PART - 2

Linear Algebra

SHIVANSH SANTOKI

5% → TUT-QUIZ
5% → ASSIGNMENTS
10% → QUIZ-2
30% → ENDSEM

MATRICES :

→ Equality of 2 Matrices :

A & B are said to be equal if they have the same order and their corresponding elements are equal.

i.e. $A = [a_{ij}]_{m\times n}$ $\quad a_{ij} = b_{ij} \ \forall\ i,j$

$B = [b_{ij}]_{m\times n}$

TYPES OF MATRICES :

1) Square Matrix $\quad m = n$ (Rows = Column)
2) Zero Matrix $\quad a_{ij} = 0 \ \forall\ i,j$
3) Diagonal Matrix $\quad a_{ij} = 0 \ \forall\ i \neq j$
   - Scalar Matrix $\quad a_{ij} = a \ ; \ a \neq 0$
     (All diagonal elements are same)
   - Identity Matrix $\quad a_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$
4) Upper Triangular Matrix : $\boxed{a_{ij} = 0 \text{ if } i > j}$
5) Lower Triangular Matrix : $\boxed{a_{ij} = 0 \text{ if } i < j}$

MATRIX OPERATION :

1) Addition of Matrices
2) Multiplication of Matrices
3) Transpose of Matrix

1) For addition $\longrightarrow$ same order $m \times n$

$C = A + B$

$C_{ij} = [a_{ij} + b_{ij}] = [a_{ij}] + [b_{ij}]$

Properties:

$(A+B) = (B+A)$

$(A+B)+C = A + (B+C)$

$k(lA) = (kl)A$

$(k+l)A = kA + lA$

2) Matrix Multiplication:

$A = [a_{ij}]_{m \times n}$ → n(column) of A

$B = [b_{ij}]_{n \times r}$ → n(Row) of B

$C = AB$

$= [c_{ij}]_{m \times r}$

$$\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}}$$

Properties:

(i) $A(BC) = (AB)C$

(ii) $A(B+C) = AB + AC$

(iii) $k(AB) = (kA)B = A(kB)$

(iv) $I_n A = A I_n = A$

where A is $n \times n$ matrix

$I_n = n \times n$ Identity matrix

(Multiplicative Identity)

Q. If A, B are square matrices of same size, $(A+B)^2 = A^2 + 2AB + B^2$?

No, $(A+B)^2 = A^2 + AB + BA + B^2$

Proof of $A(B+C)$:

$[A(B+C)]_{ij} = a_i \cdot (b_j + c_j)$

$= a_i \cdot b_j + a_i \cdot c_j = (AB)_{ij} + (AC)_{ij} = (AB+AC)_{ij}$

$\therefore \boxed{A(B+C) = AB + AC}$

Inverse of a Matrix :

If A is a $n \times n$ matrix, an inverse of A is a $n \times n$ matrix A', such that $AA' = A'A = I$

Eg: $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$$A' = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

→ Zero matrix will not have an inverse.

Result: If $A_{n \times n}$ is an invertible matrix, then the inverse is unique.

Proof: Let A' and A'' be two inverses of A

then,

$$AA' = A'A = I \rightarrow (1)$$

$$AA'' = A''A = I \rightarrow (2) \quad \text{(Associative)}$$

$$A' = A'I = A'(AA'') = (A'A)A''$$

(Identity) From (2)

$$= IA'' = A''$$

From (1)

$$\therefore \boxed{A' = A''}$$

NOTE: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

if $ad \neq bc$.

Properties:

(1) If A is an invertible matrix, A^{-1} is also invertible, then. $(A^{-1})^{-1} = A$.

(2) If A is an invertible matrix, and c is a non-zero scalar, then

$$(cA)^{-1} = \frac{1}{c}A^{-1}$$

(3) If A and B are invertible matrices of same size, then AB is also invertible, and

$$(AB)^{-1} = B^{-1}A^{-1}$$

(4) If A is invertible, then A^n is invertible for all non-negative integer n.

$$(A^n)^{-1} = (A^{-1})^n$$

Proof: (1) Let X be an inverse of A^{-1}.

$$\left.\begin{aligned} \therefore\ XA^{-1} &= A^{-1}X = I \\ \text{and, } AA^{-1} &= A^{-1}A = I \end{aligned}\right\} \boxed{(A^{-1})^{-1} = A}$$

(2) Let X be an inverse of cA.

$$\therefore X(cA) = (cA)X = I \quad \text{....(iii)}$$

$$\text{and, } \left(\tfrac{1}{c}A^{-1}\right)(cA) = \left(\tfrac{1}{c} \times c A^{-1}\right)(A) = \underbrace{A^{-1}A = I}_{(i)}$$

$$(cA)\left(\frac{1}{c}A^{-1}\right) = A\left(c\times\frac{1}{c}A^{-1}\right) = AA^{-1} = I \longrightarrow (iii)$$

From (i), (ii) and (iii),

$$\boxed{X = \frac{1}{c}A^{-1}}$$

(3) Similarly,

let $x \rightarrow$ Inverse of AB

$$\therefore X(AB) = (AB)X = I \text{ ---- (i)}$$

$$\rightarrow (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$

$$= A(IA^{-1}) = AA^{-1} = I \text{ ---- (ii)}$$

$$\rightarrow (B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$$

$$= B^{-1}IB = B^{-1}B = I \text{ ---- (iii)}$$

From (i), (ii) and (iii),

$$\boxed{X = B^{-1}A^{-1}}$$

(4) Let $X \rightarrow$ Inverse of A^n,

To prove: $XA^n = A^nX = I \Rightarrow X = (A^{-1})^n$

$\rightarrow$ Base case: $n=1$

$$AA^{-1} = A^{-1}A = I \quad (\text{True})$$

Suppose it is true for $n=k$,

$$\therefore A^k(A^{-1})^k = (A^{-1})^k A^k = I$$

i) $\therefore A\cdot A^k\cdot(A^{-1})^k\cdot(A^{-1}) = A\cdot I\cdot A^{-1} = A\cdot A^{-1} = I$

$$\therefore \boxed{A^{k+1}\cdot(A^{-1})^{k+1} = I}$$

ii) Similarly, $(A^{-1})^{k+1}\cdot A^{k+1} = I$

$\therefore$ It is true for $k+1$.

Hence, Inductively we proved that

$$A^n \cdot (A^{-1})^n = I \quad \forall \; n \geq 0$$

Elementary Matrices:

→ An elementary matrix is a matrix that can be obtained by performing elementary row operation on identity matrix.

Eg: $E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is equivalent to $3R_2$ on I_4

$E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is equivalent to $R_1 \longleftrightarrow R_3$ on I_4

$E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix}$ is equivalent to $R_4 \to R_4 - 2R_2$ on I_4

※ Inverse of an Elementary Matrix :

$\Rightarrow E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Why?

$$I_3 \xrightarrow{R_{23}} E_1 \xrightarrow{R_{23}} I_3$$

$\Rightarrow E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$I_3 \xrightarrow{R_2 \to 4R_2} E_2 \xrightarrow{R_2 \to 1/4 R_2} I_3$$

$\Rightarrow E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

$$I_3 \xrightarrow{R_3 \to R_3 - 2R_1} E_3 \xrightarrow{R_3 \to R_3 + 2R_1} I_3$$

How to find inverse ?

→ We know that $E = e(I)$

let $X =$ Inverse of E

$\therefore XE = EX = I$

$\therefore Xe(I) = e(I)X = I$

let $X = e^{-1}(I)$

$\therefore Xe(I) = e^{-1}e(I \cdot I)$

$= \boxed{I}$

$\therefore X = e^{-1}(I)$

Hence, if $E = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$e : R_2 \to R_2 + R_1$

$\therefore e^{-1} : R_2 \to R_2 - R_1$

$\boxed{E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$

Theorem:

Let E be an elementary matrix obtained by performing elementary row operation on $I_{n \times n}$.

If the same elementary row operation is done on $n \times n$ matrix A, the resultant is same as EA.

Fundamental Theorem of Invertible Matrices:

(a) A is an invertible matrix

(b) $AX = B$ has a unique solution for any $B \in R$.

(c) $AX = 0$ has only trivial solution

(d) The row reduced echelon form of A is I_n.

(e) A is a product of elementary matrices

$$(a) \longleftrightarrow (b) \longleftrightarrow (c) \longleftrightarrow (d) \longleftrightarrow (e)$$

Proof: To Prove: $(a) \rightarrow (b) \rightarrow (c) \rightarrow (d) \rightarrow (e)$

$(a) \rightarrow (b)$

Given: $AA^{-1} = A^{-1}A = I$

Proving that $AX = B$ has a solution:

Suppose $X = A^{-1}B$ is a solution.

$$\therefore A(A^{-1}B) = (AA^{-1})B = \boxed{B}$$

Now, proving that the solution is unique:

Let y be a solution of $AX = B$

$\therefore Ay = B$

$\therefore A^{-1}(Ay) = A^{-1}B$

$\therefore (A^{-1}A)y = A^{-1}B$

$\therefore Iy = \boxed{y = A^{-1}B}$ Hence, unique.

(b) $\rightarrow$ (c)

Given: $AX = B$ has a unique sol^n.

let $B = 0$

$\therefore AX = 0$ also has a unique sol^n.

A homogeneous system of linear equations always has $X = 0$ as a solution.

As, $AX = 0$ has only one unique sol^n,

$AX = 0$ only has trivial sol^n $(X = 0)$.

(c) $\rightarrow$ (d)

Let R be a row-reduced echelon form of A.

As $AX = 0$ has only $X = 0$ as it's solution,

$RX = 0$ has only trivial sol^n.

Suppose R has r non zero rows.

$\therefore \exists$ $n-r$ free scalars

$\therefore$ there will be infinite solutions,

if $n - r > 0$

$\therefore n - r \leq 0$ As $r \leq n$

$n - r = 0$

$\therefore \boxed{r = n} \Rightarrow \boxed{R = I_n}$

(d) $\rightarrow$ (e)

Given: $R = I_n$

Suppose R is obtained from A after performing n elementary row operations

$\therefore R = e_n e_{n-1} \cdots\cdots e_1(A)$

As, $E_i = e_i(I)$

$R = E_n E_{n-1} \cdots\cdots E_1(A) = I$

As elementary matrices are invertible,

$A = E_1^{-1} E_2^{-1} \cdots\cdots E_n^{-1}(I)$

As product of invertible matrices is invertible,

$\boxed{A = E_1^{-1} E_2^{-1} \cdots E_n^{-1}}$ is a product of invertible matrices.

(e) $\to$ (a)

Suppose $A = E_1 E_2 \cdots E_n$ (Product of n invertible matrices)

As E_i is invertible $\forall\ 1 \le i \le n$

$\exists\ E_i^{-1}$ s.t. $E_i^{-1}E_i = E_i E_i^{-1} = I$

$\therefore A(E_n^{-1}E_{n-1}^{-1} \cdots E_1^{-1}) = I$

let $\boxed{X = E_n^{-1}E_{n-1}^{-1} \cdots E_1^{-1}}$

$\exists\ X$ s.t. $\boxed{AX = I}$

Similarly,

$(E_n^{-1}E_{n-1}^{-1} \cdots E_1^{-1})A = I$

$\therefore \exists\ X$ s.t.

$\boxed{XA = I}$

$\therefore AX = XA = I$

$\therefore A$ is invertible.

Q: Express $A = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ as product of elementary matrices.

$A = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$I \xrightarrow[(e_1)]{R_2 \to 3R_2} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \xrightarrow[(e_2)]{R_2 \to R_2 + R_1} \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} \xrightarrow[(e_3)]{R_1 \to R_1 + R_2} \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$$

$\Rightarrow E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$E_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$E_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$\boxed{\therefore \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} = E_3 E_2 E_1}$$

Alt. method

$$\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} \xrightarrow[e_1]{R_1 \to R_1 - R_2} \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} \xrightarrow[e_2]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \xrightarrow[e_3]{R_2 \to \frac{1}{3}R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore I = E_3 E_2 E_1 A$

$\therefore \boxed{E_1^{-1}E_2^{-1}E_3^{-1} = A}$

$(e_1)^{-1} = R_1 \to R_1 + R_2$

$(e_2)^{-1} = R_2 \to R_2 + R_1$

$(e_3)^{-1} = R_2 \to 3R_2$

※ Transpose of a Matrix :

→ The transpose of a $m \times n$ matrix $A = [a_{ij}]_{m \times n}$ is a $n \times m$ matrix $A^T = [a_{ji}]_{n \times m}$ with $\boxed{a_{ji}' = a_{ij}}$

for $1 \le i \le m$, $1 \le j \le n$

Eg: $A = \begin{pmatrix} 1 & 6 \\ 8 & 2 \\ 4 & 7 \end{pmatrix}_{3 \times 2}$ $A^T = \begin{pmatrix} 1 & 8 & 4 \\ 6 & 2 & 7 \end{pmatrix}_{2 \times 3}$

→ The transpose of a matrix is exchanging the rows & columns of the matrix.

rows of A = column of A^T
columns of A = rows of A^T

⇒ If you have 2 column vectors

$$U = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix}_{n \times 1} \qquad V = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}_{n \times 1}$$

$$U \cdot V = [U_1 \cdots U_n] \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} = U^T V = U_1 V_1 + U_2 V_2 + \cdots + U_n V_n$$

$$\therefore \boxed{\vec{U} \cdot \vec{V} = \vec{U}^T \vec{V}}$$

Properties :

① $(A^T)^T = A$

② $(AB)^T = B^T A^T$

③ $(A+B)^T = A^T + B^T$

④ $(KA)^T = KA^T$

Proof of $(AB)^T = B^T A^T$:

Known : 1) $row_i(A) = column_i(A^T)$

2) $a_{ji}^T = a_{ij}$

$$\therefore [(AB)^T]_{ij} = (AB)_{ji}$$
$$= row_j(A) \cdot column_i(B)$$
$$= column_j(A^T) \cdot row_i(B^T)$$
$$= row_i(B^T) \cdot column_j(A^T)$$
$$= (B^T A^T)_{ij}$$

※ DETERMINANT :

$\rightarrow A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times 2}$

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of a 3x3 matrix :

$\rightarrow A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\det A = +a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\therefore \det A = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}$$

$$= \boxed{\sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det A_{1j}}$$

where, $\det A_{ij}$ is called the (i,j)- minor of A

Determinant of a nxn matrix :

$$\det A = |A| = a_{11}A_{11} - a_{12}A_{12} + \cdots + (-1)^{1+n} a_{1n}A_{1n}$$

$$= \boxed{\sum_{j=1}^{n} (-1)^{1+j} a_{1j} A_{1j}} \quad (A_{ij} = \text{minor of } a_{ij})$$

Cofactor :

Combine a minor with it's plus or minus sign. To this end, we define the (i,j)-cofactor of A to be:

$$\boxed{C_{ij} = (-1)^{i+j} A_{ij}}$$

$$\therefore \boxed{\det A = \sum_{j=1}^{n} a_{ij} C_{ij}}$$

※ The Laplace Expansion Theorem:

The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geq 2$, can be computed as

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$
$$= \sum_{j=1}^{n} a_{ij}C_{ij}$$

(which is the cofactor expansion along the i^{th} row) and also as

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$
$$= \sum_{i=1}^{n} a_{ij}C_{ij}$$

(the cofactor expansion along the j^{th} column).

Theorem 4.2: The determinant of a triangular matrix is the product of the entries on its main diagonal.

Specifically, if $A = [a_{ij}]$ is an $n \times n$ triangular matrix, then

$$\boxed{\det A = a_{11}a_{22} \cdots a_{nn}}$$

Properties of Determinant:

1. If A has a zero row/column, then $\det A = 0$.
2. If B is obtained by interchanging two rows/column of A, then $\det B = -\det A$
3. If A has two identical rows/columns, then $\det A = 0$.
4. If B is obtained by multiplying a row/column of A by k, then $\det B = k \det A$

⑤ If A, B and C are identical except that the i^{th} - row/column of C is the sum of the i^{th} row/column of A and B, then

$$\det C = \det A + \det B$$

⑥ If B is obtained by adding a multiple of one row/column of A to another row/column, then

$$\det B = \det A$$

Prove all of the above properties:

① Suppose i^{th} row (column) is zero in $A_{n\times n}$

Row: $\therefore \det A = \sum_{j=1}^{n} a_{ij} C_{ij}$ (As $a_{ij} = 0 \ \forall j$)

$$\boxed{\det A = 0}$$

Column: $\det A = \sum_{j=1}^{n} a_{ji} C_{ji}$ (As $a_{ji} = 0 \ \forall j$)

$$\boxed{\det A = 0}$$

③ $A \rightarrow$ has two identical rows, swap them to obtain the matrix B.

clearly, $B = A$

$\therefore \det B = \det A$

Also, by ②, $\det B = -\det A$ $\Big\}$ $\det A = -\det A$ $\boxed{\therefore \det A = 0}$

④ Suppose $b_{ij} = k a_{ij}$ for $j = 1, \ldots, n$

$\therefore$ expanding them along i^{th} row,

$$\det B = \sum_{j=1}^{n} b_{ij} C_{ij} = \sum_{j=1}^{n} k a_{ij} C_{ij} = k \sum_{j=1}^{n} a_{ij} C_{ij}$$

$$= \boxed{k \det A}$$

⑤ $c_{ij} = a_{ij} + b_{ij} \ \forall j$

else, $c_{rj} = a_{rj} = b_{rj} \ \forall r \neq i$ (the cofactors C_{ij} of the elements in the expanding along i^{th} row, i^{th} rows of A, B and C are identical.)

$$\det C = \sum_{j=1}^{n} c_{ij} C_{ij}$$

$$= \sum_{j=1}^{n} (a_{ij} + b_{ij}) C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} + \sum_{j=1}^{n} b_{ij} C_{ij}$$

$$= \boxed{\det A + \det B}$$

(6)

$R_i \to R_i + kR_r \Rightarrow B$

$$\det B = \sum_{j=1}^{n} b_{ij} C_{ij}$$

$$= \sum_{j=1}^{n} (a_{ij} + k a_{rj}) C_{ij}$$

$$= \sum_{j=1}^{n} a_{ij} C_{ij} + \boxed{k \sum_{j=1}^{n} a_{rj} C_{ij}}$$

$$= \det A + k(0) \qquad \hookrightarrow \det C$$

$$= \det A$$

where i^{th} row is replaced with r^{th} row.

So, now r^{th} and i^{th} row are identical

$\therefore \det C = 0$

(2)

Determinants of Elementary Matrices:

Theorem 4.4:

Let E be an $n \times n$ elementary matrix.

1. If E results from interchanging two rows of I_n, then $\det E = -1$.

2. If E results from multiplying one row of I_n by k, then $\det E = k$.

3. If E results from adding a multiple of one row of I_n to another row, then $\det E = 1$.

Can be directly derived from the part 2, 4 & 6 of the previous theorem.

Lemma 4.5: Let B be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix. Then

$$\det(EB) = (\det E)(\det B)$$

Theorem 4.6: A square matrix A is invertible if and only if $\det A \neq 0$.

※ CRAMER'S RULE:

→ Let A be an invertible $n \times n$ matrix, and let $b \in R^n$. Then the unique solⁿ x of the system of equation $Ax=b$ is given by

$$\boxed{x_i = \frac{\det(A_i(b))}{\det(A)}}$$ for $i = 1, 2, \ldots, n$

where $A_i(b)$ is obtained by replacing i^{th} column of A with b.

Eg: $\left.\begin{aligned} x_1 + 2x_2 &= 2 \\ -x_1 + 4x_2 &= 1 \end{aligned}\right\}$ let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore \det A = \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 4+2 = \boxed{6}$

$$\left.\begin{aligned} \therefore x_1 &= \frac{\det(A_1(b))}{\det(A)} = \frac{\begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix}}{6} = \frac{6}{6} = \boxed{1} \\ x_2 &= \frac{\det(A_2(b))}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}}{6} = \frac{3}{6} = \boxed{1/2} \end{aligned}\right\} \boxed{x = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}}$$

→ $$\boxed{A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)}$$

Eg: let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$ ⇒ $|A| = -18 - 2(-10) - 1(4)$
$= -22 + 20 = \boxed{-2}$

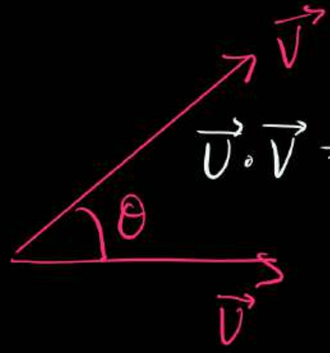
$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$

$$\operatorname{adj}(A) = \begin{bmatrix} -18 & 10 & 4 \\ 3 & -2 & -1 \\ 10 & -6 & -2 \end{bmatrix}^T = \begin{bmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{-1}{2} \begin{pmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{pmatrix} = \boxed{\begin{pmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{pmatrix}}$$

※ Inner Product Space:

Definition:



$$\vec{U}\cdot\vec{V} = |\vec{U}||\vec{V}|\cos\theta$$

An inner product on a vector space V is an operation that aligns to every pair of vectors u and v in V a real number $\langle u, v\rangle$ such that the following properties hold $\forall$ vectors u, v & $w \in V$ and all scalars c:

1. $\langle u, v\rangle = \langle v, u\rangle$
2. $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$
3. $\langle cu, v\rangle = c\langle u, v\rangle$
4. $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ iff $u = 0$

A vector space with a inner product is called an inner product space.

If v and u are two vectors is an innerproduct space V:

1. length of v is $\|v\| = \sqrt{\langle v, v\rangle}$
2. The distance b/w u & v is $d(v, u) = \|u - v\|$.
3. u. and v are orthogonal if $\langle u, v\rangle = 0$

Eg: 1] R^n is an inner product space $(u, v) = u \cdot v$

But dot product is not the only inner product that can be defined on R^n.

2]

NOTE $\{u_1, u_2, \ldots, u_k\}$ where every pair is orthogonal then, $u_i \cdot u_j = 0$ if $u_i \neq u_j$.

→ If $\{u_1, u_2, \ldots, u_n\}$ is an orthogonal set of non-zero values in R^n then they are linearly independent.

If $c_1, c_2, \ldots, c_k$ are scalars such that,

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

$$\left(\sum_{i}^{k} c_i v_i\right) \cdot v_1 = 0 \cdot v_1$$

$$= 0$$

we know that $v_i v_j = 0$ if $i \neq j$

$\therefore c_i (v_i \cdot v_i) = 0 \ \forall i$

$\therefore \boxed{c_i = 0 \ \forall i}$

Eg: $(\mathbb{R}^2, +) \to (\mathbb{R}, +, \cdot)$

$\langle u, v \rangle = 2u_1v_1 + 3u_2v_2$

Is this an inner product?

Solution:

(i) $\langle u, v \rangle$

$= 2u_1v_1 + 3u_2v_2$

$= 2v_1u_1 + 3v_2u_2$

$= \boxed{\langle v, u \rangle}$

(ii) $\langle u, v+w \rangle$

$= 2u_1(v_1+w_1) + 3u_2(v_2+w_2)$

$= 2u_1v_1 + 2u_1w_1 + 3u_2v_2 + 3u_2w_2$

$= (2u_1v_1 + 3u_2v_2) + (2u_1w_1 + 3u_2w_2)$

$= \boxed{\langle u, v \rangle + \langle u, w \rangle}$

(iii) $\langle cu, v \rangle$

$= 2(cu_1)v_1 + 3(cu_2)v_2$

$= c[2u_1v_1 + 3u_2v_2]$

$= \boxed{c\langle u, v \rangle}$

(iv) $\langle u, u \rangle$

$= \underset{\geq 0}{2u_1^2} + \underset{\geq 0}{3u_2^2}$

$\boxed{\therefore \langle u, u \rangle \geq 0}$

If $u = 0 \Rightarrow u_1 = u_2 = 0$

$\therefore \boxed{\langle u, u \rangle = 0}$

※ Complex Dot Product:

If $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ are vectors in C^n, then the complex dot product of u and v is defined by

$$\boxed{u \cdot v = \bar{u}_1 v_1 + \cdots\cdots + \bar{u}_n v_n}$$

Prove that it is a inner product:

→ The norm (length) of a complex vector v is defined as in the real case:

$$\|v\| = \sqrt{v \cdot v} \qquad \left(\text{General form: } \|v\| = \sqrt{\langle v, v \rangle}\right)$$

Distance b/w two complex vectors:

$$d(u, v) = \|u - v\|$$

Eg: let $u = \begin{pmatrix} i \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 2-3i \\ 1+5i \end{pmatrix}$

(a) $u \cdot v$

$$u \cdot v = \sum_1^2 \bar{u}_i v_i = (-i)(2-3i) + (1)(1+5i)$$
$$= -2i + 3(-1) + 1 + 5i$$
$$= \boxed{3i - 2}$$

(b) $\|u\|$

$$\|u\| = \sqrt{u \cdot u}$$
$$= \sqrt{1+1} = \boxed{\sqrt{2}}$$

(c) $\|v\|$

$$\|v\| = \sqrt{4+9+25+1} = \boxed{\sqrt{39}}$$

(d) $d(u,v)$

$$d(u,v) = \|u-v\|$$

$$= \left\| \begin{pmatrix} 4i-2 \\ -5i \end{pmatrix} \right\| = \sqrt{16+4+25}$$

$$= \boxed{\sqrt{45}}$$

NOTE: Let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ then, $\|v\| = \sqrt{|v_1^2| + |v_2^2| + \cdots + |v_n^2|}$

※ Properties:

(a) $u \cdot v = \overline{v \cdot u}$

(b) $u \cdot (v+w) = u \cdot v + u \cdot w$

(c) $(cu) \cdot v = \bar{c}(u \cdot v)$ and $u \cdot (cv) = c(u \cdot v)$

(d) $u \cdot u \geq 0$ and $u \cdot u = 0$ iff $u = 0$.

Proofs:

(a) $\langle u, v \rangle = \sum_i \bar{u}_i v_i$

As $\bar{\bar{v}} = v$

$$= \sum_i \bar{u}_i \bar{\bar{v}}_i$$

As $\overline{pq} = \bar{p}\bar{q}$

$$= \sum_i \overline{u_i \bar{v}_i}$$

As $\overline{p+q} = \bar{p} + \bar{q}$

$$= \overline{\sum_i u_i \bar{v}_i}$$

$$= \overline{\sum_i \bar{v}_i u_i}$$

$$= \boxed{\overline{\langle v, u \rangle}}$$

(b) $\langle u, v+w \rangle$

$$= \sum_i \bar{u}_i (v_i + w_i)$$

$$= \sum_i \bar{u}_i v_i + \sum_i \bar{u}_i w_i$$

$$= \boxed{\langle u, v \rangle + \langle u, w \rangle}$$

(c) $\langle cu, v \rangle = \sum_i \overline{cu_i} v_i$

$$= \sum_i \bar{c} \bar{u}_i v_i$$

$$= \bar{c} \sum_i \bar{u}_i v_i$$

$$= \boxed{\bar{c} \langle u, v \rangle}$$

(d) $\langle u, u \rangle = \sum_i \bar{u}_i u_i$

$$= \sum_i |u_i|^2 \geq 0$$

If $u = 0$

then $\sum_i |u_i|^2 = 0$

✻ Conjugate Transpose:

If A is a complex matrix, then the conjugate transpose of A is the matrix A^* defined by

$$\boxed{A^* = \bar{A}^T}$$

$\bar{A}$ refers to the matrix whose entries are the complex conjugates of the corresponding entries of A.

If $A = [a_{ij}]$

then, $\bar{A} = [\overline{a_{ij}}]$

Properties:

(a) $\bar{\bar{A}} = A$

(b) $\overline{A+B} = \bar{A} + \bar{B}$

(c) $\overline{cA} = \bar{c}\bar{A}$

(d) $\overline{AB} = \bar{A}\bar{B}$

(e) $(\bar{A})^T = (\overline{A^T})$

Proofs:

(a) let $A = [a_{ij}]$

$$\bar{A} = [\overline{a_{ij}}]$$

$$\bar{\bar{A}} = [\overline{\overline{a_{ij}}}] = [a_{ij}]$$

$$\boxed{\therefore \bar{\bar{A}} = A}$$

(b) $\overline{A+B} = [\overline{a_{ij}+b_{ij}}]$

$$= [\overline{a_{ij}} + \overline{b_{ij}}]$$

$$= [\overline{a_{ij}}] + [\overline{b_{ij}}]$$

$$\therefore \overline{A+B} = \boxed{\bar{A} + \bar{B}}$$

(d) $\overline{AB}$

$$(AB)_{ij} = \sum_k a_{ik} b_{kj}$$

$$(\overline{AB})_{ij} = \overline{\sum_k a_{ik} b_{kj}}$$

As $\forall\, p, q \in \mathbb{C}$ $(\overline{p+q} = \bar{p} + \bar{q})$ & $\overline{pq} = \bar{p}\bar{q}$

$$= \sum_k \overline{a_{ik} b_{kj}}$$

$$= \sum_k (\overline{a_{ik}})(\overline{b_{kj}})$$

$$= (\bar{A}\bar{B})_{ij}$$

$$\boxed{\therefore \overline{AB} = \bar{A}\bar{B}}$$

More Properties:

(a) $(A^*)^* = A$

(b) $(A+B)^* = A^* + B^*$

(c) $(cA)^* = \bar{c}A^*$

(d) $(AB)^* = B^*A^*$

Proofs:

(d) $(AB)^* = (\overline{AB})^T = (\bar{A}\bar{B})^T$

$$= \bar{B}^T \bar{A}^T$$

$$= \boxed{B^* A^*}$$

※ If $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

if $w_1, \dots, w_n$ are positive scalars and are vectors in R^n, then

$$\langle u, v \rangle = w_1 u_1 v_1 + \dots + w_n u_n v_n$$

This defines an inner product on R^n, called a weighted dot product.

Prove that this is a inner product:

(i) $\langle u, v \rangle = \langle v, u \rangle$

$$\Rightarrow \langle u, v \rangle = \sum_i w_i u_i v_i$$

$$= \sum_i w_i v_i u_i$$

$$= \boxed{\langle v, u \rangle}$$

(ii) $\langle u, v + w \rangle$

$$= \sum_i w_i u_i (v_i + w_i)$$

$$= \sum_i w_i u_i v_i + \sum_i w_i u_i w_i$$

$$= \boxed{\langle u, v \rangle + \langle u, w \rangle}$$

(iii) $\langle cu, v \rangle = c \langle u, v \rangle$

$$\langle cu, v \rangle = \sum_i c w_i u_i v_i$$

$$= c \sum_i w_i u_i v_i$$

$$= \boxed{c \langle u, v \rangle}$$

(iv) $\langle u, u \rangle \geq 0$

$$\langle u, u \rangle = \sum_i w_i u_i u_i$$

$$= \sum_i w_i (u_i)^2$$

$(u_i)^2 \geq 0$ & $w_i \geq 0$

$$\therefore \sum_i w_i u_i^2 \geq 0$$

$$\boxed{\therefore \langle u, u \rangle \geq 0}$$

※ Orthogonal Projections: (Pg 562)

An orthogonal set of vectors in an inner product space V is a set $\{v_1, \ldots, v_k\}$ of vectors from V such that $\langle v_i, v_j \rangle = 0$ whenever, $v_i \neq v_j$.

A set of vectors is an orthonormal set of vectors if it is an orthogonal set of unit vectors.

Theorem: (Pg: 396)

Let $\{v_1, \ldots, v_k\}$ be an orthogonal basis for a subspace W of R^n and w be any vector of W. Then there are unique scalars $c_1, c_2, \ldots, c_k$ such that,

$$w = \sum_{i=1}^{k} c_i v_i$$

where $\boxed{c_i = \frac{w \cdot v_i}{v_i \cdot v_i}}$ for $i = 1, 2, \ldots, k$

If inner product is defined in the question, then

$$\boxed{c_i = \frac{\langle w, v_i \rangle}{\langle v_i, v_i \rangle}}$$

Proof:

Since, $\{v_1, \ldots, v_k\}$ is a basis of W,

$\exists$ scalars $c_1, \ldots, c_k$ such that $w = \sum_{i=1}^{k} c_i v_i$

$$w \cdot v_i = (c_1 v_1 + \cdots + c_k v_k) v_i$$

$$= c_1 (\cancel{v_1 \cdot v_i})^{0} + \cdots + c_i (v_i \cdot v_i) + \cdots c_i (\cancel{v_k \cdot v_i})^{0}$$

$$= c_i (v_i \cdot v_i)$$

$$\therefore \boxed{c_i = \frac{w \cdot v_i}{v_i \cdot v_i}}$$

Orthogonal Projection:

→ We can define the orthogonal projection $proj_W(v)$ of a vector v onto a subspace W of an inner product space.
If $\{u_1, \ldots, u_k\}$ is an orthogonal basis for W, then

$$\boxed{proj_W(v) = \frac{\langle u_1, v\rangle}{\langle u_1, u_1\rangle} u_1 + \cdots\cdots + \frac{\langle u_k, v\rangle}{\langle u_k, u_k\rangle} u_k}$$

Then the component of v orthogonal to W is the vector:

$$\boxed{perp_W(v) = v - proj_W(v)}$$

Que: $\langle f, g\rangle = \int_a^b f(x) g(x)\, dx$

Prove that this is a inner product.

① $\langle v, u\rangle = \langle u, v\rangle$

$\therefore \int_a^b f(x) g(x) = \int_a^b g(x) f(x)\, dx$ True

② $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$

③ $\langle cu, v\rangle = c\langle u, v\rangle$ $\quad \int_a^b (cf(x)) g(x)\, dx = c\int_a^b f(x) g(x)\, dx$

④ $\langle u, u\rangle \geq 0$ $\quad \langle u, u\rangle = 0$ iff $u = 0$ $\quad \int_a^b \underbrace{f(x)^2}_{+ve}\, dx$

Q: Find an orthogonal basis for the subspace W of $\mathbb{R}^3$ given by

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x - y + 2z = 0 \right\}$$

Sol^n:

→ $x - y + 2z = 0$

$x = y - 2z$

$\therefore W = \left\{ \begin{pmatrix} y-2z \\ y \\ z \end{pmatrix} \right\} \quad y, z \in \mathbb{R}$

Let $w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$w_2 = \begin{pmatrix} y-2z \\ y \\ z \end{pmatrix}$

As W is orthogonal,

$w_1 \cdot w_2 = 0$

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} y-2z \\ y \\ z \end{pmatrix} = 0$

$\therefore y - 2z + y = 0$

$\boxed{y = z}$

$\therefore w_2 = \begin{pmatrix} -y \\ y \\ y \end{pmatrix}$

Let $y = 1$

$\boxed{\therefore w_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}$

$\left(w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right)$

$\boxed{W_B = \{w_1, w_2\}}$

↳ Orthogonal Basis of Subspace W.

※ The Gram-Schmidt Process:

→ We want to find an orthogonal basis for a subspace W of $\mathbb{R}^n$.

→ The idea is to begin with a arbitrary basis $\{x_1, \ldots, x_k\}$ and to "orthogonalize" it one vector at a time.

→ For example:

Let $W = \text{span}(x_1, x_2)$, where

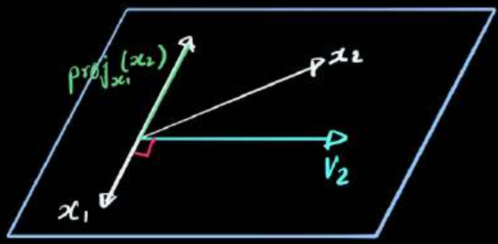
$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } x_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Construct an orthogonal basis for W.

Sol^n: Starting with x_1,

we get a second vector that is orthogonal to it by taking the component of x_2 orthogonal to x_1.

Constructing v_2 orthogonal to x_1



$x_1 \perp v_2$

Algebraically, we set $v_1 = x_1$, so

$$v_2 = \text{perp}_{x_1}(x_2) = x_2 - \text{proj}_{x_1}(x_2)$$

$$= x_2 - \left(\frac{x_1 \cdot x_2}{x_1 \cdot x_1}\right) x_1$$

$$= \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} - \left(\frac{-2}{2}\right)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\boxed{v_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}$$

Hence, $\{v_1, v_2\}$ is an orthogonal set of vectors in W.
So, it is linearly independent set.
$\rightarrow \{v_1, v_2\}$ is a basis for W, since $\dim W = 2$.

NOTE:

If we had taken $v_1 = x_2$, we would have obtained a different set of orthogonal vectors in W.

※ Theorem: The Gram-Schmidt Process:

Let $\{x_1, \ldots, x_k\}$ be a basis for a subspace W of $\mathbb{R}^n$ and define the following:

$v_1 = x_1$ $\quad\quad W_1 = \text{span}(x_1)$

$v_2 = x_2 - \text{proj}_{v_1}(x_2)$ $\quad\quad W_2 = \text{span}(x_1, x_2)$

$v_3 = x_3 - \text{proj}_{v_1}(x_3) - \text{proj}_{v_2}(x_3)$ $\quad\quad W_3 = \text{span}(x_1, x_2, x_3)$

$\vdots$

$v_k = x_k - \text{proj}_{v_1}(x_k) - \cdots - \text{proj}_{v_{k-1}}(x_k)$ $\quad\quad W_k = \text{span}(x_1, \ldots, x_k)$

Then, for each $i \in \{1, \ldots, k\}$, $\{v_1, \ldots, v_i\}$ is an orthogonal basis for W_i. In particular $\{v_1, \ldots, v_k\}$ is an orthogonal basis for W.

☆

Proof: Proof by Induction:

→ Let $v_1 = x_1$

Clearly, $\{v_1\}$ is an orthogonal basis for $W_1 = \text{span}(x_1)$.

→ Suppose,

for some $i < k$,

$\{v_1, \ldots, v_i\}$ is an orthogonal basis for W_i.

Then,

$$v_{i+1} = x_{i+1} - \text{proj}_{v_1}(x_{i+1}) - \cdots - \text{proj}_{v_i}(x_{i+1})$$

$$= x_{i+1} - \left(\frac{v_1 \cdot x_1}{v_1 \cdot v_1}\right) v_1 - \cdots - \left(\frac{v_i \cdot x_{i+1}}{v_i \cdot v_i}\right) v_i$$

By induction hypothesis,

$\{v_1, \ldots, v_i\}$ is an orthogonal basis for $\text{span}(x_1, \ldots, x_i) = W_i$

Hence, $\boxed{v_{i+1} = x_{i+1} - \text{proj}_{W_i}(x_{i+1}) = \text{perp}_{W_i}(x_{i+1})}$

→ ∴ By orthogonal Decomposition Theorem, v_{i+1} is orthogonal to W_i. — 1

By definition, $v_1, v_2, \ldots, v_i$ are linear combinations of $x_1, \ldots, x_i$ and hence, are in W_{i+1}. → 2

Also,

$v_{i+1} \neq 0$, since otherwise $x_{i+1} = \text{proj}_{W_i}(x_{i+1})$ implies x_{i+1} is in W_i.

But, since $W_i = \text{span}(x_1, \ldots, x_i)$, $[\{x_1, \ldots, x_i\}]$

this is impossible.

→ Hence, from 1 and 2,

we conclude that, $\{v_1, \ldots, v_{i+1}\}$ is a set of linearly independent vectors in W_{i+1}.

Since, $\dim W_{i+1} = i+1$,

$\{v_1, \ldots, v_{i+1}\}$ is a basis for W_{i+1}.

Hence, Proved.

Eg: Apply the Gram-Schmidt Process to construct an orthogonal basis for the subspace $W = \text{span}(x_1, x_2, x_3)$ of $\mathbb{R}^4$,

where $x_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $x_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}$

Sol^n: Let $v_1 = x_1$

Now, we compute component of x_2 orthogonal to $W = \text{span}(v_1) = v_1$

$$v_2 = \text{perp}_{W_1}(x_2)$$

$$= x_2 - \text{proj}_{W_1}(x_2)$$

$$v_2 = x_2 - \left(\frac{v_1 \cdot x_2}{v_1 \cdot v_1}\right) v_1 \quad \left[\begin{array}{l} x_1 \cdot x_2 = 2 - \cancel{1} + \cancel{1} = 2 \\ x_1 \cdot x_1 = 1+1+1+1 = 4 \end{array}\right]$$

$$= x_2 - \left(\frac{2}{4}\right) v_1$$

$$= x_2 - \tfrac{1}{2}(v_1) = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \tfrac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\boxed{v_2 = \begin{pmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{pmatrix}}$$

For easy calculation,
$\Rightarrow$ "scale" v_2
$\therefore$ let $v_2 = 2\begin{pmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{pmatrix}$

$$\boxed{v_2 = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix}}$$

Let $W_2 = \text{span}(v_1, v_2)$

$$\therefore v_3 = \text{perp}_{W_2}(x_3)$$

$$= x_3 - \text{proj}_{W_2}(x_3)$$

$$= x_3 - \left(\frac{v_1 \cdot x_3}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2 \cdot x_3}{v_2 \cdot v_2}\right) v_2$$

$v_1 \cdot x_3 = 1$
$v_1 \cdot v_1 = 4$
$v_2 \cdot x_3 = 15$
$v_2 \cdot v_2 = 20$

$$\therefore V_3 = x_3 - \left(\frac{1}{4}\right)\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} - \left(\frac{15}{20}\right)\begin{pmatrix}3\\3\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}2\\2\\1\\2\end{pmatrix} - \frac{1}{4}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} - \frac{3}{4}\begin{pmatrix}3\\3\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}2-1/4-9/4\\2+1/4-9/4\\1+1/4-3/4\\2-1/4-3/4\end{pmatrix} = \begin{pmatrix}-1/2\\0\\1/2\\1\end{pmatrix}$$

"Scale" $V_3 \Rightarrow$ Let $\boxed{V_3 = \begin{pmatrix}-1\\0\\1\\2\end{pmatrix}}$

$$\{V_1, V_2, V_3\} = \left\{\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}, \begin{pmatrix}3\\3\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\2\end{pmatrix}\right\}$$

Q: Construct an orthogonal basis for P_2 with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)(g)\,dx$$

by applying the Gram-Schmidt Process to the basis $\{1, x, x^2\}$.

Sol^n: Let $x_1 = 1$

$x_2 = x$ & $x_3 = x^2$. Also, let $V_1 = x_1$

$$\therefore \langle V_1, V_1\rangle = \int_{-1}^{1} dx = \boxed{2}$$

$$\Rightarrow \langle V_1, x_2\rangle = \int_{-1}^{1}(1)(x)\,dx$$

$$= \left.\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = \boxed{0}.$$

$$\Rightarrow V_2 = x_2 - \left(\frac{\langle V_1, x_2\rangle}{\langle V_1, V_1\rangle}\right)V_1$$

$$= x_2 - 0 = x_2 = x$$

$$\boxed{\therefore V_2 = x}$$

$$\Rightarrow V_3 = x_3 - \left(\frac{\langle V_1, x_3\rangle}{\langle V_1, V_1\rangle}\right)V_1 - \left(\frac{\langle V_2, x_3\rangle}{\langle V_2, V_2\rangle}\right)V_2$$

where,

$$\langle V_2, V_2\rangle = \int_{-1}^{1} x^2\,dx = \left.\frac{x^3}{3}\right]_{-1}^{1} = \boxed{2/3}$$

$$\langle V_1, x_3\rangle = \int_{-1}^{1}(1)(x^2)\,dx$$

$$= \left.\frac{x^3}{3}\right]_{-1}^{1} = \boxed{2/3}$$

$$\langle V_2, x_3\rangle = \int_{-1}^{1}(x)(x^2)\,dx = \boxed{0}$$

$$\therefore V_3 = x^2 - \left(\frac{2/3}{2}\right)(1) - \left(\frac{0}{2/3}\right)(x)$$

$$\boxed{V_3 = x^2 - 1/3}$$

$\therefore$ Orthogonal Basis $= \{1, x, x^2 - 1/3\}$.

※ The Cauchy-Schwarz Inequality:

→ Let u and v be vectors in an inner product space V. Then

$$|\langle u,v\rangle| \le \|u\|\,\|v\|$$

with equality holding if and only if u and v are scalar multiples of each other.

Proof: → If $u=0$, then the inequality is actually an equality, since

$$\boxed{|\langle 0,v\rangle| = 0 = \|0\|\|v\|}$$

→ If $u\neq 0$,
then let W be the subspace of V spanned by u. Since $\text{proj}_W(v) = \left(\frac{\langle u,v\rangle}{\langle u,u\rangle}\right)u$ and $\text{perp}_W(v) = v - \text{proj}_W(v)$ are othogonal,
we can apply pythagoras theorem,

$$\|v\|^2 = \|\text{proj}_W(v) + (v - \text{proj}_W(v))\|^2$$

$$= \|\text{proj}_W(v) + \text{perp}_W(v)\|^2$$

$$= \|\text{proj}_W(v)\|^2 + \|\text{perp}_W(v)\|^2$$

It follows that

$$\|v\|^2 \ge \|\text{proj}_W(v)\|^2$$

Now, $\|\text{proj}_W(v)\|^2 = \left\langle \left(\frac{\langle u,v\rangle}{\langle u,u\rangle}\right)u, \left(\frac{\langle u,v\rangle}{\langle u,u\rangle}\right)u \right\rangle = \left(\frac{\langle u,v\rangle}{\langle u,u\rangle}\right)^2 \langle u,u\rangle$

$$= \frac{\langle u,v\rangle^2}{\langle u,u\rangle}$$

$$= \boxed{\frac{\langle u,v\rangle^2}{\|u\|^2}}$$

So, we have,

$$\frac{\langle u,v\rangle^2}{\|u\|^2} \le \|v\|^2 \Rightarrow \langle u,v\rangle^2 \le \|u\|^2\|v\|^2$$

$$\boxed{\therefore\ |\langle u,v\rangle| \le \|u\|\,\|v\|}$$

Equality: iff $\|v\|^2 = \|\text{proj}_W(v)\|^2$, which is true only when $v = \text{proj}_W(v)$.
∴ If this is so, then v is a scalar multiple of u.
Conversely, if $v = cu$, ⇒ $\text{perp}_W(v) = v - \text{proj}_W(v) = cu - \left(\frac{\langle u,cu\rangle}{\langle u,u\rangle}\right)u = cu - cu = \boxed{0}$
∴ Equality holds.

※ Triangular Inequality Theorem:

$$\|u+v\| \leq \|u\| + \|v\|$$

Proof: LHS $= \|u+v\| = \sqrt{\langle u+v, u+v\rangle}$

$$\|u+v\|^2 = \langle u+v, u+v\rangle$$

$$= \langle u+v, u\rangle + \langle u+v, v\rangle$$

$$= \underbrace{\langle u,u\rangle}_{=\|u\|^2} + \underbrace{\langle u,v\rangle}_{\leq\|u\|\|v\|} + \underbrace{\langle v,u\rangle}_{\leq\|v\|\|u\|} + \underbrace{\langle v,v\rangle}_{=\|v\|^2}$$

$$\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$$

$$\|u+v\|^2 \leq (\|u\| + \|v\|)^2$$

$$\therefore \boxed{\|u+v\| \leq \|u\| + \|v\|}$$

※ The Orthogonal Decomposition Theorem:

Let W be a subspace of R^n and let v be a vector on R^n. Then there are unique vectors w in W and $w^\perp$ in $W^\perp$ such that

$$\boxed{v = w + w^\perp}$$

Proof:

To Prove: 1) Such a decomposition exist.

2) It is unique.

Part 1: Let $\{u_1, \dots, u_k\}$ be an orthogonal basis for W.

Let $w = \text{proj}_W(v)$ and,

let $w^\perp = \text{perp}_W(v)$

Then, $w + w^\perp = \text{proj}_W(v) + \text{perp}_W(v)$

$= \boxed{v}$

→ Clearly,
$w = \text{proj}_W(v)$ is in W, since it is a linear combination of basis vectors $u_1, \dots, u_k$.

→ To show that $w^\perp \in W^\perp$,
it is enough to show that $w^\perp$ is orthogonal to each of the basis vector u_i.

→ $u_i \cdot w^\perp = u_i \cdot \text{perp}_W(v)$

$= u_i \cdot (v - \text{proj}_W(v))$

$= u_i \cdot \left(v - \left(\frac{u_1 \cdot v}{u_1 \cdot u_1}\right)u_1 - \left(\frac{u_2 \cdot v}{u_2 \cdot u_2}\right)u_2 - \dots - \left(\frac{u_k \cdot v}{u_k \cdot u_k}\right)u_k\right)$

$= u_i \cdot v - \left(\frac{u_1 \cdot v}{u_1 \cdot u_1}\right)\cancel{(u_1 \cdot u_i)}^{0} - \dots - \left(\frac{u_i \cdot v}{u_i \cdot u_i}\right)(u_i \cdot u_i) - \dots - \left(\frac{u_k \cdot v}{u_k \cdot u_k}\right)\cancel{(u_k \cdot u_i)}^{0}\Big)$

Since, $u_i \cdot u_j = 0 \;\forall\; u_i \neq u_j$,

$\therefore u_i \cdot w^\perp = u_i \cdot v - 0 - \left(\frac{u_i \cdot v}{\cancel{u_i \cdot u_i}}\right)\cancel{(u_i \cdot u_i)}$

$= u_i \cdot v - u_i \cdot v$

$\boxed{\therefore u_i \cdot w^\perp = 0} \;\forall\; i$

$\therefore$ $w^\perp$ is in $W^\perp$

2) Uniqueness:

Suppose, we have another de-composition $v = w_1 + w_1^\perp$, where $w_1 \in W$ and $w_1^\perp \in W^\perp$. Then $w + w^\perp = w_1 + w_1^\perp$

$\therefore w - w_1 = w_1^\perp - w^\perp$

Since, $w - w_1 \in W$ and $w_1^\perp - w^\perp \in W^\perp$,

we know that this common vector $\in W \cap W^\perp = \{0\}$

$\therefore w - w_1 = 0$ and $w_1^\perp - w^\perp = 0$

$\therefore \boxed{w = w_1}$ and $\boxed{w^\perp = w_1^\perp}$

※ Orthogonal Components:

4

Let W be a subspace of R^n. We say that a vector v in R^n is orthogonal to W if v is orthogonal to every vector in W. The set of all vectors that are orthogonal to W is called the orthogonal complement of W, denoted by $W^\perp$.

That is,

$$\boxed{W^\perp = \{v \text{ in } R^n : v \cdot w = 0 \;\; \forall \; w \in W\}}$$

※ Theorem: Let W be a subspace of R^n.

(404)

a) $W^\perp$ is a subspace of R^n.

b) $(W^\perp)^\perp = W$

c) $W \cap W^\perp = \{0\}$

d) If $W = \text{span}(w_1, \ldots\ldots, w_k)$, then

Proof:

a) $\rightarrow$ $W^\perp$ is non-empty

$0 \cdot w = 0 \quad \forall \; w \in W$

$\boxed{0 \in W^\perp}$ —— (i)

Let $u, v \in W^\perp$

$u \cdot w = v \cdot w = 0 \quad \forall \; w \in W$

$(u+v) \cdot w = u \cdot w + v \cdot w = 0 + 0 = 0 \quad \forall \; w \in W$

$\boxed{\therefore u + v \in W^\perp}$ —— (ii)

$(cu) \cdot w = c(u \cdot w) = c(0) = 0 \quad \forall \; w \in W$

$\boxed{\therefore cu \in W^\perp}$ —— (iii)

From (i), (ii) & (iii) $W^\perp$ is a subspace.

b) let $w \in W$ and $x \in W^{\perp}$,

then $w \cdot x = 0$

this implies that, $w \in (W^{\perp})^{\perp}$

$$\boxed{\therefore W \subseteq (W^{\perp})^{\perp}} \longrightarrow (1)$$

Let $v \in (W^{\perp})^{\perp}$

By, orthogonal discomposition theorem,

$v = w + w^{\perp}$ for unique vector w in W & $w^{\perp}$ in $W^{\perp}$

$v \cdot w^{\perp} = 0$

$$\therefore (w + w^{\perp}) \cdot w^{\perp} = w \cdot w^{\perp} + w^{\perp} \cdot w^{\perp}$$

$$= 0 + w^{\perp} \cdot w^{\perp} = 0$$

$$\therefore \boxed{w^{\perp} = 0}$$

$$\therefore v = w + w^{\perp} = w + 0$$

$$\boxed{v = w}$$

$$\therefore \boxed{v \in W}$$

$$\boxed{\therefore (W^{\perp})^{\perp} \subseteq W} \longrightarrow (2)$$

From (1) and (2) $\boxed{(W^{\perp})^{\perp} = W}$

※ Theorem:

The columns of an $m \times n$ matrix Q forms an orthonormal set iff $Q^TQ = I_n$.

Proof:

To show $Q^TQ = I_n$,

$$(Q^TQ)_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Let q_i be the i^{th} column of Q (or i^{th} row of Q^T).

$$(Q^TQ)_{ij} = (i^{th} \text{ row of } Q^T) \cdot (j^{th} \text{ column of } Q)$$

$$\boxed{(Q^TQ)_{ij} = q_i \cdot q_j} \longrightarrow 1$$

Now, the columns Q form an orthogonal set iff

$$q_i \cdot q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

which by equation 1 holds iff,

$$\boxed{(Q^TQ)_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}}$$

※ Orthogonal Matrices:

An $n \times n$ matrix Q whose columns form an orthonormal set, is called orthogonal matrix.

Orthonormal:

A set of orthogonal unit vectors.

Que:

$$Q = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & x \\ 0 & 1/\sqrt{3} & y \\ -1/\sqrt{2} & 1/\sqrt{3} & z \end{pmatrix} \end{array}$$

For what values of x, y, z, Q is orthogonal matrix.

Solⁿ: $\|C\| = 1$

$\therefore \boxed{x^2+y^2+z^2 = 1} \longrightarrow (1)$

$\langle A, C \rangle = 0$

$x/\sqrt{2} - z/\sqrt{2} = 0 \Rightarrow \boxed{x = z} \longrightarrow (2)$

$\langle B, C \rangle = 0$

$x+y+z = 0 \Rightarrow \boxed{y = -2x} \longrightarrow (3)$

From (1), (2) & (3),

$x^2 + 4x^2 + x^2 = 6x^2 = 1$

$\boxed{x = \pm 1/\sqrt{6}}$

$$\boxed{\begin{aligned} \therefore (x, y, z) &= (1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6}) \\ &\text{or} \\ &(-1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6}) \end{aligned}}$$

※ Theorem: A square matrix Q is orthogonal iff $Q^{-1} = Q^T$.
(399)

Proof:

※ QR factorization :

→ Let A be a $m \times n$ matrix with linearly independent columns.
Then A can be factorized as $A = QR$ where Q is an $m \times n$ matrix with orthogonal column and R is invertible uppertriangular matrix.

→ Let $a_1, a_2, \ldots\ldots, a_n$ be the linearly independent columns of A and let $q_1, q_2, \ldots\ldots, q_n$ be the orthogonal vectors obtained by applying Gram-Schmidt process to $\{a_1, a_2, \ldots, a_n\}$ with normalization.

$$W_i = \text{span}\{a_1, a_2, \ldots, a_i\}$$
$$= \text{span}\{q_1, q_2, \ldots\ldots, q_i\}$$

∴ there are scalars $r_{1i}, r_{2i}, \ldots, r_{ii}$ such that

$$\left.\begin{aligned} a_1 &= r_{11} q_1 \\ a_2 &= r_{12} q_1 + r_{22} q_2 \\ a_3 &= r_{13} q_1 + r_{23} q_2 + r_{33} q_3 \\ &\vdots \\ a_n &= r_{1n} q_1 + r_{2n} q_2 + \cdots\cdots + r_{nn} q_n \end{aligned}\right\} \boxed{a_i = r_{1i} q_1 + r_{2i} q_2 + \cdots\cdots + r_{ii} q_i}$$

which can be written in matrix form as

$$A = [a_1\ a_2 \cdots\cdots a_n] = [q_1\ q_2 \cdots\cdots q_n] \begin{pmatrix} r_{11} & r_{12} & \cdots\cdots & r_{1n} \\ 0 & r_{22} & \cdots\cdots & r_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots\cdots & r_{nn} \end{pmatrix} = QR$$

∴ $A = QR$

As Q is an orthogonal set,
$$Q^{-1} = Q^T$$

$$\boxed{\therefore Q^T A = R}$$

Eg: Find QR factorization of

$$A = \begin{pmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Sol^n: The orthonormal basis for col(A) produced by Gram-Schmidt Process is:

$$q_1 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 3\sqrt{5}/10 \\ 3\sqrt{5}/10 \\ \sqrt{5}/10 \\ \sqrt{5}/10 \end{pmatrix}, \quad q_3 = \begin{pmatrix} -\sqrt{6}/6 \\ 0 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \end{pmatrix}$$

$$\therefore Q = \begin{pmatrix} 1/2 & 3\sqrt{5}/10 & -\sqrt{6}/6 \\ -1/2 & 3\sqrt{5}/10 & 0 \\ -1/2 & \sqrt{5}/10 & \sqrt{6}/6 \\ 1/2 & \sqrt{5}/10 & \sqrt{6}/3 \end{pmatrix}$$

$$A = QR$$

$$\therefore Q^{-1}A = QQ^{-1}R = R$$

As, Q is an orthogonal matrix,

$$Q^{-1} = Q^T$$

$$\therefore Q^{-1} = \begin{pmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 3\sqrt{5}/10 & 3\sqrt{5}/10 & \sqrt{5}/10 & \sqrt{5}/10 \\ -\sqrt{6}/6 & 0 & \sqrt{6}/6 & \sqrt{6}/3 \end{pmatrix}$$

$$\therefore R = Q^{-1}A = \begin{pmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 3\sqrt{5}/10 & 3\sqrt{5}/10 & \sqrt{5}/10 & \sqrt{5}/10 \\ -\sqrt{6}/6 & 0 & \sqrt{6}/6 & \sqrt{6}/3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 1 & 1/2 \\ 0 & \sqrt{5} & 3\sqrt{5}/2 \\ 0 & 0 & \sqrt{6}/2 \end{pmatrix}}$$

※ Eigenvalues & Eigenvectors:

Definition: Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue of A if there is a nonzero vector x such that $Ax = \lambda x$. Such a vector x is called an eigen-vector of A corresponding to λ.

Eg: Show that $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and find the corresponding eigenvalue.

Solⁿ: $Ax = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \boxed{\lambda = 4}$$

Eg: Show that 5 is an eigenvalue of $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ and determine all eigenvectors corresponding to this eigenvalue.

Eigenspace:

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The collection of all eigenvectors corresponding to λ, together with the zero vector is called the eigenspace of λ and is denoted by E_λ.

※ Eigenvalues and Eigenvectors of nxn Matrices:

→ The eigenvalues of a square matrix A are precisely the solution λ of the equation $\det(A - \lambda I) = 0$.

$$Ax = \lambda x$$
$$Ax - \lambda x = 0$$
$$\boxed{\therefore (A - \lambda)x = 0}$$

→ When we expand $\det(A - \lambda I)$, we get a polynomial in λ, called the characteristic polynomial of A.

→ The equation $\det(A - \lambda I) = 0$ is called the characteristic equation.

Steps to find the eigenvalues and eigenvectors of a matrix:

Step 1: Compute the characteristic polynomial $\det(A - \lambda I)$ of A.

Step 2: Find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$ for λ.

Step 3: For each eigenvalue λ, find the null space of the matrix $A - \lambda I$. This is the eigenspace E_λ, the nonzero vectors of which are the eigenvectors of A corresponding to λ.

Step 4: Find a basis for each eigenspan.

Eg: Find the eigenvalues and the corresponding eigenspaces of

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$$

Sol^n: let $\lambda \to$ eigenvalue

characteristic equation: $\det(A-\lambda I)=0$

$$A-\lambda I=\begin{pmatrix}0&1&0\\0&0&1\\2&-5&4\end{pmatrix}-\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}-\lambda&1&0\\0&-\lambda&1\\2&-5&4-\lambda\end{pmatrix}$$

$\Rightarrow \det(A-\lambda I)=0$

$\therefore -\lambda((\lambda-4)(\lambda)+5)-1(-2)$

$\therefore -\lambda(\lambda^2-4\lambda+5)+2$

$\therefore -\lambda^3+4\lambda^2-5\lambda+2=0$

$\therefore \boxed{\lambda^3-4\lambda^2+5\lambda-2=0}$

$\therefore \lambda^3-\lambda^2-3\lambda^2+3\lambda+2\lambda-2=0$

$\therefore (\lambda-1)(\lambda^2-3\lambda+2)=0$

$\therefore (\lambda-1)(\lambda-1)(\lambda-2)=0$

$\therefore \boxed{\lambda=1,2}$

For $\lambda=2$,

$$\begin{pmatrix}-2&1&0\\0&-2&1\\2&-5&2\end{pmatrix}\xrightarrow{R_3\to R_3+R_1}\begin{pmatrix}-2&1&0\\0&-2&1\\0&-4&2\end{pmatrix}$$

$$\downarrow R_1\to -\tfrac{1}{2}R_1$$

$$\begin{pmatrix}1&-1/2&0\\0&-2&1\\0&-4&2\end{pmatrix}\xrightarrow[\text{then, } R_3\to R_3-2R_2]{R_1\to R_1-1/4R_2}\begin{pmatrix}1&0&-1/4\\0&-2&1\\0&0&0\end{pmatrix}$$

$$\downarrow R_2\to(-1/2)R_2$$

$$\boxed{\begin{pmatrix}1&0&-1/4\\0&1&-1/2\\0&0&0\end{pmatrix}}$$

$\therefore x_1-1/4x_3=0$

$\boxed{4x_1=x_3}$

$x_2-1/2x_3=0$

$\therefore \boxed{2x_2=x_3}$

$$\therefore \boxed{x=t\begin{pmatrix}1\\2\\4\end{pmatrix}}$$

For $\lambda=1$,

$(A-\lambda I)x=0$

let $x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

$$A-\lambda I=\begin{pmatrix}-1&1&0\\0&-1&1\\2&-5&3\end{pmatrix}$$

$R_1\to -R_1$

$$\begin{pmatrix}1&-1&0\\0&-1&1\\2&-5&3\end{pmatrix}$$

$$\xrightarrow{R_3\to R_3-2R_1}\begin{pmatrix}1&-1&0\\0&-1&1\\0&-3&3\end{pmatrix}\xrightarrow{R_2\to(-1)R_2}\begin{pmatrix}1&-1&0\\0&1&-1\\0&-3&3\end{pmatrix}\xrightarrow[\text{then, } R_3\to R_3+3R_2]{R_1\to R_1+R_2}\boxed{\begin{pmatrix}1&0&-1\\0&1&-1\\0&0&0\end{pmatrix}}$$

$$\therefore \begin{pmatrix}1&0&-1\\0&1&-1\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$$

$\therefore x_1-x_3=0$

$x_2-x_3=0$

$\boxed{x_1=x_2=x_3}$

$$\boxed{\therefore x=t\begin{pmatrix}1\\1\\1\end{pmatrix}}$$

NOTE:

→ Algebraic Multiplicity:

It is multiplicity of an eigenvalue as a root of the characteristic equation.

→ Geometric Multiplicity of an eigenvalue λ is defined as the dimension of it's corresponding eigenspace.

※ Theorem:

→ A square matrix A is invertible iff 0 is not an eigenvalue of A.

Proof: We know that, a square matrix is invertible iff $\det A \neq 0$.

(→) Square matrix is invertible.

$\therefore \det A \neq 0$

$\therefore \det(A - 0I) = \boxed{\det(A) \neq 0}$

$\therefore \lambda$ is not an eigen value.

(←) Suppose zero is not an eigenvalue,

$\therefore \det(A - 0I) \neq 0$

$\therefore \det(A) \neq 0$

$\therefore$ A is invertible.

※ Theorem: Let A be a square matrix with eigenvalue λ and the corresponding eigen-vector x.

a) For any positive integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector x.

b) If A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector x.

c) If A is invertible, then for any integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector x.

Proof:

Eg: Compute $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Solⁿ: Let $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ and $x = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

We need to find $A^{10}x$.

→ The eigenvalues of A are:

$$\det(A - \lambda I) = 0$$

$$\therefore \begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(\lambda - 1)\lambda - 2 = 0$$

$$\therefore \lambda^2 - \lambda - 2 = 0$$

$$\boxed{\lambda = -1, 2}$$

Computing eigen vectors:

For $\lambda = -1$, $AV_1 = -V_1$

and for $\lambda = 2$, $AV_2 = 2V_2$

For $\lambda = -1$,

$$A - \lambda I = A + I = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \boxed{x_1 + x_2 = 0}$$

$$\therefore \boxed{x_1 = x_2 = 0} \qquad \boxed{\therefore v_1 = t_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

For $\lambda = 2$,

$$A - \lambda I = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \to \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\therefore -2x_1 + x_2 = 0$$

$$\boxed{x_2 = 2x_1} \Rightarrow \boxed{v_2 = t_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

→ Using the previous theorem(c),

if eigenvalue of A is λ with corresponding eigenvector x, for the same eigenvector, eigenvalue of A^{10} will being λ^{10} (if A is invertible).

→ Now, let $c_1, c_2 \in \mathbb{R}$ s.t.,

$$x = c_1 v_1 + c_2 v_2$$

$$\therefore \begin{pmatrix} 5 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\therefore c_1 + c_2 = 5$$

$$-c_1 + 2c_2 = 1$$

$$3c_2 = 6$$

$$\boxed{c_2 = 2} \Rightarrow \boxed{c_1 = 3}$$

$$\therefore x = 3v_1 + 2v_2$$

$$\therefore A^{10} x = 3A^{10} v_1 + 2A^{10} v_2$$

Using that theorem,

$$A^{10} v_1 = (\lambda_1)^{10} v_1$$

$$= (-1)^{10} v_1 = \boxed{v_1}$$

$$A^{10} v_2 = (\lambda_2)^{10} v_2$$

$$= (2)^{10} v_2$$

$$= \boxed{1024 v_2}$$

$$\therefore A^{10} x = 3v_1 + 2(1024) v_2$$

$$= 3\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2048 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 2051 \\ 4093 \end{pmatrix}}$$