



5% → TUT - QUIZ

5% → ASSIGNMENTS

10% → QUIZ - 2

30% → ENDSEM

MATRICES :

→ Equality of 2 Matrices :

A & B are said to be equal if they have the same order and their corresponding elements are equal.

e.g. $A = [a_{ij}]_{m \times n}$ $a_{ij} = b_{ij} \quad \forall i, j$
 $B = [b_{ij}]_{m \times n}$

Types Of Matrices:

1) Square Matrix $m = n$ (Rows = Column)

2) Zero Matrix $a_{ij} = 0 \quad \forall i, j$

3) Diagonal Matrix $a_{ij} = 0 \quad \forall i \neq j$

• Scalar Matrix $a_{ij} = a ; a \neq 0$
(All diagonal elements are same)

• Identity Matrix $a_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

4) Upper Triangular Matrix : $a_{ij} = 0 \quad \text{if } i > j$

5) Lower Triangular Matrix : $a_{ij} = 0 \quad \text{if } i < j$

MATRIX OPERATION :

- 1) Addition of Matrices
- 2) Multiplication of Matrices
- 3) Transpose of Matrix

1) For addition \rightarrow same order $m \times n$

$$C = A + B$$

$$C_{ij} = [a_{ij} + b_{ij}] = [a_{ij}] + [b_{ij}]$$

Properties :

$$(A+B) = (B+A)$$

$$(A+B)+C = A + (B+C)$$

$$k(kA) = (k\ell)A$$

$$(k+\ell)A = kA + \ell A$$

2) Matrix Multiplication :

$$A = [a_{ij}]_{m \times n} \xrightarrow{\text{No. of columns of } A}$$

$$B = [b_{ij}]_{n \times r} \xrightarrow{\text{No. of rows of } B}$$

$$C = AB$$

$$= [c_{ij}]_{m \times r}$$

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

Properties :

$$(i) A(BC) = (AB)C$$

$$(ii) A(B+C) = AB + AC$$

$$(iii) k(AB) = (kA)B = A(kB)$$

$$(iv) I_n A = A I_n = A$$

where A is $n \times n$ matrix

$I_n = n \times n$ Identity matrix

(Multiplicative Identity)

Q: If A, B are square matrices of same size,

$$(A+B)^2 = A^2 + 2AB + B^2?$$

$$\text{No, } (A+B)^2 = A^2 + AB + BA + B^2$$

Proof of $A(B+C)$:

$$\begin{aligned} [A(B+C)]_{ij} &= a_{ij} \cdot (b_j + c_j) \\ &= a_{ij} \cdot b_j + a_{ij} \cdot c_j = (AB)_{ij} + (AC)_{ij} = (AB+AC)_{ij} \\ \therefore [A(B+C)] &= AB+AC \end{aligned}$$

Inverse of a Matrix :

If A is a $n \times n$ matrix, an inverse of A is a $n \times n$ matrix A' , such that $AA' = A'A = I$

Eg: $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$$A' = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

→ Zero matrix will not have an inverse.

Result: If $A_{n \times n}$ is an invertible matrix, then the inverse is unique.

Proof: Let A' and A'' be two inverses of A . Then,

$$AA' = A'A = I \rightarrow ①$$
$$AA'' = A''A = I \rightarrow ② \quad (\text{Associative})$$

$$A' = A'I = A'(AA'') = (A'A)A''$$

(Identity) From ② $\qquad \qquad \qquad \boxed{\Rightarrow IA'' = A''}$
 $\therefore A' = A''$ From ①

Note: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
if $ad \neq bc$.

Properties:

① If A is an invertible matrix, A^{-1} is also invertible,

then $(A^{-1})^{-1} = A$.

② If A is an invertible matrix, and c is a non-zero scalar, then

$$(cA)^{-1} = \frac{1}{c} A^{-1}$$

③ If A and B are invertible matrices of same size, then AB is also invertible, and

$$(AB)^{-1} = B^{-1}A^{-1}$$

④ If A is invertible, then A^n is invertible for all non-negative integer n .

$$(A^n)^{-1} = (A^{-1})^n$$

Proof: ① Let X be an inverse of A^{-1} .

$$\begin{aligned} & \therefore XA^{-1} = A^{-1}X = I \\ & \text{and, } A(XA^{-1}) = A(A^{-1}X) = AI = I \end{aligned} \quad \boxed{(A^{-1})^{-1} = A}$$

② Let X be an inverse of cA .

$$\therefore X(cA) = (cA)X = I \quad \dots \text{(if)}$$

$$\text{and, } \left(\frac{1}{c}A^{-1}\right)(cA) = \left(\frac{1}{c} \times cA^{-1}\right)(A) = A^{-1}A = I$$

↪ (i)

$$(cA)\left(\frac{1}{c}A^{-1}\right) = A\left(c \cdot \frac{1}{c}A^{-1}\right) = AA^{-1} = I \rightarrow (i)$$

From (i), (ii) and (iii),

$$\boxed{X = \frac{1}{c}A^{-1}}$$

(3) Similarly,

Let $X \rightarrow$ inverse of AB

$$\therefore X(AB) = (AB)X = I \quad (i)$$

$$\begin{aligned} \rightarrow (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= A(IA^{-1}) = AA^{-1} = I \quad (ii) \end{aligned}$$

$$\begin{aligned} \rightarrow (B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}B)B \\ &= B^{-1}IB = B^{-1}B = I \quad (iii) \end{aligned}$$

From (i), (ii) and (iii),

$$\boxed{X = B^{-1}A^{-1}}$$

(4) Let $X \rightarrow$ inverse of A^n ,

$$\text{To prove: } XA^n = A^nX = I \Rightarrow X = (A^{-1})^n$$

\rightarrow Base case: $n=1$

$$AA^{-1} = A^{-1}A = I \quad (\underline{\text{True}})$$

Suppose it is true for $n=k$,

$$\therefore A^k(A^{-1})^k = (A^{-1})^k A^k = I$$

$$i) \quad \because A \cdot A^k \cdot (A^{-1})^k \cdot (A^{-1}) = A \cdot I \cdot A^{-1} = A \cdot A^{-1} = I$$

$$\therefore \boxed{A^{k+1} \cdot (A^{-1})^{k+1} = I}$$

$$ii) \quad \text{Similarly, } (A^{-1})^{k+1} \cdot A^{k+1} = I$$

\therefore it is true for $n+1$.

Hence, inductively we proved that

$$A^n \cdot (A^{-1})^n = I \quad \forall n \geq 0$$

Elementary Matrices :

\rightarrow An elementary matrix is a matrix that can be obtained by performing elementary row operation on identity matrix.

Eg: $E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is equivalent to
 $3R_2$ on I_4

$E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is equivalent to
 $R_1 \leftrightarrow R_3$ on I_n

$E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix}$ is equivalent to
 $R_4 \rightarrow R_4 - 2R_2$
on I_n

XXX Inverse of an Elementary Matrix :

$$\Rightarrow E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Why?

$$I_3 \xrightarrow{R_{23}} E_1 \xrightarrow{R_{23}} I_3$$

$$\Rightarrow E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \left. \begin{array}{l} \\ \end{array} \right\} I_3 \xrightarrow{R_2 \rightarrow 4R_2} E_2 \xrightarrow{R_2 \rightarrow \frac{1}{4}R_2} I_3$$

$$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \quad \left. \begin{array}{l} \\ \end{array} \right\} I_3 \xrightarrow{R_3 \rightarrow R_3 + 2R_1} E_3 \xrightarrow{R_3 \rightarrow R_3 + 2R_1} I_3$$

$$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

How to find inverse?

\Rightarrow We know that $E = e(I)$

Let $X = \text{Inverse of } E$

$$\therefore XE = EX = I$$

$$\therefore Xe(I) = e(I)X = I$$

Let $X = e^{-1}(I)$

$$\therefore Xe(I) = e^{-1}e(I \cdot I) = \boxed{I}$$

$\therefore X = e^{-1}(I)$

Hence, if $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$E: R_2 \rightarrow R_2 + R_1$$

$$\therefore e^{-1}: R_2 \rightarrow R_2 - R_1$$

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Theorem:

Let E be an elementary matrix obtained by performing elementary row operation on I_{nxn} .

If the same elementary row operation is done on nxn matrix A , the resultant is same as EA .

Fundamental Theorem of Invertible Matrices :

- (a) A is an invertible matrix
- (b) $AX = B$ has a unique solution for any $B \in R$.
- (c) $AX = 0$ has only trivial solution
- (d) The row reduced echelon form of A is I_n .
- (e) A is a product of elementary matrices

$$(a) \leftrightarrow (b) \leftrightarrow (c) \leftrightarrow (d) \leftrightarrow (e)$$

Proof: To prove: (a) \rightarrow (b) \rightarrow (c) \rightarrow (d) \rightarrow (e)

$$(a) \rightarrow (b)$$

$$\text{Given: } AA^{-1} = A^{-1}A = I$$

Proving that $AX = B$ has a solution:

Suppose $x = A^{-1}B$ is a solution.

$$\begin{aligned} \therefore A(A^{-1}B) &= (AA^{-1})B \\ &= \boxed{B} \end{aligned}$$

Now, proving that the solution is unique:

Let y be a solution of $AX = B$

$$\therefore Ay = B$$

$$\therefore A^{-1}(Ay) = A^{-1}B$$

$$\therefore (A^{-1}A)y = A^{-1}B$$

$$\therefore Iy = \boxed{y = A^{-1}B} \quad \underline{\text{Hence, unique.}}$$

(b) \rightarrow (c)

Given: $AX=B$ has a unique solⁿ.

Let $B=0$

$\therefore AX=0$ also has a unique solⁿ.

A homogeneous system of linear equations always has $X=0$ as a solution.

As, $AX=0$ has only one unique solⁿ,

$AX=0$ only has trivial solⁿ ($X=0$).

(c) \rightarrow (d)

Let R be a row-reduced echelon form of A .

As $AX=0$ has only $X=0$ as its solution,

$RX=0$ has only trivial solⁿ.

Suppose R has r non zero rows.

$\therefore r \leq n-r$ free scalars

\therefore there will be infinite solutions,

if $n-r > 0$

$\therefore n-r \leq 0$ As $r \leq n$

$$n-r=0$$

$$\therefore \boxed{r=n} \Rightarrow \boxed{R=I_n}$$

(d) \rightarrow (e)

Given: $R = I_n$

Suppose R is obtained from A after performing n elementary row operations

$$\therefore R = e_n e_{n-1} \dots e_1(A)$$

$$\text{As, } e_i = e_i(I)$$

$$R = e_n e_{n-1} \dots e_1(A) = I$$

As elementary matrices are invertible,

$$A = e_1^{-1} e_2^{-1} \dots e_n^{-1}(I)$$

As product of invertible matrices is invertible,

$$\boxed{A = e_1^{-1} e_2^{-1} \dots e_n^{-1}} \text{ is a product of invertible matrices.}$$

(c) \rightarrow (a)

Suppose $A = \epsilon_1 \epsilon_2 \dots \epsilon_n$ (product of n invertible matrices)

As ϵ_i is invertible $\forall 1 \leq i \leq n$

$\exists \epsilon_i^{-1}$ s.t. $\epsilon_i^{-1} \epsilon_i = \epsilon_i \epsilon_i^{-1} = I$

$$\therefore A(\epsilon_n^{-1} \epsilon_{n-1}^{-1} \dots \epsilon_1^{-1}) = I$$

$$\text{let } X = \boxed{\epsilon_n^{-1} \epsilon_{n-1}^{-1} \dots \epsilon_1^{-1}}$$

$$\exists X \text{ s.t. } \boxed{AX = I}$$

Similarly,

$$(\epsilon_n^{-1} \epsilon_{n-1}^{-1} \dots \epsilon_1^{-1})A = I$$

$\therefore \exists X \text{ s.t. }$

$$\boxed{XA = I}$$

$$\therefore AX = XA = I$$

$\therefore A$ is invertible.

$$\begin{aligned} & \therefore I = \epsilon_3 \epsilon_2 \epsilon_1 A \quad \left| \begin{array}{l} (\epsilon_1)^{-1} = R_1 \rightarrow R_1 + R_2 \\ (\epsilon_2)^{-1} = R_2 \rightarrow R_2 + R_1 \\ (\epsilon_3)^{-1} = R_2 \rightarrow 3R_2 \end{array} \right| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ & \therefore \boxed{\epsilon_1^{-1} \epsilon_2^{-1} \epsilon_3^{-1} = A} \quad \left| \begin{array}{l} \epsilon_3 \\ \uparrow R_2 \rightarrow \frac{1}{3} R_2 \end{array} \right. \end{aligned}$$

Alt. method $\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} \xrightarrow[\epsilon_1]{R_1 \rightarrow R_1 - R_2} \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} \xrightarrow[\epsilon_2]{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

Q. Express $A = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ as product of elementary matrices.

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} \xrightarrow[\epsilon_3]{R_1 \rightarrow R_1 + R_2} \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} \Rightarrow \epsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \uparrow R_2 \rightarrow R_2 + R_1 \quad \epsilon_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$I \xrightarrow[\epsilon_1]{R_2 \rightarrow 3R_2} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \quad \epsilon_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \boxed{\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} = \epsilon_3 \epsilon_2 \epsilon_1}$$

* Transpose of a Matrix:

→ The transpose of a $m \times n$ matrix $A = [a_{ij}]_{m \times n}$ is a $n \times m$ matrix $A^T = [a_{ji}]_{n \times m}$ with $a_{ji}^T = a_{ij}$
for $1 \leq i \leq m$, $1 \leq j \leq n$

$$\text{Eg: } A = \begin{pmatrix} 1 & 6 \\ 8 & 2 \\ 4 & 7 \end{pmatrix}_{3 \times 2} \quad A^T = \begin{pmatrix} 1 & 8 & 4 \\ 6 & 2 & 7 \end{pmatrix}_{2 \times 3}$$

→ The transpose of a matrix is exchanging the rows & columns of the matrix.

rows of A = column of A^T

columns of A = rows of A^T

→ If you have 2 column vectors

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}_{n \times 1} \quad V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}_{n \times 1}$$

$$U \cdot V = [u_1 \dots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = U^T V = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$$\therefore \boxed{\vec{U} \cdot \vec{V} = \vec{U}^T \vec{V}}$$

Proof of $(AB)^T = B^T A^T$:

known: 1) row_i(A) = column_i(A^T)

$$2) a_{ji}^T = a_{ij}$$

$$\begin{aligned} \therefore [(AB)^T]_{ij} &= (AB)_{ji} \\ &= \text{row}_j(A) \cdot \text{column}_i(B) \\ &= \text{column}_j(A^T) \cdot \text{row}_i(B^T) \\ &= \text{row}_i(B^T) \cdot \text{column}_j(A^T) \\ &= (B^T A^T)_{ij} \end{aligned}$$

Properties:

$$\textcircled{1} (A^T)^T = A \quad \textcircled{3} (A + B)^T = A^T + B^T$$

$$\textcircled{2} (AB)^T = B^T A^T \quad \textcircled{4} (KA)^T = K A^T$$

DETERMINANT:

$$\rightarrow A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2 \times 2}$$

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of a 3×3 matrix:

$$\rightarrow A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Rightarrow$$

$$\det A = +a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\therefore \det A = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}$$

$$= \boxed{\sum_{j=1}^3 (-1)^{1+j} a_{1j} \det A_{1j}}$$

where, $\det A_{ij}$ is called the (i,j) -minor of A

Determinant of a $n \times n$ matrix:

$$\det A = |A| = a_{11}A_{11} - a_{12}A_{12} + \dots + (-1)^{1+n} a_{nn}A_{nn}$$

$$= \boxed{\sum_{j=1}^n (-1)^{1+j} a_{1j} A_{1j}} \quad (A_{ij} = \text{minor of } a_{ij})$$

Cofactor:

Combine a minor with its plus or minus sign. To this end, we define the (i,j) -cofactor of A to be:

$$\boxed{C_{ij} = (-1)^{i+j} A_{ij}}$$

$$\therefore \boxed{\det A = \sum_{j=1}^n a_{1j} C_{1j}}$$

The Laplace Expansion Theorem:

The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geq 2$, can be computed as

$$\begin{aligned}\det A &= a_{11}c_{11} + a_{12}c_{12} + \dots + a_{1n}c_{1n} \\ &= \sum_{j=1}^n a_{1j}c_{1j}\end{aligned}$$

(which is the cofactor expansion along the 1st row) and also as

$$\begin{aligned}\det A &= a_{1j}c_{1j} + a_{2j}c_{2j} + \dots + a_{nj}c_{nj} \\ &= \sum_{i=1}^n a_{ij}c_{ij}\end{aligned}$$

(the cofactor expansion along the jth column).

Theorem 4.2: The determinant of a triangular matrix is the product of the entries on its main diagonal.

Specifically, if $A = [a_{ij}]$ is an $n \times n$ triangular matrix, then

$$\boxed{\det A = a_{11}a_{22}\dots a_{nn}}$$

Properties of Determinant:

- ① If A has a zero row/column, then $\det A = 0$.
- ② If B is obtained by interchanging two rows/columns of A , then $\det B = -\det A$
- ③ If A has two identical rows/columns, then $\det A = 0$.
- ④ If B is obtained by multiplying a row/column of A by k , then $\det B = k \det A$

(5) If A, B and C are identical except that the i^{th} -row/column of C is the sum of the i^{th} row/column of A and B, then

$$\det C = \det A + \det B$$

(6) If B is obtained by adding a multiple of one row/column of A to another row/column, then

$$\det B = \det A$$

Prove all of the above properties:

(1) Suppose i^{th} row(column) is zero in A.

Row: $\therefore \det A = \sum_{j=1}^n a_{ij} C_{ij}$ (As $a_{ij}=0 \forall j$)

$$\boxed{\det A = 0}$$

Column: $\det A = \sum_{j=1}^n a_{ji} C_{ji}$ (As $a_{ji}=0 \forall j$)

$$\boxed{\det A = 0}$$

(2) A \rightarrow has two identical rows, swap them to obtain the matrix B.

clearly, $B = A$

$$\therefore \det B = \det A$$

Also, by (2), $\det B = -\det A$ } $\therefore \det A = -\det A$

(3) Suppose $b_{ij} = k a_{ij}$ for $j=1, \dots, n$

\therefore expand them along i^{th} row,

$$\begin{aligned} \det B &= \sum_{j=1}^n b_{ij} C_{ij} = \sum_{j=1}^n k a_{ij} C_{ij} = k \sum_{j=1}^n a_{ij} C_{ij} \\ &= \boxed{k \det A} \end{aligned}$$

(4) $C_{ij} = a_{ij} + b_{ij} \neq j$

else, $C_{rj} = a_{rj} = b_{rj} \neq r \neq i$ (the cofactors C_{ij} of the elements in the expanding along i^{th} row, i^{th} rows of A, B and C are identical.)

$$\det C = \sum_{j=1}^n C_{ij} C_{ij}$$

$$\begin{aligned} &= \sum_{j=1}^n (a_{ij} + b_{ij}) C_{ij} = \sum_{j=1}^n a_{ij} C_{ij} + \sum_{j=1}^n b_{ij} C_{ij} \\ &= \boxed{\det A + \det B} \end{aligned}$$

(6)

$$R_i \rightarrow R_i + \kappa R_r \Rightarrow B$$

$$\begin{aligned}
 \det B &= \sum_{j=1}^n b_{ij} C_{ij} \\
 &= \sum_{j=1}^n (a_{ij} + \kappa \alpha_{rj}) C_{ij} \\
 &= \sum_{j=1}^n a_{ij} C_{ij} + \boxed{\kappa \sum_{j=1}^n \alpha_{rj} C_{ij}} \\
 &= \det A + \kappa(0) \quad \downarrow \det C \\
 &= \det A \quad \text{where } r^{\text{th}} \text{ row is replaced} \\
 &\quad \text{with } r^{\text{th}} \text{ row.} \\
 \text{So, now } r^{\text{th}} \text{ and } i^{\text{th}} \text{ row are} \\
 &\quad \text{identical} \\
 \therefore \det C &= 0
 \end{aligned}$$

(2)

Determinants of Elementary Matrices:

Theorem 4.4:

Let E be an $n \times n$ elementary matrix.

- (1) If E results from interchanging two rows of I_n , then $\det E = -1$.
- (2) If E results from multiplying one row of I_n by k , then $\det E = k$.
- (3) If E results from adding a multiple of one row of I_n to another row, then $\det E = 1$.

Can be directly derived from the part (2), (4) & (6) of the previous theorem.

Lemma 4.5: Let B be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix. Then

$$\det(EB) = (\det E)(\det B)$$

Theorem 4.6: A square matrix A is invertible if and only if $\det A \neq 0$.

CRAMER's RULE:

→ Let A be an $n \times n$ matrix, and let $b \in \mathbb{R}^n$. Then the unique sol n x of the system of equation $Ax=b$ is given by

$$x_i = \frac{\det(A_i(b))}{\det(A)}$$

for $i=1, 2, \dots, n$

where $A_i(b)$ is obtained by replacing i th column of A with b .

Eg: $\begin{cases} x_1 + 2x_2 = 2 \\ -x_1 + 4x_2 = 1 \end{cases}$ } let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\therefore \det A = \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 4+2 = 6$$

$$\begin{aligned} \therefore x_1 &= \frac{\det(A_1(b))}{\det(A)} = \frac{\begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix}}{6} = \frac{6}{6} = 1 \\ x_2 &= \frac{\det(A_2(b))}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}}{6} = \frac{3}{6} = \frac{1}{2} \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad \begin{aligned} x &= \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \end{aligned}$$

→ $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$

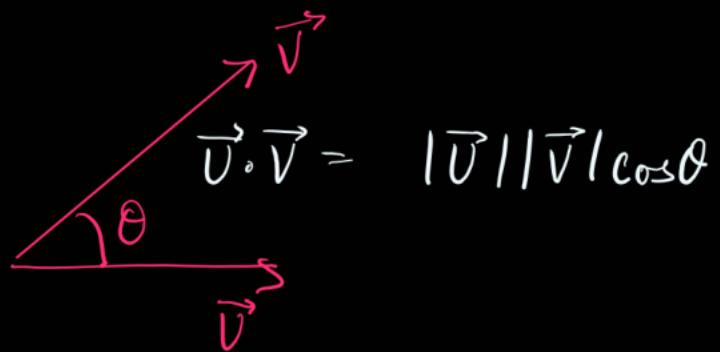
Eg: let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \Rightarrow \det(A) = -18 - 2(-10) - 1(4) = -22 + 20 = -2$

$$\text{adj}(A) = \begin{bmatrix} -18 & 10 & 4 \\ 3 & -2 & -1 \\ 10 & -6 & -2 \end{bmatrix}^T = \begin{bmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = -\frac{1}{2} \begin{pmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{pmatrix}$$

INNER PRODUCT SPACE:

Definition :



An inner product on a vector space V is an

operation that assigns to every pair of vectors U and V in V a real number $\langle U, V \rangle$ such that the following properties hold for vectors $U, V \in W \in V$ and all scalars c :

① $\langle U, V \rangle = \langle V, U \rangle$

② $\langle U, V + W \rangle = \langle U, V \rangle + \langle U, W \rangle$

③ $\langle cU, V \rangle = c\langle U, V \rangle$

④ $\langle U, U \rangle \geq 0$ and $\langle U, U \rangle = 0$ iff $U = 0$

A vector space with an inner product is called an inner product space.

If v and u are two vectors in an innerproduct space V :

① length of v is $\|v\| = \sqrt{\langle v, v \rangle}$

② The distance b/w $U \notin V$ is $d(U, V) = \|U - V\|$.

③ U , and V are orthogonal if $\langle U, V \rangle = 0$

Eg: \mathbb{R}^n is an inner product space $(v_i \cdot v_j) = v_i \cdot v_j$

But dot product is not the only inner product that can be defined on \mathbb{R}^n .

2]

NOTE: $\{v_1, v_2, \dots, v_n\}$ where every pair is orthogonal
then, $v_i \cdot v_j = 0$ if $v_i \neq v_j$

→ If $\{v_1, v_2, \dots, v_n\}$ is an orthogonal set of non-zero vectors in \mathbb{R}^n then they are linearly independent.

If c_1, c_2, \dots, c_n are scalars such that,

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

$$\left(\sum_i c_i v_i \right) \cdot v_i = 0 \cdot v_i \\ = 0$$

we know that $v_i \cdot v_j = 0$ if $i \neq j$

$$\therefore c_i (v_i \cdot v_i) = 0 \quad \forall i$$

$$\therefore \boxed{c_i = 0 \quad \forall i}$$

$$\text{Def: } (\mathbb{R}^2, +) \rightarrow (\mathbb{R}, +, \cdot)$$

$$\langle u, v \rangle = 2u_1v_1 + 3u_2v_2$$

Is this an inner product?

Solution:

$$\begin{aligned} \text{(i)} \quad & \langle u, v \rangle \\ &= 2u_1v_1 + 3u_2v_2 \\ &= 2v_1u_1 + 3v_2u_2 \\ &= \boxed{\langle v, u \rangle} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad & \langle u, v+w \rangle \\ &= 2u_1(v_1+w_1) + 3u_2(v_2+w_2) \\ &= 2u_1v_1 + 2u_1w_1 + 3u_2v_2 + 3u_2w_2 \\ &= (2u_1v_1 + 3u_2v_2) + (2u_1w_1 + 3u_2w_2) \\ &= \boxed{\langle u, v \rangle + \langle u, w \rangle} \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad & \langle cv, v \rangle \\ &= 2(cv_1)v_1 + 3(cv_2)v_2 \\ &= c[2u_1v_1 + 3u_2v_2] \\ &= \boxed{c\langle u, v \rangle} \end{aligned} \quad \left| \begin{array}{l} \text{(iv)} \quad \langle u, u \rangle \\ = 2u_1^2 + 3u_2^2 \\ > 0 \quad > 0 \\ \therefore \langle u, u \rangle \geq 0 \end{array} \right.$$

If $u = 0 \Rightarrow u_1 = u_2 = 0$

$\therefore \boxed{\langle u, u \rangle = 0}$

Complex Dot Product:

If $U = \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{pmatrix}$ and $V = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix}$ are vectors in C^n , then the

complex dot product of U and V is defined by

$$U \cdot V = \bar{U}_1 V_1 + \dots + \bar{U}_n V_n$$

Prove that it is a inner product:

→ The norm (length) of a complex vector V is defined as in the real case:

$$\|V\| = \sqrt{V \cdot V} \quad \left(\text{General form: } \|V\| = \sqrt{\langle V, V \rangle} \right)$$

Distance b/w two complex vectors:

$$d(U, V) = \|U - V\|$$

Eg: let $U = \begin{pmatrix} i \\ 1 \end{pmatrix}$ and $V = \begin{pmatrix} 2-3i \\ 1+5i \end{pmatrix}$

(a) $U \cdot V$

$$\begin{aligned} U \cdot V &= \sum_i \bar{U}_i V_i = (-i)(2-3i) + (1)(1+5i) \\ &= -2i + 3(-i) + 1 + 5i \\ &= 3i - 2 \end{aligned}$$

(b) $\|U\|$

$$\begin{aligned} \|U\| &= \sqrt{U \cdot U} \\ &= \sqrt{1+1} = \sqrt{2} \end{aligned}$$

(c) $\|v\|$

$$\|v\| = \sqrt{4+9+25+1} = \boxed{\sqrt{39}}$$

(d) $d(u, v)$

$$\begin{aligned} d(u, v) &= \|u - v\| \\ &= \left\| \begin{pmatrix} 4 & -2 \\ -5 & 1 \end{pmatrix} \right\| = \sqrt{16+4+25} \\ &= \boxed{\sqrt{45}} \end{aligned}$$

NOTE: Let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ then, $\|v\| = \sqrt{|v_1|^2 + |v_2|^2 + \dots + |v_n|^2}$

Properties :

(a) $U \circ V = \overline{V \circ U}$

(b) $U \circ (V + W) = U \circ V + U \circ W$

(c) $(cU) \circ V = \bar{c}(U \circ V)$ and $U \circ (cv) = c(U \circ v)$

(d) $U \circ U \geq 0$ and $U \circ U = 0$ iff $U = 0$.

Proofs:

(a) $\langle u, v \rangle = \sum_i \bar{u}_i v_i$

As $\bar{\bar{v}} = v$

$$= \sum_i \bar{u}_i \bar{v}_i$$

As $\bar{\bar{p}} = \bar{p}$

$$= \sum_i \bar{u}_i \bar{v}_i$$

As $\bar{p+q} = \bar{p} + \bar{q}$
= $\overline{\sum_i u_i v_i}$
= $\overline{\sum_i \bar{v}_i u_i}$
= $\boxed{\langle v, u \rangle}$

$$\begin{aligned}
 (b) \quad & \langle v, v+w \rangle \\
 &= \sum_i \bar{v}_i (v_i + w_i) \\
 &= \sum_i \bar{v}_i v_i + \sum_i \bar{v}_i w_i \\
 &= \boxed{\langle v, v \rangle + \langle v, w \rangle} \\
 \end{aligned}
 \qquad\qquad\qquad
 \begin{aligned}
 (c) \quad & \langle cv, v \rangle = \sum_i \bar{c} \bar{v}_i v_i \\
 &= \sum_i \bar{c} \bar{v}_i v_i \\
 &= \bar{c} \sum_i \bar{v}_i v_i \\
 &= \boxed{\bar{c} \langle v, v \rangle}
 \end{aligned}$$

$$\begin{aligned}
 (d) \quad & \langle v, v \rangle = \sum_i \bar{v}_i v_i \\
 &= \sum_i |v_i|^2 \geq 0
 \end{aligned}$$

If $v=0$
then $\sum_i |v_i|^2 = 0$

Conjugate Transpose:

If A is a complex matrix, then the conjugate transpose of A is the matrix A^* defined by

$$A^* = \bar{A}^T$$

\bar{A} refers to the matrix whose entries are the complex conjugates of the corresponding entries of A .

$$\text{If } A = [a_{ij}]$$

$$\text{then, } \bar{A} = [\bar{a}_{ij}]$$

Properties:

$$(a) \bar{\bar{A}} = A$$

$$(b) \overline{A+B} = \bar{A} + \bar{B}$$

$$(c) \overline{cA} = \bar{c}\bar{A}$$

$$(d) \overline{AB} = \bar{A}\bar{B}$$

$$(e) (\bar{A})^T = (\bar{A}^T)$$

Proofs:

$$(a) \text{let } A = [a_{ij}]$$

$$\bar{A} = [\bar{a}_{ij}]$$

$$\bar{\bar{A}} = [\bar{\bar{a}}_{ij}] = [a_{ij}]$$

$$\therefore \bar{\bar{A}} = A$$

$$(b) \overline{A+B} = [a_{ij} + b_{ij}]$$

$$= [\bar{a}_{ij} + \bar{b}_{ij}]$$

$$= [\bar{a}_{ij}] + [\bar{b}_{ij}]$$

$$\therefore \overline{A+B} = \boxed{\bar{A} + \bar{B}}$$

$$(d) \overline{AB}$$

$$(AB)_{ij} = \sum_k a_{ik} b_{kj}$$

$$(\bar{AB})_{ij} = \sum_k \bar{a}_{ik} \bar{b}_{kj}$$

$$\text{As } p, q \in C \quad (\bar{p+q}) = \bar{p} + \bar{q}$$

$$pq = \bar{p}\bar{q} \quad \therefore = \sum_k (\bar{a}_{ik})(\bar{b}_{kj})$$

$$= (\bar{A}\bar{B})_{ij}$$

$$\therefore \overline{AB} = \boxed{\bar{A}\bar{B}}$$

More Properties:

$$(a) (A^*)^* = A$$

$$(b) (A+B)^* = A^* + B^*$$

$$(c) (cA)^* = \bar{c}A^*$$

$$(d) (AB)^* = B^*A^*$$

Proofs:

$$(d) (AB)^* = (\bar{AB})^T = (\bar{A}\bar{B})^T$$

$$= \bar{B}^T \bar{A}^T$$

$$= \boxed{B^*A^*}$$

* If $U = \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{pmatrix}$ and $V = \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix}$

if w_1, \dots, w_n are positive scalars and are vectors in R^n , then

$$\langle U, V \rangle = w_1 U_1 V_1 + \dots + w_n U_n V_n$$

This defines an inner product on R^n , called a weighted dot product.

Prove that this is an inner product:

$$\begin{aligned} (i) \quad & \langle U, V \rangle = \langle V, U \rangle \\ \Rightarrow & \langle U, V \rangle = \sum_i w_i U_i V_i \\ & = \sum_i w_i V_i U_i \\ & = \boxed{\langle V, U \rangle} \end{aligned}$$

$$\begin{aligned} (ii) \quad & \langle U, V + W \rangle \\ & = \sum_i w_i U_i (V_i + W_i) \\ & = \sum_i w_i U_i V_i + \sum_i w_i U_i W_i \\ & = \boxed{\langle U, V \rangle + \langle U, W \rangle} \end{aligned}$$

$$\begin{aligned} (iii) \quad & \langle cU, V \rangle = c \langle U, V \rangle \\ \langle cU, V \rangle & = \sum_i c w_i U_i V_i \\ & = c \sum_i w_i U_i V_i \\ & = \boxed{c \langle U, V \rangle} \end{aligned}$$

$$\begin{aligned} (iv) \quad & \langle U, U \rangle \geq 0 \\ \langle U, U \rangle & = \sum_i w_i U_i U_i \\ & = \sum_i w_i (U_i)^2 \\ & \quad (U_i)^2 \geq 0 \quad w_i \geq 0 \\ & \therefore \sum_i w_i U_i^2 \geq 0 \\ & \boxed{\therefore \langle U, U \rangle \geq 0} \end{aligned}$$

Orthogonal Projections: (Pg 562)

An orthogonal set of vectors in an inner product space V is a set $\{v_1, \dots, v_k\}$ of vectors from V such that $\langle v_i, v_j \rangle = 0$ whenever, $v_i \neq v_j$.

A set of vectors is an orthonormal set of vectors if it is an orthogonal set of unit vectors.

Theorem: (Pg: 396)

Let $\{v_1, \dots, v_k\}$ be an orthogonal basis for a subspace W of R^n and w be any vector of W . Then there are unique scalars c_1, c_2, \dots, c_k

such that,

$$w = \sum_{i=1}^k c_i v_i$$

where
$$c_i = \frac{w \cdot v_i}{v_i \cdot v_i} \quad \text{for } i = 1, 2, \dots, k$$

Proof:

Since, $\{v_1, \dots, v_k\}$ is a basis of W ,

Find scalars c_1, \dots, c_k such that $w = \sum_{i=1}^k c_i v_i$

$$w \cdot v_i = (c_1 v_1 + \dots + c_k v_k) \cdot v_i$$

$$= c_1 (v_1 \cdot v_i) + \dots + c_i (v_i \cdot v_i) + \dots + c_k (v_k \cdot v_i)$$

$$= c_i (v_i \cdot v_i)$$

If inner product is defined in the question, then

$$c_i = \frac{\langle w, v_i \rangle}{\langle v_i, v_i \rangle}$$

$$\therefore c_i = \frac{w \cdot v_i}{v_i \cdot v_i}$$

Orthogonal Projection:

→ We can define the orthogonal projection $\text{proj}_W(v)$ of a vector v onto a subspace W of an inner product space.

If $\{v_1, \dots, v_k\}$ is an orthogonal basis for W , then

$$\boxed{\text{proj}_W(v) = \frac{\langle v_1, v \rangle}{\langle v_1, v_1 \rangle} v_1 + \dots + \frac{\langle v_k, v \rangle}{\langle v_k, v_k \rangle} v_k}$$

Then the component of v orthogonal to W is the vector:

$$\boxed{\text{perp}_W(v) = v - \text{proj}_W(v)}$$

Ques: $\langle f, g \rangle = \int_a^b f(x)g(x) dx$

Prove that this is an inner product.

(1) $\langle v, v \rangle = \langle v, v \rangle$

$$\therefore \int_a^b f(x)g(x) dx = \int_a^b g(x)f(x) dx \quad \underline{\text{True}}$$

(2) $\langle v, v+w \rangle = \langle v, v \rangle + \langle v, w \rangle$

(3) $\langle cv, v \rangle = c \langle v, v \rangle \quad \int_a^b (cf(x))g(x) dx = c \int_a^b f(x)g(x) dx$

(4) $\langle v, v \rangle \geq 0 \quad \langle v, v \rangle = 0 \quad \text{iff} \quad v = 0 \quad \int_a^b f(x)^2 dx \geq 0$

Q) Find an orthogonal basis for the subspace W of \mathbb{R}^3 given by

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x - y + 2z = 0 \right\}$$

Sol:

$$\rightarrow x - y + 2z = 0 \quad \therefore W = \left\{ \begin{pmatrix} y-2z \\ y \\ z \end{pmatrix} \right\} \quad y, z \in \mathbb{R}$$

$$x = y - 2z$$

$w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $w_2 = \begin{pmatrix} y-2z \\ y \\ z \end{pmatrix}$ $\therefore w_2 = \begin{pmatrix} -y \\ y \\ y \end{pmatrix}$	As W is orthogonal, $w_1 \cdot w_2 = 0$ $\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} y-2z \\ y \\ z \end{pmatrix} \right) = 0$ $\therefore y-2z + y = 0$ $\boxed{y=2z}$	Let $y=1$ $\therefore w_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ $(w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix})$ $W_b = \{w_1, w_2\}$ ↳ Orthogonal Basis of Subspace W .
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* The Gram-Schmidt Process:

- We want to find an orthogonal basis for a subspace W of \mathbb{R}^n .
- The idea is to begin with a arbitrary basis $\{n_1, \dots, n_k\}$ and to "orthogonalize" it one vector at a time.
- For example:

let $W = \text{span}(n_1, n_2)$, where

$$n_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } n_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Construct an orthogonal basis for W .

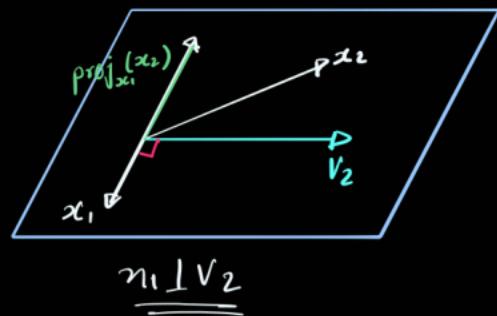
Solⁿ: Starting with x_1 , we get a second vector that is orthogonal to it by taking the component of x_2 orthogonal to x_1 .

Constructing v_2 orthogonal to n_1 .

Algebraically, we set $v_1 = n_1$, so

$$\begin{aligned} v_2 &= \text{proj}_{n_1}(x_2) = n_2 - \text{proj}_{n_1}(x_2) \\ &= x_2 - \left(\frac{n_1 \cdot x_2}{n_1 \cdot n_1} \right) n_1 \\ &= \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} - \left(\frac{-2}{2} \right) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

$$v_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$



$$\underline{n_1 \perp v_2}$$

Hence, $\{v_1, v_2\}$ is an orthogonal set of vectors in W .
So, it is linearly independent set.

$\rightarrow \{v_1, v_2\}$ is a basis for W , since $\dim W = 2$.

NOTE:

If we had taken $v_1 = x_2$, we would have obtained a different set of orthogonal vectors in W .

※ Theorem: The Gram-Schmidt Process

Let $\{x_1, \dots, x_n\}$ be a basis for a subspace W of R^n and define the following:

$$v_1 = x_1 \quad W_1 = \text{span}(v_1)$$

$$v_2 = x_2 - \text{proj}_{v_1}(x_2) \quad W_2 = \text{span}(v_1, v_2)$$

$$v_3 = x_3 - \text{proj}_{v_1}(x_3) - \text{proj}_{v_2}(x_3) \quad W_3 = \text{span}(v_1, v_2, v_3)$$

⋮

$$v_n = x_n - \text{proj}_{v_1}(x_n) - \dots - \text{proj}_{v_{n-1}}(x_n) \quad W_n = \text{span}(v_1, \dots, v_n)$$

Then, for each $i \in \{1, \dots, n\}$, $\{v_1, \dots, v_i\}$ is an orthogonal basis for W_i . In particular $\{v_1, \dots, v_n\}$ is an orthogonal basis for W .



Proof: Proof by Induction:

→ let $v_1 = x_i$,

Clearly, $\{v_1\}$ is an orthogonal basis for $W_i = \text{span}(x_i)$.

→ Suppose,

for some $i < k$,

$\{v_1, \dots, v_i\}$ is an orthogonal basis for W_i .

Then,

$$\begin{aligned} v_{i+1} &= x_{i+1} - \text{proj}_{v_1}(x_{i+1}) - \dots - \text{proj}_{v_i}(x_{i+1}) \\ &= x_{i+1} - \left(\frac{v_1 \cdot x_{i+1}}{v_1 \cdot v_1} \right) v_1 - \dots - \left(\frac{v_i \cdot x_{i+1}}{v_i \cdot v_i} \right) v_i \end{aligned}$$

By induction hypothesis,

$\{v_1, \dots, v_i\}$ is an orthogonal basis for $\text{span}(x_1, \dots, x_i) = W_i$

Hence,
$$v_{i+1} = x_{i+1} - \text{proj}_{W_i}(x_{i+1}) = \text{perp}_{W_i}(x_{i+1})$$

→ By orthogonal Decomposition Theorem, v_{i+1} is orthogonal to W_i . —①

By definition, v_1, v_2, \dots, v_i are linear combinations of x_1, \dots, x_i and hence, are in W_i . —②

Also,

$v_{i+1} \neq 0$, since otherwise $x_{i+1} = \text{proj}_{W_i}(x_{i+1})$ implies x_{i+1} is in W_i .

But, since $W_i = \text{span}(x_1, \dots, x_i)$, $\{x_1, \dots, x_i\}$
this is impossible.

→ Hence, from ① and ②,

we conclude that, $\{v_1, \dots, v_{i+1}\}$ is a set of linearly independent vectors in W_{i+1} .

Since, $\dim W_{i+1} = i+1$,

$\{v_1, \dots, v_{i+1}\}$ is a basis for W_{i+1} .

Hence, Proved.

Eg: Apply the Gram-Schmidt Process to construct an orthogonal basis for the subspace $W = \text{span}(n_1, n_2, n_3)$ of \mathbb{R}^4 .

where $n_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $x_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}$

Sol: Let $v_1 = x_1$,

Now, we compute component of x_2 orthogonal to $W = \text{span}(v_1) = V_1$

$$V_2 = \text{perp}_{W_1}(x_2)$$

$$= x_2 - \text{proj}_{W_1}(x_2)$$

$$V_2 = x_2 - \left(\frac{V_1 \cdot n_2}{V_1 \cdot V_1} \right) V_1 \quad \begin{bmatrix} n_1 \cdot n_2 = 2 - 1 + 1 = 2 \\ n_1 \cdot n_1 = 1 + 1 + 1 + 1 = 4 \end{bmatrix}$$

$$= n_2 - \left(\frac{2}{4} \right) V_1$$

$$= x_2 - P_2(V_1) = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} - P_2 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\boxed{V_2 = \begin{pmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{pmatrix}}$$

For easy calculation,
⇒ "scale" V_2
∴ let $V_2 = 2 \begin{pmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{pmatrix}$

Let $W_2 = \text{span}(V_1, V_2)$

$$\therefore V_3 = \text{perp}_{W_2}(n_3)$$

$$= x_3 - \text{proj}_{W_2}(n_3)$$

$$= x_3 - \left(\frac{V_1 \cdot x_3}{V_1 \cdot V_1} \right) V_1 - \left(\frac{V_2 \cdot x_3}{V_2 \cdot V_2} \right) V_2$$

$$\boxed{V_2 = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}}$$

$$\left| \begin{array}{l} V_1 \cdot x_3 = 1 \\ V_1 \cdot V_1 = 4 \\ V_2 \cdot x_3 = 15 \\ V_2 \cdot V_2 = 20 \end{array} \right.$$

$$\begin{aligned}
 V_3 &= xC_3 - \left(\frac{1}{4}\right) \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} - \left(\frac{15}{20}\right) \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} - \frac{3}{4} \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} 2 - \frac{1}{4} - \frac{9}{4} \\ 2 + \frac{1}{4} - \frac{9}{4} \\ 1 + \frac{1}{4} - \frac{3}{4} \\ 2 - \frac{1}{4} - \frac{3}{4} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{pmatrix} \quad \left\{ V_1, V_2, V_3 \right\} \\
 &\quad = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}
 \end{aligned}$$

"Scale" $V_3 \Rightarrow$ let $\boxed{V_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}}$

 Q: Construct an orthogonal basis for P_2 with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^1 f(n)g(n) dn$$

by applying the Gram-Schmidt Process to the basis $\{1, x, x^2\}$.

Solⁿ: Let $n_1 = 1$
 $n_2 = n$ & $n_3 = x^2$. Also, let $v_1 = x$,

$$\begin{aligned}
 \therefore \langle v_1, v_1 \rangle &= \int_{-1}^1 1^2 dn = \boxed{2} \\
 \therefore \langle v_1, n_2 \rangle &= \int_{-1}^1 (1)(n) dn \\
 &= \left[\frac{n^2}{2} \right]_{-1}^1 = \frac{1}{2} - \frac{-1}{2} = \boxed{0} \\
 \therefore v_2 &= n_2 - \left(\frac{\langle v_1, n_2 \rangle}{\langle v_1, v_1 \rangle} \right) v_1 \\
 &= n_2 - 0 = n_2 = x^2 \\
 \therefore v_2 &= x^2
 \end{aligned}$$

$$\begin{aligned}
 \therefore V_3 &= xC_3 - \left(\frac{\langle v_1, n_3 \rangle}{\langle v_1, v_1 \rangle} \right) v_1 - \left(\frac{\langle v_2, n_3 \rangle}{\langle v_2, v_2 \rangle} \right) v_2 \\
 \text{where, } \quad \langle v_2, v_2 \rangle &= \int_{-1}^1 n^2 dn = \left[\frac{n^3}{3} \right]_{-1}^1 = \boxed{\frac{2}{3}} \\
 \langle v_1, n_3 \rangle &= \int_{-1}^1 (1)(x^2) dn \\
 &= \left[\frac{x^3}{3} \right]_{-1}^1 = \boxed{\frac{2}{3}} \\
 \langle v_2, n_3 \rangle &= \int_{-1}^1 (n)(x^2) dn = \boxed{0} \\
 \therefore V_3 &= x^2 - \left(\frac{\frac{2}{3}}{2} \right)(1) - \left(\frac{0}{\frac{2}{3}} \right)(n) \\
 &= \boxed{x^2 - \frac{1}{3}}
 \end{aligned}$$

\therefore Orthogonal Basis = $\{1, n, n^2 - \frac{1}{3}\}$.

The Cauchy-Schwarz Inequality:

→ let u and v be vectors in an inner product space V . Then

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

with equality holding if and only if u and v are scalar multiples of each other.

Proof: → If $u=0$, then the inequality is actually an equality, since

$$|\langle 0, v \rangle| = 0 = \|0\| \|v\|$$

→ If $u \neq 0$, then let W be the subspace of V spanned by u . Since $\text{proj}_W^{(u)} = \left(\frac{\langle u, v \rangle}{\langle u, u \rangle} \right) u$ and $\text{perp}_W^{(u)} = v - \text{proj}_W^{(u)}$ are orthogonal, we can apply Pythagoras theorem,

$$\begin{aligned} \|v\|^2 &= \|\text{proj}_W^{(u)} + (v - \text{proj}_W^{(u)})\|^2 \\ &= \|\text{proj}_W^{(u)} + \text{perp}_W^{(u)}\|^2 \\ &= \|\text{proj}_W^{(u)}\|^2 + \|\text{perp}_W^{(u)}\|^2 \end{aligned}$$

It follows that

$$\begin{aligned} \|v\|^2 &\geq \|\text{proj}_W^{(u)}\|^2 \\ \text{Now, } \|\text{proj}_W^{(u)}\|^2 &= \left\langle \left(\frac{\langle u, v \rangle}{\langle u, u \rangle} \right) u, \left(\frac{\langle u, v \rangle}{\langle u, u \rangle} \right) u \right\rangle = \left(\frac{\langle u, v \rangle}{\langle u, u \rangle} \right)^2 \langle u, v \rangle \\ &= \frac{\langle u, v \rangle^2}{\langle u, u \rangle} \\ &= \boxed{\frac{\langle u, v \rangle^2}{\|u\|^2}} \end{aligned}$$

So, we have,

$$\frac{\langle u, v \rangle^2}{\|u\|^2} \leq \|v\|^2 \Rightarrow \langle u, v \rangle^2 \leq \|u\|^2 \|v\|^2$$

$$\therefore |\langle u, v \rangle| \leq \|u\| \|v\|$$

Equality: iff $\|v\|^2 = \|\text{proj}_W^{(u)}\|^2$, which is true only when $v = \text{proj}_W^{(u)}$.

∴ If this is so, then v is a scalar multiple of u .

Conversely, if $v=cu$, $\Rightarrow \text{perp}_W^{(u)} = v - \text{proj}_W^{(u)} = cu - \left(\frac{\langle u, cu \rangle}{\langle u, u \rangle} \right) u = cu - cu = \boxed{0}$

∴ Equality holds.

* Triangular Inequality Theorem:

$$\|u+v\| \leq \|u\| + \|v\|$$

Proof: LHS = $\|u+v\| = \sqrt{\langle u+v, u+v \rangle}$

$$\|u+v\|^2 = \langle u+v, u+v \rangle$$

$$= \langle u+v, u \rangle + \langle u+v, v \rangle$$

$$= \underbrace{\langle u, u \rangle}_{=\|u\|^2} + \underbrace{\langle u, v \rangle}_{\leq \|u\| \|v\|} + \underbrace{\langle v, u \rangle}_{\leq \|v\| \|u\|} + \underbrace{\langle v, v \rangle}_{=\|v\|^2}$$

$$\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$$

$$\|u+v\|^2 \leq (\|u\| + \|v\|)^2$$

$\therefore \boxed{\|u+v\| \leq \|u\| + \|v\|}$

* The Orthogonal Decomposition Theorem:

Let W be a subspace of R^n and let v be a vector in R^n . Then there are unique vectors w in W and w^\perp in W^\perp such that

$$\boxed{v = w + w^\perp}$$

Proof:

- To Prove: 1) Such a decomposition exist.
2) It is unique.

Part 1: Let $\{v_1, \dots, v_k\}$ be an orthogonal basis for W .

Let $w = \text{proj}_W(v)$ and,

let $w^\perp = \text{perp}_W(v)$

$$\begin{aligned} \text{Then, } w + w^\perp &= \text{proj}_W(v) + \text{perp}_W(v) \\ &= \boxed{v} \end{aligned}$$

→ Clearly,

$w = \text{proj}_W(v)$ is in W , since it is a linear combination of basis vectors v_1, \dots, v_k .

→ To show that $w^\perp \in W^\perp$,

it is enough to show that w^\perp is orthogonal to each of the basis vector v_i .

$$\begin{aligned} \rightarrow v_i \cdot w^\perp &= v_i \cdot \text{perp}_W(v) \\ &= v_i \cdot (v - \text{proj}_W(v)) \end{aligned}$$

$$\begin{aligned} &= v_i \cdot \left(v - \left(\frac{v_i \cdot v}{v_i \cdot v_i} \right) v_i - \left(\frac{v_2 \cdot v}{v_2 \cdot v_2} \right) v_2 - \dots - \left(\frac{v_k \cdot v}{v_k \cdot v_k} \right) v_k \right) \\ &= v_i \cdot v - \left(\frac{v_i \cdot v}{v_i \cdot v_i} \right) (v_i \cdot v_i) - \dots - \left(\frac{v_i \cdot v}{v_i \cdot v_i} \right) (v_i \cdot v_i) - \dots - \left(\frac{v_k \cdot v}{v_k \cdot v_k} \right) (v_k \cdot v_i) \end{aligned}$$

Since, $v_i \cdot v_j = 0 \quad \forall \quad v_i \neq v_j$,

$$\therefore v_i \cdot w^\perp = v_i \cdot v - 0 - \left(\frac{v_i \cdot v}{v_i \cdot v_i} \right) (v_i \cdot v_i)$$

$$= v_i \cdot v - v_i \cdot v =$$

$$\boxed{\therefore v_i \cdot w^\perp = 0 \quad \forall i}$$

w^\perp is in W^\perp

2) Uniqueness:

Suppose, we have another de-composition $v = w_i + w_i^\perp$, where $w_i \in W$ and $w_i^\perp \in W^\perp$. Then $w + w^\perp = w_i + w_i^\perp$

$$\therefore w - w_i = w_i^\perp - w^\perp$$

Since, $w - w_i \in W$ and $w_i^\perp - w^\perp \in W^\perp$,

we know that this common vector $\in W \cap W^\perp = \{0\}$

$$\therefore w - w_i = 0 \quad \text{and} \quad w_i^\perp - w^\perp = 0$$

$$\therefore \boxed{w = w_i} \quad \text{and} \quad \boxed{w^\perp = w_i^\perp}$$

Orthogonal Components: (4)

Let W be a subspace of \mathbb{R}^n . We say that a vector v in \mathbb{R}^n is orthogonal to W if v is orthogonal to every vector in W . The set of all vectors that are orthogonal to W is called the orthogonal complement of W , denoted by W^\perp .

That is,

$$W^\perp = \{v \in \mathbb{R}^n : v \cdot w = 0 \text{ } \forall w \in W\}$$

Theorem: Let W be a subspace of \mathbb{R}^n .

(a) a) W^\perp is a subspace of \mathbb{R}^n .

$$b) (W^\perp)^\perp = W$$

$$c) W \cap W^\perp = \{0\}$$

d) If $W = \text{span}(w_1, \dots, w_k)$, then

Proof:

a) $\rightarrow W^\perp$ is non-empty

$$0 \cdot w = 0 \quad \forall w \in W$$

$$[0 \in W^\perp] \quad (i)$$

$$\text{Let } u, v \in W^\perp$$

$$u \cdot w = v \cdot w = 0 \quad \forall w \in W$$

$$(u+v) \cdot w = u \cdot w + v \cdot w = 0 + 0 = 0 \quad \forall w \in W$$

$$[\therefore u+v \in W^\perp] \quad (ii)$$

$$(cu) \cdot w = c(u \cdot w) = c(0) = 0 \quad \forall w \in W$$

$$[\therefore cu \in W^\perp] \quad (iii)$$

From (i), (ii) & (iii) W^\perp is a subspace.

b) let $w \in W$ and $x \in W^\perp$,

then $w \cdot x = 0$

this implies that, $w \in (W^\perp)^\perp$

$$\boxed{\therefore w \in (W^\perp)^\perp} \rightarrow \textcircled{1}$$

let $v \in (W^\perp)^\perp$

By, orthogonal decomposition theorem,

$V = W + W^\perp$ for unique vector w in W & w^\perp in W^\perp

$$v \cdot w^\perp = 0$$

$$\begin{aligned} \therefore (w + w^\perp) \cdot w^\perp &= w \cdot w^\perp + w^\perp \cdot w^\perp \\ &= 0 + w^\perp \cdot w^\perp = 0 \\ \therefore \boxed{w^\perp = 0} \end{aligned}$$

$$\therefore V = w + w^\perp = w + 0$$

$$\boxed{V = w}$$

$$\therefore \boxed{v \in W}$$

$$\boxed{\therefore (W^\perp)^\perp \subseteq W} \rightarrow \textcircled{2}$$

From \textcircled{1} and \textcircled{2} $\boxed{(W^\perp)^\perp = W}$

*Theorem:

The columns of an $m \times n$ matrix Q forms an orthonormal set iff $Q^T Q = I_n$.

Proof:

To show $Q^T Q = I_n$,

$$(Q^T Q)_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Let q_i be the i^{th} column of Q (or i^{th} row of Q^T).

$$(Q^T Q)_{ij} = (\text{i^{th} row of Q^T}) \cdot (\text{j^{th} column of Q})$$

$$\boxed{(Q^T Q)_{ij} = q_i \cdot q_j} \rightarrow ①$$

Now, the columns Q form an orthogonal set iff

$$q_i \cdot q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

which by equation ① holds iff,

$$\boxed{(Q^T Q)_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}}$$

*Orthogonal Matrices:

An $n \times n$ matrix Q whose columns form an orthonormal set, is called orthogonal matrix.

Orthonormal:

A set of orthogonal unit vectors.

Qve: $Q = \begin{pmatrix} A & B & C \\ 4\sqrt{2} & \frac{1}{\sqrt{3}} & x \\ 0 & \frac{1}{\sqrt{3}} & y \\ -4\sqrt{2} & \frac{1}{\sqrt{3}} & z \end{pmatrix}$

For what values of x, y, z , Q is orthogonal matrix.

Sol: $\|c\| = 1$

$$\boxed{x^2 + y^2 + z^2 = 1} \rightarrow (1)$$

$$\langle A, c \rangle = 0$$

$$4\sqrt{2}x - \frac{1}{\sqrt{3}}z = 0 \Rightarrow \boxed{x = \frac{z}{4\sqrt{6}}} \rightarrow (2)$$

$$\langle B, c \rangle = 0$$

$$0 + y + z = 0 \Rightarrow \boxed{y = -z} \rightarrow (3)$$

From (1), (2) & (3),

$$x^2 + 4z^2 + z^2 = 6z^2 = 1$$

$$\boxed{z = \pm \frac{1}{\sqrt{6}}}$$

$$\therefore (x, y, z) = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right)$$

or

$$\left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right)$$

Theorem: A square matrix Q is orthogonal iff $Q^{-1} = Q^T$.
(399)

Proof:

QR factorization :

→ Let A be a $m \times n$ matrix with linearly independent columns.

Then A can be factorized as $A = QR$ where Q is an $m \times n$ matrix with orthogonal column and R is invertible uppertriangular matrix.

→ Let a_1, a_2, \dots, a_n be the linearly independent columns of A and let q_1, q_2, \dots, q_n be the orthogonal vectors obtained by applying Gram-Schmidt process to $\{a_1, a_2, \dots, a_n\}$ with normalization.

$$w_i = \text{span} \{a_1, a_2, \dots, a_i\}$$

$$= \text{span} \{q_1, q_2, \dots, q_i\}$$

∴ there are scalars $r_{11}, r_{21}, \dots, r_{ii}$ such that

$$a_1 = r_{11}q_1$$

$$a_2 = r_{12}q_1 + r_{22}q_2$$

$$a_3 = r_{13}q_1 + r_{23}q_2 + r_{33}q_3$$

⋮

$$a_n = r_{1n}q_1 + r_{2n}q_2 + \dots + r_{nn}q_n$$

$$\left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \boxed{a_i = r_{1i}q_1 + r_{2i}q_2 + \dots + r_{ii}q_i}$$

which can be written in matrix form as

$$A = [a_1 \ a_2 \ \dots \ a_n] = [q_1 \ q_2 \ \dots \ q_n] \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \dots & r_{nn} \end{pmatrix} = QR$$

$$\therefore A = QR$$

As Q is an orthogonal set,

$$Q^{-1} = Q^T$$

$$\boxed{\therefore Q^T A = R}$$

Eg: Find QR factorization of

$$A = \begin{pmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Solⁿ: The orthonormal basis for $\text{col}(A)$ produced by Gram-Schmidt

process is:

$$q_1 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}, q_2 = \begin{pmatrix} 3\sqrt{5}/10 \\ 3\sqrt{5}/10 \\ \sqrt{5}/10 \\ \sqrt{5}/10 \end{pmatrix}, q_3 = \begin{pmatrix} -\sqrt{6}/6 \\ 0 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \end{pmatrix}$$

$$\therefore Q = \begin{pmatrix} 1/2 & 3\sqrt{5}/10 & -\sqrt{6}/6 \\ -1/2 & 3\sqrt{5}/10 & 0 \\ -1/2 & \sqrt{5}/10 & \sqrt{6}/6 \\ 1/2 & \sqrt{5}/10 & \sqrt{6}/3 \end{pmatrix}$$

$$A = QR$$

$$\therefore Q^{-1}A = QQ^{-1}R = R$$

As, Q is an orthogonal matrix,

$$Q^{-1} = Q^T$$

$$\therefore Q^{-1} = \begin{pmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 3\sqrt{5}/10 & 3\sqrt{5}/10 & \sqrt{5}/10 & \sqrt{5}/10 \\ -\sqrt{6}/6 & 0 & \sqrt{6}/6 & \sqrt{6}/3 \end{pmatrix}$$

$$\therefore R = Q^{-1}A = \begin{pmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 3\sqrt{5}/10 & 3\sqrt{5}/10 & \sqrt{5}/10 & \sqrt{5}/10 \\ -\sqrt{6}/6 & 0 & \sqrt{6}/6 & \sqrt{6}/3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 1 & 1/2 \\ 0 & \sqrt{5} & 3\sqrt{5}/2 \\ 0 & 0 & \sqrt{6}/2 \end{pmatrix}}$$

Eigenvalues & Eigenvectors:

Definition: Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue of A if there is a nonzero vector x such that $Ax = \lambda x$. Such a vector x is called an eigen-vector of A corresponding to λ .

Eg: Show that $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and find the corresponding eigenvalue.

$$\text{Soln: } Ax = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \boxed{\lambda = 4}$$

Eg: Show that 5 is an eigenvalue of $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ and determine all eigenvectors corresponding to this eigenvalue.

Eigenspace:

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A . The collection of all eigenvectors corresponding to λ , together with the zero vector is called the eigenspace of λ and is denoted by ℓ_λ .

Eigenvalues and Eigenvectors of $n \times n$ Matrices:

→ The eigenvalues of a square matrix A are precisely the solution λ of the equation $\det(A - \lambda I) = 0$.

$$\begin{aligned} Ax &= \lambda x \\ Ax - \lambda x &= 0 \\ \boxed{(A - \lambda)x} &= 0 \end{aligned}$$

→ When we expand $\det(A - \lambda I)$, we get a polynomial in λ , called the characteristic polynomial of A .

→ The equation $\det(A - \lambda I) = 0$ is called the characteristic equation.

Steps to find the Eigenvalues and eigenvectors of a matrix:

Step 1: Compute the characteristic polynomial $\det(A - \lambda I)$ of A .

Step 2: Find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$ for λ .

Step 3: For each eigenvalue λ , find the null space of the matrix $A - \lambda I$. This is the eigenspace ℓ_λ , the nonzero vectors of which are the eigenvectors of A corresponding to λ .

Step 4: Find a basis for each eigenspace.

Eg: Find the eigenvalues and the corresponding eigenspaces of

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$$

Solⁿ: Let $\lambda \rightarrow$ eigenvalue

Characteristic equation: $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{pmatrix}$$

$$\rightarrow \det(A - \lambda I) = 0$$

$$\therefore -\lambda((\lambda-4)(\lambda)+5) - 1(-2)$$

$$\therefore -\lambda(\lambda^2 - 4\lambda + 5) + 2$$

$$\therefore -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0$$

$$\therefore \boxed{\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0}$$

$$\therefore \lambda^3 - \lambda^2 - 3\lambda^2 + 3\lambda + 2\lambda - 2 = 0$$

$$\therefore (\lambda-1)(\lambda^2 - 3\lambda + 2) = 0$$

$$\therefore (\lambda-1)(\lambda-1)(\lambda-2) = 0$$

$$\therefore \boxed{\lambda = 1, 2}$$

For $\lambda = 2$,

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 2 & -5 & 2 \end{pmatrix} \xrightarrow[R_3 \rightarrow R_3 + R_1]{R_1 \rightarrow \frac{1}{2}R_1} \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -4 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -4 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xleftarrow[R_3 \rightarrow R_3 - 2R_2]{\text{Then, } R_1 \rightarrow R_1 - 4R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\downarrow R_2 \rightarrow -4R_2 \quad \left\{ \begin{array}{l} n_1 - 4n_2 n_3 = 0 \\ 4x_1 = x_3 \end{array} \right.$$

$$\boxed{\begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$\left\{ \begin{array}{l} n_2 - 4n_3 = 0 \\ 2x_2 = x_3 \end{array} \right.$$

$$\left. \begin{array}{l} x = t \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \\ \boxed{x = t \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}} \end{array} \right\}$$

For $\lambda = 1$,

$$(A - \lambda I)x = 0$$

$$\text{Let } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{pmatrix}$$

$$\xrightarrow[R_1 \rightarrow -R_1]{R_1 \rightarrow -R_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\therefore n_1 - n_3 = 0$$

$$n_2 - n_3 = 0$$

$$\boxed{n_1 = n_2 = x_3}$$

$$\therefore x = t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\xrightarrow[R_3 \rightarrow R_3 - 2R_1]{R_2 \rightarrow -R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 3 \end{pmatrix} \xrightarrow[R_1 \rightarrow R_1 + R_2]{\text{Then, } R_3 \rightarrow R_3 + 3R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

NOTE:

→ Algebraic Multiplicity:

It is multiplicity of an eigenvalue as a root of the characteristic equation.

→ Geometric Multiplicity of an eigenvalue λ is defined as the dimension of its corresponding eigenspace.

Theorem:

→ A square matrix A is invertible iff 0 is not an eigenvalue of A .

Proof: We know that, a square matrix is invertible iff $\det A \neq 0$.

(\Rightarrow) Square matrix is invertible.

$$\therefore \det A \neq 0$$

$$\therefore \det(A - 0I) = \boxed{\det(A)} \neq 0$$

$\therefore \lambda$ is not an eigen value.

(\Leftarrow) Suppose zero is not an eigenvalue,

$$\therefore \det(A - 0I) \neq 0$$

$$\therefore \det(A) \neq 0$$

$\therefore A$ is invertible.

Theorem: Let A be a square matrix with eigenvalue λ and the corresponding eigen-vector x .

a) for any positive integer n , λ^n is an eigenvalue of A^n with corresponding eigenvector x .

b) If A is invertible, then λ^{-1} is an eigenvalue of A^{-1} with corresponding eigenvector x .

c) If A is invertible, then for any integer n , λ^n is an eigenvalue of A^n with corresponding eigenvector x .

Proof:

Eg: Compute $\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}^10 \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

Soln: Let $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ and $x = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

We need to find $A^{10}x$.

→ The eigenvalues of A are:

$$\begin{aligned} \det(A - \lambda I) &= 0 && \text{Computing eigen vectors:} \\ \therefore \begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} &= 0 && \text{For } \lambda = -1, AV_1 = -V_1 \\ (\lambda-1)\lambda - 2 &= 0 && \text{and for } \lambda = 2, AV_2 = 2V_2 \\ \therefore \lambda^2 - \lambda - 2 &= 0 \\ \boxed{\lambda = -1, 2} \end{aligned}$$

For $\lambda = -1$,

$$A - \lambda I = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad n_1 + n_2 = 0$$

$$\therefore n_1 = n_2 = 0$$

$$\therefore v_1 = t_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda = 2$,

$$A - \lambda I = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\therefore \begin{cases} -2n_1 + n_2 = 0 \\ n_2 = 2n_1 \end{cases} \Rightarrow v_2 = t_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

\rightarrow Using the previous theorem(c),

if eigenvalue of A is λ with corresponding eigenvector being x , for the same eigenvector, eigenvalue of A^{10} will be λ^{10} (if A is invertible).

\rightarrow Now, let $c_1, c_2 \in \mathbb{R}$ s.t.,

$$x = c_1 v_1 + c_2 v_2$$

$$\therefore \begin{pmatrix} 5 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{aligned} \therefore c_1 + c_2 &= 5 \\ \underline{c_1 + 2c_2 &= 1} \\ \underline{3c_2 &= 6} \\ \underline{c_2 &= 2} \end{aligned} \Rightarrow c_1 = 3$$

$$\begin{aligned} \therefore x &= 3v_1 + 2v_2 \\ \therefore A^{10}x &= 3A^{10}v_1 + 2A^{10}v_2 \\ \text{Using that theorem,} \\ A^{10}v_1 &= (\lambda_1)^{10}v_1 \\ &= (-1)^{10}v_1 = v_1 \\ A^{10}v_2 &= (\lambda_2)^{10}v_2 \\ &= (2)^{10}v_2 \\ &= 1024v_2 \end{aligned}$$

$$\begin{aligned} \therefore A^{10}x &= 3v_1 + 2(1024)v_2 \\ &= 3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2048 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2051 \\ 4093 \end{pmatrix} \end{aligned}$$