EE 046202 - Technion - Unsupervised Learning & Data Analysis

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Tutorial 03 - Classical Methods in Statistical Inference - Hypothesis Testing 1

Agenda

In [1]:

```
# imports for the tutorial
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
%matplotlib notebook
```

Hypothesis Testing

Let's begin with an example: consider a radar system that uses radio waves to detect aircrafts. The system receives a signal, and based on the received signal, it needs to decide whether an aircraft is present or not. We formulate two hypotheses:

1. H_0: No aircraft is present.
2. H_1: An aircraft is present.

H_0 is called the **null hypothesis** (also: the *default hypothesis*) and H_1 is called the **alternative hypothesis**. We initially assume that H_0 is **true** and based on the observed data we need to decide whether or not to accept H_1 or reject it.

Hypothesis Testing Steps

Step 1

Null & Alternative Hypotheses - Formulate the null hypothesis $H_0: \theta \in \Theta_0$ (that the observations are the result of pure chance) and the alternative hypothesis $H_1: \theta \in \Theta_1$ (that the observations show a real effect combined with a component of chance variation).

Step 2

Test Statistic - Identify a test statistic that can be used to assess the truth of the null hypothesis. It is a value computed from sample data. The test statistic is used to assess the strength of evidence in support of a null hypothesis.

- A **statistic** is a real-valued function of the data. For example, the sample mean: $W(X_1, X_2, \dots, X_n) = \frac{X_1+X_2+\dots+X_n}{n}$ is a statistic.
- A **test satistic** is a statistic on we which we build our test.
- **Acceptence Region** A - A *set* $A \subset \mathbb{R}$ is defined to be the set of all possible values of the test statistic for which we accept H_0.
- **Rejection Region** R - A *set* $R = \mathbb{R} - A$ is defined to be the set of all possible values of the test statistic for which we reject H_0 and accept H_1.

Step 3

P-value & Interpretation - Compute the P-value, which is the probability that a test statistic, at least as significant as the one observed, would be obtained assuming that the null hypothesis were true. The smaller the P-value, the stronger the evidence **against** the null hypothesis.

Step 4

Significance Level - Compare the p-value to an acceptable significance value α (sometimes called an α value, a probability threshold below which the null hypothesis will be rejected. Common values are 5% and 1%.). If $p \leq \alpha$ (the observed effect is statistically significant), the null hypothesis is ruled out, and the alternative hypothesis is valid.

The Two Competing Theories

- **NULL** Hypothesis - H_0 - any observed deviation from what we expect to see is due to chance variability.
- **ALTERNATIVE** Hypothesis - H_a - 'claim', or a theory you wish to test (the reason for the observed statistic).

H_0 is assumed **true** until enough evidence goes against it (we then refute it and believe the alternative H_a).

Error Types

False Positive False Negative

- **Type I Error (False Positive)** - the incorrect rejection of a true null hypothesis. Usually a type I error leads one to conclude that a supposed effect or relationship exists when in fact it doesn't.
  - For example, a test that shows a patient to have a disease when in fact the patient does not have the disease, a fire alarm going on indicating a fire when in fact there is no fire, or an experiment indicating that a medical treatment should cure a disease when in fact it does not.
  - The chance of **rejecting the null hypthesis H_0, when it is TRUE**, denoted by α
  - $\rightarrow$ the chance of **accepting the null hypthesis H_0, when it is TRUE** is $1-\alpha$
  - Formally:
    - Denote a *test statistic* as W
    - $P(\text{Type 1 Error}|\theta) = P(\text{Reject } H_0|\theta) = P(W \in R|\theta), \theta \in \Theta_0$
      - If $P(\text{Type 1 Error}|\theta) \leq \alpha, \forall\theta \in \Theta_0$, we say that the test has **significance level** α.

- **Type II Error (False Negative)** - the failure to reject a false null hypothesis.
  - For example, a blood test failing to detect the disease it was designed to detect, in a patient who really has the disease; a fire breaking out and the fire alarm does not ring; or a clinical trial of a medical treatment failing to show that the treatment works when really it does.
  - The chance of **not rejecting the null hypothesis H_0, when it is FALSE**, denoted by β
  - $\rightarrow$ the chance of **rejecting the null hypthesis H_0, when it is FALSE** is $1-\beta$ (also called **power**)
  - Since the alternative hypothesis, H_1, usually contains more than one value of θ, the probability of type II error is usually a **function of** θ, and denoted by β.
  - Formally: $\beta(\theta) = P(\text{Accept } H_0|\theta),\ \text{for } \theta \in \Theta_1$

Example - Error Types

- Hypothesis: "A patient's symptoms improve after treatment A more rapidly than after a placebo treatment."
- Null hypothesis (H_0): "A patient's symptoms after treatment A are indistinguishable from a placebo."
- A Type I error would falsely indicate that treatment A is more effective than the placebo, whereas a Type II error would be a failure to demonstrate that treatment A is more effective than placebo even though it actually is more effective.

Example - Body Weight - Hypothesis Testing for the Mean

The following example will be used to demonstrate the statistic process:

In the 1970s, 20–29 year old men in the U.S. had a mean μ body weight of 170 pounds (77 kg). Standard deviation σ was 40 pounds (18 kg). We test whether mean body weight in the population is **bigger** now.

1- Null & Alternative Hypotheses

- Under the **null hypothesis** there is no difference in the mean body weight between then and now, in which case μ would still equal 170 pounds:
$$H_0 : \mu = 170$$
- Under the **alternative hypothesis**, the mean weight has increased:
$$H_a : \mu > 170$$
  - This statement of the alternative hypothesis is **one-sided**. That is, it looks only for values larger than stated under the null hypothesis.
  - There is another way to state the alternative hypothesis. We could state it in a **"two-sided"** manner, looking for values that are either higher- or lower-than expected. For the current illustrative example, the two-sided alternative is $H_a : \mu \neq 170$. Although for the current illustrative example, this seems unnecessary, two-sided alternative offers several advantages and are much more common in practice.

2- Test Statistic (TS)

- It is a measure of how far the observed data is from what is expected under the null hypothesis H_0
  - Compute the value of a test statistic (TS) from the data
- The particular TS computed depends on the tested parameter
  - For example, to test the population mean, the TS is the sample mean (or standardized sample mean), if the variance is known.
  - It is very similar to the process we did in point estimation, for choosing the correct estimator.
- The null hypothesis H_0 is rejected if the TS falls in a user-specified rejection region.
- Different hypothesis tests use different test statistics based on the probability model assumed in the null hypothesis. Common tests and their test statistics include:

Hypothesis Test	Test Statistic
Z-test	Z-statistic
t-tests	t-statistic
ANOVA	F-statistic
Chi-square tests	Chi-square statistic

Example - TS - Z-statistic

The Z-statistic has the standard normal distribution under the null hypothesis. It is a **mean** test when σ is known. We will use this statistic to test the problem:
$$z_{stat} = \frac{\overline{x} - \mu_0}{\sigma_{\overline{x}}}$$

Assumptions:

- μ_0 is the **population mean** assuming H_0 is **true**
- $\overline{x} = \frac{x_1+x_2+..+x_n}{n}$ is the sample mean.
- $\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}}$

Solve for the "Body Weight" problem:

- $\mu_0 = 170$
- $\sigma = 40$
- We'll take sample size of $n = 64$ samples, $\rightarrow \sqrt{n} = 8$
- $\sigma_{\overline{x}} = \frac{40}{8} = 5$
  - If the **variance is unknown** (the variance of X_i): we use the **sample standard deviation** S instead of σ (unbiased)
$$S = \sqrt{\frac{1}{n-1}\sum_{k=1}^{n}(X_k - \overline{X})^2} = \sqrt{\frac{1}{n-1}\left(\sum_{k=1}^{n} X_k^2 - n\overline{X}^2\right)}$$
  - Nore that you first estimate the variance, and then still devide by $\sqrt{n}$ to get the sample STD. [Read more (https://www.probabilitycourse.com/chapter8/8_4_3_hypothesis_testing_for_mean.php)](https://www.probabilitycourse.com/chapter8/8_4_3_hypothesis_testing_for_mean.php).

Now, let's assume we found a sample mean of **173**, then:

- $z_{stat} = \frac{\overline{x}-\mu_0}{\sigma_{\overline{x}}} = \frac{173-170}{5} = 0.6$

Now, let's assume we found a sample mean of **185**, then:

- $z_{stat} = \frac{\overline{x}-\mu_0}{\sigma_{\overline{x}}} = \frac{185-170}{5} = 3$

Reminder: The Central Limit Theorem (CLT)

- The CLT states that given a sufficiently large sample size from a population with a finite level of variance, the mean of all samples from the same population will be approximately equal to the mean of the population.
- When n is large, the distribution of the **sample means** will approach a normal distribution. More formally:

If $X_1, X_2, \ldots, X_n$ is a random sample of size n taken from a population with mean μ and variance σ^2, and if $\overline{X}$ is the sample mean, the limiting form of the distribution:

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

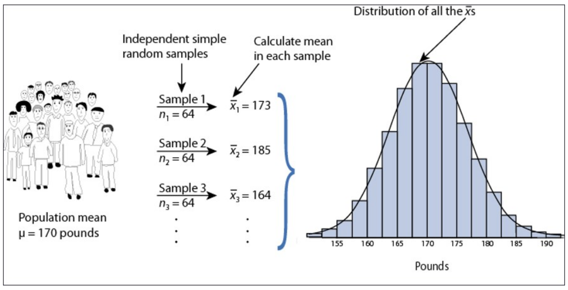
as $n \rightarrow \infty$, is the **standard normal distribution**

[CLT DEMO (http://onlinestatbook.com/stat_sim/sampling_dist/)](http://onlinestatbook.com/stat_sim/sampling_dist/)

Reasoning Behind z_{stat}

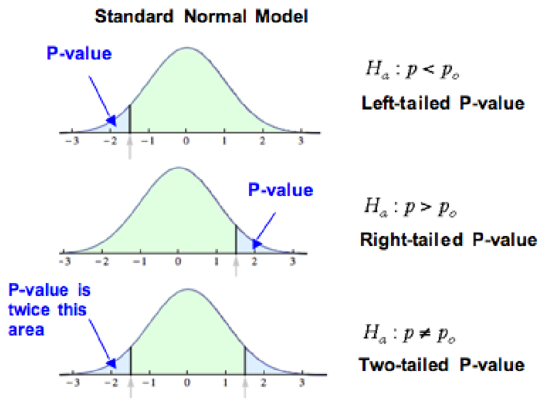
Sampling distribution of $\overline{x}$ under H_0:

$$\overline{x} \sim N(170, 5)$$



3 - P-value & Interpretation

- All hypothesis tests ultimately use a **p-value** to weigh the strength of the evidence (what the data are telling you about the population). The p-value is a number between 0 and 1, and is **the probability of the observed test statistic (or one more extreme) when H_0 is true**
  - **P-value** is the *lowest* significance level α that results in rejecting the null hypothesis.
- It corresponds to the **Area Under the Curve (AUC)** in the tail of the Standard Normal Distribution beyond the z_{stat}
- Converting Z-statistics to *P-value*:
$$For\ H_1: \mu > \mu_0 \rightarrow P = Pr(Z > z_{stat}) = right\text{-}tail\ beyond\ z_{stat}$$

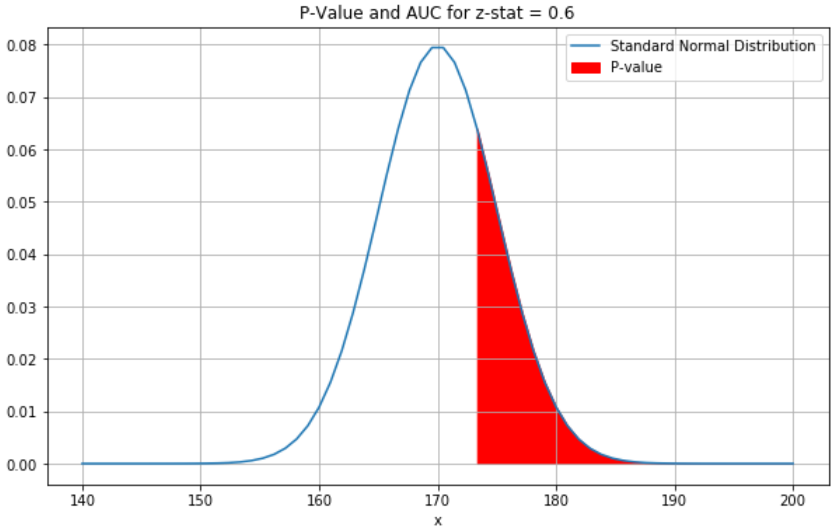


- [Image Source (https://courses.lumenlearning.com/wmopen-concepts-statistics/chapter/hypothesis-test-for-a-population-proportion-2-of-3/)](https://courses.lumenlearning.com/wmopen-concepts-statistics/chapter/hypothesis-test-for-a-population-proportion-2-of-3/)

In [3]:
```
# let's see for the body weight problem
x = np.linspace(140, 200, 64)
mu = 170  # H_0 is true!
sigma = 5  # calcualted for 64 samples
f_x = (1 / np.sqrt(2 * np.pi * sigma ** 2)) * np.exp(- (x - mu) ** 2 / (2 * sigma ** 2))
x_normed = (x - mu) / sigma
```

In [4]:
```
fig = plt.figure(figsize=(10,6))
ax = fig.add_subplot(1,1,1)
ax.plot(x, f_x, label='Standard Normal Distribution')
ax.fill_between(x[np.where(x.astype(int)==173)[0][0]:], y1=f_x[np.where(x.astype(int)==173)[0][0]:],
                color='red', label="P-value")
ax.grid()
ax.legend()
ax.set_xlabel('x')
ax.set_title('P-Value and AUC for z-stat = 0.6')
p_val = np.sum(f_x[np.where(x.astype(int)==173)[0][0]:])
print('p-val (AUC) = {:.3f}'.format(p_val))
```

```
p-val (AUC) = 0.298
```

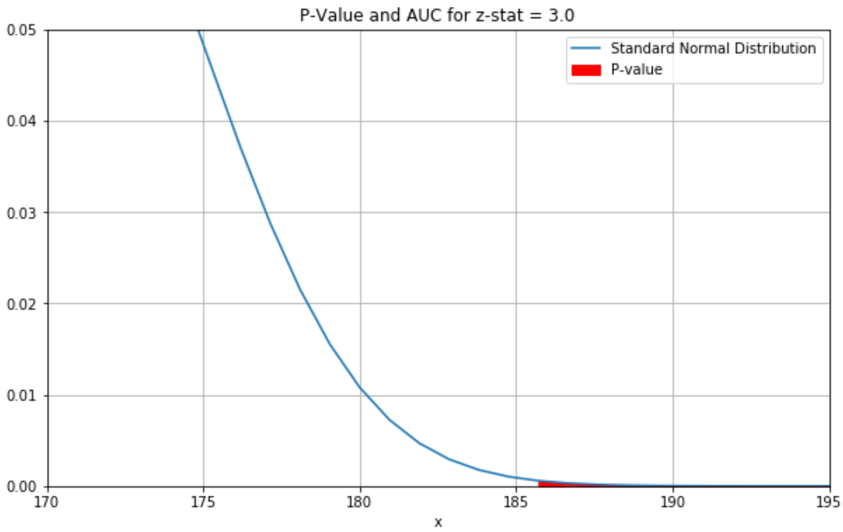


```
In [9]: # Let's see for the body weight problem
x = np.linspace(140, 200, 64)
mu = 170  # H_0 is true!
sigma = 5  # calcualted for 64 samples
f_x = (1 / np.sqrt(2 * np.pi * sigma ** 2)) * np.exp(- (x - mu) ** 2 / (2 * sigma ** 2))
x_normed = (x - mu) / sigma

def plot_p_auc():
    fig = plt.figure(figsize=(10,6))
    ax = fig.add_subplot(1,1,1)
    ax.plot(x, f_x, label='Standard Normal Distribution')
    ax.fill_between(x[np.where(x.astype(int)==185)[0][0]:], y1=f_x[np.where(x.astype(int)==185)[0][0]:],
                    color='red', label="P-value")
    ax.grid()
    ax.legend()
    ax.set_xlim([170, 195])
    ax.set_ylim([0, 0.05])
    ax.set_xlabel('x')
    ax.set_title('P-Value and AUC for z-stat = 3.0')
    p_val = np.sum(f_x[np.where(x.astype(int)==185)[0][0]:])
    print('p-val (AUC) = {:.3f}'.format(p_val))
```

```
In [10]: plot_p_auc()
```

```
p-val (AUC) = 0.001
```



Interpretation

- A small p-value (typically ≤ 0.05) indicates **strong evidence against the null hypothesis** H_0, so you reject the null hypothesis.
- A large p-value (> 0.05) indicates weak evidence against the null hypothesis, so you fail to reject the null hypothesis.
- p-values very close to the cutoff (0.05) are considered to be marginal (could go either way).

4- Significance Level (α)

- It is the degree of certainty required in order to **reject** the null hypothesis H_0.
- A test statistic, TS, with p-value **less** than some pre-determined false positive (or size) is said to be statistically significant at that level.
- Commonly used p-values:

P-Value	Wording
$p > 0.05$	Not Significant
$0.01 \leq p \leq 0.05$	Significant
$0.001 \leq p < 0.01$	Very Significant
$p < 0.001$	Extremely Significant

Formalization

Let's design a level α test to choose between:

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0$$

We initially assume H_0, thus $z_{stat} \sim \mathcal{N}(0,1)$. We will choose a threshold c. If $|z_{stat}| \leq c$, we **accept** H_0, and if $|z_{stat}| > c$, accept H_1.

To choose c:

$$P(|z_{stat}| > c|H_0) = \alpha$$

Since the standard normal PDF is **symetric** around 0, we have:

$$P(|z_{stat}| > c|H_0) = 2P(z_{stat} > c|H_0) \rightarrow P(z_{stat} > c|H_0) = \frac{\alpha}{2} \rightarrow c = z_{\frac{\alpha}{2}}$$

Therefore, we accept H_0 if

$$|\frac{\overline{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}| \leq z_{\frac{\alpha}{2}}$$

and reject it otherwise.

Relation to Confidence Intervals

Notice that saying we accept H_0 if

$$|\frac{\overline{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}| \leq z_{\frac{\alpha}{2}}$$

can be interpreted as the following **acceptance region** for μ_0:

$$\mu_0 \in [\overline{X} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}, \overline{X} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}]$$

Which is the $(1-\alpha)$ confidence interval for μ_0.

Exercise - Hypothesis Testing

We continue with the radar example. Recall that the system receives a signal and based upon that signal it decides whether an aircraft is present or not. We denote:

- X - the received signal (R.V., sampled)
- We suppose that:

$$X = W, \text{ if no aircraft is present}$$
$$X = 1 + W, \text{ if an aircraft is present}$$
$$W \sim \mathcal{N}(0, \sigma^2 = \frac{1}{9})$$

- We can write instead:

$$X = \theta + W$$

  where $\theta = 0$ if there is no aircraft and $\theta = 1$ otherwise.
- The **hypotheses**:
  - H_0 (null): No aircraft is present
  - H_1 (alternative): An aircraft is present

1. Write H_0 and H_1 in terms of possible values of θ.
2. Suggest a *simple* test statistic with level $\alpha = 0.05$ to decide between H_0 and H_1.
3. Find the probability of missing a present aircraft, that is, find β (the probability of type 2 error).
4. If we observe $X = 0.6$, is there enough evidence to reject H_0 at a *significance level* $\alpha = 0.01$?
5. For a probability less than 5% to miss a present aircraft, what is the **smallest** significance level that we can achieve?

- Reminder:

$$W \sim \mathcal{N}(\mu, \sigma^2) \rightarrow \frac{W - \mu}{\sigma} \sim \mathcal{N}(0,1)$$

Solution

Section 1 - Write H_0 and H_1 in terms of possible values of θ.

The **hypotheses**:

- H_0 (null): No aircraft is present: $\theta = 0$
- H_1 (alternative): An aircraft is present : $\theta = 1$

Section 2 - Suggest a *simple* test statistic with level $\alpha = 0.05$ to decide between H_0 and H_1.

- The *observed* data is a random variable X.
- Under H_0 we have $X \sim \mathcal{N}(0, \frac{1}{9})$ and under H_1 we have $X \sim \mathcal{N}(1, \frac{1}{9})$.
- We suggest the following test: set a threshold c. If the obsereved value of X is less than c, choose H_0 ($\theta = \mathbb{E}[X] = 0$), otherwise, choose H_1 ($\theta = \mathbb{E}[X] = 1$).
- To find the best c we use the required α, that is, we demand:
$$P(\text{type I error}) = P(\text{Reject } H_0|H_0) = P(X > c|H_0) = P(W > c) = P(\frac{Z}{3} > c) = 1 - \phi(3c) = \alpha$$
(the last transition is due to the fact that we assume H_0 and $X \sim \mathcal{N}(0, \frac{1}{9})$, which is not the standard distribution).
- It holds that $P(\text{type I error}) = \alpha$, thus we get:
$$c = \frac{1}{3}\phi^{-1}(1-\alpha) = \frac{1}{3}\phi^{-1}(1-0.05) = \frac{1}{3}\phi^{-1}(0.95) = 0.548$$

Section 3 - Find the probability of missing a present aircraft, that is, find β (the probability of type 2 error).

Note that the *alternative* hypothesis is simple, that is, it contains only one value ($\theta = 1$), so $\beta(\theta) = \beta$ and we can write:
$$\beta = P(\text{type II error}) = P(\text{Accept } H_0|H_1) = P(X < c|H_1) = P(1 + W < c) = P(W < c - 1) = \phi(3(c-1))$$
Since we found out that for the given α, $c = 0.548$ then $\beta = 0.088$.

Section 4 - If we observe $X = 0.6$, is there enough evidence to reject H_0 at a *significance level* $\alpha = 0.01$?

- For $\alpha = 0.01$ we get $c = \frac{1}{3}\phi^{-1}(1-\alpha) = \frac{1}{3}\phi^{-1}(1-0.01) = \frac{1}{3}\phi^{-1}(0.99) = 0.775$, which is **larger** than 0.6.
- Thus, we **cannot** reject H_0 at significance level $\alpha = 0.01$.

Section 5 - For a probability less than 5% to miss a present aircraft, what is the smallest significance level that we can achieve?

We want $\beta = 0.05$, and from (3) we deduce that $c = 1 + \frac{1}{3}\phi^{-1}(\beta) = 0.452$. Thus, we need $c \leq 0.452$ to obtain $\beta \leq 0.05)$. Let's calculate α:
$$P(\text{type I error}) = P(\text{Reject } H_0|H_0) = 1 - \phi(3c) = 0.0875$$
which means that the smallest significance level that we can achieve is 0.0875.

Hypothesis Testing for the Mean Summary

All expansions can be found HERE (https://www.probabilitycourse.com/chapter8/8_4_3_hypothesis_testing_for_mean.php).

- 2-sided hypothesis testing for the mean: $H_0 : \mu = \mu_0$, $H_1 : \mu \neq \mu_0$

Case	Test Statistic	Acceptance Region
$X_i \sim \mathcal{N}(\mu, \sigma)$, σ **known**	$W = \frac{\overline{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$	$\lvert W \rvert \leq z_{\frac{\alpha}{2}}$
n large, X_i non-normal	$W = \frac{\overline{X} - \mu_0}{\frac{S}{\sqrt{n}}}$	$\lvert W \rvert \leq z_{\frac{\alpha}{2}}$

- 1-sided hypothesis testing for the mean: $H_0 : \mu \leq \mu_0, H_1 : \mu > \mu_0$

Case	Test Statistic	Acceptance Region
$X_i \sim \mathcal{N}(\mu, \sigma)$, σ **known**	$W = \frac{\overline{X}-\mu_0}{\frac{\sigma}{\sqrt{n}}}$	$W \leq z_\alpha$
n large, X_i non-normal	$W = \frac{\overline{X}-\mu_0}{\frac{S}{\sqrt{n}}}$	$W \leq z_\alpha$

- The only difference is the *absolute* sign on W

Recommended Videos

Warning!

- These videos do not replace the lectures and tutorials.
- Please use these to get a better understanding of the material, and not as an alternative to the written material.

Video By Subject

- Hypothesis Testing - Hypothesis Testing Statistics Problems & Examples (https://www.youtube.com/watch?v=VK-rnA3-41c)
- p-Value - Understanding the p-value - Statistics Help (https://www.youtube.com/watch?v=eyknGvncKLw)
  - What is a P Value? What does it tell us? (https://www.youtube.com/watch?v=-MKT3yLDkqk)
- Test Statistics (t-stat is covered in the next tutorial):
  - Test Statistics: Crash Course Statistics (https://www.youtube.com/watch?v=QZ7kgmhdIwA)
  - Z-statistics vs. T-statistics (https://www.youtube.com/watch?v=5ABpqVSx33I)

Credits

- Examples, exercises and definitions from Introduction to Probability, Statistics and Random Processes (https://probabilitycourse.com/) - https://probabilitycourse.com (https://probabilitycourse.com)
- Icons from Icon8.com (https://icons8.com/) - https://icons8.com (https://icons8.com)
- Datasets from Kaggle (https://www.kaggle.com/) - https://www.kaggle.com/ (https://www.kaggle.com/)