Assignment 1 Solutions

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Due Sept.27/17

Question 1; For each of the three given functions decide whether it is a solution of the respective ODE:

i)

$$y_1(x) = e^x, y_1'(x) = e^x \rightarrow e^x \neq e^x + e^x$$

Therefore $y_1(x)$ is not a solution of the DE.

$$y_2(x) = xe^x, y_2' = x + xe^x \rightarrow e^x + xe^x = xe^x + e^x$$

Therefore $y_2(x)$ is a solution to the DE.

$$y_3(x) = x, y_3'(x) = 1 \rightarrow 1 \neq x + e^x$$

Therefore $y_3(x)$ is not a solution to the DE.

ii)

$$y_1(x) = -\frac{x^2}{4}, y_1'(x) = -\frac{x}{2} \rightarrow \frac{x^2}{4} + x(-\frac{x}{2}) + \frac{1}{4}x^2 = \frac{x^2}{2} - \frac{x^2}{2} = 0$$

Therefore $y_1(x)$ is a solution to the DE.

$$y_2(x) = x + 1, y_2'(x) = 1 \rightarrow 1 + x(x+1) - (x+1) \neq 0$$

Therefore $y_2(x)$ is not a solution to the DE.

$$y_3(x) = x^2 + 1, y_3'(x) = 2x \rightarrow 4x^2 + 2x^2 - (x^2+1) \neq 0$$

Therefore $y_3(x)$ is not a solution to the DE.

iii)

$$y_1(x) = x \rightarrow y''(x) DNE$$

Therefore, $y_1(x)$ is not a solution to the DE.

$y_2(x) = x^2, y_2'(x) = 2x, y_2''(x) = 2 \rightarrow (1-x^2)(2)-(x^2)(2x)+2 = 2-2x^2-2x^2+2 = 4-4x^2 \neq 0$

Therefore, $y_2(x)$ is not a solution to the DE.

$y_3(x) = (1-x^2)^{1/2}, y_3'(x) = x(1-x^2)^{-1/2}, y_3''(x) = -(1-x^2)^{-3/2} \rightarrow$
$(1-x^2)(-1)(1-x^2)^{-3/2} + x(x)(1-x^2)^{-1/2} + (1-x^2)^{1/2}$
$= -(1-x^2)^{-1/2}+x^2 1-x^2)^{-1/2} + (1-x^2)^{1/2} = -(1-x^2)(1-x^2)^{-1/2} + (1-x^2)^{1/2} = 0$
Therefore, $y_3(x)$ is a solution to the DE.

Question 2; Solve the IVP for y=y(x),

$$y' = 6y - 3y^2, y(1) = 1,$$

and determine $lim_{x\to\infty} y(x)$. Re-do your calculation with $y(2) = 3$ instead of $y(1) = 1$.

$\frac{dy}{dx} = 6y - 3y^2 \rightarrow \frac{dy}{6y-3y^2} = dx$
Using partial fraction decomposition:
$\frac{1}{6y-3y^2} = \frac{A(2-y)}{(3y)(2-y)} + \frac{B(3y)}{(3y)(2-y)}$
Let A = 1/2 and B = 1/6.
$\frac{1}{6y-3y^2} = \frac{1}{6y} - \frac{1}{6(2-y)}$
It then follows that:
$\int \frac{1}{6y-3y^2} = \int \frac{1}{6y} - \int \frac{1}{6(2-y)}$
$= \frac{ln|y|}{6} - \frac{ln|2-y|}{6} + C$
$\therefore \int \frac{dy}{6y-3y^2} = \frac{ln|y|}{6} - \frac{ln|2-y|}{6} + C = x$

Rearranging for y=f(x) format below:
$x - c = (1/6)(ln|y| - ln|2-y|)$
$6(x+c) = ln|\frac{y}{2-y}|$
$e^{6(x+c)} = \frac{y}{2-y}$
$2e^{6(x+c)} - ye^{6(x+c)} = y$
$\frac{2e^{6x+6c}}{1+e^{6x+6c}} = y$

Using the initial values$(x,y) = (1,1)$,
$1 = \frac{2e^{6-c}}{1+e^{6-c}}$
$2e^{6-6c} = 1 + e^{6-6c}$
$e^{6-6c} = 1$
$6 - 6c = 0$
$c = 1$
$\therefore$ The function when $y(1) = 1$ is,
$y = \frac{2e^{6x-6}}{1+e^{6x-6}}$

Using the initial conditions $(x,y) = (2,3)$

$\frac{2e^{12-6c}}{1+e^{12-6c}} = 3$
$2e^{12-6c} = 3 + 3e^{12-6c}$
$-e^{12-6c} = 3$

The above equation has no real solution for c.

Question 3; Find the general solution of $y' - \frac{2xy}{x^2+1} = 1$.

Begin by multiplying μ on both sides.

$$\mu y' - \frac{2xy}{x^2+1}\mu = \mu$$

Notice $(\mu y)' = y'\mu + \mu' y. Then : \mu y' - \frac{2xy\mu}{x^2+1} = \mu y' + y\mu'$

$$\to -\frac{2xy\mu}{x^2+1} = y\mu'$$

$$\to \int \frac{d\mu}{\mu} = -\int \frac{2x}{x^2+1}$$

Let $a = x^2+1; da = 2x$. Then

$$ln(\mu) = -\int \frac{da}{a}$$

$$\to ln(\mu) = ln(a^{-1})$$

$$\to \mu = \frac{1}{x^2+1}$$

It follows that:

$$\int (\mu y)' = \int (x^2+1)^{-1}$$

Thus:

$$y = (x^2+1)(\arctan(x) + c)$$

Question 4; Determine the most general function $N = N(x, y)$ such that the equation

$$(ye^{xy} - 4x^3y + 2)dx + N(x, y)dy = 0$$

is exact.

Using **Theorem 2.16**:
Suppose N(x,y) is twice continuously differentiable in $\omega \subset^2$. Suppose the equation above is exact. It then follows that:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$e^{xy} + xye^{xy} - 4x^3 = \frac{\partial N}{\partial x}$$

Treating y as a constant and taking the integral of x on both sides leads to:

$$\int e^{xy} + \int yxe^{xy} - \int 4x^3 = N(x,y)$$

The first integral and third integral are trivial to solve. The third integral requires technique. I will choose to use integration by parts.
Let $u = yx, du = y, dv = e^{yx}, v = \frac{e^{xy}}{y}. It follows that: \int yxe^{xy} = xe^{xy} - \int \frac{e^{xy}}{y} * y = xe^{xy} - \frac{e^{xy}}{y}$ Combining all integrals to form N(x,y):

$$\frac{e^{xy}}{y} + xe^{xy} - \frac{e^{xy}}{y} - x^4 = N(x,y) \to xe^{xy} - x^4 = N(x,y)$$

Therefore, $N(x,y) = xe^{xy} - x^4$.

Question 5; Solve the IVP $(2y - x)y' + 2x = y$ using the initial conditions $y(1) = 3$.

Rearranging the equation:

$$(2y - x)y' + 2x = y \to (2y - x)dy + (2x - y)dx$$

Let $M(x,y) = 2x - y$, and let $N(x,y) = 2y - x$. It follows that:

$$\frac{\partial N}{\partial x} = -1 = \frac{\partial M}{\partial y}$$

The equation is exact. By **theorem 2.16**, there exists a function F such that:

$$\frac{\partial F}{\partial y} = N = 2y - x \to F = y^2 - xy + g(x)$$

It follows that:

$$\frac{\partial F}{\partial x} = M = 2x - y = \frac{\partial y^2 - xy + g(x)}{\partial x} = -y + g'(x)$$

Then:

$$2x - y = -y + g'(x) \to 2x = g'(x) \to g(x) = x^2 + k$$

Substituting $g(x)$ into F:

$$F = y^2 - xy + x^2 + k$$

Therefore, the solution to the DE in implicit form is:

$$y^2 - xy + x^2 = c$$

Using the initial conditions $(x,y) = (1,3)$:

$$3^2 - 3(1) + 1^2 = 7 = c$$

Therefore, the solution to the IVP is:

$$y^2 - xy + x^2 = 7$$

Question 6; Demonstrate first that the ODE $e^x(x+1) + (ye^6 - xe^x)y' = 0$ is *not* exact. Then show that $\mu(x, y) = e^{-y}$ is an integrating factor, i.e., multiplication by μ makes the ODE exact. Use this to find the general solution.

Rearranging the equation:

$$e^x(x+1) + (-ye^y + xe^x)y' = 0 \to e^x(x+1)dx + (ye^y - xe^x)dy = 0$$

Let $N = ye^y - xe^x$, and $M = e^x(x+1)$. It follows that:

$$\frac{\partial M}{\partial y} = 0, \frac{\partial N}{\partial x} = xe^x + e^x$$

It is obvious that they are not equivalent, so the equation is not exact. However, if the equation is multiplied by $\mu(x, y) = e^{-y}$:

$$(e^{x-y}(x+1))dx + (y - xe^{x-y})dy = 0 \to \frac{\partial N}{\partial x} = -xe^{x-y} - e^{x-y} = \frac{\partial M}{\partial y}$$

Since the equation is now exact, it follows that:

$$\frac{\partial F}{\partial y} = y - xe^{x-y} \to F = y^2 + xe^{x-y} + g(x)$$

Note that:

$$\frac{\partial F}{\partial x} = xe^{x-y} + e^{x-y} + g'(x) = e^{x-y}(x+1)$$

It then follows that:

$$g'(x) = 0 \to g(x) = c$$

It follows that:

$$F = \frac{y^2}{2} + xe^{x-y} + c$$

Then, the general solution is:

$$\frac{y^2}{2} + xe^{x-y} = c_1$$

Question 7; Find the general solution of $2xydx + (y^2 - 3x^2)dy = 0$.

Begin by multiplying the equation by the integrating factor μ:

$$\mu 2xydx + \mu(y^2 - 3x^2)dy = 0$$

Assume that $\frac{\partial \mu(2xy)}{\partial x} = \frac{\partial \mu(-3x^2)}{\partial y}. Then : \frac{\partial \mu}{\partial y}(2xy) + \mu(2x) = \frac{\partial \mu}{\partial x}(y^2 - 3x^2) + \mu(-6x)$

$$\frac{\partial \mu}{\partial y}(2xy) - \frac{\partial \mu}{\partial x}(y^2 - 3x^2) = \mu(-8x)$$

Assume μ is a function of y only.

$$\frac{d\mu}{dy} = \mu(-8x) \rightarrow \frac{d\mu}{\mu} = \frac{-4dy}{y} \rightarrow ln(\mu) = -4ln(y) \rightarrow \mu = y^{-4}$$

Since the equation is now exact, it then follows that:

$$\frac{\partial F}{\partial y} = y^{-2} - 3x^2y^{-4} \rightarrow F = -y^{-1} + x^2y^{-3} + g(x)$$

It then follows that:

$$\frac{\partial F}{\partial x} = 2xy^{-3} + g'(x) = 2xy^{-3} \rightarrow g(x) = c$$

Therefore, the general solution to the DE is:

$$-y^{-1} + x^2y^{-3} = c$$