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Stronge's hypothesis-based solution to the planar collision-with-friction problem

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Abstract This paper deals with collision with friction. Differential equations governing a one-point collision of planar, simple non-holonomic systems are generated. Expressions for quantities of interest (e.g., normal and tangential impulses, normal and tangential relative velocities of the colliding points, and the change of the system mechanical energy), are written for *five* types of collision (i.e., sticking in compression, forward sliding, etc.) associated with Stronge's collision hypothesis and Coulomb's coefficient of friction, in conjunction with a two-integration procedure. These expressions, together with Routh's semi-graphical method are used to show that the algebraic signs of *four* configuration-related parameters span *five* cases of system configuration. For each, the ratio between the tangential and normal components of the velocity of approach, called α, determine the type of collision which, once found, allows the evaluation of the changes in the motion variables. The analysis of these cases indicates that an algebraic, Stronge's hypothesis-based solution to the planar collision-with-friction problem always exists, and is unique, coherent and energy consistent. Finally, substitutions are found which transform the Stronge's hypothesis-based solution to a Poisson's hypothesis-based and to a Newton's hypothesis-based solutions appearing in the literature.

Keywords Collision · Collision with friction · Stronge's collision hypothesis · Newton's collision hypothesis · Poisson's collision hypothesis · Routh's graph · Coulomb's coefficient of friction

1 Introduction

It is customary to classify collision problems in three categories according to the deformation involved and the tools required to develop solutions, as indicated, e.g., by Najafabadi et al. in [1] and by Chatterjee and Ruina in [2]. 'Large' deformation collision (i.e., large when compared to the size of the colliding bodies) usually requires continuum mechanics

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tools, ‘small’ deformation collision (sometimes called ‘soft’ collision) can be dealt with by force-deflection laws, and ‘extremely small’ deformation collision (sometimes called ‘hard’ collision) can do with algebraic collision laws. If, in the framework of multibody dynamics, one wishes to augment an existing simulation with a collision-solver, then the last two categories come into play. The task of the analyst in connection with the ‘soft’ category is to formulate collision force-deflection relationships, as done, for example, by Flores et al., using the Hertz contact model in [3], and by Erickson et al. developing a spring-dashpot contact model in [4]. The ‘hard’ category is associated with algebraic solutions, usually based on the classical collision hypotheses by Newton [5], Poisson [6], and Stronge [7]. Solutions within the realm of ‘soft’ collision involve time integration of the system equations of motion, augmented with the additional collision forces which appear when collision is detected. Solutions within the realm of ‘hard’ collision involve algebraic calculations of changes in the motion variables conducted after the completion of an integration step following collision detection. The latter are more computationally efficient, hence the motivation for their development.

With reference to the ‘hard’ collision category, a number of difficulties present themselves, discussed here briefly in connection with the three classical hypotheses applied to a one-point collision. First, there is no closed form general solution to 3D problems as pointed out by, e.g., Bhatt and Koechling [8], Batlle [9], and Brach [10]. One can develop solutions to special cases, as done by Whittaker in [11], where he solves a two-body, frictionless collision problem, or by Brach [10], solving the problem of a spherical pendulum hitting a plane. One can develop approximations as the one by Zhen and Liu in [12] who replaced integration with a difference method, or try new collision hypotheses suggested, e.g., by Chatterjee and Ruina [2] and by Rubin [13]. Finally, one can resort to solutions involving integration (see, e.g., Keller [14]), in which case one looses the computational advantage of this category.

With regards to 2D problems, the use of Newton’s hypothesis suffers from energy-related inconsistencies (first pointed out by Kane [15]). Concerning Poison’s hypothesis, some authors (e.g., Najafabadi et al. [1], Ivanov [16] and Stronge [17]) indicate that its physical soundness is questionable. Najafabadi et al. [1] point out the fact that Stronge’s hypothesis does not account for phenomena associated with collision, primarily wave propagation and structural damping (an equally valid claim in connection with Newton’s and Poisson’s hypotheses). Smith and Liu [18] and Chatterjee and Ruina [2] point out that none of the hypotheses capture tangential compliance, whereas Stoianovici and Hurmuzlu [19] found that the coefficient of restitution is configuration-dependent.

In spite of these shortcomings, the ‘hard’ collision category is widely used. For example, Kane and Levinson solve their double pendulum problem in [20, p. 348], using Newton’s collision hypothesis. Routh [21] used Poisson’s hypothesis in connection with the planar one-point, two-body collision, introducing his graphical approach. Wang and Mason reconsidered the same problem in [22], using both Newton’s and Poisson’s hypotheses. Stronge applied his hypothesis to two examples in [23]—the compound pendulum and the double pendulum. Finally, Smith and Liu [18] examined all three hypotheses by reference to a body hitting a plane.

At this point, the following question suggests its, namely, can a *complete* solution be set forth, for which *existence, uniqueness, coherence, and energy-consistenc*e are guaranteed (a coherent solution is one ensuring a positive normal impulse, positive normal velocity of separation, and a positive, zero or negative tangential velocity of separation for forward sliding, sticking and reverse sliding, respectively) for a broad class of systems, preferably simple non-holonomic.

A number of authors offer such general solutions based upon Poisson's hypothesis. For example, Lankarani [24] suggests a solution for systems with open or closed kinematical chains. Djerassi [26] presents a solution for simple, non-holonomic systems, proven to be 'complete'. Accordingly, it is the purpose of this work to formulate a Stronge's hypothesis-based solution for a one-point collision of planar, simple non-holonomic systems, which can be readily implemented in an existing simulation code; and prove its 'completeness'. Such a task has not been undertaken to date.

To this end, equations underlying a one-point collision of a simple, non-holonomic system are developed in Sect. 2. Sections 3 and 4 are devoted to the development of a Stronge's hypothesis-based collision theory, and to the analysis of the associated five types of collision and their regions of application. Next, it is shown in Sect. 5 that the mapping of all possible configurations of planar systems conducted in [26] with the aid of Routh's semi-graphical method, is applicable to Stronge's hypothesis-based solution, and can be used to determine the type of collision. Kane's double-pendulum of [20] is used as an illustrative example in Sect. 6. A discussion of the one- and two-integration procedures is conducted in Sect. 7, with reference to the two-sled collision example of [26]. A short summary in Sect. 8 concludes this work.

2 Preliminaries

Let

$$F_r + F_r^* = 0 \quad (r = 1, \dots, p) \tag{1}$$

be Kane's equations of motion for S, a simple, non-holonomic system of v particles P_i $(i = 1, \dots, v)$ of mass m_i, possessing p independent generalized speeds $u_1, \dots, u_p$ and n $(n > p)$ generalized coordinates $q_1, \dots, q_n$, where F_r and F_r^* are, respectively, the rth generalized active force and the rth generalized inertia force for S (Kane and Levinson [20]). $\mathbf{v}^{P_i}$, the velocity of P_i in N, a Newtonian reference frame, can be expressed in terms of $u_1, \dots, u_p, q_1, \dots, q_n$ and time t as

$$\mathbf{v}^{P_i} = \sum_{r=1}^{P} \mathbf{v}_r^{P_i} u_r + \mathbf{v}_t^{P_i} \quad (i = 1, \dots, v) \tag{2}$$

where $\mathbf{v}_r^{P_i}$, called the rth partial velocity of P_i, and $\mathbf{v}_t^{P_i}$, called the remainder partial velocity of P_i, are functions of $q_1, \dots, q_n$ and t. Let B and B' be bodies of S, and let P be a point of body B coming into contact with point P' of body B' during the collision of B with B' occurring between two instants t_1 and t_2. Let $\mathbf{v}^R$ be the relative velocity of points P and P', defined as

$$\mathbf{v}^R \triangleq \mathbf{v}^P - \mathbf{v}^{P'}, \tag{3}$$

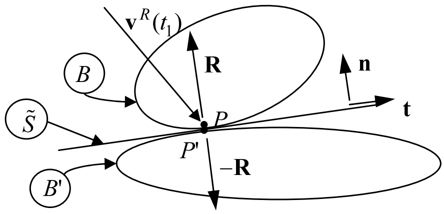
and note that $\mathbf{v}^R$ can be written similarly to $\mathbf{v}^{P_i}$ in (2), hence that

$$\mathbf{v}_r^R = \mathbf{v}_r^P - \mathbf{v}_r^{P'}, \tag{4a}$$

$$\mathbf{v}_t^R = \mathbf{v}_t^P - \mathbf{v}_t^{P'}, \tag{4b}$$

where $\mathbf{v}_r^R$ is the coefficient of u_r in $\mathbf{v}^R$. Suppose that, during collision, P' exerts on P a force $\mathbf{R}$, so that P exerts on P' a force $-\mathbf{R}$. Then (1) give way to equations that bring into

Fig. 1 2D collision



evidence the contributions of $\mathbf{R}$, i.e.,

$$F_r + F_r^* + \mathbf{R} \cdot \mathbf{v}_r^P - \mathbf{R} \cdot \mathbf{v}_r^{P'} = 0 \quad (r = 1, \dots, p; t_1 \le t \le t_2) \tag{5}$$

or, in view of (4a),

$$F_r + F_r^* + \mathbf{R} \cdot \mathbf{v}_r^R = 0 \quad (r = 1, \dots, p). \tag{6}$$

During the collision, P is assumed to maintain contact with P', i.e., to coincide with P'; and a plane $\tilde{S}$ exists which passes through P ($\equiv P'$) and is tangent to B and B' at P if both are locally smooth, or to B' if only B' is locally smooth. Name B and B' such that $\mathbf{n}$, a unit vector perpendicular to $\tilde{S}$, makes $\mathbf{v}^R(t_1) \cdot \mathbf{n}$ a non-positive quantity. Aline $\mathbf{t}$, a unit vector lying in $\tilde{S}$, with the projection of $\mathbf{v}^R(t_1)$ on $\tilde{S}$, making $\mathbf{v}^R(t_1) \cdot \mathbf{t}$ a non-negative quantity (see Fig. 1). Then

$$\mathbf{v}^R(t) = \mathbf{v}^R(t) \cdot \mathbf{nn} + \mathbf{v}^R(t) \cdot \mathbf{tt}, \tag{7a}$$

$$\mathbf{v}^R(t_1) \cdot \mathbf{n} \le 0, \tag{7b}$$

$$\mathbf{v}^R(t_1) \cdot \mathbf{t} \ge 0. \tag{7c}$$

$\mathbf{v}^R(t_1)$ and $\mathbf{v}^R(t_2)$ are called velocity of approach and velocity of separation, respectively (as in [20]). Equation (7) makes it possible to replace (6) with

$$F_r + F_r^* + \mathbf{R} \cdot \mathbf{n} \mathbf{v}_r^R \cdot \mathbf{n} + \mathbf{R} \cdot \mathbf{t} \mathbf{v}_r^R \cdot \mathbf{t} = 0 \quad (r = 1, \dots, p; t_1 \le t \le t_2). \tag{8}$$

If it is assumed that $t_2 - t_1$ is 'small' compared to time constants associated with the motion of S, and that consequently, $q_1, \dots, q_n$ and t remain constants between t_1 and t_2, then both sides of (8) can be integrated from t_1 to $t \le t_2$, yielding

$$\sum_{s=1}^{P} m_{rs} \Delta u_s + I_n \mathbf{v}_r^R \cdot \mathbf{n} + I_t \mathbf{v}_r^R \cdot \mathbf{t} = 0 \quad (r = 1, \dots, p). \tag{9}$$

Here, I_n and I_t are the normal and tangential impulses at time t, defined as

$$I_n \triangleq \left(\int_{t_1}^{t} \mathbf{R} \, \mathrm{d}t \right) \cdot \mathbf{n}, \qquad I_t \triangleq \left(\int_{t_1}^{t} \mathbf{R} \, \mathrm{d}t \right) \cdot \mathbf{t} \quad (t < t_2), \tag{10}$$

$\mathbf{v}_r^R \triangleq \mathbf{v}_r^R(t)$ $(t_1 \le t \le t_2, r = 1, \dots, p)$ (see (4a)), Δu_s $(s = 1, \dots, p)$ are defined as

$$\Delta u_s \triangleq u_s(t) - u_s(t_1) \quad (s = 1, \dots, p; t \le t_2), \tag{11}$$

and m_{rs} is the entry in row r, column s of the mass matrix $\mathbf{M}$ associated with (1) (see (A.1), Appendix A). Note that $\mathbf{n}$ and $\mathbf{t}$, defined only for $t_1 \le t \le t_2$, remain fixed in N during the collision. Also note that $\mathbf{R} \cdot \mathbf{n}$ $(t_1 \leq t \leq t_2) > 0$ (P' cannot 'pull' P), hence $I_n > 0$. The matrix form of (9), solved for Δu_s $(s = 1, \ldots, p)$, reads

$$|\Delta u_1 \cdots \Delta u_p| = -\left|\mathbf{v}_1^R \cdot \mathbf{n} \cdots \mathbf{v}_p^R \cdot \mathbf{n}\right|\mathbf{M}^{-1} I_n - \left|\mathbf{v}_1^R \cdot \mathbf{t} \cdots \mathbf{v}_p^R \cdot \mathbf{t}\right|\mathbf{M}^{-1} I_t, \tag{12}$$

where $\mathbf{M}$ and $\mathbf{M}^{-1}$ are negative definite matrices (see Appendix A). Denoting

$$\mathbf{V}_n \triangleq \left|\mathbf{v}_1^R \cdot \mathbf{n} \cdots \mathbf{v}_p^R \cdot \mathbf{n}\right|, \qquad \mathbf{V}_t \triangleq \left|\mathbf{v}_1^R \cdot \mathbf{t} \cdots \mathbf{v}_p^R \cdot \mathbf{t}\right|, \tag{13}$$

$$\mathbf{u} = \left|u_1(t) \cdots u_p(t)\right|, \tag{14a}$$

$$\mathbf{u}_1 = \left|u_1(t_1) \cdots u_p(t_1)\right|, \tag{14b}$$

$$\mathbf{u}_2 = \left|u_1(t_2) \cdots u_p(t_2)\right|, \tag{14c}$$

one can replace (12) with

$$\mathbf{u} - \mathbf{u}_1 = -\mathbf{V}_n \mathbf{M}^{-1} I_n - \mathbf{V}_t \mathbf{M}^{-1} I_t. \tag{15}$$

Now, the quantities of interest in the context of simulations of motion of multibody systems undergoing collisions are the entries of $\mathbf{u}_2$ (equation (14c)), which play the role of initial conditions for the post-collision motion. However, these cannot be obtained from (15) straightforwardly since I_n and I_t are not known. Two additional conditions must therefore be imposed. Here, it is shown that Stronge's hypothesis and Coulomb's coefficient of friction, in conjunction with the 'two-integration' procedure discussed in Sect. 7, lead to closed form solutions $\mathbf{u}_2$ for simple non-holonomic planar systems which are unique, coherent, and energy consistent. To this end, two instants t_C and t_S associated with inequalities (7b) and (7c) are introduced as follows. Inequality (7b) indicates that every collision starts with 'compression' (initial negative normal velocity), which comes to an end at time t_C. Then 'restitution' prevails from t_C to t_2. Similarly, inequality (7c) indicates that every collision starts with a forward sliding (initial positive tangential velocity), which may come to an end at time t_S. Then either sticking or reverse sliding prevail from t_S to t_2. Accordingly, a piecewise continuity of I_n and I_t is implied, which, in turn, gives rise to five types of collision, hence to five different expressions for $\mathbf{u}_2$. These can be generated with the aid of the differential form of (15), which reads

$$\mathrm{d}\mathbf{u} = -\mathbf{V}_n \mathbf{M}^{-1}\,\mathrm{d}I_n - \mathbf{V}_t \mathbf{M}^{-1}\,\mathrm{d}I_t; \qquad \mathrm{d}\mathbf{u} \triangleq \left|du_1(t) \cdots du_p(t)\right|. \tag{16}$$

Before this objective is pursued, it is noted that if, and only if

$$\sum_{i=1}^{v} m_i \mathbf{v}_t^{P_i} \cdot \left[\mathbf{v}^{P_i}(t_2) - \mathbf{v}^{P_i}(t_1)\right] = 0, \tag{17}$$

then ΔE, the change in the mechanical energy of the system, can be written as

$$\Delta E = 1/2\mathbf{u}_1 \mathbf{M} \mathbf{u}_1^T - 1/2\mathbf{u}_2 \mathbf{M} \mathbf{u}_2^T, \tag{18}$$

a matrix form of (A.5) in Appendix A. If $\mathbf{v}_t^R(t_1) = 0$ and $\mathbf{v}_t^{P_i}(t_1) = 0$ $(i = 1, \ldots, v)$, then by (17) above and (28)–(29) in [25]

$$\Delta E = 1/2 I_n(t_2)\left[\mathbf{v}^R(t_2) + \mathbf{v}^R(t_1)\right] \cdot \mathbf{n} + 1/2 I_t(t_2)\left[\mathbf{v}^R(t_2) + \mathbf{v}^R(t_1)\right] \cdot \mathbf{t}.$$

3 A collision theory with Stronge's hypothesis and Coulomb's friction coefficient

Let ΔE_{nR} (>0) and ΔE_{nC} (<0) be the works done by the normal contact forces (exerted by the colliding points on one another) during the restitution phase (increasing the system kinetic energy) and during the compression phase (decreasing the system kinetic energy), respectively. Then in accordance with Stronge's hypothesis [7], the ratio $-\Delta E_{nR}/\Delta E_{nC}$ is constant, and equals the square of a quantity e_e called Stronge's coefficient of restitution, namely

$$e_e^2 \triangleq -\Delta E_{nR}/\Delta E_{nC}, \quad 0 \le e_e \le 1. \tag{19}$$

Define ΔI_n and ΔI_t as $\Delta I_n \triangleq (\int_{t'}^{t''} \mathbf{R}\,\mathrm{d}t)\cdot\mathbf{n}$, $\Delta I_t \triangleq (\int_{t'}^{t''} \mathbf{R}\,\mathrm{d}t)\cdot\mathbf{t}$, $t_1 \le t' \le t'' \le t_2$, and let I_n and I_t be continuous in the range $t' - t''$. Then the theory in question stipulates that if

$$|\Delta I_t| < \mu \Delta I_n \tag{20}$$

where μ is Coulomb's static coefficient of friction, then sticking occurs, i.e.,

$$\mathbf{v}^R(t)\cdot\mathbf{t} = 0 \quad (t' \le t \le t''). \tag{21}$$

If inequality (20) is violated, then forward sliding or reverse sliding take place, hence

$$\Delta I_t \triangleq -\mu' \Delta I_n \mathbf{v}^R(t)\cdot\mathbf{t}/\left|\mathbf{v}^R(t)\cdot\mathbf{t}\right| \quad (t' \le t \le t''), \tag{22}$$

where μ' is Coulomb's dynamic coefficient of friction. In this work no distinction is made between μ' and μ (see, e.g., Whittaker's [11] and Keller's [14] analyses) as it may lead to inconclusive solutions discussed in Sect. 4.2 of [25].

Equation (16) makes it possible to express $\mathbf{u}_2$ in terms of m_{nn}, m_{tt} and m_{nt}, configuration dependent quantities defined as

$$m_{nn} \triangleq -\mathbf{V}_n\mathbf{M}^{-1}\mathbf{V}_n^T > 0, \tag{23a}$$

$$m_{nt} \triangleq -\mathbf{V}_n\mathbf{M}^{-1}\mathbf{V}_t^T, \tag{23b}$$

$$m_{tt} \triangleq -\mathbf{V}_t\mathbf{M}^{-1}\mathbf{V}_t^T > 0 \tag{23c}$$

in conjunction with g, h, p, q, Δ, r_m and α, quantities defined as

$$\begin{aligned} &g \triangleq \mu m_{tt} - m_{nt}, \qquad h \triangleq m_{nn} - \mu m_{nt}, \\ &p \triangleq \mu m_{tt} + m_{nt}, \qquad q \triangleq m_{nn} + \mu m_{nt}, \qquad \Delta \triangleq m_{nn}m_{tt} - m_{nt}^2, \end{aligned} \tag{24}$$

$$r_m \triangleq g/h = (\mu m_{tt} - m_{nt})/(m_{nn} - \mu m_{nt}), \qquad \alpha \triangleq \mathrm{v}_t^A/\left|\mathrm{v}_n^A\right| \quad (0 < \alpha < \infty). \tag{25}$$

Accordingly, $\mathbf{u}_2$ can be formulated for five types of collision in terms of the indicated system parameters and of $\mathbf{u}_1$, a task undertaken next. With

$$\mathrm{v}_n \triangleq \mathbf{u}\mathbf{V}_n^T, \qquad \mathrm{v}_t \triangleq \mathbf{u}\mathbf{V}_t^T, \tag{26}$$

the following notation will be used: $\mathbf{u}_S = \mathbf{u}(t_S)$ and $\mathbf{u}_C = \mathbf{u}(t_C)$ designate the values of $\mathbf{u}$ at t_S and t_C; $\mathrm{v}_{n1} = \mathbf{u}_1\mathbf{V}_n^T$, $\mathrm{v}_{tS} = \mathbf{u}_S\mathbf{V}_t^T$, $I_{nC} = I_n(t_C)$ and $I_{t2} = I_1(t_2)$ designate the values of v_n, v_t, I_n and I_t at t_1, t_S, t_C and t_2, respectively, etc. Finally, E_{nS-C} designates the work done by the normal impulsive force between t_S and t_C, E_{tC-2} designates the work done by tangential impulsive force between t_C and t_2, etc.

4 Types of collision

Type 1. *Sticking in compression*, comprising sliding ($t_1 \le t \le t_S$) and sticking in compression ($t_S \le t \le t_C$) and sticking in restitution ($t_C \le t \le t_2$). Here,

$$\mathrm{v}_{nC} = 0, \tag{27a}$$

$$\mathrm{v}_{tS} = 0, \tag{27b}$$

$$\mathrm{v}_{tC} = 0, \tag{27c}$$

$$\mathrm{v}_{t2} = 0 \tag{27d}$$

(Fig. 2). Integrating both sides of (16) from t_1 to t $(\le t_S)$ from t_S to t $(\le t_C)$ and from t_C to t $(\le t_2)$, one obtains the relations

$$t_1 \le t \le t_S \quad \mathbf{u} = \mathbf{u}_1 - (\mathbf{V}_n - \mu \mathbf{V}_t)\mathbf{M}^{-1} I_n \quad (I_t = -\mu I_n), \tag{28}$$

$$t_S \le t \le t_C \quad \mathbf{u} = \mathbf{u}_S - \mathbf{V}_n \mathbf{M}^{-1}(I_n - I_{nS}) - \mathbf{V}_t \mathbf{M}^{-1}(I_t - I_{tS}), \tag{29}$$

$$t_C \le t \le t_2 \quad \mathbf{u} = \mathbf{u}_C - \mathbf{V}_n \mathbf{M}^{-1}(I_n - I_{nC}) - \mathbf{V}_t \mathbf{M}^{-1}(I_t - I_{tC}) \tag{30}$$

which when post-multiplied throughout by $\mathbf{V}_n^T$ and $\mathbf{V}_t^T$, lead to

$$t_1 \le t \le t_S \quad \mathrm{v}_n \underset{(28)}{=} \mathrm{v}_{n1} + h I_n,^{1} \tag{31}$$

$$\mathrm{v}_t \underset{(28)}{=} \mathrm{v}_{t1} - g I_n; \tag{32}$$

$$t_S \le t \le t_C \quad \mathrm{v}_n \underset{(29)}{=} \mathrm{v}_{nS} + m_{nn}(I_n - I_{nS}) + m_{nt}(I_t - I_{tS}), \tag{33}$$

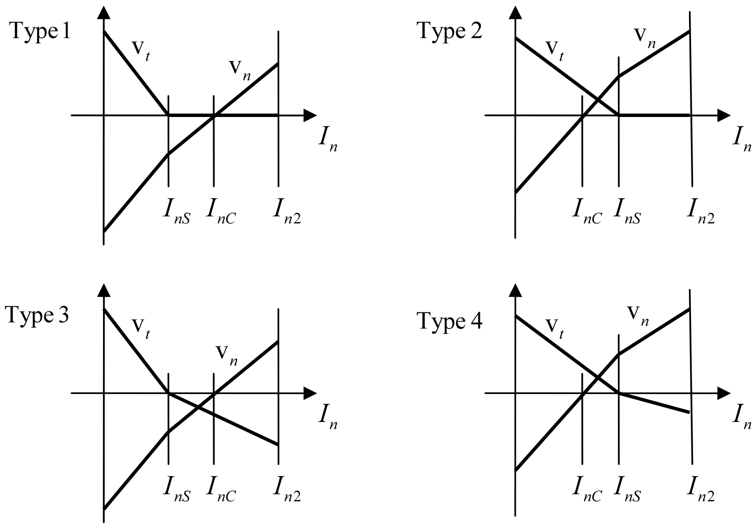


Fig. 2 v_n and v_t vs. I_n for collision Types 1–4

[1]Numbers appearing under equal signs refer to equations numbered correspondingly.

$$\mathrm{v}_t(=0) \underset{(29)}{=} \mathrm{v}_{tS}(\underset{(27b)}{=} 0) + m_{nt}(I_n - I_{nS}) + m_{tt}(I_t - I_{tS}); \tag{34}$$

$$t_C \le t \le t_2 \quad \mathrm{v}_n \underset{(30)}{=} \mathrm{v}_{nC}(\underset{(27a)}{=} 0) + m_{nn}(I_n - I_{nC}) + m_{nt}(I_t - I_{tC}), \tag{35}$$

$$\mathrm{v}_t(=0) \underset{(30)}{=} \mathrm{v}_{tC}(\underset{(27c)}{=} 0) + m_{nt}(I_n - I_{nC}) + m_{tt}(I_t - I_{tC}) \tag{36}$$

(see (23)–(26)). With E_{n1-S}, E_{nS-C} and E_{nC-2} of Appendix B, (19) becomes

$$e_e^2 = -\Delta E_{nR}/\Delta E_{nC} = -E_{nC-2}/(E_{n1-S} + E_{nS-C}); \tag{37}$$

and the use of (B.1)–(B.3) of Appendix B in (37) yields, for I_{n2}, the expression

$$I_{n2} = I_{nC} - \mathrm{v}_{n1} e_e/\Delta\sqrt{m_{tt}\big(m_{tt} + 2\alpha m_{nt} - m_{nt}/r_m\alpha^2\big)}. \tag{38}$$

$\mathbf{u}_S$, $\mathbf{u}_C$, and $\mathbf{u}_2$ are obtained from (28)–(30). Successive substitutions lead to

$$\mathbf{u}_2 = \mathbf{u}_1 - (\mathbf{V}_n - \mu\mathbf{V}_t)\mathbf{M}^{-1} I_{nS} - (\mathbf{V}_n - m_{nt}/m_{tt}\mathbf{V}_t)\mathbf{M}^{-1}(I_{n2} - I_{nS}) \tag{39}$$

in view of (34) and (36). With (31)–(34) and (27) one can show that

$$I_{nS} \underset{(32)}{=} \mathrm{v}_{t1}/g, \tag{40a}$$

$$I_{nC} \underset{(33),(34)}{=} -\mathrm{v}_{n1}(m_{tt} + \alpha m_{nt})/\Delta, \tag{40b}$$

$$\mathrm{v}_{nS} \underset{(31)}{=} -\mathrm{v}_{n1}(\alpha/r_m - 1). \tag{40c}$$

Post-multiplying (39) throughout with $\mathbf{V}_n^T$ and $\mathbf{V}_t^T$, one obtains for $t = t_2$

$$\mathrm{v}_{n2} \underset{(26)}{=} \mathbf{u}_2\mathbf{V}_n^T = \mathrm{v}_{n1} + hI_{nS} + \Delta/m_{tt}(I_{n2} - I_{nS}), \tag{41a}$$

$$\mathrm{v}_{t2} \underset{(40),(27d)}{=} \mathbf{u}_2\mathbf{V}_t^T = 0. \tag{41b}$$

By (40) $I_{nS} > 0$ and $\mathrm{v}_{nS} < 0$ if, and only if

$$g \underset{(40a)}{>} 0, \qquad h > 0 \text{ and } 0 < \alpha \underset{(40c)}{<} r_m, \qquad h < 0\ (\Rightarrow m_{nt} > 0) \text{ and } \alpha \underset{(40c)}{>} 0. \tag{42}$$

These inequalities are called coherence conditions; and their satisfaction guarantees

$$I_{n2} \underset{(38)}{>} I_{nC} \underset{(40b)}{>} I_{nS} \underset{(40a)}{>} 0, \qquad \mathrm{v}_{n2} \underset{(41a)}{>} 0. \tag{43}$$

Type 2. *Sticking in restitution*, comprising sliding in compression ($t_1 \le t \le t_C$) and sliding ($t_C \le t \le t_S$) and sticking in restitution ($t_S \le t \le t_2$). Here,

$$\mathrm{v}_{nC} = 0, \tag{44a}$$

$$\mathrm{v}_{tS} = 0, \tag{44b}$$

$$\mathrm{v}_{t2} = 0 \tag{44c}$$

(Fig. 2). Integrating both sides of (16) from t_1 to t $(\leq t_C)$ from t_C to t $(\leq t_S)$ and from t_S to t $(\leq t_2)$, one obtains the relations

$$t_1 \leq t \leq t_C \quad \mathbf{u} = \mathbf{u}_1 - (\mathbf{V}_n - \mu \mathbf{V}_t)\mathbf{M}^{-1} I_n \quad (I_t = -\mu I_n), \tag{45}$$

$$t_C \leq t \leq t_S \quad \mathbf{u} = \mathbf{u}_C - (\mathbf{V}_n - \mu \mathbf{V}_t)\mathbf{M}^{-1}(I_n - I_{nC}) \quad \left[I_t - I_{tC} = -\mu(I_n - I_{nC})\right], \tag{46}$$

$$t_S \leq t \leq t_2 \quad \mathbf{u} = \mathbf{u}_S - \mathbf{V}_n \mathbf{M}^{-1}(I_n - I_{nS}) - \mathbf{V}_t \mathbf{M}^{-1}(I_t - I_{tS}) \tag{47}$$

which, when post-multiplied throughout by $\mathbf{V}_n^T$ and $\mathbf{V}_t^T$, lead to

$$t_1 \leq t \leq t_C \quad \mathrm{v}_n \underset{(45)}{=} \mathrm{v}_{n1} + h I_n, \tag{48}$$

$$\mathrm{v}_t \underset{(45)}{=} \mathrm{v}_{t1} - g I_n; \tag{49}$$

$$t_C \leq t \leq t_S \quad \mathrm{v}_n \underset{(46)}{=} \mathrm{v}_{nC}(\underset{(44a)}{=} 0) + h(I_n - I_{nC}), \tag{50}$$

$$\mathrm{v}_t \underset{(46)}{=} \mathrm{v}_{tC} - g(I_n - I_{nC}); \tag{51}$$

$$t_S \leq t \leq t_2 \quad \mathrm{v}_n \underset{(47)}{=} \mathrm{v}_{nS} + m_{nn}(I_n - I_{nS}) + m_{nt}(I_t - I_{tS}), \tag{52}$$

$$\mathrm{v}_t(=0) \underset{(47)}{=} \mathrm{v}_{tS}(\underset{(44b)}{=} 0) + m_{nt}(I_n - I_{nS}) + m_{tt}(I_t - I_{tS}) \tag{53}$$

(see (23)–(26)). With E_{n1-C}, E_{nC-S} and E_{nS-2}, of Appendix B, (19) becomes

$$e_e^2 = -\Delta E_{nR}/\Delta E_{nC} = -(E_{nC-S} + E_{nS-2})/E_{n1-C}; \tag{54}$$

and the use of (B.8)–(B.10) of Appendix B in (54) yields, for I_{n2}, the expression

$$I_{n2} = I_{nS} + \mathrm{v}_{n1}(\alpha/r_m - 1)m_{tt}/\Delta - \mathrm{v}_{n1}\sqrt{\left[(\alpha/r_m - 1)m_{tt}/\Delta\right]^2 - m_{tt}/\Delta\left[(\alpha/r_m - 1)^2 - e_e^2\right]/h}. \tag{55}$$

$\mathbf{u}_S$, $\mathbf{u}_C$, and $\mathbf{u}_2$ are obtained from (45)–(47). Successive substitutions lead to

$$\mathbf{u}_2 = \mathbf{u}_1 - (\mathbf{V}_n - \mu \mathbf{V}_t)\mathbf{M}^{-1} I_{nS} - (\mathbf{V}_n - m_{nt}/m_{tt}\mathbf{V}_t)\mathbf{M}^{-1}(I_{n2} - I_{nS}) \tag{56}$$

in view of (53). With (48)–(51) and (44) one can show that

$$I_{nS} \underset{(51),(49)}{=} \mathrm{v}_{t1}/g, \tag{57a}$$

$$I_{nC} \underset{(48)}{=} \mathrm{v}_{n1}/h, \tag{57b}$$

$$\mathrm{v}_{nS} \underset{(50)}{=} -\mathrm{v}_{n1}(\alpha/r_m - 1). \tag{57c}$$

Post-multiplying (56) throughout with $\mathbf{V}_n^T$ and $\mathbf{V}_t^T$, one obtains for $t = t_2$

$$\mathrm{v}_{n2} = \mathbf{u}_2\mathbf{V}_n^T = \mathrm{v}_{n1} + h I_{nS} + \Delta/m_{tt}(I_{n2} - I_{nS}), \tag{58a}$$

$$\mathrm{v}_{t2} \underset{(57),(44c)}{=} \mathbf{u}_2\mathbf{V}_t^T = 0. \tag{58b}$$

By (57) and (55) $I_{nS} > 0$, $I_{nC} > 0$, $\mathrm{v}_{nS} > 0$ and $I_{n2} > I_{nS}$ if, and only if

$$g \underset{(57a)}{>} 0, \qquad h \underset{(57b)}{>} 0, \qquad r_m \underset{(57c)}{<} \alpha \underset{(55)}{<} (1+e_e)r_m \tag{59}$$

(coherence conditions); and the satisfaction of inequalities (59) guarantees

$$I_{n2} \underset{(55)}{>} I_{nS} \underset{(57b)}{>} I_{nC} \underset{(57a)}{>} 0, \qquad \mathrm{v}_{n2} \underset{(58a)}{>} 0. \tag{60}$$

Type 3. *Reverse sliding in compression*, comprising sliding ($t_1 \le t \le t_S$) and reverse sliding in compression ($t_S \le t \le t_C$) and reverse sliding in restitution ($t_C \le t \le t_2$). Here,

$$\mathrm{v}_{nC} = 0, \tag{61a}$$

$$\mathrm{v}_{tS} = 0 \tag{61b}$$

(Fig. 2). Integrating both sides of (16) from t_1 to t $(\le t_S)$ from t_S to t $(\le t_C)$ and from t_C to t $(\le t_2)$, one obtains the relations

$$t_1 \le t \le t_S \quad \mathbf{u} = \mathbf{u}_1 - (\mathbf{V}_n - \mu\mathbf{V}_t)\mathbf{M}^{-1} I_n \quad (I_t = -\mu I_n), \tag{62}$$

$$t_S \le t \le t_C \quad \mathbf{u} = \mathbf{u}_S - (\mathbf{V}_n + \mu\mathbf{V}_t)\mathbf{M}^{-1}(I_n - I_{nS}) \quad \left[I_t - I_{tS} = \mu(I_n - I_{nS})\right], \tag{63}$$

$$t_C \le t \le t_2 \quad \mathbf{u} = \mathbf{u}_C - (\mathbf{V}_n + \mu\mathbf{V}_t)\mathbf{M}^{-1}(I_n - I_{nC}) \quad \left[I_t - I_{tC} = \mu(I_n - I_{nC})\right] \tag{64}$$

which, when post-multiplied by $\mathbf{V}_n^T$ and $\mathbf{V}_t^T$, lead to

$$t_1 \le t \le t_S \quad \mathrm{v}_n \underset{(62)}{=} \mathrm{v}_{n1} + hI_n, \tag{65}$$

$$\mathrm{v}_t \underset{(62)}{=} \mathrm{v}_{t1} - gI_n; \tag{66}$$

$$t_S \le t \le t_C \quad \mathrm{v}_n \underset{(63)}{=} \mathrm{v}_{nS} + q(I_n - I_{nS}), \tag{67}$$

$$\mathrm{v}_t \underset{(63)}{=} \mathrm{v}_{tS}(\underset{(61b)}{=} 0) + p(I_n - I_{nS}); \tag{68}$$

$$t_C \le t \le t_2 \quad \mathrm{v}_n \underset{(64)}{=} \mathrm{v}_{nC}(\underset{(61a)}{=} 0) + q(I_n - I_{nC}), \tag{69}$$

$$\mathrm{v}_t \underset{(64)}{=} \mathrm{v}_{tC} + p(I_n - I_{nC}) \tag{70}$$

(see (23)–(26)). With E_{n1-S}, E_{nS-C} and E_{nC-2}, of Appendix B, (19) becomes

$$e_e^2 = -\Delta E_{nR}/\Delta E_{nC} = -E_{nC-2}/(E_{n1-S} + E_{nS-C}); \tag{71}$$

and the use of (B.15)–(B.17) of Appendix B in (71) yield, for I_{n2}, the expression

$$I_{n2} = I_{nC} - \mathrm{v}_{n1} e_e/q\sqrt{1 + 4\mu m_{nt}\alpha/g - 2\mu m_{nt}\alpha^2/(g r_m)}. \tag{72}$$

$\mathbf{u}_S$, $\mathbf{u}_C$, and $\mathbf{u}_2$ are obtained from (62)–(64). Successive substitutions lead to

$$\mathbf{u}_2 = \mathbf{u}_1 - (\mathbf{V}_n - \mu\mathbf{V}_t)\mathbf{M}^{-1} I_{nS} - (\mathbf{V}_n + \mu\mathbf{V}_t)\mathbf{M}^{-1}(I_{n2} - I_{nS}). \tag{73}$$

With (65)–(68) and (61) one can show that

$$I_{nS} \underset{(66)}{=} \mathrm{v}_{t1}/g, \tag{74a}$$

$$I_{nC} \underset{(67)}{=} -\mathrm{v}_{n1}(1+2\mu m_{nt}\alpha/g)/q, \tag{74b}$$

$$\mathrm{v}_{nS} \underset{(65)}{=} -\mathrm{v}_{n1}(\alpha/r_m - 1). \tag{74c}$$

Post-multiplying (73) throughout with $\mathbf{V}_n^T$ and $\mathbf{V}_t^T$, one obtains for $t = t_2$

$$\mathrm{v}_{n2} = \mathbf{u}_2\mathbf{V}_n^T = \mathrm{v}_{n1} + hI_{nS} + q(I_{n2} - I_{nS}), \tag{75a}$$

$$\mathrm{v}_{t2} = \mathbf{u}_2\mathbf{V}_t^T = p(I_{n2} - I_{nS}). \tag{75b}$$

Assume that $I_{n2} > I_{nS}$. Then by (74)–(75) $I_{nS} > 0$, $\mathrm{v}_{nS} < 0$, $\mathrm{v}_{t2} < 0$ and $I_{nC} > 0$ if, and only if

$$g \underset{(74a)}{>} 0, \quad 0 \underset{(74c)}{<} \alpha < r_m, \quad p \underset{(75b)}{<} 0\,(\Rightarrow m_{nt} < 0 \Rightarrow) \quad h > 0, \quad q \underset{(74b)}{>} 0 \tag{76}$$

(coherence conditions); and the satisfaction of inequalities (76) guarantees

$$I_{n2} \underset{(72)}{>} I_{nC} \underset{(74b)}{>} I_{nS} \underset{(74a)}{>} 0, \qquad \mathrm{v}_{n2} \underset{(75a)}{>} 0. \tag{77}$$

Type 4. *Reverse sliding in restitution*, comprising sliding in compression ($t_1 \le t \le t_C$), sliding ($t_C \le t \le t_S$) and reverse sliding in restitution ($t_S \le t \le t_2$). Here,

$$\mathrm{v}_{nC} = 0, \tag{78a}$$

$$\mathrm{v}_{tS} = 0 \tag{78b}$$

(Fig. 2). Integrating both sides of (16) from t_1 to t $(\le t_S)$ from t_S to t $(\le t_C)$ and from t_C to t $(\le t_2)$, one obtains the relations

$$t_1 \le t \le t_C \quad \mathbf{u} = \mathbf{u}_1 - (\mathbf{V}_n - \mu\mathbf{V}_t)\mathbf{M}^{-1}I_n \quad (I_t = -\mu I_n), \tag{79}$$

$$t_C \le t \le t_S \quad \mathbf{u} = \mathbf{u}_C - (\mathbf{V}_n - \mu\mathbf{V}_t)\mathbf{M}^{-1}(I_n - I_{nC}) \quad \big[I_t - I_{tC} = -\mu(I_n - \mu I_{nC})\big], \tag{80}$$

$$t_S \le t \le t_2 \quad \mathbf{u} = \mathbf{u}_S - (\mathbf{V}_n - \mu\mathbf{V}_t)\mathbf{M}^{-1}(I_n - I_{nS}) \quad \big[I_t - I_{tS} = -\mu(I_n - \mu I_{nS})\big] \tag{81}$$

which, when post-multiplied by $\mathbf{V}_n^T$ and $\mathbf{V}_t^T$, lead to

$$t_1 \le t \le t_C \quad \mathrm{v}_n \underset{(79)}{=} \mathrm{v}_{n1} + hI_n, \tag{82}$$

$$\mathrm{v}_t \underset{(79)}{=} \mathrm{v}_{t1} - gI_n; \tag{83}$$

$$t_C \le t \le t_S \quad \mathrm{v}_n \underset{(80)}{=} \mathrm{v}_{nC}(\underset{(78a)}{=} 0) + h(I_n - I_{nC}), \tag{84}$$

$$\mathrm{v}_t \underset{(80)}{=} \mathrm{v}_{tC} - g(I_n - I_{nC}); \tag{85}$$

$$t_S \le t \le t_2 \quad \mathrm{v}_n \underset{(81)}{=} \mathrm{v}_{nS} + q(I_n - I_{nS}), \tag{86}$$

$$\mathrm{v}_t \underset{(81)}{=} \mathrm{v}_{tS}(\underset{(78b)}{=} 0) + p(I_n - I_{nS}) \tag{87}$$

(see (23)–(26)). With E_{n1-C}, E_{nC-S} and E_{nS-2} of Appendix B, (19) becomes

$$e_e^2 = -\Delta E_{nR}/\Delta E_{nC} = -(E_{nC-S} + E_{nS-2})/E_{n1-C}; \tag{88}$$

and the use of (B.22)–(B.24) of Appendix B in (88) yields, for I_{n2}, the expression

$$\begin{aligned} I_{n2} &= I_{nS} + \mathrm{v}_{n1}(\alpha/r_m - 1)/q \\ &\quad - \mathrm{v}_{n1}\sqrt{\left[(\alpha/r_m - 1)/q\right]^2 - \left[(\alpha/r_m - 1)^2 - e_e^2\right]/(qh)}. \end{aligned} \tag{89}$$

$\mathbf{u}_S$, $\mathbf{u}_C$, and $\mathbf{u}_2$ are obtained from (79)–(81). Successive substitutions lead to

$$\mathbf{u}_2 = \mathbf{u}_1 - (\mathbf{V}_n - \mu\mathbf{V}_t)\mathbf{M}^{-1}I_{nS} - (\mathbf{V}_n + \mu\mathbf{V}_t)\mathbf{M}^{-1}(I_{n2} - I_{nS}). \tag{90}$$

With (82)–(85) and (78) one can show that

$$I_{nS} \underset{(83),(85)}{=} \mathrm{v}_{t1}/g, \tag{91a}$$

$$I_{nC} \underset{(82)}{=} -\mathrm{v}_{n1}/h, \tag{91b}$$

$$\mathrm{v}_{nS} \underset{(84)}{=} -\mathrm{v}_{n1}(\alpha/r_m - 1). \tag{91c}$$

Post-multiplying (90) throughout with $\mathbf{V}_n^T$ and $\mathbf{V}_t^T$, one obtains for $t = t_2$

$$\mathrm{v}_{n2} = \mathbf{u}_2\mathbf{V}_n^T = \mathrm{v}_{n1} + hI_{nS} + q(I_{n2} - I_{nS}), \tag{92a}$$

$$\mathrm{v}_{t2} = \mathbf{u}_2\mathbf{V}_t^T = p(I_{n2} - I_{nS}). \tag{92b}$$

Assume that $I_{n2} > I_{nS}$. Then, by (91), (92) and (89) $I_{nS} > 0$, $I_{nC} > 0$, $\mathrm{v}_{t2} < 0$, $\mathrm{v}_{n2} > 0$ and $\mathrm{v}_{nS} > 0$ if, and only if

$$g \underset{(91a)}{>} 0, \quad h \underset{(91b)}{>} 0, \quad p \underset{(92b)}{<} 0, \quad q \underset{(92a)}{>} 0, \quad r_m \underset{(91c)}{<} \alpha \underset{(89)}{<} (1 + e_e)r_m \tag{93}$$

(coherence conditions); and the satisfaction of inequalities (93) guarantees

$$I_{n2} \underset{(89)}{>} I_{nS} \underset{(91a)}{>} I_{nC} \underset{(91b)}{>} 0, \qquad \mathrm{v}_{n2} \underset{(92a)}{>} 0. \tag{94}$$

Type 5. *Forward sliding* ($t_1 \le t_2$).

Integrating both sides of (16) from t_1 to t ($\le t_C$) and from t_C to t ($\le t_2$), one obtains

$$t_1 \le t \le t_C \quad \mathbf{u} = \mathbf{u}_1 - (\mathbf{V}_n - \mu\mathbf{V}_t)\mathbf{M}^{-1}I_n \quad (I_t = -\mu I_n), \tag{95}$$

$$t_C \le t \le t_2 \quad \mathbf{u} = \mathbf{u}_C - (\mathbf{V}_n - \mu\mathbf{V}_t)\mathbf{M}^{-1}(I_n - I_{nC}) \quad \left[I_t - I_{tC} = -\mu(I_n - I_{nC})\right] \tag{96}$$

which, when post-multiplied by $\mathbf{V}_n^T$ and V_t^T, lead to

$$t_1 \le t \le t_C \quad \mathrm{v}_n \underset{(95)}{=} \mathrm{v}_{n1} + hI_n, \tag{97}$$

$$\mathrm{v}_t \underset{(95)}{=} \mathrm{v}_{t1} - gI_n; \tag{98}$$

$$t_C \le t \le t_2 \quad \mathrm{v}_n \underset{(96)}{=} \mathrm{v}_{nC}(=0) + h(I_n - I_{nC}), \tag{99}$$

$$\mathrm{v}_t \underset{(96)}{=} \mathrm{v}_{tC} - g(I_n - I_{nC}) \tag{100}$$

(see (23)–(26)). With E_{n1-C} and E_{nC-2} of Appendix B, (19) becomes

$$e_e^2 = -\Delta E_{nR}/\Delta E_{nC} = -E_{nC-2}/E_{n1-C}; \tag{101}$$

and the use of (B.29)–(B.30) of Appendix B in (101) yields, for I_{n2}, the expression

$$I_{n2} = -(1+e_e)\mathrm{v}_{n1}/h. \tag{102}$$

$\mathbf{u}_C$ and $\mathbf{u}_2$ are obtained from (95) and (96). Successive substitutions lead to

$$\mathbf{u}_2 = \mathbf{u}_1 - (\mathbf{V}_n - \mu\mathbf{V}_t)\mathbf{M}^{-1}I_{n2}. \tag{103}$$

With (97) one can show that

$$I_{nC} = -\mathrm{v}_{n1}/h. \tag{104}$$

Post-multiplying (103) throughout with $\mathbf{V}_n^T$ and $\mathbf{V}_t^T$, one obtains for $t = t_2$

$$\mathrm{v}_{n2} = \mathbf{u}_2\mathbf{V}_n^T = \mathrm{v}_{n1} + hI_{n2}, \tag{105a}$$

$$\mathrm{v}_{t2} = \mathbf{u}_2\mathbf{V}_t^T = -\mathrm{v}_{n1}\big[\alpha - (1+e_e)r_m\big]. \tag{105b}$$

By (104) and (105) $I_{nC} > 0$ and $\mathrm{v}_{t2} > 0$ if, and only if

$$h \underset{(104)}{>} 0, \quad g > 0 \text{ and } \alpha \underset{(105b)}{>} (1+e_e)r_m, \quad g < 0 \text{ and } \alpha > 0 \tag{106}$$

(coherence conditions); and the satisfaction of inequalities (106) guarantees

$$I_{n2} \underset{(102)}{>} I_{nC} \underset{(104)}{>} 0, \qquad \mathrm{v}_{n2} \underset{(105a)}{>} 0. \tag{107}$$

It is worth noting that if $g < 0$ then, by (98) and (100) v_t increases with I_n.

5 Determination of the type of collision

5.1 Routh semi-graphical method

Routh's impulse diagram, described in [25] and [26], is built on 'forward slide', 'sticking', 'reverse slide' and 'maximum compression' lines, all valid in connection with the integration of both sides of (16). In fact, (28b) in Type 1 (or (44b) in Type 2) comprises the forward sliding line. The sticking line can be obtained from (34) and (40a) in Type 1, (or (53) and (57a) in Type 2, etc.). The reverse sliding line is obtained form the (coordinates of the) intersection point of the forward sliding and the sticking lines, in conjunction with the slope μ; and the maximum compression line can be obtained from (33) and (31) in Type 1 (or (52) and (48) in Type 2, etc.). The line-equations do not involve the coefficient of restitution, hence the μ-bounds given by inequalities (59)–(61) of [25] remain valid:

$$\text{Types 1, 2:} \quad g > 0, \ p > 0; \qquad \text{Type 3, 4:} \quad p < 0; \qquad \text{Type 5:} \quad \text{no limit.} \tag{108}$$

With regards to the α-regions in [26], inequalities (56) and (57) for Types 1 and 3 and the second of inequalities (53) for Types 2 and 4 in [26] remain valid since they do not involve the coefficient of restitution. Moreover, $\alpha \le (1+e_e)r_m$ for Types 2 and 4 ensure

$I_{n2} \geq I_{nS}$ ((55) and (89)), in agreement with the first of inequalities (53) of [26]. Lastly, forward sliding, for which Newton's, Poisson's, and Stronge's collision hypotheses lead to identical results (Comment f hereinafter), prevails when $\alpha \geq (1+e_e)r_m$ (in agreement with inequality (54) of [26]), bordering the α-regions of Types 2 and 4 from above. Thus, the α-regions in Sect. 3.1 of [26] remain valid:

$$\begin{aligned}&\text{Type 1:}\quad h>0 \text{ and } 0<\alpha<r_m,\ h<0 \text{ and } \alpha>0;\\&\text{Type 3:}\quad 0<\alpha<r_m; \qquad \text{Types 2, 4:}\quad r_m<\alpha<(1+e_e)r_m;\\&\text{Type 5:}\quad g>0 \text{ and } \alpha>(1+e_e)r_m,\ g<0 \text{ and } \alpha>0.\end{aligned} \tag{109}$$

5.2 The determination of the type of collision

Unifying inequalities (108) (μ-bounds), (109) (α-regions) and (42), (59), (76), (93), and (106) (coherence conditions) for the five types of collision, one has

$$\begin{aligned}&\text{Type 1:}\quad g>0,\ h>0 \text{ and } 0<\alpha<r_m,\ h<0 \text{ and } \alpha>0,\ p>0;\\&\text{Type 2:}\quad g>0,\ h>0,\ r_m<\alpha<(1+e_e)r_m,\ p>0;\\&\text{Type 3:}\quad g>0,\ h>0,\ 0<\alpha<r_m,\ q>0,\ p<0;\\&\text{Type 4:}\quad g>0,\ h>0,\ p<0,\ q>0,\ r_m<\alpha<(1+e_e)r_m;\\&\text{Type 5:}\quad h>0,\ g>0 \text{ and } \alpha>(1+e_e)r_m,\ g<0 \text{ and } \alpha>0.\end{aligned}$$

It was shown in [25] that there are only five admissible combinations of the algebraic signs of g, h, p, and q. Arranging these in a table together with the α-regions of the five types of collision, one obtained Table 1, which is identical with Table 1 in [26] (and also with Table 1 in [25] if, as with Newton's hypothesis-based solution, no distinction is made between Types 1 and 2, jointly called sticking, and between Types 3 and 4, jointly called reverse sliding). By Table 1, a solution always exists, and is unique and coherent. Energy consistence is proven in Appendix B for the five types of collision.

An algorithm for the solution of a collision problem can now be established, provided μ and e are specified, and m_{nn}, m_{tt}, m_{nt} (equation (23)), and then g, h, p, q Δ, r_m and α (equations (24)–(25)) are calculated for t_1. The collision type is determined with the aid of Table 1, and used to evaluate $\mathbf{u}_2$ with (39), (56), (73), (90), or (103).

Table 1 Admissible sign variations of g, h, p and q; T1, T2, … refer to Types 1, 2, … etc.

No	g	h	p	q	Sticking (T1&T2)	R. Sliding (T3&T4)	F. Sliding (T5)
1	>0	>0	>0	>0	T1 $0<\alpha<r_m$ T2 $r_m<\alpha<(1+e_e)r_m$	–	$\alpha>(1+e_e)r_m$
2	>0	>0	>0	<0	T1 $0<\alpha<r_m$ T2 $r_m<\alpha<(1+e_e)r_m$	–	$\alpha>(1+e_e)r_m$
3	>0	>0	<0	>0	–	T3 $0<\alpha<r_m$ T4 $r_m<\alpha<(1+e_e)r_m$	$\alpha>(1+e_e)r_m$
4	>0	<0	>0	>0	T1 $\alpha>0$	–	–
5	<0	>0	>0	>0	–	–	$\alpha>0$

5.3 Comments

(a) I_{n2}, I_{t2}, v_{n2} and v_{t2} are continuous functions of α. This continuity is maintained through the passages from one type of collision to another, occurring at $\alpha = r_m$ and $\alpha = (1+e_e)r_m$; i.e., $I_n[\alpha = (1+e_e)r_m]_{\text{Type 2}} = I_n[\alpha = (1+e_e)r_m]_{\text{Type 5}}$, etc. Consequently, equal signs can be added to the inequalities defining the regions of α in Table 1.

(b) If $\mu = 0$, then $g = -m_{nt}$ and $p = m_{nt}$, and Cases 1, 2, and 4 in Table 1 vanish. Also, $r_m \underset{(25)}{=} -m_{nt}/m_{nn}$. By inequalities (108) $m_{nt} = 0$ and $r_m = 0$ for Types 1 and 2, $m_{nt} < 0 \Rightarrow r_m > 0$ for Types 3 and 4, and $m_{nt} < 0$ or $m_{nt} > 0$ for Type 5. By inequalities (109) one has Types 1 and 2 for $\alpha = 0$, Type 3 for $0 < \alpha < r_m$, Type 4 for $r_m < \alpha < (1+e_e)r_m$, and Type 5 for $\alpha > (1+e_e)r_m$, as in Case 3. If $m_{nt} > 0$, then $r_m < 0$, hence $\alpha > 0$, in accordance with Case 5.

(c) $\mathrm{v}_{n2}|_{e_e=0} = 0$ for all the types of collisions (see (41), (58), (75), (92), and (105)). In connection with Types 2 and 4, note that if $e_e = 0$, then $\alpha = r_m$, the region of α reduces to zero, and $\mathrm{v}_{n2}|_{\text{Type 2}} \underset{(58),(41)}{=} = \mathrm{v}_{n2}|_{\text{Type 1}}$ and $\mathrm{v}_{n2}|_{\text{Type 4}} \underset{(92),(75)}{=} = \mathrm{v}_{n2}|_{\text{Type 3}}$ (see comment (a)). Moreover, it can be verified that $\partial I_{n2}/\partial e_e > 0$ and $\partial \mathrm{v}_{n2}/\partial e_e > 0$ for all types of collision.

(d) Direct impact ($\mathrm{v}_{n1} < 0$, $\mathrm{v}_{t1} = 0$, $\alpha \to 0$) can be followed by collision Types 1, 3 and 5 (not 2 and 4, limited by inequalities (59) and (93) to $0 < r_m < \alpha < (1+e_e)r_m$). Expressions for I_{n2}, I_{nS} and I_{nC} are obtained by the substitution of $\mathrm{v}_t^A = 0$ (or $\alpha = 0$) in (38), (40) (Type 1) (72), (74) (Type 3) (102) and (104) (Type 5). Grazing ($\mathrm{v}_{n1} = 0$, $\mathrm{v}_{t1} > 0$, $\alpha \to \infty$) can be followed only by collision Type 5 ($\alpha > (1+e_e)r_m$), however, substitutions of $\mathrm{v}_{n1} = 0$ in (102) and (105a) reveal that $I_{n2} = 0$ and $\mathrm{v}_{n2} = 0$, i.e., there is no collision.

(e) When $m_{nt} = 0$ ('balanced collision'), no reverse sliding is possible since, by inequalities (108), $m_{tt} < 0$ ($p < 0$), in contrast with (23c), which requires $m_{tt} > 0$.

(f) Newton's, Poisson's, and Stronge's hypotheses lead to identical results for forward sliding ((52), (30), (53), and (67) in [25], (49) (and (48)), (50), (51), and (58e) in [26], and (102), (105a), (105b), and (B.33) in Appendix B, for I_{n2}, $\mathbf{v}_{n2}$, $\mathbf{v}_{t2}$ and ΔE, respectively, are identical).

(g) Solutions require $g \neq 0$ in Types 1, 2, 3, and 4 ((40a), (57a), (74a), and (91a)), $h \neq 0$ in Types 2, 4 and 5 ((57b), (91b), and (104)) and $p \neq 0$ and $q \neq 0$ in Types 3 and 4 ((75b) and (92b), and (72) and (89)), or else I_{nC}, I_{nS} and/or I_{n2} go to infinity, and/or v_{t2} vanish. Therefore, equal signs cannot be added to the ranges of g, h, p, and q in Table 1. Nevertheless, solutions with $g = 0$, $h = 0$, $p = 0$ and/or $q = 0$ are possible.

6 Kane and Levinson's example ([20], p. 348)

Figure 3 shows a double pendulum S consisting of uniform rods A and B, each of length l and mass m. Let q_1 and q_2 be the orientation angles of the rods, and let $u_i = \dot{q}_i$ $(i = 1, 2)$. Suppose that at time t_1 the endpoint $\overline{B}$ of B strikes H, a flat surface, and that at t_1 $q_1 = 20$, $q_2 = 30$ deg and $u_1 = -0.1$, $u_2 = -0.2$ rad/sec. It is required to evaluate the change in the kinetic energy of S following the collision, for $m = 3$ kg and $l = 2$ m. To this end, $\mathbf{n}$ and $\mathbf{t}$ are identified as $\mathbf{n} = -\mathbf{n}_1$ and $\mathbf{t} = -\mathbf{n}_2$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit vectors shown in Fig. 3. Next, the velocity of $\overline{B}$ at t_1, which is the velocity of approach, and the equation of motion

Fig. 3 Double pendulum

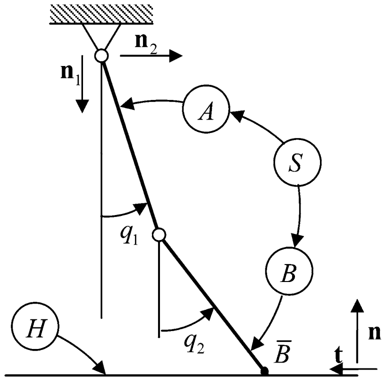


Table 2 The double pendulum collision problem

	e_e	μ	r_m	$(1+e_e)r_m$	Type	$[\Delta u_1, \Delta u_2]$ [r/s, r/s]	v_{n2} [m/s]	v_{t2} [m/s]	ΔE [J]
1	0.5	0.2	1.529	2.294	4	$[-0.11, 0.46]$	0.11	-0.047	-0.105
2	0.5	0.5	1.549	2.323	4	$[-0.10, 0.42]$	0.09	-0.013	-0.124
3	0.3	0.5	1.549	2.013	4	$[-0.09, 0.41]$	0.09	-0.408	-0.133
4	0.7	0.5	1.549	2.633	4	$[-0.11, 0.44]$	0.10	-0.028	-0.111
5	0.9	0.5	1.549	2.943	4	$[-0.12, 0.47]$	0.11	-0.047	-0.095

of S, can be written as

$$\mathbf{v}^{\overline{B}}(t_1) = \mathbf{v}^R(t_1) = -0.2684\mathbf{n} + 0.5343\mathbf{t},$$
$$-1/3ml^2\big[4\dot{u}_1 + 3/2\cos(q_1 - q_2)\dot{u}_2 + 3/2\sin(q_1 - q_2)u_2^2\big] = 0,$$
$$-1/3ml^2\big[3/2\cos(q_1 - q_2)\dot{u}_1 + \dot{u}_2 - 3/2\sin(q_1 - q_2)u_1^2\big] = 0.$$

Substitutions in (23)–(25) yield $m_{nn} = 0.3365$, $m_{tt} = 0.8134$, $m_{nt} = -0.5071$, $\Delta = 0.0166$ and $\alpha = 1.9908$. Rows 1–5 of Table 2 show cases with different values of e_e and μ similar to those in [25] (Rows 1–4 correspond to the four cases of [20]).

7 Newton's and Poisson's hypotheses and the two-integration procedure

The collision theories developed in [25] and [26] involve a one-integration procedure, namely the integration of the system equations of motion for the collision time (leading to (12) in [25], which are identical with (9)). The theory developed in this paper involves an additional integration for impulses work evaluation (Stronge [7], Sect. 2.3), carried out in Appendix B. Now, let I_{nR} be the normal impulse during restitution, given by $I_{nR} = I_{n2} - I_{nC}$, and note that Newton's and Poisson's coefficients of restitution e_v and e_i are defined $e_v \triangleq -v_{n2}/v_{n1}$ and $e_i \triangleq I_{nR}/I_{nC}$. Then one may conclude, with the aid of (38) and (40b), (55) and (57b), (72) and (74b), (89) and (91b), and (102) and (104) that there exist relations between e_e and e_v and, with the aid of (41a), (58a), (75a), (92a), and (105a) between e_e and e_i. These can be expressed, for the different types of collision, as

$$\text{Type 1} \quad e_e = e_v m_{tt} / \sqrt{m_{tt}\left(m_{tt} + 2\alpha m_{nt} - m_{nt}/r_m \alpha^2\right)}, \tag{110}$$

$$\text{Type 2} \quad e_e^2 = \left(g m_{nt} a^2 + h m_{tt} e_v^2\right)/\Delta, \tag{111}$$

$$\text{Type 3} \quad e_e = e_v / \sqrt{1 + 4\mu m_{nt}\alpha/g - 2\mu m_{nt}\alpha^2/(g r_m)}, \tag{112}$$

$$\text{Type 4} \quad e_e^2 = \left(2\mu m_{nt} a^2 + h e_v^2\right)/q, \tag{113}$$

$$\text{Type 5} \quad e_e = e_v, \tag{114}$$

where $a = (\alpha/r_m - 1)$; and

$$\text{Type 1} \quad e_e = e_i(m_{tt} + \alpha m_{nt}) / \sqrt{m_{tt}\left(m_{tt} + 2\alpha m_{nt} - m_{nt}/r_m \alpha^2\right)}, \tag{115}$$

$$\text{Type 2} \quad e_e^2 = e_i^2 + g m_{nt}(e_i - a)^2/(h m_{tt}), \tag{116}$$

$$\text{Type 3} \quad e_e = e_i(1 + 2\mu m_{nt}\alpha/g) / \sqrt{1 + 4\mu m_{nt}\alpha/g - 2\mu m_{nt}\alpha^2/(g r_m)}, \tag{117}$$

$$\text{Type 4} \quad e_e^2 = e_i^2 + 2\mu m_{nt}(e_i - a)^2/h, \tag{118}$$

$$\text{Type 5} \quad e_e = e_i. \tag{119}$$

One can thus specify e_v or e_i, evaluate e_e, and, regarding e_e as a parameter, apply the procedure of Sect. 5.2. Now, suppose that e_e from (115) is substituted in (38). Then simplifications show that the latter becomes $I_{n2} = -(1+e_i)(m_{tt} + \alpha m_{nt})\mathrm{v}_{n1}/\Delta$, an equation *identical* with (26a) in [26]. Indeed, it is a straightforward matter to show that substitutions of e_e from (116), (117), (118), and (119) in (55), (72), (89), and (104), respectively, lead to (29a), (36a), (44a), and (49a) for I_{n2} in [26]. The effect of these substitutions is the transformation of the Stronge's hypothesis-based two-integration procedure to the Poisson's hypothesis-based one-integration procedure, which was proven in [26] to be 'complete'. One can thus check that when e_i is given the values in the second column of Table 2, then e_e, now a parameter, becomes 0.4814, 0.4587, 0.2997, 0.5851, and, 0.6967, respectively; and, when used in (89) and then in (39) and (18) to evaluate (I_{n2}, $\mathbf{u}_2$ and) ΔE leads to -0.1063, -0.1265, -0.1337, -0.1192, and -0.1115 J, the *exact* values obtained in [26]. Similar substitutions of e_e from (110)–(114) transform the Stronge's hypothesis-based two-integration procedure to the Newton's hypothesis-based one-integration procedure (shown in [25] to suffer from energy inconsistencies). For example, substitutions from (110) in (38) and from (111) in (55) lead to (47a) of [25]; and from (112) in (72) and from (113) in (89) lead to (49a) of [25]. Indeed, one can check that when e_v is given the values in the second column of Table 2, then e_e becomes 0.6222, 1.131, 0.3766, 1.728, and 2.294, respectively, and when used in (89), (39), and (18) to evaluate (I_{n2}, $\mathbf{u}_2$ and) ΔE, leads to -0.0948, -0.0731, -0.1306, -0.00196 and 0.0829 J, the *exact* values obtained in [25]. One can thus exploit Newton's and Poisson's hypotheses-based algorithms either directly, as in [25] and [26], or indirectly, as outlined here, making it a simple matter to switch between solutions based on the different hypotheses, in the context of simulations codes designed to deal with collisions. These results are further illustrated by the two-sled collision example of [26] solved here in Appendix C.

The ratios e_e/e_v and e_e/e_i ((110)–(119)) can be plotted vs. α as in Fig. 4. Setting $\partial e_e/\partial\alpha = 0$ in (110)–(119) one can show straightforwardly that e_e/e_v ((110)–(113)) achieves an extremum for $\alpha = r_m$, whereas e_e/e_i ((115)–(118)) achieves an extremum for $\alpha = 0$ (Types 1 and 3) and for $\alpha = (1 + e_e)r_m$ (Types 2 and 4). Thus, it may occur that

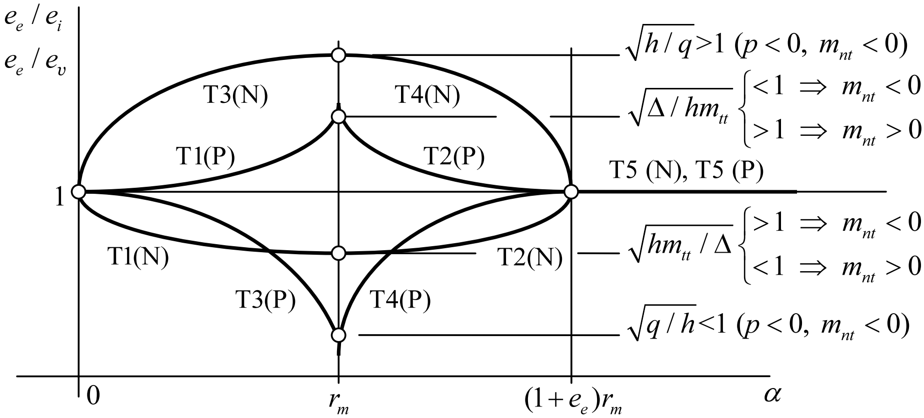


Fig. 4 e_e/e_i and e_e/e_v vs. α. T1, ..., T5 stand for Type 1, ..., Type 5; N and P stand for Newton's and Poisson's hypotheses-related ratios. T1(N), T2(N), T1(P), and T2(N) are drawn for $m_{nt} > 0$

$0 < e_i < 1$ leads to $e_e > 1$ $(E_{nR} > |E_{nC}|)$; and that, $0 < e_e < 1$ leads to $e_i > 1$ $(I_{nR} > I_{nC})$, in contrast with what 'feels right'. Nevertheless, the use of both e_e in Sect. 5.2 and e_i in [26] retains the overall energy consistency.

8 Summary

Stronge's collision hypothesis, together with Coulomb's coefficient of friction, Routh's semi-graphical method and the two-integration procedure were used to generate an analytical solution to the problem of a single-point collision with friction of simple, non-holonomic planar systems; and show that a solution always exists, which is unique, coherent and energy consistent. Moreover, it was shown that the use of Stronge's 'equivalent' of Poisson's coefficient of restitution transforms the Stronge's hypothesis-based solution to the Poisson's hypothesis-based solution of [26]; and that similar transformation is obtained in connection with Newton's hypothesis-based solution of [25]. Thus, three algorithms, that can be readily implemented in simulation of systems undergoing collisions, are at one's disposal. The ones based on Newton's and Poisson's hypotheses can be obtained directly, as in [25] and [26], or from that based on Stronge's hypothesis by the indicated simple transformation, a choice that makes it simple to switch between the different solutions within a single simulation code.

Appendix A

The entry m_{rs} in row r, column s of the $p \times p$ (negative definite) mass matrix $\mathbf{M}$ associated with (1) is the coefficient of $\dot{u}_s$ in the rth equation, and is given by

$$m_{rs} \triangleq -\sum_{i=1}^{v} m_i \mathbf{v}_r^{P_i} \cdot \mathbf{v}_s^{P_i} \quad (r, s = 1, \ldots, p). \tag{A.1}$$

This can be shown formally if $\mathbf{a}^{P_i}$, the acceleration of P_i in N, is expressed as

$$\mathbf{a}^{P_i} = \sum_{r=1}^{p} \mathbf{v}_r^{P_i} \dot{u}_r + \dot{\mathbf{v}}_r^{P_i} u_r + \dot{\mathbf{v}}_t^{P_i} \quad (i = 1, \ldots, v) \tag{A.2}$$

(see (2)). The inertia force associated with P_i equals $-m_i \mathbf{a}^{P_i}$ hence its contribution to the rth of (1) is given by $-m_i \mathbf{a}^{P_i} \cdot \mathbf{v}_r^{P_i}$. Summation of such contributions from all the particles comprise the rth generalized inertia force, namely, $F_r^* = \sum_{i=1}^{v} -m_i \mathbf{a}^{P_i} \cdot \mathbf{v}_r^{P_i}$. Substitutions from (A.2) for $i = 1, \ldots, v$ make it possible to verify that coefficient of $\dot{u}_s$ is the one given by (A.1). Note that $m_{rs} = m_{sr}$, hence that the mass matrix is symmetric. Next, the kinetic energy of a simple non-holonomic system defined in Sect. 1 is given by

$$E = 1/2 \sum_{i=1}^{v} m_i \left[\sum_{r=1}^{p} \mathbf{v}_r^{P_i} u_r + \mathbf{v}_t^{P_i} \right] \left[\sum_{s=1}^{p} \mathbf{v}_s^{P_i} u_s + \mathbf{v}_t^{P_i} \right] \tag{A.3}$$

or, when use is made of m_{rs} defined in (A.1),

$$E = -1/2 \sum_{r=1}^{p} \sum_{s=1}^{p} m_{rs} u_r u_s + \sum_{i=1}^{v} m_i \mathbf{v}_t^{P_i} \mathbf{v}^{P_i} - 1/2 \sum_{i=1}^{v} m_i \left(\mathbf{v}_t^{P_i}\right)^2. \tag{A.4}$$

The loss of mechanical energy during collision is $\Delta E \triangleq E(t_2) - E(t_1)$, and since $\mathbf{v}_t^{P_i}(t_2) = \mathbf{v}_t^{P_i}(t_1)$ $(i = 1, \ldots, v)$, then

$$\begin{aligned} \Delta E = {} & -1/2 \sum_{r=1}^{p} \sum_{s=1}^{p} m_{rs} u_r(t_2) u_s(t_2) + 1/2 \sum_{r=1}^{p} \sum_{s=1}^{p} m_{rs} u_r(t_1) u_s(t_1) \\ & + \sum_{i=1}^{v} m_i \mathbf{v}_t^{P_i} \cdot \left[\mathbf{v}^{P_i}(t_2) - \mathbf{v}^{P_i}(t_1)\right]. \end{aligned} \tag{A.5}$$

Appendix B

The evaluation of the work done on the system by the normal and tangential forces during the different segments of collision is conducted here for the five types of collision with an additional integration (i.e., additional to that leading to (9)):
Type 1: $\Delta E_n = E_{n1-S} + E_{nS-C} + E_{nC-2}$, $\Delta E_t = E_{t1-S} + E_{tS-C} + E_{tC-2}$, where

$$E_{n1-S} = \int_0^{I_{nS}} \mathrm{v}_n \, \mathrm{d}I_n \underset{(31)}{=} \int_0^{I_{nS}} (\mathrm{v}_{n1} + hI_n) \, \mathrm{d}I_n = \mathrm{v}_{n1} I_{nS} + hI_{nS}^2/2, \tag{B.1}$$

$$\begin{aligned} E_{nS-C} &= \int_{I_{nS}}^{I_{nC}} \mathrm{v}_n \, \mathrm{d}I_n \underset{(33),(34)}{=} \int_0^{I_{nS}} (\mathrm{v}_{nS} - \Delta/m_{tt} I_{nS} + \Delta/m_{tt} I_n) \, \mathrm{d}I_n \\ &= -\Delta/m_{tt} (I_{nC} - I_{nS})^2/2, \end{aligned} \tag{B.2}$$

$$\begin{aligned} E_{nC-2} &= \int_{I_{nC}}^{I_{n2}} \mathrm{v}_n \, \mathrm{d}I_n \underset{(35),(36)}{=} \int_{I_{nC}}^{I_{n2}} \Delta/m_{tt} (I_n - I_{nC}) \, \mathrm{d}I_n \\ &= \Delta/m_{tt} (I_{n2} - I_{nC})^2/2, \end{aligned} \tag{B.3}$$

$$E_{t1-S} = \int_0^{I_{tS}} \mathrm{v}_t \,\mathrm{d}I_t \underset{(32)}{=} -\mu \int_0^{I_{nS}} (\mathrm{v}_{t1} - gI_n)\,\mathrm{d}I_n = -\mu \mathrm{v}_{t1} I_{nS} + \mu g I_{nS}^2/2, \tag{B.4}$$

$$E_{tS-C} = \int_{I_{tS}}^{I_{tC}} \mathrm{v}_t \,\mathrm{d}I_t \underset{(34)}{=} 0, \tag{B.5}$$

$$E_{tC-2} = \int_{I_{tC}}^{I_{t2}} \mathrm{v}_t \,\mathrm{d}I_t \underset{(36)}{=} 0. \tag{B.6}$$

Substitutions of $\mathbf{u}_2$ from (39) in (18), on the one hand, and of I_{nS} and I_{nC} form (40) and I_{n2} from (38) in (B.1)–(B.6), on the other, reveal that

$$\Delta E = E_{n1-S} + E_{nS-C} + E_{nC-2} + E_{11-S} + E_{tS-C} + E_{tC-2}. \tag{B.7}$$

Type 2: $\Delta E_n = E_{n1-C} + E_{nC-S} + E_{nS-2}$, $\Delta E_t = E_{t1-C} + E_{tC-S} + E_{tS-2}$, where

$$E_{n1-C} = \int_0^{I_{nC}} \mathrm{v}_n \,\mathrm{d}I_n \underset{(48)}{=} \int_0^{I_{nC}} (\mathrm{v}_{n1} + hI_n)\,\mathrm{d}I_n = \mathrm{v}_{n1} I_{nC} + hI_{nC}^2/2, \tag{B.8}$$

$$E_{nC-S} = \int_{I_{nC}}^{I_{nS}} \mathrm{v}_n \,\mathrm{d}I_n \underset{(50)}{=} \int_{I_{nC}}^{I_{nS}} h(I_n - I_{nC})\,\mathrm{d}I_n = h(I_{nS} - I_{nC})^2/2, \tag{B.9}$$

$$\begin{aligned} E_{nS-2} &= \int_{I_{nS}}^{I_{n2}} \mathrm{v}_n \,\mathrm{d}I_n \underset{(52),(53)}{=} \int_{I_{nS}}^{I_{n2}} (\mathrm{v}_{nS} - \Delta/m_{tt} I_{nS} + \Delta/m_{tt} I_n)\,\mathrm{d}I_n \\ &= \mathrm{v}_{nS}(I_{n2} - I_{nS}) + \Delta/m_{tt}(I_{n2} - I_{nS})^2/2; \end{aligned} \tag{B.10}$$

$$E_{t1-C} = \int_0^{I_{tC}} \mathrm{v}_t \,\mathrm{d}I_t \underset{(49)}{=} -\mu \int_0^{I_{nC}} (\mathrm{v}_{t1} - gI_n)\,\mathrm{d}I_n = -\mu \mathrm{v}_{t1} I_{nC} + \mu g I_{nC}^2/2, \tag{B.11}$$

$$\begin{aligned} E_{tC-S} &= \int_{I_{tC}}^{I_{tS}} \mathrm{v}_t \,\mathrm{d}I_t \underset{(51)}{=} -\mu \int_{I_{nC}}^{I_{nS}} (\mathrm{v}_{tC} + gI_{nC} - gI_n)\,\mathrm{d}I_t \\ &= -\mu \mathrm{v}_{tC}(I_{nS} - I_{nC}) + \mu g(I_{nS} - I_{nC})^2/2, \end{aligned} \tag{B.12}$$

$$E_{tS-2} = \int_{I_{tS}}^{I_{t2}} \mathrm{v}_t \,\mathrm{d}I_t \underset{(53)}{=} 0. \tag{B.13}$$

Substitutions of $\mathbf{u}_2$ from (56) in (18), on the one hand, and of I_{nS} and I_{nC} form (57) and I_{n2} from (55) in (B.8)–(B.13), on the other, reveal that

$$\Delta E = E_{n1-C} + E_{nC-S} + E_{nS-2} + E_{t1-C} + E_{tC-S} + E_{tS-2}. \tag{B.14}$$

Type 3: $\Delta E_n = E_{n1-S} + E_{nS-C} + E_{nC-2}$, $\Delta E_t = E_{t1-S} + E_{tS-C} + E_{tC-2}$, where

$$E_{n1-S} = \int_0^{I_{nS}} \mathrm{v}_n \,\mathrm{d}I_n \underset{(65)}{=} \int_0^{I_{nS}} (\mathrm{v}_{n1} + hI_n)\,\mathrm{d}I_n = \mathrm{v}_{n1} I_{nS} + hI_{nS}^2/2, \tag{B.15}$$

$$E_{nS-C} = \int_{I_{nS}}^{I_{nC}} \mathrm{v}_n \,\mathrm{d}I_n \underset{(67)}{=} \int_{I_{nS}}^{I_{nC}} (\mathrm{v}_{nS} - qI_{nS} + qI_n)\,\mathrm{d}I_n = -q(I_{nC} - I_{nS})^2/2, \tag{B.16}$$

$$E_{nC-2} = \int_{I_{nC}}^{I_{n2}} \mathrm{v}_n \,\mathrm{d}I_n \underset{(69)}{=} \int_{I_{nC}}^{I_{n2}} q(I_n - I_{nS})\,\mathrm{d}I_n = q(I_{n2} - I_{nC})^2/2; \tag{B.17}$$

$$E_{t1-S} = \int_0^{I_{tS}} \mathrm{v}_t \, \mathrm{d}I_t \underset{(66)}{=} -\mu \int_0^{I_{nS}} (\mathrm{v}_{t1} - gI_n) \, \mathrm{d}I_n = -\mu \mathrm{v}_{t1} I_{nS} + \mu g I_{nS}^2/2, \tag{B.18}$$

$$E_{tS-C} = \int_{I_{tS}}^{I_{tC}} \mathrm{v}_t \, \mathrm{d}I_t \underset{(68)}{=} \mu \int_{I_{nS}}^{I_{nC}} p(I_n - I_{nS}) \, \mathrm{d}I_n = \mu p (I_{nC} - I_{nS})^2/2, \tag{B.19}$$

$$\begin{aligned} E_{tC-2} &= \int_{I_{tC}}^{I_{t2}} \mathrm{v}_t \, \mathrm{d}I_t \underset{(70)}{=} \mu \int_{I_{nC}}^{I_{n2}} (\mathrm{v}_{tC} - pI_{nc} + pI_n) \, \mathrm{d}I_n \\ &= \mu p (I_{nC} - I_{nS})(I_{n2} - I_{nC}) + \mu p (I_{n2} - I_{nC})^2/2. \end{aligned} \tag{B.20}$$

Substitutions of $\mathbf{u}_2$ from (73) in (18), on the one hand, and of I_{nS} and I_{nC} form (74) and I_{n2} from (72) in (B.15)–(B.20), on the other, reveal that

$$\Delta E = E_{n1-S} + E_{nS-C} + E_{nC-2} + E_{t1-S} + E_{tS-C} + E_{tC-2}. \tag{B.21}$$

Type 4: $\Delta E_n = E_{n1-C} + E_{nC-S} + E_{nS-2}$, $\Delta E_t = E_{t1-C} + E_{tC-S} + E_{tS-2}$, where

$$E_{n1-C} = \int_0^{I_{nC}} \mathrm{v}_n \, \mathrm{d}I_n \underset{(82)}{=} \int_0^{I_{nC}} (\mathrm{v}_{n1} + hI_n) \, \mathrm{d}I_n = \mathrm{v}_{n1} I_{nC} + h I_{nC}^2/2, \tag{B.22}$$

$$E_{nC-S} = \int_{I_{nC}}^{I_{nS}} \mathrm{v}_n \, \mathrm{d}I_n \underset{(84)}{=} \int_{I_{nC}}^{I_{nS}} h(I_n - I_{nC}) \, \mathrm{d}I_n = h(I_{nS} - I_{nC})^2/2, \tag{B.23}$$

$$\begin{aligned} E_{nS-2} &= \int_{I_{nS}}^{I_{n2}} \mathrm{v}_n \, \mathrm{d}I_n \underset{(86)}{=} \int_{I_{nS}}^{I_{n2}} (\mathrm{v}_{nS} - qI_{nS} + qI_n) \, \mathrm{d}I_n \\ &= \mathrm{v}_{nS}(I_{n2} - I_{nS}) + q(I_{n2} - I_{nS})^2/2; \end{aligned} \tag{B.24}$$

$$E_{t1-C} = \int_0^{I_{tC}} \mathrm{v}_t \, \mathrm{d}I_t \underset{(83)}{=} -\mu \int_0^{I_{nC}} (\mathrm{v}_{t1} - gI_n) \, \mathrm{d}I_n = -\mu \mathrm{v}_{t1} I_{nC} + \mu g I_{nC}^2/2, \tag{B.25}$$

$$\begin{aligned} E_{tC-S} &= \int_{I_{tC}}^{I_{tS}} \mathrm{v}_t \, \mathrm{d}I_t \underset{(85)}{=} -\mu \int_{I_{nC}}^{I_{nS}} (\mathrm{v}_{tC} + gI_{nC} - gI_n) \, \mathrm{d}I_n \\ &= -\mu g (I_{nS} - I_{nC})^2/2, \end{aligned} \tag{B.26}$$

$$E_{tS-2} = \int_{I_{tS}}^{I_{t2}} \mathrm{v}_t \, \mathrm{d}I_t \underset{(87)}{=} \mu \int_{I_{nS}}^{I_{n2}} p(I_n - I_{nS}) \, \mathrm{d}I_t = \mu p (I_{n2} - I_{nS})^2/2. \tag{B.27}$$

Substitutions of $\mathbf{u}_2$ from (90) in (18), on the one hand, and of I_{nS} and I_{nC} form (91) and I_{n2} from (89) in (B.22)–(B.27), on the other, reveal that

$$\Delta E = E_{n1-C} + E_{nC-S} + E_{nS-2} + E_{t1-C} + E_{tC-S} + E_{tS-2}. \tag{B.28}$$

Type 5: $\Delta E_n = E_{n1-C} + E_{nC-2}$, $\Delta E_t = E_{t1-C} + E_{tC-2}$, where

$$E_{n1-C} = \int_0^{I_{nC}} \mathrm{v}_n \, \mathrm{d}I_n \underset{(97)}{=} \int_0^{I_{nC}} (\mathrm{v}_{n1} + hI_n) \, \mathrm{d}I_n = \mathrm{v}_{n1} I_{nC} + h I_{nC}^2/2, \tag{B.29}$$

$$\begin{aligned} E_{nC-2} &= \int_{I_{nC}}^{I_{n2}} \mathrm{v}_n \, \mathrm{d}I_n \underset{(99)}{=} \int_{I_{nC}}^{I_{n2}} (\mathrm{v}_{n1} + hI_n) \, \mathrm{d}I_n \\ &= \mathrm{v}_{n1}(I_{n2} - I_{nC}) + h\left(I_{n2}^2 - I_{nC}^2\right)/2; \end{aligned} \tag{B.30}$$

$$E_{t1-C} = \int_0^{I_{tC}} \mathrm{v}_t \,\mathrm{d}I_t \underset{(98)}{=} -\mu \int_0^{I_{nC}} (\mathrm{v}_{t1} - gI_n)\,\mathrm{d}I_n = -\mu \mathrm{v}_{t1} I_{nC} + \mu h I_{nC}^2/2, \quad \text{(B.31)}$$

$$E_{tC-2} = \int_{I_{tC}}^{I_{t2}} \mathrm{v}_t \,\mathrm{d}I_t \underset{(100)}{=} -\mu \int_{I_{nC}}^{I_{n2}} (\mathrm{v}_{t1} - gI_n)\,\mathrm{d}I_n$$
$$= -\mu \mathrm{v}_{t1}(I_{n2} - I_{nC}) + \mu g\big(I_{n2}^2 - I_{nC}^2\big)/2. \quad \text{(B.32)}$$

Substitutions of $\mathbf{u}_2$ from (103) in (18), on the one hand, and of I_{nC} form (104) and I_{n2} from (102) in (B.29)–(B.32), on the other, reveal that

$$\Delta E = E_{n1-C} + E_{nC-2} + E_{t1-C} + E_{tC-2}. \quad \text{(B.33)}$$

ΔE comprises the contributions $\Delta E_n (= \Delta E_{nC} + \Delta E_{nR})$ and ΔE_t from the normal and the tangential impulses, respectively. By (19) $\Delta E_n|_{e_e=1} = 0$, hence $\Delta E_n|_{e_e<1} < 0$; and, since $\Delta E_t < 0$, as can be shown for all types of collision, then $\Delta E < 0$.

Appendix C

Regarding the two-sled collision problem discussed in [26], one has $m_{nn} = 0.9097$, $m_{nt} = 0.0836$ and $m_{tt} = 0.3713$ for Row 1, and $m_{nn} = 0.8063$, $m_{nt} = -0.3127$ and $m_{tt} = 0.7777$ for Rows 2 and 3 of each of the hypotheses in Table 3. Collision Types 5, 3 and 1 (or 3, 2, and 1 with Newton's hypothesis) are identified with the aid of Table 1. Stronge's hypothesis leads to the results reported in the last three rows of Table 3. Poisson's hypothesis with $e_i = 0.8$, in conjunction with (119), (117) and (115) leads to the results reported in the three mid-rows of Table 3, which are identical with their counterparts in Table 3 of [26]. Similarly, Newton's hypothesis with $e_v = 0.8$, in conjunction with (114), (112), and (110), leads to the results reported in the first three rows of Table 3, which are identical with their counterparts in Table 3 of [26].

Table 3 The two-sled collision problem of [26]: three solutions

e_e	μ	γ	δ	Type	$[\Delta u_1, \Delta u_3, \Delta u_4, \Delta u_6]$ [m/s, r/s, m/s, r/s]	v_n^A, v_n^S [m/s]	v_t^A, v_t^S [m/s]	ΔE [J]
Newton's hypothesis; e_e is Stronge's equivalent to Newton's $e_v = 0.8$								
0.800	0.3	0.2	0.20	3	[−0.209, 0.984, −0.667, −0.052]	−1.08, 0.86	0.77, 0.710	−0.724
0.839	0.3	0.2	0.85	2	[−0.193, 1.197, −0.405, −0.329]	−1.08, 0.86	0.16, −0.186	−0.292
0.848	0.7	0.2	0.85	1	[−0.286, 1.232, −0.342, −0.242]	−1.08, 0.86	0.16, 0.000	−0.224
Poisson's hypothesis; e_e is Stronge's equivalent to Poissons's $e_i = 0.8$								
0.800	0.3	0.2	0.20	5	[−0.209, 0.984, −0.667, −0.052]	−1.08, 0.86	0.77, 0.710	−0.724
0.796	0.3	0.2	0.85	3	[−0.187, 1.169, −0.396, −0.322]	−1.08, 0.82	0.16, −0.181	−0.341
0.797	0.7	0.2	0.85	1	[−0.276, 1.198, −0.334, −0.238]	−1.08, 0.81	0.16, 0.000	−0.288
Stronge's hypothesis								
0.800	0.3	0.2	0.20	5	[−0.209, 0.984, −0.667, −0.052]	−1.08, 0.86	0.77, 0.710	−0.724
0.800	0.3	0.2	0.85	3	[−0.188, 1.172, −0.397, −0.323]	−1.08, 0.82	0.16, −0.182	−0.336
0.800	0.7	0.2	0.85	1	[−0.277, 1.200, −0.334, −0.238]	−1.08, 0.81	0.16, 0.000	−0.284

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