Asymmetric Encryption

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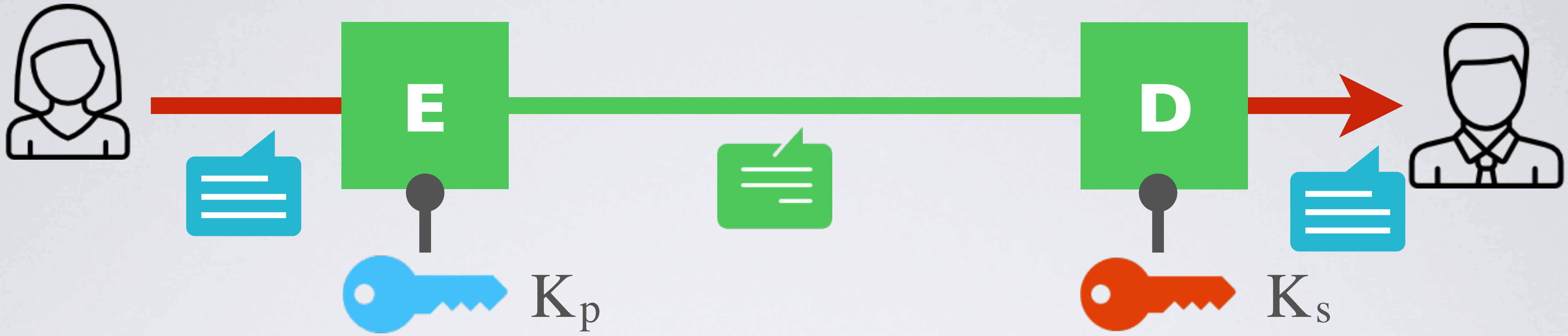
Functional Requirements



- ➡ The public key K_p for encryption
- ➡ The private key K_s for decryption

1. $D_{ks}(E_{kp}(m))=m$ for every pair (K_p, K_s)
2. $E_{kp}(m)$ is easy to compute (either polynomial or linear)
3. $D_{ks}(C)$ is easy to compute (either polynomial or linear)
4. $p = D_{ks}(C)$ finding m is hard without K_s (exponential)
5. Generating a pair (K_p, K_s) is easy to compute (polynomial)
6. Finding a matching key K_s for a given K_p is hard (exponential)

RSA - Rivest, Shamir and Alderman

Key Size	1024 - 4096
Speed	~ factor of 10^6 cycles / operation
Mathematical Foundation	Prime number theory

Number Theory - Prime numbers

Prime Numbers

- p is prime if 1 and p are its only divisors e.g $3, 5, 7, 11 \ldots$
- p and q are relatively prime (a.k.a. coprime) if $\gcd(p,q) = 1$
  e.g $\gcd(4,5) = 1$

➡ There are infinitely many primes

Euler-Fermat Theorem

If $n = p \,.\, q$ and $z = (p-1).(q-1)$

and a such that a and n are relative primes

Then $a^z \equiv 1 \pmod{n}$

Computational Complexity

Easy problems with prime numbers

- Generating a prime number p
- Addition, multiplication, exponentiation
- Inversion, solving linear equations

Hard problem with prime numbers

- Factoring primes
  e.g. given $\mathbf{n}$ find $\mathbf{p}$ and $\mathbf{q}$ such that $\mathbf{n} = \mathbf{p} \cdot \mathbf{q}$

RSA - generating the key pair

1. Pick p and q two large prime numbers and calculate $n = p \cdot q$ (see primality tests)
2. Compute $z = (p-1).(q-1)$
3. Pick a prime number $e < z$ such that e and z are relative primes

➡ (e,n) is the **public key**

4. Solve the linear equation $e * d = 1 \pmod{z}$ to find d

➡ (d,n) is the **private key**
however p and q must be kept secret too

RSA - encryption and decryption

Given $K_p = (e, n)$ and $K_s = (d,n)$

➡ Encryption : $E_{kp}(m) = m^e \bmod n = c$

➡ Decryption : $D_{ks}(c) = c^d \bmod n = m$

➡ $\mathbf{(m^e)^d \bmod n = (m^d)^e \bmod n = m}$

The security of RSA

RSA Labs Challenge : factoring primes set

Key length	Year	Time
140	1999	1 month
155	1999	4 months
160	2003	20 days
200	2005	18 months
768	2009	3 years

Challenges are no longer active

Key length and Key n-bit security

- RSA has very long keys, 1024, 2048 and 4096 are common
- Is it more secure than asymmetric crypto with key lengths of 56, 128, 192, 256 ?

➡ Key lengths **do not compare !**

RSA Key length	Effective key length
1,024	80
2,048	112
3,072	128
7,680	192
15,360	256

Asymmetric vs Symmetric

	Symmetric	Asymmetric
pro	Fast	No key agreement
cons	Key agreement	Very slow

The best of both worlds

- ➡ Use RSA to encrypt a shared key
- ➡ Use AES to encrypt message

$$E(m) = RSA_{Kp}(k), AES_{k}(m)$$

Other asymmetric cryptography schemes

Diffie-Hellman (precursor)

➡ No Authentication but good for key-exchange

El-Gamal

➡ Good properties for homomorphic encryption

Elliptic Curve Cryptography (trending nowadays)

➡ Fast and small keys (190 bits equivalent to 1024 bits RSA)