



5% → TUT - QUIZ

5% → ASSIGNMENTS

10% → QUIZ - 2

30% → ENDSEM

MATRICES :

→ Equality of 2 Matrices :

A & B are said to be equal if they have the same order and their corresponding elements are equal.

e.g. $A = [a_{ij}]_{m \times n}$ $a_{ij} = b_{ij} \quad \forall i, j$
 $B = [b_{ij}]_{m \times n}$

Types Of Matrices:

- 1) Square Matrix $m = n$ (Rows = Column)
- 2) Zero Matrix $a_{ij} = 0 \quad \forall i, j$
- 3) Diagonal Matrix $a_{ij} = 0 \quad \forall i \neq j$

• Scalar Matrix $a_{ij} = a ; a \neq 0$
 (All diagonal elements are same)

• Identity Matrix $a_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

4) Upper Triangular Matrix : $a_{ij} = 0 \quad \text{if } i > j$

5) Lower Triangular Matrix : $a_{ij} = 0 \quad \text{if } i < j$

MATRIX OPERATION :

- 1) Addition of Matrices
- 2) Multiplication of Matrices
- 3) Transpose of Matrix

1) For addition \rightarrow same order $m \times n$

$$C = A + B$$

$$C_{ij} = [a_{ij} + b_{ij}] = [a_{ij}] + [b_{ij}]$$

Properties :

$$(A+B) = (B+A)$$

$$(A+B)+C = A + (B+C)$$

$$k(kA) = (k\ell)A$$

$$(k+\ell)A = kA + \ell A$$

2) Matrix Multiplication :

$$A = [a_{ij}]_{m \times n} \xrightarrow{\text{No. of columns of } A}$$

$$B = [b_{ij}]_{n \times r} \xrightarrow{\text{No. of rows of } B}$$

$$C = AB$$

$$= [c_{ij}]_{m \times r}$$

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

Properties :

$$(i) A(BC) = (AB)C$$

$$(ii) A(B+C) = AB + AC$$

$$(iii) k(AB) = (kA)B = A(kB)$$

$$(iv) I_n A = A I_n = A$$

where A is $n \times n$ matrix

$I_n = n \times n$ Identity matrix

(Multiplicative Identity)

Q: If A, B are square matrices of same size,

$$(A+B)^2 = A^2 + 2AB + B^2?$$

$$\text{No, } (A+B)^2 = A^2 + AB + BA + B^2$$

Proof of $A(B+C)$:

$$\begin{aligned} [A(B+C)]_{ij} &= a_{ij} \cdot (b_j + c_j) \\ &= a_{ij} \cdot b_j + a_{ij} \cdot c_j = (AB)_{ij} + (AC)_{ij} = (AB+AC)_{ij} \\ \therefore [A(B+C)] &= AB+AC \end{aligned}$$

Inverse of a Matrix :

If A is a $n \times n$ matrix, an inverse of A is a $n \times n$ matrix A' , such that $AA' = A'A = I$

Eg: $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$$A' = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

→ Zero matrix will not have an inverse.

Result: If $A_{n \times n}$ is an invertible matrix, then the inverse is unique.

Proof: Let A' and A'' be two inverses of A . Then,

$$AA' = A'A = I \rightarrow ①$$
$$AA'' = A''A = I \rightarrow ② \quad (\text{Associative})$$

$$A' = A'I = A'(AA'') = (A'A)A''$$

(Identity) From ② $\qquad \qquad \qquad \boxed{\Rightarrow IA'' = A''}$
 $\therefore A' = A''$ From ①

Note: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
if $ad \neq bc$.

Properties:

① If A is an invertible matrix, A^{-1} is also invertible,

then $(A^{-1})^{-1} = A$.

② If A is an invertible matrix, and c is a non-zero scalar, then

$$(cA)^{-1} = \frac{1}{c} A^{-1}$$

③ If A and B are invertible matrices of same size, then AB is also invertible, and

$$(AB)^{-1} = B^{-1}A^{-1}$$

④ If A is invertible, then A^n is invertible for all non-negative integer n .

$$(A^n)^{-1} = (A^{-1})^n$$

Proof: ① Let X be an inverse of A^{-1} .

$$\begin{aligned} & \therefore XA^{-1} = A^{-1}X = I \\ & \text{and, } A(XA^{-1}) = A(A^{-1}X) = AI = I \end{aligned} \quad \boxed{(A^{-1})^{-1} = A}$$

② Let X be an inverse of cA .

$$\therefore X(cA) = (cA)X = I \quad \dots \text{(if)}$$

$$\text{and, } \left(\frac{1}{c}A^{-1}\right)(cA) = \left(\frac{1}{c} \times cA^{-1}\right)(A) = A^{-1}A = I$$

$\hookrightarrow (i)$

$$(cA)\left(\frac{1}{c}A^{-1}\right) = A\left(c \cdot \frac{1}{c}A^{-1}\right) = AA^{-1} = I \rightarrow (\text{ii})$$

From (i), (ii) and (iii),

$$\boxed{x = \frac{1}{c}A^{-1}}$$

(3) Similarly,

Let $x \rightarrow$ inverse of AB

$$\therefore x(AB) = (AB)x = I \quad \text{--- (i)}$$

$$\begin{aligned} \rightarrow (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= A(IA^{-1}) = AA^{-1} = I \quad \text{--- (ii)} \end{aligned}$$

$$\begin{aligned} \rightarrow (B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}B)B \\ &= B^{-1}IB = B^{-1}B = I \quad \text{--- (iii)} \end{aligned}$$

From (i), (ii) and (iii),

$$\boxed{x = B^{-1}A^{-1}}$$

* (4) Let $x \rightarrow$ inverse of A^n ,

$$\text{To prove: } xA^n = A^n x = I \Rightarrow x = (A^{-1})^n$$

\rightarrow Base case: $n=1$

$$AA^{-1} = A^{-1}A = I \quad \underline{\text{True}}$$

Suppose it is true for $n=k$,

$$\therefore A^k(A^{-1})^k = (A^{-1})^k A^k = I$$

$$\text{i) } \because A \cdot A^k \cdot (A^{-1})^k \cdot (A^{-1}) = A \cdot I \cdot A^{-1} = A \cdot A^{-1} = I$$

$$\therefore \boxed{A^{k+1} \cdot (A^{-1})^{k+1} = I}$$

$$\text{ii) Similarly, } (A^{-1})^{k+1} \cdot A^{k+1} = I$$

\therefore it is true for $n+1$.

Hence, inductively we proved that

$$A^n \cdot (A^{-1})^n = I \quad \forall n \geq 0$$

Elementary Matrices :

\rightarrow An elementary matrix is a matrix that can be obtained by performing elementary row operation on identity matrix.

Eg: $E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is equivalent to
 $3R_2$ on I_4

$E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is equivalent to
 $R_1 \leftrightarrow R_3$ on I_n

$E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix}$ is equivalent to
 $R_4 \rightarrow R_4 - 2R_2$
on I_n

XXX Inverse of an Elementary Matrix :

$$\Rightarrow E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Why?

$$I_3 \xrightarrow{R_{23}} E_1 \xrightarrow{R_{23}} I_3$$

$$\Rightarrow E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \left. \begin{array}{l} \\ \end{array} \right\} I_3 \xrightarrow{R_2 \rightarrow 4R_2} E_2 \xrightarrow{R_2 \rightarrow \frac{1}{4}R_2} I_3$$

$$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \quad \left. \begin{array}{l} \\ \end{array} \right\} I_3 \xrightarrow{R_3 \rightarrow R_3 + 2R_1} E_3 \xrightarrow{R_3 \rightarrow R_3 + 2R_1} I_3$$

$$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

How to find inverse?

→ We know that $E = e(I)$

Let $X = \text{Inverse of } E$

$$\therefore XE = EX = I$$

$$\therefore Xe(I) = e(I)X = I$$

Let $X = e^{-1}(I)$

$$\therefore Xe(I) = e^{-1}e(I \cdot I) = \boxed{I}$$

$\therefore X = e^{-1}(I)$

Hence, if $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$E: R_2 \rightarrow R_2 + R_1$$

$$\therefore e^{-1}: R_2 \rightarrow R_2 - R_1$$

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Theorem:

Let E be an elementary matrix obtained by performing elementary row operation on I_{nxn} .

If the same elementary row operation is done on nxn matrix A , the resultant is same as EA .

Fundamental Theorem of Invertible Matrices :

- (a) A is an invertible matrix
- (b) $AX = B$ has a unique solution for any $B \in R$.
- (c) $AX = 0$ has only trivial solution
- (d) The row reduced echelon form of A is I_n .
- (e) A is a product of elementary matrices

$$(a) \leftrightarrow (b) \leftrightarrow (c) \leftrightarrow (d) \leftrightarrow (e)$$

Proof: To prove: (a) \rightarrow (b) \rightarrow (c) \rightarrow (d) \rightarrow (e)

$$(a) \rightarrow (b)$$

$$\text{Given: } AA^{-1} = A^{-1}A = I$$

Proving that $AX = B$ has a solution:

Suppose $x = A^{-1}B$ is a solution.

$$\begin{aligned} \therefore A(A^{-1}B) &= (AA^{-1})B \\ &= \boxed{B} \end{aligned}$$

Now, proving that the solution is unique:

Let y be a solution of $AX = B$

$$\therefore Ay = B$$

$$\therefore A^{-1}(Ay) = A^{-1}B$$

$$\therefore (A^{-1}A)y = A^{-1}B$$

$$\therefore Iy = \boxed{y = A^{-1}B} \quad \underline{\text{Hence, unique.}}$$

(b) \rightarrow (c)

Given: $AX=B$ has a unique solⁿ.

Let $B=0$

$\therefore AX=0$ also has a unique solⁿ.

A homogeneous system of linear equations always has $X=0$ as a solution.

As, $AX=0$ has only one unique solⁿ,

$AX=0$ only has trivial solⁿ ($X=0$).

(c) \rightarrow (d)

Let R be a row-reduced echelon form of A .

As $AX=0$ has only $X=0$ as its solution,

$RX=0$ has only trivial solⁿ.

Suppose R has r non zero rows.

$\therefore r \leq n-r$ free scalars

\therefore there will be infinite solutions,

if $n-r > 0$

$\therefore n-r \leq 0$ As $r \leq n$

$$n-r=0$$

$$\therefore \boxed{r=n} \Rightarrow \boxed{R=I_n}$$

(d) \rightarrow (e)

Given: $R = I_n$

Suppose R is obtained from A after performing n elementary row operations

$$\therefore R = e_n e_{n-1} \dots e_1(A)$$

$$\text{As, } e_i = e_i(I)$$

$$R = e_n e_{n-1} \dots e_1(A) = I$$

As elementary matrices are invertible,

$$A = e_1^{-1} e_2^{-1} \dots e_n^{-1}(I)$$

As product of invertible matrices is invertible,

$$\boxed{A = e_1^{-1} e_2^{-1} \dots e_n^{-1}} \text{ is a product of invertible matrices.}$$

(c) \rightarrow (a)

Suppose $A = \epsilon_1 \epsilon_2 \dots \epsilon_n$ (product of n invertible matrices)

As ϵ_i is invertible $\forall 1 \leq i \leq n$

$\exists \epsilon_i^{-1}$ s.t. $\epsilon_i^{-1} \epsilon_i = \epsilon_i \epsilon_i^{-1} = I$

$$\therefore A(\epsilon_n^{-1} \epsilon_{n-1}^{-1} \dots \epsilon_1^{-1}) = I$$

$$\text{let } X = \boxed{\epsilon_n^{-1} \epsilon_{n-1}^{-1} \dots \epsilon_1^{-1}}$$

$$\exists X \text{ s.t. } \boxed{AX = I}$$

Similarly,

$$(\epsilon_n^{-1} \epsilon_{n-1}^{-1} \dots \epsilon_1^{-1})A = I$$

$\therefore \exists X \text{ s.t. }$

$$\boxed{XA = I}$$

$$\therefore AX = XA = I$$

$\therefore A$ is invertible.

$$\begin{aligned} & \therefore I = \epsilon_3 \epsilon_2 \epsilon_1 A \quad \left| \begin{array}{l} (\epsilon_1)^{-1} = R_1 \rightarrow R_1 + R_2 \\ (\epsilon_2)^{-1} = R_2 \rightarrow R_2 + R_1 \\ (\epsilon_3)^{-1} = R_2 \rightarrow 3R_2 \end{array} \right| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ & \therefore \boxed{\epsilon_1^{-1} \epsilon_2^{-1} \epsilon_3^{-1} = A} \quad \left| \begin{array}{l} \epsilon_3 \\ \uparrow R_2 \rightarrow \frac{1}{3} R_2 \end{array} \right. \end{aligned}$$

Alt. method $\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} \xrightarrow[\epsilon_1]{R_1 \rightarrow R_1 - R_2} \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} \xrightarrow[\epsilon_2]{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

Q. Express $A = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ as product of elementary matrices.

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} \xrightarrow[\epsilon_3]{R_1 \rightarrow R_1 + R_2} \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} \Rightarrow \epsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \uparrow R_2 \rightarrow R_2 + R_1 \quad \epsilon_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$I \xrightarrow[\epsilon_1]{R_2 \rightarrow 3R_2} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \quad \epsilon_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \boxed{\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} = \epsilon_3 \epsilon_2 \epsilon_1}$$

* Transpose of a Matrix:

→ The transpose of a $m \times n$ matrix $A = [a_{ij}]_{m \times n}$ is a $n \times m$ matrix $A^T = [a_{ji}]_{n \times m}$ with $a_{ji}^T = a_{ij}$
for $1 \leq i \leq m$, $1 \leq j \leq n$

$$\text{Eg: } A = \begin{pmatrix} 1 & 6 \\ 8 & 2 \\ 4 & 7 \end{pmatrix}_{3 \times 2} \quad A^T = \begin{pmatrix} 1 & 8 & 4 \\ 6 & 2 & 7 \end{pmatrix}_{2 \times 3}$$

→ The transpose of a matrix is exchanging the rows & columns of the matrix.

rows of A = column of A^T

columns of A = rows of A^T

→ If you have 2 column vectors

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}_{n \times 1} \quad V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}_{n \times 1}$$

$$U \cdot V = [u_1 \dots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = U^T V = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$$\therefore \boxed{\vec{U} \cdot \vec{V} = \vec{U}^T \vec{V}}$$

Proof of $(AB)^T = B^T A^T$:

known: 1) row_i(A) = column_i(A^T)

$$2) a_{ji}^T = a_{ij}$$

$$\begin{aligned} \therefore [(AB)^T]_{ij} &= (AB)_{ji} \\ &= \text{row}_j(A) \cdot \text{column}_i(B) \\ &= \text{column}_j(A^T) \cdot \text{row}_i(B^T) \\ &= \text{row}_i(B^T) \cdot \text{column}_j(A^T) \\ &= (B^T A^T)_{ij} \end{aligned}$$

Properties:

$$\textcircled{1} (A^T)^T = A \quad \textcircled{3} (A + B)^T = A^T + B^T$$

$$\textcircled{2} (AB)^T = B^T A^T \quad \textcircled{4} (KA)^T = K A^T$$

DETERMINANT:

$$\rightarrow A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2 \times 2}$$

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of a 3×3 matrix:

$$\rightarrow A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Rightarrow$$

$$\det A = +a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\therefore \det A = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}$$

$$= \boxed{\sum_{j=1}^3 (-1)^{1+j} a_{1j} \det A_{1j}}$$

where, $\det A_{ij}$ is called the (i,j) -minor of A

Determinant of a $n \times n$ matrix:

$$\det A = |A| = a_{11}A_{11} - a_{12}A_{12} + \dots + (-1)^{1+n} a_{nn}A_{nn}$$

$$= \boxed{\sum_{j=1}^n (-1)^{1+j} a_{1j} A_{1j}} \quad (A_{ij} = \text{minor of } a_{ij})$$

Cofactor:

Combine a minor with its plus or minus sign. To this end, we define the (i,j) -cofactor of A to be:

$$\boxed{C_{ij} = (-1)^{i+j} A_{ij}}$$

$$\therefore \boxed{\det A = \sum_{j=1}^n a_{1j} C_{1j}}$$

The Laplace Expansion Theorem:

The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geq 2$, can be computed as

$$\begin{aligned}\det A &= a_{11}c_{11} + a_{12}c_{12} + \dots + a_{1n}c_{1n} \\ &= \sum_{j=1}^n a_{1j}c_{1j}\end{aligned}$$

(which is the cofactor expansion along the 1st row) and also as

$$\begin{aligned}\det A &= a_{1j}c_{1j} + a_{2j}c_{2j} + \dots + a_{nj}c_{nj} \\ &= \sum_{i=1}^n a_{ij}c_{ij}\end{aligned}$$

(the cofactor expansion along the jth column).

Theorem 4.2: The determinant of a triangular matrix is the product of the entries on its main diagonal.

Specifically, if $A = [a_{ij}]$ is an $n \times n$ triangular matrix, then

$$\boxed{\det A = a_{11}a_{22}\dots a_{nn}}$$

Properties of Determinant:

- ① If A has a zero row/column, then $\det A = 0$.
- ② If B is obtained by interchanging two rows/columns of A , then $\det B = -\det A$
- ③ If A has two identical rows/columns, then $\det A = 0$.
- ④ If B is obtained by multiplying a row/column of A by k , then $\det B = k \det A$

(5) If A, B and C are identical except that the i^{th} -row/column of C is the sum of the i^{th} row/column of A and B, then

$$\det C = \det A + \det B$$

(6) If B is obtained by adding a multiple of one row/column of A to another row/column, then

$$\det B = \det A$$

Prove all of the above properties:

(1) Suppose i^{th} row(column) is zero in A.

Row: $\therefore \det A = \sum_{j=1}^n a_{ij} C_{ij}$ (As $a_{ij}=0 \forall j$)

$$\boxed{\det A = 0}$$

Column: $\det A = \sum_{j=1}^n a_{ji} C_{ji}$ (As $a_{ji}=0 \forall j$)

$$\boxed{\det A = 0}$$

(2) A \rightarrow has two identical rows, swap them to obtain the matrix B.

clearly, $B = A$

$$\therefore \det B = \det A$$

Also, by (2), $\det B = -\det A$ } $\therefore \det A = -\det A$

(3) Suppose $b_{ij} = k a_{ij}$ for $j=1, \dots, n$

\therefore expand them along i^{th} row,

$$\begin{aligned} \det B &= \sum_{j=1}^n b_{ij} C_{ij} = \sum_{j=1}^n k a_{ij} C_{ij} = k \sum_{j=1}^n a_{ij} C_{ij} \\ &= \boxed{k \det A} \end{aligned}$$

(4) $C_{ij} = a_{ij} + b_{ij} \neq j$

else, $C_{rj} = a_{rj} = b_{rj} \neq r \neq i$ (the cofactors C_{ij} of the elements in the expanding along i^{th} row, i^{th} rows of A, B and C are identical.)

$$\det C = \sum_{j=1}^n C_{ij} C_{ij}$$

$$\begin{aligned} &= \sum_{j=1}^n (a_{ij} + b_{ij}) C_{ij} = \sum_{j=1}^n a_{ij} C_{ij} + \sum_{j=1}^n b_{ij} C_{ij} \\ &= \boxed{\det A + \det B} \end{aligned}$$

(6)

$$R_i \rightarrow R_i + \kappa R_r \Rightarrow B$$

$$\begin{aligned}
 \det B &= \sum_{j=1}^n b_{ij} C_{ij} \\
 &= \sum_{j=1}^n (a_{ij} + \kappa \alpha_{rj}) C_{ij} \\
 &= \sum_{j=1}^n a_{ij} C_{ij} + \boxed{\kappa \sum_{j=1}^n \alpha_{rj} C_{ij}} \\
 &= \det A + \kappa(0) \quad \downarrow \det C \\
 &= \det A \quad \text{where } r^{\text{th}} \text{ row is replaced} \\
 &\quad \text{with } r^{\text{th}} \text{ row.} \\
 \text{So, now } r^{\text{th}} \text{ and } i^{\text{th}} \text{ row are} \\
 &\quad \text{identical} \\
 \therefore \det C &= 0
 \end{aligned}$$

(2)

Determinants of Elementary Matrices:

Theorem 4.4:

Let E be an $n \times n$ elementary matrix.

- (1) If E results from interchanging two rows of I_n , then $\det E = -1$.
- (2) If E results from multiplying one row of I_n by k , then $\det E = k$.
- (3) If E results from adding a multiple of one row of I_n to another row, then $\det E = 1$.

Can be directly derived from the part (2), (4) & (6) of the previous theorem.

Lemma 4.5: Let B be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix. Then

$$\det(EB) = (\det E)(\det B)$$

Theorem 4.6: A square matrix A is invertible if and only if $\det A \neq 0$.

CRAMER's RULE:

→ Let A be an $n \times n$ matrix, and let $b \in \mathbb{R}^n$. Then the unique sol n x of the system of equation $Ax=b$ is given by

$$x_i = \frac{\det(A_i(b))}{\det(A)}$$

for $i=1, 2, \dots, n$

where $A_i(b)$ is obtained by replacing i th column of A with b .

Eg: $\begin{cases} x_1 + 2x_2 = 2 \\ -x_1 + 4x_2 = 1 \end{cases}$ } let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\therefore \det A = \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 4+2 = 6$$

$$\begin{aligned} \therefore x_1 &= \frac{\det(A_1(b))}{\det(A)} = \frac{\begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix}}{6} = \frac{6}{6} = 1 \\ x_2 &= \frac{\det(A_2(b))}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}}{6} = \frac{3}{6} = \frac{1}{2} \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad \begin{aligned} x &= \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \end{aligned}$$

→ $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$

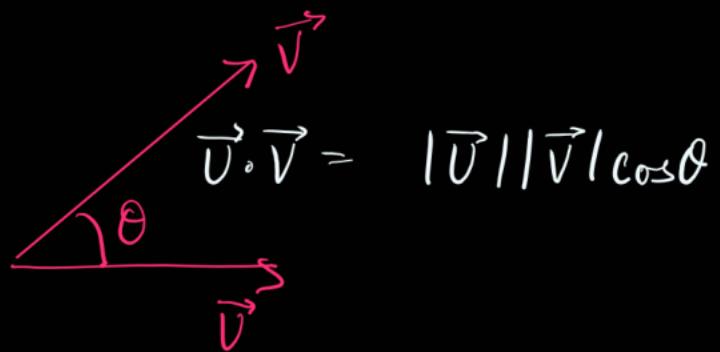
Eg: let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \Rightarrow \det(A) = -18 - 2(-10) - 1(4) = -22 + 20 = -2$

$$\text{adj}(A) = \begin{bmatrix} -18 & 10 & 4 \\ 3 & -2 & -1 \\ 10 & -6 & -2 \end{bmatrix}^T = \begin{bmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = -\frac{1}{2} \begin{pmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{pmatrix}$$

INNER PRODUCT SPACE:

Definition :



An inner product on a vector space V is an

operation that assigns to every pair of vectors U and V in V a real number $\langle U, V \rangle$ such that the following properties hold for vectors $U, V \in W \in V$ and all scalars c :

① $\langle U, V \rangle = \langle V, U \rangle$

② $\langle U, V + W \rangle = \langle U, V \rangle + \langle U, W \rangle$

③ $\langle cU, V \rangle = c\langle U, V \rangle$

④ $\langle U, U \rangle \geq 0$ and $\langle U, U \rangle = 0$ iff $U = 0$

A vector space with an inner product is called an inner product space.

If v and u are two vectors in an innerproduct space V :

① length of v is $\|v\| = \sqrt{\langle v, v \rangle}$

② The distance b/w $U \notin V$ is $d(U, V) = \|U - V\|$.

③ U , and V are orthogonal if $\langle U, V \rangle = 0$

Eg: \mathbb{R}^n is an inner product space $(v_i \cdot v_j) = v_i \cdot v_j$

But dot product is not the only inner product that can be defined on \mathbb{R}^n .

2]

NOTE: $\{v_1, v_2, \dots, v_n\}$ where every pair is orthogonal
then, $v_i \cdot v_j = 0$ if $v_i \neq v_j$

→ If $\{v_1, v_2, \dots, v_n\}$ is an orthogonal set of non-zero vectors in \mathbb{R}^n then they are linearly independent.

If c_1, c_2, \dots, c_n are scalars such that,

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

$$\left(\sum_i c_i v_i \right) \cdot v_i = 0 \cdot v_i \\ = 0$$

we know that $v_i \cdot v_j = 0$ if $i \neq j$

$$\therefore c_i (v_i \cdot v_i) = 0 \quad \forall i$$

$$\therefore \boxed{c_i = 0 \quad \forall i}$$

$$\text{Def: } (\mathbb{R}^2, +) \rightarrow (\mathbb{R}, +, \cdot)$$

$$\langle u, v \rangle = 2u_1v_1 + 3u_2v_2$$

Is this an inner product?

Solution:

$$\begin{aligned} \text{(i)} \quad & \langle u, v \rangle \\ &= 2u_1v_1 + 3u_2v_2 \\ &= 2v_1u_1 + 3v_2u_2 \\ &= \boxed{\langle v, u \rangle} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & \langle u, v+w \rangle \\ &= 2u_1(v_1+w_1) + 3u_2(v_2+w_2) \\ &= 2u_1v_1 + 2u_1w_1 + 3u_2v_2 + 3u_2w_2 \\ &= (2u_1v_1 + 3u_2v_2) + (2u_1w_1 + 3u_2w_2) \\ &= \boxed{\langle u, v \rangle + \langle u, w \rangle} \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad & \langle cv, v \rangle \\ &= 2(cv_1)v_1 + 3(cv_2)v_2 \\ &= c[2u_1v_1 + 3u_2v_2] \\ &= \boxed{c\langle u, v \rangle} \end{aligned} \quad \left| \begin{array}{l} \text{(iv)} \quad \langle u, u \rangle \\ = 2u_1^2 + 3u_2^2 \\ > 0 \quad > 0 \\ \therefore \langle u, u \rangle \geq 0 \end{array} \right.$$

If $u = 0 \Rightarrow u_1 = u_2 = 0$

$\therefore \boxed{\langle u, u \rangle = 0}$

Complex Dot Product:

If $U = \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{pmatrix}$ and $V = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix}$ are vectors in C^n , then the

complex dot product of U and V is defined by

$$U \cdot V = \bar{U}_1 V_1 + \dots + \bar{U}_n V_n$$

Prove that it is a inner product:

→ The norm (length) of a complex vector V is defined as in the real case:

$$\|V\| = \sqrt{V \cdot V} \quad \left(\text{General form: } \|V\| = \sqrt{\langle V, V \rangle} \right)$$

Distance b/w two complex vectors:

$$d(U, V) = \|U - V\|$$

Eg: let $U = \begin{pmatrix} i \\ 1 \end{pmatrix}$ and $V = \begin{pmatrix} 2-3i \\ 1+5i \end{pmatrix}$

(a) $U \cdot V$

$$\begin{aligned} U \cdot V &= \sum_i \bar{U}_i V_i = (-i)(2-3i) + (1)(1+5i) \\ &= -2i + 3(-i) + 1 + 5i \\ &= 3i - 2 \end{aligned}$$

(b) $\|U\|$

$$\begin{aligned} \|U\| &= \sqrt{U \cdot U} \\ &= \sqrt{1+1} = \sqrt{2} \end{aligned}$$

(c) $\|v\|$

$$\|v\| = \sqrt{4+9+25+1} = \boxed{\sqrt{39}}$$

(d) $d(u, v)$

$$\begin{aligned} d(u, v) &= \|u - v\| \\ &= \left\| \begin{pmatrix} 4 & -2 \\ -5 & 1 \end{pmatrix} \right\| = \sqrt{16+4+25} \\ &= \boxed{\sqrt{45}} \end{aligned}$$

NOTE: Let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ then, $\|v\| = \sqrt{|v_1|^2 + |v_2|^2 + \dots + |v_n|^2}$

Properties :

(a) $U \circ V = \overline{V \circ U}$

(b) $U \circ (V + W) = U \circ V + U \circ W$

(c) $(cU) \circ V = \bar{c}(U \circ V)$ and $U \circ (cv) = c(U \circ v)$

(d) $U \circ U \geq 0$ and $U \circ U = 0$ iff $U = 0$.

Proofs:

(a) $\langle u, v \rangle = \sum_i \bar{u}_i v_i$

As $\bar{\bar{v}} = v$

$$= \sum_i \bar{u}_i \bar{v}_i$$

As $\bar{\bar{p}} = \bar{p}$

$$= \sum_i \bar{u}_i \bar{v}_i$$

As $\bar{p+q} = \bar{p} + \bar{q}$
= $\overline{\sum_i u_i v_i}$
= $\overline{\sum_i \bar{v}_i u_i}$
= $\boxed{\langle v, u \rangle}$

$$(b) \langle v, v+w \rangle$$

$$= \sum_i \bar{v}_i (v_i + w_i)$$

$$= \sum_i \bar{v}_i v_i + \sum_i \bar{v}_i w_i$$

$$= \boxed{\langle v, v \rangle + \langle v, w \rangle}$$

$$(c) \langle cv, v \rangle = \sum_i c \bar{v}_i v_i$$

$$= \sum_i c \bar{v}_i v_i$$

$$= c \sum_i \bar{v}_i v_i$$

$$= \boxed{c \langle v, v \rangle}$$

$$(d) \langle v, v \rangle = \sum_i \bar{v}_i v_i$$

$$= \sum_i |v_i|^2 \geq 0$$

If $v=0$

$$\text{then } \sum_i |v_i|^2 = 0$$

Conjugate Transpose:

If A is a complex matrix, then the conjugate transpose of A is the matrix A^* defined by

$$A^* = \bar{A}^T$$

\bar{A} refers to the matrix whose entries are the complex conjugates of the corresponding entries of A .

$$\text{If } A = [a_{ij}]$$

$$\text{then, } \bar{A} = [\bar{a}_{ij}]$$

Properties:

$$(a) \bar{\bar{A}} = A$$

$$(b) \overline{A+B} = \bar{A} + \bar{B}$$

$$(c) \overline{cA} = \bar{c}\bar{A}$$

$$(d) \overline{AB} = \bar{A}\bar{B}$$

$$(e) (\bar{A})^T = (\bar{A}^T)$$

Proofs:

$$(a) \text{let } A = [a_{ij}]$$

$$\bar{A} = [\bar{a}_{ij}]$$

$$\bar{\bar{A}} = [\bar{\bar{a}}_{ij}] = [a_{ij}]$$

$$\therefore \bar{\bar{A}} = A$$

$$(b) \overline{A+B} = [a_{ij} + b_{ij}]$$

$$= [\bar{a}_{ij} + \bar{b}_{ij}]$$

$$= [\bar{a}_{ij}] + [\bar{b}_{ij}]$$

$$\therefore \overline{A+B} = \boxed{\bar{A} + \bar{B}}$$

$$(d) \overline{AB}$$

$$(AB)_{ij} = \sum_k a_{ik} b_{kj}$$

$$(\bar{AB})_{ij} = \sum_k \bar{a}_{ik} \bar{b}_{kj}$$

$$\text{As } \bar{p+q} = \bar{p} + \bar{q} \quad \Rightarrow \quad = \sum_k \bar{a}_{ik} \bar{b}_{kj}$$

$$pq = \bar{p}\bar{q} \quad \Rightarrow \quad = \sum_k (\bar{a}_{ik})(\bar{b}_{kj})$$

$$= (\bar{A}\bar{B})_{ij}$$

$$\therefore \overline{AB} = \boxed{\bar{A}\bar{B}}$$

More Properties:

$$(a) (A^*)^* = A$$

$$(b) (A+B)^* = A^* + B^*$$

$$(c) (cA)^* = \bar{c}A^*$$

$$(d) (AB)^* = B^*A^*$$

Proofs:

$$(d) (AB)^* = (\bar{AB})^T = (\bar{A}\bar{B})^T$$

$$= \bar{B}^T \bar{A}^T$$

$$= \boxed{B^*A^*}$$

* If $U = \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{pmatrix}$ and $V = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix}$

if w_1, \dots, w_n are positive scalars and are vectors in R^n , then

$$\langle U, V \rangle = w_1 U_1 V_1 + \dots + w_n U_n V_n$$

This defines an inner product on R^n , called a weighted dot product.

Prove that this is an inner product:

$$\begin{aligned} (i) \quad & \langle U, V \rangle = \langle V, U \rangle \\ \Rightarrow & \langle U, V \rangle = \sum_i w_i U_i V_i \\ & = \sum_i w_i V_i U_i \\ & = \boxed{\langle V, U \rangle} \end{aligned}$$

$$\begin{aligned} (ii) \quad & \langle U, V + W \rangle \\ & = \sum_i w_i U_i (V_i + W_i) \\ & = \sum_i w_i U_i V_i + \sum_i w_i U_i W_i \\ & = \boxed{\langle U, V \rangle + \langle U, W \rangle} \end{aligned}$$

$$\begin{aligned} (iii) \quad & \langle cU, V \rangle = c \langle U, V \rangle \\ \langle cU, V \rangle & = \sum_i c w_i U_i V_i \\ & = c \sum_i w_i U_i V_i \\ & = \boxed{c \langle U, V \rangle} \end{aligned}$$

$$\begin{aligned} (iv) \quad & \langle U, U \rangle \geq 0 \\ \langle U, U \rangle & = \sum_i w_i U_i U_i \\ & = \sum_i w_i (U_i)^2 \\ & \quad (U_i)^2 \geq 0 \quad w_i \geq 0 \\ & \therefore \sum_i w_i U_i^2 \geq 0 \\ & \boxed{\therefore \langle U, U \rangle \geq 0} \end{aligned}$$

Orthogonal Projections: (Pg 562)

An orthogonal set of vectors in an inner product space V is a set $\{v_1, \dots, v_k\}$ of vectors from V such that $\langle v_i, v_j \rangle = 0$ whenever, $v_i \neq v_j$.

A set of vectors is an orthonormal set of vectors if it is an orthogonal set of unit vectors.

Theorem: (Pg: 396)

Let $\{v_1, \dots, v_k\}$ be an orthogonal basis for a subspace W of R^n and w be any vector of W . Then there are unique scalars c_1, c_2, \dots, c_k

such that,

$$w = \sum_{i=1}^k c_i v_i$$

where
$$c_i = \frac{w \cdot v_i}{v_i \cdot v_i} \quad \text{for } i = 1, 2, \dots, k$$

Proof:

Since, $\{v_1, \dots, v_k\}$ is a basis of W ,

Find scalars c_1, \dots, c_k such that $w = \sum_{i=1}^k c_i v_i$

$$\begin{aligned} w \cdot v_i &= (c_1 v_1 + \dots + c_k v_k) \cdot v_i \\ &= c_1 (v_1 \cdot v_i) + \dots + c_i (v_i \cdot v_i) + \dots + c_k (v_k \cdot v_i) \\ &= c_i (v_i \cdot v_i) \end{aligned}$$

$$\therefore c_i = \frac{w \cdot v_i}{v_i \cdot v_i}$$

If inner product is defined in the question, then

$$c_i = \frac{\langle w, v_i \rangle}{\langle v_i, v_i \rangle}$$

Orthogonal Projection:

→ We can define the orthogonal projection $\text{proj}_W(v)$ of a vector v onto a subspace W of an inner product space.

If $\{v_1, \dots, v_k\}$ is an orthogonal basis for W , then

$$\boxed{\text{proj}_W(v) = \frac{\langle v_1, v \rangle}{\langle v_1, v_1 \rangle} v_1 + \dots + \frac{\langle v_k, v \rangle}{\langle v_k, v_k \rangle} v_k}$$

Then the component of v orthogonal to W is the vector:

$$\boxed{\text{perp}_W(v) = v - \text{proj}_W(v)}$$

Ques: $\langle f, g \rangle = \int_a^b f(x)g(x) dx$

Prove that this is an inner product.

(1) $\langle v, v \rangle = \langle v, v \rangle$

$$\therefore \int_a^b f(x)g(x) dx = \int_a^b g(x)f(x) dx \quad \underline{\text{True}}$$

(2) $\langle v, v+w \rangle = \langle v, v \rangle + \langle v, w \rangle$

(3) $\langle cv, v \rangle = c \langle v, v \rangle \quad \int_a^b (cf(x))g(x) dx = c \int_a^b f(x)g(x) dx$

(4) $\langle v, v \rangle \geq 0 \quad \langle v, v \rangle = 0 \quad \text{iff} \quad v = 0 \quad \int_a^b f(x)^2 dx \geq 0$

Q) Find an orthogonal basis for the subspace W of \mathbb{R}^3 given by

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x - y + 2z = 0 \right\}$$

Sol:

$$\rightarrow x - y + 2z = 0 \quad \therefore W = \left\{ \begin{pmatrix} y-2z \\ y \\ z \end{pmatrix} \right\} \quad y, z \in \mathbb{R}$$

$$x = y - 2z$$

$w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$	As W is orthogonal, $w_1 \cdot w_2 = 0$	$\therefore w_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $(w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix})$ $W_b = \{w_1, w_2\}$ \hookrightarrow Orthogonal Basis of Subspace W .
$w_2 = \begin{pmatrix} y-2z \\ y \\ z \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} y-2z \\ y \\ z \end{pmatrix} = 0$ $\therefore y-2z + y = 0$ $\boxed{y=2z}$	
$\therefore w_2 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$		

The Gram-Schmidt Process:

- We want to find an orthogonal basis for a subspace W of \mathbb{R}^n .
- The idea is to begin with a arbitrary basis $\{v_1, \dots, v_m\}$ and to "orthogonalize" it one vector at a time.
- For example:

let $W = \text{span}(v_1, v_2)$, where

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Construct an orthogonal basis for W .

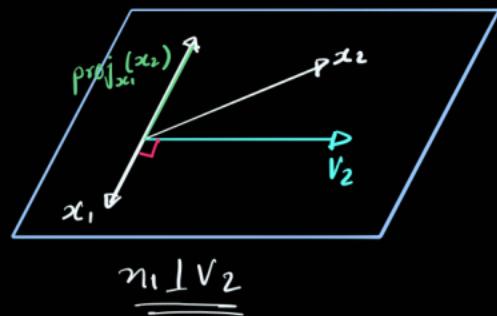
Solⁿ: Starting with x_1 , we get a second vector that is orthogonal to it by taking the component of x_2 orthogonal to x_1 .

Constructing v_2 orthogonal to n_1 .

Algebraically, we set $v_1 = n_1$, so

$$\begin{aligned} v_2 &= \text{proj}_{n_1}(x_2) = n_1 - \text{proj}_{n_1}(x_2) \\ &= x_2 - \left(\frac{n_1 \cdot x_2}{n_1 \cdot n_1} \right) n_1 \\ &= \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} - \left(\frac{-2}{2} \right) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

$$v_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$



$$\underline{n_1 \perp v_2}$$

Hence, $\{v_1, v_2\}$ is an orthogonal set of vectors in W .
So, it is linearly independent set.

$\rightarrow \{v_1, v_2\}$ is a basis for W , since $\dim W = 2$.

NOTE:

If we had taken $v_1 = x_2$, we would have obtained a different set of orthogonal vectors in W .

※ Theorem: The Gram-Schmidt Process

Let $\{x_1, \dots, x_n\}$ be a basis for a subspace W of R^n and define the following:

$$v_1 = x_1 \quad W_1 = \text{span}(v_1)$$

$$v_2 = x_2 - \text{proj}_{v_1}(x_2) \quad W_2 = \text{span}(v_1, v_2)$$

$$v_3 = x_3 - \text{proj}_{v_1}(x_3) - \text{proj}_{v_2}(x_3) \quad W_3 = \text{span}(v_1, v_2, v_3)$$

⋮

$$v_n = x_n - \text{proj}_{v_1}(x_n) - \dots - \text{proj}_{v_{n-1}}(x_n) \quad W_n = \text{span}(v_1, \dots, v_n)$$

Then, for each $i \in \{1, \dots, n\}$, $\{v_1, \dots, v_i\}$ is an orthogonal basis for W_i . In particular $\{v_1, \dots, v_n\}$ is an orthogonal basis for W .



Proof: Proof by Induction:

→ let $v_1 = x_i$,

Clearly, $\{v_1\}$ is an orthogonal basis for $W_i = \text{span}(x_i)$.

→ Suppose,

for some $i < k$,

$\{v_1, \dots, v_i\}$ is an orthogonal basis for W_i .

Then,

$$\begin{aligned} v_{i+1} &= x_{i+1} - \text{proj}_{v_1}(x_{i+1}) - \dots - \text{proj}_{v_i}(x_{i+1}) \\ &= x_{i+1} - \left(\frac{v_1 \cdot x_{i+1}}{v_1 \cdot v_1} \right) v_1 - \dots - \left(\frac{v_i \cdot x_{i+1}}{v_i \cdot v_i} \right) v_i \end{aligned}$$

By induction hypothesis,

$\{v_1, \dots, v_i\}$ is an orthogonal basis for $\text{span}(x_1, \dots, x_i) = W_i$

Hence,
$$v_{i+1} = x_{i+1} - \text{proj}_{W_i}(x_{i+1}) = \text{perp}_{W_i}(x_{i+1})$$

→ By orthogonal Decomposition Theorem, v_{i+1} is orthogonal to W_i . —①

By definition, v_1, v_2, \dots, v_i are linear combinations of x_1, \dots, x_i and hence, are in W_i . —②

Also,

$v_{i+1} \neq 0$, since otherwise $x_{i+1} = \text{proj}_{W_i}(x_{i+1})$ implies x_{i+1} is in W_i .

But, since $W_i = \text{span}(x_1, \dots, x_i)$, $\{x_1, \dots, x_i\}$
this is impossible.

→ Hence, from ① and ②,

we conclude that, $\{v_1, \dots, v_{i+1}\}$ is a set of linearly independent vectors in W_{i+1} .

Since, $\dim W_{i+1} = i+1$,

$\{v_1, \dots, v_{i+1}\}$ is a basis for W_{i+1} .

Hence, Proved.

Eg: Apply the Gram-Schmidt Process to construct an orthogonal basis for the subspace $W = \text{span}(n_1, n_2, n_3)$ of \mathbb{R}^4 .

where $n_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $x_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}$

Sol: Let $v_1 = x_1$,

Now, we compute component of x_2 orthogonal to $W = \text{span}(v_1) = V_1$

$$v_2 = \text{perp}_{W_1}(x_2)$$

$$= x_2 - \text{proj}_{W_1}(x_2)$$

$$v_2 = x_2 - \left(\frac{v_1 \cdot x_2}{v_1 \cdot v_1} \right) v_1 \quad \begin{bmatrix} n_1 \cdot n_2 = 2 - 1 + 1 = 2 \\ n_1 \cdot n_1 = 1 + 1 + 1 + 1 = 4 \end{bmatrix}$$

$$= x_2 - \left(\frac{2}{4} \right) v_1$$

$$= x_2 - \text{proj}_2(v_1) = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \text{proj}_2 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\boxed{v_2 = \begin{pmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{pmatrix}}$$

For easy calculation,
⇒ "scale" v_2
∴ let $v_2 = 2 \begin{pmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{pmatrix}$

Let $W_2 = \text{span}(v_1, v_2)$

$$\therefore v_3 = \text{perp}_{W_2}(x_3)$$

$$= x_3 - \text{proj}_{W_2}(x_3)$$

$$= x_3 - \left(\frac{v_1 \cdot x_3}{v_1 \cdot v_1} \right) v_1 - \left(\frac{v_2 \cdot x_3}{v_2 \cdot v_2} \right) v_2$$

$$\boxed{v_2 = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}}$$

$$\left| \begin{array}{l} v_1 \cdot x_3 = 1 \\ v_1 \cdot v_1 = 4 \\ v_2 \cdot x_3 = 15 \\ v_2 \cdot v_2 = 20 \end{array} \right.$$

$$\begin{aligned}
 V_3 &= xC_3 - \left(\frac{1}{4}\right) \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} - \left(\frac{15}{20}\right) \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} - \frac{3}{4} \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} 2 - \frac{1}{4} - \frac{9}{4} \\ 2 + \frac{1}{4} - \frac{9}{4} \\ 1 + \frac{1}{4} - \frac{3}{4} \\ 2 - \frac{1}{4} - \frac{3}{4} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{pmatrix} \quad \left\{ V_1, V_2, V_3 \right\} \\
 &\quad = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}
 \end{aligned}$$

"Scale" $V_3 \Rightarrow$ let $\boxed{V_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}}$

 Q: Construct an orthogonal basis for P_2 with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^1 f(n)g(n) dn$$

by applying the Gram-Schmidt Process to the basis $\{1, x, x^2\}$.

Solⁿ: Let $n_1 = 1$
 $n_2 = n$ & $n_3 = x^2$. Also, let $v_1 = x$,

$$\begin{aligned}
 \therefore \langle v_1, v_1 \rangle &= \int_{-1}^1 1^2 dn = \boxed{2} \\
 \therefore \langle v_1, n_2 \rangle &= \int_{-1}^1 (1)(n) dn \\
 &= \left[\frac{n^2}{2} \right]_{-1}^1 = \frac{1}{2} - \frac{-1}{2} = \boxed{0} \\
 \therefore v_2 &= n_2 - \left(\frac{\langle v_1, n_2 \rangle}{\langle v_1, v_1 \rangle} \right) v_1 \\
 &= n_2 - 0 = n_2 = x^2 \\
 \therefore v_2 &= x^2
 \end{aligned}$$

$$\begin{aligned}
 \therefore V_3 &= xC_3 - \left(\frac{\langle v_1, n_3 \rangle}{\langle v_1, v_1 \rangle} \right) v_1 - \left(\frac{\langle v_2, n_3 \rangle}{\langle v_2, v_2 \rangle} \right) v_2 \\
 \text{where, } \quad \langle v_2, v_2 \rangle &= \int_{-1}^1 n^2 dn = \left[\frac{n^3}{3} \right]_{-1}^1 = \boxed{\frac{2}{3}} \\
 \langle v_1, n_3 \rangle &= \int_{-1}^1 (1)(x^2) dn \\
 &= \left[\frac{x^3}{3} \right]_{-1}^1 = \boxed{\frac{2}{3}} \\
 \langle v_2, n_3 \rangle &= \int_{-1}^1 (n)(x^2) dn = \boxed{0} \\
 \therefore V_3 &= x^2 - \left(\frac{\frac{2}{3}}{2} \right)(1) - \left(\frac{0}{\frac{2}{3}} \right)(n) \\
 &= \boxed{x^2 - \frac{1}{3}}
 \end{aligned}$$

\therefore Orthogonal Basis = $\{1, n, n^2 - \frac{1}{3}\}$.

The Cauchy-Schwarz Inequality:

→ let u and v be vectors in an inner product space V . Then

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

with equality holding if and only if u and v are scalar multiples of each other.

Proof: → If $u=0$, then the inequality is actually an equality, since

$$|\langle 0, v \rangle| = 0 = \|0\| \|v\|$$

→ If $u \neq 0$,

then let W be the subspace of V spanned by u . Since $\text{proj}_W^{(u)} = \left(\frac{\langle u, v \rangle}{\langle u, u \rangle} \right) u$ and $\text{perp}_W^{(u)} = v - \text{proj}_W^{(u)}$ are orthogonal, we can apply Pythagoras theorem,

$$\begin{aligned} \|v\|^2 &= \|\text{proj}_W^{(u)} + (v - \text{proj}_W^{(u)})\|^2 \\ &= \|\text{proj}_W^{(u)} + \text{perp}_W^{(u)}\|^2 \\ &= \|\text{proj}_W^{(u)}\|^2 + \|\text{perp}_W^{(u)}\|^2 \end{aligned}$$

It follows that

$$\|v\|^2 \geq \|\text{proj}_W^{(u)}\|^2$$

$$\begin{aligned} \text{Now, } \|\text{proj}_W^{(u)}\|^2 &= \left\langle \left(\frac{\langle u, v \rangle}{\langle u, u \rangle} \right) u, \left(\frac{\langle u, v \rangle}{\langle u, u \rangle} \right) u \right\rangle = \left(\frac{\langle u, v \rangle}{\langle u, u \rangle} \right)^2 \langle u, v \rangle \\ &= \frac{\langle u, v \rangle^2}{\langle u, u \rangle} \\ &= \frac{\langle u, v \rangle^2}{\|u\|^2} \end{aligned}$$

So, we have,

$$\frac{\langle u, v \rangle^2}{\|u\|^2} \leq \|v\|^2 \Rightarrow \langle u, v \rangle^2 \leq \|u\|^2 \|v\|^2$$

$$\therefore |\langle u, v \rangle| \leq \|u\| \|v\|$$

Equality: iff $\|v\|^2 = \|\text{proj}_W^{(u)}\|^2$, which is true only when $v = \text{proj}_W^{(u)}$.

∴ If this is so, then v is a scalar multiple of u .

Conversely, if $v=cu$, $\Rightarrow \text{perp}_W^{(u)} = v - \text{proj}_W^{(u)} = cu - \left(\frac{\langle u, cu \rangle}{\langle u, u \rangle} \right) u = cu - cu$

∴ Equality holds.

* Triangular Inequality Theorem:

$$\|u+v\| \leq \|u\| + \|v\|$$

Proof: LHS = $\|u+v\| = \sqrt{\langle u+v, u+v \rangle}$

$$\|u+v\|^2 = \langle u+v, u+v \rangle$$

$$= \langle u+v, u \rangle + \langle u+v, v \rangle$$

$$= \underbrace{\langle u, u \rangle}_{=\|u\|^2} + \underbrace{\langle u, v \rangle}_{\leq \|u\| \|v\|} + \underbrace{\langle v, u \rangle}_{\leq \|v\| \|u\|} + \underbrace{\langle v, v \rangle}_{=\|v\|^2}$$

$$\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$$

$$\|u+v\|^2 \leq (\|u\| + \|v\|)^2$$

$\therefore \boxed{\|u+v\| \leq \|u\| + \|v\|}$

* The Orthogonal Decomposition Theorem:

Let W be a subspace of R^n and let v be a vector in R^n . Then there are unique vectors w in W and w^\perp in W^\perp such that

$$\boxed{v = w + w^\perp}$$

Proof:

- To Prove: 1) Such a decomposition exist.
2) It is unique.

Part 1: Let $\{v_1, \dots, v_k\}$ be an orthogonal basis for W .

Let $w = \text{proj}_W(v)$ and,

let $w^\perp = \text{perp}_W(v)$

$$\begin{aligned} \text{Then, } w + w^\perp &= \text{proj}_W(v) + \text{perp}_W(v) \\ &= \boxed{v} \end{aligned}$$

→ Clearly,

$w = \text{proj}_W(v)$ is in W , since it is a linear combination of basis vectors v_1, \dots, v_k .

→ To show that $w^\perp \in W^\perp$,

it is enough to show that w^\perp is orthogonal to each of the basis vector v_i .

$$v_i \cdot w^\perp = v_i \cdot \text{perp}_W(v)$$

$$= v_i \cdot (v - \text{proj}_W(v))$$

$$= v_i \cdot \left(v - \left(\frac{v_1 \cdot v}{v_1 \cdot v_1} \right) v_1 - \left(\frac{v_2 \cdot v}{v_2 \cdot v_2} \right) v_2 - \dots - \left(\frac{v_k \cdot v}{v_k \cdot v_k} \right) v_k \right)$$

$$= v_i \cdot v - \left(\frac{v_1 \cdot v}{v_1 \cdot v_1} \right) (v_i \cdot v_1) - \dots - \left(\frac{v_i \cdot v}{v_i \cdot v_i} \right) (v_i \cdot v_i) - \dots - \left(\frac{v_k \cdot v}{v_k \cdot v_k} \right) (v_i \cdot v_k)$$

Since, $v_i \cdot v_j = 0 \neq v_i \neq v_j$,

$$\therefore v_i \cdot w^\perp = v_i \cdot v - 0 - \left(\frac{v_i \cdot v}{v_i \cdot v_i} \right) (v_i \cdot v_i)$$

$$= v_i \cdot v - v_i \cdot v -$$

$$\boxed{\therefore v_i \cdot w^\perp = 0} \quad \forall i$$

∴ w^\perp is in W^\perp

2) Uniqueness:

Suppose, we have another de-composition $v = w_i + w_i^\perp$, where $w_i \in W$ and $w_i^\perp \in W^\perp$. Then $w + w^\perp = w_i + w_i^\perp$

$$\therefore w - w_i = w_i^\perp - w^\perp$$

Since, $w - w_i \in W$ and $w_i^\perp - w^\perp \in W^\perp$,

we know that this common vector $\in W \cap W^\perp = \{0\}$

$$\therefore w - w_i = 0 \text{ and } w_i^\perp - w^\perp = 0$$

$$\therefore \boxed{w = w_i} \text{ and } \boxed{w^\perp = w_i^\perp}$$

Orthogonal Complement:

Let W be a subspace of \mathbb{R}^n . We say that a vector v in \mathbb{R}^n is orthogonal to W if v is orthogonal to every vector in W . The set of all vectors that are orthogonal to W is called the orthogonal complement of W , denoted by W^\perp .

That is,

$$W^\perp = \{v \in \mathbb{R}^n : v \cdot w = 0 \text{ } \forall w \in W\}$$

Theorem: Let W be a subspace of \mathbb{R}^n .

(to prove) a) W^\perp is a subspace of \mathbb{R}^n .

$$b) (W^\perp)^\perp = W$$

$$c) W \cap W^\perp = \{0\}$$

$$d) \text{ If } W = \text{span}(w_1, \dots, w_k), \text{ then } \quad \textcircled{1}$$

Proof:

a) $\rightarrow W^\perp$ is non-empty

$$0 \cdot w = 0 \quad \forall w \in W$$

$$[0 \in W^\perp] \quad \text{--- (i)}$$

$$\text{Let } u, v \in W^\perp$$

$$u \cdot w = v \cdot w = 0 \quad \forall w \in W$$

$$(u+v) \cdot w = u \cdot w + v \cdot w = 0 + 0 = 0 \quad \forall w \in W$$

$$[\therefore u+v \in W^\perp] \quad \text{--- (ii)}$$

$$(cu) \cdot w = c(u \cdot w) = c(0) = 0 \quad \forall w \in W$$

$$[\therefore cu \in W^\perp] \quad \text{--- (iii)}$$

From (i), (ii) & (iii) W^\perp is a subspace.

b) let $w \in W$ and $x \in W^\perp$,

then $w \cdot x = 0$

this implies that, $w \in (W^\perp)^\perp$

$$\boxed{\therefore w \in (W^\perp)^\perp} \rightarrow \textcircled{1}$$

let $v \in (W^\perp)^\perp$

By, orthogonal decomposition theorem,

$V = W + W^\perp$ for unique vector w in W & w^\perp in W^\perp

$$v \cdot w^\perp = 0$$

$$\begin{aligned} \therefore (w + w^\perp) \cdot w^\perp &= w \cdot w^\perp + w^\perp \cdot w^\perp \\ &= 0 + w^\perp \cdot w^\perp = 0 \\ \therefore \boxed{w^\perp = 0} \end{aligned}$$

$$\therefore V = w + w^\perp = w + 0$$

$$\boxed{V = w}$$

$$\therefore \boxed{v \in W}$$

$$\boxed{\therefore (W^\perp)^\perp \subseteq W} \rightarrow \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$ $\boxed{(W^\perp)^\perp = W}$

*Theorem:

The columns of an $m \times n$ matrix Q forms an orthonormal set iff $Q^T Q = I_n$.

Proof:

To show $Q^T Q = I_n$,

$$(Q^T Q)_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Let q_i be the i^{th} column of Q (or i^{th} row of Q^T).

$$(Q^T Q)_{ij} = (\text{i^{th} row of Q^T}) \cdot (\text{j^{th} column of Q})$$

$$\boxed{(Q^T Q)_{ij} = q_i \cdot q_j} \rightarrow ①$$

Now, the columns Q form an orthogonal set iff

$$q_i \cdot q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

which by equation ① holds iff,

$$\boxed{(Q^T Q)_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}}$$

*Orthogonal Matrices:

An $n \times n$ matrix Q whose columns form an orthonormal set, is called orthogonal matrix.

Orthonormal:

A set of orthogonal unit vectors.

Qve: $Q = \begin{pmatrix} A & B & C \\ 4\sqrt{2} & \frac{1}{4}\sqrt{3} & x \\ 0 & \frac{1}{4}\sqrt{3} & y \\ -4\sqrt{2} & \frac{1}{4}\sqrt{3} & z \end{pmatrix}$

For what values of x, y, z , Q is orthogonal matrix.

Sol: $\|c\| = 1$

$$\boxed{x^2 + y^2 + z^2 = 1} \rightarrow (1)$$

$$\langle A, c \rangle = 0$$

$$4\sqrt{2}x - \frac{1}{4}\sqrt{3} = 0 \Rightarrow \boxed{x = \frac{1}{16}\sqrt{3}} \rightarrow (2)$$

$$\langle B, c \rangle = 0$$

$$4\sqrt{2}y - \frac{1}{4}\sqrt{3} = 0 \Rightarrow \boxed{y = -2\sqrt{2}x} \rightarrow (3)$$

From (1), (2) & (3),

$$x^2 + 4x^2 + y^2 = 6x^2 = 1$$

$$\boxed{x = \pm \frac{1}{\sqrt{6}}}$$

$$\therefore (x, y, z) = \left(\frac{1}{\sqrt{6}}, -2\sqrt{2}x, \frac{1}{4}\sqrt{3} \right)$$

or

$$\left(-\frac{1}{\sqrt{6}}, 2\sqrt{2}x, \frac{1}{4}\sqrt{3} \right)$$

Theorem: A square matrix Q is orthogonal iff $Q^{-1} = Q^T$.
(399)

Proof:

(2)

QR factorization :

→ Let A be a $m \times n$ matrix with linearly independent columns.

Then A can be factorized as $A = QR$ where Q is an $m \times n$ matrix with orthogonal column and R is invertible uppertriangular matrix.

→ Let a_1, a_2, \dots, a_n be the linearly independent columns of A and let q_1, q_2, \dots, q_n be the orthogonal vectors obtained by applying Gram-Schmidt process to $\{a_1, a_2, \dots, a_n\}$ with normalization.

$$w_i = \text{span} \{a_1, a_2, \dots, a_i\}$$

$$= \text{span} \{q_1, q_2, \dots, q_i\}$$

∴ there are scalars $r_{11}, r_{21}, \dots, r_{ii}$ such that

$$a_1 = r_{11}q_1$$

$$a_2 = r_{12}q_1 + r_{22}q_2$$

$$a_3 = r_{13}q_1 + r_{23}q_2 + r_{33}q_3$$

⋮

$$a_n = r_{1n}q_1 + r_{2n}q_2 + \dots + r_{nn}q_n$$

$$\left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \boxed{a_i = r_{1i}q_1 + r_{2i}q_2 + \dots + r_{ii}q_i}$$

which can be written in matrix form as

$$A = [a_1 \ a_2 \ \dots \ a_n] = [q_1 \ q_2 \ \dots \ q_n] \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \dots & r_{nn} \end{pmatrix} = QR$$

∴ $A = QR$

As Q is an orthogonal set,

$$Q^{-1} = Q^T$$

$$\boxed{\therefore Q^T A = R}$$

Eg: Find QR factorization of

$$A = \begin{pmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Solⁿ: The orthonormal basis for $\text{col}(A)$ produced by Gram-Schmidt

process is:

$$q_1 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}, q_2 = \begin{pmatrix} 3\sqrt{5}/10 \\ 3\sqrt{5}/10 \\ \sqrt{5}/10 \\ \sqrt{5}/10 \end{pmatrix}, q_3 = \begin{pmatrix} -\sqrt{6}/6 \\ 0 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \end{pmatrix}$$

$$\therefore Q = \begin{pmatrix} 1/2 & 3\sqrt{5}/10 & -\sqrt{6}/6 \\ -1/2 & 3\sqrt{5}/10 & 0 \\ -1/2 & \sqrt{5}/10 & \sqrt{6}/6 \\ 1/2 & \sqrt{5}/10 & \sqrt{6}/3 \end{pmatrix}$$

$$A = QR$$

$$Q^T A = Q^T Q R = R$$

$$\therefore Q^T = \begin{pmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 3\sqrt{5}/10 & 3\sqrt{5}/10 & \sqrt{5}/10 & \sqrt{5}/10 \\ -\sqrt{6}/6 & 0 & \sqrt{6}/6 & \sqrt{6}/3 \end{pmatrix}$$

$$\therefore R = Q^T A = \begin{pmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 3\sqrt{5}/10 & 3\sqrt{5}/10 & \sqrt{5}/10 & \sqrt{5}/10 \\ -\sqrt{6}/6 & 0 & \sqrt{6}/6 & \sqrt{6}/3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 1 & 1/2 \\ 0 & \sqrt{5} & 3\sqrt{5}/2 \\ 0 & 0 & \sqrt{6}/2 \end{pmatrix}}$$

Eigenvalues & Eigenvectors:

Definition: Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue of A if there is a nonzero vector x such that $Ax = \lambda x$. Such a vector x is called an eigen-vector of A corresponding to λ .

Eg: Show that $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and find the corresponding eigenvalue.

$$\text{Soln: } Ax = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore \boxed{\lambda = 4}$$

Eg: Show that 5 is an eigenvalue of $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ and determine all eigenvectors corresponding to this eigenvalue.

$$\Rightarrow Ax = \lambda x$$

$$(A - \lambda I)x = 0$$

for $x \neq 0$,

$$\det(A - \lambda I) = 0$$

$$\therefore \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$$\therefore (\lambda-1)(\lambda-3) - 8 = 0$$

$$\therefore \lambda^2 - 4\lambda + 3 - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda-5)(\lambda+1) = 0$$

$$\boxed{\lambda = -1, 5}$$

$$\text{for } \lambda = 5$$

$$\text{let } x = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

$$(A - \lambda I)x = 0$$

$$\therefore \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} 4n_1 &= 2n_2 & \Rightarrow \left\{ t \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid t \in \mathbb{R} \right\} \\ 2n_2 &= 2n_1 \end{aligned}$$

Eigenspace:

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A . The collection of all eigenvectors corresponding to λ , together with the zero vector is called the eigenspace of λ and is denoted by ℓ_λ .

Eigenvalues and Eigenvectors of $n \times n$ Matrices:

→ The eigenvalues of a square matrix A are precisely the solution λ of the equation $\det(A - \lambda I) = 0$.

$$\begin{aligned} Ax &= \lambda x \\ Ax - \lambda x &= 0 \\ \boxed{(A - \lambda)x} &= 0 \end{aligned}$$

→ When we expand $\det(A - \lambda I)$, we get a polynomial in λ , called the characteristic polynomial of A .

→ The equation $\det(A - \lambda I) = 0$ is called the characteristic equation.

Steps to find the Eigenvalues and eigenvectors of a matrix:

Step 1: Compute the characteristic polynomial $\det(A - \lambda I)$ of A .

Step 2: Find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$ for λ .

Step 3: For each eigenvalue λ , find the null space of the matrix $A - \lambda I$. This is the eigenspace ℓ_λ , the nonzero vectors of which are the eigenvectors of A corresponding to λ .

Step 4: Find a basis for each eigenspace.

Eg: Find the eigenvalues and the corresponding eigenspaces of

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$$

Solⁿ: Let $\lambda \rightarrow$ eigenvalue

Characteristic equation: $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{pmatrix}$$

$$\Rightarrow \det(A - \lambda I) = 0$$

$$\therefore -\lambda((\lambda-4)(\lambda)+5) - 1(-2)$$

$$\therefore -\lambda(\lambda^2 - 4\lambda + 5) + 2$$

$$\therefore -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0$$

$$\therefore \boxed{\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0}$$

$$\therefore \lambda^3 - \lambda^2 - 3\lambda^2 + 3\lambda + 2\lambda - 2 = 0$$

$$\therefore (\lambda-1)(\lambda^2 - 3\lambda + 2) = 0$$

$$\therefore (\lambda-1)(\lambda-1)(\lambda-2) = 0$$

$$\therefore \boxed{\lambda = 1, 2}$$

For $\lambda = 2$,

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 2 & -5 & 2 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 + R_1]{R_1 \rightarrow \frac{1}{2}R_1} \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -4 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -4 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xleftarrow[R_3 \rightarrow R_3 - 2R_2]{\text{then, } R_1 \rightarrow R_1 - 4R_2} \begin{pmatrix} 1 & -4 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\downarrow R_2 \rightarrow -4R_2 \quad \left\{ \begin{array}{l} n_1 - 4n_2 n_3 = 0 \\ 4x_1 = x_3 \end{array} \right.$$

$$\begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left\{ \begin{array}{l} n_2 - 4n_3 = 0 \\ 2x_2 = x_3 \end{array} \right. \quad \left. \begin{array}{l} n_1 - 4n_2 n_3 = 0 \\ 4x_1 = x_3 \\ n_2 - 4n_3 = 0 \\ 2x_2 = x_3 \end{array} \right\} \quad \left. \begin{array}{l} x = t \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \end{array} \right.$$

For $\lambda = 1$,

$$(A - \lambda I)x = 0$$

$$\text{Let } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{pmatrix}$$

$$\xrightarrow[R_1 \rightarrow -R_1]{R_3 \rightarrow R_3 - 2R_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\therefore n_1 - n_3 = 0$$

$$n_2 - n_3 = 0$$

$$\boxed{n_1 = n_2 = x_3}$$

$$\therefore x = t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\xrightarrow[R_3 \rightarrow R_3 - 2R_1]{R_2 \rightarrow -R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 3 \end{pmatrix} \xrightarrow[R_1 \rightarrow R_1 + R_2]{\text{then, } R_3 \rightarrow R_3 + 3R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

NOTE:

→ Algebraic Multiplicity:

It is multiplicity of an eigenvalue as a root of the characteristic equation.

→ Geometric Multiplicity of an eigenvalue λ is defined as the dimension of its corresponding eigenspace.

Theorem:

→ A square matrix A is invertible iff 0 is not an eigenvalue of A .

Proof: We know that, a square matrix is invertible iff $\det A \neq 0$.

(\Rightarrow) Square matrix is invertible.

$$\therefore \det A \neq 0$$

$$\therefore \det(A - 0I) = \boxed{\det(A)} \neq 0$$

$\therefore \lambda$ is not an eigen value.

(\Leftarrow) Suppose zero is not an eigenvalue,

$$\therefore \det(A - 0I) \neq 0$$

$$\therefore \det(A) \neq 0$$

$\therefore A$ is invertible.

Theorem: Let A be a square matrix with eigenvalue λ and the corresponding eigen-vector x .

a) for any positive integer n , λ^n is an eigenvalue of A^n with corresponding eigenvector x .

b) If A is invertible, then λ^{-1} is an eigenvalue of A^{-1} with corresponding eigenvector x .

c) If A is invertible, then for any integer n , λ^n is an eigenvalue of A^n with corresponding eigenvector x .

Proof:

Eg: Compute $\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}^10 \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

Solⁿ: Let $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ and $x = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

We need to find $A^{10}x$.

→ The eigenvalues of A are:

$$\begin{array}{|c|c|} \hline \det(A - \lambda I) = 0 & \text{Computing eigen vectors:} \\ \hline \therefore \begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = 0 & \text{For } \lambda = -1, AV_1 = -V_1 \\ (\lambda-1)\lambda - 2 = 0 & \text{and for } \lambda = 2, AV_2 = 2V_2 \\ \therefore \lambda^2 - \lambda - 2 = 0 & \\ \boxed{\lambda = -1, 2} & \end{array}$$

For $\lambda = -1$,

$$A - \lambda I = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad n_1 + n_2 = 0$$

$$\therefore n_1 = n_2 = 0$$

$$\therefore v_1 = t_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda = 2$,

$$A - \lambda I = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\therefore \begin{cases} -2n_1 + n_2 = 0 \\ n_2 = 2n_1 \end{cases} \Rightarrow v_2 = t_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

\rightarrow Using the previous theorem(c),

if eigenvalue of A is λ with corresponding eigenvector being x , for the same eigenvector, eigenvalue of A^{10} will be λ^{10} (if A is invertible).

\rightarrow Now, let $c_1, c_2 \in \mathbb{R}$ s.t.,

$$x = c_1 v_1 + c_2 v_2$$

$$\therefore \begin{pmatrix} 5 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{aligned} \therefore c_1 + c_2 &= 5 \\ \underline{c_1 + 2c_2 &= 1} \\ \underline{3c_2 &= 6} \\ \underline{c_2 &= 2} \end{aligned} \Rightarrow c_1 = 3$$

$$\begin{aligned} \therefore x &= 3v_1 + 2v_2 \\ \therefore A^{10}x &= 3A^{10}v_1 + 2A^{10}v_2 \\ \text{Using that theorem,} \\ A^{10}v_1 &= (\lambda_1)^{10}v_1 \\ &= (-1)^{10}v_1 = v_1 \\ A^{10}v_2 &= (\lambda_2)^{10}v_2 \\ &= (2)^{10}v_2 \\ &= 1024v_2 \end{aligned}$$

$$\begin{aligned} \therefore A^{10}x &= 3v_1 + 2(1024)v_2 \\ &= 3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2048 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2051 \\ 4093 \end{pmatrix} \end{aligned}$$

Theorem: Let A be an $n \times n$ matrix & let $\lambda_1, \lambda_2, \dots, \lambda_m$ be distinct eigen vectors of A with corresponding eigen vectors v_1, \dots, v_m , then v_1, \dots, v_m are linearly independent.

Proof: Let v_1, \dots, v_m are linearly dependent.

Let v_{k+1} be the first vector which is linearly dependent on $\{v_1, \dots, v_k\}$.

\therefore If c_1, \dots, c_k (scalars) such that,

$$v_{k+1} = \sum_{i=1}^k c_i v_i \quad \rightarrow \textcircled{1}$$

Multiplying eqⁿ \textcircled{1} by A on both side

$$AV_{k+1} = A(c_1 v_1 + \dots + c_k v_k)$$

$$\therefore \lambda V_{k+1} = c_1(\lambda v_1) + c_2(\lambda v_2) + \dots + c_k(\lambda v_k)$$

$$\therefore \lambda V_{k+1} = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \dots + c_k \lambda_k v_k \rightarrow \textcircled{2}$$

Multiplying eqⁿ(1) by λ_{K+1} on both sides,

$$\lambda_{K+1} V_{K+1} = c_1 \lambda_{K+1} v_1 + \dots + c_K \lambda_{K+1} v_K \rightarrow (3)$$

eqⁿ(3) - eqⁿ(2),

$$\therefore 0 = c_1 v_1 (\lambda_{K+1} - \lambda_1) + c_2 v_2 (\lambda_{K+1} - \lambda_2) + \dots + c_K v_K (\lambda_{K+1} - \lambda_K)$$

$$\therefore \sum_{i=1}^K c_i v_i (\lambda_{K+1} - \lambda_i) = 0 \quad \text{As } \{v_1, \dots, v_K\} \text{ are linearly independent}$$

$$\therefore c_i (\lambda_{K+1} - \lambda_i) = 0 \quad \forall 1 \leq i \leq K$$

$$\therefore \boxed{c_i = 0} \quad \forall 1 \leq i \leq K$$

$$\therefore V_{K+1} = \sum_{i=1}^K c_i v_i = \boxed{0}$$

It is impossible since eigenvectors cannot be zero.

∴ Contradiction.

Hence, $\{v_1, \dots, v_K\}$ are linearly independent.

Similar Matrices

Definition Let A and B be $n \times n$ matrices. We say that A is similar to B if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$. If A is similar to B , we write $A \sim B$.

$$P^{-1}AP = B \Rightarrow A \sim B$$

→ If A is similar to B , $P^{-1}AP = B$

$$\therefore PP^{-1}APP^{-1} = PBP^{-1}$$

$$\therefore \boxed{A = PBP^{-1}}$$

$$\therefore \boxed{AP = PB}$$

Theorem 4.21

Let A , B , and C be $n \times n$ matrices.

- a. $A \sim A$
- b. If $A \sim B$, then $B \sim A$.
- c. If $A \sim B$ and $B \sim C$, then $A \sim C$.

Proof: (a) $A = IAI$ $[I^{-1} = I]$

$$\therefore A = I^{-1}AI$$

$$\therefore \boxed{A \sim A}$$

(b) $A \sim B$

\exists invertible matrix P such that,

$$P^{-1}AP = B$$

$$\therefore PP^{-1}APP^{-1} = PBP^{-1}$$

$$\therefore A = PBP^{-1}$$

$$\text{Let } P_1 = P^{-1}$$

$$P_1^{-1} = (P^{-1})^{-1} = P$$

$$\therefore \boxed{A = P_1^{-1}BP_1}$$

$$\therefore \boxed{B \sim A}$$

(c) Given $A \sim B$

$$B \sim C$$

$$\therefore P_1^{-1}AP_1 = B$$

$$\text{and, } P_2^{-1}BP_2 = C$$

$$\therefore P_1^{-1}P_1^{-1}AP_1P_2 = P_2^{-1}BP_2$$

$$\text{Let } P_3 = P_1P_2$$

$$\therefore P_3^{-1} = P_2^{-1}P_1^{-1}$$

$$\therefore P_3^{-1}AP_3 = P_2^{-1}BP_2 = C$$

$$\therefore \boxed{P_3^{-1}AP_3 = C}$$

$$\therefore \boxed{A \sim C}$$

$$\text{Q.E.D. } A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}$$

Find the matrix P such that $A \sim B$

Solⁿ: Let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$AP = PB$$

$$\therefore \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} a+2c & b+2d \\ -c & -d \end{pmatrix} = \begin{pmatrix} a-2b & -b \\ c-2d & -d \end{pmatrix}$$

$$\left. \begin{array}{l} a+2c = a-2b \\ c = -b \end{array} \right\} \rightarrow ① \quad \left. \begin{array}{l} b+2d = -b \\ 2d = -2b \\ d = -b \end{array} \right\} \rightarrow ② \quad \left. \begin{array}{l} -c = c-2d \\ 2c = 2d \\ c = d \end{array} \right\} \rightarrow ③$$

$$\therefore \boxed{d = c = -b}$$

$$\therefore P = \begin{pmatrix} a & b \\ -b & -b \end{pmatrix}$$

$$\text{Let } a=1 \quad \Rightarrow P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Theorem 4.22

Let A and B be $n \times n$ matrices with $A \sim B$. Then

- a. $\det A = \det B$
- b. A is invertible if and only if B is invertible.
- c. A and B have the same rank.
- d. A and B have the same characteristic polynomial.
- e. A and B have the same eigenvalues.
- f. $A^m \sim B^m$ for all integers $m \geq 0$.
- g. If A is invertible, then $A^m \sim B^m$ for all integers m .

Proof: (a) $\det A = \det B$

$$\Rightarrow P^{-1}AP = B$$

$$\det(P^{-1}) \det(A) \det(P) = \det(B)$$

$$\therefore \frac{1}{\det(P)} \det(A) \det(P) = \det(B)$$

$$\therefore \boxed{\det(A) = \det B}$$

$$\begin{aligned}
 (d) \quad \det(B - \lambda I) &= \det(P^{-1}AP - \lambda I) \\
 &= \det(P^{-1}AP - \lambda(P^{-1}IP)) \\
 &= \det(P^{-1}AP - P^{-1}(\lambda I)P) \\
 &= \det(P^{-1}(A - \lambda I)P) \\
 &= \det(P^{-1}) \det(A - \lambda I) \det(P)
 \end{aligned}$$

$$\boxed{\det(B - \lambda I) = \det(A - \lambda I)}$$

*Diagonalisation of Matrices

Definition An $n \times n$ matrix A is **diagonalizable** if there is a diagonal matrix D such that A is similar to D —that is, if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = D$.

An $n \times n$ matrix A is diagonalizable, if \exists a diagonal matrix D , such that $\boxed{A \sim D}$.

$$\therefore \exists P \text{ s.t. } \boxed{P^{-1}AP = D}$$

Theorem 4.23

Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

More precisely, there exist an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors in P in the same order.

Proof: \rightarrow Suppose $A \sim D$,

$$\exists P \text{ st. } \therefore P^{-1}AP = D \Rightarrow P \boxed{AP = PD}$$

Let the columns of P be p_1, \dots, p_n
and the diagonal entries of D be $\lambda_1, \dots, \lambda_n$.

$$A[p_1, \dots, p_n] = P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

$$\therefore [AP_1, \dots, AP_n] = [p_1, \dots, p_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

$$\therefore [AP_1, \dots, AP_n] = [\lambda_1 p_1, \dots, \lambda_n p_n]$$

$$\therefore AP_i = \lambda_i p_i$$

This proves that the column vectors of P are eigenvectors of A , whose corresponding eigenvalues are $\lambda_1, \dots, \lambda_n$.

Since P is invertible,

its column vectors are linearly independent.

[By Fundamental Theorem of Invertible Matrices].

Hence A has n linearly independent eigen vectors.

Fundamental Theorem of Invertible Matrices :

$A \rightarrow nxn$ matrix.

The following are equivalent:

(a) A is an invertible matrix

(b) $AX=B$ has a unique solution for any $B \in R$.

(c) $AX=0$ has only trivial solution

(d) The row reduced echelon form of A is I_n .

(e) A is a product of elementary matrices

(f) rank of $A = n$

(g) nullity of $A = 0$

(h) The column vectors of A are linearly independent.

Q: If possible, find a matrix D that diagonalises

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix}$$

Soln: Eigen values:

$$\det(A - \lambda I) = 0$$

$$\therefore \begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\therefore -(1+\lambda)(\lambda(1+\lambda)) + ((0+\lambda)) = 0$$

$$\therefore \lambda = (\lambda+1)^2(\lambda)$$

$$\therefore \lambda[(\lambda+1)^2 - 1] = 0$$

$$\therefore \lambda[\lambda^2 + 2\lambda] = 0$$

$$\therefore \lambda^2(\lambda+2) = 0$$

$$\therefore \lambda = 0, -2$$

$$n_1 = n_3$$

$$\hookrightarrow \left\{ t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$t, s \in \mathbb{R}$

for $\lambda = -2$,

$$\begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0$$

$$\therefore x_1 = -x_3$$

$$\hookrightarrow 3x_1 + 2x_2 - 3x_3 = 0$$

$$\therefore 6n_3 = 2n_2 \rightarrow n_2 = 3n_3$$

$$n_1 = t$$

$$n_3 = -t \Rightarrow n_2 = -3t$$

$$\therefore \ell_{-2} = \left\{ t \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \right\}.$$

For $\lambda = 0$,

$$\begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \text{eigen vectors} \Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \right\}$$

$w_1 \quad w_2 \quad w_3$

$\therefore w_1, w_2 \text{ and } w_3$ are linearly independent.

Hence, A is diagonalizable

$$\Rightarrow P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -1 \end{pmatrix}$$

$$P^{-1} = \frac{\text{adj}(P)}{\det(P)} = \frac{1}{2} \begin{pmatrix} -1 & 3 & -1 \\ 0 & -2 & 0 \\ -1 & 3 & 1 \end{pmatrix}^T$$

$$= \boxed{\begin{pmatrix} 1/2 & 0 & 1/2 \\ -3/2 & 1 & -3/2 \\ 1/2 & 0 & -1/2 \end{pmatrix}}$$

$$\Rightarrow \boxed{P = P^{-1}AP}$$

Q If possible find a matrix P that diagonalizes

$$(a) A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{vmatrix} = 0$$

$$\therefore -\lambda(\lambda(\lambda-4) + 5) - 1(0 - 2) = 0$$

$$\therefore -\lambda(\lambda^2 - 4\lambda + 5) + 2 = 0$$

$$\therefore -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0$$

$$\therefore \lambda^3 - 4\lambda^2 + 5\lambda - 2$$

$$\lambda^3 - \lambda^2 - 3\lambda^2 + 3\lambda + 2\lambda - 2 = 0$$

$$(\lambda^2 - 3\lambda + 2)(\lambda - 1) = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 1) = 0$$

$$\boxed{(\lambda - 1)^2(\lambda - 2) = 0} \Rightarrow \boxed{\lambda = 1, 2}$$

\rightarrow For $\lambda = 1$,

$$A - \lambda I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{pmatrix}$$

Let v be the linearly independent vector

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow (A - \lambda I)v = 0$$

$$\therefore \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0$$

$$\therefore \left. \begin{array}{l} n_1 = n_2 \\ n_2 = n_3 \end{array} \right\} \quad \begin{array}{l} n_1 - 5n_2 + 3n_3 = 0 \checkmark \\ \therefore \text{Basis} = \left\{ t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\} \end{array}$$

$$\lambda = 2, \quad (A - \lambda I)v = 0$$

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 2 & -5 & 2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0$$

$$\therefore \left. \begin{array}{l} n_2 = 2n_1 \\ n_3 = 2n_2 \end{array} \right\} \quad \left. \begin{array}{l} n_3 = 4n_1 \end{array} \right\}$$

$$\therefore \text{Basis} = \left\{ t \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

\therefore Only 2 linearly independent eigen vectors.

\therefore Not diagonalizable.

$$(b) \quad A = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\therefore \begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\therefore -(1+\lambda)[\lambda(1+\lambda) - 0] + 1(0 + \lambda) = 0$$

$$\therefore \lambda[(1+\lambda)^2 + 1] = 0$$

$$\therefore \lambda[\lambda^2 + 2\lambda] = 0 \Rightarrow \lambda^2(\lambda + 2) = 0$$

$$\lambda = 0, -2$$

For $\lambda = 0$,

$$A - \lambda I = A = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0$$

$$\textcircled{n_1 = n_3} \quad \therefore \text{Basis} = \left\{ t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, s \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mid t, s \in \mathbb{R} \right\} \quad \boxed{\dim = 2}$$

for $\lambda = -2$,

$$A - \lambda I = A + 2I = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0$$

$$\therefore \boxed{n_1 = -n_3} \quad \left\{ \begin{array}{l} 3n_1 + 2n_2 - 3(-n_1) = 0 \\ 6n_1 + 2n_2 = 0 \end{array} \right.$$

$$\boxed{n_2 = -3n_1}$$

$$\text{Basis} = \left\{ t \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

$$\boxed{\dim = 1}$$

Hence, there are 3 linearly independent eigen vectors.

$\therefore P = [P_1 \ P_2 \ P_3]$ Column vectors are
 \hookrightarrow linearly independent eigen
 vectors

$$\therefore P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow \det P = -1 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \boxed{-2}$$

$$\therefore P^{-1} = \frac{\text{adj}(P)}{|P|}$$

$$= \frac{1}{-2} \begin{pmatrix} -3 & -1 & 1 \\ -2 & 0 & 0 \\ 3 & -1 & -1 \end{pmatrix}^T$$

$$= -\frac{1}{2} \begin{pmatrix} -3 & -2 & 3 \\ -1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} = \boxed{\begin{pmatrix} 3/2 & 1 & -3/2 \\ 1/2 & 0 & 1/2 \\ -1/2 & 0 & 1/2 \end{pmatrix}}$$

$$\boxed{D = P^{-1}AP}$$

Steps: Given $A (n \times n)$

- ① Find all eigen values of A .
 - ② Then find all corresponding eigen vectors.
 - ③ If there are n linearly independent eigen vectors of A , then compute P .
- $P = [P_1 \dots P_n]$
- ④ Compute $P^{-1}AP = D$ to verify.

Q Find the values of $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10}$ using diagonalization.

Solⁿ: $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$

$$\det(A - \lambda I) = 0$$

$$\therefore \begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda(\lambda - 1) - 2 = 0$$

$$\therefore \lambda^2 - \lambda - 2 = 0$$

$$\therefore (\lambda - 2)(\lambda + 1) = 0$$

$\lambda = 1$
 $\lambda = 2$
 $1 - \lambda$

$\lambda = -1, 2$

$$\Rightarrow \lambda = -1$$

$$(A - \lambda I)v = 0 \Rightarrow \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$u_1 = -u_2$

$$\therefore \text{Basis} = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

$$\lambda = 2$$

$$(A - \lambda I)v = 0$$

$$\therefore \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left. \begin{array}{l} \therefore -2u_1 + u_2 = 0 \\ \boxed{u_2 = 2u_1} \end{array} \right\} \quad \text{Basis} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

$$\therefore P = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \rightarrow P^{-1} = \frac{1}{2+1} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{pmatrix}}$$

We know that,

$$P^{-1}AP = D \quad \left(\text{where } P = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \right)$$

$$D^{10} = (P^{-1}AP)^{10} = P^{-1}AP \underbrace{(P^{-1}AP)}_{2} \cdots \underbrace{(P^{-1}AP)}_{10}$$

$$\therefore P^{-1} A^{10} P = D^{10}$$

$$\therefore \boxed{A^{10} = P D^{10} P^{-1}}$$

$$A^{10} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}^{10} \begin{pmatrix} 2\gamma_3 & -\gamma_3 \\ \gamma_3 & \gamma_3 \end{pmatrix}$$

$$A^{10} = \gamma_3 \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (2)^{10} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \gamma_3 \begin{pmatrix} 1 & 2^{10} \\ -1 & (2)^{10} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \gamma_3 \begin{pmatrix} 2+2^{10} & 2^{10}-1 \\ 2^{10}-2 & 2^{10}+1 \end{pmatrix}$$

$$= \gamma_3 \begin{pmatrix} 1026 & 1023 \\ 2046 & 2049 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 342 & 341 \\ 682 & 683 \end{pmatrix}}$$

Orthogonal Diagonalization of Symmetric Matrices :

Definition A square matrix A is *orthogonally diagonalizable* if there exists an orthogonal matrix Q and a diagonal matrix D such that $Q^T A Q = D$.

Q: If possible, diagonalize the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$$

Sol^u: $\det(A - \lambda I) = 0$

$$\therefore \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\therefore -(1-\lambda)(2+\lambda) - 4 = 0$$

$$(1-1)(\lambda+2) - 4 = 0$$

$$\lambda^2 + \lambda - 2 - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$\boxed{(\lambda+3)(\lambda-2)=0}$$

$$\lambda = 2,$$

$$(A - \lambda I)v = 0$$

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = 0$$

$$\boxed{n_1 = 2n_2}$$

$$\therefore \text{Basis} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

$$\lambda = -3,$$

$$(A - \lambda I)v = 0$$

$$\therefore \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = 0 \Rightarrow \boxed{n_2 = -2n_1}$$

$$\therefore \text{Basis} = \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

$$\therefore P = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \quad \left| \quad \det Q = \frac{1}{\sqrt{5}} (-5) = \boxed{-\sqrt{5}} \right.$$

$$\therefore Q = \frac{1}{\sqrt{5+1}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \quad \left| \quad Q^T = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \right.$$

$$= \boxed{\frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}}$$

$$\Rightarrow D = Q^T A Q$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ 2 & 6 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 10 & 0 \\ 0 & -15 \end{pmatrix}$$

$$D = \boxed{\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}}$$

Theorem 5.17

If A is orthogonally diagonalizable, then A is symmetric.

Proof: If A is orthogonally diagonalizable, then there exists an orthogonal matrix Q and a diagonal matrix D such that

$$Q^T A Q = D$$

$$\text{Since. } Q^{-1} = Q^T$$

$$\text{we have } Q^T Q = I = Q Q^T$$

$$\therefore Q Q^T A Q Q^T = Q D Q^T$$

$$\therefore \boxed{A = Q D Q^T}$$

$$\begin{aligned} A^T &= (Q^T)^T D^T Q^T \\ &= Q D^T Q^T \end{aligned}$$

Since, D is a diagonal matrix, $D = D^T$

$$\therefore \boxed{A^T = Q D Q^T = A}$$

Hence, A is symmetric.

Theorem 5.18

If A is a real symmetric matrix, then the eigenvalues of A are real.

Recall that the complex conjugate of a complex number $z = a + bi$ is the number

Proof: Suppose that λ is an eigenvalue of A with corresponding eigenvector v . Then $Av = \lambda v$,

and taking complex conjugates, we have $\overline{Av} = \overline{\lambda v}$

But then, as $A = \bar{A}$ (A is real)

$$A\bar{v} = \bar{A}\bar{v} = \overline{Av} = \overline{\lambda v} = \bar{\lambda}\bar{v}$$

Using the fact that A is symmetric,

$$(A\bar{v})^T = (\bar{\lambda}\bar{v})^T$$

$$\therefore \bar{V}^T A^T = \bar{V}^T A = \bar{\lambda}(\bar{v})^T$$

$$\left[\begin{array}{l} A = \bar{A} \\ A = A^T \end{array} \right]$$

$$\begin{aligned}
 \therefore \lambda(\bar{V}^T V) &= (\bar{V}^T)(\lambda V) = \bar{V}^T(AV) = (\bar{V}^T A)V \\
 &= (\bar{\lambda} \bar{V}^T)V \\
 &= \bar{\lambda}(\bar{V}^T V) \\
 \therefore \lambda(\bar{V}^T V) &= \bar{\lambda}(\bar{V}^T V) \\
 \therefore \lambda(\bar{V}^T V) - \bar{\lambda}(\bar{V}^T V) &= 0 \\
 \therefore (\lambda - \bar{\lambda})(\bar{V}^T V) &= 0 \\
 \therefore \boxed{\lambda = \bar{\lambda}} \Rightarrow \underline{\lambda \text{ is Real}}.
 \end{aligned}$$

Theorem 5.19

If A is a symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

Proof: Let $v_1, v_2 \rightarrow$ eigenvectors corresponding to distinct eigenvalues ($\lambda_1 \neq \lambda_2$)

$$\therefore Av_1 = \lambda_1 v_1 \text{ and } Av_2 = \lambda_2 v_2$$

→ Using $A^T = A$ and $x \cdot y = x^T y$,

$$\begin{aligned}
 \lambda_1(v_1 \cdot v_2) &= (\lambda_1 v_1) \cdot v_2 = (Av_1) \cdot v_2 \\
 &= (Av_1)^T v_2 \\
 &= v_1^T A^T v_2 \\
 &= v_1^T (Av_2) \\
 &= v_1^T (\lambda_2 v_2) \\
 &= \lambda_2(v_1^T v_2) \\
 &= \lambda_2(v_1 \cdot v_2)
 \end{aligned}
 \quad \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array} \right\} \begin{aligned}
 \lambda_1(v_1 \cdot v_2) &= \lambda_2(v_1 \cdot v_2) \\
 \therefore (\lambda_1 - \lambda_2)(v_1 \cdot v_2) &= 0 \\
 \text{As } \lambda_1 \neq \lambda_2 \quad \boxed{v_1 \cdot v_2 = 0}
 \end{aligned}$$

Q. Verify the result of Theorem 5.19 for

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\underline{\underline{SOL^u}}: \det(A - \lambda I) = 0$$

$$\therefore \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore (2-\lambda)[(2-\lambda)^2 - 1] - 1[2-\lambda - 1] + 1[1 - (2-\lambda)] = 0$$

$$\therefore (2-\lambda)[4+\lambda^2 - 4\lambda - 1] - 2(1-\lambda) = 0$$

$$\therefore (2-\lambda)(\lambda^2 - 4\lambda + 3) + 2(\lambda - 1) = 0$$

$$\therefore (2-\lambda)(\lambda^2 - 4\lambda + 3) + 2(\lambda - 1) = 0$$

$$\therefore (\lambda - 1)[(\lambda - 3)(2 - \lambda) + 2] = 0$$

$$\therefore (\lambda - 1)[\lambda^2 - 5\lambda + 6] = 0$$

$$\therefore \boxed{(\lambda - 1)^2(\lambda - 6) = 0}$$

For $\lambda = 1$,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0 \quad \therefore \text{Basis} = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$\therefore \boxed{n_1 + n_2 + n_3 = 0}$$

For $\lambda = 6$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0$$

$$\begin{aligned} \therefore n_2 + n_3 &= 2n_1 \\ n_1 + n_3 &= 2n_2 \\ n_1 + n_2 &= 2n_3 \end{aligned} \quad \left\{ \begin{array}{l} n_1 = n_2 = n_3 \end{array} \right.$$

$$\text{Basis} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

v_3

Verifying,

$$v_2 \cdot v_3 = 1 - 1 = 0$$

$$v_1 \cdot v_3 = 1 - 1 = 0$$

Hence, verified.

Theorem 5.20

The Spectral Theorem

Let A be an $n \times n$ real matrix. Then A is symmetric if and only if it is orthogonally diagonalizable.

$$Q = [q_1 \ q_2 \ \dots \ q_n]$$

$$Q^T = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} \rightarrow A = [q_1 \dots q_n] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix}$$

$$A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \dots + \lambda_n q_n q_n^T$$

↳ Spectral decomposition form
of a symmetric matrix.

Eg: $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ Orthogonally diagonalize the matrix.

Solⁿ: $\det(A - \lambda I) = 0$

$\hookrightarrow \lambda = 1, 1, 1$

$$e_4 = \left\{ t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

$$e_1 = \left\{ p \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + q \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, p, q \in \mathbb{R} \right\}$$

$$\rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

\rightarrow Gram-Schmidt process:

v_2 and v_3 are not orthogonal

$$\therefore \text{let } v_1' = v_1$$

$$v_2' = v_2$$

$$\begin{aligned} (\because v_1 \cdot v_3 = 0) \& \quad v_3' = v_3 - \left(\frac{v_2 \cdot v_3}{v_2 \cdot v_2} \right) v_2 \\ &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \left(\frac{1}{2} \right) \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \end{pmatrix} \\ &= \boxed{\begin{pmatrix} -1/2 \\ 1 \\ 1/2 \end{pmatrix}} \end{aligned}$$

$$\begin{aligned} &\therefore \{v_1, v_2, v_3'\} \text{ is} \\ &\quad \text{orthogonal} \\ &\quad \text{but not orthonormal} \\ &\therefore \{q_1, q_2, q_3\} \\ &= \left\{ \left(\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix} \right) \right\} \end{aligned}$$

$$\rightarrow q_1 q_1^T = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$\rightarrow q_2 q_2^T = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}$$

$$\rightarrow q_3 q_3^T = \begin{pmatrix} 1/6 & -1/3 & -1/6 \\ -1/3 & 2/3 & 1/3 \\ -1/6 & 1/3 & 1/6 \end{pmatrix}$$

$$A = q_1 q_1^T + q_2 q_2^T + q_3 q_3^T$$

Q Find a 2×2 matrix with eigen-values $\lambda_1 = 3, \lambda_2 = -2$ and corresponding eigenvectors

$$V_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad V_2 = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

Sol: As $V_1 \cdot V_2 = 0$,
if $Q \in \mathbb{R}^{2 \times 2}$ such that,

$$Q^T A Q = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\text{Let } Q = (q_1 \ q_2)$$

$$\therefore A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T$$

$$q_1 = \begin{pmatrix} 3/5 \\ 1/5 \end{pmatrix} \quad q_2 = \begin{pmatrix} -4/5 \\ 3/5 \end{pmatrix}$$

$$\therefore A = 3 \begin{pmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{pmatrix} + (-2) \begin{pmatrix} 16/25 & -12/25 \\ -12/25 & 9/25 \end{pmatrix}$$

$$A = \begin{pmatrix} -1/5 & 12/5 \\ 12/5 & 6/5 \end{pmatrix}$$

*Quadratic Form:

→ An expression of the form

$$ax^2 + by^2 + cxy$$

is called a quadratic form of x, y .

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz$$

↳ of x, y, z

$$\Rightarrow = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Definition: A quadratic form in n variable is a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ of the form

$$f(x) = x^T A x$$

where A is a symmetric $n \times n$ matrix and x is in \mathbb{R}^n . We refer to A as the matrix associated with f .

Q: What is the quadratic form with associated matrix $A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

Sol: $\begin{bmatrix} a & c_1 \\ c_1 & b \end{bmatrix}$

$$\left. \begin{array}{l} c = f \\ a = 2 \\ b = 5 \end{array} \right\} \boxed{2x^2 + 5y^2 - 6xy}$$

Q: Find the matrix associated with the quadratic form.

$$f(x_1, x_2, x_3) = 2x_1^2 - x_2^2 + 5x_3^2 + 6x_1x_2 - 3x_1x_3$$

$$\left. \begin{array}{l} a = 2 \\ b = -1 \\ c = 5 \\ d = 6 \\ f = -3 \\ e = 0 \end{array} \right\} \begin{bmatrix} 2 & 3 & -3 \\ 3 & -1 & 0 \\ -3 & 0 & 5 \end{bmatrix}$$

* Principal Axis Theorem:

Let A be a $n \times n$ symmetric matrix associated with a quadratic form $x^T A x$.

If Q is orthogonal matrix such that $Q^T A Q = D$,
 (D is a diagonal matrix), then $x = Qy$

transforms the quadratic form to $y^T D y$.

Proof: Pg 436 (let $f(x) = \dots$)

Q: Find a change in variable that transforms the quadratic form $f(n_1, n_2) = 5n_1^2 + 4n_1n_2 + 2n_2^2$ into one with no cross-product terms.

Sol^u: Corresponding matrix,

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$$

$$\Rightarrow \boxed{\det(A - \lambda I) = 0} \quad \boxed{\lambda_1 = 6} \quad \boxed{\lambda_2 = 1}$$

\Rightarrow The corresponding eigenvectors are:

$$\text{Also, } D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2\sqrt{5} \\ 1\sqrt{5} \end{pmatrix} \text{ and } \begin{pmatrix} 1\sqrt{5} \\ -2\sqrt{5} \end{pmatrix}$$

$$\lambda = 6 \quad \lambda = 1$$

If we set $Q = \begin{pmatrix} 2\sqrt{5} & 1\sqrt{5} \\ 1\sqrt{5} & -2\sqrt{5} \end{pmatrix}$ then $D = Q^T A Q$ will be a diagonal matrix.

Since we don't want any cross-pdt. terms, the change of variable $x = Qy$ will create the Diagonal Matrix D such that,

$$f(y) = y^T D y.$$

$$\text{Let } Y = [y_1 \ y_2]$$

$$= [y_1 \ y_2] \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \boxed{6y_1^2 + y_2^2}$$

- * A quadratic form $f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$ is classified as one of the following: symmetric \mathbf{A} is also called:
- Positive definite if $f(\mathbf{x}) > 0 \forall \mathbf{x} \neq 0$
 - Positive semidefinite if $f(\mathbf{x}) \geq 0 \forall \mathbf{x}$
 - Negative definite if $f(\mathbf{x}) < 0 \forall \mathbf{x} \neq 0$
 - Negative semidefinite if $f(\mathbf{x}) \leq 0 \forall \mathbf{x}$.
 - Indefinite if $f(\mathbf{x})$ takes on both positive and negative values.

Theorem 5.22

Let A be an $n \times n$ symmetric matrix. The quadratic form $f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$ is

- positive definite if and only if all of the eigenvalues of A are positive.
- positive semidefinite if and only if all of the eigenvalues of A are nonnegative.
- negative definite if and only if all of the eigenvalues of A are negative.
- negative semidefinite if and only if all of the eigenvalues of A are nonpositive.
- indefinite if and only if A has both positive and negative eigenvalues.

Eg: Classify $f(x, y, z) = 3x^2 + 3y^2 + 3z^2 - 2xy - 2xz - 2yz$
 as
 +ve definite
 -ve definite
 indefinite
 none of these

Soln:

$$A = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \quad \Rightarrow \det(A - \lambda I) = 0$$

$$\therefore \begin{vmatrix} 3-\lambda & -1 & -1 \\ -1 & 3-\lambda & -1 \\ -1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(\lambda+2)(\lambda+4) = 0$$

$$\lambda = 3, -2, -4 \quad \therefore \text{Indefinite} \quad \therefore (3-\lambda)^3 - (3-\lambda) + 1 \left((\lambda-3)+1 \right) - 1 \left(1+(\lambda-3) \right)$$

\otimes Singular Values of a Matrix.

Definition If A is an $m \times n$ matrix, the **singular values** of A are the square roots of the eigenvalues of $A^T A$ and are denoted by $\sigma_1, \dots, \sigma_n$. It is conventional to arrange the singular values so that $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$.

For any $m \times n$ matrix A , $A^T A$ is a $n \times n$ sym. matrix.

By spectral theorem, we can orthogonally diagonalize $A^T A$.

All eigenvalues of $A^T A$ are real and non-negative.
(let λ be the eigen value of $A^T A$ with corresponding unit eigen vector v)

$$\begin{aligned} 0 &\leq \|Av\|^2 = (Av) \cdot (Av) \\ &= (Av)^T (Av) \\ &= v^T A^T A v \quad (\lambda \rightarrow \text{eigenvalue of } A^T A) \\ &= v^T (\lambda v) \quad \therefore A^T A v = \lambda v \\ &= \lambda v^T v \\ &= \lambda (v \cdot v) = \lambda \|v\|^2 = \boxed{\lambda} \end{aligned}$$

$$\therefore \boxed{\lambda \geq 0}$$

The Singular Value Decomposition:

$A \rightarrow m \times n$ matrix

$\therefore A^T A \rightarrow n \times n$ matrix

let $\lambda_1, \dots, \lambda_n \rightarrow$ eigenvalues

\therefore Singular values: $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}$

$$\therefore \sigma_1, \dots, \sigma_n$$

$$\therefore \lambda_i = \|A\mathbf{v}_i\|^2$$

$$\therefore \sigma_i = \sqrt{\|A\mathbf{v}_i\|^2}$$

$$\boxed{\sigma_i = \|A\mathbf{v}_i\|}$$

i.e. the singular values of A are the lengths of the vectors $A\mathbf{v}_1, \dots, A\mathbf{v}_n$.

Now, we want to show that $m \times n$ matrix A can be factored as $A = U\Sigma V^T$

where U is $m \times m$ orthogonal matrix, V is $n \times n$ orthogonal matrix and Σ is an $m \times n$ diagonal matrix.

\rightarrow If the non-zero singular values of A are

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$$

$$\text{and } \sigma_{r+1} = \sigma_{r+2} = \dots = \sigma_n = 0$$

then Σ is of the form:

$$\Sigma = \begin{bmatrix} D & \begin{matrix} \underbrace{\quad}_{n-r} \\ \vdots \\ 0 \end{matrix} \\ \hline \begin{matrix} \overbrace{0}^r \\ \vdots \\ 0 \end{matrix} & \begin{matrix} \underbrace{0}_{m-r} \\ \vdots \\ 0 \end{matrix} \end{bmatrix}$$

where,

$$D = \begin{pmatrix} \sigma_1 & & & & 0 \\ 0 & \sigma_2 & & & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & \cdots & \sigma_r \end{pmatrix}$$

Eg: (of Σ matrix): for $r=2$,

$$\Sigma = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}, \Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}, \Sigma = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Steps:

- To construct the orthogonal matrix V , we first find an orthonormal basis $\{v_1, \dots, v_n\}$ for \mathbb{R}^n consisting of eigenvectors of the $n \times n$ symmetric matrix $A^T A$.

Then $V = [v_1 \dots v_n]$

is an orthogonal $n \times n$ matrix.

② For the orthogonal matrix U , we first note that $\{Av_1, \dots, Av_n\}$ is an orthogonal set of vectors in \mathbb{R}^n .

For $i \neq j$,

$$\begin{aligned}
 (Av_i) \cdot (Av_j) &= (Av_i)^T (Av_j) \\
 &= v_i^T A^T A v_j \\
 &\stackrel{\text{def}}{=} \lambda_j v_i^T v_j \\
 &= v_i^T (\lambda_j v_i) \\
 &= \lambda_j (v_i^T v_i) \\
 &= \lambda_j (v_i \cdot v_i) \\
 &= \boxed{0} \quad (\text{since the eigenvectors } v_i \text{ are orthogonal}).
 \end{aligned}$$

$$\rightarrow \sigma_i = \|Av_i\| \quad (\text{let first } r \text{ of these be nonzero})$$

$$\therefore \text{Let } U_i = \frac{1}{\sigma_i} Av_i \quad (i = \{1, \dots, r\})$$

This guarantees that $\{U_1, \dots, U_r\}$ is an orthonormal set in \mathbb{R}^m , but if $r < m$, it will not be a basis for \mathbb{R}^m .

In this case, we extend the set $\{U_1, \dots, U_r\}$ to an orthonormal basis $\{U_1, \dots, U_m\}$ for \mathbb{R}^m .

Then we set,

$$\boxed{U = [U_1 \ U_2 \ \dots \ U_m]}$$

Proof: (of the above method)

$V \rightarrow n \times n$ orthogonal

$$\therefore V^T = V^{-1}$$

$$\therefore A = U \Sigma V^T$$

$$\therefore AV = U \Sigma$$

We know that,

$$AV_i = \sigma_i v_i \text{ for } i=1, \dots, r$$

$$\text{and } \|AV_i\| = \sigma_i$$

$$\therefore AV_i = 0 \text{ for } i > r$$

$$\begin{aligned} \therefore AV &= A[v_1 \dots v_n] \\ &= [AV_1 \dots AV_n] \\ &= [AV_1 \dots AV_r 0 \dots 0] \end{aligned}$$

$$= [\sigma_1 v_1 \dots \sigma_r v_r 0 \dots 0]$$

$$= [v_1 \dots v_m] \left(\begin{array}{c|c} \sigma_1 & 0 \\ \vdots & \vdots \\ 0 & \sigma_r \\ \hline 0 & 0 \end{array} \right)$$

$$\boxed{AV = U \Sigma} \quad \text{as required.}$$

This was the proof of The Singular Value Decomposition Theorem.

Theorem 7.13 The Singular Value Decomposition

Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ and $\sigma_{r+1} = \sigma_{r+2} = \dots = \sigma_n = 0$. Then there exist an $m \times m$ orthogonal matrix U , an $n \times n$ orthogonal matrix V , and an $m \times n$ matrix Σ of the form shown in Equation (1) such that

$$A = U\Sigma V^T$$

Q Find a singular value decomposition for the following matrices.

$$(a) A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{2 \times 3}$$

$$A^T A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(A^T A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore (1-\lambda)^3 - 1((1-\lambda)^2) = 0$$

$$\therefore (1-\lambda)((1-\lambda)^2 - 1) = 0$$

$$\therefore (1-\lambda)(\lambda^2 - 2\lambda + 1 - 1) = 0$$

$$\therefore \lambda(\lambda-1)(\lambda-2) = 0$$

$$\boxed{\lambda = 0, 1, 2}$$

$$\begin{aligned} & \lambda_1 \geq \lambda_2 \geq \lambda_3 \\ & \therefore \lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 0 \end{aligned}$$

for $\lambda_3 = 0$,

$$\text{let } V_3 = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$A^T A V_3 = \lambda_3 V_3$$

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{l} \therefore n_1 = -n_2 \\ n_3 = 0 \end{array} \quad \left\{ \begin{array}{l} V_3 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \end{array} \right.$$

Similarly,

for $\lambda_2 = 1$,

$$\text{let } V_2 = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$A^T A V_2 = \lambda_2 V_2$$

$$\begin{array}{l} n_1 + n_2 = n_1 \\ n_2 = 0 \end{array} \quad \left| \quad \begin{array}{l} n_1 + n_2 = n_2 \\ n_1 = 0 \end{array} \right.$$

$$\therefore V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Similarly for

for $\lambda_1 = 2$, $(A^T A - \lambda_1 I) V_1 = 0$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \begin{array}{l} n_1 = n_2 \\ n_3 = 0 \end{array} \quad \left\{ \begin{array}{l} V_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \end{array} \right.$$

$$\therefore V = [V_1 \ V_2 \ V_3] = \boxed{\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}}$$

For U ,

$$U_1 = \frac{1}{\sigma_1} AV_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$
$$= \boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$U_2 = \frac{1}{\sigma_2} AV_2 = \frac{1}{1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$= \boxed{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

These vectors already form an orthonormal basis of \mathbb{R}^2 .

\therefore we have $U = [U_1 \ U_2]$

$$= \boxed{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$\therefore A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$$
$$= \boxed{U \Sigma V^\top}$$

Theorem 7.14

The Outer Product Form of the SVD

Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ and $\sigma_{r+1} = \sigma_{r+2} = \dots = \sigma_n = 0$. Let $\mathbf{u}_1, \dots, \mathbf{u}_r$ be left singular vectors and let $\mathbf{v}_1, \dots, \mathbf{v}_r$ be right singular vectors of A corresponding to these singular values. Then

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \dots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T$$

\Rightarrow If

$$\begin{aligned}
A &= U\Sigma V^T = [\mathbf{u}_1 \ \cdots \ \mathbf{u}_m] \begin{bmatrix} \sigma_1 & \cdots & 0 & O \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \sigma_r & O \\ O & & & O \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \\
&= [\mathbf{u}_1 \ \cdots \ \mathbf{u}_r \ \mathbf{u}_{r+1} \ \cdots \ \mathbf{u}_m] \begin{bmatrix} \sigma_1 & \cdots & 0 & O \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \sigma_r & O \\ O & & & O \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_r^T \\ \mathbf{v}_{r+1}^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \\
&= [\mathbf{u}_1 \ \cdots \ \mathbf{u}_r] \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_r^T \end{bmatrix} + [\mathbf{u}_{r+1} \ \cdots \ \mathbf{u}_m] [O] \begin{bmatrix} \mathbf{v}_{r+1}^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} \\
&= [\mathbf{u}_1 \ \cdots \ \mathbf{u}_r] \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_r^T \end{bmatrix} \\
&= [\sigma_1 \mathbf{u}_1 \ \cdots \ \sigma_r \mathbf{u}_r] \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_r^T \end{bmatrix} \\
&= \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T
\end{aligned}$$

Q Find Outer Product form of $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow A^T A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\det(A^T A - \lambda I) = 0$$

$$\therefore \lambda_1 = 3 \quad \lambda_2 = 1$$

$$\sigma_1 = \sqrt{3} \quad \sigma_2 = 1$$

\Rightarrow For $\lambda_1 = 3$,

Let x_1 be the eigen-vector

$$x_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\therefore (A^T A - \lambda_1 I) x_1 = 0$$

$$\therefore \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \boxed{n_1 = n_2}$$

$$E_3 = \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

$$\text{Let } v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$v_1 = \frac{1}{\sqrt{2}} A v_1, \quad v_2 = \frac{1}{\sqrt{2}} A v_2$$

\Rightarrow For $\lambda = 1$,

$$(A^T A - \lambda I) x_2 = 0$$

$$\therefore \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \boxed{n_1 = -n_2}$$

$$\therefore E_1 = \left\{ t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

$$\begin{aligned} &= \frac{1}{\sqrt{2}\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \frac{1}{1\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} &= \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \end{aligned}$$

\therefore Outer Product form

$$= \sigma_1 v_1 v_1^T + \sigma_2 v_2 v_2^T$$

$$= \sqrt{3} \begin{pmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 1/\sqrt{2} \end{pmatrix}$$

$$= \sqrt{3} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

Pseudo Inverse :

Definition Let $A = U\Sigma V^T$ be an SVD for an $m \times n$ matrix A , where $\Sigma = \begin{bmatrix} D & O \\ O & O \end{bmatrix}$ and D is an $r \times r$ diagonal matrix containing the nonzero singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ of A . The *pseudoinverse* (or *Moore-Penrose inverse*) of A is the $n \times m$ matrix A^+ defined by

$$A^+ = V\Sigma^+U^T$$

where Σ^+ is the $n \times m$ matrix

$$\Sigma^+ = \begin{bmatrix} D^{-1} & O \\ O & O \end{bmatrix}$$

$$Q: A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Find the pseudo inverse of A .

$$\text{Sol: } A^+ = U\Sigma^+V^T$$

$$A^TA = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rightarrow \det(A^TA - \lambda I) = 0$$

$$\therefore (1-\lambda)^3 - 1((1-\lambda)^2 - 0) = 0$$

$$\therefore (1-\lambda)[(1-\lambda)^2 - 1] = 0 \Rightarrow (\lambda-1)(\lambda^2 - 2\lambda + 1 - 1) = 0$$

$$\therefore [\lambda(\lambda-1)(\lambda-2) = 0]$$

$$\therefore \lambda_1 = 2, \lambda_2 = 1 \text{ and } \lambda_3 = 0$$

$$\therefore \sigma_1 = \sqrt{2}, \sigma_2 = 1$$

$$\therefore D = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}$$

\rightarrow for $\lambda_1 = 2$,

$$(A^T A - \lambda I)x = 0$$

$$\therefore \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \begin{cases} n_1 = n_2 \\ n_3 = 0 \end{cases} \quad \boxed{\ell_2 = \left\{ t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \mid t \in R \right\}}$$

\rightarrow for $\lambda = 1$,

$$Q: A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ Find Pseudo Inverse.}$$

Solⁿ: As solved earlier,

$$\Sigma = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \Sigma^+ = \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \quad V = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$A^+ = V \Sigma^+ U^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} U^T$$

$$= \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1/\sqrt{3} & 2/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{3} & -1/\sqrt{3} & 2/\sqrt{3} \end{pmatrix}}$$

* Hermitian Matrix :

→ A square matrix A_{nn} is said to be Hermitian iff

$$A^* = A$$

Q. $A = \begin{pmatrix} 1 & 2+i \\ 2-i & i \end{pmatrix}$ Hermitian?

$$\rightarrow A^* = \bar{A}^T$$

$$= \begin{pmatrix} 1 & 2-i \\ 2+i & -i \end{pmatrix}^T = \boxed{\begin{pmatrix} 1 & 2+i \\ 2-i & -i \end{pmatrix} \neq A} \quad \times$$

$$(ii) \quad A = \begin{pmatrix} 3 & 1+i \\ 1-i & 4 \end{pmatrix}$$

$$A^* = \begin{pmatrix} 3 & 1-i \\ i+1 & 4 \end{pmatrix}^T = \boxed{\begin{pmatrix} 3 & 1+i \\ 1-i & 4 \end{pmatrix}}$$

* Unitary Matrix :

→ A square matrix is unitary if $U^* = U^{-1}$.

$$U^*U = UU^* = I ?$$

→ A square complex matrix A is called unitarily diagonalizable if there exists a unitary matrix U and a diagonal matrix D s.t.

$$D = U^*AU$$

Step 1: Compute the eigen-values of the matrix A.

Step 2: Find the basis for each eigenspace.

Step 3: Ensure that each eigen space contain orthonormal eigen vectors. [Use Gram-Schmidt process if needed]

Step 4:

Q: Find the unitary matrix U and diagonal matrix D such that $U^*AU = D$.

(a) $A = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}$

$$\left. \begin{array}{l} \det(A - \lambda I) = 0 \\ \therefore \begin{vmatrix} 2-\lambda & i \\ -i & 2-\lambda \end{vmatrix} = 0 \\ \therefore (2-\lambda)^2 + i^2 = 0 \\ \lambda^2 - 4\lambda + 4 - 1 = 0 \\ \therefore (\lambda-3)(\lambda-1) = 0 \\ \underline{\lambda_1 = 3} \quad \underline{\lambda_2 = 1} \end{array} \right| \begin{array}{l} \text{for } \lambda_1 = 3 \\ \begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \therefore \boxed{n_1 = i n_2} \quad \& \quad \boxed{n_2 = -i n_1} \\ \therefore e_1 = \begin{cases} 1 \\ i \end{cases} \text{ 1 tec} \end{array}$$

For $\lambda_2 = 1$

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore n_1 + i n_2 = 0$$

$$\Rightarrow \mathcal{C}_2 = \left\{ t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mid t \in \mathbb{C} \right\}$$

$$\begin{aligned}\Rightarrow \langle V_1, V_2 \rangle &= \overline{V_1} \cdot V_2 \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 - 1 \\ &= \boxed{0}\end{aligned}$$