(\frac{n}{8})$. Hence,

$$x[n] = \Delta\left(\frac{n-8}{8}\right) + \Delta\left(\frac{n+8}{8}\right)$$

The DTFT of $\Delta\left(\frac{n}{8}\right)$ can be found in several way. Here we shall use the method of convolution. The reader can verify that the convolution $w[n]$ shown in Figure S9.3-6c with $w[-n]$ yields $\Delta\left(\frac{n}{8}\right)$.
From pair 7, we obtain

$$q[n] \Longleftrightarrow \frac{\sin 2\Omega}{\sin 0.5\Omega} e^{-j1.5\Omega}$$

Therefore

$$q[-n] \Longleftrightarrow \frac{\sin(-2\Omega)}{\sin(-0.5\Omega)} e^{j1.5\Omega} = \frac{\sin 2\Omega}{\sin 0.5\Omega} e^{j1.5\Omega}$$

Because

$$\Delta\left(\frac{n}{8}\right) = q[n] * q[-n]$$

$$\Delta\left(\frac{n}{8}\right) \Longleftrightarrow \frac{\sin^2(2\Omega)}{\sin^2(0.5\Omega)}$$

and

$$X(\Omega) = \frac{2\sin^2(2\Omega)}{\sin^2(0.5\Omega)} \cos 8\Omega$$

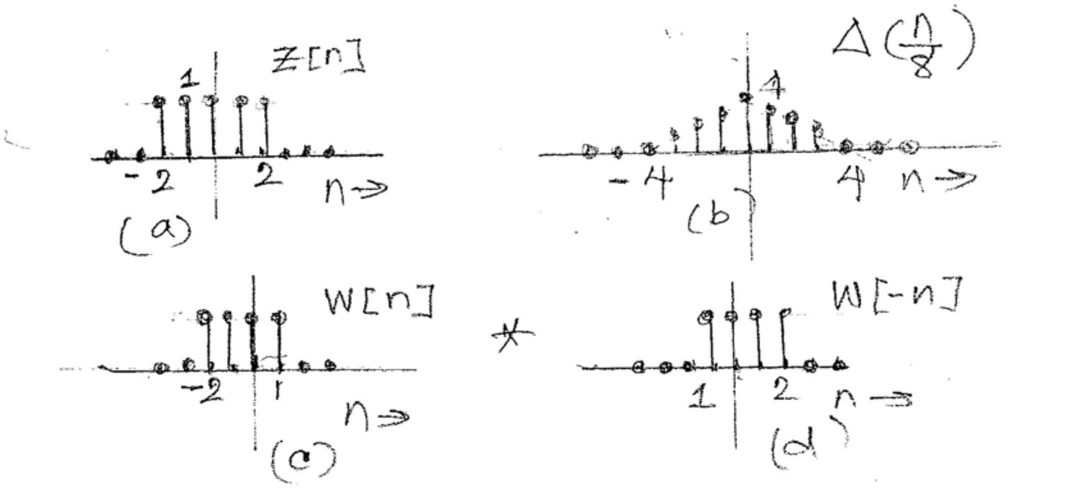

Figure S9.3-6

9.3-7. Use of time-shift property yields

$$x[n \pm k] \Longleftrightarrow X(\Omega) e^{\pm jk\Omega}$$

Therefore

$$x[n+k] - x[n-k] = X(\Omega)\left[e^{jk\Omega} - e^{-jk\Omega}\right] = 2X(\Omega)\sin k\Omega$$

From pair 7 (Table 9.1)

$$w[n] = u[n] - u[n-5] \Longleftrightarrow \frac{\sin 2.5\Omega}{\sin 0.5\Omega} e^{-j2\Omega}$$

and

$$x[n] = -\{w[n-2] - w[n+2]\} \Longleftrightarrow -\frac{\sin 2.5\Omega}{\sin 0.5\Omega} \sin 2\Omega e^{-j2\Omega}$$

9.3-8. If

$$W(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - \gamma}$$

then

$$X(\Omega) = W(\Omega)W(\Omega)$$

and

$$\begin{aligned} x[n] &= \gamma^n u[n] * \gamma^n u[n] \\ &= \sum_{m=0}^{n} \gamma^m \gamma^{n-m} \qquad n \geq 0 \\ &= \gamma^n \sum_{m=0}^{n} 1 = \begin{cases} n\gamma^n & n \geq 0 \\ 0 & n < 0 \end{cases} \\ &= n\gamma^n u[n] \end{aligned}$$

9.3-9. Pair 2:

$$\gamma^n u[n] = \delta[n] + \gamma\delta[n-1] + \gamma^2\delta[n-2] + \ldots + \ldots$$

Therefore

$$\begin{aligned} \gamma^n u[n] &\Longleftrightarrow 1 + \gamma e^{-j\Omega} + \gamma^2 e^{-j2\Omega} + \gamma^3 e^{-j3\Omega} + \ldots + \ldots \\ &= \sum_{m=0}^{\infty} \left(\gamma e^{-j\Omega}\right)^m \\ &= \frac{1}{1 - \gamma e^{-j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega} - \gamma} \end{aligned}$$

Pair 3:
Use time inversion property to obtain

$$\lambda^{-n} u[-n] \Longleftrightarrow \frac{e^{-j\Omega}}{e^{-j\Omega} - \lambda} \qquad |\lambda| > 1$$

and

$$\lambda^{-n} u[-(n+1)] = \lambda^{-n} u[-n] - \delta[n]$$

Hence

$$-\lambda^{-n} u[-(n+1)] = -\lambda^{-n} u[-n] + \delta[n] \Longleftrightarrow \frac{-e^{-j\Omega}}{e^{-j\Omega} - \gamma} + 1 = \frac{-\lambda}{e^{-j\Omega} - \gamma} \qquad |\lambda| > 1$$

Letting $\lambda = 1/\gamma$, we obtain

$$-\gamma^n u[-(n+1)] \Longleftrightarrow \frac{-1/\gamma}{e^{j\Omega} - 1/\gamma} = \frac{e^{j\Omega}}{e^{j\Omega} - \gamma} \qquad |\gamma| < 1$$

Pair 4:

$$\gamma^{|n|} = \gamma^n u[n] - \left(\frac{1}{\gamma}\right)^n u[-(n+1)]$$

Hence

$$\gamma^{|n|} \Longleftrightarrow \frac{e^{j\Omega}}{e^{j\Omega}-\gamma} - \frac{e^{j\Omega}}{e^{j\Omega}-1/\gamma} = \frac{1-\gamma^2}{1-2\gamma\cos\Omega+\gamma^2}$$

Pair 5:

Apply 'multiplication by n' property [Eq. (9.50)] to pair 2 to obtain

$$n\gamma^n u[n] \Longleftrightarrow j\frac{d}{d\Omega}\left[\frac{e^{j\Omega}}{e^{j\Omega}-\gamma}\right] = \frac{\gamma e^{j\Omega}}{(e^{j\Omega}-\gamma)^2}$$

Pair 6:

$$\begin{aligned}
\gamma^n \cos[\Omega_0+\theta]u[n] &= \frac{1}{2}\left[\gamma^n e^{j(\Omega_0 n+\theta)} + \gamma^n e^{-j(\Omega_0 n+\theta)}\right]u[n] \\
&= \frac{1}{2}\left[e^{j\theta}\left(\gamma e^{j\Omega_0}\right)^n u[n] + e^{-j\theta}\left(\gamma e^{-j\Omega_0}\right)^n u[n]\right]
\end{aligned}$$

Hence

$$\begin{aligned}
X(\Omega) &= \frac{1}{2}\left[e^{j\theta}\frac{e^{j\Omega}}{e^{j\Omega}-\gamma e^{j\Omega_0}} + e^{-j\theta}\frac{e^{j\Omega}}{e^{j\Omega}-\gamma e^{-j\Omega_0}}\right] \\
&= \frac{e^{j\Omega}\left[e^{j\Omega}\cos\theta - \gamma\cos(\Omega_0-\theta)\right]}{e^{j2\Omega} - 2\gamma\cos\Omega_0 e^{j\Omega} + \gamma^2}
\end{aligned}$$

Pair 7:

$$x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots + \delta[n-M+1]$$

and

$$\begin{aligned}
X(\Omega) &= 1 + e^{-j\Omega} + e^{-j2\Omega} + \cdots + e^{-j(M-1)\Omega} \\
&= \sum_{k=0}^{M-1} e^{-jk\Omega} = \frac{e^{-jM\Omega}-1}{e^{-j\Omega}-1} = \frac{e^{-\frac{jM\Omega}{2}}\left(e^{-\frac{jM\Omega}{2}} - e^{\frac{jM\Omega}{2}}\right)}{e^{-\frac{j\Omega}{2}}\left(e^{-\frac{j\Omega}{2}} - e^{\frac{j\Omega}{2}}\right)} \\
&= \frac{\sin(M\Omega/2)}{\sin(\Omega/2)} e^{-j(M-1)\Omega/2}
\end{aligned}$$

9.3-10. We shall consider the spectrum only within the band $|\Omega| \leq \pi$.

$$x[n] = e^{j\Omega n} = e^{j\frac{\Omega_0 n}{2}} \times e^{j\frac{\Omega_0 n}{2}}$$

Use of frequency convolution property yields

$$\begin{aligned}
X(\Omega) &= \frac{1}{2\pi}\left[2\pi\delta\left(\Omega - \frac{\Omega_0}{2}\right) * 2\pi\delta\left(\Omega - \frac{\Omega_0}{2}\right)\right] \qquad |\Omega| \leq \pi \\
&= 2\pi\delta\left(\Omega - \frac{\Omega_0}{2}\right) * \delta\left(\Omega - \frac{\Omega_0}{2}\right) \qquad |\Omega| \leq \pi \\
&= 2\pi\int_{-\infty}^{\infty}\delta\left(x - \frac{\Omega_0}{2}\right)\delta\left(\Omega - \Omega + \frac{\Omega_0}{2}\right)dx \qquad |\Omega| \leq \pi
\end{aligned}$$

From the sampling property [Eq. (1.24)] of the impulse, we obtain

$$X(\Omega) = 2\pi\delta(\Omega - \Omega_0) \qquad |\Omega| \le \pi$$

9.3-11. (a) Let

$$x[n] = \text{sinc}(\Omega_c n)$$

From pair 8, we have

$$X(\Omega) = \frac{\pi}{\Omega_c}\text{rect}\left(\frac{\Omega}{2\Omega_c}\right) \qquad |\Omega| \le \pi$$

But

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

Hence

$$X(0) = \sum_{n=-\infty}^{\infty} x[n] = \sum_{n=-\infty}^{\infty} \text{sinc}(\Omega_c n)$$

But $X(0) = \frac{\pi}{\Omega_c}$, which proves the result.

(b) Let

$$x[n] = (-1)^n \text{sinc}(\Omega_c n)$$

First, we prove that if $x[n] \Longleftrightarrow X(\Omega)$, then $(-1)^n x[n] \Longleftrightarrow X(\Omega - \pi)$. This follows from the definition of the DTFT

$$\begin{aligned}
(-1)^n x[n] \quad &\Longleftrightarrow \quad \sum_{n=-\infty}^{\infty} (-1)^n x[n] e^{-j\Omega n} \\
&= \quad \sum_{n=-\infty}^{\infty} x[n] e^{-j\pi n} e^{j\Omega n} \\
&= \quad \sum_{n=-\infty}^{\infty} x[n] e^{-j(\Omega-\pi)n} \\
&= \quad X(\Omega - \pi)
\end{aligned}$$

Hence, we have

$$(-1)^n \text{sinc}(\Omega_c n) \Longleftrightarrow \frac{\pi}{\Omega_c}\text{rect}\left(\frac{\Omega - \pi}{2\Omega_c}\right) \qquad |\Omega| \le \pi$$

Moreover

$$\text{rect}\left(\frac{\Omega - \pi}{2\Omega_c}\right) = 0 \qquad \text{at} \qquad \Omega = 0 \qquad \text{if} \qquad \Omega_c < \pi$$

Hence

$$X(0) = \sum_{n=-\infty}^{\infty} (-1)^n \text{sinc}(\Omega_c n) = 0$$

(c)

$$\text{sinc}^2(\Omega_c n) \Longleftrightarrow \frac{\pi}{\Omega_c}\Delta\left(\frac{\Omega}{4\Omega_c}\right) \qquad |\Omega| \le \frac{\pi}{2}$$

Using the argument in part (a), we obtain the desired result.

(d) Using the argument in part (b) and (c), we obtain the desired result.

(e)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega$$

Hence

$$x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)d\Omega \tag{1}$$

If we left-shift x[n] in pair 7 by $\frac{M-1}{2}$ units, we obtain

$$x[n] = u\left[n+\frac{M-1}{2}\right] - u\left[n+\frac{M+1}{2}\right] \Longleftrightarrow \frac{\sin(M\Omega/2)}{\sin(\Omega/2)}$$

Moreover $x[0]=1$. Hence use of Eq. (1) above yields

$$2\pi = \int_{-\pi}^{\pi}\frac{\sin(M\Omega/2)}{\sin(\Omega/2)}d\Omega$$

(f) From pair 9, we have

$$\text{sinc}^2(\Omega_c n) \Longleftrightarrow \frac{\pi}{\Omega_c}\Delta\left(\frac{\omega}{4\Omega_c}\right) \qquad |\Omega| < \frac{\pi}{2}$$

Application of Parseval's theorem [Eq. (9.60)] yields

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}\left|\text{sinc}^2(\Omega_c n)\right|^2 &= \frac{2}{2\pi}\frac{\pi^2}{\Omega_c^2}\int_0^{2\Omega_c}\left|1-\frac{\Omega}{\Omega_c}\right|^2 d\Omega \\
&= \frac{\pi}{\Omega_c^2}\int_0^{2\Omega_c}\left(1-\frac{2\Omega}{\Omega_c}+\frac{\Omega^2}{\Omega_c^2}\right)d\Omega \\
&= \frac{2\pi}{3\Omega_c}
\end{aligned}$$

9.3-12.

$$\begin{aligned}
E_{x_c} &= \int_{-\infty}^{\infty}|x_c(t)|^2 dt = \int_{-\infty}^{\infty}x_c(t)x_c^*(t)dt \\
&= \int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty}x_c(nT)\text{sinc}(2\pi Bt-n\pi)\right]\left[\sum_{m=-\infty}^{\infty}x_c^*(mT)\text{sinc}(2\pi Bt-m\pi)\right]dt
\end{aligned}$$

Because of orthogonality property of the sinc function, stated in the problem, all the cross-product terms, for $m \neq n$, vanish. Moreover when $m=n$, the integral is $1/2B$. Recall also that $x_c(kT)=x[k]$. Hence

$$E_{x_c} = \frac{1}{2B}\sum_{n=-\infty}^{\infty}|x[n]|^2 = T\sum_{n=-\infty}^{\infty}|x[n]|^2 = TE_x$$

9.4-1.

$$\begin{aligned}
X(\Omega) &= \frac{1}{1+0.5e^{-j\Omega}} = \frac{e^{j\Omega}}{e^{j\Omega}+0.5} \\
Y(\Omega) &= X(\Omega)H(\Omega) = \frac{e^{j\Omega}\left(e^{j\Omega}+0.32\right)}{\left(e^{j\Omega}+0.5\right)\left(e^{j\Omega}+0.8\right)\left(e^{j\Omega}+0.2\right)} \\
\frac{Y(\Omega)}{e^{j\Omega}} &= \frac{e^{j\Omega}+0.32}{\left(e^{j\Omega}+0.5\right)\left(e^{j\Omega}+0.8\right)\left(e^{j\Omega}+0.2\right)} \\
&= \frac{2}{e^{j\Omega}+0.5} - \frac{8/3}{e^{j\Omega}+0.8} + \frac{2/3}{e^{j\Omega}+0.2} \\
Y(\Omega) &= 2\frac{e^{j\Omega}}{e^{j\Omega}+0.5} - \frac{8}{3}\frac{e^{j\Omega}}{e^{j\Omega}+0.8} + \frac{2}{3}\frac{e^{j\Omega}}{e^{j\Omega}+0.2} \\
y[n] &= \left[2\left(-0.5\right)^n - \frac{8}{3}\left(-0.8\right)^n + \frac{2}{3}\left(-0.2\right)^n\right]u[n]
\end{aligned}$$

9.4-2.

$$Y(\Omega) = \left[\frac{e^{j\Omega}+0.32}{(e^{j\Omega}+0.2)(e^{j\Omega}+0.8)}\right]\left[\pi\delta(\Omega) + \frac{e^{j\Omega}}{e^{j\Omega}-1}\right]$$

We use the fact that $f(x)\delta(x) = f(0)\delta(x)$ to obtain

$$Y(\Omega) = \frac{1.32\pi}{2.16}\delta(\Omega) + \frac{e^{j\Omega}(e^{j\Omega}+0.32)}{(e^{j\Omega}-1)(e^{j\Omega}+0.2)(e^{j\Omega}+0.8)}$$

Using partial fraction expansion, we obtain

$$Y(\Omega) = \frac{1.32\pi}{2.16}\delta(\Omega) + \frac{1.32}{2.16}\frac{e^{j\Omega}}{e^{j\Omega}-1} - \frac{1}{6}\frac{e^{j\Omega}}{e^{j\Omega}+0.2} - \frac{4}{9}\frac{e^{j\Omega}}{e^{j\Omega}+0.8}$$

and

$$y[n] = 0.611u[n] - \left[\frac{1}{6}(-0.2)^n + \frac{4}{9}(-0.8)^n\right]u[n]$$

9.4-3.

$$\begin{aligned}
X(\Omega) &= \frac{e^{j\Omega}}{e^{j\Omega}-0.8} - \frac{2e^{j\Omega}}{e^{j\Omega}-2} \\
Y(\Omega) &= X(\Omega)H(\Omega) = \frac{e^{j2\Omega}}{\left(e^{j\Omega}-0.5\right)\left(e^{j\Omega}-0.8\right)} - \frac{2e^{j2\Omega}}{\left(e^{j\Omega}-0.5\right)\left(e^{j\Omega}-2\right)} \\
&= \frac{-5/3}{e^{j\Omega}-0.5} + \frac{8/3}{e^{j\Omega}-0.8} + \frac{2/3}{e^{j\Omega}-0.5} - \frac{8/3}{e^{j\Omega}-2} \\
&= \frac{-1}{e^{j\Omega}-0.5} + \frac{8/3}{e^{j\Omega}-0.8} - \frac{8/3}{e^{j\Omega}-2} \\
Y(\Omega) &= -\frac{e^{j\Omega}}{e^{j\Omega}-0.5} + \frac{8}{3}\frac{e^{j\Omega}}{e^{j\Omega}-0.8} - \frac{8}{3}\frac{e^{j\Omega}}{e^{j\Omega}-2} \\
y[n] &= \left[-\left(0.5\right)^n + \frac{8}{3}\left(0.8\right)^n\right]u[n] + \frac{8}{3}\left(2\right)^n u\left[-\left(n+1\right)\right]
\end{aligned}$$

9.4-4. (a) When $x[n] = \delta[n]$, the output is $h[n]$, given by (assuming causal accumulator)

$$h[n] = \sum_{k=0}^{n} \delta[k] = u[n]$$

and

$$H(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - 1} + \pi\delta(\Omega)$$

(b) If this accumulator is used as a digital processor for the digital integrator, discussed in example 3.7, and the input is $x(t) = u(t)$, then the sampled $x(t)$ yields the digital input $x[n] = x(nT) = u[n]$

9.4-5. (a)

$$\text{sinc}\left(\frac{\pi n}{2}\right) \Longleftrightarrow 2\text{rect}\left(\frac{\Omega}{\pi}\right)$$

Hence the output corresponding to this input is

$$Y(\Omega) = 2\text{rect}^2\left(\frac{\Omega}{\pi}\right) e^{-j2\Omega} = 2\text{rect}\left(\frac{\Omega}{\pi}\right) e^{-j2\Omega}$$

Therefore

$$y[n] = \text{sinc}\left[\frac{\pi(n-2)}{2}\right]$$

(b)

$$\text{sinc}(\pi n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$\text{sinc}(\pi n) = \delta[n] \Longleftrightarrow 1$$

The output corresponding to this input is

$$Y(\Omega) = \text{rect}\left(\frac{\Omega}{\pi}\right) e^{-j2\Omega}$$

and

$$y[n] = \frac{1}{2}\text{sinc}\left[\frac{\pi(n-2)}{2}\right]$$

(c)

$$\text{sinc}^2\left(\frac{\pi n}{4}\right) \Longleftrightarrow 4\Delta\left(\frac{\Omega}{\pi}\right)$$

The output corresponding to this input is

$$\begin{aligned} Y(\Omega) &= 4\Delta\left(\frac{\Omega}{\pi}\right) \text{rect}\left(\frac{\Omega}{\pi}\right) e^{-j2\Omega} \\ &= 4\Delta\left(\frac{\Omega}{\pi}\right) e^{-j2\Omega} \end{aligned}$$

and

$$y(n) = \text{sinc}^2\left[\frac{\pi(n-2)}{4}\right]$$

9.4-6. (a) Let

$$y[n] = (-1)^n x[n] = e^{-j\pi n}x[n]$$

Use of frequency-shifting property [Eq. (9.45)] yields

$$Y(\Omega) = X(\Omega - \pi)$$

(b) Figure S9.4-6a shows $\gamma^n u[n]$ and $(-1)\gamma^n u[n]$. The spectra for $(-1)^n\gamma^n u[n]$ are the same as those for $\gamma^n u[n]$ (Figure S9.4-6 b and c) but shifted by π, as shown in Figure S9.4-6 b over the fundamental band $|\Omega| \leq \pi$.

(c)

$$h_{LP}[n] = \frac{\Omega_c}{\pi}\text{sinc}(\Omega_c n)$$

The frequency response of $(-1)^n h_{LP}[n] = \frac{\Omega_c}{\pi}(-1)^n\text{sinc}(\Omega_c n)$ is $\text{rect}\left(\frac{\Omega}{2\Omega_c}\right)$ frequency-shifted by π is $\text{rect}\left(\frac{\Omega-\pi}{2\Omega_c}\right)$, as shown in Figure S9.4-6c. It is clear that this is a highpass filter.

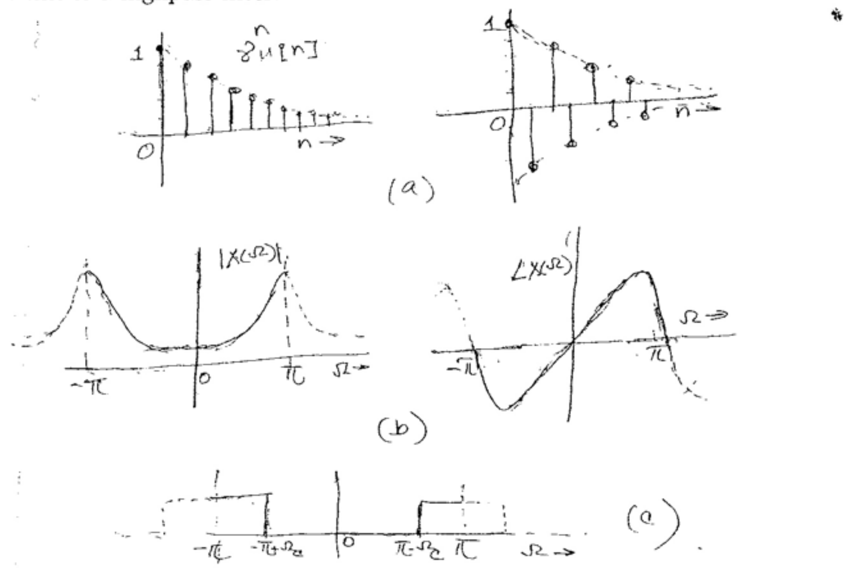

Figure S9.4-6

9.4-7.

$$w[n] = (-1)^n x[n]$$

Hence $W(\Omega) = X(\Omega - \pi)$, as shown in Problem 9.4-6a.

$$Q(\Omega) = X(\Omega - \pi)H(\Omega)$$

and because

$$y[n] = (-1)^n q[n]$$

$$
\begin{aligned}
Y(\Omega) = Q(\Omega - \pi) &= X(\Omega - 2\pi)H(\Omega - \pi) \\
&= X(\Omega)H(\Omega - \pi) \\
&= X(\Omega)H_1(\Omega)
\end{aligned}
$$

Therefore

$$H_1\Omega = H(\Omega - \pi)$$

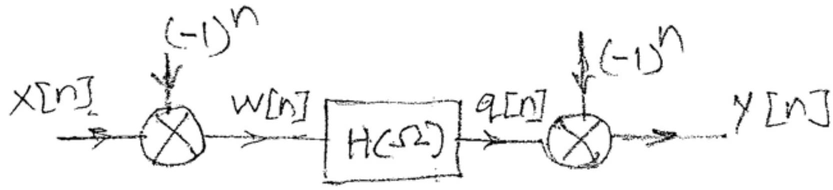

Figure S9.4-7

9.4-8. (a) System $\mathcal{S}_1$ can be described by Eq. (3.17)b as

$$y_1[n] + \sum_{k=1}^{N} a_k y_1[n-k] = \sum_{k=0}^{N} b_k x[n-k]$$

Take DTFT of this equation to obtain

$$Y_1(\Omega)\left[1 + \sum_{k=1}^{N} a_k e^{-jk\Omega}\right] = X(\Omega)\sum_{k=0}^{N} b_k e^{-jk\Omega}$$

and the frequency response of this system is given by

$$H_1(\Omega) = \frac{Y_1(\Omega)}{X(\Omega)} = \frac{\sum_{k=0}^{N} b_k e^{-jk\Omega}}{1 + \sum_{k=1}^{N} a_k e^{-jk\Omega}}$$

Now consider the system $\mathcal{S}_2$, whose a_i and b_i coefficients are $(-1)^k$ times the corresponding coefficients of $\mathcal{S}_1$. Hence

$$H_2(\Omega) = \frac{Y_2(\Omega)}{X(\Omega)} = \frac{\sum_{k=0}^{N}(-1)^k b_k e^{-jk\Omega}}{1 + \sum_{k=1}^{N}(-1)^k a_k e^{-jk\Omega}} = \frac{\sum_{k=0}^{N} b_k e^{-jk(\Omega-\pi)}}{1 + \sum_{k=1}^{N} a_k e^{-jk(\Omega-\pi)}} = H_1(\Omega-\pi)$$

(b) As shown in Prob. 9.4-6, if $H_1(\Omega)$ is a lowpass filter, then $H_2(\Omega) = H_1(\Omega - \pi)$ is a highpass filter.

(c) The DTFT of the system equation $y[n] - 0.8y[n-1] = x[n]$ is

$$Y_1(\Omega)\left[1 - 0.8e^{-j\Omega}\right] = X(\Omega)$$

and

$$H_1(\Omega) = \frac{Y_1(j\Omega)}{X(j\Omega)} = \frac{1}{1 - 0.8e^{-j\Omega}}$$

The frequency spectra for this system are shown in Figure 9.4, which shows that this is a lowpass system.
The DTFT of the system equation $y[n] + 0.8y[n-1] = x[n]$ is

$$Y_2(\Omega)\left[1 + 0.8e^{-j\Omega}\right] = X(\Omega)$$

and

$$H_2(\Omega) = \frac{Y_2(j\Omega)}{X(j\Omega)} = \frac{1}{1+0.8e^{-j\Omega}} = \frac{1}{1-0.8e^{-j(\Omega-\pi)}} = H_1(\Omega-\pi)$$

Hence the frequency response spectra for $H_2(\Omega)$ are same as those for $H_1(\Omega)$ frequency-shifted by π. Hence, $H_2(\Omega)$ is a highpass system.

9.4-9. (a) Let us compute the response $h[n]$ to the unit impulse input $\delta[n]$. Because the system contains time varying multipliers, however, we must also test whether it is a time varying or a time-invariant system. It is therefore appropriate to consider the system response to an input $\delta[n-k]$. This is an impulse at $n = k$. Using the fact that $x[n]\,\delta[n-k] = x[n]\delta[n-k]$, we can express the signals at various points as follows:

at a_1	$2\cos\,(\Omega_c k)\,\delta[n-k]$
a_2	$2\sin\,(\Omega_c k)\,\delta[n-k]$
b_1	$2\cos\,(\Omega_c k)h_o[n-k]$
b_2	$2\sin\,(\Omega_c k)h_o[n-k]$
c_1	$2\cos\,(\Omega_c k)\cos\,(\Omega_c n)h_o[n-k]$
c_2	$2\sin\,(\Omega_c k)\sin\,(\Omega_c n)h_o[n-k]$
d	$2h_o[n-k]\;[\cos\,(\Omega_c k)\cos\,(\Omega_c n) + \sin(\Omega_c k)\sin\,(\Omega_c n)]$
	$= 2h_o[n-k]\cos\,(\Omega_c[n-k])$

Thus, the system response to the input $\delta[n-k]$ is $2h_o[n-k]\cos[\Omega_c[n-k]]$. Clearly, the system is linear time-invariant, with impulse response

$$h[n] = 2h_o[n]\,\cos\Omega_c n$$

(b) From the modulation property, it follows that

$$H(\Omega) = H_0(\Omega-\Omega_c) + H_0(\Omega+\Omega_c)$$

If $H_0(\Omega) = (rect)(\Omega/2W)$, then

$$H(\omega) = \text{rect}\left(\frac{\Omega-\Omega_c}{2W}\right) + \text{rect}\left(\frac{\Omega+\Omega_c}{2W}\right)$$

The transfer function $H(\omega)$ [Figure S9.4-9b] represents an ideal bandpass filter when $\Omega_c + W \leq \pi$.

9.M-1. (a) The inverse DTFS is given by $x[n] = \sum_{r=0}^{N_0-1} \mathcal{D}_r e^{jr\Omega_0 n}$. Like the DTFS, the IDTFS can be computed using a matrix based approach. First, define $W_{N_0} = e^{j\Omega_0}$, which is a constant for a given N_0. Substituting W_{N_0} into the IDTFS equation yields $x[n] = \sum_{n=0}^{N_0-1} \mathcal{D}_r W_{N_0}^{nr}$. An inner product of two vectors computes $x[n]$.

$$x[n] = \left[1, W_{N_0}^{n}, W_{N_0}^{2n}, \cdots, W_{N_0}^{(N_0-1)r}\right] \begin{bmatrix} \mathcal{D}_0 \\ \mathcal{D}_1 \\ \mathcal{D}_2 \\ \vdots \\ \mathcal{D}_{N_0-1} \end{bmatrix}.$$

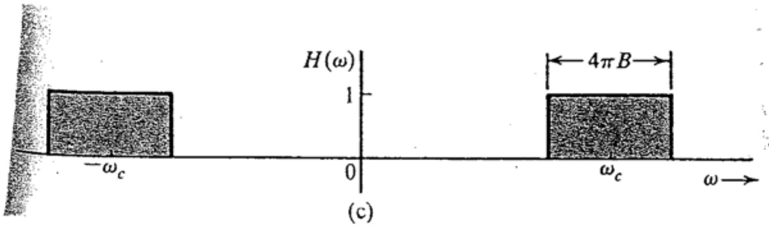

Figure S9.4-9b

Stacking the results for all n yields:

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N_0-1] \end{bmatrix} = \begin{bmatrix} 1, & 1, & 1, & \cdots, & 1 \\ 1, & W_{N_0}^1, & W_{N_0}^2, & \cdots, & W_{N_0}^{(N_0-1)} \\ 1, & W_{N_0}^2, & W_{N_0}^4, & \cdots, & W_{N_0}^{2(N_0-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1, & W_{N_0}^{(N_0-1)}, & W_{N_0}^{2(N_0-1)}, & \cdots, & W_{N_0}^{(N_0-1)^2} \end{bmatrix} \begin{bmatrix} \mathcal{D}_0 \\ \mathcal{D}_1 \\ \mathcal{D}_2 \\ \vdots \\ \mathcal{D}_{N_0-1} \end{bmatrix}.$$

In matrix notation, the IDTFS is compactly written as $\mathbf{x} = \mathbf{W}_{N_0}^* \boldsymbol{\mathcal{D}}$. Notice, $\mathbf{W}_{N_0}^*$ is just the conjugate of the DFT matrix $\mathbf{W}_{N_0}$.
In MATLAB, the N_0-by-N_0 IDTFS matrix is easily computed according to

```
>> Wconj= (exp(j*2*pi/N_0)).^((0:N_0-1)'*(0:N_0-1));
```

(b) MATLAB code, similar to that presented in MATLAB session 9, is used to test the execution speed of the matrix IDTFS approach to the inverse FFT approach. First, test vectors and IDTFS matrices are created.

```
>> X10 = fft(randn(10,1));
>> X100 = fft(randn(100,1));
>> X1000 = fft(randn(1000,1));
>> W10 = (exp(j*2*pi/10)).^((0:10-1)'*(0:10-1));
>> W100 = (exp(j*2*pi/100)).^((0:100-1)'*(0:100-1));
>> W1000 = (exp(j*2*pi/1000)).^((0:1000-1)'*(0:1000-1));
\end{verbatim}
Next, execution speeds are measured by repeating calculations within a loop.
Notice, output from MATLAB's {\tt ifft} command must be scaled by
$1/N_0$ to compute the IDTFS.
\begin{verbatim}
>> tic; for t=1:50000, ifft(X10)/10; end; T10ifft =toc;
>> tic; for t=1:50000, W10*X10; end; T10mat = toc;
>> tic; for t=1:5000, ifft(X100)/100; end; T100ifft =toc;
```

```
>> tic; for t=1:5000, W100*X100; end; T100mat = toc;
>> tic; for t=1:500, ifft(X1000)/1000; end; T1000ifft =toc;
>> tic; for t=1:500, W1000*X1000; end; T1000mat = toc;
>> [T10mat/T10ifft, T100mat/T100ifft, T1000mat/T1000ifft]
ans = 1.0323    3.5000   101.4754
```

For these trials, these results indicate that the IFFT approach is about as fast as the matrix approach for $N_0 = 10$, about an order of magnitude faster than the matrix approach for $N_0 = 100$, and about two orders of magnitude faster than the matrix approach for $N_0 = 1000$. While actual times will vary considerably from computer to computer and from trial to trial, the general trend is clear: the matrix based approach is less efficient than the inverse FFT approach, and this difference grows rapidly as N_0 increases.

(c) Substituting $\mathcal{D} = \frac{1}{N_0}\mathbf{W}_{N_0}\mathbf{x}$ into $\mathbf{x} = \mathbf{W}^*_{N_0}\mathcal{D}$ yields $\mathbf{x} = \mathbf{W}^*_{N_0}\frac{1}{N_0}\mathbf{W}_{N_0}\mathbf{x} = \frac{1}{N_0}\mathbf{W}^*_{N_0}\mathbf{W}_{N_0}\mathbf{x}$. For equality, $\frac{1}{N_0}\mathbf{W}^*_{N_0}\mathbf{W}_{N_0} = I_{N_0}$, where I_{N_0} is the N_0-by-N_0 identity matrix. Thus, $\mathbf{W}^*_{N_0}\mathbf{W}_{N_0} = N_0 I_{N_0} = \mathbf{W}_{N_0}\mathbf{W}^*_{N_0}$.

Thus, multiplying the DFT matrix $\mathbf{W}_{N_0}$ by the inverse DTFS matrix $\mathbf{W}^*_{N_0}$, or vice versa, yields the scaled identity matrix $N_0 I_{N_0}$:

$$\mathbf{W}^*_{N_0}\mathbf{W}_{N_0} = \mathbf{W}_{N_0}\mathbf{W}^*_{N_0} = N_0 I_{N_0}.$$

This result is consistent with the fact that the DTFS represents a signal using an orthogonal set of basis functions. Since the columns (or rows) of $\mathbf{W}_{N_0}$ are orthogonal, $\mathbf{W}^*_{N_0}\mathbf{W}_{N_0}$ must be a diagonal matrix. The scale factor of N_0 results from mixing matrices from the DFT and DTFS (recall, the DFT is N_0 times the DFTS).

9.M-2. (a) We know that $|H(e^{j\Omega_c})|^2 = \frac{1}{2} = \frac{(1+\alpha)^2}{4}\frac{|1-e^{-j\Omega}|^2}{|1-\alpha e^{-j\Omega}|^2} = \frac{1+2\alpha+\alpha^2}{4}\frac{(1-\cos(-\Omega_c))^2+(-\sin(-\Omega_c))^2}{(1-\alpha\cos(-\Omega_c))^2+(\alpha\sin(-\Omega_c))^2} = \frac{1+2\alpha+\alpha^2}{4}\frac{1-2\cos(\Omega_c)+\cos^2(\Omega_c)+\sin^2(\Omega_c)}{1-2\alpha\cos(\Omega_c)+\alpha^2\cos^2(\Omega_c)+\alpha^2\sin^2(\Omega_c)} = \frac{1+2\alpha+\alpha^2}{4}\frac{2-2\cos(\Omega_c)}{1+\alpha^2-2\alpha\cos(\Omega_c)} = \frac{1+2\alpha+\alpha^2}{4}\frac{2-2\cos(\Omega_c)}{1+\alpha^2-2\alpha\cos(\Omega_c)}$. Thus, $2(1+\alpha^2-2\alpha\cos(\Omega_c)) = (1+2\alpha+\alpha^2)(2-2\cos(\Omega_c))$ or $2+2\alpha^2-4\alpha\cos(\Omega_c) = 2+4\alpha+2\alpha^2-2\cos(\Omega_c)-4\alpha\cos(\Omega_c)-2\alpha^2\cos(\Omega_c)$. This simplifies to $0 = -2\alpha^2\cos(\Omega_c)+4\alpha-2\cos(\Omega_c)$ or $\cos(\Omega_c)\alpha^2-2\alpha+\cos(\Omega_c)$. Solving with the quadratic formula yields $\alpha = \frac{2\pm\sqrt{4-4\cos^2(\Omega_c)}}{2\cos(\Omega_c)} = \frac{1\pm\sin(\Omega_c)}{\cos(\Omega_c)}$. For $0 \leq \Omega_c \leq \pi$, $\left|\frac{1+\sin(\Omega_c)}{\cos(\Omega_c)}\right| \geq 1$ and $\left|\frac{1-\sin(\Omega_c)}{\cos(\Omega_c)}\right| \leq 1$. Since a stable system is desired,

$$\alpha = \frac{1-\sin(\Omega_c)}{\cos(\Omega_c)}.$$

(b) For this part, $\Omega_c = \frac{2\pi}{5}$. Using MATLAB to solve:

```
>> Omega_c = 2*pi/5; alpha = (1-sin(Omega_c))/cos(Omega_c)
alpha = 0.1584
```

The corresponding difference equation is determined from $H(z) = \frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)} = \left(\frac{1+\alpha}{2}\right)\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$. Using MATLAB:

```
>> B = (1+alpha)/2*[1,-1], A = [1,-alpha]
B = 0.5792   -0.5792
A = 1.0000   -0.1584
```

Thus, the difference equation is

$$y[n] - 0.1584y[n-1] = 0.5792x[n] - 0.5792x[n-1].$$

This first order system has one pole at $z = \alpha = 0.1584$. Since this pole is inside the unit circle, the system is stable. Frequency response is computed using the signal processing toolbox function `freqz`:

```
>> Omega = linspace(0,pi,1001); H = freqz(B,A,Omega);
>> H3dB = freqz(B,A,[-Omega_c,Omega_c]);
>> plot(Omega,(abs(H)),'k',...
   [0,pi],[1/sqrt(2),1/sqrt(2)],'k:',...
   [Omega_c,Omega_c],[0,1],'k:');
>> axis([0,pi,0,1]);
>> xlabel('\Omega'); ylabel('|H(e^{j\Omega})|');
>> set(gca,'xtick',[0:pi/5:pi],'xticklabel',['  0 ';...
   ' p/5';'2p/5';'3p/5';'4p/5';'  p '],...
   'fontname','symbol');
```

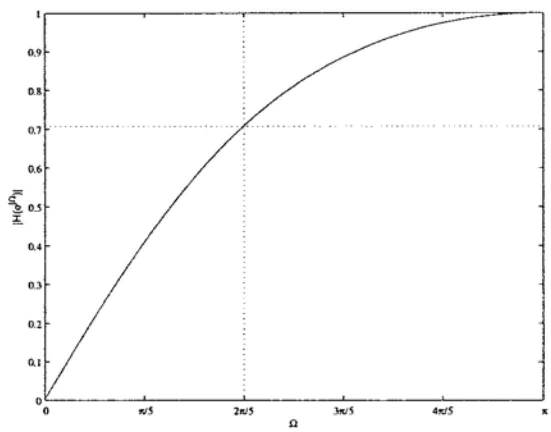

Figure S9.M-2b: $|H(e^{j\Omega})|$ for digital HPF with $\Omega_c = 2\pi/5$.

As seen if Figure S9.M-2b, the filter is highpass with a cutoff frequency $\Omega_c = 2\pi/5$, as desired.

(c) Since α, and therefore $H(z)$, is held constant, the cutoff frequency Ω_c remains constant as well. That is, changing the sampling frequency does not affect the digital cutoff frequency Ω_c of the filter. However, the cut-off frequency expressed in hertz scales directly with the sampling frequency. That is, as $\mathcal{F}_s$ is increased to $\mathcal{F}_s = 50\text{kHz}$, $f_c = 1\text{kHz}$ is increased to $f_c = 10\text{kHz}$.

(d) The inverse to $H(z) = \left(\frac{1+\alpha}{2}\right)\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$ is $H^{-1}(z) = \left(\frac{2}{1+\alpha}\right)\left(\frac{1-\alpha z^{-1}}{1-z^{-1}}\right)$. Since the inverse has a root on the unit circle, it is not BIBO stable and therefore not well behaved.

(e) For $\Omega_c = \pi/2$, $\alpha = \frac{1-\sin(\Omega_c}{\cos(\Omega_c)} = \frac{1-1}{0}$, which is indeterminant. Using L'Hospital's rule, $\lim_{\Omega_c \to \pi/2} \frac{1-\sin(\Omega_c}{\cos(\Omega_c)} = \lim_{\Omega_c \to \pi/2} \frac{d}{dt}\left(\frac{1-\sin(\Omega_c}{\cos(\Omega_c)}\right) = \lim_{\Omega_c \to \pi/2} \frac{-\cos(\Omega_c}{-\sin(\Omega_c)} = \frac{0}{1} = 0$. Using $\alpha = 0$, the system function is $H(z) = 0.5(1 - z^{-1})$. What is particularly interesting is that when $\Omega_c = \pi/2$ this normally IIR filter $H(z)$ becomes an FIR filter! Notice that the impulse response is $h[n] = 0.5\delta[n] - 0.5\delta[n-1]$, which is a finite duration signal.

9.M-3. (a) It would be unlikely if not impossible to achieve this exact magnitude response with a practical FIR filter. Since the magnitude response has points of derivative discontinuities, an infinite length filter would be required, which is not practical.

(b) MATLAB is used to design a length-31 FIR filter that reasonably approximates the desired response. Different approximations are easily accomplished by changing N. To perform computations, program MS9P3 and MS5P1 are utilized. Notice, it is important to define Ω over $[0, 2\pi)$ not $[-\pi, \pi)$.

```
>> H_d = inline(['((mod(Omega,2*pi)>=0)&(mod(Omega,2*pi)<pi/4))',...
   '.*(4*mod(Omega,2*pi)/pi)+',...
   '((mod(Omega,2*pi)>=pi/4)&(mod(Omega,2*pi)<pi/2))',...
   '.*(2-4*mod(Omega,2*pi)/pi)+',...
   '((mod(Omega,2*pi)>7*pi/4)&(mod(Omega,2*pi)<=2*pi))',...
   '.*(-4*(mod(Omega,2*pi)-2*pi)/pi)+',...
   '((mod(Omega,2*pi)>3*pi/2)&(mod(Omega,2*pi)<=7*pi/4))',...
   '.*(2+4*(mod(Omega,2*pi)-2*pi)/pi)']);
>> N = 31; h = MS9P3(N,H_d);
>> Omega = linspace(0,2*pi,1001);
>> Omega_samples = linspace(0,2*pi*(1-1/N),N)';
>> H = MS5P1(h,1,Omega);
>> subplot(2,1,1); stem([0:N-1],h,'k');
>> xlabel('n'); ylabel('h[n]');
>> subplot(2,1,2); plot(Omega_samples,H_d(Omega_samples),'ko',...
   Omega,H_d(Omega),'k:',Omega,abs(H),'k');
>> axis([0 2*pi -0.1 1.3]); xlabel('\Omega'); ylabel('|H(\Omega)|');
>> legend('Samples','Desired','Actual',0);
>> set(gca,'xtick',[0:pi/2:2*pi],'xticklabel',['  0 ';...
   ' p/2';'';'  p ';'3p/2';' 2p '],...
   'fontname','symbol');
```

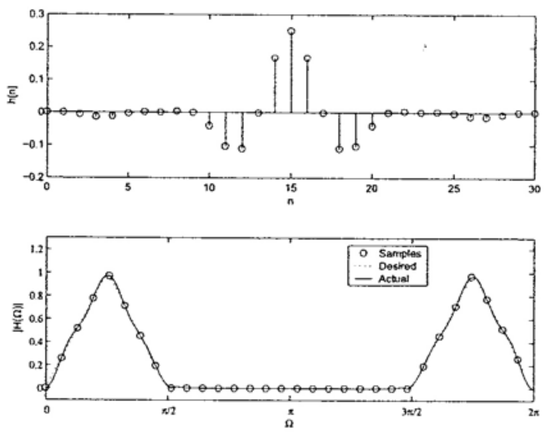

Figure S9.M-3b: Length-31 FIR "triangle" bandpass filter.

Repeating the above code using $N = 101$ yields a much closer approximation.

9.M-4. A simple first-order highpass filter is given by $H_{\text{HP}}(z) = k(1 - z^{-1})$. To achieve a gain of 3, solve $3 = k|1 - e^{-j\pi}| = 2k$. Thus, $k = 3/2$. To realize the desired comb filter, the

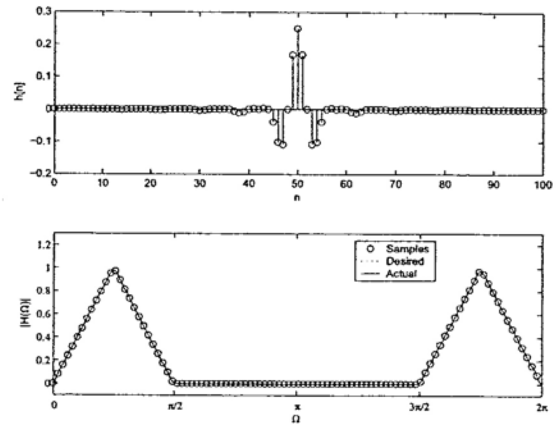

Figure S9.M-3b: Length-101 FIR "triangle" bandpass filter.

HPF response needs to be compressed by a factor of 4, which effectively replicates the original response four times over $[0, 2\pi)$. Compression is achieved by letting $z = z^4$. Thus,

$$H_{\text{comb}}(z) = 1.5(1 - z^{-4}).$$

The corresponding impulse response is

$$h_{\text{comb}} = 1.5\delta[n] - 1.5\delta[n-4].$$

MATLAB is used to verify operation:

```
>> Omega = linspace(0,2*pi,1001);
>> H = MS5P1([3/2 0 0 -3/2],1,Omega);
>> plot(Omega,abs(H),'k');
>> axis([0 2*pi -0.1 3.1]); xlabel('\Omega'); ylabel('|H(\Omega)|');
>> set(gca,'xtick',[0:pi/2:2*pi],'xticklabel',[' 0 ';...
    ' p/2';'';' p ';'3p/2';' 2p '],...
    'fontname','symbol');
```

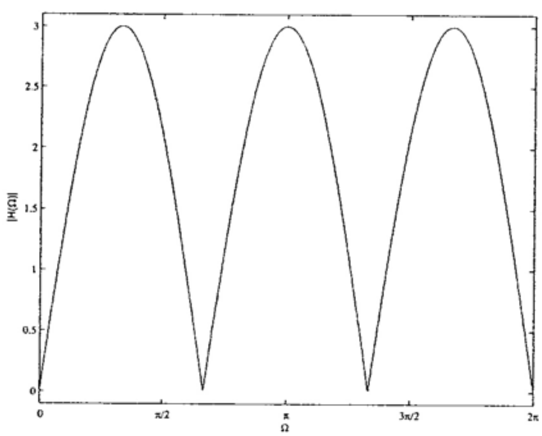

Figure S9.M-4: $|H(\Omega)|$ for FIR comb filter.

9.M-5. (a) Reversing the order of the elements of column vector $\mathbf{x}$ can be accomplished using a N_0-by-N_0 permutation matrix R_{N_0} that is simply a 90-degree rotated N_0-by-N_0 identity matrix. For example, R_5 is

$$R_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) Let integer $i \in \{0, 1, \ldots, N_0 - 1\}$ be used to designate row or column of $\mathbf{W}_{N_0}$. Row i of $\mathbf{W}_{N_0}$ is represented as $\mathbf{r}_i = e^{j2\pi[0,1,\ldots,N_0-1]i/N_0}$. Column i of $\mathbf{W}_{N_0}$ is represented as $\mathbf{c}_i = e^{j2\pi[0,1,\ldots,N_0-1]^T i/N_0}$. For $i \geq 1$, notice that column $N_0 - i$ is $\mathbf{c}_{N_0-i} = e^{j2\pi[0,1,\ldots,N_0-1]^T(N_0-i)/N_0} = e^{j2\pi[0,1,\ldots,N_0-1]^T} e^{-j2\pi[0,1,\ldots,N_0-1]^T i/N_0} = e^{-j2\pi[0,1,\ldots,N_0-1]^T i/N_0} = \mathbf{r}_i^H$. That is, for $i \geq 1$, column $(N_0 - i)$ is the complex-conjugate transpose of row i. Also notice that $\mathbf{W}_{N_0}$ is composed of orthogonal rows,

$$\mathbf{r}_i \mathbf{r}_k^H = \begin{cases} 0 & i \neq k \\ N_0 & i = k \end{cases}.$$

Combining these facts yields

$$\mathbf{W}_{N_0}^2 = N_0 \begin{bmatrix} 1 & 0 \; \ldots \; 0 \\ 0 & \\ \vdots & R_{N_0-1} \\ 0 & \end{bmatrix}.$$

For example, if $N_0 = 5$ then

$$\mathbf{W}_5^2 = 5 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

By inspection, it is clear that $\mathbf{W}_{N_0}^2$ is a scaled permutation matrix. The operation $\mathbf{W}_{N_0}^2\mathbf{x}$ scales and reorders the vector $\mathbf{x}$: the first element of $\mathbf{x}$ is not moved, but the order of the remaining $N_0 - 1$ elements are reversed.

MATLAB is used to confirm these conclusions.

```
>> x = [1 2 3 4 5]';
>> W_5 = dftmtx(5);
>> real(W_5*W_5*x)'
ans = 5.0000   25.0000   20.0000   15.0000   10.0000
```

The last line includes the `real` command to remove minute imaginary components that result due to computer round-off. As expected, vector $\mathbf{x}$ is scaled by $N_0 = 5$ and the order of the last four elements is reversed.

(c) Using the previous result, $\mathbf{W}_{N_0}^4\mathbf{x} = (\mathbf{W}_{N_0}^2)(\mathbf{W}_{N_0}^2)\mathbf{x} = N_0^2\mathbf{x}$. The first multiplication by $(\mathbf{W}_{N_0}^2)$ scales $\mathbf{x}$ by N_0 and reverses the order of the last $N_0 - 1$ elements. The second multiplication again scales $\mathbf{x}$ by N_0 for a total of N_0^2 and reverses the previously reversed last $N_0 - 1$ elements, effectively leaving the order of $\mathbf{x}$ unchanged. MATLAB is used to confirm these conclusions.

```
>> x = [1 2 3 4 5]';
>> W_5 = dftmtx(5);
>> real(W_5*W_5*W_5*W_5*x)'
ans = 25.0000   50.0000   75.0000  100.0000  125.0000
```

The result is just x scaled by $N_0^2 = 25$.