

# Numerical Methods HW1

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## 1

By the chain rule differentiation, the following can be obtained:

$$\Phi_x = \frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \Phi}{\partial \eta} \frac{\partial \eta}{\partial x} = \Phi_\xi \xi_x + \Phi_\eta \eta_x$$

Similarly for  $\Phi_y$ :

$$\Phi_y = \Phi_\xi \xi_y + \Phi_\eta \eta_y$$

For  $\Phi_{xx}$ :

$$\begin{aligned}\Phi_{xx} &= \frac{\partial}{\partial x} \left( \frac{\partial \Phi}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial \Phi}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \Phi}{\partial \eta} \frac{\partial \eta}{\partial x} \right) \\ &= \frac{\partial^2 \Phi}{\partial \xi^2} \left( \frac{\partial \xi}{\partial x} \right)^2 + \frac{\partial^2 \Phi}{\partial \xi \partial \eta} \frac{\partial \eta}{\partial x} \frac{\partial \xi}{\partial x} + \frac{\partial \Phi}{\partial \xi} \frac{\partial^2 \xi}{\partial x^2} \\ &\quad + \frac{\partial^2 \Phi}{\partial \eta^2} \left( \frac{\partial \eta}{\partial x} \right)^2 + \frac{\partial^2 \Phi}{\partial \eta \partial \xi} \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial \Phi}{\partial \eta} \frac{\partial^2 \eta}{\partial x^2} \\ &= \Phi_{\xi\xi} \xi_x^2 + \Phi_{\xi\eta} \eta_x \xi_x + \Phi_\xi \xi_{xx} + \Phi_{\eta\eta} \eta_x^2 + \Phi_{\eta\xi} \xi_x \eta_x + \Phi_\eta \eta_{xx} \\ &= \Phi_{\xi\xi} \xi_x^2 + 2\Phi_{\xi\eta} \eta_x \xi_x + \Phi_{\eta\eta} \eta_x^2 + \Phi_\xi \xi_{xx} + \Phi_\eta \eta_{xx}.\end{aligned}$$

Similarly,

$$\Phi_{yy} = \Phi_{\xi\xi} \xi_y^2 + 2\Phi_{\xi\eta} \eta_y \xi_y + \Phi_{\eta\eta} \eta_y^2 + \Phi_\xi \xi_{yy} + \Phi_\eta \eta_{yy}.$$

For  $\Phi_{xy}$ :

$$\begin{aligned}\Phi_{xy} &= \frac{\partial}{\partial x} \left( \frac{\partial \Phi}{\partial y} \right) = \frac{\partial}{\partial x} (\Phi_\xi \xi_y + \Phi_\eta \eta_y) \\ &= \Phi_{\xi\xi} \xi_x \xi_y + \Phi_{\xi\eta} \eta_x \xi_y + \Phi_\xi \xi_{xy} + \Phi_{\eta\xi} \xi_x \eta_y + \Phi_{\eta\eta} \eta_x \eta_y + \Phi_\eta \eta_{xy}.\end{aligned}$$

Plugging the above results back into the PDE, the following can be obtain:

$$\begin{aligned}& (a\xi_x^2 + b\xi_x \xi_y + c\xi_y^2) \Phi_{\xi\xi} + [2a\xi_x \eta_x + b(\xi_x \eta_y + \xi_y \eta_x) + 2c\xi_y \eta_y] \Phi_{\xi\eta} + (a\eta_x^2 + b\eta_x \eta_y + c\eta_y^2) \Phi_{\eta\eta} \\ &= g(x, y) - (a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y) \Phi_\xi - (a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y) \Phi_\eta - f\Phi.\end{aligned}$$

Therefore:

$$A = a\xi_x^2 + b\xi_x \xi_y + c\xi_y^2,$$

$$B = 2a\xi_x \eta_x + b(\xi_x \eta_y + \xi_y \eta_x) + 2c\xi_y \eta_y,$$

$$C = a\eta_x^2 + b\eta_x \eta_y + c\eta_y^2,$$

$$H = g(x, y) - (a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y) \Phi_\xi - (a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y) \Phi_\eta - f\Phi.$$

2

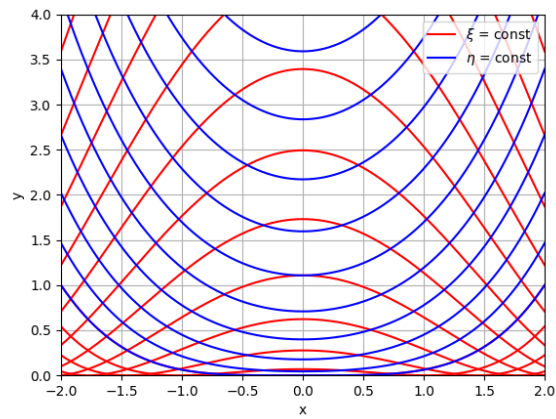
a)

$$D = 0 - 4(-x^2y) = 4x^2y > 0 \implies \text{Hyperbolic}$$

$$(\psi_x)^2 - x^2y(\psi_y)^2 = 0 \implies \left(\frac{\psi_x}{\psi_y}\right)^2 = x^2y \implies \frac{\psi_x}{\psi_y} = \pm x\sqrt{y}$$

$$\implies \psi = 2\sqrt{y} \mp \frac{x^2}{2}$$

$$\xi = 2\sqrt{y} + \frac{x^2}{2}, \quad \eta = 2\sqrt{y} - \frac{x^2}{2}$$

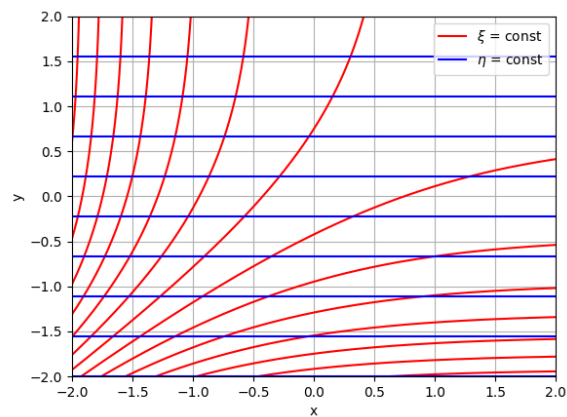


b)

$$D = 4e^{2x+2y} - 4e^{2x}e^{2y} = 0 \implies \text{Parabolic}$$

$$\frac{\xi_x}{\xi_y} = -\frac{-e^y}{-e^x}$$

$$\implies \xi = e^{-x} - e^{-y}, \eta = y$$

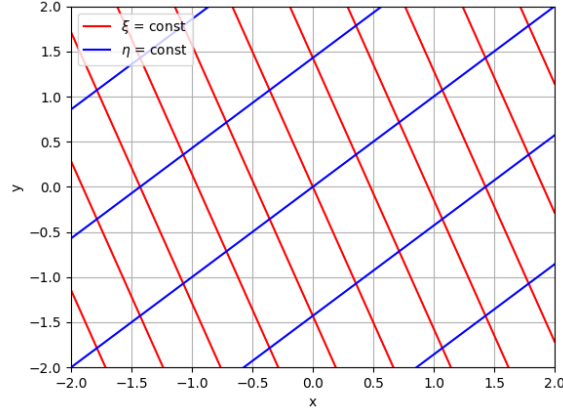


c)

$$D = 16 - 4 \times 2 \times (-6) = 64 > 0 \implies \text{Hyperbolic}$$

$$\frac{\psi_x}{\psi_y} = 1 \pm 2$$

$$\xi = y + 3x, \quad \eta = y - x$$



**3**

a)

By Taylor Series Expansion, the following can be obtained:

$$T_{i+1} = T_i + \Delta x_i \left. \frac{dT}{dx} \right|_i + \frac{\Delta x_i^2}{2!} \left. \frac{d^2T}{dx^2} \right|_i + \frac{\Delta x_i^3}{3!} \left. \frac{d^3T}{dx^3} \right|_i + \dots$$

$$T_{i-1} = T_i - \Delta x_{i-1} \left. \frac{dT}{dx} \right|_i + \frac{\Delta x_{i-1}^2}{2!} \left. \frac{d^2T}{dx^2} \right|_i - \frac{\Delta x_{i-1}^3}{3!} \left. \frac{d^3T}{dx^3} \right|_i + \dots$$

$$T_{i+1} + T_{i-1} = 2T_i + (\Delta x_i - \Delta x_{i-1}) \left. \frac{dT}{dx} \right|_i + \frac{\Delta x_i^2 + \Delta x_{i-1}^2}{2!} \left. \frac{d^2T}{dx^2} \right|_i + \frac{\Delta x_i^3 - \Delta x_{i-1}^3}{3!} \left. \frac{d^3T}{dx^3} \right|_i + \dots$$

$$\implies \left. \frac{dT}{dx} \right|_i = \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x_i - \Delta x_{i-1}} - \frac{\Delta x_i^2 + \Delta x_{i-1}^2}{2!(\Delta x_i - \Delta x_{i-1})} \left. \frac{d^2T}{dx^2} \right|_i - \frac{\Delta x_i^3 - \Delta x_{i-1}^3}{3!(\Delta x_i - \Delta x_{i-1})} \left. \frac{d^3T}{dx^3} \right|_i - \dots$$

Therefore,

$$\left. \frac{dT}{dx} \right|_i \approx \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x_i - \Delta x_{i-1}}$$

b)

$$\text{Truncation Error} = -\left(\frac{\Delta x_i^2 + \Delta x_{i-1}^2}{2!(\Delta x_i - \Delta x_{i-1})} \frac{d^2 T}{dx^2} \Big|_i + \frac{\Delta x_i^3 - \Delta x_{i-1}^3}{3!(\Delta x_i - \Delta x_{i-1})} \frac{d^3 T}{dx^3} \Big|_i - \dots\right)$$

$$\text{Order of Error} = O\left(\frac{\Delta x_i^2 + \Delta x_{i-1}^2}{\Delta x_i - \Delta x_{i-1}}\right) = O(h)$$

## 4

First, write out  $u_{i-2}, u_{i-1}, u_{i+1}, u_{i+2}$  with Taylor Series Expansion:

$$u_{i-2} = u_i - 2\Delta x \frac{du}{dx} \Big|_i + \frac{(2\Delta x)^2}{2!} \frac{d^2 u}{dx^2} \Big|_i - \frac{(2\Delta x)^3}{3!} \frac{d^3 u}{dx^3} \Big|_i + \frac{(2\Delta x)^4}{4!} \frac{d^4 u}{dx^4} \Big|_i - \frac{(2\Delta x)^5}{5!} \frac{d^5 u}{dx^5} \Big|_i + \dots$$

$$u_{i-1} = u_i - \Delta x \frac{du}{dx} \Big|_i + \frac{(\Delta x)^2}{2!} \frac{d^2 u}{dx^2} \Big|_i - \frac{(\Delta x)^3}{3!} \frac{d^3 u}{dx^3} \Big|_i + \frac{(\Delta x)^4}{4!} \frac{d^4 u}{dx^4} \Big|_i - \frac{(\Delta x)^5}{5!} \frac{d^5 u}{dx^5} \Big|_i + \dots$$

$$u_{i+1} = u_i + \Delta x \frac{du}{dx} \Big|_i + \frac{(\Delta x)^2}{2!} \frac{d^2 u}{dx^2} \Big|_i + \frac{(\Delta x)^3}{3!} \frac{d^3 u}{dx^3} \Big|_i + \frac{(\Delta x)^4}{4!} \frac{d^4 u}{dx^4} \Big|_i + \frac{(\Delta x)^5}{5!} \frac{d^5 u}{dx^5} \Big|_i + \dots$$

$$u_{i+2} = u_i + 2\Delta x \frac{du}{dx} \Big|_i + \frac{(2\Delta x)^2}{2!} \frac{d^2 u}{dx^2} \Big|_i + \frac{(2\Delta x)^3}{3!} \frac{d^3 u}{dx^3} \Big|_i + \frac{(2\Delta x)^4}{4!} \frac{d^4 u}{dx^4} \Big|_i + \frac{(2\Delta x)^5}{5!} \frac{d^5 u}{dx^5} \Big|_i + \dots$$

Then, cancel out the first and second order derivatives with the following:

$$u_{i+1} - u_{i-1} = 2\Delta x \frac{du}{dx} \Big|_i + \frac{2(\Delta x)^3}{3!} \frac{d^3 u}{dx^3} \Big|_i + \frac{2(\Delta x)^5}{5!} \frac{d^5 u}{dx^5} \Big|_i + \dots$$

$$u_{i+2} - (u_{i+1} - u_{i-1}) = u_i + \frac{(2\Delta x)^2}{2!} \frac{d^2 u}{dx^2} \Big|_i + \frac{6(\Delta x)^3}{3!} \frac{d^3 u}{dx^3} \Big|_i + \frac{(4\Delta x)^4}{4!} \frac{d^4 u}{dx^4} \Big|_i + \frac{30(\Delta x)^5}{5!} \frac{d^5 u}{dx^5} \Big|_i + \dots$$

$$u_{i-2} + (u_{i+1} - u_{i-1}) = u_i + \frac{(2\Delta x)^2}{2!} \frac{d^2 u}{dx^2} \Big|_i - \frac{6(\Delta x)^3}{3!} \frac{d^3 u}{dx^3} \Big|_i + \frac{(4\Delta x)^2}{4!} \frac{d^4 u}{dx^4} \Big|_i - \frac{30(\Delta x)^5}{5!} \frac{d^5 u}{dx^5} \Big|_i$$

$$[u_{i+2} - (u_{i+1} - u_{i-1})] - [u_{i-2} + (u_{i+1} - u_{i-1})] = \frac{12(\Delta x)^3}{3!} \frac{d^3 u}{dx^3} \Big|_i + \frac{60(\Delta x)^5}{5!} \frac{d^5 u}{dx^5} \Big|_i + \dots$$

$$\Rightarrow \frac{d^3 u}{dx^3} = \frac{u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}}{2(\Delta x)^3} - \frac{30(\Delta x)^2}{5!} \frac{d^5 u}{dx^5} \Big|_i - \dots$$

$$\text{Truncation Error} = \frac{30(\Delta x)^2}{5!} \frac{d^5 u}{dx^5} \Big|_i + \dots = \frac{(\Delta x)^2}{4} \frac{d^5 u}{dx^5} \Big|_i + \dots$$

## 5

$$f_{i-2} = f_i - 2\Delta x \left. \frac{df}{dx} \right|_i + \frac{(2\Delta x)^2}{2!} \left. \frac{d^2 f}{dx^2} \right|_i - \frac{(2\Delta x)^3}{3!} \left. \frac{d^3 f}{dx^3} \right|_i + \frac{(2\Delta x)^4}{4!} \left. \frac{d^4 f}{dx^4} \right|_i - \frac{(2\Delta x)^5}{5!} \left. \frac{d^5 f}{dx^5} \right|_i + \dots \quad (1)$$

$$f_{i-1} = f_i - \Delta x \left. \frac{df}{dx} \right|_i + \frac{(\Delta x)^2}{2!} \left. \frac{d^2 f}{dx^2} \right|_i - \frac{(\Delta x)^3}{3!} \left. \frac{d^3 f}{dx^3} \right|_i + \frac{(\Delta x)^4}{4!} \left. \frac{d^4 f}{dx^4} \right|_i - \frac{(\Delta x)^5}{5!} \left. \frac{d^5 f}{dx^5} \right|_i + \dots \quad (2)$$

$$f_{i+1} = f_i + \Delta x \left. \frac{df}{dx} \right|_i + \frac{(\Delta x)^2}{2!} \left. \frac{d^2 f}{dx^2} \right|_i + \frac{(\Delta x)^3}{3!} \left. \frac{d^3 f}{dx^3} \right|_i + \frac{(\Delta x)^4}{4!} \left. \frac{d^4 f}{dx^4} \right|_i + \frac{(\Delta x)^5}{5!} \left. \frac{d^5 f}{dx^5} \right|_i + \dots \quad (3)$$

$$f_{i+2} = f_i + 2\Delta x \left. \frac{df}{dx} \right|_i + \frac{(2\Delta x)^2}{2!} \left. \frac{d^2 f}{dx^2} \right|_i + \frac{(2\Delta x)^3}{3!} \left. \frac{d^3 f}{dx^3} \right|_i + \frac{(2\Delta x)^4}{4!} \left. \frac{d^4 f}{dx^4} \right|_i + \frac{(2\Delta x)^5}{5!} \left. \frac{d^5 f}{dx^5} \right|_i + \dots \quad (4)$$

By (3) the first-order forward difference approximation can be derived:

$$\left. \frac{df}{dx} \right|_i = \frac{f_{i+1} - f_i}{\Delta x} - \frac{\Delta x}{2!} \left. \frac{d^2 f}{dx^2} \right|_i - \dots$$

By (2) and (3) the second-order central difference approximation can be derived:

$$\begin{aligned} f_{i+1} - f_{i-1} &= 2\Delta x \left. \frac{df}{dx} \right|_i + \frac{2(\Delta x)^3}{3!} \left. \frac{d^3 f}{dx^3} \right|_i + \frac{2(\Delta x)^5}{5!} \left. \frac{d^5 f}{dx^5} \right|_i \dots \\ \Rightarrow \left. \frac{df}{dx} \right|_i &= \frac{f_{i+1} - f_{i-1}}{2\Delta x} - \frac{(\Delta x)^2}{3!} \left. \frac{d^3 f}{dx^3} \right|_i - \dots \end{aligned} \quad (5)$$

The fourth-order central difference approximation can be derived with (1), (4), and (5):

$$f_{i+2} - f_{i-2} = 4\Delta x \left. \frac{df}{dx} \right|_i + \frac{2(2\Delta x)^3}{3!} \left. \frac{d^3 f}{dx^3} \right|_i + \frac{2(2\Delta x)^5}{5!} \left. \frac{d^5 f}{dx^5} \right|_i + \dots \quad (6)$$

Eliminating the 3rd order derivative of  $f$  at  $i$  with (5) and (6) gives:

$$\begin{aligned} f_{i+2} - f_{i-2} - 8(f_{i+1} - f_{i-1}) &= -12\Delta x \left. \frac{df}{dx} \right|_i + \frac{48(\Delta x)^5}{5!} \left. \frac{d^5 f}{dx^5} \right|_i + \dots \\ \Rightarrow \left. \frac{df}{dx} \right|_i &= \frac{-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2}}{12\Delta x} + \frac{4(\Delta x)^4}{5!} \left. \frac{d^5 f}{dx^5} \right|_i + \dots \end{aligned}$$

Finally, the truncation errors for  $x = 4.0$  are plotted with 100 grid sizes evenly distributed between 0.05 and 1.

