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Analysis and Simulation of Mechanical
Systems with Multiple Frictional
Contacts

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**Analysis and Simulation Of Mechanical Systems
With Multiple Frictional Contacts**

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1 Introduction

1.1 General

In many engineering applications such as assembly of mechanical components, gripping, fixturing and part feeding, there are situations in which a rigid body is subject to multiple frictional contacts with other bodies. It is proposed to develop a systematic method for the analysis and simulation of such systems.

1.2 Literature

In most of the previous work on dynamic analysis and simulation of mechanical systems, the emphasis is on the dynamics of linkages which, for the most part, are characterized by bilateral, holonomic constraints [24,7]. But there is much less literature about systems in which there are multiple contacts between rigid bodies.

There are at least two key factors that make this a challenging research problem. Firstly, since the mechanical system is characterized by unilateral, frictional constraints, the topology of the system varies with time. That is, each time when a contact is formed or broken, or when a rolling contact changes to a sliding contact, the mobility of the mechanical system and the structure of the differential equations that characterize the system change. Secondly, empirical frictional laws, such as Coulomb's law, engender inconsistencies and ambiguities when they are used in conjunction with the equations of rigid body mechanics. This is shown in a later section.

It is convenient to use rigid body model instead of a more elaborate and perhaps more accurate elastic or elastic-plastic model. This is because the governing equations for rigid body models are

simpler -- they are described by a system of ordinary differential equations.

When solving the ordinary differential equations for mechanical systems with constraints, there are methods in analytical mechanics to reduce the dimension of the system of ordinary differential equations to the degree of freedom of the mechanical system [24]. For problems involving unilateral contacts, it is necessary to reformulate the system of equations each time a new contact is established or an old contact disappears. This is quite cumbersome if the number of contacts is large. For such mechanical systems, the Lagrange multipliers approach is much more flexible and efficient. Furthermore, the multipliers are directly proportional to the contact and friction forces. Lotstedt [18] presented a method to find the normal forces and the friction forces by reducing the problem to a quadratic programming problem.

Lotstedt [17,18] developed a computer algorithm for simulation of systems with unilateral constraints. However, impacts were modeled using Newton's law for direct central impacts which states the velocity of separation after impact is a fraction of the velocity of approach before the impact. Wang and Mason [28] and Gilmore [6] presented an approach derived from Routh's graphical technique and Possion's impulse hypothesis which raise one solution from zero. However, the bases for the impulse hypothesis and Newton's law are unclear and the solution can be shown to violate the energy conservation law for some special cases [27].

Inconsistencies and ambiguities in rigid bodies analysis in problems with friction are well known [3,17,20]. There are situations

in which no feasible solution for contact forces exist, and others in which multiple solutions for system accelerations exist.

The use of non-rigid or compliant contact models provide an alternative method for solving the problem with inconsistencies. Furthermore, such models also resolve the static indeterminacy that is inherent in such systems. Goldsmith [5] models the contact force by a parallel linear spring-damper element called the Kelvin-Voigt model. The coefficients of stiffness and damping have been assumed to be known parameters. Khulief and Shabana [11] applied the Kelvin-Voigt model to multibody systems. They compensated for the existence of and changes in the joint forces and determined the material compliance and the damping coefficient from energy relations. Hunt and Grossley [8] showed that the linear spring-damper model does not represent the physical nature of energy transfer process. In their study, an estimate of the dissipated energy during impact was obtained by a damping force which was function of the elastic penetration between the colliding surfaces. The analysis was confined to two free bodies impacting at fairly low velocities. Lankarani and Nikravesh [15] presented a continuous analysis method for two spheres having a direct-central impact based on the Hertz contact model. A hysteresis damping function is used for this model. The parameters in this model and the effective coefficient of restitution are determined based on the geometric and material properties. This model is restricted to one dimensional impact of two bodies and no tangential impulses are analyzed. Sinha and Abel [26] analyzed robotic grasping by modeling each contact with a finite element grid. This allows them to implement nonlinear models for

friction and deformations. But they restrict the analysis to quasi-static problems. The mechanics of contact and the relevant principles of continuum mechanics are discussed in [9]. Applications to the mechanics of rail-wheel interaction are investigated by Paul and coworkers [16,23].

Recent work on the mechanics of dry friction is also important in this research. Oden [22] propose a new friction contact model by considering the nonlinear and nonlocal frictional model instead of Coulomb's empirical model.

1.3 Objectives

It is proposed to analyze and simulate mechanical systems with multiple frictional contacts. In this work, three important subproblems will be studied:

- (1) A method for the simulation of mechanical systems with changing topologies will be developed.
- (2) Impact hypotheses for rigid body models will be analyzed. The problems with inconsistencies and ambiguities will be studied carefully. The objective is to develop a model for impacts with friction which is free from inconsistencies. That is, energy conservation should not be violated and unique solution should exist for all feasible states and inputs.
- (3) Compliant contact models will be analyzed and compared with approximate rigid body models.
- (4) Visualization via computer animation on a three-dimensional graphical display will be investigated. And applications in robotics will be demonstrated.

1.4 Organization

This paper will be organized in the following manner. The problem, a brief introduction and the scope of the paper are presented in chapter 1. Chapter 2 is an overview of the problem formulation for the dynamic system and modeling of collisions. Several impact hypotheses are discussed in chapter 3, while an approach to modeling contact compliance is described in chapter 4. Finally the impact of the proposed work and possible avenues for future research are described in chapter 5.

2 Problem formulation

2.1 Mathematical modeling

The equations of motion for a mechanical system with rigid bodies in a d -dimensional space are a set of nonlinear coupled differential equations:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{c}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{f} + \mathbf{G}\lambda \quad (2.1)$$

where \mathbf{q} is the vector of n generalized coordinates for the system, $\mathbf{M}(\mathbf{q})$ is a symmetric, positive definite $n \times n$ inertia matrix with masses and moments of inertia of the different bodies, \mathbf{c} is a $n \times 1$ vector including inertial forces that are nonlinear function of velocities, \mathbf{f} is a $n \times 1$ vector of external forces, λ is the $k \times 1$ vector of multipliers or constraint forces, and \mathbf{G} is a $n \times k$ Jacobian matrix whose columns represent the directions of the k constraints.

Since simulation of bilaterally and holonomically constrained dynamic systems has been extensively studied [24], this is not discussed any further here. However, we do allow for m unilateral constraints, $\Phi(\mathbf{q}) \geq 0$, and l non-integrable, nonholonomic constraints, $\Psi(\mathbf{q}) \dot{\mathbf{q}} \geq 0$. Finally, we restrict the treatment here to planar systems.

Consider a system with m unilateral constraints $\Phi(\mathbf{q}) \geq 0$ and let k of the the m constraints be active. The Jacobian matrix is determined by the k active constraints:

$$\mathbf{G} = [\mathbf{g}_1 \ \mathbf{g}_2 \ \dots \ \mathbf{g}_k]$$

where $\mathbf{g}_i = \begin{bmatrix} \frac{\partial \phi_i}{\partial q_1} & \frac{\partial \phi_i}{\partial q_2} & \dots & \frac{\partial \phi_i}{\partial q_k} \end{bmatrix}^T$

In equation (2.1), the state of the system and external forces are given. The objective is to find $\ddot{\mathbf{q}}$ and if possible, λ . It is convenient to lump the

known vectors, \mathbf{c} and \mathbf{f} , into a single vector \mathbf{b} . Thus equation (2.1) can be rewritten in the form:

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{b} + \mathbf{G}\lambda, \lambda \geq 0 \quad (2.2)$$

If a constraint remains active for a certain time interval, it can be treated as a holonomic, bilateral constraint in that interval. That is, if $\phi_i = 0$, by differentiating it twice with respect to time, we obtain:

$$\zeta_i = \mathbf{g}_i^T \dot{\mathbf{q}} = 0 \text{ and } \eta_i = \mathbf{g}_i^T \ddot{\mathbf{q}} + \dot{\mathbf{g}}_i^T \dot{\mathbf{q}} = 0 \quad (2.3)$$

Thus, if all the k constraints remain active through a finite time interval, we get

$$\boldsymbol{\eta} = [\eta_1 \ \eta_2 \ \dots \ \eta_m]^T = \mathbf{G}^T \ddot{\mathbf{q}} + \dot{\mathbf{G}}^T \dot{\mathbf{q}} = 0 \text{ or } \mathbf{G}^T (\mathbf{M}^{-1}[\mathbf{b} + \mathbf{G}\lambda]) + \dot{\mathbf{G}}^T \dot{\mathbf{q}} = 0 \quad (2.4)$$

If the constraints are linearly independent, the constraint force vector, λ , can be obtained from equation (2.4) and $\ddot{\mathbf{q}}$ can be obtained by substituting for λ in (2.2). If they are linearly dependent, $\mathbf{G}\lambda$ can still be obtained by solving the k dependent equations in (2.4) [17], but λ can not be determined uniquely. In the latter case, the system is statically indeterminate. Nevertheless, it is possible to substitute for $\mathbf{G}\lambda$ in (2.2) and solve for $\ddot{\mathbf{q}}$.

2.1.1 System with changing topologies

Active unilateral constraint becomes inactive

Consider a previously active constraint ϕ_i which becomes inactive at time t_n , so that the corresponding constraint force λ_i vanishes. Assume that the constraint is active at t_{n-1} . Since $\phi_i > 0$ at t_n , there must be small intervals Δt and $\Delta t'$ ($0 < \Delta t < \Delta t'$) such that

$$\zeta_i = 0 \text{ for } t_{n-1} < t \leq t_n - \Delta t, \ \zeta_i > 0 \text{ for } t_n - \Delta t < t \leq t_n$$

$$\eta_i = 0 \text{ for } t_{n-1} < t \leq t_n - \Delta t', \ \eta_i > 0 \text{ for } t_n - \Delta t' < t \leq t_n$$

For all the active constraints, the normal displacements, velocities and accelerations together with the constraint forces satisfy the complementarity relations:

$$\lambda_i \phi_i = 0, \lambda_i \geq 0, \phi_i \geq 0, \text{ for } 0 \leq t < \infty \quad (2.5a)$$

$$\lambda_i \zeta_i = 0, \lambda_i \geq 0, \zeta_i \geq 0, \text{ for } t_{n-1} \leq t < t_n - \Delta t \quad (2.5b)$$

$$\lambda_i \eta_i = 0, \lambda_i \geq 0, \eta_i \geq 0, \text{ for } t_{n-1} \leq t < t_n - \Delta t' \quad (2.5c)$$

In a finite time interval ($t_{n-1} \leq t < t_n$), if no passive constraints become active, but we allow for the possibility of active constraints becoming passive at t_n , equation (2.5c) can be written for each k active constraints, so that:

$$\lambda^T \eta = 0, \lambda \geq 0, \eta \geq 0, \text{ for } t_{n-1} \leq t \leq t_n - \Delta t'$$

Now, equations (2.2) and (2.4) will be:

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1} [\mathbf{b} + \mathbf{G}\lambda]$$

$$\mathbf{G}^T (\mathbf{M}^{-1} [\mathbf{b} + \mathbf{G}\lambda]) + \dot{\mathbf{G}}^T \dot{\mathbf{q}} = \eta \geq 0$$

$$\text{where } \lambda^T \eta = 0, \lambda \geq 0$$

This is a linear complementarity problem in λ and η . It is equivalent to the following quadratic programming problem (QP) [21]:

$$\min_{\lambda \geq 0} \frac{1}{2} \lambda^T \mathbf{G}^T \mathbf{M}^{-1} \mathbf{G} \lambda + \lambda^T (\mathbf{G}^T \mathbf{M}^{-1} \mathbf{b} + \dot{\mathbf{G}}^T \dot{\mathbf{q}}) \quad (2.6)$$

The algorithms used to solve the QP problem will be discussed in next section.

Inactive unilateral constraint becomes active

If a constraint is inactive at t_{n-1} , that is, $\phi_i > 0$, but at $t = t_n$, $\phi_i = 0$. In this case, there is an impact and therefore a discontinuity in the velocities. Let $t^- < t_n < t^+$ and let $\dot{\mathbf{q}}(t^+) = \dot{\mathbf{q}}^+$ and $\dot{\mathbf{q}}(t^-) = \dot{\mathbf{q}}^-$. Let i be the constraint that is added, and let \mathbf{g}_i be the constraint direction and λ_i be the contact force. The other $(k-1)$ constraints are described by the Jacobian \mathbf{G}^* and multipliers λ^* so that

$$\mathbf{G} = [\mathbf{G}^* \mathbf{g}_i], \lambda = [\lambda^* \lambda_i]$$

Integration of equation (2.2) proceeds as follows:

$$\begin{aligned}\dot{\mathbf{q}}^+ &= \dot{\mathbf{q}}^- + \mathbf{M}^{-1} \int \mathbf{g}_i \lambda_i dt + \mathbf{M}^{-1} \int \mathbf{G}^* \lambda^* dt \\ &= \dot{\mathbf{q}}^- + \mathbf{M}^{-1} \mathbf{g}_i \Lambda_i + \mathbf{M}^{-1} \mathbf{G}^* \Lambda^*\end{aligned}$$

where Λ denotes impulses. At this point it is not possible to proceed without a suitable impact model. For example, if we apply Poisson's hypothesis of impact with Coulomb's frictional law and assume the coefficient of restriction is e , then Λ is the optimal solution to the QP problem:

$$\min \frac{1}{2} \Lambda^T \mathbf{G}^T \mathbf{M}^{-1} \mathbf{G} \Lambda + \Lambda^T \mathbf{G}^T \mathbf{M}^{-1} \Lambda^* \quad (2.7)$$

For all the unilateral constraints, the condition $\Lambda_i \geq 0$ applied. Further discussion on impact hypotheses is presented in chapter 3.

Frictional constraints

To illustrate the formulation with frictional constraints, it is assumed that Coulomb's model is valid. The frictional constraints are:

$$\begin{aligned}-\mu \lambda_{Ni} &\leq \lambda_{Fi} \leq \mu \lambda_{Ni} \\ \zeta_{Fi} (\mu \lambda_{Ni} - |\lambda_{Fi}|) &= 0\end{aligned}$$

$$\lambda_{Fi} \zeta_{Fi} \leq 0$$

where $\zeta_{Fi} = \mathbf{g}_{Fi}^T \dot{\mathbf{q}}$. The variables λ_{Fi} and λ_{Ni} are the tangential constraint force and the normal constraint force, respectively, and μ is the coefficient of friction.

Let us renumber the constraints such that first f constraints are the frictional constraints, the next r constraints are rolling constraints followed by the $k-f-r$ smooth frictionless constraints. We introduce

$$\mathbf{F} = [\mathbf{g}_{F,r+1} \dots \mathbf{g}_{F,r+f}], \text{ for all the frictional constraints,}$$

$\mathbf{G}_1 = [\mathbf{g}_{N1} \dots \mathbf{g}_{Nk}; \mathbf{g}_{F1} \dots \mathbf{g}_{Fr}]$, for the normal constraint forces and the r rolling constraints and construct a $f \times (k+r)$ matrix χ such that, equation (2.2) can be expressed as:

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{b} + [\mathbf{G}_1 + \mathbf{F}\chi] \lambda_1 \quad (2.8)$$

where $\lambda_1 = [\lambda_{N1} \dots \lambda_{Nk}; \lambda_{F1} \dots \lambda_{Fr}]^T$. In addition, we have the complementarity relations:

if $\zeta_{Ni} = 0$,

$$\lambda_{Ni} \eta_{Ni} = 0, \eta_{Ni} = \mathbf{g}_{Ni}^T \ddot{\mathbf{q}} + \dot{\mathbf{g}}_{Ni}^T \dot{\mathbf{q}} \geq 0, i = 1, 2, \dots, k \quad (2.9a)$$

$$\mu \lambda_{Ni} - \left| \lambda_{Fi} \right| \geq 0, i = 1, 2, \dots, r+f \quad (2.9b)$$

if $\zeta_{Fi} = \mathbf{g}_{Fi}^T \dot{\mathbf{q}} \neq 0$,

$$\lambda_{Fi} = -j_i \mu \lambda_{Ni},$$

else

$$\eta_{Fi} = \mathbf{g}_{Fi}^T \ddot{\mathbf{q}} + \dot{\mathbf{g}}_{Fi}^T \dot{\mathbf{q}} \geq 0, \eta_{Fi}(\mu \lambda_{Ni} - \left| \lambda_{Fi} \right|) = 0, i = 1, 2, \dots, r+f \quad (2.9c)$$

where j_i is the sign of sliding velocity.

If the constraints are known and they remain unchanged through a finite interval, substituting for $\ddot{\mathbf{q}}$ we obtain:

$$\eta_1 = \mathbf{G}_1^T (\mathbf{M}^{-1} [\mathbf{b} + (\mathbf{G}_1 + \mathbf{F}\chi) \lambda_1]) + \dot{\mathbf{G}}_1^T \dot{\mathbf{q}} = 0 \quad (2.10)$$

This can be solved for λ_1 , or at least for $(\mathbf{G}_1 + \mathbf{F}\chi)\lambda_1$, and substituting back to yield a solution for $\ddot{\mathbf{q}}$. However, because the Coulomb frictional law is an empirical law, a few problems arise when we consider the uniqueness and existence of such solution.

2.1.2 Static indeterminacy

In equation (2.2), if the constraints are linearly dependent (the column vector in the Jacobian matrix, \mathbf{G} are linearly dependent), it is not possible

to find a unique solution for λ . However, it is possible to show that there always exists a unique solution for $\bar{\omega} = G\lambda$ [17], and therefore, a unique solution for $\ddot{\mathbf{q}}$. When there is friction at the contacts, and the Coulomb model is used, the uniqueness and existence are not guaranteed.

2.1.3 Inconsistencies with approximate contact models

Inconsistencies in rigid body analysis in problems with friction have been well known to Delassus, Klein, von Mises and Bouligand [17, 20]. There are situations in which no solution for λ exist, and others in which multiple solutions for $\ddot{\mathbf{q}}$ exist. More recently, Mason and Wang [20] and Featherstone [3] have studied the inconsistencies in a rod sliding along a rough surface. Although the inconsistencies have been attributed to the approximate nature of Coulomb's law [20] and the inadequacy of rigid body models, no clear explanation have been found.

In order to eliminate inconsistencies, Kilmister and Reeve [12] propose an impulse model which postulates that impulses shall act in order to maintain the constraints, whenever and only when inconsistencies are encountered. Although this principle can be used to resolve ambiguities and inconsistencies, there is no basis for this hypothesis.

Methods involving the use of penalty functions instead of Lagrange Multipliers offer an alternative. This only requires that the constraints be satisfied approximately (for example, penetration of rigid bodies is allowed). Goldsmith [5] and Haug [7] have used linear springs (and if necessary, linear dampers in order to dissipate the impact energy) at each contact. Although such methods do yield solutions, the validity of the solutions is not clear. Further, the problems of uniqueness and existence in the presence of Coulomb frictional constraints still remain unresolved.

2.2 Simulation

The basic steps for the simulation are summarized below (see Figure 2.1):

1. Initialize the state $(\mathbf{q}, \dot{\mathbf{q}})$.
2. Solve the QP problem to find λ based on the state from the previous time step.
3. Integrate through time interval Δt , and solve for the positions and velocities for next step.
4. Check for possible changes in the topology of the system by using *collision detector* (by velocity and/or position criterions). If no change, go to step 2. If there is a change, find the time of event, dt ($< \Delta t$). (There are different recursive routines applied to find the time of event and an integration function embedded in each routine to integrate through dt .)
6. Update the status of the constraints in the system and go to step 2.

The subroutines in dash window are new techniques to handle cases includeing separations, impacts, rolling and sliding, and compliant impacts. Therefore, the simulation routine is a integration of traditional techniques and new techniques.

When solving for \mathbf{q} and $\dot{\mathbf{q}}$ by integration, the use of a linear multistep method [4] is more efficient and accurate than the conventional Runge-Kutta integration method.

The Quadratic programming problem [29]

Consider the problem of finding the minimum of a multivariable quadratic objective function of the form:

$$\min \frac{1}{2} \mathbf{x}^T \mathbf{A}^T \mathbf{A} \mathbf{x} + \mathbf{x}^T \mathbf{A}^T \mathbf{d} \quad (2.11a)$$

subject to linear constraints:

$$H_{ji}x_i \leq K_j, j = 1 \text{ to } m \quad (2.11b)$$

$$x_i \geq 0, i = 1 \text{ to } n \quad (2.11c)$$

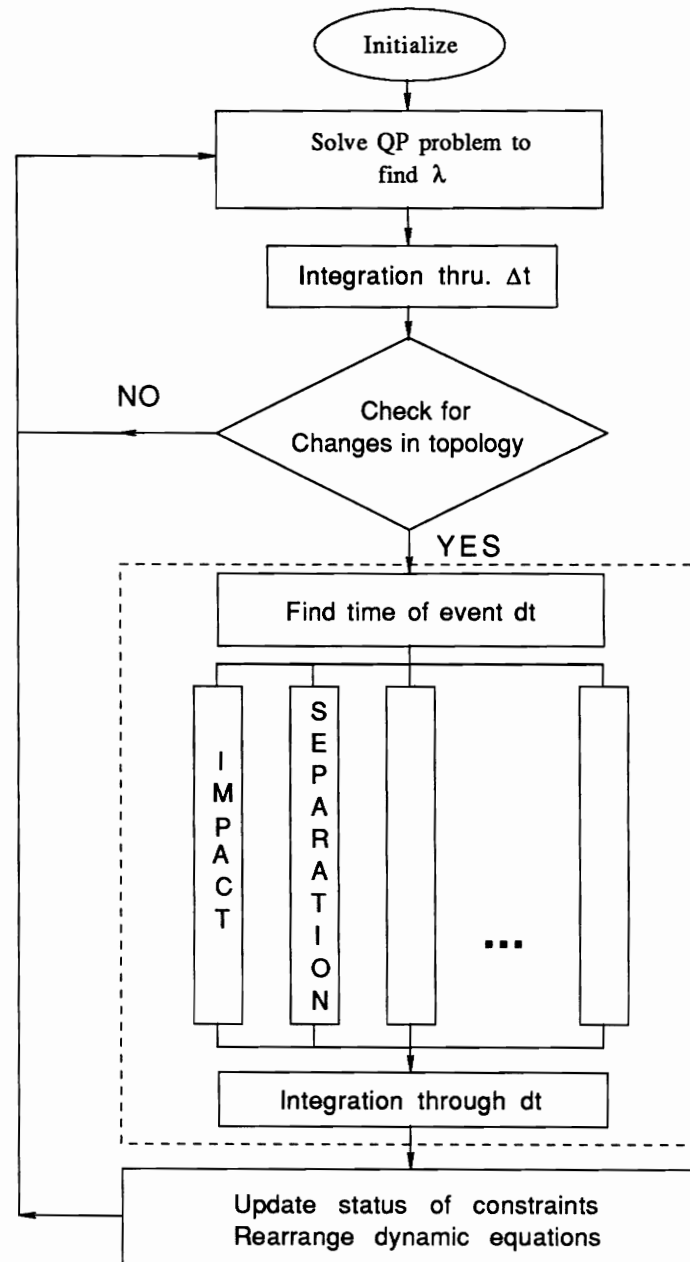


Figure 2.1: Flow chart of simulation

This is the quadratic programming problem. The Lagrangian for equation (2.11) is as following:

$$\Phi(\mathbf{x}, \mathbf{u}) = \frac{1}{2} \mathbf{x}^T \mathbf{A}^T \mathbf{A} \mathbf{x} + \mathbf{x}^T \mathbf{A}^T \mathbf{d} + \mathbf{u}^T (\mathbf{H} \mathbf{x} - \mathbf{K}) \quad (2.12)$$

where \mathbf{u} is a set of Lagrange multipliers. By the Kuhn-Tucker Condition [13], the Lagrangian satisfies the complementarity condition,

$$\begin{aligned} \frac{\partial \Phi}{\partial x_i} &\geq 0, x_i \geq 0, x_i \left(\frac{\partial \Phi}{\partial x_i} \right) = 0, \\ \frac{\partial \Phi}{\partial u_j} &\geq 0, u_j \geq 0, u_j \left(\frac{\partial \Phi}{\partial u_j} \right) = 0. \end{aligned}$$

We can reduce the QP problem to linearly programming problem [14] as

$$\mathbf{H} \mathbf{x} = \mathbf{K} \quad (2.13a)$$

$$\mathbf{A}^T \mathbf{A} \mathbf{x} - \mathbf{v} + \mathbf{H}^T \mathbf{u} = -\mathbf{A}^T \mathbf{d} \quad (2.13b)$$

$$\mathbf{x} \geq 0, \quad \mathbf{v} \geq 0 \quad (2.13c)$$

$$\mathbf{x}^T \mathbf{v} = 0 \quad (2.13d)$$

Totally, there are $m+n$ equations with $m+2n$ variables. This can be solved using a modified simplex method which was developed by Wolfe [14]. This algorithm is directly applicable to equation (2.6) with $\mathbf{A} = \mathbf{D}^T \mathbf{G}_n$, $\mathbf{x} = \lambda_n$, $\mathbf{d} = \mathbf{D} \mathbf{c}_n / h_n$, where $\mathbf{M} = \mathbf{D}^T \mathbf{D}$, and h_n is the integration time step.

3 Rigid body impact models

3.1 Introduction

When two objects collide at contact point c_1 on object 1 and c_2 on object 2, Figure 3.1, we can apply principle of impulse and momentum [1] to obtain the final velocities of each object after collision.

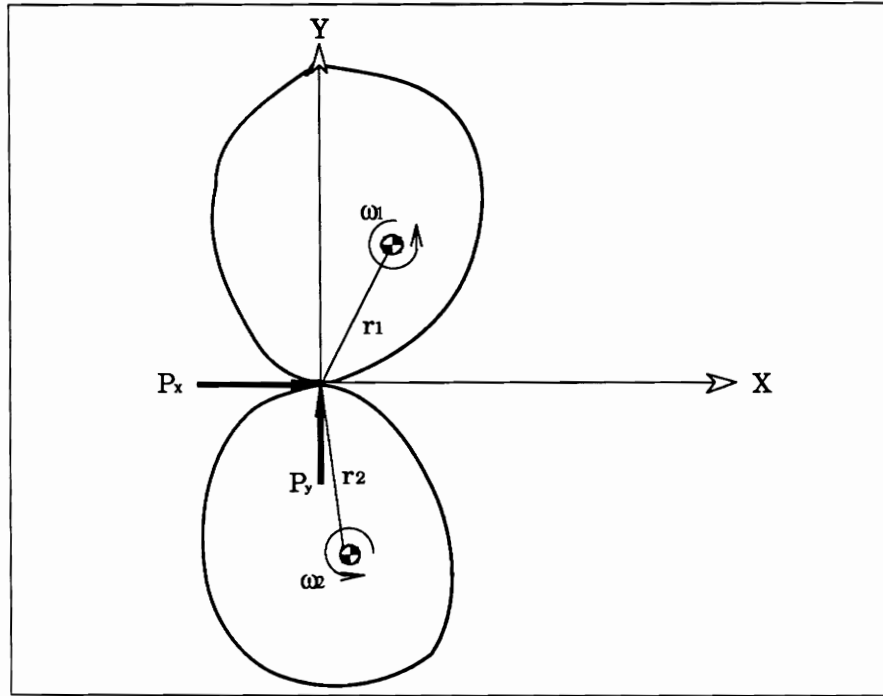


Figure 3.1: Two-body collision

The principle of impulse and momentum provides the following relations:

$$M_1(\bar{v}_{1x} - \bar{V}_{1x}) = P_x \quad (3.1a)$$

$$M_1(\bar{v}_{1y} - \bar{V}_{1y}) = P_y \quad (3.1b)$$

$$I_1(\omega_1 - \Omega_1) = r_{1y}P_x - r_{1x}P_y \quad (3.1c)$$

$$M_2(\bar{v}_{2x} - \bar{V}_{2x}) = -P_x \quad (3.1d)$$

$$M_2(\bar{v}_{2y} - \bar{V}_{2y}) = -P_y \quad (3.1e)$$

$$I_2(\omega_2 - \Omega_2) = r_{2x}P_y - r_{2y}P_x \quad (3.1f)$$

where \bar{V} and \bar{v} are the initial and terminal velocities of mass center, respectively. Similarly, W and w are initial and terminal angular velocities of mass center, $P_y(t)$ and $P_x(t)$ are the normal and tangential impulse. There are six equations and eight unknowns, including six velocity unknowns and two impulse unknowns. Two more equations from the constraints of the normal impulse and frictional impulse are required in order to solve this problem. We can construct these two constraint equations by considering the relative sliding velocity $S(t)$ and the relative compression velocity $C(t)$ of the contact point:

$$\begin{aligned} S &= v_{1x} - v_{2x} = (\bar{v}_{1x} + r_{1y}\omega_1) - (\bar{v}_{2x} + r_{2y}\omega_2) \\ &= S_0 + \frac{P_x}{m_1} - \frac{P_y}{m_3} \end{aligned} \quad (3.2a)$$

$$\begin{aligned} C &= v_{1y} - v_{2y} = (\bar{v}_{1y} - r_{1x}\omega_1) - (\bar{v}_{2y} - r_{2x}\omega_2) \\ &= C_0 - \frac{P_x}{m_3} + \frac{P_y}{m_2} \end{aligned} \quad (3.2b)$$

where $S_0 = S(t_0)$, $C_0 = C(t_0)$,

$$\begin{aligned} \frac{1}{m_1} &= \frac{1}{M_1} + \frac{1}{M_2} + \frac{r_{1y}^2}{I_1} + \frac{r_{2y}^2}{I_2}, \\ \frac{1}{m_2} &= \frac{1}{M_1} + \frac{1}{M_2} + \frac{r_{1x}^2}{I_1} + \frac{r_{2x}^2}{I_2} \quad \text{and} \quad \frac{1}{m_3} = \frac{r_{1x}r_{1y}}{I_1} + \frac{r_{2x}r_{2y}}{I_2}. \end{aligned}$$

where the effective mass m_1 , m_2 and m_3 are independent of velocity. By Coulomb's law of friction, the tangential impulse is related to the normal impulse [25] by

$$\frac{|dP_x|}{dP_y} \leq \mu \quad (3.2c)$$

where $\mu = \mu_s$ when sticking occurs and $\mu = \mu_k$ for sliding. The equations in (3.2) provide two additional impulse constraints.

In general, an impact may be considered to occur in two phases, the compression phase, and restitution phase. During the compression phase,

the two bodies deform in a direction normal to the impact surface, and the relative velocities of the contact points on the two bodies in normal direction is reduced to zero. The end of compression phase is also referred to as the instant of maximum compression. The restitution phase starts at this point and lasts until the two bodies separate.

Routh [25] presented a graphical technique (Figure 3.2) for solving the impact problem. The final impulse can be determined by the following procedure. At the end of compression phase, the relative compression velocity $C(t)$ vanishes and this is represented by a straight line in the P_x - P_y plane. We call this the *line of maximum compression*:

$$C_0 + \frac{P_x}{m_1} - \frac{P_y}{m_3} = 0 \quad (3.3a)$$

Similarly, when the relative sliding velocity vanishes, we have the *line of sticking*:

$$S_0 - \frac{P_x}{m_3} + \frac{P_y}{m_2} = 0 \quad (3.3b)$$

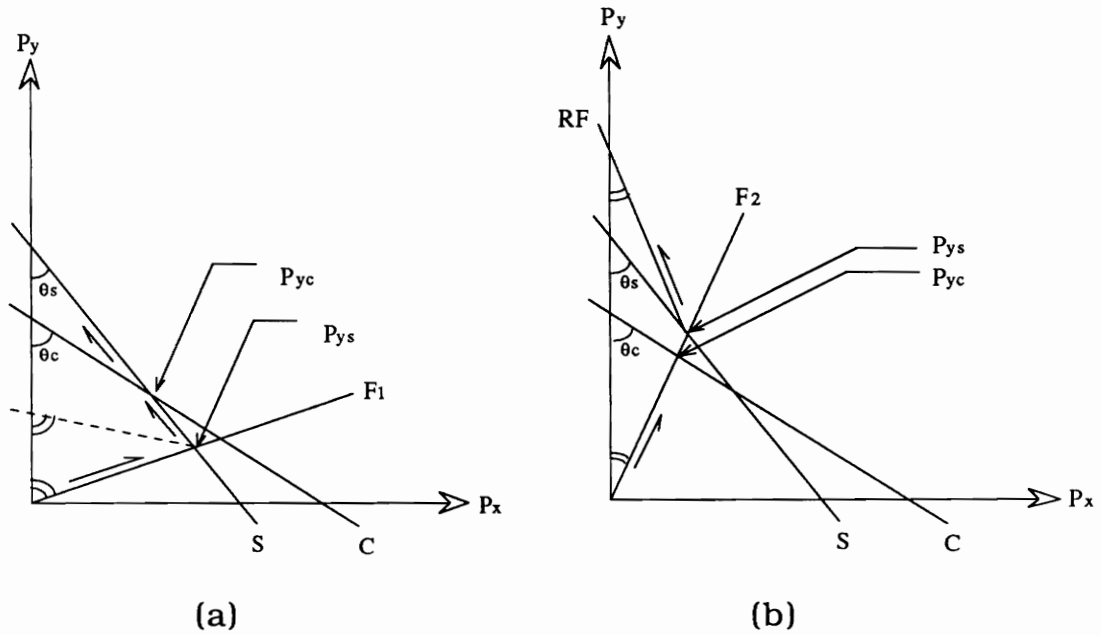


Figure 3.2: Routh's graphical method

The two constraint lines $C=0$ and $S=0$ (equations (3.3a) and (3.3b)) are shown on Figure 3.2. The slopes of the lines are

$$\tan\theta_s = \frac{m_1}{|m_3|}, \quad \tan\theta_c = \frac{|m_3|}{m_2}$$

since $m_1 m_2 < m_3^2$, the angle θ_c is greater than θ_s .

If the bodies slide over each other at the beginning of impact, the impulse will develop along the *line of limiting*, F . This line satisfies the extension of Coulomb's law of impulse: $P_x = \mu P_y$. This line intersects C at the ordinate P_{yc} and S at P_{ys} . In equation (3.3b), if the initial velocity $S_0 = 0$, the impact is said to be a direct impact. If $S_0 \neq 0$, the impact is an oblique impact. If $r_{1x} = 0$ and $r_{2x} = 0$, the impact is said to be a central impact. Otherwise, the impact is said to be an eccentric impact. With different ranges of P_{yc} , P_{ys} and S , all five possible cases of impact [28] are summarized in Table 3.1. They are (a) sliding and reversed sliding on compression; (b) sliding and reversed sliding on restitution; (c) sliding and sticking on compression; (d) sliding and sticking on restitution; (e) forward sliding.

In each case, the final velocities and impulses can be determined by solving equation (3.1) and (3.2), assuming an appropriate impact law is available.

Energy dissipation

The energy dissipation is the negative of the work done by the impulses. In order to find the energy dissipation, we plot the relative velocities at the contact point versus the normal impulse, as shown in Figure 3.3. Equation (3.2) can be rewritten as:

$$S = S_0 - \frac{j\mu P_y}{m_x} \quad (3.4a)$$

$$C = C_0 - \frac{P_y}{m_y} \quad (3.4b)$$

Table 3.1: Cases of collision

$\begin{matrix} S_0 \\ \backslash \\ S_f \quad P_{ys} \end{matrix}$	Oblique impact: $S_0 \neq 0$			Direct impact: $S_0=0$
	$0 < P_{ys} < P_{yc}$ or $P_{yc} < 0$	$0 < P_{yc} < P_{ys}$ $\leq (1+e)P_{yc}$	$0 < (1+e)P_{yc} < P_{ys}$ or $P_{ys} < 0$	$P_{ys}=0$
$S_f = 0$	Sliding and sticking on compression	Sliding and sticking on restitution	Sliding	Sliding and sticking
$j S_f < 0$	Sliding and reverse sliding on compression	Sliding and reverse sliding on restitution		Sliding

where the effective masses m_x and m_y at the contact point for the normal and tangential reactions are defined as:

$$m_x = \left[\frac{k_1^2 + r_{1y}^2 + j\mu^{-1} r_{1x}r_{1y}}{k_1^2 M_1} + \frac{k_2^2 - r_{2y}^2 + j\mu^{-1} r_{2x}r_{2y}}{k_2^2 M_2} \right]^{-1},$$

$$m_y = - \left[\frac{k_1^2 + r_{1x}^2 + j\mu r_{1x}r_{1y}}{k_1^2 M_1} + \frac{k_2^2 + r_{2x}^2 + j\mu r_{2x}r_{2y}}{k_2^2 M_2} \right]^{-1}$$

where k_1 and k_2 are the radius of gyration of object 1 and object 2, respectively. We denote j as the direction of initial sliding velocity which is equal to $\frac{S_0}{|S_0|}$. Note that the effective masses m_x and m_y depend on the direction of slip.

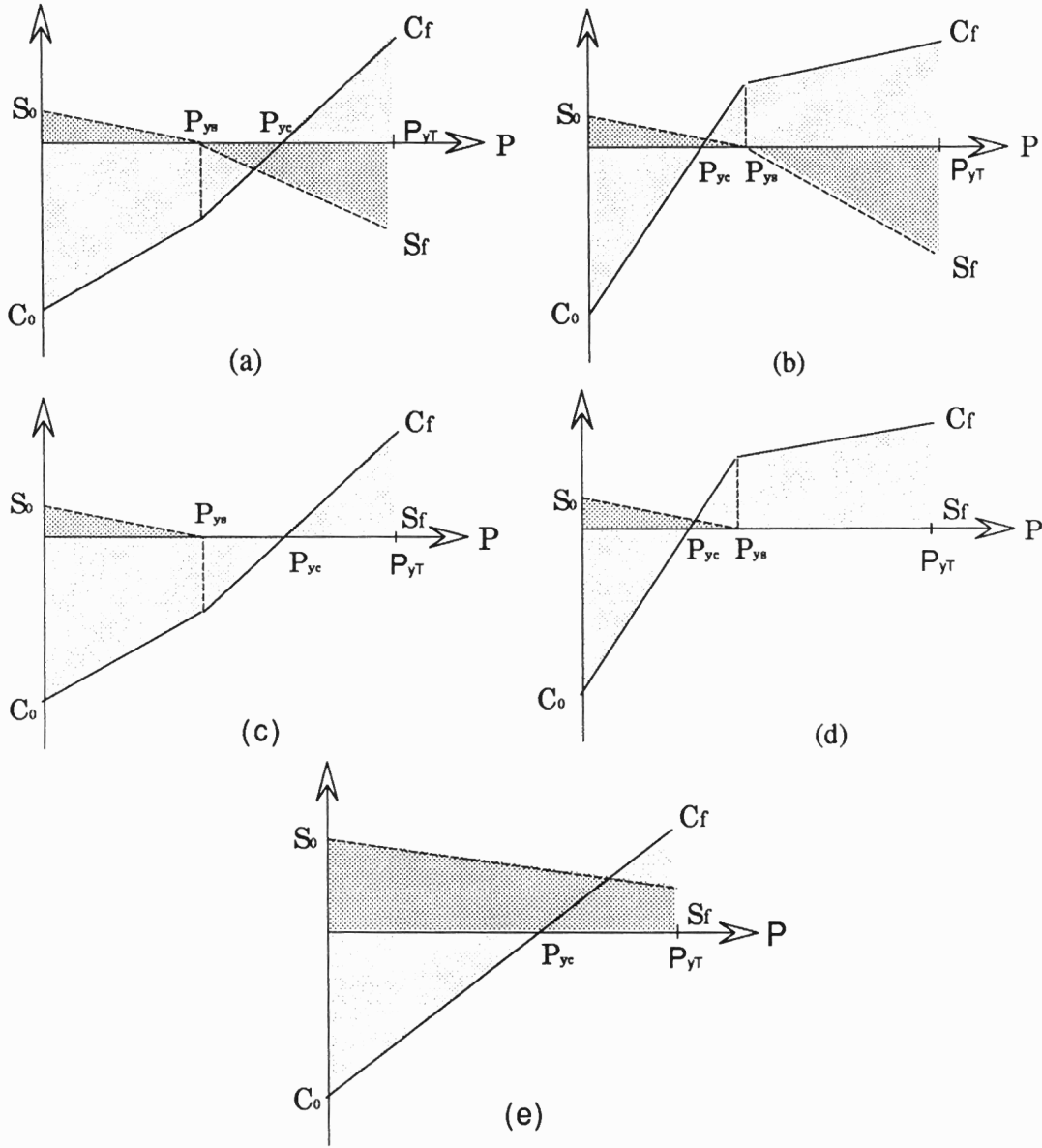


Figure 3.3: Cases of collision

The final velocities can be obtained as following:

$$S_f = \begin{cases} -\frac{j\mu(P_{yT}-P_{ys})}{m_x} & \text{reverse sliding} \\ 0 & \text{slip stopped} \end{cases} \quad (3.5a)$$

$$C_f = C_0 - \frac{P_{ys}}{m_y^+} - \frac{(P_{yT} - P_{ys})}{m_y^-} \quad (3.5b)$$

where m_x^+ and m_y^+ are the effective masses before slip-stop, and m_x^- and m_y^- are the effective masses after slip-stop.

The energy dissipation of the system is defined as

$$D = - \int \mathbf{v} \cdot d\mathbf{P} \quad (3.6)$$

A component of dissipation for any period of slip is equal to the area between the line S or C and the abscissa, as shown in Figure 3.3:

The normal dissipation D_n and tangential impulse D_t are found for each case as:

(a) Slip reversed on compression:

$$\begin{aligned} D_n = & -\frac{1}{2} \left(2C_0 - \frac{P_{ys}}{m_y^+} \right) P_{ys} - \frac{1}{2} \left(C_0 - \frac{P_{ys}}{m_y^+} \right) (P_{yc} - P_{ys}) \\ & - \frac{1}{2} \left(C_0 - \frac{P_{ys}}{m_y^+} - \frac{(P_{yT} - P_{ys})}{m_y^-} \right) (P_{yT} - P_{yc}) \end{aligned} \quad (3.7a)$$

$$D_t = \frac{1}{2} S_0 j \mu P_{ys} - \frac{1}{2} j \frac{\mu^2}{m_x^-} (P_{yT} - P_{ys})^2 \quad (3.7b)$$

(b) Slip reversed on restitution:

$$\begin{aligned} D_n = & -\frac{1}{2} C_0 P_{yc} - \frac{1}{2} \left(C_0 - \frac{P_{ys}}{m_y^+} \right) (P_{ys} - P_{yc}) \\ & - \frac{1}{2} \left(2C_0 - \frac{2P_{ys}}{m_y^+} - \frac{(P_{yT} - P_{ys})}{m_y^-} \right) (P_{yT} - P_{ys}) \end{aligned} \quad (3.8a)$$

$$D_t = \text{(same as equation (3.7b))} \quad (3.8b)$$

(c) Sliding and sticking on compression:

$$D_n = \text{(same as equation (3.7a) with } m_y^- \text{ substituted by } m_y^\sigma) \quad (3.9a)$$

$$D_t = \frac{1}{2} S_0 j \mu P_{ys} \quad (3.9b)$$

(d) Sliding and sticking on restitution:

$$D_n = (\text{same as equation (3.8a) with } \bar{m}_y \text{ substituted by } m_y^\sigma) \quad (3.10a)$$

$$D_t = (\text{same as equation (3.9b)}) \quad (3.10b)$$

(e) Forward sliding:

$$D_n = -\frac{1}{2} C_0 P_{yc} - \frac{1}{2} \left(C_0 - \frac{P_{yT}}{m_y^+} \right) (P_{yT} - P_{yc}) \quad (3.11a)$$

$$D_t = \frac{1}{2} (2S_0 - \frac{j\mu P_{yT}}{m_x^+}) j\mu P_{ys} \quad (3.11b)$$

where P_{yT} is the total impulse of impact whose value depends on the impact model, and

$$m_y^\sigma = - \left[\frac{k_1^2 + r_{1x}^2 + \frac{m_1}{m_3} r_{1x} r_{1y}}{k_1^2 M_1} + \frac{k_2^2 + r_{2x}^2 + \frac{m_1}{m_3} r_{2x} r_{2y}}{k_2^2 M_2} \right]^{-1}$$

3.2 Impact laws

There are three types of impact hypothesis that have been applied to collision systems. These are Newton's kinematic hypothesis, Poisson's impulse hypothesis [12] and Stronge's internal dissipation hypothesis [27].

Kinematic hypothesis: Newton's law of impact says *the ratio of normal velocity after impact to the normal velocity before impact is equal to e* . We can find the final impulse for each case as following:

$$e = - \frac{C_f}{C_0} \quad (3.12)$$

Thus for cases (a)-(e), expressions for P_{yT} can be derived:

(a) and (b) sliding and reversed sliding:

$$P_{yT} = (1 + e) C_0 \bar{m}_y - \frac{\bar{m}_y P_{ys}}{m_y^+} + P_{ys} \quad (3.13)$$

(c) and (d) sliding and sticking:

$$P_{yT} = (1 + e) C_0 m_y^\sigma - \frac{m_y^\sigma P_{ys}}{m_y^+} + P_{ys} \quad (3.14)$$

(e) forward sliding:

$$P_{yT} = (1 + e) C_0 m_y^+ \quad (3.15)$$

Impulse hypothesis: Kilmister and Reeve [12] propose the principle of constraints:

Constraints shall be maintained by forces, so long as this is possible; otherwise, and only otherwise, by impulses.

They also advocate, Poisson's hypothesis that can be stated as:

The impulse in restitution period is e times that in compression period.

The final impulse is determined as:

$$e = \frac{(P_{yT} - P_{yc})}{P_{yc}} \quad (3.16a)$$

In cases (a), (b), (c), (d) and (e):

$$P_{yT} = (1 + e) P_{yc} \quad (3.16b)$$

Internal dissipation hypothesis: *The square of coefficient of restitution e^2 is the ratio of elastic strain energy released at contact point during restitution to the energy absorbed by deformation during compression.*

The final impulse P_{yT} is given by:

(a) and (c) sliding and reversed sliding:

$$e^2 = \frac{\frac{(P_{yT} - P_{yc})^2}{(P_{yc} - P_{ys})} (C_0 - \frac{P_{ys}}{m_y^+})}{(2 C_0 - \frac{P_{ys}}{m_y^+}) P_{ys} + (C_0 - \frac{P_{ys}}{m_y^+}) (P_{yc} - P_{ys})} \quad (3.17a)$$

$$P_{yT} = P_{yc} + e \left[\frac{(P_{yc} - P_{ys}) (C_0 P_{ys} + C_0 P_{yc} - \frac{P_{ys} P_{yc}}{m_y^+})}{C_0 - \frac{P_{ys}}{m_y^+}} \right]^{\frac{1}{2}} \quad (3.17b)$$

(b) and (d) sliding and sticking:

$$e^2 = \frac{(-C_0 + \frac{P_{ys}}{m_y^+})(P_{ys} - P_{yc}) + (2C_0 - \frac{2P_{ys}}{m_y^+} - \frac{(P_{yT} - P_{ys})}{m_y^-})(P_{yT} - P_{ys})}{C_0 P_{yc}} \quad (3.18a)$$

$$P_{yT} = (C_0 m_y^- + P_{ys} - \frac{m_y^- P_{ys}}{m_y^+}) + A^{\frac{1}{2}} \quad (3.18b)$$

where

$$A = \left[\left(\frac{m_y^- P_{ys}}{m_y^+} - C_0 m_y^- - P_{ys} \right)^2 - \left(C_0 m_y^- P_{yc} - C_0 e^2 m_y^- P_{yc} + C_0 m_y^- P_{ys} \right) - \frac{m_y^- P_{yc} P_{yc}}{m_y^+} + P_{ys}^2 - \frac{m_y^- P_{ys}^2}{m_y^+} \right]^{\frac{1}{2}}$$

(e) forward sliding:

$$e^2 = \frac{(C_0 - P_{yT})(P_{yT} - P_{yc})}{-C_0 P_{yc}} \quad (3.19a)$$

$$P_{yT} = \frac{1}{2} (C_0 m_y^+ + P_{yc}) + \left[(C_0 m_y^+ P_{yc})^2 - 4 C_0 (1 - e^2) m_y^+ P_{yc} \right]^{\frac{1}{2}} \quad (3.19b)$$

For a simple collinear impact of two smooth bodies dissipation hypothesis, Poisson's hypothesis and Newton's law of impact give the same results; and this is about the whole extent of their agreement.

My preliminary investigation has shown that:

- (1) Newton's kinematic hypothesis and Poisson's impulse hypothesis yield results that do not satisfy energy conservation principles.
- (2) None of the three hypothesis offer an expression of cases in which there is no feasible solution to an initial value problem.

4 Compliant model of collision

4.1 Introduction

There are several approaches to modeling the contact compliance depending on the material properties and the geometry of the contacting surface. We assume that linear elasticity provides a sufficiently accurate model. Secondly, since the objective is to incorporate a contact model into a computer simulation, we do not pursue analytical solutions. While the elastic half space theory, Boussinesq's influence functions and Hertz' contact model do lead to analytical solutions [9,16,23], they do so only in the simplest of geometries.

4.2 Finite dimensional model

The basic approach is the one adapted by Sinha and Abel [26], we discretize the contact area into n_e small elements or contact patches with lumped stiffness, as shown in Figure 4.1. The contact area and the deformations are small compared to the gross dimensions of the contacting object. At the j^{th} contact patch for the i^{th} contact, the normal and tangential forces are N_{ij} and T_{ij} respectively. That is,

$$\lambda_{Ni} = \sum_{j=1}^{n_e} N_{ij} \quad (4.1a)$$

$$\lambda_{Fi} = \sum_{j=1}^{n_e} T_{ij} \quad (4.1b)$$

Let δ denote the relative rigid body displacement in the normal direction at the i^{th} contact. Since the i^{th} constraint is ϕ_i , clearly $\delta = -\phi_i$. Let the profiles of the two contacting bodies be given by $f_1(x)$ and $f_2(x)$. If $u_{in}^1(x)$ and $u_{in}^2(x)$

are the deformations in normal direction for the two bodies, and s is the separation between two bodies at contact point i ,

$$s_i = f_1 + f_2 + u_{in}^1 + u_{in}^2 - \delta \quad (4.2)$$

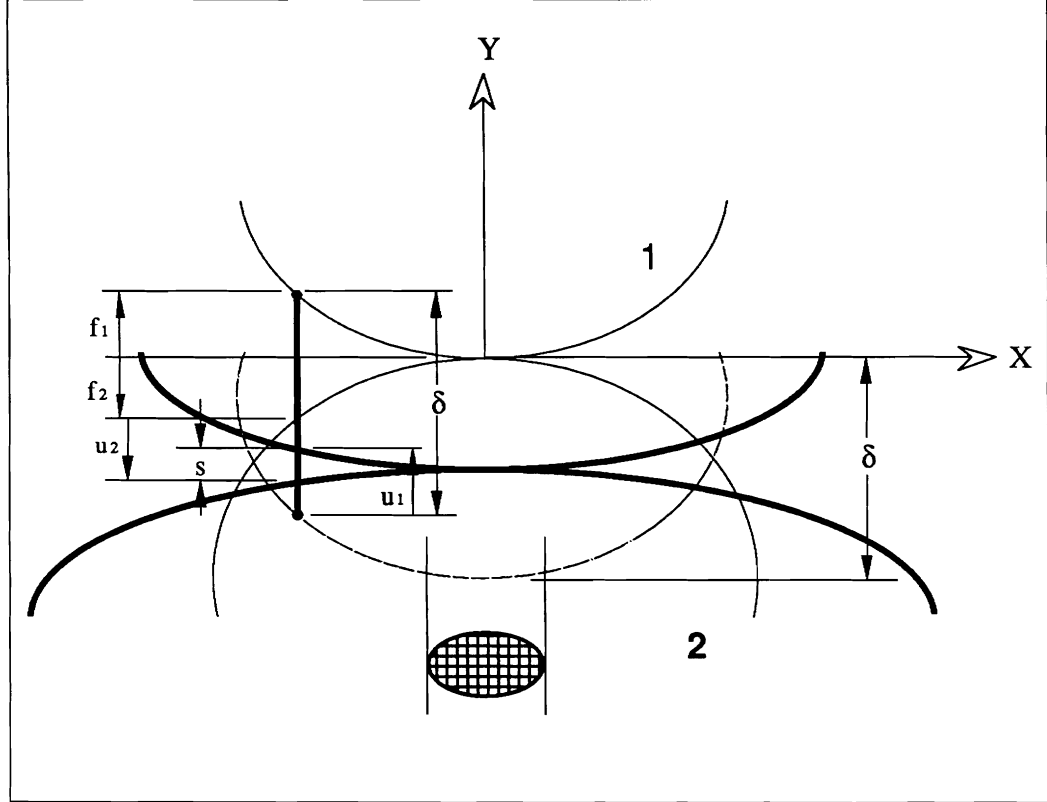


Figure 4.1: Two-body contact with compliance

where $s_i = 0$, $N_{ik} \neq 0$ is inside the contact area; $s_i > 0$, $N_{ik} = 0$, $T_{ik} = 0$ is outside the contact area.

The displacement u_{in}^1 (u_{in}^2) is related to the pressure on body 1 (body 2) by

the expression

$$u_{in}^1 = \sum_k^{n_e} \left[\xi_{jk}^n N_{ik} + \xi_{jk}^t T_{ik} \right] \quad (4.3)$$

where the influence functions ξ_{jk}^n is the normal displacement at contact patch j , due to a unit normal force at contact patch k , and ξ_{jk}^t is the tangential displacement due to a unit tangential force at contact patch k . These influence function are Green's functions [23] which depend on the contact geometry and the material properties.

Similar analysis can be done in tangential direction,

$$u_{it}^1 = \sum_k^{n_e} \left[\gamma_{jk}^n N_{ik} + \gamma_{jk}^t T_{ik} \right] \quad (4.4)$$

If the contact is counterformal, that is the dimensions of the contact patch remain small compared to the radii of curvatures of the undeformed surfaces, it is appropriate to use elastic half space theory and influence functions by Boussinesq [19]. However, in conformal contact, the influence functions may not be found analytically; therefore, they must be generated numerically such as finite element method [23], or else be approximated by some convenient mathematical expressions.

The normal and tangential forces are subject to frictional constraints. The simplest constraint is generated by a point-wise application of Coulomb's law of friction:

$$\mu N_{ij} - |T_{ij}| \geq 0 \quad (4.5a)$$

$$s_{ij} \left(\mu N_{ij} - |T_{ij}| \right) = 0 \quad (4.5b)$$

$$T_{ij} s_{ij} \leq 0 \quad (4.5c)$$

Where s_{ij} is the separation at the j^{th} contact patch of the i^{th} contact. Assume that the rigid body relative motion (δ_{in} and δ_{it}) and the geometry are known. Equations (4.2) can be written for all the n_e contact patches at the i^{th} contact:

$$\mathbf{U}_i = \mathbf{A}_i \mathbf{N}_i + \mathbf{B}_i \mathbf{T}_i + \mathbf{C}_i \quad (4.6a)$$

$$\mathbf{U}_i^T \mathbf{N}_i = 0 \quad (4.6b)$$

$$\mathbf{U}_i \geq 0 \quad (4.6c)$$

$$\mathbf{N}_i \geq 0 \quad (4.6d)$$

where \mathbf{U}_i , \mathbf{N}_i and \mathbf{T}_i are $n_e \times 1$ vectors containing separation s_{ij} , \mathbf{N}_{ij} and \mathbf{T}_{ij} respectively, \mathbf{A}_i and \mathbf{B}_i are $n_e \times n_e$ matrices containing the influence coefficients while \mathbf{C}_i consists of known constants δ_{in} , δ_{it} , f_1 and f_2 . Clearly, if there is no friction and $T = 0$, this is a LCP as equation (2.6), and is solved by considering a QP problem of the type:

$$\min_{\mathbf{N}_i \geq 0} \frac{1}{2} \mathbf{N}_i^T \mathbf{A} \mathbf{N}_i + \mathbf{N}_i^T \mathbf{C}_i \quad (4.7)$$

The objective function can be identified as the potential energy of the system and the minimization is the application of the minimum potential energy theorem.

5 Concluding remarks

5.1 Impact

The proposed work deals with the analysis and simulation of mechanical systems with changing topologies and multiple unilateral frictional constraints. The computer simulation will be interfaced with a three dimensional graphical display on a high resolution workstation. The resulting CAD tool will be directly applicable to the design of manufacturing process such as assembly of mechanical components, gripping, fixturing and part feeding. It can be used to analyze complex systems such as multifingered hands or locomotion systems. The simulation can be used to evaluate robot control algorithms. The simulation routine provides several new techniques of handling impact and separation cases. Finally, the proposed study will improve the understanding of impact mechanics and the rigid interaction with multiple unilateral frictional contacts.

5.2 Future Work

- (1) Impact and friction models developed by previous researchers will be investigated. In particular, energy conservation or dissipation during impact will be studied. This will lead to the development of a satisfactory model for the analysis of systems with unilateral constraints.
- (2) A computer simulation package for the analysis of dynamic system with variable topology will be developed. The package will also process a three dimensional graphical display. This will be done by the animation package, the Jack, and the IRIS machines available in the Graphics Laboratory.

- (3) A model of contact compliance will be developed using the approach described in the previous section. It will be compared and validated with analytical continuous contact force models such as the model described in reference [15], and finite element models using ABAQUS.
- (4) A variety of nonlocal and nonlinear friction models will be investigated.

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