Yildirim Hurmuzlu
Dan B. Marghitu
Mechanical Engineering Department
Southern Methodist University
Dallas, Texas 75275

Rigid Body Collisions of Planar Kinematic Chains With Multiple Contact Points

Abstract

This article deals with the rigid body collisions of planar kinematic chains with an external surface while in contact with other surfaces. Two solution procedures that cast the impact equations in differential and algebraic forms are developed to solve the general problem. The differential formulation can be used to obtain three sets of solutions based on the kinematic, kinetic, and energetic definitions of the coefficient of restitution, whereas the algebraic formulation can be used to obtain solutions based on the approaches presented in Whittaker (1904) and Brach (1991). A specific example of a planar three-link chain with two contact points is studied to compare the outcomes predicted by each approach. A particular emphasis is placed on the energy loss that results from the application of each solution scheme. The circumstances in which various methods lead to identical or distinct outcomes are investigated. Most importantly, the study elaborates on the rebounds at the noncolliding ends, a phenomenon that is observed only in multicontact collisions. The interaction of the chain with the contact surfaces at the noncolliding contact points is examined, and the differences in the prediction of rebounds that arise from using various methods are investigated.

1. Introduction

The origin of the present approaches to solve rigid body collision problems dates back to Newton. In 1686, Newton generalized the partial results of his predecessors when he presented the third law of motion and its relation to partly elastic collisions. Subsequently, Routh (1905) presented a graphic method based on Poisson's hypothesis to treat collision problems. Also at the turn of the twentieth century, Whittaker (1904) expanded Newton's method to account for frictional impulse. Whittaker's and Routh's approaches fundamentally differ in the treatment of motion in the normal and tangential directions at the point of collision. In the former method, the coefficient of restitution e, is a kinematic quantity that is used to derive a relation between the normal components of the approach and departure velocities at the contact point. The latter method, however, divides the collision period into the compression and restitution phases. Poisson's hypothesis defines e as a kinetic quantity that relates the normal impulses at the contact point that occur during each phase. The approaches also evolved differently in the treatment of the motion in the tangential direction at the point of contact. Whittaker's method allows slippage when the ratio of the normal and tangential impulses are greater than the coefficient of friction μ. On the other hand, Routh solves for the slip velocity during collision and introduces the possibility of changes in slip direction during contact. In general, it has been thought that the two approaches can be reconciled. The engineering community accepted Whittaker's method, because it yields algebraic equations compared to the graphic makeup of Routh's method, which largely remained unnoticed. A typical example can be found in the undergraduate dynamics book of Beer and Johnston (1988), where the derivation of impact equations for colliding rigid bodies is carried out by using Poisson's hypothesis (without specifically referring to it) and leading to Newton's kinematic definition of e. Then the derivation follows along the lines of Whittaker's method to resolve the motion in the tangential direction when friction is present.

Through a simple example problem that considers a two-link pendulum striking a flat surface, Kane and Levinson (1985) pointed out that the classic solution of rigid body impact problems using Newtonian mechanics produces energetically inconsistent results. As evidenced by recent publications, Kane and Levinson's remarks sparked a remarkable interest in a problem that was thought to have been solved long ago. Keller (1986) attributed this paradoxic behavior to slip reversals during collision subject to frictional effects. The Newtonian approach ignores the changes in the direction of slip, leading to overestimation of the rebound velocity as a result of impact (Stronge 1990). Keller introduced a revised formulation of rigid body collision equations based on Poisson's hypothesis such that impact never increases

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energy. Yet Stronge (1990) has exposed energy inconsistencies in solutions using Poisson's hypothesis when e is assumed not to depend on the coefficient of friction. He divided the energy that is dissipated during collision into two portions: dissipation owing to frictional impulse and dissipation owing to normal impulse. Then he demonstrated that Poisson's hypothesis does not lead to vanishing dissipation owing to normal impulse when the coefficient of restitution is unity (perfectly elastic impact). He proposed an energetic definition for e. This definition equates the square of the coefficient of restitution to the ratio of elastic strain energy released at the contact point during restitution to the energy absorbed by deformation during compression. Brach (1989) has proposed a solution scheme based on revising Whittaker's method to rid energy increases from resulting solutions. The approach treats the tangential impulse as a constant fraction μ of the normal impulse. Then energy loss is examined to determine the appropriate value of μ that can be used in the actual solution. He expanded his approach in 1991 (Brach 1991) to treat contacts that take place over finite areas and introduced a moment coefficient e_m to solve the collision problem. He also considered the possibility of a tangential coefficient of restitution when hard objects strike relatively compliant surfaces.

The aforementioned studies consider problems where the rigid bodies are not in contact with external surfaces before impact. However, more general circumstances can be conceived, such as a multibody system that is in unconstrained contact with several surfaces when collision with an external object occurs. In such circumstances the issue of rebound is not only confined to the point where collision occurs, but should also be considered at the other contact points. Such general cases have theoretical importance because they provide rigorous testing platforms to demonstrate the utility and soundness of the basic methods. The recent interest in this area, after all, was sparked by a kinematic chain problem. Furthermore, a special class of such collision problems arises in the study of walking machines. During gait, one limb contacts the walking surface while others are on the ground and are free to detach. A particular case where one end of a planar kinematic chain strikes a horizontal surface while the other end is stationary on the surface is considered in Hurmuzlu and Chang (1992). The solution method presented in Hurmuzlu and Chang builds on the basic approach presented in Brach (1989). The method successfully predicts rebounds at both ends and yields physically consistent transitions among the various cases associated with lateral and horizontal motions at the contact points.

To the best of our knowledge, a systematic approach to the solution of multicontact collisions of kinematic chains does not exist. The first objective of the present article is to develop a procedure that can be followed to solve the general planar problem. We consider a planar multibody system with frictionless revolute joints, where one end strikes a surface while k+1 ends are in contact with other external surfaces. However, in the light of recent developments in the area, there are several available methods to treat the rigid body collision problems. The approach that is taken here is to first develop solution schemes to solve the general problem by using five different methods. Then a specific example is used to study the agreements and disagreements among the outcomes predicted by various schemes. Accordingly, two procedures are developed to solve the problem using the kinematic, kinetic, and energetic definitions of the coefficient of restitution. The first procedure casts the impact equations in differential form and solves the problem by using either one of the three definitions of the coefficient of restitution. The second procedure casts the equations in algebraic form and is based on the kinematic definition of the coefficient of restitution. The solution can be obtained either by directly using Whittaker's method or by using the approach presented in Brach (1991) to account for energy increases. Then a planar three-link chain with two contact points is considered to study the outcomes that are predicted by the application of various solution schemes.

2. Description of the Problem

A general representation of the kinematic chains that are considered here is shown in Figure 1. Immediately before impact, the n-link kinematic chain is in contact with k+1 surfaces, S_i at the points, A_i. Here the term *contact* is used to denote that the relative distances between the ends A_i of the chain and surfaces S_i are zero at the onset of the collision. The impact is initiated when end A_c of the chain strikes surface S_c. Without loss of generality, the coordinate axes are aligned with surface S_0, whereas surfaces S_i are taken at angles θ_i with the horizontal. The normal and tangential directions at A_i are defined as shown. The coefficients of friction between the chain and surfaces are given as μ_i, whereas the coefficient of restitution at A_c is e. The collision at A_c may lead to several outcomes depending on the initial conditions, the coefficients of friction among the surfaces and the chain, and the coefficient of restitution at A_c. The contacting ends may stay on the respective surfaces or rebound as a result of the collision. Rebound at the end A_c is directly related to the coefficient of restitution at this point, whereas the rebounds at other contact points depend on the events that occur during collision.

Throughout the collision period, the motion of an end point of the chain at any given contact point can be described by one of the following three cases:

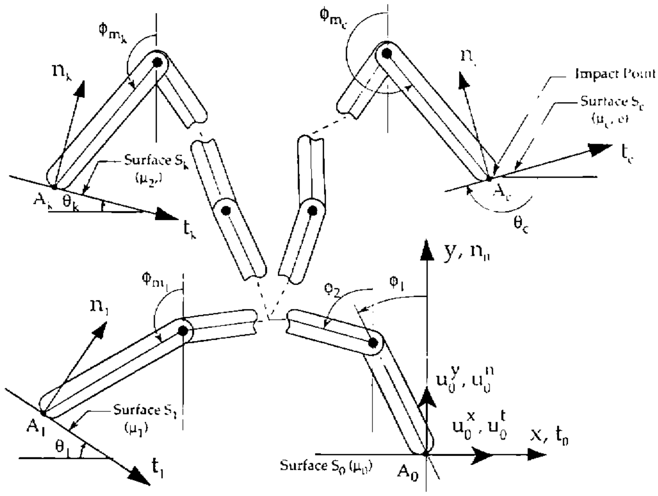

Fig. 1. Multicontact, planar chains.

Case 1: The end is slipping along the surface with interaction in the normal direction (nonzero normal impulse between the chain and the surface).

Case 2: The end does not slip along surface but interacts with it in the normal direction (nonzero normal impulse between the chain and the surface).

Case 3: The end does not interact with the surface.

In general, during collision, a particular end may undergo successive motions that can be described by a certain combination of these three cases. The treatment of motion in the tangential direction relies heavily on the approach that is taken in solving the collision problem. The equations to solve the rigid body impact problems can be formulated in two forms: the algebraic formulation and the differential formulation. The solutions proposed by Whittaker (1904) and Brach (1989) fall in the former category. The algebraic formulation casts the equations in terms of velocities and impulses at the onset and at the end of collision period. The approach does not provide any information about the events that occur during collision. On the other hand, Keller (1986) and Stronge (1990) formulate the impact equations in differential form. With these approaches, one can specifically determine the events that occur during collision with the price tag of more complex analysis (i.e., solving differential equations and incorporating slip and stop conditions). In the next section, we develop the equations that can be used in solving the multicontact collision problems of kinematic chains by expanding the basic approaches that are proposed in the aforementioned articles. We present solution procedures that yield the postimpact velocities and the final outcomes in terms of slippage and rebound at the contacting ends.

3. Development of the Impact Equations

3.1. Differential Formulation

3.1.1. Normal and Tangential Velocities of the Contacting Ends

The equations of motion for the chain shown in Figure 1 can be written in the following general form:

$$\mathbf{M}(\mathbf{x})\ddot{\mathbf{x}} + \mathbf{C}(\mathbf{x},\dot{\mathbf{x}}) + \mathbf{G}(\mathbf{x}) = \begin{bmatrix}\mathbf{T}\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}\mathbf{D}_1(\mathbf{x})\\ \mathbf{D}_2(\boldsymbol{\theta})\end{bmatrix}\mathbf{F}, \qquad (1)$$

where $\mathbf{x} = (\phi_1, \ldots, \phi_n, u_0^t, u_0^n)^T$ is the (n+2)×1–dimensional vector of generalized coordinates; ϕ_i are absolute rotations measured from the vertical, as shown in Figure 1, $\dot{\mathbf{x}} = (\dot\phi_1, \ldots, \dot\phi_n, \dot u_0^t, \dot u_0^n)^T$ is the (n+2)×1–dimensional vector of generalized velocities; $\ddot{\mathbf{x}} = (\ddot\phi_1, \ldots, \ddot\phi_n, \ddot u_0^t, \ddot u_0^n)^T$ is the (n+2)×1–dimensional vector of generalized accelerations; $\mathbf{M}(\ddot{\mathbf{x}})$ is the (n+2)×(n+2) mass matrix; $\mathbf{C}(\mathbf{x},\dot{\mathbf{x}})$ is a (n+2)×1 vector; $\mathbf{G}(\mathbf{x})$ is the (n+2)×1 vector of gravity terms; $\mathbf{T}$ is the vector of joint moments; $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k, \theta_c)^T$ is the vector of surface inclination angles; $\mathbf{D}_1(\mathbf{x})$ is an n×(2k+4) matrix; $\mathbf{D}_2(\boldsymbol{\theta})$ is an 2×(2k+4) matrix; and $\mathbf{F} = (F_0^t, F_0^n, \ldots, F_k^t, F_k^n, F_c^t, F_c^n)^T$ is the 1×(2k+4) vector of contact forces. Here, the superscripts t and n denote the normal and tangential components, respectively.

Solving for the acceleration vector in eq. (1) yields

$$\ddot{\mathbf{x}} = \mathbf{M}(\mathbf{x})^{-1}\left\{-\mathbf{C}(\mathbf{x},\dot{\mathbf{x}}) - \mathbf{G}(\mathbf{x}) + \begin{bmatrix}\mathbf{T}\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}\mathbf{D}_1(\mathbf{x})\\ \mathbf{D}_2(\boldsymbol{\theta})\end{bmatrix}\mathbf{F}\right\}. \qquad (2)$$

On the other hand, kinematic relations and eq. (2) can be used to obtain the following relation for the normal and tangential components of the linear accelerations of the contacting ends of the chain

$$\mathbf{a}_t = \begin{bmatrix}\mathbf{0} \;\; 1 \;\; 1\\ \mathbf{H}_1(\mathbf{x})\end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix}\mathbf{0}\\ \mathbf{H}_2(\mathbf{x},\dot{\mathbf{x}})\end{bmatrix}, \qquad (3)$$

where $\mathbf{a}_t = (a_0^t, a_0^n, \ldots, a_k^t, a_k^n, a_c^t, a_c^n)^T$ is the (2 k+4)×1 acceleration vector. Combining eqs. (2) and (3) yields

$$\begin{aligned}\mathbf{a}_t = \begin{bmatrix}\mathbf{0}\\ \mathbf{H}_2(\mathbf{x},\dot{\mathbf{x}})\end{bmatrix} + \begin{bmatrix}\mathbf{0} \;\; 1 \;\; 1\\ \mathbf{H}_1(\mathbf{x})\end{bmatrix}\\ \cdot \mathbf{M}(\mathbf{x})^{-1}\Bigg\{ - \mathbf{C}(\mathbf{x},\dot{\mathbf{x}}) - \mathbf{G}(\mathbf{x}) + \begin{bmatrix}\mathbf{T}\\ 0\\ 0\end{bmatrix}\\ + \begin{bmatrix}\mathbf{D}_1(\mathbf{x})\\ \mathbf{D}_2(\boldsymbol{\theta})\end{bmatrix}\mathbf{F}\Bigg\}. \qquad (4)\end{aligned}$$

Proceeding along the lines of Keller's method, we eliminate the nonimpulsive terms from eq. (4) (the first term and the first three terms between the braces in eq. (4)). Furthermore, assuming that the generalized coordinates do not change during impact, we let the generalized position vector be the constant vector $\bar{\mathbf{x}} = (\bar{\phi}_1, \ldots, \bar{\phi}_n, 0, 0)^T$. This yields

$$\mathbf{a}_t = \frac{d\mathbf{v}_t}{dt} = \begin{bmatrix} \mathbf{0}\ 1\ 1 \\ \mathbf{H}_1(\bar{\mathbf{x}}) \end{bmatrix} \mathbf{M}(\bar{\mathbf{x}})^{-1} \begin{bmatrix} \mathbf{D}_1(\bar{\mathbf{x}}) \\ \mathbf{D}_2(\boldsymbol{\theta}) \end{bmatrix} \mathbf{F} = \boldsymbol{\Gamma}(\bar{\mathbf{x}}, \boldsymbol{\theta})\mathbf{F}, \tag{5}$$

where $\mathbf{v}_t$ is the velocity vector of the contact points of the chain and $\boldsymbol{\Gamma}(\bar{\mathbf{x}}, \boldsymbol{\theta})$ is a constant (2 k+4)×(2 k+4) matrix that depends on preimpact positions and inclination angles of the contact surfaces.

The impulses at the contact points are given by the following relations:

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_0^t \\ \tau_0^n \\ \vdots \\ \tau_k^t \\ \tau_k^n \\ \tau_c^t \\ \tau_c^n \end{bmatrix} = \begin{bmatrix} \int_0^t F_0^t dt \\ \int_0^t F_0^n dt \\ \vdots \\ \int_0^t F_k^t dt \\ \int_0^t F_k^n dt \\ \int_0^t F_c^t dt \\ \int_0^t F_c^n dt \end{bmatrix}. \tag{6}$$

Using the last rows of the vectors in eq. (6) we let

$$\frac{d}{dt} = F_c^n \frac{d}{d\eta}, \tag{7}$$

where $\eta \equiv \tau_c^n$.

Additional equations can be obtained by considering the relative motions of the contacting ends with respect to the respective surfaces. Accordingly, the equations that correspond to the three possible cases of the motion of the end points of the chain at contact points are given by

Case 1: Because contact is maintained and slipping occurs, the normal and tangential components of the contact forces can be represented as

$$F_j^t = -\mu_j \mathrm{Sign}(v_j^t) F_j^n \quad \text{and} \quad a_j^n = v_j^n = 0$$

with $j = 0, 1, \ldots, k$

$$F_c^t = -\mu_c \mathrm{Sign}(v_c^t) F_c^n \quad \text{when } \mathrm{A}_j = \mathrm{A}_c$$

Case 2: Conversely, when the end is not slipping, we have

$$a_j^t = v_j^t = 0 \quad \text{and} \quad a_j^n = v_j^n = 0$$

subject to $\left|F_j^t / F_j^n\right| \le \mu_j$ with $j = 0, 1, \ldots, k$

$$a_c^t = v_c^t = 0 \tag{9}$$

subject to $\left|F_c^t / F_c^n\right| \le \mu_c$ when $\mathrm{A}_j = \mathrm{A}_c$

Case 3: On the other hand, when there is no interaction at A_j, the contact forces become

$$F_j^n = F_j^t = 0 \tag{10}$$

with $j = 0, 1, \ldots, k$. Also, during collision $F_c^n \neq 0$, since impact is initiated at A_c.

Once the relative motions at the contacting ends are defined, one can obtain 2k+1 relations that will be of the form of the equations given by eqs. (7) through (10). Then these additional relations can be used along with the 2k+2 relations of eq. (5) to solve for the contact forces and accelerations in terms of F_c^n; this yields

$$\begin{bmatrix} \mathbf{F} \\ d\mathbf{v}_t/dt \end{bmatrix} = \begin{bmatrix} d\boldsymbol{\tau}/dt \\ d\mathbf{v}_t/dt \end{bmatrix} = F_c^n \begin{bmatrix} \boldsymbol{\Psi}_1(\bar{\mathbf{x}}, \boldsymbol{\theta}) \\ \boldsymbol{\Psi}_2(\bar{\mathbf{x}}, \boldsymbol{\theta}) \end{bmatrix}. \tag{11}$$

Dividing eq. (11) by F_c^n and using eq. (7) yields

$$\begin{bmatrix} d\boldsymbol{\tau}/d\eta \\ d\mathbf{v}_t/d\eta \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Psi}_1(\bar{\mathbf{x}}, \boldsymbol{\theta}) \\ \boldsymbol{\Psi}_2(\bar{\mathbf{x}}, \boldsymbol{\theta}) \end{bmatrix}. \tag{12}$$

Noting that the (k+2)×1 vectors $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$ do not depend on η, we integrate eq. (12) to get

$$\boldsymbol{\tau}(\eta) = \boldsymbol{\Psi}_1(\bar{\mathbf{x}}, \boldsymbol{\theta})(\eta - \eta_0) + \boldsymbol{\tau}(\eta_0) \tag{13}$$

$$\mathbf{v}_t(\eta) = \boldsymbol{\Psi}_2(\bar{\mathbf{x}}, \boldsymbol{\theta})(\eta - \eta_0) + \mathbf{v}_t(\eta_0) \tag{14}$$

3.1.2. *Slippage and Stoppage in the Tangential Direction*

So far, we have derived the equations that can be used when the relative motions at the contacting ends are known. However, during collision, changes may occur to cause the relative motion of a contacting end to transfer from one case to another. For example, although initially the end may be slipping in a particular direction, it may stop and/or slip in the opposite direction. In this section we present a set of equations that can be used to detect changes in the relative tangential motion when there is interaction between the chain and a contact surface (cases 1 and 2).

Case 1: For this case the end A_j is slipping initially (i.e., $v_j^t(\eta_0) \neq 0$). Thus, the end may stop slipping during collision. To detect this event, we set the left side of the $(2j-1)$st row of eq. (14) equal to zero and solve for η to obtain

$$\eta_j^* = -v_j^t(\eta_0)/\boldsymbol{\Psi}_2^{2j-1} + \eta_0, \tag{15}$$

where η_j^* is the normal impulse at A_c when end j stops slipping, and the superscript on $\boldsymbol{\Psi}_2$ denotes the row number.

Case 2: For this case the end A_j is not slipping initially (i.e., $v_j^t(\eta_0) = 0$), but it may slip again. Slippage at a particular end occurs when the friction condition in eq. (9) is violated.

3.1.3. Separation and Attachment in the Normal Direction

The second factor that should be considered here is the motion of the contacting ends in the normal directions. An end that may initially be interacting with the surface may detach during collision and may reattach again. Accordingly we derive the following relations for the respective cases:

Cases 1 and 2: The end A_j stops interacting with the surface when the normal acceleration a_j^n becomes positive as a result of a case change at any of the contacting ends. In the absence of such changes, A_j will not detach from the surface S_j until the end of collision period.

Case 3: For this case the initial normal velocity of the contacting end is directed away from the surface (i.e., $v_j^n(\eta_0) \neq 0$). When this velocity vanishes, interaction reoccurs at A_j. To detect this event, we set the left side of the $(2j)$th row of eq. (14) equal to zero and solve for η to obtain

$$\eta_j^{**} = -v_j^n(\eta_0)/{\boldsymbol{\Psi}_2}^{2j} + \eta_0, \tag{16}$$

where η_j^{**} is the normal impulse at A_c when end j starts interacting with surface S_j, and the superscript on $\boldsymbol{\Psi}_2$ denotes the row number.

3.1.4. End of the Compression and Restitution Phases

The compression phase ends when the normal velocity at A_c vanishes. Accordingly, $\eta^\dagger$, the value of the normal impulse at A_c that marks the end of the restitution phase can be calculated from

$$\eta^\dagger = -v_{k+2}^n(\eta_{tr})/{\boldsymbol{\Psi}_2}^{k+2} + \eta_{tr}, \tag{17}$$

where the superscript on $\boldsymbol{\Psi}_2$ denotes the row number, and η_{tr} is the normal impulse at A_c corresponding to the last case change at the contacting ends that occurs during compression.

The value of η at the end of the restitution phase (i.e., end of the collision) η_f, however, depends on the particular definition that is used for the coefficient of restitution. As a matter of fact, the only computational difference between using the three definitions of the coefficient of restitution arises in this calculation. Computation of η_f for each definition can be described as follows:

1. *Kinematic Definition:* According to this definition, η_f is given by

$$\eta_f^{(1)} = [-ev_{k+2}^n(0) - v_{k+2}^n(\eta_{tr})]/{\boldsymbol{\Psi}_2}^{k+2} + \eta_{tr}. \tag{18}$$

Here η_{tr} is the normal impulse at A_c corresponding to the last case change at the contacting ends that occurs during collision.

2. *Kinetic Definition:* Using the kinetic definition, η_f is obtained from

$$\eta_f^{(2)} = (e+1)\eta^\dagger. \tag{19}$$

3. *Energetic Definition:* This definition requires computation of the work done by the normal component of the contact force at A_c during compression and restitution, respectively. Work done by the normal force at the impact point is given by,

$$\triangle W^n = -\int F_c^n v_c^n dt = -\int v_c^n(\eta) d\eta. \tag{20}$$

Then, using eqs. (14) and (20) and the energetic definition, we obtain $\eta_f^{(3)}$ as

$$\eta_f^{(3)} = \eta_{tr} + \frac{\left\{(v_{k+2}^n(\eta_{tr}))^2 - 2{\boldsymbol{\Psi}_2}^{k+2}[e^2 \triangle W^n(\eta^\dagger) + \triangle W^n(\eta_{tr})]\right\}^{\frac{1}{2}} - v_{k+2}^n(\eta_{tr})}{{\boldsymbol{\Psi}_2}^{k+2}}, \tag{21}$$

where η_{tr} marks the last case transfer during restitution.

3.1.5. Changes in the Generalized Velocities

Finally, we should establish the relation among the changes that occur in the generalized velocity vector $\dot{\mathbf{x}}$ and in the impulse vector $\boldsymbol{\tau}$. Using the conservation of linear and angular impulse and momentum for the entire chain, one can write the following general relation:

$$\dot{\mathbf{x}}(\eta_2) - \dot{\mathbf{x}}(\eta_1) = \mathbf{M}^{-1}(\bar{\mathbf{x}}) \begin{bmatrix} \mathbf{D}_1(\bar{\mathbf{x}}) \\ \mathbf{D}_2(\boldsymbol{\theta}) \end{bmatrix} [\boldsymbol{\tau}(\eta_2) - \boldsymbol{\tau}(\eta_1)]. \tag{22}$$

3.1.6. Solution Procedure

In this section we outline a solution procedure to compute the velocity vector at the end of the collision period, $\dot{\mathbf{x}}(\eta_f)$, given the velocity vector $\dot{\mathbf{x}}(0)$ at the onset of the collision. We note that the normal impulse at A_c is equal to zero at the onset of collision and equal to η_f when the collision ends.

The procedure described here treats the motion during collision in two phases: the phases of compression and restitution at A_c. Furthermore, each phase may consist of several stages that arise as a result of case transfers at one of the contact points with the surfaces (for example, end A_j stops slipping). At any instant during collision, the motion at every contact point can be identified with one of the three cases described earlier. This identification is necessary to construct the proper set of equations to solve the problem. On the other hand, a solution should be obtained to check the conditions that are required to classify the motions of the contact points. Therefore, the problem should be solved repeatedly until one identifies the correct solution that satisfies all the necessary conditions for the validity of the results. This process can be described as follows:

Step 1: Set $\eta_{tr} = 0$, $\boldsymbol{\tau}(0) = \mathbf{0}$, $\mathbf{v}_t(0) = \dot{\mathbf{x}}(0)$, and $\triangle W^n(0) = 0$.

Step 2: Create a list L_c that includes the contact points where the chain interacts with the respective contact surfaces. Add A_c to the list L_c (there is always interaction at A_c).

Step 3: Set $F_i^t = F_i^n = 0 \; \forall \; A_i \notin L_c$. Create a new empty list L_s that includes the contact points $A_i \in L_c$ having zero tangential velocities ($v_i^t(\eta_{tr}) = 0$) at η_{tr} but slip for $\eta > \eta_{tr}$.

Step 4: Set $F_i^t = -\mu_i \mathrm{Sign}(v_i^t(\eta_{tr}))F_i^n \; \forall \; A_i \in L_c$ but $\notin L_s$.

Step 5: Set $F_i^t = \mu_i \mathrm{Sign}(\mu_i^a)F_i^n \; \forall \; A_i \in L_s$.

Step 6: Set $a_i^t = 0$ for the remaining points in L_c.

Step 7: Set $a_i^n = 0 \; \forall \; A_i \in L_c$.

Step 8: Using the relations defined in steps 3, 4, 5, 6, and 7, solve eq. (5) for $\mathbf{a}_t$ and $\mathbf{F}$ in terms of F_c^n.

Step 9: Check the following condition for the last member of the list L_s:

$$\mathrm{Sign}[a_i^t] \neq \mathrm{Sign}[\mu_i^a].$$

Proceed to the next step if the test does not fail; otherwise, repeat step 8 by reversing the sign of the expression used for F_i^t at the contact point corresponding to the last member of L_s.

Step 10: Compute the force ratios $\mu_i^a = F_i^t/F_i^n$—note that the ratios F_i^t/F_i^n do not depend on F_c^n—at the contact points that are included in step 6. If all $\mu_i^a < \mu_i$, proceed to the next step. Otherwise, add the point A_j to the list L_s where $\mu_j^a = \max[|\mu_{i_1}^a|, \ldots, |\mu_{i_l}^a|]$, such that $|\mu_{i_1}^a| > \mu_{i_1}, \ldots, |\mu_{i_l}^a| > \mu_{i_l}$. Then, go to step 5.

Step 11: Check if all $a_i^n \geq 0$ for $A_i \notin L_c$ and $v_i^n(\eta_{tr}) = 0$; if true, proceed to the next step. Otherwise, add A_j to the list L_c, where $a_j^n = \max[|a_{i_1}^n|, \ldots, |a_{i_m}^n|]$, such that $a_{i_1}^n, \ldots, a_{i_m}^n < 0$. Then, go to step 3. Note that it is possible to check for positiveness of the normal components of the accelerations and find the maximum of their absolute values, because these accelerations are in the form of F_c^n (always positive) multiplied by a positive or a negative number.

Step 12: Calculate the vector $\boldsymbol{\Psi}_2$ and use eq. (14) to solve for the velocity vector $\mathbf{v}_t(\eta)$. Use eq. (15) to solve for η_j^* at all contacting ends including A_c and eq. (16) to solve for η_j^{**} for $j = 0, \ldots, k$ only. Compute $\eta^\dagger$, the normal impulse at A_c where the normal velocity at this point vanishes (end of the restitution phase) by using the following equation:

$$\eta^\dagger = -v_{k+2}^n(\eta_{tr})/\boldsymbol{\Psi}_2{}^{k+2} + \eta_{tr}, \tag{23}$$

where the superscript on $\boldsymbol{\Psi}_2$ denotes the row number.

Step 13: Compute $\eta_{\text{new}} = \min[\eta^\dagger, \eta_0^*, \ldots, \eta_k^*, \eta_c^*, \eta_0^{**}, \ldots, \eta_k^{**}]$ subject to $\eta_{\text{new}} > \eta_{tr}$. Compute $\boldsymbol{\tau}(\eta_{\text{new}})$, $\mathbf{v}_t(\eta_{\text{new}})$, and $\triangle W^n(\eta_{\text{new}})$ using

$$\boldsymbol{\tau}(\eta_{\text{new}}) = \boldsymbol{\Psi}_1(\bar{\mathbf{x}}, \boldsymbol{\theta})(\eta_{\text{new}} - \eta_{tr}) + \boldsymbol{\tau}(\eta_{tr}) \tag{24}$$

$$\mathbf{v}_t(\eta_{\text{new}}) = \boldsymbol{\Psi}_2(\bar{\mathbf{x}}, \boldsymbol{\theta})(\eta_{\text{new}} - \eta_{tr}) + \mathbf{v}_t(\eta_{tr}) \tag{25}$$

$$\triangle W^n(\eta_{\text{new}}) = \boldsymbol{\Psi}_2{}^{k+2}(\bar{\mathbf{x}}, \boldsymbol{\theta})\frac{(\eta_{\text{new}} - \eta_{tr})^2}{2} + v_{k+2}^n(\eta_{tr})(\eta_{\text{new}} - \eta_{tr}) + \triangle W^n(\eta_{tr}) \tag{26}$$

and let $\eta_{tr} = \eta_{\text{new}}$. If $\eta_{\text{new}} \neq \eta^\dagger$ go to step 2, otherwise proceed to the next step.

Step 14: Set $\triangle W^n(\eta^\dagger) = \triangle W^n(\eta_{\text{new}})$, $\triangle W^n(\eta_{\text{new}}) = 0$, and go to step 2. Henceforth, in step 12 skip the calculation of $\eta^\dagger$ and when using:

1. The kinematic definition, calculate $\eta_f^{(1)}$ from eq. (18).
2. The kinetic definition, calculate $\eta_f^{(2)}$ from eq. (19).
3. The energetic definition, calculate $\eta_f^{(3)}$ from eq. (21).

Then, carry out the computations of step 13 with $\eta_f^{(i)}$ instead of $\eta^\dagger$, then skip this step and proceed to step 15.

Step 15: Compute $\dot{\mathbf{x}}(\eta_f)$, the velocity vector at the end of the collision period using

$$\dot{\mathbf{x}}(\eta_f) = \dot{\mathbf{x}}(0) + \mathbf{M}^{-1}(\bar{\mathbf{x}}) \begin{bmatrix} \mathbf{D}_1(\bar{\mathbf{x}}) \\ \mathbf{D}_2(\boldsymbol{\theta}) \end{bmatrix} \boldsymbol{\tau}(\eta_f). \tag{27}$$

Furthermore, the relative motions of the contacting ends can be resolved as follows:

- *Contact points that rebound from the respective surfaces:* All A_i such that $A_i \notin L_c$ and A_c if $e \neq 0$.
- *Contact points that neither rebound nor slip:* All A_i such that $A_i \in L_c$ but $\notin L_s$.
- *Contact points that do not rebound but slip:* All A_i that are not included in one of the categories above.

The outlined procedure divides the collision period into several stages that are marked by initiation of impact, end of compression phase, or case changes at the contacting ends. At the onset of each stage, we presume that there is no interaction at the contacting ends except at A_c. Therefore, in step 2, the list of contacting ends L_c includes only this point. Then, in step 11, we check whether interaction occurs at the points where a particular end has no velocity in the normal direction. Interacting points are added one at a time based on the magnitudes of the normal accelerations. Each time a contact point is added to L_c, the slippage is checked again for all interacting points in step 10. The final value of the normal impulse at A_c is computed in step 2 according to the approach that is taken. Then, in step 15, the final solution of the impact problem is determined.

Finally, we note that no specific methods were provided here for the computation of the matrices $\mathbf{M}(\mathbf{x})$, $\mathbf{D}_1(\mathbf{x})$, $\mathbf{D}_2(\mathbf{x})$, $\mathbf{H}_1(\mathbf{x})$ and $\mathbf{H}_2(\mathbf{x})$. Numerous methods of formulation of the kinematic relations and derivation of the equations of motion of kinematic chains have been published in the literature. For an excellent survey of the existing approaches, we refer the reader to Huston (1991) and the references therein.

3.2. Algebraic Formulation

The algebraic formulation is suitable when the kinematic definition of the coefficient of restitution is used. In this section we formulate two solution schemes to solve the present problem by using the approaches proposed in Whittaker (1904) and Brach (1989).

In the present formulation the solution of the impact problem is obtained by using the conservation of linear and angular impulse and momentum equations given by eq. (22) to obtain

$$\dot{\mathbf{x}}(\eta_f) - \dot{\mathbf{x}}(0) = \dot{\mathbf{x}}^+ - \dot{\mathbf{x}}^- = \mathbf{M}^{-1}(\bar{\mathbf{x}}) \begin{bmatrix} \mathbf{D}_1(\bar{\mathbf{x}}) \\ \mathbf{D}_2(\boldsymbol{\theta}) \end{bmatrix} \hat{\boldsymbol{\tau}}, \quad (28)$$

where $\dot{\mathbf{x}}^+$ and $\dot{\mathbf{x}}^-$ are the $2\,n\times 1$ velocity vectors immediately before and after impact respectively and $\hat{\boldsymbol{\tau}}$ is the $(2\,k+2)\times 1$ impulse vector for the entire collision. Equation (28) should be supplemented by an additional $2k+2$ equations to solve for the vectors $\dot{\mathbf{x}}^+$ and $\hat{\boldsymbol{\tau}}$. These additional equations are obtained by considering slippage and rebounds at the respective contact surfaces.

3.2.1. Slippage in the Tangential Direction

According to Whittaker's method, a contacting end slips when,

$$\left|\hat{\tau}_j^t/\hat{\tau}_j^n\right| > \mu_j \quad \text{for} \quad j = 0, 1, \ldots, k$$

$$\text{and} \quad \left|\hat{\tau}_c^t/\hat{\tau}_c^n\right| > \mu_c \quad \text{when} \quad A_j = A_c \quad (29)$$

Brach (1990), modifies the slippage condition such that the impulse ratio at the impact point is given by

$$\left|\hat{\tau}_c^t/\hat{\tau}_c^n\right| = \min\left[\,|\mu_0|\,, |\mu_c|\,, |\mu_t|\,\right], \quad (30)$$

where μ_0 is the impulse ratio when A_c does not slip, and μ_t is the impulse ratio that produces zero overall energy loss as a result of collision. The two slippage conditions are equivalent when the energy loss $\triangle KE = \mathrm{KE}(0) - \mathrm{KE}(\eta_f)$ is positive or zero but distinct otherwise. With this modification the revised method does not lead to energy increases in the resulting solutions.

3.2.2. Detachment in the Normal Direction

The postimpact, normal velocity at the colliding end depends directly on the coefficient of restitution and can be obtained from

$$v_c^{n\,(+)} = -e v_c^{n\,(-)}. \quad (31)$$

For the other contacting ends, equations for the normal velocities can be obtained from

$$v_j^{n\,(+)} > 0 \;\text{ and }\; \hat{\tau}_j^n = 0 \;\text{ for } j = 0, 1, \ldots, k, \quad (32)$$

when the end rebounds, or

$$v_j^{n\,(+)} = 0 \;\text{ and }\; \hat{\tau}_j^n > 0 \;\text{ for } j = 0, 1, \ldots, k, \quad (33)$$

when the end interacts with the surface and does not detach as a result of the collision. Treatment of the rebound at the contacting ends in this fashion leads to two possibilities: the end rebounds without interaction or it interacts with the surface and does not rebound. The scheme eliminates the possibility of rebounds from the surface when the interaction occurs. This limitation is not present in the solutions obtained using the differential formulation. There the solution scheme takes into account the events during collision, and normal accelerations are used to resolve the motion along the normal direction (see step 11 of the solution method outlined earlier). The algebraic formulation, however, does not permit the consideration of cases in which rebound occurs with interaction with the surface. In such cases, an additional equation cannot be obtained from the relative motion in the normal direction, leading to fewer equations than unknowns. As will be demonstrated in Section 4 later, in a special set of circumstances, the solutions obtained by using the two formulations disagree in terms of prediction of rebounds at the contacting ends.

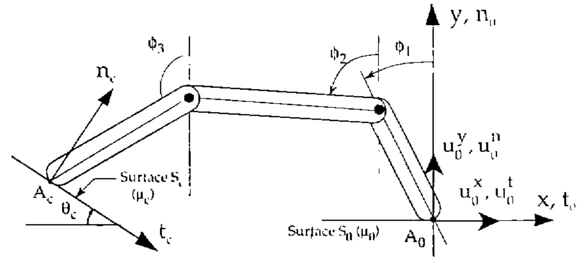

Fig. 2. Three-link chain with two contact points.

3.2.3. Solution Procedure

The solution method presented here is based on repeatedly solving the problem until the proper solution is identified. The procedure can be described by the following steps:

Step 1: Obtain $4^k \times 3$ sets of 2k + 1 equations by considering all possible combinations of the following cases at each of the contacting ends and the colliding end:

a. A_i detaches without interaction: $\hat{\tau}_i^t = \hat{\tau}_i^n = 0$ for $i = 0, \ldots, k$.
b. A_i does not detach and does not slip: $v_i^{n(+)} = v_i^{t(+)} = 0$ for $i = 0, \ldots, k$ and $v_c^{t(+)} = 0$ for A_c.
c. A_i does not detach but slips in the positive direction of the tangential coordinate: $\hat{\tau}_i^t / \hat{\tau}_i^n = \mu_i$ for $i = 0, \ldots, k$ and $\hat{\tau}_c^t / \hat{\tau}_c^n = \mu_c$ for A_c.
d. A_i does not detach but slips in the negative direction of the tangential coordinate: $\hat{\tau}_i^t / \hat{\tau}_i^n = -\mu_i$ for $i = 0, \ldots, k$ and $\hat{\tau}_c^t / \hat{\tau}_c^n = -\mu_c$ for A_c.

Step 2: Use eqs. (28) and (31) along with the equations obtained in the previous step to obtain $4^k \times 3$ sets of solutions.

Step 3: Eliminate the solution sets that fail to satisfy the following conditions at every contacting point:

a. $v_i^{n(+)} > 0$ for $i = 0, \ldots, k$ if the solution is obtained by assuming that A_i detaches without interaction.
b. $\hat{\tau}_i^n > 0$ and $|\hat{\tau}_i^t / \hat{\tau}_i^n| \leq \mu_i$ for $i = 0, \ldots, k$; and $|\hat{\tau}_c^t / \hat{\tau}_c^n| \leq \mu_c$ if the solution is obtained by assuming that A_i does not detach and does not slip.
c. $\hat{\tau}_i^n > 0$ and $\text{sign}[\hat{\tau}_i^t] \neq \text{sign}[v_i^{t(+)}]$ for $i = 0, \ldots, k$; and $\text{sign}[\hat{\tau}_c^t] \neq \text{sign}[v_c^{t(+)}]$ if the solution is obtained by assuming that A_i does not detach but slips in the positive direction of the tangential coordinate.
d. $\hat{\tau}_i^n > 0$ and $\text{sign}[\hat{\tau}_i^t] \neq \text{sign}[v_i^{t(+)}]$ for $i = 0, \ldots, k$; and $\text{sign}[\hat{\tau}_c^t] \neq \text{sign}[v_c^{t(+)}]$ if the solution is obtained by assuming that A_i does not detach but slips in the negative direction of the tangential coordinate.

Step 4: Stop when a solution is found that satisfies all the conditions given in step 3.

Step 5: Check the energy loss for the solution found in the previous step, if $\triangle KE \geq 0$ stop otherwise proceed to the next step.

Step 6: Obtain 4^k sets of new solutions by using the relations given in step 1 for the contacting ends and compute the impulse ratio at the colliding end by setting the energy loss equal to zero.

Step 7: Perform the tests that are given in step 3 for each contacting end, and identify the proper solution.

4. Application to a Three-Link Chain With Two Contact Points

In this section we apply the procedures proposed above to the three-link, two-contact-point kinematic chain show in Figure 2. Slender members with 1-m lengths, 1-kg masses, and moments of inertia of $1/12 \quad \text{kg} - \text{m}^2$ that are connected with revolute joints are assumed. The angle θ_c is taken as zero. The joint coordinates at impact are

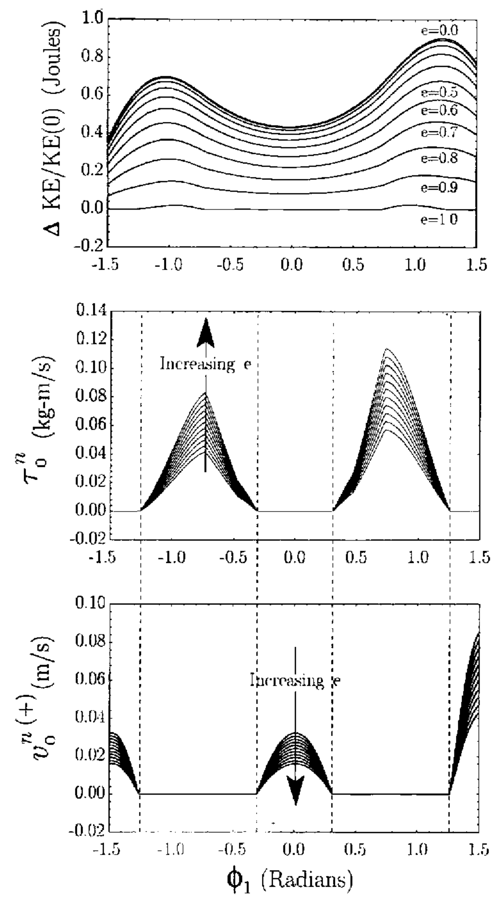

Fig. 3. Impact with a smooth surface. A, Energy loss. B, Normal impulse at A_0. *C, Normal velocity at* A_0.

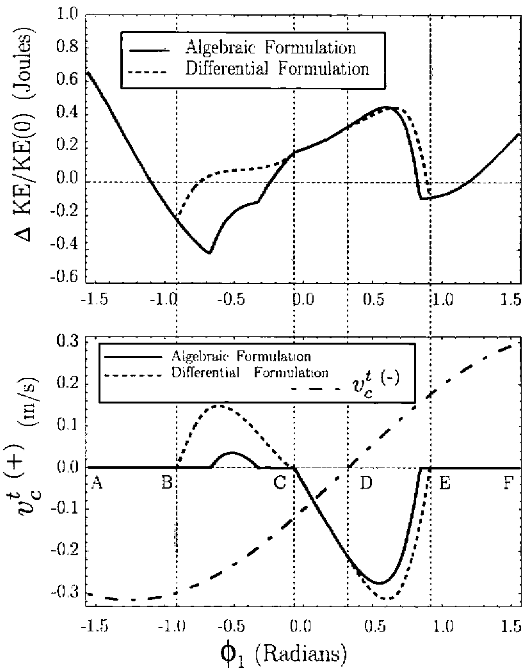

Fig. 4. Impact with friction using the kinematic definition of the coefficient of restitution. A, Energy loss. B, Tangential velocity at A_c.

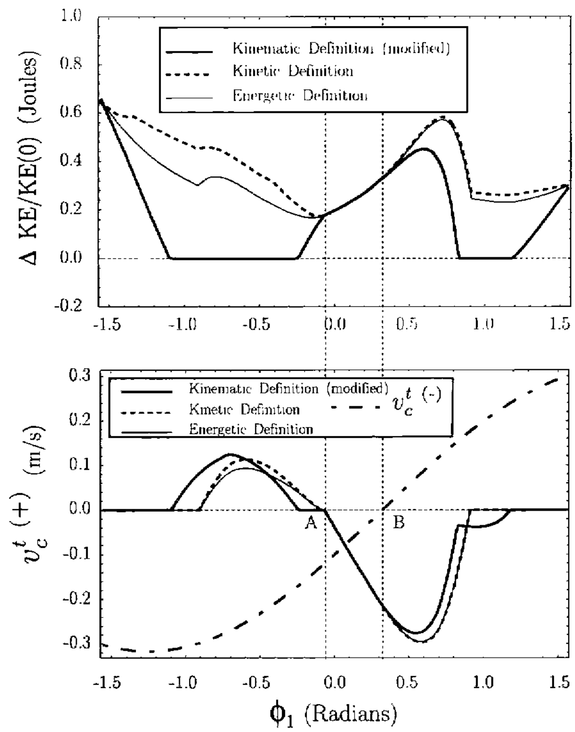

Fig. 5. Impact with friction using the three solution schemes that do not lead to energy increases due to impact. A, Energy loss. B, Tangential velocity at A_c.

selected as $\phi_2 = \pi/2$, $\phi_3 = \pi - \phi_1$, while ϕ_1 varies between $-\pi/2$ and $\pi/2$. The preimpact velocities are selected as $\dot{u}_0^t(0) = \dot{u}_0^n(0) = 0$, $\dot{\phi}_1(0) = 0.1$ rad/s, $\dot{\phi}_2(0) = 0.2$ rad/s, and $\dot{\phi}_3(0) = 0.3$ rad/s.

Figure 3 depicts the results obtained by letting $\mu_0 = 0.5$ and $\mu_c = 0$ and changing the coefficient of restitution from 0 to 1. When friction is absent at the colliding end, the differential and algebraic methods yield exactly the same results. Figure 3*A* shows the energy loss owing to impact. The energy loss is zero for perfectly elastic collisions ($e = 1$) when the end A_0 does not slip. As e is increased, the system loses more energy as a result of collision and the results are energetically consistent. Figures 3*B* and 3*C* depict the normal impulse and the rebound velocity, respectively, at the noncolliding end. Comparing the two figures, we observe that when the normal velocity is nonzero, the normal impulse is zero, and when the normal impulse is nonzero, the normal velocity is zero. Thus, when friction is absent, the end does not rebound if impulse is imparted to the chain at A_0. Furthermore, increasing the coefficient of restitution at A_c causes higher normal impulse values but lower rebound velocities at A_0.

Figure 4 depicts the results obtained by using the kinematic definition and solving the problem by differential and the algebraic formulations. The coefficients of friction are selected as $\mu_0 = \mu_c = 0.5$ and the coefficient of restitution is 0.9. Although the same definition of the coefficient of restitution is used, we observe that the two approaches lead to dissimilar outcomes for certain configurations. As can be seen from Figure 4*A*, both formulations may lead to energy gains as a result of collision. This is an inherent characteristic of the solutions obtained using Newton's definition of the coefficient of restitution. The solutions differ in the two intervals marked by B–C and D–E on the figure. We can observe from Figure 4*B* that the two formulations lead to distinct outcomes when the tangential velocity at the colliding end is reversed. Furthermore, as it can be seen from the tangential velocities in the two intervals B–C and D–E, the differential formulation is more likely to predict slippage. This arises because the algebraic formulation overestimates the friction impulse when velocity reversals take place. On the other hand, when both solutions predict stoppage or constant slip direction, the results become identical (intervals A–B, C–D, and E–F).

Figure 5 shows the results obtained by using the three approaches that do not lead to energy increases as a result of collision. The coefficients of friction are selected as $\mu_0 = \mu_c = 0.5$, and the coefficient of restitution is 0.9. The results obtained by the three approaches are dissimilar when the slip stops or reverses at the impact point. The solutions coincide only when the end slips remains in the same direction (interval A–B). Furthermore,

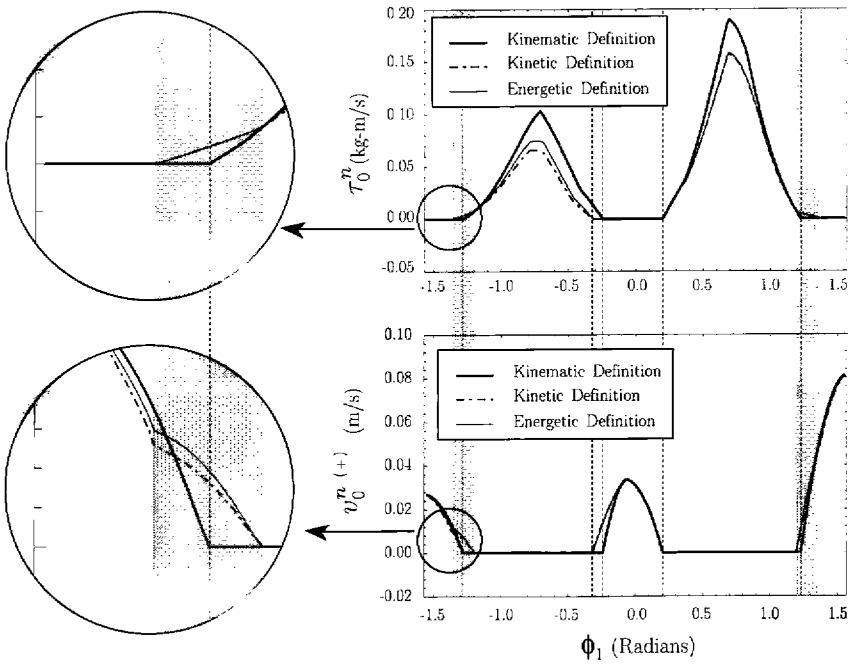

Fig. 6. Rebound at A_0. *A, Normal impulse. B, Rebound velocity.*

among the three methods, the solutions obtained by the kinetic and energetic definitions are in closest agreement with each other when compared with the solutions obtained using the method proposed in Brach (1991). In Figure 6 the rebounds at the end A_0 for the three methods are considered. The most notable difference among the three approaches is that the kinematic definition is used with the algebraic formulation and the other two are based on the differential formulation. For a wide variety of circumstances, the solutions predict rebound at A_0 without interaction with the contact surface, the assumption implicit in the algebraic formulation. However, when friction is present, the differential methods may predict rebound with interaction with the surface for a special set of conditions. The two shaded regions in the figure are the intervals where such rebounds are observed. They are located at the transition regions from rebound at A_0 to no rebound. In the two other transition regions, the differential methods do not yield rebounds with interaction.

Figure 7 depicts the results obtained for perfectly plastic collisions ($e = 0$) and selecting the friction coefficients as $\mu_0 = \mu_c = 0.5$. For this case the outcomes are different only when different formulations are used. More specifically, the differential solutions produce the same outcomes regardless of the definition of the coefficient of restitution. The algebraic solutions are the same, as Whittaker's method never leads to energy increases. The difference in energy curves are almost indistinguishable, but the two sets of solutions disagree in the prediction of rebounds at the noncontacting end. As can be seen from Figures 7*B* and 7*C*, the differential methods predict rebounds with interaction, while the algebraic methods don't.

5. Conclusion

This article considers the multicontact, rigid body collisions of planar kinematic chains in the presence of friction. Two general solution procedures that are based on the formulation of the equations of impact in algebraic and differential forms are presented in full detail. The algebraic formulation can be used to obtain solutions based on the kinematic definition of coefficient of restitution and treating the tangential motion at the point of impact by using the approach in either Whittaker (1904) or Brach (1990). The differential formulation can be used to solve the problem subject to the kinematic (Newton 1686), kinetic (Routh 1905; Keller 1986), and the energetic (Stronge 1990) definitions of the coefficient of restitution.

The results verify the previous observations that the solutions obtained are independent of the method of approach when friction is absent at the collision point or when the slip at this point does not change direction because of impact. However, when friction is present, the predicted outcomes rely heavily on the particular formulation of the impact equations and the definition of the coefficient of restitution.

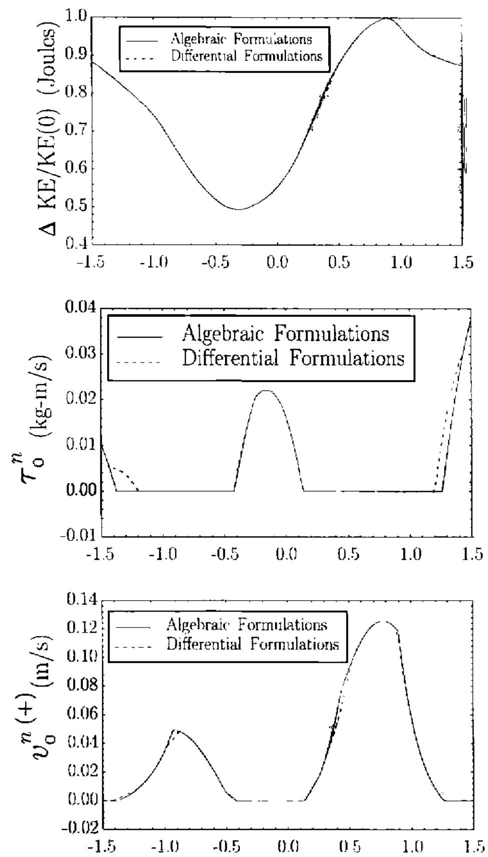

Fig. 7. Perfectly plastic impacts. A, Energy loss. B, Normal impulse at A_0. *C, Normal velocity at* A_0.

Considering the energy loss owing to impact, the results predicted by the energetic definition of the coefficient of restitution is the most consistent, because the approach properly accounts for the energy loss owing to friction and to plastic effects at the collision point. If we confine our observations only to energy loss, we can judge the consistency of various approaches by comparing the energy losses obtained by using each approach with the results produced using the energetic definition. Accordingly, we deduce that the kinetic definition overestimates the energy loss but produces more consistent results than the other three methods. Brach's approach overestimates the energy loss but resolves the energy inconsistencies in Whittaker's method. The kinematic definition used with the differential formulation leads to energy increases as a result of collision. Surprisingly, Whittaker's method, the most widely used of the five, produces the energetically least consistent results.

The prediction of rebounds at the nonimpacting contact points also relies on the particular method of approach. The algebraic methods preclude the possibility of rebounds when the chain interacts with the surface (nonzero normal impulse). On the other hand, differential methods may predict rebound with interaction when friction is present at the impact point and the slip stops or reverses direction at this point. However, for a variety of circumstances, all methods may exhibit the transitional behavior where rebounds occur only when the chain does not interact with the surface. Although such transitions are intuitively difficult to accept, they occur smoothly and, analytically speaking, are consistent, because the solutions meet at a point where the normal impulse and velocity are both equal to zero. Therefore, we are unable to refute this phenomenon based on solely theoretical results. This issue may provide an important factor in conducting an experimental study to assess the validity of such predictions.

Finally, we note that the approaches presented here should be used with caution when applied to kinematic chains with very large numbers of elements or very long members. In such cases, the analyses presented here may not be proper, because they neglect the impulses owing to the weights of the members and presume simultaneous collisions at all ends.

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