100

Numerical Methods HW2

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1

(a)

$$f_i' + a_0 f_i + a_1 f_{i+1} + a_2 f_{i+2} + a_3 f_{i+1}' + a_4 f_{i+2}' = 0$$

	f_i	f_i'	f_i''	f_i'''	f_i^{iv}	f_i^{v}
f_i'	0	1	0	0	0	0
$a_0 f_i$	a_0	0	0	0	0	0
$a_1 f_{i+1}$	a_1	$a_1 h$	$\frac{a_1 h^2}{2}$	$\frac{a_1 h^3}{6}$	$\frac{a_1 h^4}{24}$	$\frac{a_1 h^5}{120}$
$a_2 f_{i+2}$	a_2	$a_2(2h)$	$\frac{a_2 (2h)^2}{2}$	$\frac{a_2 (2h)^3}{6}$	$\frac{a_2 (2h)^4}{24}$	$\frac{a_2 (2h)^5}{120}$
$a_3 f_{i+1}'$	0	a_3	$a_3 h$	$\frac{a_3 h^2}{2}$	$\frac{a_3 h^3}{6}$	$\frac{a_3 h^4}{24}$
$a_4 f_{i+2}'$	0	a_4	$a_4(2h)$	$\frac{a_4 (2h)^2}{2}$	$\frac{a_4 (2h)^3}{6}$	$\frac{a_4 (2h)^4}{24}$

Table 1: Taylor Table for (a)

$$\begin{cases} a_0 + a_1 + a_2 = 0 \\ a_1 h + 2a_2 h + a_3 + a_4 = -1 \\ a_1 h + 4a_2 h + 2a_3 + 4a_4 = 0 \\ a_1 h + 8a_2 h + 3a_3 + 12a_4 = 0 \\ a_1 h + 16a_2 h + 4a_3 + 32a_4 = 0 \end{cases} \implies \begin{cases} a_0 = \frac{3}{h} \\ a_1 = 0 \\ a_2 = -\frac{3}{h} \\ a_3 = 4 \\ a_4 = 1 \end{cases}$$

$$\implies f_i' + 4f_{i+1}' + f_{i+2}' \approx \frac{3}{h}(f_{i+2} - fi)$$

$$\text{Truncation Error} = \left(\frac{a_1 h^5}{120} + \frac{32 a_2 h^5}{120} + \frac{a_3 h^4}{24} + \frac{16 a_4 h^4}{24}\right) f_i^v = \frac{1}{30} h^4 f_i^v$$

(b)

$$f_i' + a_0 f_i + a_1 f_{i-1} + a_2 f_{i+1} + a_3 f_{i-1}' + a_4 f_{i+1}' = 0$$

	f_i	f_i'	f_i''	f_i'''	f_i^{iv}	f_i^{v}
f_i'	0	1	0	0	0	0
$a_0 f_i$	a_0	0	0	0	0	0
$a_1 f_{i-1}$	a_1	$-a_1 h$	$\frac{a_1 h^2}{2}$	$-\frac{a_1 h^3}{6}$	$\frac{a_1 h^4}{24}$	$-\frac{a_1 h^5}{120}$
$a_2 f_{i+1}$	a_2	$a_2 h$	$\frac{a_2 h^2}{2}$	$\frac{a_2 h^3}{6}$	$\frac{a_2 h^4}{24}$	$\frac{a_2 h^5}{120}$
$a_3 f_{i-1}'$	0	a_3	$-a_3 h$	$\frac{a_3 h^2}{2}$	$-\frac{a_3 h^3}{6}$	$\frac{a_3 h^4}{24}$
$a_4 f_{i+1}'$	0	a_4	$a_4 h$	$\frac{a_4 h^2}{2}$	$\frac{a_4 h^3}{6}$	$\frac{a_4 h^4}{24}$

Table 2: Taylor Table for (b)

$$\begin{cases} a_0 + a_1 + a_2 = 0 \\ -a_1 h + a_2 h + a_3 + a_4 = -1 \\ a_1 h + a_2 h - 2a_3 + 2a_4 = 0 \\ -a_1 h + a_2 h + 3a_3 + 3a_4 = 0 \\ a_1 h + a_2 h - 4a_3 + 4a_4 = 0 \end{cases} \implies \begin{cases} a_0 = 0 \\ a_1 = \frac{3}{4h} \\ a_2 = \frac{-3}{4h} \\ a_3 = \frac{1}{4} \\ a_4 = \frac{1}{4} \end{cases}$$

$$\implies f_{i-1}' + 4f_i' + f_{i+1}' \approx \frac{3}{h}(f_{i+1} - f_{i-1})$$

$$\text{Truncation Error} = 4\left(-\frac{a_1 h^5}{120} + \frac{a_2 h^5}{120} + \frac{a_3 h^4}{24} + \frac{a_4 h^4}{24}\right) f_i^v = \frac{1}{30} h^4 f_i^v$$

In summary, both compact different methods have the same truncation error and scheme structure.

2

a)

a.1

In the $(-\pi, \pi)$ range, the function takes the following form:

$$F(x) = \begin{cases} \frac{x}{2\pi} + 1, & -\pi < x < 0 \\ \frac{x}{2\pi}, & 0 \le x < \pi \end{cases}$$

The Fourier transform of F can therefore be obtained by the following:

First, put $n = 0$ and obtain A_0:

$$\begin{aligned} 2\pi A_0 &= \int_{-\pi}^{\pi} F(x)dx \\ &= \int_{-\pi}^{\pi} \frac{x}{2\pi} dx + \int_{-\pi}^{0} dx \\ &= \pi \end{aligned}$$

$$\implies A_0 = \frac{1}{2}$$

Put $n \neq 0$:

$$
\begin{aligned}
2\pi A_n &= \int_{-\pi}^{0} (\frac{x}{2\pi} + 1)e^{-inx}dx + \int_{0}^{\pi} \frac{x}{2\pi}e^{-inx}dx \\
&= \int_{-\pi}^{\pi} \frac{x}{2\pi}e^{-inx}dx + \int_{-\pi}^{0} e^{-inx}dx \\
&= \frac{i}{2n\pi}\int_{-\pi}^{\pi} xde^{-inx} + \frac{i}{n}e^{-inx}\Big|_{-\pi}^{0} \\
&= \frac{i}{2n\pi}\left[(x - \frac{i}{n})e^{-inx}\right]\Big|_{-\pi}^{\pi} + \frac{i}{n}(1 - e^{in\pi}) \\
&= \frac{i}{2n}(e^{-in\pi} + e^{in\pi}) + \frac{1}{2n^2\pi}(e^{-in\pi} - e^{in\pi}) + \frac{i}{n}(1 - e^{in\pi}) \\
&\text{Noting that: } e^{-in\pi} = e^{in\pi} = (-1)^n \\
&= \frac{i}{n}
\end{aligned}
$$

$$
\implies A_n = \frac{i}{2\pi n}, n \neq 0
$$

Therefore:

$$
A_n = \begin{cases} \frac{1}{2}, & n = 0 \\ \frac{i}{2\pi n}, & n \neq 0 \end{cases}
$$

$$
F(x) = \sum_{n=-\infty}^{\infty} \left(A_n e^{inx}\right)
$$

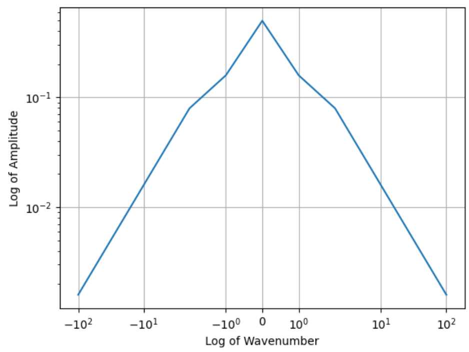

Figure 1: loglog plot of Fourier Spectrum for a.1

a.2

In the $(-\pi, \pi)$ range, the function takes the following form:

$$
F(x) = \begin{cases} -\frac{x}{\pi}, & -\pi < x < 0 \\ \frac{x}{\pi}, & 0 \leq x < \pi \end{cases}
$$

The Fourier transform of F can therefore be obtained by the following:

First, put $n = 0$ and obtain A_0:

$$
\begin{aligned}
2\pi A_0 &= \int_{-\pi}^{0} (-\frac{x}{\pi})dx + \int_{0}^{\pi} \frac{x}{\pi}dx \\
&= \pi
\end{aligned}
$$

$$\implies A_0 = \frac{1}{2}$$

Then, put $n \neq 0$:

$$\begin{aligned} 2\pi A_n &= \int_{-\pi}^{0} (-\frac{x}{\pi}) e^{-inx} dx + \int_{0}^{\pi} \frac{x}{\pi} e^{-inx} dx \\ &= -\frac{1}{\pi}\left((x - \frac{i}{n})e^{-inx}\right)\Big|_{-\pi}^{0} + \frac{1}{\pi}(x - \frac{i}{n})e^{-inx}\Big|_{0}^{\pi} \\ &= -\frac{2}{\pi n^2}(1 - (-1)^n) \end{aligned}$$

$$\implies A_n = -\frac{1}{\pi^2 n^2}(1 - (-1)^n), n \neq 0$$

Therefore:

$$A_n = \begin{cases} \frac{1}{2}, & n = 0 \\ -\frac{1}{\pi^2 n^2}(1 - (-1)^n), & n \neq 0 \end{cases}$$

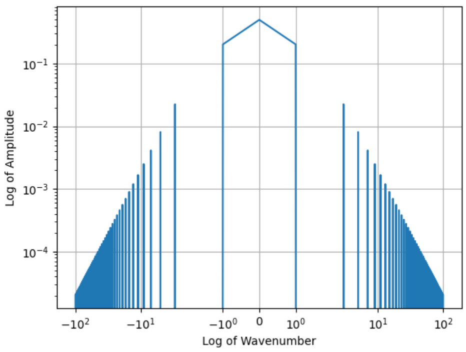

Figure 2: loglog plot of Fourier Spectrum for a.1

a.3

In the $(-\pi, \pi)$ range, the function takes the following form:

$$F(x) = \begin{cases} -\sin(\frac{x}{2}), & -\pi < x < 0 \\ \sin(\frac{x}{2}), & 0 \leq x < \pi \end{cases}$$

The Fourier transform of F can therefore be obtained by the following:

First, put $n = 0$ and obtain A_0:

$$\begin{aligned} 2\pi A_0 &= \int_{-\pi}^{0} (-\sin\frac{x}{2}) dx + \int_{0}^{\pi} \sin\frac{x}{2} dx \\ &= 4 \end{aligned}$$

$$\implies A_0 = \frac{2}{\pi}$$

Then, put $n \neq 0$ and obtain:

$$
\begin{aligned}
2\pi A_n &= \int_{-\pi}^{0} (-\sin\frac{x}{2}) \exp(-inx)dx + \int_{0}^{\pi} \sin\frac{x}{2} \exp(-inx)dx \\
&\text{Noting that } \sin\frac{x}{2} = -\frac{i}{2}[\exp(i\frac{x}{2}) - \exp(-i\frac{x}{2})] \\
&= \frac{i}{2}\int_{-\pi}^{0} \Big\{\exp\big(ix[\frac{1}{2} - n]\big) - \exp\big(-ix[\frac{1}{2} + n]\big)\Big\}dx \\
&\quad - \frac{i}{2}\int_{0}^{\pi} \Big\{\exp\big(ix[\frac{1}{2} - n]\big) - \exp\big(-ix[\frac{1}{2} + n]\big)\Big\}dx \\
&= \frac{i}{2}\Big\{\frac{i}{n-\frac{1}{2}}[1 + (-1)^n i] - \frac{i}{n+\frac{1}{2}}[1 - (-1)^n i]\Big\} \\
&\quad - \frac{i}{2}\Big\{\frac{i}{n-\frac{1}{2}}[(-1)^n i - 1] - \frac{i}{n+\frac{1}{2}}[-(-1)^n i - 1]\Big\} \\
&= -\frac{4}{(2n+1)(2n-1)}
\end{aligned}
$$

$$
\implies A_n = -\frac{2}{\pi(2n+1)(2n-1)}, n \neq 0
$$

Therefore:

$$
A_n = \begin{cases} \frac{2}{\pi}, & n = 0 \\ -\frac{2}{\pi(2n+1)(2n-1)}, & n \neq 0 \end{cases}
$$

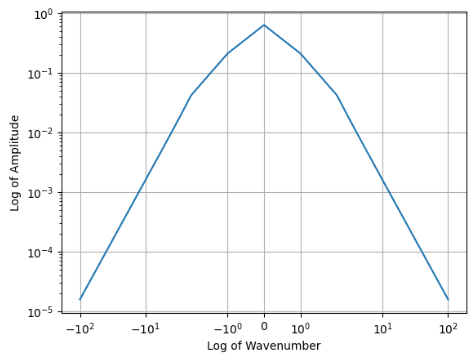

Figure 3: loglog plot of Fourier Spectrum for a.3

a.4

In the $(-\pi, \pi)$ range, the function takes the following form:

$$
F(x) = \begin{cases} -\sin(2x), & -\pi < x < 0 \\ \sin(x), & 0 \leq x < \pi \end{cases}
$$

The Fourier transform of F can therefore be obtained by the following:

First, put $n = 0$ and obtain A_0:

$$
\begin{aligned}
2\pi A_0 &= \int_{-\pi}^{0} (-\sin 2x)dx + \int_{0}^{\pi} \sin x dx \\
&= 2
\end{aligned}
$$

$$\Longrightarrow A_0 = \frac{1}{\pi}$$

Then, put $n \neq 0$ and obtain:

$$\begin{aligned}
2\pi A_n &= \int_{-\pi}^{0} (-\sin 2x)\exp(-inx)dx + \int_{0}^{\pi} \sin x \exp(-inx)dx \\
&= \frac{i}{2}\int_{-\pi}^{0} \Big\{\exp[ix(2-n)] - \exp[-ix(2+n)]\Big\}dx \\
&\quad - \frac{i}{2}\int_{0}^{\pi} \Big\{\exp[ix(1-n)] - \exp[-ix(1+n)]\Big\}dx \\
&= \frac{i}{2}\Big\{\frac{i}{n-2}[1-(-1)^n] - \frac{i}{n+2}[1-(-1)^n]\Big\} \\
&\quad - \frac{i}{2}\Big\{\frac{i}{n-1}[-(-1)^n - 1] - \frac{i}{n+1}[-(-1)^n - 1]\Big\} \\
&= -\frac{2[1-(-1)^n]}{(n-2)(n+2)} - \frac{1+(-1)^n}{(n-1)(n+1)}
\end{aligned}$$

Put $n = 1$:

$$\begin{aligned}
2\pi A_1 &= \int_{-\pi}^{0} (-\sin 2x)\exp(-ix)dx + \int_{0}^{\pi} \sin x \exp(-ix)dx \\
&= \frac{4}{3} - \frac{i}{2}\int_{0}^{\pi}\Big[1 - \exp(-i2x)\Big]dx \\
&= \frac{4}{3} - \frac{i\pi}{2}
\end{aligned}$$

$$\Longrightarrow A_1 = \frac{2}{3\pi} - \frac{i}{4}$$

Put $n = -1$:

$$\begin{aligned}
2\pi A_{-1} &= \int_{-\pi}^{0} (-\sin 2x)\exp(ix)dx + \int_{0}^{\pi} \sin x \exp(ix)dx \\
&= \frac{4}{3} - \frac{i}{2}\int_{0}^{\pi}\Big[exp(i2x) - 1\Big]dx \\
&= \frac{4}{3} + \frac{i\pi}{2}
\end{aligned}$$

$$\Longrightarrow A_{-1} = \frac{2}{3\pi} + \frac{i}{4}$$

Put $n = 2$:

$$\begin{aligned}
2\pi A_2 &= \int_{-\pi}^{0} (-\sin 2x)\exp(-i2x)dx + \int_{0}^{\pi} \sin x \exp(-i2x)dx \\
&= \frac{i}{2}\int_{-\pi}^{0}\Big[1 - exp(-i4x)\Big]dx - \frac{2}{3} \\
&= -\frac{2}{3} + \frac{i\pi}{2}
\end{aligned}$$

$$\Longrightarrow A_2 = -\frac{1}{3\pi} + \frac{i}{4}$$

Put $n = -2$:

$$\begin{aligned}
2\pi A_{-2} &= \int_{-\pi}^{0} (-\sin 2x)\exp(i2x)dx + \int_{0}^{\pi} \sin x \exp(i2x)dx \\
&= \frac{i}{2}\int_{-\pi}^{0}\Big[exp(i4x) - 1\Big]dx - \frac{2}{3} \\
&= -\frac{2}{3} - \frac{i\pi}{2}
\end{aligned}$$

$$\implies A_{-2} = -\frac{1}{3\pi} - \frac{i}{4}$$

Therefore:

$$A_n = \begin{cases} \frac{1}{\pi}, & n = 0 \\ -\frac{1}{3\pi} - \frac{i}{4}, & n = -2 \\ \frac{2}{3\pi} + \frac{i}{4}, & n = -1 \\ \frac{2}{3\pi} - \frac{i}{4}, & n = 1 \\ -\frac{1}{3\pi} + \frac{i}{4}, & n = 2 \\ -\frac{2[1-(-1)^n]}{(n-2)(n+2)} - \frac{1+(-1)^n}{(n-1)(n+1)}, & \text{otherwise} \end{cases}$$

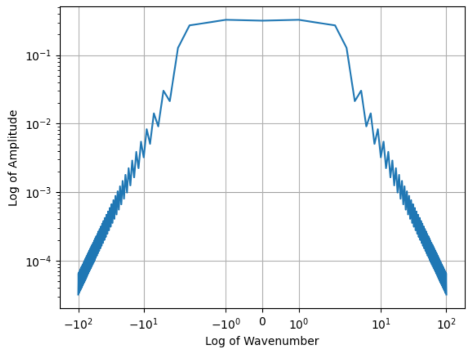

Figure 4: loglog plot of Fourier Spectrum for a.4

b)

$|k| \leq 8$

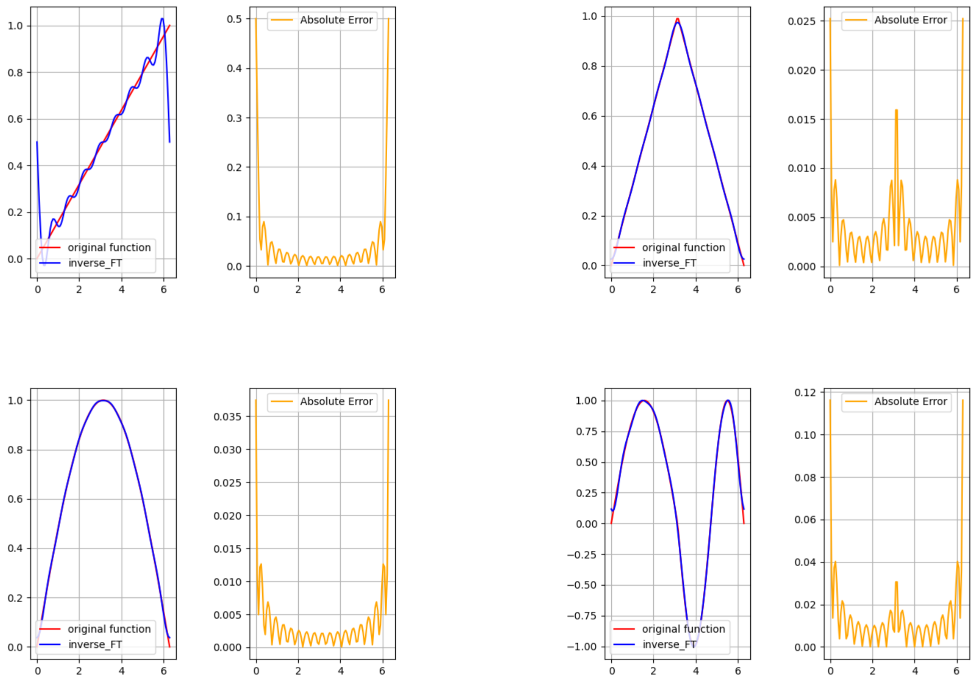

Figure 5: Original function, its inverse FT, and their difference, $|k| \leq 8$

$|k| \leq 16$

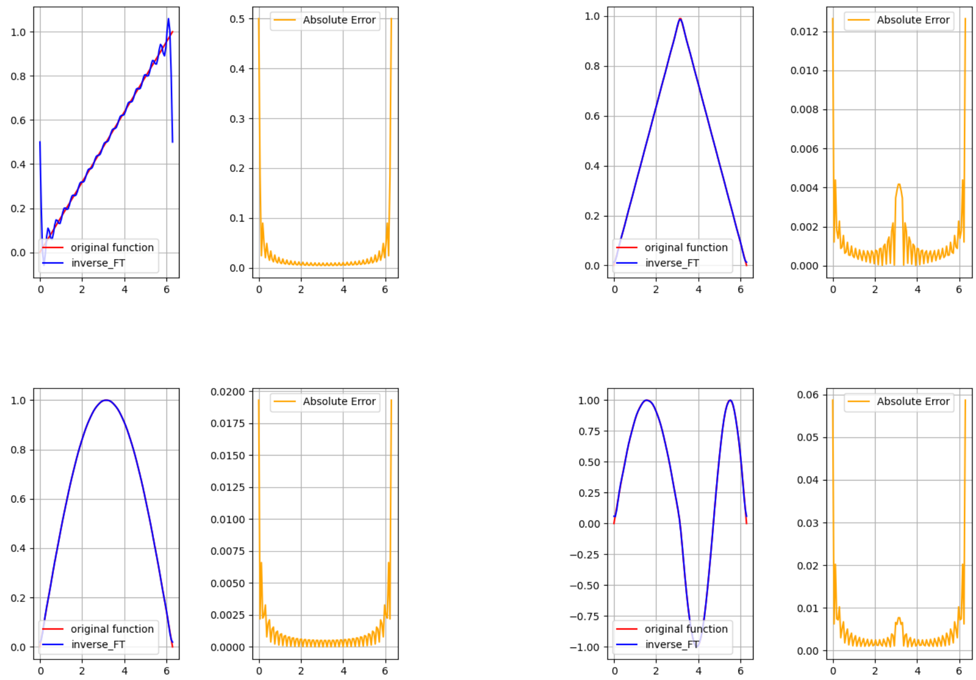

Figure 6: Original function, its inverse FT, and their difference, $|k| \leq 16$

$|k| \leq 32$

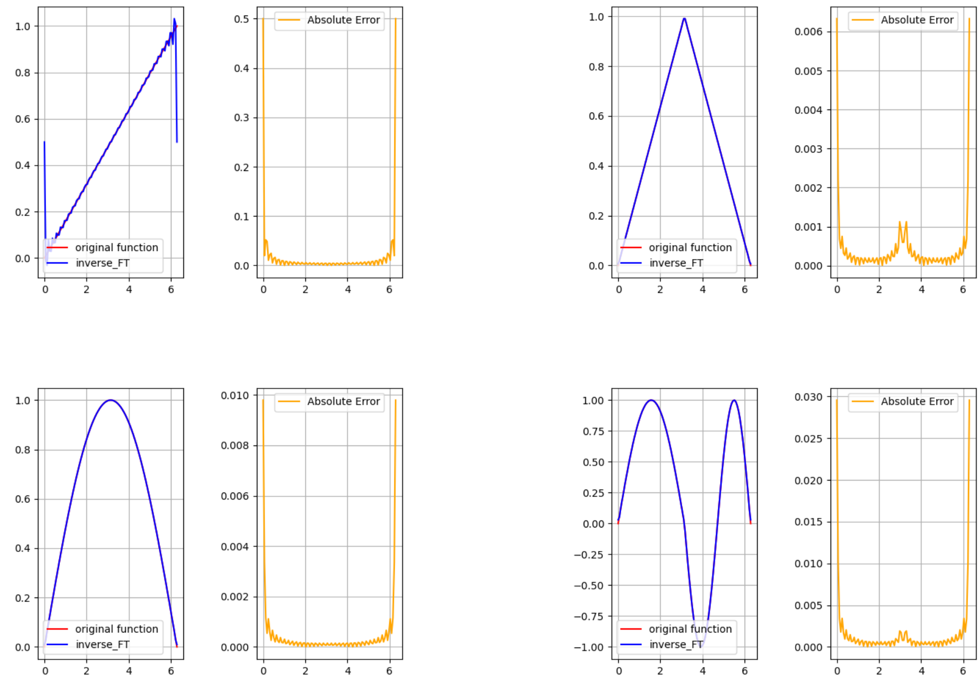

Figure 7: Original function, its inverse FT, and their difference, $|k| \leq 32$

Comment: for each of the 4 functions, the order of the absolute difference between the inverse FT and the original function reduces as the maximum modes increase. Furthermore, it is observed that the absolute differences increase significantly where there are discontinuities in the original functions or their derivatives. For example, for the first function, the error increases dramatically as x approaches 0 and 2π because there are jump discontinuities of the original function. As for the fourth function, the error increases around $x = \pi$ as well because there is a jump discontinuity of the first derivative of the original function. The similar goes for other functions as well.