SOLUTION

MID-SEM-I (2023-24-I)

1. A particle of mass m moving along the x-axis has the following potential energy

$$V(x) = \begin{cases} \infty & x \le 0 \\ \frac{1}{2} M\Omega^2 x^2 & x > 0 \end{cases}$$

(a) Plot the potential energy. **(1)**

(b) Find the time-period T and fundamental frequency ω_0 of the motion performed by the particle in terms of Ω (***Hint:*** *use* $\mathrm{dt} = \frac{\mathrm{dx}}{\mathrm{v}}$) **(3)**

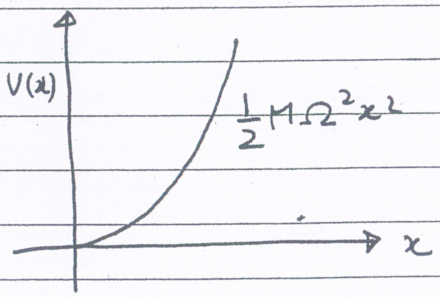
(c) Plot displacement $x(t)$ versus time t graph and write its functional form taking $x(0) = 0$. **(2)**

(d) Find the energy of n^{th} quantum level using the Bohr-Sommerfeld quantum condition. **(2)**

(e) If we write $x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t)$, find $A_k = \frac{\omega_0}{\pi}\int_0^T x(t)\cos k\omega_0 t\, dt$ for $k \neq 0$. On the basis of this, what values of m will be allowed if the particle makes a transition from n^{th} level to m^{th} level? **(4)**

(f) Making use of the correspondence principle, find the lifetime of $n = 3$ state. **(3)**

SOLUTION: (a)



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(b) Let the maximum distance travelled by the particle be A so that energy $E = \frac{1}{2}M\Omega^2A^2$

$$v(x) = \frac{1}{M}\sqrt{2M\left(E - \frac{1}{2}M\Omega^2x^2\right)}$$

$$= \Omega\sqrt{A^2 - x^2}$$

$$T = 2\int_0^A \frac{dx}{\Omega\sqrt{A^2-x^2}}$$

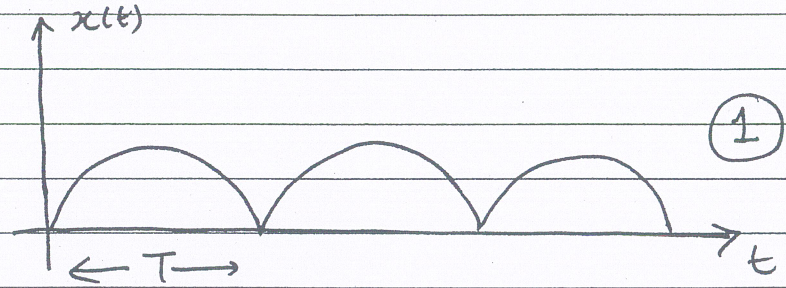
Take $x = A\sin\theta \quad dx = A\cos\theta\, d\theta$

$$T = \frac{2}{\Omega}\int_0^{\pi/2} \frac{\cancel{A\cos\theta}\, d\theta}{\cancel{A\cos\theta}}$$

$$= \frac{\pi}{\Omega} \quad \text{——} \quad 2$$

$$\omega_0 = \frac{2\pi}{T} = 2\Omega \quad \text{——} \quad 1$$

(c)



$$x(t) = A|\sin\Omega t| \quad \text{——} \quad 1$$

(d) Action $= 2\int_0^A \sqrt{2M\left(\frac{1}{2}M\Omega^2A^2 - \frac{1}{2}M\Omega^2x^2\right)}\, dx$

$$= 2M\Omega \int_0^A \sqrt{A^2 - x^2}\, dx$$

$$x = A\sin\theta$$

$$\text{Action} = 2M\Omega A^2 \int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= 2M\Omega A^2 \times \frac{\pi}{4}$$

$$= \left(\frac{1}{2}M\Omega A^2\right)\pi$$

$= nh$ by quantum condition

$\Rightarrow$ Energy $E_n = \frac{1}{2} M \Omega^2 A^2 = \frac{n h \Omega}{\pi}$

2 $= 2n\hbar\Omega$

$(n = +1, +2, \ldots)$

(e) $$A_k = \frac{\omega_0}{\pi} \int_0^T A (\sin \Omega t) \cos(k\omega_0 t)\, dt$$

$$= \frac{\omega_0 A}{\pi} \int_0^T \sin\left(\frac{\omega_0}{2} t\right) \cos(k\omega_0 t)\, dt$$

2 $$= \frac{\omega_0 A}{2\pi} \int_0^T \left\{ \sin\left(k+\tfrac{1}{2}\right)\omega_0 t - \sin\left(k-\tfrac{1}{2}\right)\omega_0 t \right\} dt$$

$$= \left(\frac{\omega_0 A}{2\pi}\right) \frac{1}{\omega_0} \left[-\frac{\cos(k+1/2)\omega_0 t}{(k+\frac{1}{2})} + \frac{\cos(k-\frac{1}{2})\omega_0 t}{(k-\frac{1}{2})} \right]_0^T$$

$$= \frac{\omega_0 A}{2\pi} \cdot \frac{1}{\omega_0} \left[\frac{2}{2k+1} + \frac{2}{2k+1} - \frac{2}{(2k-1)} - \frac{2}{2k-1} \right]$$

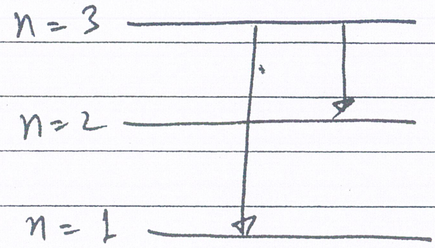
$$= \frac{4A}{2\pi} \left[\frac{1}{(2k+1)} - \frac{1}{(2k-1)} \right]$$

1 $$= \frac{-4A}{\pi(4k^2-1)}$$ for all k's

Therefor all $|n-m| = 1, 2 \ldots$ are allowed

1

(f) From $n=3$ to $n=2$ and $n=1$ transitions are allowed. So total probability of transition



$$A = A_{32} + A_{31} \quad (2)$$

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{q^2}{4\pi\epsilon_0 c^3}$$

$$\times \left[\omega_{32}^4 \times \left(\frac{4}{\pi} \frac{A_3}{3} \right)^2 + \omega_{31}^4 \left(\frac{4}{\pi} \frac{A_3}{15} \right)^2 \right]$$

For $3 \to 2$ transition $k=1$
For $3-1$ transition $k=2$

$$\frac{1}{2} M \Omega^2 A_3^2 = E_3 = 6 \hbar \Omega$$

$$A_3^2 = \frac{12\hbar}{M\Omega}$$

So $$A = \frac{e^2}{12\pi\epsilon_0 c^3} \times \frac{16}{\pi^2} \times \frac{12\hbar}{M\Omega} \left(\frac{\omega_{32}^4}{9} + \frac{\omega_{31}^4}{225} \right)$$

$$\omega_{32} = \frac{6\hbar\Omega - 4\hbar\Omega}{\hbar} = 2\Omega$$

$$\omega_{31} = \frac{6\hbar\Omega - 2\hbar\omega}{\hbar} = 4\Omega$$

$$A = \frac{16 e^2 \hbar}{6\pi^3 c^3 M\Omega} \times 16\Omega^4 \left(\frac{1}{9} + \frac{16}{225} \right)$$

$$= \frac{256 e^2 \hbar \Omega^3}{6\pi^3 c^3 M} \times \left(\frac{241}{2025} \right)$$

Life time

$$\tau = A^{-1} = \frac{6\pi^3 c^3 M}{256\, e^2 \hbar\, \Omega^3}\left(\frac{2025}{241}\right) \quad \text{— } 1$$

2. Take the entropy $S(T)$ of radiation at temperature T as $S = V\int s_\nu(T)d\nu$ in a cavity of volume V, where $s_\nu(T)$ is the entropy density at T per unit frequency.
   (a) At equilibrium $\Delta S = 0$ for small changes $\Delta u_\nu(T)$ in the spectral energy density distribution. Show if the total energy $U = \int u_\nu(T)d\nu$ remains constant, then $\Delta S = 0 \Rightarrow \left(\frac{\partial s_\nu}{\partial u_\nu}\right)_V = \text{constant} = \beta$. **(5)**
   (b) Taking $\beta = \frac{1}{T}$, find the expression for $s_\nu(T)$ in terms of $u_\nu(T)$ using Wien's distribution formula $u_\nu(T) = \frac{8\pi h\nu^3}{c^3}e^{-\frac{h\nu}{k_BT}}$. **(7)**
   (c) Using the expression for $s_\nu(T)$ derived in part (b), show that the change in entropy $S = Vs_\nu(T)\Delta\nu$ between frequencies ν and $\nu + \Delta\nu$ of radiation when the volume of the cavity is changed from V_1 to V_2 keeping the energy $U = Vu_\nu(T)\Delta\nu$ unchanged is $\Delta S = Nk_B \ln\left(\frac{V_2}{V_1}\right)$, where $N = \frac{Vu_\nu(T)\Delta\nu}{h\nu}$. **(3)**

<u>SOLUTION:</u> (a) $S = V\int s_\nu(T)\,d\nu$

$$\Delta S = V\int \left(\frac{\partial s_\nu}{\partial u_\nu}\right)_V \Delta u_\nu(T)\,d\nu = 0 \quad \text{— } 2$$

Now $\int \Delta u_\nu(T)\,d\nu = 0$ So — 1

$\left(\frac{\partial s_\nu}{\partial u_\nu}\right)_V = \text{Constant} = \beta$ make ΔS vanish $\Rightarrow \left(\frac{\partial s_\nu}{\partial u_\nu}\right)_V = \beta$ — 2

with logic

(b) $\beta = \frac{1}{T} \Rightarrow \frac{\partial s_\nu}{\partial u_\nu} = \frac{1}{T}$

From Klien's formula

$$u_\nu = \frac{8\pi h\nu^3}{c^3} e^{-h\nu/k_B T}$$

$$\frac{1}{T} = + \frac{k_B}{h\nu} \ln\left(\frac{8\pi h\nu^3}{u_\nu c^3}\right)$$

As $$\frac{\partial s_\nu}{\partial u_\nu} = \frac{k_B}{h\nu} \ln\left(\frac{8\pi h\nu^3}{u_\nu c^3}\right) \quad (2)$$

$$\Rightarrow s_\nu = \frac{k_B}{h\nu}\left[u_\nu \ln\left(\frac{8\pi h\nu^3}{c^3}\right) - u_\nu \ln u_\nu + u_\nu \right]$$

$$= \frac{k_B}{h\nu} u_\nu \left[\ln\left(\frac{8\pi h\nu^3}{u_\nu c^3}\right) + 1 \right] \quad (2)$$

$$V s_\nu \Delta\nu = k_B \frac{V u_\nu \Delta\nu}{h\nu}\left[\ln\left(\frac{8\pi h\nu^3 V\Delta\nu}{V u_\nu \Delta\nu\, c^3}\right) + 1 \right]$$

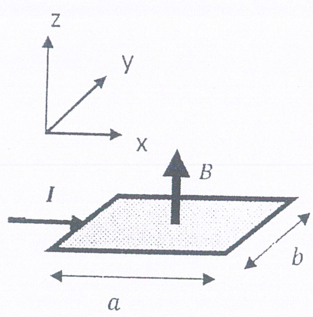
Take $E_{\nu,\nu+\Delta\nu} = V u_\nu \Delta\nu$ = Energy in the range $\nu \rightarrow \nu + \Delta\nu$

As $$S_{\nu,\nu+\Delta\nu} = k_B \frac{E_{\nu,\nu+\Delta\nu}}{h\nu}\left[\ln\left(\frac{8\pi h\nu^3 \Delta\nu\, V}{E_{\nu,\nu+\Delta\nu}\, c^3}\right) + 1 \right] \quad (2)$$

Keeping the energy constant, if entropy change is calculated

$$\left.\begin{aligned} \Delta S_{\nu,\nu+\Delta\nu} &= k_B N \ln\left(\frac{V_2}{V_1}\right) \\ N &= \left(\frac{E_{\nu,\nu+\Delta\nu}}{h\nu}\right) \end{aligned}\right\} \quad (1)$$

3. **Towards understanding Quantum Hall effect:** Shown in the figure is a two-dimensional strip of a conducting material in the x-y plane. It has free electron density of n_e per unit area. It has current I going through it in the x-direction and a magnetic field B is applied perpendicular to it in the z-direction. Take electronic charge to be of the magnitude e.



(a) What is the current density j_x per unit length and the drift speed of the electrons in the strip? **(2)**

(b) Obtain the Hall electric field E_y and the Hall resistivity $\frac{E_y}{j_x}$ in the strip. Show that the Hall resistivity is the same as Hall resistance? **(4)**

Now we consider quantized energy levels of electrons in the field.

(c) Find the quantized energy levels and area ΔS between orbits of two consecutive levels by applying Bohr-Sommerfeld quantization condition. The number of electrons in each energy level is equal to the area of the strip divided by this area, i.e., $\frac{ab}{\Delta S}$. Hence find the electron number density per unit area if N number of these orbits are filled. **(6)**

(d) Using the result of part (c), show that Hall resistance is quantized. **(3)**

SOLUTION: (a) $j_x = \frac{I}{b} = n_e v_d e$ — 1

$\Rightarrow \quad v_d = \frac{I}{b n_e e}$ — 1

(b) $E_y = v_d B$ (by balancing the force)

$= \frac{j_x B}{n_e e}$ — 2

Hall resistance $= \frac{V_{Hall}}{I}$

2 — $= \frac{E_y b}{j_x b} = \frac{E_y}{j_x}$

$=$ Resistivity

(C) Quantum Condition in magnetic field

$$(2\pi R) m v - e\Phi = nh$$

$$\frac{mv^2}{R} = qvB = evB$$

$$\Rightarrow mv = eBR$$

$$e\Phi = e\pi R^2 B = \pi m v R$$

So $$2\pi R mv - \pi m v R = nh$$

$$\frac{1}{2} m v = \frac{nh}{2\pi R} = \frac{n\hbar}{R}$$

$$\frac{1}{2} m v^2 = n\hbar \left(\frac{v}{R}\right) = n\hbar\omega$$

where $$\omega = \frac{v}{R} = \left(\frac{eB}{m}\right)$$

So $$E_n = n\hbar\left(\frac{eB}{m}\right) = n\hbar\omega \quad \text{— } 2$$

$$\Delta S = \pi\left(R_{n+1}^2 - R_n^2\right)$$

$$= \pi\left(\frac{m^2 v_{n+1}^2}{e^2B^2} - \frac{m^2 v_n^2}{e^2 B^2}\right)$$

$$= \frac{2\pi m}{e^2B^2}\left(E_{n+1} - E_n\right)$$

$$= \frac{2\pi \cancel{m}}{e^2B^2} \times \frac{\hbar eB}{\cancel{m}}$$

$$= \left(\frac{h}{eB}\right) \quad \text{— } 2$$

Therefore number of electrons n_e per unit area if N such orbits (energy levels) are filled

$$n_e = \frac{1}{ab} N \left[\frac{ab}{(h/eB)} \right] = \frac{NeB}{h} \quad \text{--- (2)}$$

(d) Hall resistance

$$= \frac{E_Y}{J_x}$$

$$= \frac{B}{n_e e}$$

$$= \frac{B}{N.\frac{eB}{h}.e}$$

$$= \frac{1}{N}\left(\frac{h}{e^2}\right) \quad \text{--- (3)}$$

4. (a) Write the form of the solution for displacement $\psi(x, t)$ (disturbance) of a stationary wave and the corresponding wave equation for a single angular frequency ω. Take the speed of the wave to be v. **(4)**

(b) Show using the quantum condition that the matrices representing x and p in quantum mechanics should be infinite-dimensional. **(4)**

(c) What is the volume in phase space occupied by photons of frequency in the range ν to $\nu + \Delta\nu$ in a cavity of volume V? **(2)**

(d) Make a schematic plot of the ratio $\frac{C_V}{R}$ of the specific heat C_V and gas constant R for hydrogen in the range of temperature 0 K to 400 K. **(2)**

(e) What do you think is a possible fundamental reason why a particle performing simple harmonic motion has zero-point energy. **(3)**

SOLUTION: (a) $\psi(x,t) = \psi(x)\, T(t)$ — (1)

Wave equation for single frequency

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \psi(x,t)}{\partial t^2} = 0$$ with $T(t) = \sin\omega t$

(1) —

So

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{v^2}\frac{\partial \psi}{\partial t^2}$$

$$= \left(\frac{d^2\psi}{dx^2}\right) T - \frac{\omega^2}{v^2}\psi\, T = 0$$

$$\Rightarrow \quad \frac{d^2\psi}{dx^2} + \frac{\omega^2}{v^2}\psi = 0$$

Taking $\frac{\omega}{v} = k$ we get

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$

2

(b) $xp - px = i\hbar I$

Taking diagonal element (nn)

$$(xp - px)_{nn} = i\hbar$$

$$\Rightarrow \quad \sum (xp - px)_{nn} = i\hbar \sum 1$$

$$\sum (xp)_{nn} - \sum (px)_{nn} = i\hbar \sum 1$$ 1

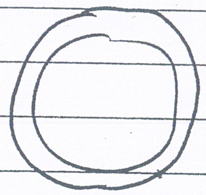
But $\sum (xp)_{nn} = \sum (px)_{nn}$

so LHS $= 0$ — 1

$0 =$ finite number

This is a contradiction $\Rightarrow$ matrices cannot be finite — 2

(c) Photon momentum $= \frac{h\nu}{c}$

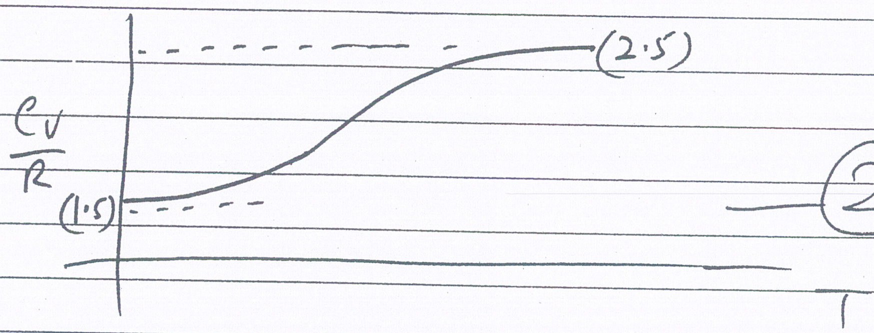


So volume in phase space occupied $=$ V. volume in momentum space

$$= V.\ 4\pi \left(\frac{h\nu}{c}\right)^2 d\left(\frac{h\nu}{c}\right)$$

$$= V.\ \frac{4\pi h^3 \nu^2}{c^3} \Delta\nu \quad \text{— } 2$$

(d)



— 2

(e) Uncertainty prevents the particle to have precise $x=0$ so it has some uncertainty in position Δx. That gives Δp

$$\frac{\Delta p^2}{2m} + \frac{\Delta x^2}{2} > 0$$

So minimum energy cannot be zero.

Fundamental Reason: Uncertainty principle

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