Homework 1
Numerical Method
EN 530.766

1) Consider the following PDE

$$a\Phi_{xx} + b\Phi_{xy} + c\Phi_{yy} + d\Phi_x + e\Phi_y + f\,\Phi = g(x,y).$$

Transform the above equation from (x,y) to (ξ,η) and show that the transformed equation can be written as

$$A\Phi_{\xi\xi} + B\Phi_{\xi\eta} + C\Phi_{\eta\eta} = H(\Phi_\xi, \Phi_\eta, \Phi, \xi, \eta)$$

Obtain expressions for $A, B, C, \& H$. You should use chain-rule differentiation to transform the equations.

2) Classify and determine the characteristics of:

a) $u_{xx} - x^2 y u_{yy} = 0 \quad y > 0$

b) $e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0$

c) $2u_{xx} - 4u_{xy} - 6u_{yy} + u_x = 0$

Plot the family (or families) of characteristics.

3) Develop finite-difference approximations for dT/dx at (i) for a non-uniform grid. Assume that $\Delta x_i = x_{i+1} - x_i$, $\Delta x_{i-1} = x_i - x_{i-1}$, and $\Delta x_{i-1} \neq \Delta x_i$.
   a. Develop expressions that employ the following stencils: (T_{i-1}, T_i, T_{i+1}).
      Hint: use Taylor series expansion
   b. Show the first two terms in the truncation error and determine the order of the truncation error.

4) Develop a second-order accurate finite difference formulation for $\left. d^3u/dx^3 \right)_i$ on a uniform grid using central differencing. Show the leading-order term in the truncation error.

5) Consider the function

$$f(x) = \frac{\sin x}{x^3}$$

Derive the first-order forward difference, second-order central difference and fourth-order central-difference approximations to the first derivative. Plot the absolute value of the difference between computed and exact derivative (i.e. the truncation error) for x=4.0 for different grid sizes (Δx) and show that the error changes with grid size as expected (order of accuracy). Employ at least five different grid sizes.

Note: A log-log plot is the most appropriate way of showing the order of accuracy.