Deep Generative Models

Lecture 2

Roman Isachenko

Moscow Institute of Physics and Technology
Yandex School of Data Analysis

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Recap of Previous Lecture

We're given **finite** number of i.i.d. samples $\{\mathbf{x}_i\}_{i=1}^n \subset \mathbb{R}^m$ drawn from an **unknown** distribution $p_{\text{data}}(\mathbf{x})$.

Objective

Our aim is to learn a distribution $p_{\text{data}}(\mathbf{x})$ that allows us to:

- Generate new samples from $p_{\text{data}}(\mathbf{x})$ (sample $\mathbf{x} \sim p_{\text{data}}(\mathbf{x})$) — **generation**.
- Evaluate $p_{\text{data}}(\mathbf{x})$ on novel data (answering "How likely is an object $\mathbf{x}$?") — **density estimation**;

Divergence Minimization Task

- $D(\pi \| p) \geq 0$ for all $\pi, p \in \mathcal{P}$;
- $D(\pi \| p) = 0$ if and only if $\pi \equiv p$.

$$\min_{\boldsymbol{\theta}} D(p_{\text{data}} \| p_{\boldsymbol{\theta}})$$

Recap of Previous Lecture

Forward KL Divergence

$$\mathrm{KL}(p_{\text{data}} \| p_{\boldsymbol{\theta}}) = \int \pi(\mathbf{x}) \log \frac{p_{\text{data}}(\mathbf{x})}{p_{\boldsymbol{\theta}}(\mathbf{x})} d\mathbf{x} \rightarrow \min_{\boldsymbol{\theta}}$$

Reverse KL Divergence

$$\mathrm{KL}(p_{\boldsymbol{\theta}} \| p_{\text{data}}) = \int p_{\boldsymbol{\theta}}(\mathbf{x}) \log \frac{p_{\boldsymbol{\theta}}(\mathbf{x})}{p_{\text{data}}(\mathbf{x})} d\mathbf{x} \rightarrow \min_{\boldsymbol{\theta}}$$

Maximum Likelihood Estimation (MLE)

$$\boldsymbol{\theta}^* = \arg\max_{\boldsymbol{\theta}} \prod_{i=1}^{n} p_{\boldsymbol{\theta}}(\mathbf{x}_i) = \arg\max_{\boldsymbol{\theta}} \sum_{i=1}^{n} \log p_{\boldsymbol{\theta}}(\mathbf{x}_i)$$

Maximum likelihood estimation is equivalent to minimizing the Monte Carlo estimate of the forward KL divergence.

Recap of Previous Lecture

Likelihood as Product of Conditionals

Let $\mathbf{x} = (x_1, \ldots, x_m)$, and define $\mathbf{x}_{1:j} = (x_1, \ldots, x_j)$. Then,

$$p_{\boldsymbol{\theta}}(\mathbf{x}) = \prod_{j=1}^{m} p_{\boldsymbol{\theta}}(x_j | \mathbf{x}_{1:j-1}), \quad \log p_{\boldsymbol{\theta}}(\mathbf{x}) = \sum_{j=1}^{m} \log p_{\boldsymbol{\theta}}(x_j | \mathbf{x}_{1:j-1})$$

MLE for Autoregressive Models

$$\boldsymbol{\theta}^* = \arg\max_{\boldsymbol{\theta}} \sum_{i=1}^{n} \sum_{j=1}^{m} \log p_{\boldsymbol{\theta}}(x_{ij} | \mathbf{x}_{i,1:j-1})$$

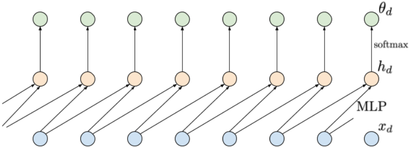
Sampling

$$\hat{x}_1 \sim p_{\boldsymbol{\theta}}(x_1), \quad \hat{x}_2 \sim p_{\boldsymbol{\theta}}(x_2 | \hat{x}_1), \quad \ldots, \quad \hat{x}_m \sim p_{\boldsymbol{\theta}}(x_m | \hat{\mathbf{x}}_{1:m-1})$$

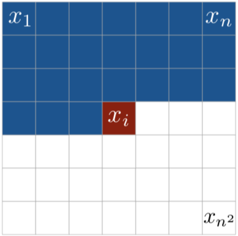
The generated sample is $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_m)$.

Recap of Previous Lecture

Autoregressive MLP



Autoregressive Transformer



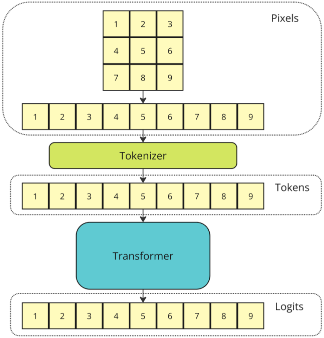


Image credit: https://jmtomczak.github.io/blog/2/2_ARM.html
Chen M. et al. Generative Pretraining from Pixels, 2020

Outline

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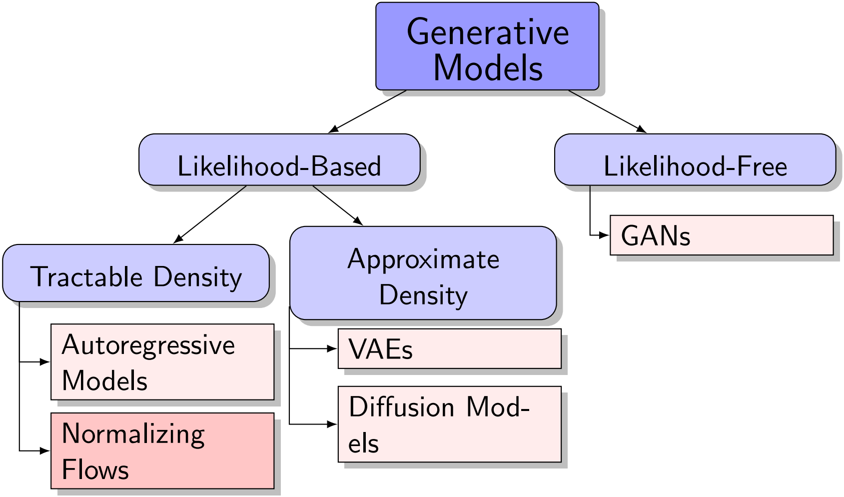
1. Normalizing Flows (NF)

2. NF Examples
    - Linear Normalizing Flows
    - Gaussian Autoregressive NF
    - Coupling Layer (RealNVP)

3. Latent Variable Models (LVM)

Generative Models Zoo



Normalizing Flows: Prerequisites

Jacobian Matrix

Let $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^m$ be a differentiable function.

$$\mathbf{z} = \mathbf{f}(\mathbf{x}), \quad \mathbf{J} = \frac{\partial \mathbf{z}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \cdots & \frac{\partial z_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_m}{\partial x_1} & \cdots & \frac{\partial z_m}{\partial x_m} \end{pmatrix} \in \mathbb{R}^{m \times m}$$

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Change of Variables Theorem (CoV)

Let $\mathbf{x} \in \mathbb{R}^m$ be a random vector with density $p(\mathbf{x})$, and let $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^m$ be a C^1-diffeomorphism ($\mathbf{f}$ and $\mathbf{f}^{-1}$ are continuously differentiable mappings). If $\mathbf{z} = \mathbf{f}(\mathbf{x})$, then

$$p(\mathbf{x}) = p(\mathbf{z}) |\det(\mathbf{J_f})| = p(\mathbf{z}) \left| \det \left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}} \right) \right|$$

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$$p(\mathbf{z}) = p(\mathbf{x}) |\det(\mathbf{J}_{\mathbf{f}^{-1}})| = p(\mathbf{x}) \left| \det \left(\frac{\partial \mathbf{x}}{\partial \mathbf{z}} \right) \right| = p(\mathbf{f}^{-1}(\mathbf{z})) \left| \det \left(\frac{\partial \mathbf{f}^{-1}(\mathbf{z})}{\partial \mathbf{z}} \right) \right|$$

Jacobian Determinant

Inverse Function Theorem

If the function $\mathbf{f}$ is invertible and its Jacobian is continuous and non-singular, then

$$\mathbf{J}_{\mathbf{f}^{-1}} = \mathbf{J}_{\mathbf{f}}^{-1};$$

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- $\mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x})$ is a parameterized transformation.

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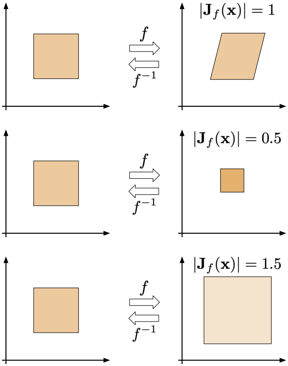
- $\mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x})$ is a parameterized transformation.
- The determinant of the Jacobian $\mathbf{J} = \frac{\partial \mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x})}{\partial \mathbf{x}}$ quantifies how the volume is changed by the transformation.



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Fitting Normalizing Flows

MLE Problem

$$p_{\boldsymbol{\theta}}(\mathbf{x}) = p(\mathbf{z}) \left| \det \left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}} \right) \right| = p(\mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x})) \left| \det \left(\frac{\partial \mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x})}{\partial \mathbf{x}} \right) \right|$$

Dinh L., Sohl-Dickstein J., Bengio S. Density Estimation Using Real NVP, 2016

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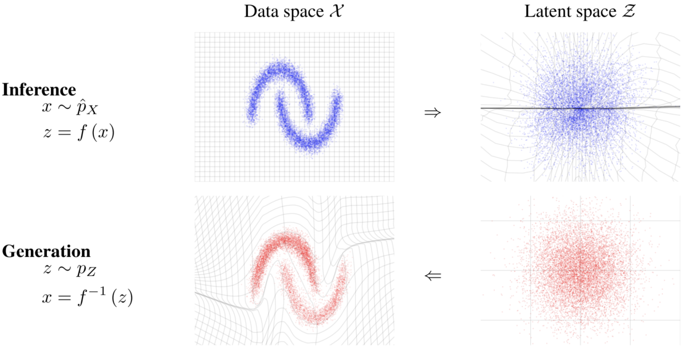

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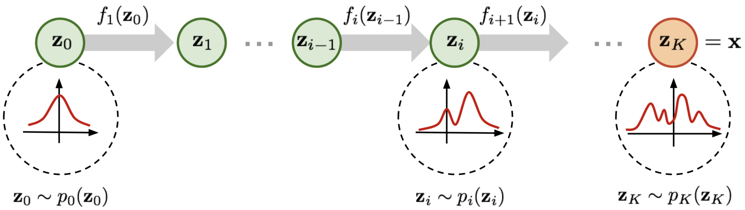
Composition of Normalizing Flows



https://lilianweng.github.io/lil-log/2018/10/13/flow-based-deep-generative-models.html

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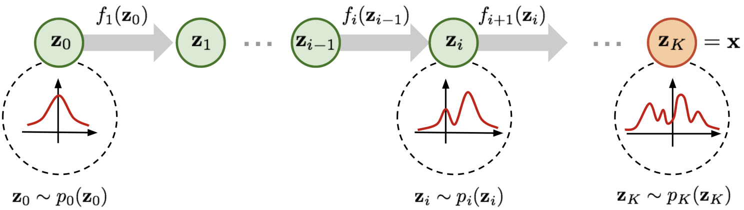

Theorem

If every $\{\mathbf{f}_k\}_{k=1}^{K}$ satisfies the conditions of the change-of-variables theorem, then the composition $\mathbf{f}(\mathbf{x}) = \mathbf{f}_K \circ \ldots \circ \mathbf{f}_1(\mathbf{x})$ also satisfies them.

$$p_\theta(\mathbf{x}) = p(\mathbf{f}(\mathbf{x})) \left| \det \left(\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \right) \right|$$

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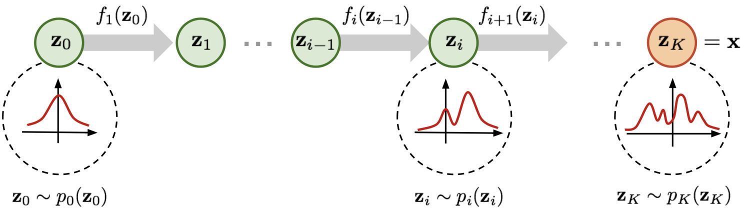

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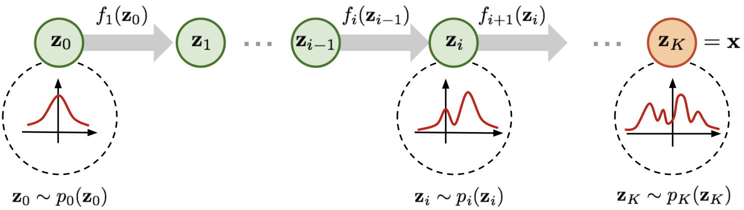

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Normalizing Flows (NF)

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Definition

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- **Normalizing** refers to mapping samples from $p_{\text{data}}(\mathbf{x})$ to a base distribution $p(\mathbf{z})$.
- **Flow** describes the sequence of transformations that maps samples from $p(\mathbf{z})$ to the target, more complex distribution.

$$\mathbf{z} = \mathbf{f}_K \circ \ldots \circ \mathbf{f}_1(\mathbf{x}); \quad \mathbf{x} = \mathbf{f}_1^{-1} \circ \ldots \circ \mathbf{f}_K^{-1}(\mathbf{z})$$

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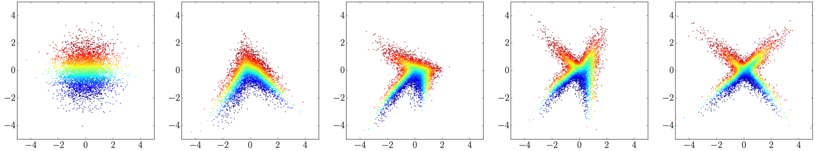
Note: Here we consider only **continuous** random variables.

Normalizing Flows

Example: 4-Step NF



Papamakarios G. et al. *Normalizing Flows for Probabilistic Modeling and Inference, 2019*

Normalizing Flows

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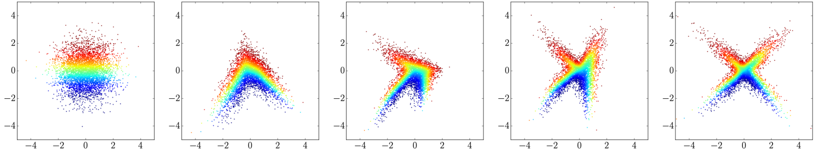

NF Log-Likelihood

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What's the computational complexity of evaluating this determinant?

Papamakarios G. et al. Normalizing Flows for Probabilistic Modeling and Inference, 2019

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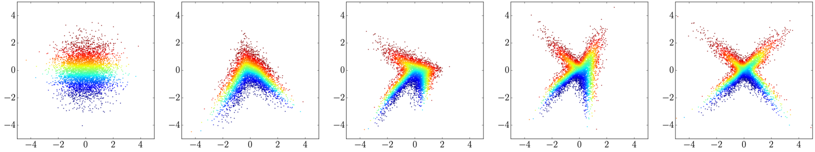

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Requirements

- Efficient computation of the Jacobian $\mathbf{J}_{\mathbf{f}} = \frac{\partial \mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x})}{\partial \mathbf{x}}$
- Efficient inversion of the transformation $\mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x})$

Papamakarios G. et al. Normalizing Flows for Probabilistic Modeling and Inference, 2019

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What is $\det(\mathbf{J})$ in These Cases?

Consider a linear layer $\mathbf{z} = \mathbf{W}\mathbf{x}$, $\mathbf{W} \in \mathbb{R}^{m \times m}$.

1. $\mathbf{z}$ is a permutation of $\mathbf{x}$.

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$$\log \left| \det \left(\frac{\partial \mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x})}{\partial \mathbf{x}} \right) \right| = \log \left| \prod_{j=1}^{m} \frac{\partial f_{j,\boldsymbol{\theta}}(x_j)}{\partial x_j} \right| = \sum_{j=1}^{m} \log \left| \frac{\partial f_{j,\boldsymbol{\theta}}(x_j)}{\partial x_j} \right|$$

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$$\log \left| \det \left(\frac{\partial \mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x})}{\partial \mathbf{x}} \right) \right| = \log \left| \prod_{j=1}^{m} \frac{\partial f_{j,\boldsymbol{\theta}}(x_j)}{\partial x_j} \right| = \sum_{j=1}^{m} \log \left| \frac{\partial f_{j,\boldsymbol{\theta}}(x_j)}{\partial x_j} \right|$$

3. z_j depends only on $\mathbf{x}_{1:j}$ (autoregressive dependency).

Linear Normalizing Flows

$$\mathbf{z} = \mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x}) = \mathbf{W}\mathbf{x}, \quad \mathbf{W} \in \mathbb{R}^{m \times m}, \quad \boldsymbol{\theta} = \mathbf{W}, \quad \mathbf{J}_{\mathbf{f}} = \mathbf{W}^T$$

In general, matrix inversion has computational complexity $O(m^3)$.

Papamakarios G. et al. Normalizing Flows for Probabilistic Modeling and Inference, 2019

Linear Normalizing Flows

$$\mathbf{z} = \mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x}) = \mathbf{W}\mathbf{x}, \quad \mathbf{W} \in \mathbb{R}^{m \times m}, \quad \boldsymbol{\theta} = \mathbf{W}, \quad \mathbf{J}_{\mathbf{f}} = \mathbf{W}^T$$

In general, matrix inversion has computational complexity $O(m^3)$.

Invertibility

- Diagonal matrix: $O(m)$.
- Triangular matrix: $O(m^2)$.
- Directly parameterizing all invertible matrices in a continuous way is infeasible
  (there is not surjective function from $\mathbb{R}^{m^2}$ to the set of all invertible matrices of size $m \times m$).

Papamakarios G. et al. Normalizing Flows for Probabilistic Modeling and Inference, 2019

Linear Normalizing Flows

$$\mathbf{z} = \mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x}) = \mathbf{W}\mathbf{x}, \quad \mathbf{W} \in \mathbb{R}^{m \times m}, \quad \boldsymbol{\theta} = \mathbf{W}, \quad \mathbf{J}_{\mathbf{f}} = \mathbf{W}^T$$

Kingma D. P., et al. Glow: Generative Flow with Invertible 1x1 Convolutions, 2018
Hoogeboom E., et al. Emerging Convolutions for Generative Normalizing Flows, 2019

Linear Normalizing Flows

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Matrix Decompositions

- **LU Decomposition:**

$$\mathbf{W} = \mathbf{P}\mathbf{L}\mathbf{U},$$

where $\mathbf{P}$ is a permutation matrix, $\mathbf{L}$ is lower triangular with positive diagonal, and $\mathbf{U}$ is upper triangular with positive diagonal.

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Linear Normalizing Flows

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Matrix Decompositions

- **LU Decomposition:**

$$\mathbf{W} = \mathbf{PLU},$$

  where $\mathbf{P}$ is a permutation matrix, $\mathbf{L}$ is lower triangular with positive diagonal, and $\mathbf{U}$ is upper triangular with positive diagonal.

- **QR Decomposition:**

$$\mathbf{W} = \mathbf{QR},$$

  where $\mathbf{Q}$ is orthogonal, and $\mathbf{R}$ is upper triangular with positive diagonal.

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Hoogeboom E., et al. Emerging Convolutions for Generative Normalizing Flows, 2019

Linear Normalizing Flows

$$\mathbf{z} = \mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x}) = \mathbf{W}\mathbf{x}, \quad \mathbf{W} \in \mathbb{R}^{m \times m}, \quad \boldsymbol{\theta} = \mathbf{W}, \quad \mathbf{J}_{\mathbf{f}} = \mathbf{W}^T$$

Matrix Decompositions

- **LU Decomposition:**

  $$\mathbf{W} = \mathbf{P}\mathbf{L}\mathbf{U},$$

  where $\mathbf{P}$ is a permutation matrix, $\mathbf{L}$ is lower triangular with positive diagonal, and $\mathbf{U}$ is upper triangular with positive diagonal.
- **QR Decomposition:**

  $$\mathbf{W} = \mathbf{Q}\mathbf{R},$$

  where $\mathbf{Q}$ is orthogonal, and $\mathbf{R}$ is upper triangular with positive diagonal.

Decomposition is performed only at initialization; the decomposed matrices ($\mathbf{P}, \mathbf{L}, \mathbf{U}$ or $\mathbf{Q}, \mathbf{R}$) are optimized during training.

Kingma D. P., et al. Glow: Generative Flow with Invertible 1x1 Convolutions, 2018
Hoogeboom E., et al. Emerging Convolutions for Generative Normalizing Flows, 2019

Outline

Gaussian Autoregressive Model

Consider the autoregressive model:

$$p_{\boldsymbol{\theta}}(\mathbf{x}) = \prod_{j=1}^{m} p_{\boldsymbol{\theta}}(x_j|\mathbf{x}_{1:j-1}), \quad p_{\boldsymbol{\theta}}(x_j|\mathbf{x}_{1:j-1}) = \mathcal{N}\left(\mu_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1}), \sigma^2_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1})\right)$$

Kingma D. P. et al. Improving Variational Inference with Inverse Autoregressive Flow, 2016

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Sampling

$$x_j = \sigma_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1}) \cdot z_j + \mu_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1}), \quad z_j \sim \mathcal{N}(0, 1)$$

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Sampling

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Inverse Transformation

$$z_j = \frac{x_j - \mu_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1})}{\sigma_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1})}$$

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Sampling

$$x_j = \sigma_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1}) \cdot z_j + \mu_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1}), \quad z_j \sim \mathcal{N}(0, 1)$$

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$$z_j = \frac{x_j - \mu_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1})}{\sigma_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1})}$$

- This gives an C^1-**diffeomorphism** from $p(\mathbf{z})$ to $p_{\boldsymbol{\theta}}(\mathbf{x})$ (assume that $\sigma_j \neq 0$).

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Sampling

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Inverse Transformation

$$z_j = \frac{x_j - \mu_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1})}{\sigma_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1})}$$

- This gives an C^1-**diffeomorphism** from $p(\mathbf{z})$ to $p_{\boldsymbol{\theta}}(\mathbf{x})$ (assume that $\sigma_j \neq 0$).
- This model is called an autoregressive (AR) NF with base distribution $p(\mathbf{z}) = \mathcal{N}(0, \mathbf{I})$.

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Gaussian Autoregressive Model

Consider the autoregressive model:

$$p_{\boldsymbol\theta}(\mathbf{x}) = \prod_{j=1}^{m} p_{\boldsymbol\theta}(x_j|\mathbf{x}_{1:j-1}), \quad p_{\boldsymbol\theta}(x_j|\mathbf{x}_{1:j-1}) = \mathcal{N}\left(\mu_{j,\boldsymbol\theta}(\mathbf{x}_{1:j-1}), \sigma^2_{j,\boldsymbol\theta}(\mathbf{x}_{1:j-1})\right)$$

Sampling

$$x_j = \sigma_{j,\boldsymbol\theta}(\mathbf{x}_{1:j-1}) \cdot z_j + \mu_{j,\boldsymbol\theta}(\mathbf{x}_{1:j-1}), \quad z_j \sim \mathcal{N}(0,1)$$

Inverse Transformation

$$z_j = \frac{x_j - \mu_{j,\boldsymbol\theta}(\mathbf{x}_{1:j-1})}{\sigma_{j,\boldsymbol\theta}(\mathbf{x}_{1:j-1})}$$

- This gives an C^1-**diffeomorphism** from $p(\mathbf{z})$ to $p_{\boldsymbol\theta}(\mathbf{x})$ (assume that $\sigma_j \neq 0$).
- This model is called an autoregressive (AR) NF with base distribution $p(\mathbf{z}) = \mathcal{N}(0, \mathbf{I})$.
- The Jacobian matrix of this transformation is triangular.

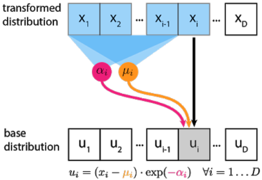
Kingma D. P. et al. Improving Variational Inference with Inverse Autoregressive Flow, 2016

Gaussian Autoregressive NF

Forward Transformation: $\mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x})$

$$\mathbf{z} = \mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x})$$

$$z_j = \frac{x_j - \mu_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1})}{\sigma_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1})}$$



Papamakarios G., Pavlakou T., Murray I. Masked Autoregressive Flow for Density Estimation, 2017

Gaussian Autoregressive NF

Forward Transformation: $\mathbf{f}_\theta(\mathbf{x})$

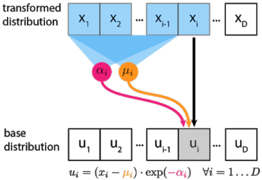
$$\mathbf{z} = \mathbf{f}_\theta(\mathbf{x})$$

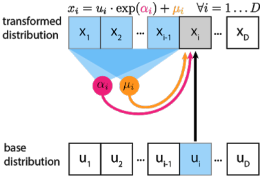
$$z_j = \frac{x_j - \mu_{j,\theta}(\mathbf{x}_{1:j-1})}{\sigma_{j,\theta}(\mathbf{x}_{1:j-1})}$$

Inverse Transformation: $\mathbf{f}_\theta^{-1}(\mathbf{z})$

$$\mathbf{x} = \mathbf{f}_\theta^{-1}(\mathbf{z})$$

$$x_j = \sigma_{j,\theta}(\mathbf{x}_{1:j-1}) \cdot z_j + \mu_{j,\theta}(\mathbf{x}_{1:j-1})$$





Papamakarios G., Pavlakou T., Murray I. Masked Autoregressive Flow for Density Estimation, 2017

Gaussian Autoregressive NF

Forward Transformation: $\mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x})$

$$\mathbf{z} = \mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x})$$

$$z_j = \frac{x_j - \mu_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1})}{\sigma_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1})}$$

Inverse Transformation: $\mathbf{f}_{\boldsymbol{\theta}}^{-1}(\mathbf{z})$

$$\mathbf{x} = \mathbf{f}_{\boldsymbol{\theta}}^{-1}(\mathbf{z})$$

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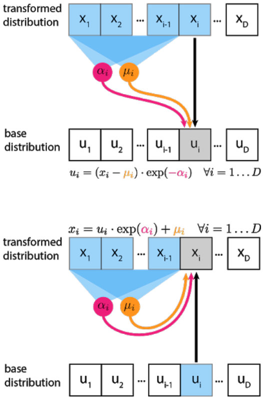

- Sampling must be done sequentially, but density estimation can be parallelized.
- The forward KL divergence is a natural objective for training.

Papamakarios G., Pavlakou T., Murray I. Masked Autoregressive Flow for Density Estimation, 2017

Outline

RealNVP

Split $\mathbf{x}$ and $\mathbf{z}$ into two parts:

$$\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2] = [\mathbf{x}_{1:d}, \mathbf{x}_{d+1:m}]; \quad \mathbf{z} = [\mathbf{z}_1, \mathbf{z}_2] = [\mathbf{z}_{1:d}, \mathbf{z}_{d+1:m}]$$

Dinh L., Sohl-Dickstein J., Bengio S. Density Estimation Using Real NVP, 2016

RealNVP

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Coupling Layer

$$\begin{cases} \mathbf{x}_1 = \mathbf{z}_1 \\ \mathbf{x}_2 = \mathbf{z}_2 \odot \boldsymbol{\sigma}_{\boldsymbol{\theta}}(\mathbf{z}_1) + \boldsymbol{\mu}_{\boldsymbol{\theta}}(\mathbf{z}_1) \end{cases}$$

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Split $\mathbf{x}$ and $\mathbf{z}$ into two parts:

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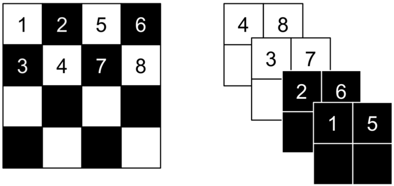
Coupling Layer

$$\begin{cases} \mathbf{x}_1 = \mathbf{z}_1 \\ \mathbf{x}_2 = \mathbf{z}_2 \odot \boldsymbol{\sigma}_\theta(\mathbf{z}_1) + \boldsymbol{\mu}_\theta(\mathbf{z}_1) \end{cases} \qquad \begin{cases} \mathbf{z}_1 = \mathbf{x}_1 \\ \mathbf{z}_2 = (\mathbf{x}_2 - \boldsymbol{\mu}_\theta(\mathbf{x}_1)) \odot \frac{1}{\boldsymbol{\sigma}_\theta(\mathbf{x}_1)} \end{cases}$$

Image Partitioning



- Checkerboard ordering corresponds to masking.
- Channelwise ordering relies on splitting.

Dinh L., Sohl-Dickstein J., Bengio S. Density Estimation Using Real NVP, 2016

RealNVP

Coupling Layer

$$\begin{cases} \mathbf{x}_1 = \mathbf{z}_1 \\ \mathbf{x}_2 = \mathbf{z}_2 \odot \boldsymbol{\sigma}_{\boldsymbol{\theta}}(\mathbf{z}_1) + \boldsymbol{\mu}_{\boldsymbol{\theta}}(\mathbf{z}_1) \end{cases} \qquad \begin{cases} \mathbf{z}_1 = \mathbf{x}_1 \\ \mathbf{z}_2 = (\mathbf{x}_2 - \boldsymbol{\mu}_{\boldsymbol{\theta}}(\mathbf{x}_1)) \odot \frac{1}{\boldsymbol{\sigma}_{\boldsymbol{\theta}}(\mathbf{x}_1)} \end{cases}$$

In both training and sampling, only a single forward pass is needed!

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In both training and sampling, only a single forward pass is needed!

Jacobian

$$\det\left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}}\right) = \det\begin{pmatrix} \mathbf{I}_d & 0_{d \times m-d} \\ \frac{\partial \mathbf{z}_2}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{z}_2}{\partial \mathbf{x}_2} \end{pmatrix}$$

Dinh L., Sohl-Dickstein J., Bengio S. Density Estimation Using Real NVP, 2016

RealNVP

Coupling Layer

$$\begin{cases} \mathbf{x}_1 = \mathbf{z}_1 \\ \mathbf{x}_2 = \mathbf{z}_2 \odot \boldsymbol{\sigma}_{\boldsymbol{\theta}}(\mathbf{z}_1) + \boldsymbol{\mu}_{\boldsymbol{\theta}}(\mathbf{z}_1) \end{cases} \qquad \begin{cases} \mathbf{z}_1 = \mathbf{x}_1 \\ \mathbf{z}_2 = (\mathbf{x}_2 - \boldsymbol{\mu}_{\boldsymbol{\theta}}(\mathbf{x}_1)) \odot \frac{1}{\boldsymbol{\sigma}_{\boldsymbol{\theta}}(\mathbf{x}_1)} \end{cases}$$

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RealNVP

Coupling Layer

$$\begin{cases} \mathbf{x}_1 = \mathbf{z}_1 \\ \mathbf{x}_2 = \mathbf{z}_2 \odot \boldsymbol{\sigma}_{\boldsymbol{\theta}}(\mathbf{z}_1) + \boldsymbol{\mu}_{\boldsymbol{\theta}}(\mathbf{z}_1) \end{cases} \qquad \begin{cases} \mathbf{z}_1 = \mathbf{x}_1 \\ \mathbf{z}_2 = (\mathbf{x}_2 - \boldsymbol{\mu}_{\boldsymbol{\theta}}(\mathbf{x}_1)) \odot \frac{1}{\boldsymbol{\sigma}_{\boldsymbol{\theta}}(\mathbf{x}_1)} \end{cases}$$

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Gaussian AR NF

$$\mathbf{x} = \mathbf{f}_{\boldsymbol{\theta}}^{-1}(\mathbf{z}) \quad \Rightarrow \quad x_j = \sigma_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1}) \cdot z_j + \mu_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1})$$

$$\mathbf{z} = \mathbf{f}_{\boldsymbol{\theta}}(\mathbf{x}) \quad \Rightarrow \quad z_j = (x_j - \mu_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1})) \cdot \frac{1}{\sigma_{j,\boldsymbol{\theta}}(\mathbf{x}_{1:j-1})}.$$

How can the RealNVP layer be derived as a special instance of the <u>Gaussian autoregressive</u> NF?

Dinh L., Sohl-Dickstein J., Bengio S. Density Estimation Using Real NVP, 2016

Outline

Bayesian Framework

Bayes' Theorem

$$p(\boldsymbol{\theta}|\mathbf{x}) = \frac{p(\mathbf{x}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathbf{x})} = \frac{p(\mathbf{x}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{\int p(\mathbf{x}|\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}}$$

- $\mathbf{x}$: observed variables;
- $\boldsymbol{\theta}$: unknown latent variables/parameters;
- $p_{\boldsymbol{\theta}}(\mathbf{x}) = p(\mathbf{x}|\boldsymbol{\theta})$: likelihood;
- $p(\mathbf{x}) = \int p(\mathbf{x}|\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}$: evidence;
- $p(\boldsymbol{\theta})$: prior distribution;
- $p(\boldsymbol{\theta}|\mathbf{x})$: posterior distribution.

Bayesian Framework

Bayes' Theorem

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- $p(\boldsymbol{\theta})$: prior distribution;
- $p(\boldsymbol{\theta}|\mathbf{x})$: posterior distribution.

Interpretation

- We begin with unknown variables $\boldsymbol{\theta}$ and a prior belief $p(\boldsymbol{\theta})$.
- Once data $\mathbf{x}$ is observed, the posterior $p(\boldsymbol{\theta}|\mathbf{x})$ incorporates both prior beliefs and evidence from the data.

Bayesian Framework

Consider the case where the unobserved variables $\boldsymbol{\theta}$ are model parameters (i.e., $\boldsymbol{\theta}$ are random variables).

- $\mathbf{X} = \{\mathbf{x}_i\}_{i=1}^n$: observed samples;
- $p(\boldsymbol{\theta})$: prior distribution.

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Posterior Distribution

$$p(\boldsymbol{\theta}|\mathbf{X}) = \frac{p(\mathbf{X}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\mathbf{X})} = \frac{p(\mathbf{X}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{\int p(\mathbf{X}|\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}}$$

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If the evidence $p(\mathbf{X})$ is intractable (due to high-dimensional integration), the posterior cannot be computed exactly.

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If the evidence $p(\mathbf{X})$ is intractable (due to high-dimensional integration), the posterior cannot be computed exactly.

Maximum a Posteriori (MAP) Estimation

$$\boldsymbol{\theta}^* = \arg\max_{\boldsymbol{\theta}} p(\boldsymbol{\theta}|\mathbf{X}) = \arg\max_{\boldsymbol{\theta}}(\log p(\mathbf{X}|\boldsymbol{\theta}) + \log p(\boldsymbol{\theta}))$$

Latent Variable Models (LVM)

Maximum Likelihood Extimation (MLE) Problem

$$\boldsymbol{\theta}^* = \arg\max_{\boldsymbol{\theta}} p_{\boldsymbol{\theta}}(\mathbf{X}) = \arg\max_{\boldsymbol{\theta}} \prod_{i=1}^{n} p_{\boldsymbol{\theta}}(\mathbf{x}_i) = \arg\max_{\boldsymbol{\theta}} \sum_{i=1}^{n} \log p_{\boldsymbol{\theta}}(\mathbf{x}_i).$$

Latent Variable Models (LVM)

Maximum Likelihood Extimation (MLE) Problem

$$\boldsymbol{\theta}^* = \arg\max_{\boldsymbol{\theta}} p_{\boldsymbol{\theta}}(\mathbf{X}) = \arg\max_{\boldsymbol{\theta}} \prod_{i=1}^{n} p_{\boldsymbol{\theta}}(\mathbf{x}_i) = \arg\max_{\boldsymbol{\theta}} \sum_{i=1}^{n} \log p_{\boldsymbol{\theta}}(\mathbf{x}_i).$$

The distribution $p_{\boldsymbol{\theta}}(\mathbf{x})$ can be highly complex and often intractable (just like the true data distribution $p_{\text{data}}(\mathbf{x})$).

Latent Variable Models (LVM)

Maximum Likelihood Extimation (MLE) Problem

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The distribution $p_{\boldsymbol{\theta}}(\mathbf{x})$ can be highly complex and often intractable (just like the true data distribution $p_{\text{data}}(\mathbf{x})$).

Extended Probabilistic Model

Introduce a latent variable $\mathbf{z}$ for each observed sample $\mathbf{x}$:

$$p_{\boldsymbol{\theta}}(\mathbf{x}, \mathbf{z}) = p_{\boldsymbol{\theta}}(\mathbf{x}|\mathbf{z})p(\mathbf{z}); \quad \log p_{\boldsymbol{\theta}}(\mathbf{x}, \mathbf{z}) = \log p_{\boldsymbol{\theta}}(\mathbf{x}|\mathbf{z}) + \log p(\mathbf{z}).$$

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Motivation

Both $p_{\boldsymbol{\theta}}(\mathbf{x}|\mathbf{z})$ and $p(\mathbf{z})$ are usually much simpler than $p_{\boldsymbol{\theta}}(\mathbf{x})$.

Summary

- The CoV theorem provides a method for computing a random variable's density under an invertible transformation.
- Normalizing flows transform a simple base distribution into a complex one via a sequence of invertible mappings, each with efficient Jacobian determinants.
- Linear NFs capture invertible matrices by using matrix decompositions.
- Gaussian autoregressive NFs are AR models with triangular Jacobians.
- The RealNVP coupling layer provides an efficient normalizing flow (a special case of AR NF), supporting fast inference and sampling.
- The Bayesian framework generalizes nearly all standard machine learning methods.