Question 2: Evaluate the limit or show that it does not exist:

a)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+2y^2}$$

Let $y = mx, m \in R, x = x$:

$$\lim_{x\to 0} \frac{m^2x^4}{x^2+2m^2x^2} = \lim_{x\to 0} \frac{m^2x^2}{1+2m^2} = 0$$

Since $m \in R$, it is obvious that the above conclusion passes the two path test, showing that the limit exists. Assume the limit $L = 0$. Let $x = r\cos\theta$ and let $y = r\sin\theta$. It follows that:

$$\lim_{(x,y)\to(0,0)} \frac{r^4cos^2(\theta)sin^2(\theta)}{r^2+r^2sin^2(\theta)}$$

$$\frac{r^2cos^2(\theta)sin^2(\theta)}{1+sin^2(\theta)} \le r^2cos^2(\theta)sin^2(\theta) \to 0 \le r^2cos^2(\theta)sin^2(\theta) \le r^2$$

It follows that:

$$\lim_{r\to 0} r^2 = 0$$

Therefore, by Squeeze Theorem,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+2y^2} = 0$$

b)

$$\lim_{(x,y)\to(0,0)} \frac{2xy\cos y}{x^2+2y^2}$$

It is obvious that:

$$\frac{2xy\cos(y)}{x^2+2y^2} \le 2xy\cos y \to -2xy \le 2xy\cos y \le 2xy$$

It follows that:

$$\lim_{(x,y)\to(0,0)} -2xy = \lim_{(x,y)\to(0,0)} 2xy = 0$$

Thus, by Squeeze Theorem:

$$\lim_{(x,y)\to(0,0)} \frac{2xy\cos y}{x^2+2y^2} = 0$$