EE 046202 - Technion - Unsupervised Learning & Data Analysis

- Formerly 046193

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Tutorial 06 - Dimensionality Reduction - Kernels & KPCA

Agenda

- Recap: PCA Algorithm
- Kernels
    - The Kernel Trick
- NLPCA- Nonlinear PCA
    - Kernel PCA

In [1]:
```
# imports for the tutorial
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
from sklearn.decomposition import PCA, KernelPCA
from sklearn.datasets import make_circles, make_moons
%matplotlib notebook
```

PCA Algorithm

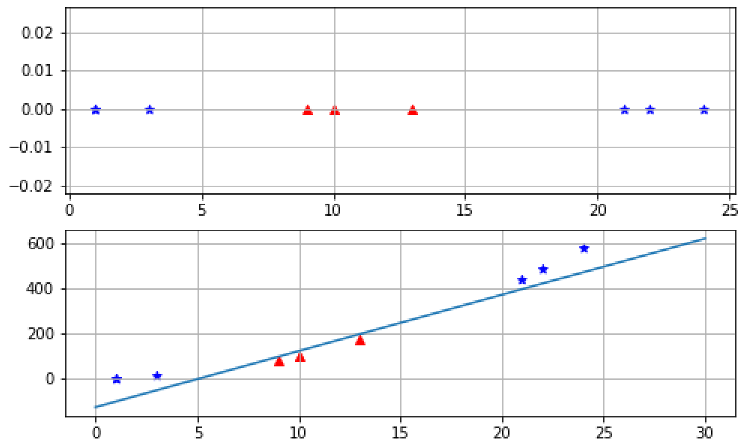
1. **Normalize/Standartize** (if we use features of different scales, we may get misleading components) and center the data. Given data $X \in \mathcal{R}^{m \times N}$, where m is the number of features and N is the number of samples, normalization:
$$\tilde{X} = X - \overline{X}$$
Standartization:
$$\tilde{X} = \frac{X - \overline{X}}{\overline{\sigma}_x}$$
Where $\overline{\sigma}_x$ is the empirical standard deviation (the square root of the emprical variance).
2. Calculate the covariance matrix X of data points:
$$P = \tilde{X}\tilde{X}^T \in \mathcal{R}^{m \times m}$$
3. Calculate eigenvectors and corresponding eigenvalues.
4. Sort the eigenvectors according to their eigenvalues in decreasing order.
5. Choose first k largest eigenvectors and that will be the new k dimensions.
6. Transform the original d dimensional data points into k dimensions.

Kernels

- The main shortcoming of PCA is that it is unable to capture nonlinear structures in the data.
- Consider the following example of linearly inseparatable 1-D set of exmaples and then extracting polynimial (second order) features:

In [2]:
```python
def plot_kernel_example():
    x_1 = np.random.randint(0,6, size=(3,))
    x_2 = np.random.randint(8,14, size=(3,))
    x_3 = np.random.randint(20,25, size=(3,))

    x_1_p = x_1 ** 2
    x_2_p = x_2 ** 2
    x_3_p = x_3 ** 2

    x_class = np.linspace(0, 30, 400)
    y_class = 25 * x_class - 130

    fig = plt.figure(figsize=(8,5))
    ax_1 = fig.add_subplot(211)
    ax_1.scatter(x_1, np.zeros_like(x_1), marker='*', color='b')
    ax_1.scatter(x_2, np.zeros_like(x_2), marker='^', color='r')
    ax_1.scatter(x_3, np.zeros_like(x_3), marker='*', color='b')
    ax_1.grid()
    ax_2 = fig.add_subplot(212)
    ax_2.scatter(x_1, x_1_p, marker='*', color='b')
    ax_2.scatter(x_2, x_2_p, marker='^', color='r')
    ax_2.scatter(x_3, x_3_p, marker='*', color='b')
    ax_2.plot(x_class, y_class)
    ax_2.grid()
```

In [3]:
```python
plot_kernel_example()
```



- Adding polynomial features is simple to implement and can work great with all sorts of ML algorithms.
- At a **low polynomial** degree it cannot deal with more complex datasets.
- At a **high polynomial** degree there are a lot of features, which makes the computation very slow.
  - Computation in the feature space can be costly because it is high dimensional (even go to infinity).
- **The Kernel Trick** comes to the rescue!
  - It makes it possible to get the same result as if you added many features (even in high dimension), **without actually adding them**!
    - So there is no computational disaster resulting from the large number of features.

The Kernel Trick

- When the **data points**, $\{x^{(i)}\}_{i=1}^{m}$, only appear as **inner (dot) products** - $(x^{(i)})^T x^{(j)}$, we can apply the kernel trick.
- As long as we can calculate the inner product in the **feature space**, we don't need the mapping explicitly.
  - The kernel function, denoted $K(x^{(i)}, x^{(j)}) = \phi(x^{(i)})^T \phi(x^{(j)})$, stands for the inner product in the feature space, and can easily plugged into the optimization.
  - If we could find a function $K(x^{(i)}, x^{(j)})$ that is simple enough, we can actually save us the huge number of calculations required to calculate $\phi(x^{(i)})^T \phi(x^{(j)})$, or even prevent us from the feature extraction itself, $\phi(x)$, this is the **kernel trick**.

- There are no restrictions on the form of $x^{(i)}, x^{(j)}$, but there are restrictions on the kernel function $K(x^{(i)}, x^{(j)})$
  - $K(x^{(i)}, x^{(j)})$ must satisfy the **Mercer Condition** - the $n \times n$ kernel matrix $K(x^{(i)}, x^{(j)})$ is **PSD**
  - This implies that the quadratic optimization is convex
    - It has a uniuqe solution and can be solved efficiently
- **Mercer's Theorem**: *Suppose that $k(x, y)$ is a symmetric real valued function such that $k(x, y) \leq C$ for some $C > 0$. Suppose the linear operator*

$$\mathcal{L}(f)(x) = \int_{\mathcal{R}^D} k(x, y) f(y) dy$$

is PSD, that is

$$\int \int_{\mathcal{R}^D} k(x, y) f(y) dy dx \geq 0$$

Let ψ_i be the normalized orthogonal eigenfunctions of $\mathcal{L}$, associated with eigenvalues $\lambda_i > 0$, sorted in non-decreasing order, and let m be the number of non-zero eigenvalues. Then, the sequence of eigenvalues is summable, namely $\sum_i \mid \lambda_i \mid < \infty$, and the kernel can be expressed as

$$k(x, y) = \sum_{i=1}^{m} \lambda_i \psi_i(x) \psi_i(y)$$

We thus see that by choosing $\phi_i(x) = \sqrt{\lambda_i} \psi_i(x)$, a feature based representation is obtained.

The Kernel Trick - Polynomial Kernel Example

- We will now demonstrate the kernel trick for a 2^{nd}-degree polynomial mapping
- Suppose the feature mapping (or extraction) is defined as follows:

$$\phi(u) = \phi(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}) = \begin{bmatrix} 1 \\ \sqrt{2}u_1 \\ \sqrt{2}u_2 \\ \sqrt{2}u_1u_2 \\ u_1^2 \\ u_2^2 \end{bmatrix}$$

- The inner product $\phi(u)^T \phi(v)$:

$$\phi(u)^T \phi(v) = 1 + 2u_1v_1 + 2u_2v_2 + 2u_1u_2v_1v_2 + u_1^2v_1^2 + u_2^2v_2^2 = (1 + u_1v_1 + u_2v_2)^2 = (1 + u^Tv)^2$$

  - **HOW ABOUT THAT?** - We just found out that we can calculate $\phi(u)^T \phi(v)$ just by calculating the input inner product u^Tv and plugging it in $(1 + u^Tv)^2$, thus saving us the whole feature mapping calculation process!
- We define **the 2^{nd} polynomial kernel**:

$$K(u, v) = (1 + u^Tv)^2$$

  - No need to calculate $\phi(u)^T \phi(v)$ explicitly!
  - The use of kernel function to avoid computing $\phi(\cdot)$ explicitly is known as **the kernel trick**

Other Kernel Functions

- Linear kernel:

$$K(u, v) = u^Tv$$

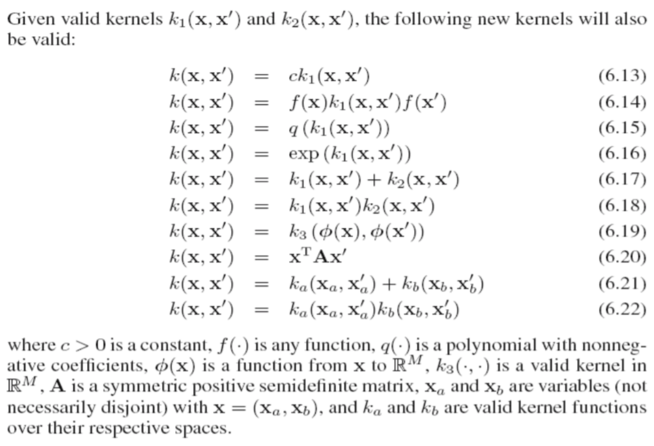
- Polynomial kernel with degree d:

$$K(u, v) = (\alpha u^Tv + \beta)^d$$

  - The feature space is all monomials up to degree d

- **Radial Basis Function** (Gaussian RBF) kernel with width σ:
$$K(u,v)=e^{-\frac{||u-v||^2}{2\sigma^2}}=e^{-\gamma||u-v||^2}=exp(-\gamma||u-v||^2)$$
    - It is also a type of a *similarity function* that measures *how far are two points?*
    - RBF covers the space with "balls" of a fixed radius, where the centers are the support vectors (in SVM).
    - The radius of the "balls" is determined by the parameter $\gamma=\frac{1}{2\sigma^2}$
        - A **smaller** γ means a larger radius, a lower "model complexity"
        - A **larger** γ means a smaller radius, a finer grain coverage which may possibly lead to overfitting.
    - The feature space is **infinite-dimensional** (hint for HW: expand the exponential to an infinite series using Taylor).
- Sigmoid kernel, with parameters κ and θ:
$$K(u,v)=tanh(\kappa u^Tv+\theta)$$
    - It does not satisfy the *Mercer* condition on all κ,θ

- Techniques for constructing new kernels (from *Pattern Recognition and Machine Learning*, C.Bishop):



Given valid kernels $k_1(\mathbf{x},\mathbf{x}')$ and $k_2(\mathbf{x},\mathbf{x}')$, the following new kernels will also be valid:

$$\begin{aligned}
k(\mathbf{x},\mathbf{x}') &= ck_1(\mathbf{x},\mathbf{x}') && (6.13)\\
k(\mathbf{x},\mathbf{x}') &= f(\mathbf{x})k_1(\mathbf{x},\mathbf{x}')f(\mathbf{x}') && (6.14)\\
k(\mathbf{x},\mathbf{x}') &= q\left(k_1(\mathbf{x},\mathbf{x}')\right) && (6.15)\\
k(\mathbf{x},\mathbf{x}') &= \exp\left(k_1(\mathbf{x},\mathbf{x}')\right) && (6.16)\\
k(\mathbf{x},\mathbf{x}') &= k_1(\mathbf{x},\mathbf{x}')+k_2(\mathbf{x},\mathbf{x}') && (6.17)\\
k(\mathbf{x},\mathbf{x}') &= k_1(\mathbf{x},\mathbf{x}')k_2(\mathbf{x},\mathbf{x}') && (6.18)\\
k(\mathbf{x},\mathbf{x}') &= k_3\left(\phi(\mathbf{x}),\phi(\mathbf{x}')\right) && (6.19)\\
k(\mathbf{x},\mathbf{x}') &= \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}' && (6.20)\\
k(\mathbf{x},\mathbf{x}') &= k_a(\mathbf{x}_a,\mathbf{x}'_a)+k_b(\mathbf{x}_b,\mathbf{x}'_b) && (6.21)\\
k(\mathbf{x},\mathbf{x}') &= k_a(\mathbf{x}_a,\mathbf{x}'_a)k_b(\mathbf{x}_b,\mathbf{x}'_b) && (6.22)
\end{aligned}$$

where $c>0$ is a constant, $f(\cdot)$ is any function, $q(\cdot)$ is a polynomial with nonnegative coefficients, $\phi(\mathbf{x})$ is a function from $\mathbf{x}$ to $\mathbb{R}^M$, $k_3(\cdot,\cdot)$ is a valid kernel in $\mathbb{R}^M$, $\mathbf{A}$ is a symmetric positive semidefinite matrix, $\mathbf{x}_a$ and $\mathbf{x}_b$ are variables (not necessarily disjoint) with $\mathbf{x}=(\mathbf{x}_a,\mathbf{x}_b)$, and k_a and k_b are valid kernel functions over their respective spaces.

Exercise - Valid Kernels

Let $k_1(x,y)$ and $k_2(x,y)$ be valid kernels according to Mercer's theorem (that is, they represent an inner product in some transformed space). Prove that the following functions are valid kernels as well:

1. Prove:
$$k(x,y)=k_1(x,y)+k_2(x,y)$$
2. Prove:
$$k(x,y)=f(x)k_1(x,y)f(y)\ \text{ where }\ f:\mathcal{R}^D\to\mathcal{R}$$

Solution

The basic trick here is using the initial knowledge that both kernels are valid according to Mercer's theorem.

Section 1

- Prove for:
$$k(x,y)=k_1(x,y)+k_2(x,y)$$

Using Mercer's theorem we know that there exists mappings $\phi_1(x)$ and $\phi_2(x)$ such that:
$$k_1(x,y)=\phi_1(x)^T\phi_1(y)$$
$$k_2(x,y)=\phi_2(x)^T\phi_2(y)$$
Thus,
$$\begin{aligned}k(x,y)&=k_1(x,y)+k_2(x,y)\\&=\phi_1(x)^T\phi_1(y)+\phi_2(x)^T\phi_2(y)\\&=[\phi_1(x),\phi_2(x)]^T[\phi_1(y),\phi_2(y)]=\Phi(x)^T\Phi(y)\end{aligned}$$

- $[\phi_1(x),\phi_2(x)]$ is a concatentanion of the vectors $\phi_1(x),\phi_2(x)$:
$$\begin{bmatrix}\phi_{11}(x) & \dots & \phi_{1n}(x) & \phi_{21}(x) & \dots & \phi_{2m}(x)\end{bmatrix}^T$$
- Thus, $k(x,y)$ represents an inner product in some transformed space.

Section 2

- Prove for:

$$k(x, y) = f(x)k_1(x, y)f(y) \text{ where } f : \mathcal{R}^D \to \mathcal{R}$$

Again, using Mercer's theorem:

$$\begin{aligned} k(x, y) &= f(x)k_1(x, y)f(y) \\ &= f(x)\phi_1(x)^T\phi_1(y)f(y) \\ &= \Phi(x)^T\Phi(y) \end{aligned}$$

where $\Phi(x) = f(x)\phi_1(x)$.

- Recall that $\phi_1(x)$ is a vector and $f(x)$ is a **scalar**.
- Thus, $k(x, y)$ represents an inner product in some transformed space.

Nonlinear PCA - NLPCA (& Kernel PCA)

- The main shortcoming of PCA is that it is **unable to capture nonlinear structures in the data**.
- The basic idea behind Kernel PCA is that by embedding the data into a high-dimensional space for which linear sub-spaces capture the relavant structures.
- Mathematically:

$$x \in \mathcal{R}^D \xrightarrow{\text{nonlinear}} \phi(x) \in \mathcal{R}^m \xrightarrow{\text{linear}} y \in \mathcal{R}^d, m \geq D > d$$

  (and usually $m >> D$)

- The basic steps of Nonlinear PCA:
  - Using the mapping $\phi(x)$, we map each data point $x_i \to \phi(x_i),\ i = 1, \dots, n$.
  - As in regular PCA, we center (or normalize/standartize) the data. We define:

$$\Phi = \left(\phi(x_1) - \overline{\phi}, \dots, \phi(x_n) - \overline{\phi}\right) \in \mathcal{R}^{m \times n}$$
$$\overline{\phi} = \frac{1}{n}\sum_{j=1}^{n}\phi(x_j)$$

  - Define the sample covariance matrix:

$$\hat{P}_{\phi(x)} = \frac{1}{n}\sum_{j=1}^{n}\left(\phi(x_j) - \overline{\phi}\right)\left(\phi(x_j) - \overline{\phi}\right)^T = \frac{1}{n}\Phi\Phi^T \in \mathcal{R}^{m \times m}$$

  - Perform the *eigen-decomposition* of the sample covariance matrix:

$$\hat{P}_{\phi(x)}u_i = \lambda_i u_i,\ i = 1, \dots, m$$

    - The d *nonlinear* Principal Components (PCs):

$$y_i = u_i^T\left(\phi(x) - \overline{\phi}\right) \in \mathcal{R},\ u_i \in \mathcal{R}^m,\ i = 1, \dots, d$$

- Compuational cost:
  - Computing the eigenvectors of a $m \times m$ matrix requires (in general) $O(m^3)$ steps - very expensive for large m!
- **Note** - the features $\phi(x_i)$ need to be centered even if the x_i are centered, since the mapping ϕ is **nonlinear**.
- Reducing the computational cost using the **Transpose Trick**:
  - The dimension m of the feature space may be very large which may be very expensive computationally.
  - Using the *transpose trick* which is based on the relation between the eigen-decomposition of $\Phi\Phi^T$ and $\Phi^T\Phi$, we can reduce the cost.
    - Recall the the **eigenvalues** of $\Phi\Phi^T$ and $\Phi^T\Phi$ are equal.
  - Doing the same derivation as before (look up...) we get that if w is an eigen-vector of $\Phi^T\Phi$ and v is an eigen-vector of $\Phi\Phi^T$ then:

$$u = \lambda^{-1}\Phi w$$

    - It is much cheaper to compute as the number of samples $n << m$.
  - The d *nonlinear* Principal Components (PCs):

$$y_i = u_i^T\left(\phi(x) - \overline{\phi}\right) = \lambda_i^{-1}w_i^T\Phi^T\left(\phi(x) - \overline{\phi}\right) \in \mathcal{R}, u_i \in \mathcal{R}^m,\ i = 1, \dots, d$$

Kernel PCA

- Notice that the nonlinear PCs only require the computation of *inner products* between the features ϕ. This indicates that we can use **the kernel trick** to calculate these inner products.
- We define the kernel and *centered kernel functions*:

$$k(x,y)=\phi(x)^T\phi(y)$$
$$\tilde{k}(x,y)=\left(\phi(x)-\bar{\phi}\right)^T\left(\phi(y)-\bar{\phi}\right)$$

  - Note that:

$$\tilde{k}(x,y)=k(x,y)-\frac{1}{n}\sum_{j=1}^{n}k(x,x_j)-\frac{1}{n}\sum_{i=1}^{n}k(x_i,y)+\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}k(x_i,y_j)$$

  - Define the kernel and centered *kernel matrices*, which are defined w.r.t. the data points:

$$K=\left[k_{ij}\right],\ \ k_{ij}=k(x_i,x_j)$$
$$\tilde{K}=\Phi^T\Phi=\left[\tilde{k}_{ij}\right],\ \ \tilde{k}_{ij}=\tilde{k}(x_i,x_j)$$

Claim

The nonlinear PCs can be computed directly in terms of the matrix $\tilde{K}$.

- It can be shown that:

$$\tilde{K}=\left(I-\frac{1}{n}11^T\right)K\left(I-\frac{1}{n}11^T\right)=JKJ$$

  - $1=(1,1,\dots,1)^T\in\mathcal{R}^n$

- We also define the vectors:

$$k_x=[k(x_1,x),\dots,k(x_n,x)]^T$$
$$\tilde{k_x}=\Phi^T(\phi(x)-\bar{\phi})=[\tilde{k}(x_1,x),\dots,\tilde{k}(x_n,x)]^T$$

- The nonlinear PCs can then be computed as:

$$y_i=w_i^T\Phi^T\left(\phi(x)-\bar{\phi}\right)=w_i^T\tilde{k}_x,\ \ i=1,2,\dots,d$$

  - w_i are the **eignevectors** of $\tilde{K}$:

$$\tilde{K}w_i=\lambda_i w_i$$
$$||w_i||^2=\lambda_i^{-1},\ \ [w_1,\dots,w_d]=V_d\Lambda_d^{-\frac{1}{2}}$$

  - V_d and Λ_d are obtained from the top-d eigenvectors of $\tilde{K}$'s eigen-decomposition:

$$\tilde{K}=V_{\tilde{K}}\Lambda_{\tilde{K}}V_{\tilde{K}}^T$$

  - Concatenating the components y_i we get (the projection):

$$Y=(y_1,y_2,\dots,y_d)^T=\Lambda_d^{-\frac{1}{2}}V_d^T\tilde{K}$$

    - Notice that $\Lambda_d^{-\frac{1}{2}}$ is the normaliztion factor (recall that Λ_d is a diagonal matrix with the eigenvalues on the diagonal).

- The low-dimensional nonlinear representation is obtained from the top-d eigenvectors of the matrix $\tilde{K}$.
  - The problem has been solved in the high-dimensional feature space ϕ while not explicitly computing ϕ or even working in $\mathcal{R}^m$

NLPCA Algorithm (Kernel PCA - KPCA)

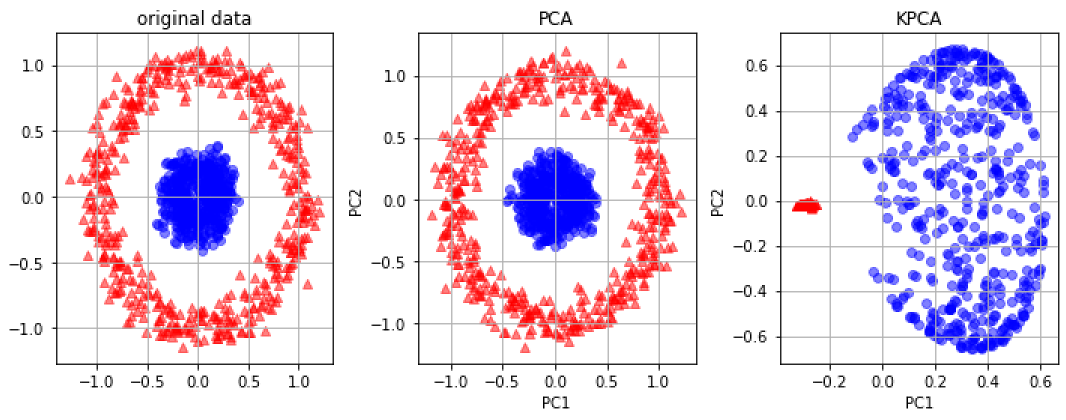
- **Input**: A set of points $\mathcal{X}=\{x_1,x_2,\dots,x_N\}\subset\mathcal{R}^D$, and a map $\phi:\mathcal{R}^D\to\mathcal{R}^M$ **or** a symmetric positive definite kernel function $k:\mathcal{R}^D\times\mathcal{R}^D\to\mathcal{R}$

1. Compute $\overline{\phi} = \frac{1}{N}\sum_j \phi(x_j)$ and the centered embedded data matrix Φ **or** the kernel matrix K.
2. Compute the centered kernel *matrix*:
$$\tilde{\mathcal{K}} = \Phi^T\Phi \text{ or } \tilde{\mathcal{K}} = \left(I - \frac{1}{n}11^T\right)K\left(I - \frac{1}{n}11^T\right) = JKJ \text{ or } \left(\tilde{k}(x_i, x_j)\right) \in \mathcal{R}^{N\times N}$$
3. Compute the eigenvectors $w_i \in \mathcal{R}^N$ of $\tilde{\mathcal{K}}$:
$$\tilde{\mathcal{K}}w_i = \lambda_i w_i$$
and normalize so that $||w_i||^2 = \lambda_i^{-1}$ (can be done as in step 4).
   - For every data point x, its i^{th} nonlinear principal component is given by:
$$y_i = w_i^T\Phi^T(\phi(x) - \overline{\phi}) \text{ or } w_i^T[\tilde{k}(x_1, x), \dots, \tilde{k}(x_N, x)]^T$$
   for $i = 1, 2, \dots, d$
4. The projection is given by:
$$Y = (y_1, y_2, \dots, y_d)^T = \Lambda_d^{-\frac{1}{2}}V_d^T\tilde{K}$$

- **Output**: A set of points $\{y_j\}_{j=1}^N$ lying in $\mathcal{R}^d$, where y_{ij} in the i^{th} nonlinear principal component of x_j for $i = 1, \dots, d$ and $j = 1, \dots, N$

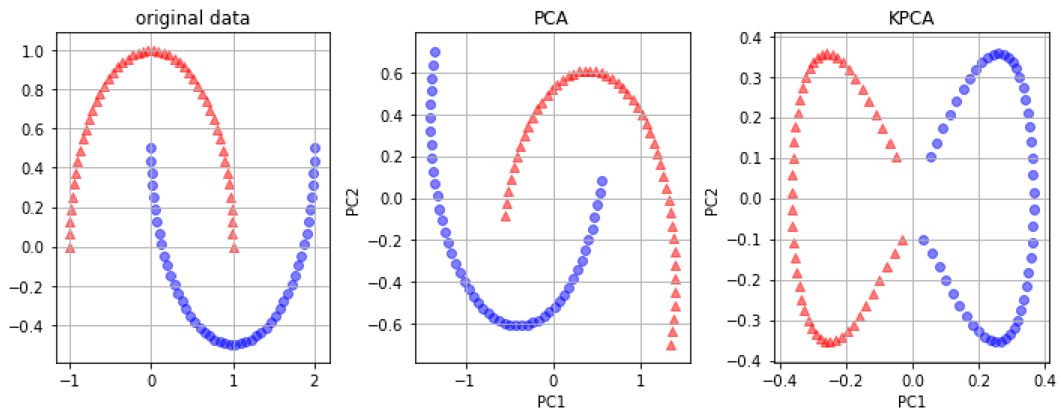
In [4]:
```python
def plot_circles_example():
    # example - circles
    X, y = make_circles(n_samples=1000, random_state=123, noise=0.1, factor=0.2)
    pca = PCA(n_components=2)
    kpca = KernelPCA(n_components=2, kernel='rbf', gamma=15)  # Radial Basis Function kernel
    X_pca = pca.fit_transform(X)
    X_kpca = kpca.fit_transform(X)

    # plot
    fig = plt.figure(figsize=(10,4))

    # original data
    ax1 = fig.add_subplot(1,3,1)
    ax1.scatter(X[y==0, 0], X[y==0, 1], color='red', marker='^', alpha=0.5)
    ax1.scatter(X[y==1, 0], X[y==1, 1], color='blue', marker='o', alpha=0.5)
    ax1.grid()
    ax1.set_title("original data")

    # pca
    ax2 = fig.add_subplot(1,3,2)
    ax2.scatter(X_pca[y==0, 0], X_pca[y==0, 1], color='red', marker='^', alpha=0.5)
    ax2.scatter(X_pca[y==1, 0], X_pca[y==1, 1], color='blue', marker='o', alpha=0.5)
    ax2.grid()
    ax2.set_title("PCA")
    ax2.set_xlabel("PC1")
    ax2.set_ylabel("PC2")

    # kpca
    ax3 = fig.add_subplot(1,3,3)
    ax3.scatter(X_kpca[y==0, 0], X_kpca[y==0, 1], color='red', marker='^', alpha=0.5)
    ax3.scatter(X_kpca[y==1, 0], X_kpca[y==1, 1], color='blue', marker='o', alpha=0.5)
    ax3.grid()
    ax3.set_title("KPCA")
    ax3.set_xlabel("PC1")
    ax3.set_ylabel("PC2")

    plt.tight_layout()
```

In [5]:
```
plot_circles_example()
```



In [6]:
```
def plot_moons_example():
    # example - moons
    X, y = make_moons(n_samples=100, random_state=123)
    pca = PCA(n_components=2)
    kpca = KernelPCA(n_components=2, kernel='rbf', gamma=15)  # Radial Basis Function kernel
    X_pca = pca.fit_transform(X)
    X_kpca = kpca.fit_transform(X)

    # plot
    fig = plt.figure(figsize=(10,4))

    # original data
    ax1 = fig.add_subplot(1,3,1)
    ax1.scatter(X[y==0, 0], X[y==0, 1], color='red', marker='^', alpha=0.5)
    ax1.scatter(X[y==1, 0], X[y==1, 1], color='blue', marker='o', alpha=0.5)
    ax1.grid()
    ax1.set_title("original data")

    # pca
    ax2 = fig.add_subplot(1,3,2)
    ax2.scatter(X_pca[y==0, 0], X_pca[y==0, 1], color='red', marker='^', alpha=0.5)
    ax2.scatter(X_pca[y==1, 0], X_pca[y==1, 1], color='blue', marker='o', alpha=0.5)
    ax2.grid()
    ax2.set_title("PCA")
    ax2.set_xlabel("PC1")
    ax2.set_ylabel("PC2")

    # kpca
    ax3 = fig.add_subplot(1,3,3)
    ax3.scatter(X_kpca[y==0, 0], X_kpca[y==0, 1], color='red', marker='^', alpha=0.5)
    ax3.scatter(X_kpca[y==1, 0], X_kpca[y==1, 1], color='blue', marker='o', alpha=0.5)
    ax3.grid()
    ax3.set_title("KPCA")
    ax3.set_xlabel("PC1")
    ax3.set_ylabel("PC2")

    plt.tight_layout()
```

In [7]:
```
plot_moons_example()
```



Recommended Videos

Warning!

- These videos do not replace the lectures and tutorials.
- Please use these to get a better understanding of the material, and not as an alternative to the written material.

Video By Subject

- The Kernel Trick - Machine Learning Fundamentals - The Kernel Trick (https://www.youtube.com/watch?v=JiM_LXpAtLc)
- Kernel PCA - David Thompson: Nonlinear Dimensionality Reduction: KPCA (https://www.youtube.com/watch?v=HbDHohXPLnU)

Credits

- Icons from Icon8.com (https://icons8.com/) - https://icons8.com (https://icons8.com)
- Datasets from Kaggle (https://www.kaggle.com/) - https://www.kaggle.com/ (https://www.kaggle.com/)