Multi Agent Systems Seminar

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The gathering problem

N agents located in the plane in $\mathbb{R}^2$, with positions: $\{V_i \,|\, (x_i, y_i)\, 1 \le i \le N\}$

Let P be a probability function, $P : \mathbb{R} \to [0,1]$.

P is well defined and have derevatives in all of her domain,

P is a strictly deccreasing function ($d_1 < d_2 \Rightarrow P(d_1) > P(d_2)$)

V_i is a neighbor of V_j ($V_j \in N(V_i)$) under some probability $P(D_{ij})$ where D_{ij} is $\|V_i - V_j\|$

The probability is a function that depends on the distance between the agents

and a certain distance R exists such that if $D_{ij} \le R$, $P(D_{ij}) = 1$

by that, if the agents are very close they will never lose connectivity.

note that $P(D_{ij}) = P(D_{ji})$, and the neighboring is mutual, so a lost ot exist link applies to both of the agents.

Every agent has information about the bearing and the distances to his neighbors.

Every agent can move one step that is limited by the constant R;

The system work under a global clock (synchronous) when at every point the link still exists or lost.

Secotr

Let v_i be an agent and $v_j \in N(v_i)$ and the distance between those agents is L

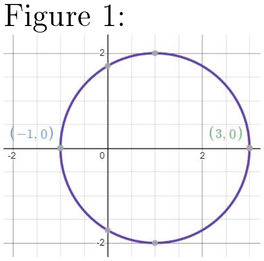
We will define $\text{Sector}_i(v_j)$ as the circle that his diameter is $\|V_i - V_j\|$ and both agents coordinates are on the circle

1. the circle center is located at $\frac{V_i+V_j}{2}$
2. the circle radius is $\frac{\|V_i-V_j\|}{2}$

so the sector is represented by the circle that both of the agents are on it:

$$\begin{cases} (x-x_i)^2 + (y-y_i)^2 \le R^2 \\ (x-x_j)^2 + (y-y_j)^2 \le L^2 \end{cases}$$

Figure 1:



Motion Rule

For v_i agent with neighbors $v_1, v_2, ..$ so the sectors Sector$_i(v_j)$ are as previously defined under "Sector"

Let S be the intersection of all the sectors of $v_i's$ neighbors $S = \cap_{i=1}^{N(v_i)} Sector_i(v_j)$.

So, S is defined by the common solution of all of $Sector_i(v_j)$ equations (domains)

An agent will be called "Surrounded" if the only common domain for the system equations is $S = \{(x_0, y_0)\}$

If there is a common domain of all Sector$_i(v_j)$ so it can be represented by the inequality equations:

$x_{min} \leq x \leq x_{max}$, $y_{min} \leq y \leq y_{max}$

where x_{min} is the smallest x in common, x_{max} is the max x that is common, and respectly in y.

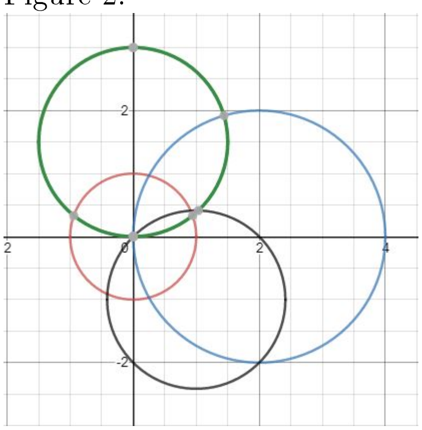
please note that at least one of each coordinates in the inequality equations will be our agent's coordinates:

x_0, y_0

The motion rule will be:

- if the agent is surrounded, stay put
- goto $(\frac{x_{min}+x_{max}}{2}, \frac{y_{min}+y_{max}}{2})$

Figure 2:



Claim 1: If the angle between 3 agents $\neq 180$ then the sector intersection is not empty

We will show that for every $\theta \neq 0$ there are 2 points of intersection between the circle of the pivot agent and the tangent to the sections

hence there is an area $\neq 0$ in which the agent can move.

Let v_i be an agent with $v_j, v_k \in N(v_i)$ with sectors $Sector_i(v_j)$ and $Sector_i(v_k)$ the circle that represents them with $R_j, R_k > 0$.

We will note v_i cooardinates as $(0,0)$, that will be only for conveniency.

Another assumption is that v_k is fixed with cooardinated $(0, y_k)$, so $R_k = |y_k|$

We will call L_j be the tangent to $Sector_i(v_j)$, so (0,0) $\in L_j$

Let m be the clockwise angle between the L_j and the x axis.

m is orthogonal to (x_j, y_j) , vector from (0,0) to $v_j's$ cooardinates, let α be the angle between v_j and the x-axis

And $m = tan(\alpha - 90)$

So, the linear equation that describes L_j is $y = mx$.

The intersection will not be empty if there is a meeting point $\neq 0$ between L_j and the $Sector_i(v_k)$.

$$\begin{cases} y = mx \\ x^2 + y^2 = R^2 \\ x^2 + (y - y_k)^2 = R_k^2 \end{cases}$$

$x^2 + y^2 - 2y_k + y_k^2 = R_k^2$

$(*)\ R_k = |y_k| \Rightarrow R_k^2 = y_k^2$

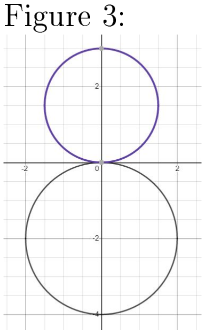
$x^2 + y^2 - 2mxy_k = 0$

$R^2 - 2mxy_k = 0$

$x = \frac{R^2}{2my_k}$

For every $m \neq 0$ there is a meeting point of L_j,the tangent to $Sector_i(v_j)$ and $Sector_i(v_k)$, hence the intersection is not empty

Figure 3:



Claim 2: if $v_j \in N(v_i)$ in t_0 and in t_1 and $t_0 < t_1$, $d_{ij}(t_0) \geq d_{ij}(t_1)$

Moving according to the Motion Rule may only decrease the distances between connected agents.

let $(x_i^0, y_i^0), (x_j^0, y_j^0)$ be the coardinates of v_i, v_j in t_0, with distance $L = \sqrt{(x_i^0 - x_j^0)^2 + (y_i^0 - y_j^0)^2}$ and v_i, v_j are neighbors with sector intersection S_i and S_j respectively.

$(x_i^0, y_i^0) \in S_i, (x_j^0, y_j^0) \in S_j$

S_i is a subset of $sector_i(v_j)$, within that sector the initial coradintaes (x_i^0, y_i^0) are the fathest from (x_j^0, y_j^0) in the sector (please note that the sector is defined by that distance).

So, for every $(x_i, y_i) \in sector_i(v_j)$: $L^i = \sqrt{(x_i - x_j^0)^2 + (y_i - y_j^0)^2} \leq L$

$S_i \subseteq sector_i(v_j) \Rightarrow \sqrt{(x_i - x_j^0)^2 + (y_i - y_j^0)^2} \leq L \ \forall x_i, y_i \in S_i$

$\Delta^i = L - L^i$,$\Delta^i \geq 0$

This happens respectively for v_j,and we will use $\Delta^j = L - L^j$ in the same notation it was previously defined with i.

Please note that $\Delta^j \leq \frac{L}{2}$ under the motion rule that was previously defined

Let $(x_i^1, y_i^1) \in S_i, (x_j^1, y_j^1) \in S_j$ be the coardinted that were chosen in t_0.

so the distance in $t_1 = L' = L - \Delta_i - \Delta_j \Rightarrow$

(1) $L' \leq L$.

(2) if Δ_i or Δ_j are larger ($>$) than 0 $\Rightarrow L' < L$

Claim 3: if the system is connected, there will always be an agent with improving step

Under the assumption that the system is connected, Let's assume that there is no agent with improving step.

we will define a group of functions:

$D = \{d_{ij}(t) \mid d_{ij}(t) = \begin{cases} \|V_i - V_j\|_2 & v_j \in N(v_i) \\ \infty & o.w \end{cases} \}$

$\exists t : \Sigma_{i=1}^{N} \Sigma_{j \in N(v_i)} d_{ij}(t) \leq \Sigma_{i=1}^{N} \Sigma_{j \in N(v_i)} d_{ij}(t+1)$

since losing neighbors may only decrease the right side of the equation value, we will assume that no connection is lost.

please note that if we will not use this assumption the right side of the inequality may only decrease due to the use os the sum of the neighbors only.

1. According to claim 2, every step that holds $\Delta > 0$ will decrease the distance. so the only way for that equation is to be true is if there is no movement is the system.
2. Hence, all of the agents have to stay put in order for the equation to hold
   (a) For $N < 3$ agents, the system can't stay put (when it's connected) because it either has one agent (hence gathered) or has two agents connected to each other with one sector for each, so the intersection (S) can't be empty so they have an improving step that will decrease the distances, So for $N < 3$ there is no t that can hold the equation under the assumption of connectivity.
   (b) for $N \geq 3$, the system will stay put if all of the agents are surrounded.
      i. if the connectivity graph is open, there exist at least one agent with 1 neighbor, then he is not surrounded and can move hence the equality can't hold.
      ii. if the connectivity graph is closed then there exist at least 3 agents that the angle between them is different than 180^o. from claim 1 we get that at least one agent can't be surrounded hence he can move.

$\Rightarrow$ there is no t for which the equation is true, there will always be an agent with improving step

If we won't assume that the system remains connected the inequality may be true.

for example, 2 agents that are not connected, in t and $t+1$ the left side of the equation will be equal zero, so does the right side of the equation.

Gathering in infinite time under the assumption that the connectivity remains

As shown in claim 2, as long as the connectivity remains, there will always be an improving step.

Also, please note the Probability function $P(d)$ is well defined and has a derivative in her domain.

We will define a group of helper functions:

$D = \{d_{ij}(t) \,|= \begin{cases} \|V_i - V_j\|_2 & v_j \in N(v_i) \\ \infty & o.w \end{cases}\}$

All of those functions are well defined (simple distance without being dependent on the probability

The Lyapunov function will be defined:

$L(t) = \Sigma_{i=1}^{N} \Sigma_{j \in N(v_i)}((1 - P(d_{ij}(t)))$

The Lyapunov is well defined as it is comprised of well-defined functions with complete derivatives.

We will derivate by the chain rule:

$\frac{\partial}{\partial t} L(t) = \Sigma_{i=1}^{N} \Sigma_{j \in N(v_i)} - \frac{\partial P}{\partial d_{ij}}(d_{ij}(t)) \cdot \frac{\partial d}{\partial t}(t)$

Please note that we use the notation ∂ instead of d only to avoid confusion with the functions d_{ij}.

1. according to claim 2, d_{ij} is decreasing hence $\frac{\partial d}{\partial t}(t) \leq 0$
2. For $[t, t']$, time which the distance d_{ij} did not decrease: $d_{ij}(t) = d_{ij}(t')$:
   $P(d_{ij}(t)) = P(d_{ij}(t')) \Rightarrow \frac{\partial P}{\partial d_{ij}}(d_{ij}(t)) = \frac{\partial P}{\partial d_{ij}}(d_{ij}(t')) = 0$
3. P is a strictly decreasing function (for larger distance it gives smaller chance) , hence $\frac{\partial P}{\partial d_{ij}} < 0$
   (a) $\frac{\partial d}{\partial t} \leq 0$
   (b) $\frac{\partial P}{\partial d_{ij}} < 0$

According to claim 2, those are the only considiration we might take,
hence $\forall i, j$ we get that $-\frac{\partial P}{\partial d_{ij}}(d_{ij}(t)) \cdot \frac{\partial d}{\partial t}(t) \leq 0$.
since $L(t)$ is the sum of the individual distance functions we get that $\frac{\partial L}{\partial t}(t) \leq 0$.

Now we left to proof that $\frac{\partial L}{\partial t}(t) < 0$ and L will reach zero.

According to claim 3, there is always some agent with improving step in time t (will note him as agent i)

so, $-\frac{\partial P}{\partial d_{ij}}(d_{ij}(t)) \cdot \frac{\partial d}{\partial t}(t) < 0$. even if all the other agents stay put, we still get $\frac{\partial L}{\partial t}(t) < 0$

that happens because if there is no change in the distance in time t for an agent the donation to $\frac{\partial L}{\partial t}(t)$ is zero (to the sum).

$\Rightarrow \frac{\partial L}{\partial t}(t) < 0$, and since the $N < \infty$ there exist t wich L will reach zero.

If the connectivity does not remain, we get that there exist t such that for all $t' > t$ $\frac{\partial L}{\partial t}(t) = 0$ so L will never reach zero

Lower bound for the gathering probability

Since the connectivity heavily depends on the system topology, it is not trivial to calculate the general probability of the connectivity of the system at any given moment. Here we suggest a lower bound for any connected state of the system, that (with Claim 3) will guarantee the in a probability higher than that, the graph will gather.

for some system configuration where the graph that represents the agent connection is connected.

the probability of v_i to lose all of his connections is noted by: $C(v_i) = \Pi_{v_j \in N(v_i)}(1 - P(d_{ij}))$ where $P(d_{ij})$ is the probability to hold the connection

let v^* be the agent with the minimal C function value, if there is more than one, choose randomly between them.

so for any agent v: $C(v) \leq C(v^*)$.

Hence, for each agent, the upper bound of the probability of losing all of his neighbors (and so does connectivity) is $C(v^*)$.

So, the probability to lose connectivity in the graph will be $\Sigma_{i=1}^{N} C(v_i) \leq \Sigma_{i=1}^{N} C(v^*) = N \cdot C(v^*)$.

Meaning that the probability of staying connected is $\geq$ then $N \cdot C(v^*)$

and from the previous claim we know that if the graph stays connected it will gather, and it will happen in a probability of at least $N \cdot C(v^*)$.

The lower bound of the gathering is therefore $N \cdot C(v^*)$.