$100 - 5 = 95$

Numerical Methods HW1

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1

By the chain rule differentiation, the following can be obtained:

$$\Phi_x = \frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \Phi}{\partial \eta}\frac{\partial \eta}{\partial x} = \Phi_\xi \xi_x + \Phi_\eta \eta_x$$

Similarly for Φ_y:

$$\Phi_y = \Phi_\xi \xi_y + \Phi_\eta \eta_y$$

For Φ_{xx}:

$$\begin{aligned}
\Phi_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \Phi}{\partial \eta}\frac{\partial \eta}{\partial x}\right) \\
&= \frac{\partial^2 \Phi}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{\partial^2 \Phi}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial \Phi}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} \\
&\quad + \frac{\partial^2 \Phi}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial^2 \Phi}{\partial \eta \partial \xi}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \Phi}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} \\
&= \Phi_{\xi\xi}\xi_x^2 + \Phi_{\xi\eta}\eta_x\xi_x + \Phi_\xi \xi_{xx} + \Phi_{\eta\eta}\eta_x^2 + \Phi_{\eta\xi}\xi_x\eta_x + \Phi_\eta \eta_{xx} \\
&= \Phi_{\xi\xi}\xi_x^2 + 2\Phi_{\xi\eta}\eta_x\xi_x + \Phi_{\eta\eta}\eta_x^2 + \Phi_\xi \xi_{xx} + \Phi_\eta \eta_{xx}.
\end{aligned}$$

Similarly,

$$\Phi_{yy} = \Phi_{\xi\xi}\xi_y^2 + 2\Phi_{\xi\eta}\eta_y\xi_y + \Phi_{\eta\eta}\eta_y^2 + \Phi_\xi \xi_{yy} + \Phi_\eta \eta_{yy}.$$

For Φ_{xy}:

$$\begin{aligned}
\Phi_{xy} &= \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial y}\right) = \frac{\partial}{\partial x}(\Phi_\xi \xi_y + \Phi_\eta \eta_y) \\
&= \Phi_{\xi\xi}\xi_x\xi_y + \Phi_{\xi\eta}\eta_x\xi_y + \Phi_\xi \xi_{xy} + \Phi_{\eta\xi}\xi_x\eta_y + \Phi_{\eta\eta}\eta_x\eta_y + \Phi_\eta \eta_{xy}.
\end{aligned}$$

Plugging the above results back into the PDE, the following can be obtain:

$$\begin{aligned}
&(a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2)\Phi_{\xi\xi} + [2a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + 2c\xi_y\eta_y]\Phi_{\xi\eta} + (a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2)\Phi_{\eta\eta} \\
=&g(x,y) - (a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y)\Phi_\xi - (a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y)\Phi_\eta - f\Phi.
\end{aligned}$$

Therefore:

$A = a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2,$
$B = 2a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + 2c\xi_y\eta_y,$
$C = a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2,$
$H = g(x,y) - (a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y)\Phi_\xi - (a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y)\Phi_\eta - f\Phi.$

2

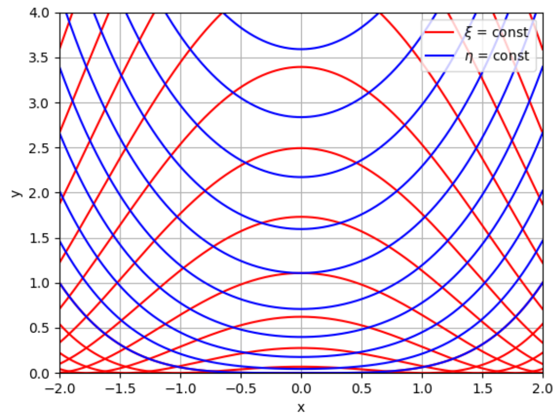
a)

$$D = 0 - 4(-x^2y) = 4x^2y > 0 \quad \Longrightarrow \text{Hyperbolic}$$ when $x \neq 0$

$$(\psi_x)^2 - x^2y(\psi_y)^2 = 0 \Rightarrow \left(\frac{\psi_x}{\psi_y}\right)^2 = x^2y \Rightarrow \frac{\psi_x}{\psi_y} = \pm x\sqrt{y}$$ $x=0 \Rightarrow$ parabolic

$$\Longrightarrow \psi = 2\sqrt{y} \mp \frac{x^2}{2}$$

$$\xi = 2\sqrt{y} + \frac{x^2}{2}, \quad \eta = 2\sqrt{y} - \frac{x^2}{2}$$



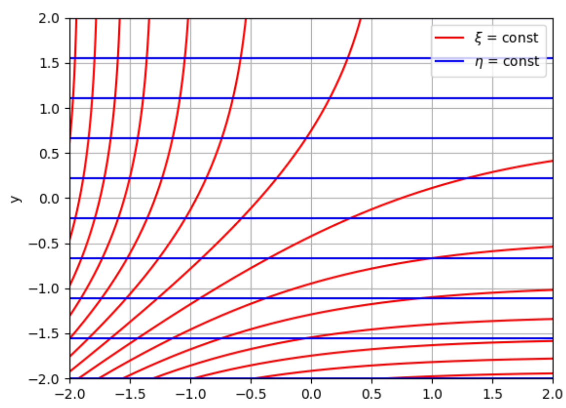
b)

$$D = 4e^{2x+2y} - 4e^{2x}e^{2y} = 0 \quad \Longrightarrow \text{Parabolic}$$

$$\frac{\xi_x}{\xi_y} = -\frac{-e^y}{-e^x}$$

parabolic eq. has only 1 chara. curve group

$$\Longrightarrow \quad \xi = e^{-x} - e^{-y}, \cancel{\eta = y}$$

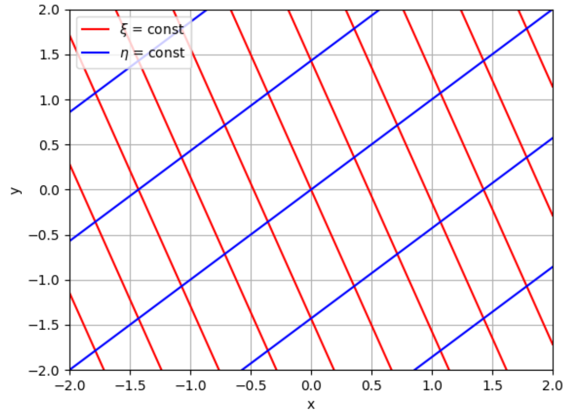


c)

$$D = 16 - 4 \times 2 \times (-6) = 64 > 0 \quad \Longrightarrow \text{Hyperbolic}$$

$$\frac{\psi_x}{\psi_y} = 1 \pm 2$$

$$\xi = y + 3x, \quad \eta = y - x$$



3

a)

By Taylor Series Expansion, the following can be obtained:

$$T_{i+1} = T_i + \Delta x_i \frac{dT}{dx}\bigg|_i + \frac{\Delta x_i^2}{2!}\frac{d^2T}{dx^2}\bigg|_i + \frac{\Delta x_i^3}{3!}\frac{d^3T}{dx^3}\bigg|_i + \ldots$$

$$T_{i-1} = T_i - \Delta x_{i-1} \frac{dT}{dx}\bigg|_i + \frac{\Delta x_{i-1}^2}{2!}\frac{d^2T}{dx^2}\bigg|_i - \frac{\Delta x_{i-1}^3}{3!}\frac{d^3T}{dx^3}\bigg|_i + \ldots$$

$$T_{i+1} + T_{i-1} = 2T_i + (\Delta x_i - \Delta x_{i-1})\frac{dT}{dx}\bigg|_i + \frac{\Delta x_i^2 + \Delta x_{i-1}^2}{2!}\frac{d^2T}{dx^2}\bigg|_i + \frac{\Delta x_i^3 - \Delta x_{i-1}^3}{3!}\frac{d^3T}{dx^3}\bigg|_i + \ldots$$

$$\Longrightarrow \frac{dT}{dx}\bigg|_i = \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x_i - \Delta x_{i-1}} - \frac{\Delta x_i^2 + \Delta x_{i-1}^2}{2!(\Delta x_i - \Delta x_{i-1})}\frac{d^2T}{dx^2}\bigg|_i - \frac{\Delta x_i^3 - \Delta x_{i-1}^3}{3!(\Delta x_i - \Delta x_{i-1})}\frac{d^3T}{dx^3}\bigg|_i - \ldots$$

Therefore,

$$\frac{dT}{dx}\bigg|_i \approx \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x_i - \Delta x_{i-1}}$$

the intent to use i-1,i,i+1 3 points
is to derive a second order scheme,
plus this scheme is not a robust one,
given that in reality,
though different, dxi is still close to dxi-1,
but this scheme approaches singularity

−2

b)

$$\text{Truncation Error} = -(\frac{\Delta x_i^2 + \Delta x_{i-1}^2}{2!(\Delta x_i - \Delta x_{i-1})}\frac{d^2T}{dx^2}\bigg|_i + \frac{\Delta x_i^3 - \Delta x_{i-1}^3}{3!(\Delta x_i - \Delta x_{i-1})}\frac{d^3T}{dx^3}\bigg|_i - ...)$$

$$\text{Order of Error} = O\left(\frac{\Delta x_i^2 + \Delta x_{i-1}^2}{\Delta x_i - \Delta x_{i-1}}\right) = O\left(h\right)$$

4

First, write out $u_{i-2}, u_{i-1}, u_{i+1}, u_{i+2}$ with Taylor Series Expansion:

$$u_{i-2} = u_i - 2\Delta x\frac{du}{dx}\bigg|_i + \frac{(2\Delta x)^2}{2!}\frac{d^2u}{dx^2}\bigg|_i - \frac{(2\Delta x)^3}{3!}\frac{d^3u}{dx^3}\bigg|_i + \frac{(2\Delta x)^4}{4!}\frac{d^4u}{dx^4}\bigg|_i - \frac{(2\Delta x)^5}{5!}\frac{d^5u}{dx^5}\bigg|_i + ...$$

$$u_{i-1} = u_i - \Delta x\frac{du}{dx}\bigg|_i + \frac{(\Delta x)^2}{2!}\frac{d^2u}{dx^2}\bigg|_i - \frac{(\Delta x)^3}{3!}\frac{d^3u}{dx^3}\bigg|_i + \frac{(\Delta x)^4}{4!}\frac{d^4u}{dx^4}\bigg|_i - \frac{(\Delta x^5)}{5!}\frac{d^5u}{dx^5}\bigg|_i + ...$$

$$u_{i+1} = u_i + \Delta x\frac{du}{dx}\bigg|_i + \frac{(\Delta x)^2}{2!}\frac{d^2u}{dx^2}\bigg|_i + \frac{(\Delta x)^3}{3!}\frac{d^3u}{dx^3}\bigg|_i + \frac{(\Delta x)^4}{4!}\frac{d^4u}{dx^4}\bigg|_i + \frac{(\Delta x^5)}{5!}\frac{d^5u}{dx^5}\bigg|_i + ...$$

$$u_{i+2} = u_i + 2\Delta x\frac{du}{dx}\bigg|_i + \frac{(2\Delta x)^2}{2!}\frac{d^2u}{dx^2}\bigg|_i + \frac{(2\Delta x)^3}{3!}\frac{d^3u}{dx^3}\bigg|_i + \frac{(2\Delta x)^4}{4!}\frac{d^4u}{dx^4}\bigg|_i + \frac{(2\Delta x)^5}{5!}\frac{d^5u}{dx^5}\bigg|_i + ...$$

Then, cancel out the first and second order derivatives with the following:

$$u_{i+1} - u_{i-1} = 2\Delta x\frac{du}{dx}\bigg|_i + \frac{2(\Delta x)^3}{3!}\frac{d^3u}{dx^3}\bigg|_i + \frac{2(\Delta x)^5}{5!}\frac{d^5u}{dx^5}\bigg|_i + ...$$

$$u_{i+2} - (u_{i+1} - u_{i-1}) = u_i + \frac{(2\Delta x)^2}{2!}\frac{d^2u}{dx^2}\bigg|_i + \frac{6(\Delta x)^3}{3!}\frac{d^3u}{dx^3}\bigg|_i + \frac{(4\Delta x)^4}{4!}\frac{d^4u}{dx^4}\bigg|_i + \frac{30(\Delta x)^5}{5!}\frac{d^5u}{dx^5}\bigg|_i + ...$$

$$u_{i-2} + (u_{i+1} - u_{i-1}) = u_i + \frac{(2\Delta x)^2}{2!}\frac{d^2u}{dx^2}\bigg|_i - \frac{6(\Delta x)^3}{3!}\frac{d^3u}{dx^3}\bigg|_i + \frac{(4\Delta x)^2}{4!}\frac{d^4u}{dx^4}\bigg|_i - \frac{30(\Delta x)^5}{5!}\frac{d^5u}{dx^5}\bigg|_i$$

$$[u_{i+2} - (u_{i+1} - u_{i-1})] - [u_{i-2} + (u_{i+1} - u_{i-1})] = \frac{12(\Delta x)^3}{3!}\frac{d^3u}{dx^3}\bigg|_i + \frac{60(\Delta x)^5}{5!}\frac{d^5u}{dx^5}\bigg|_i + ...$$

$$\implies \frac{d^3u}{dx^3} = \frac{u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}}{2(\Delta x)^3} - \frac{30(\Delta x)^2}{5!}\frac{d^5u}{dx^5}\bigg|_i - ...$$

$$\text{Truncation Error} = \frac{30(\Delta x)^2}{5!}\frac{d^5u}{dx^5}\bigg|_i + ... = \frac{(\Delta x)^2}{4}\frac{d^5u}{dx^5}\bigg|_i + ...$$

5

$$f_{i-2} = f_i - 2\Delta x \frac{df}{dx}\bigg|_i + \frac{(2\Delta x)^2}{2!}\frac{d^2 f}{dx^2}\bigg|_i - \frac{(2\Delta x)^3}{3!}\frac{d^3 f}{dx^3}\bigg|_i + \frac{(2\Delta x)^4}{4!}\frac{d^4 f}{dx^4}\bigg|_i - \frac{(2\Delta x)^5}{5!}\frac{d^5 f}{dx^5}\bigg|_i + \dots \quad (1)$$

$$f_{i-1} = f_i - \Delta x \frac{df}{dx}\bigg|_i + \frac{(\Delta x)^2}{2!}\frac{d^2 f}{dx^2}\bigg|_i - \frac{(\Delta x)^3}{3!}\frac{d^3 f}{dx^3}\bigg|_i + \frac{(\Delta x)^4}{4!}\frac{d^4 f}{dx^4}\bigg|_i - \frac{(\Delta x^5)}{5!}\frac{d^5 f}{dx^5}\bigg|_i + \dots \quad (2)$$

$$f_{i+1} = f_i + \Delta x \frac{df}{dx}\bigg|_i + \frac{(\Delta x)^2}{2!}\frac{d^2 f}{dx^2}\bigg|_i + \frac{(\Delta x)^3}{3!}\frac{d^3 f}{dx^3}\bigg|_i + \frac{(\Delta x)^4}{4!}\frac{d^4 f}{dx^4}\bigg|_i + \frac{(\Delta x^5)}{5!}\frac{d^5 f}{dx^5}\bigg|_i + \dots \quad (3)$$

$$f_{i+2} = f_i + 2\Delta x \frac{df}{dx}\bigg|_i + \frac{(2\Delta x)^2}{2!}\frac{d^2 f}{dx^2}\bigg|_i + \frac{(2\Delta x)^3}{3!}\frac{d^3 f}{dx^3}\bigg|_i + \frac{(2\Delta x)^4}{4!}\frac{d^4 f}{dx^4}\bigg|_i + \frac{(2\Delta x)^5}{5!}\frac{d^5 f}{dx^5}\bigg|_i + \dots \quad (4)$$

By (3) the first-order forward difference approximation can be derived:

$$\frac{df}{dx}\bigg|_i = \frac{f_{i+1} - f_i}{\Delta x} - \frac{\Delta x}{2!}\frac{d^2 f}{dx^2}\bigg|_i - ...$$

By (2) and (3) the second-order central difference approximation can be derived:

$$f_{i+1} - f_{i-1} = 2\Delta x \frac{df}{dx}\bigg|_i + \frac{2(\Delta x)^3}{3!}\frac{d^3 f}{dx^3}\bigg|_i + \frac{2(\Delta x)^5}{5!}\frac{d^5 f}{dx^5}\bigg|_i ... \quad (5)$$

$$\Longrightarrow \frac{df}{dx}\bigg|_i = \frac{f_{i+1} - f_{i-1}}{2\Delta x} - \frac{(\Delta x)^2}{3!}\frac{d^3 f}{dx^3}\bigg|_i - ...$$

The fourth-order central difference approximation can be derived with (1), (4), and (5):

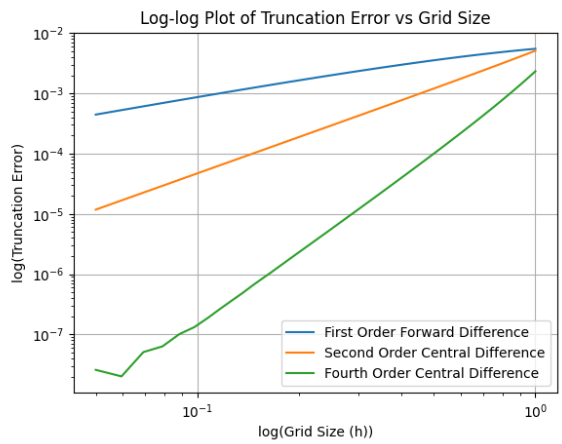
$$f_{i+2} - f_{i-2} = 4\Delta x \frac{df}{dx}\bigg|_i + \frac{2(2\Delta x)^3}{3!}\frac{d^3 f}{dx^3}\bigg|_i + \frac{2(2\Delta x)^5}{5!}\frac{d^5 f}{dx^5}\bigg|_i + ... \quad (6)$$

Eliminating the 3rd order derivative of f at i with (5) and (6) gives:

$$f_{i+2} - f_{i-2} - 8(f_{i+1} - f_{i-1}) = -12\Delta x \frac{df}{dx}\bigg|_i + \frac{48(\Delta x)^5}{5!}\frac{d^5 f}{dx^5}\bigg|_i + ...$$

$$\Longrightarrow \frac{df}{dx}\bigg|_i = \frac{-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2}}{12\Delta x} + \frac{4(\Delta x)^4}{5!}\frac{d^5 f}{dx^5}\bigg|_i + ...$$

Finally, the truncation errors for $x = 4.0$ are plotted with 100 grid sizes evenly distributed between 0.05 and 1.



need to show the slope = 1, 2, 4

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