Computer Graphics (CS 543)
Lecture 3c: Linear Algebra for Graphics (Points, Scalars, Vectors)

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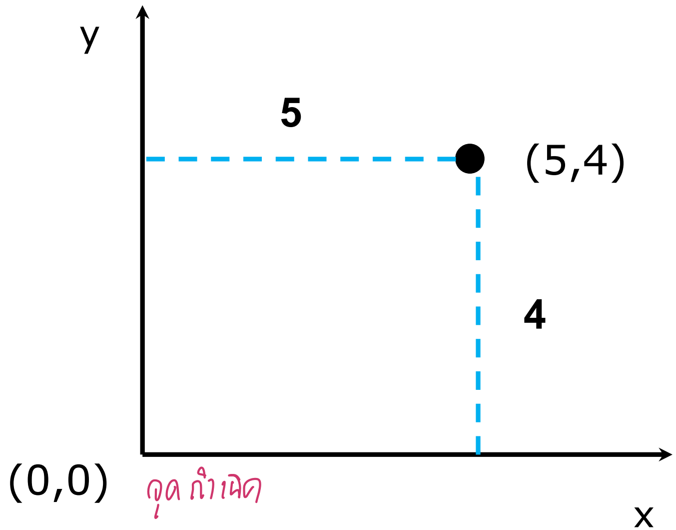
Points, Scalars and Vectors

- Points, vectors defined relative to a coordinate system
- **Point:** Location in coordinate system

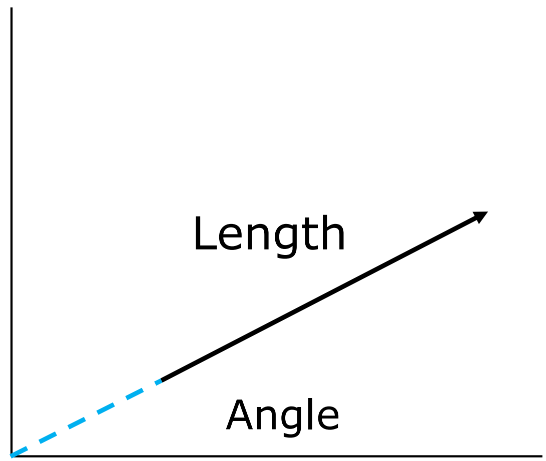
  space ปริภูมิ
- Example: Point (5,4)
- Cannot add or scale **points**



Vectors

- บอกทิศทาง
- ขนาด, ความยาว vectors

- Magnitude
- Direction ทิศทาง
- **NO** position
- Can be added, scaled, rotated
- CG vectors: 2, 3 or 4 dimensions



Vector-Point Relationship

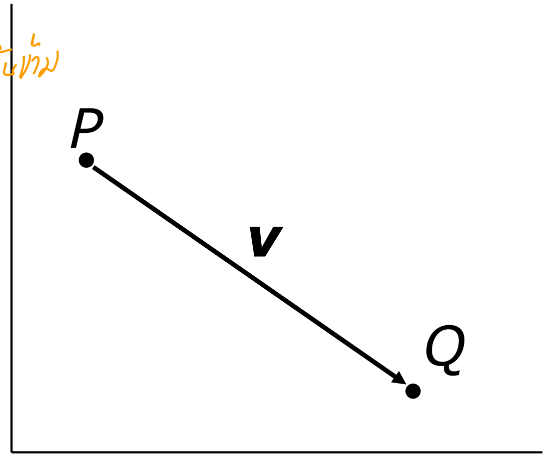
- Subtract **2 points** = **vector**

$$\mathbf{v} = Q - P$$ $\neq P - Q$ ทิศทางตรงกันข้าม

หางเวกเตอร์

- point + vector = point

$$P + \mathbf{v} = Q$$ หาจุด Q ได้



Vector Operations

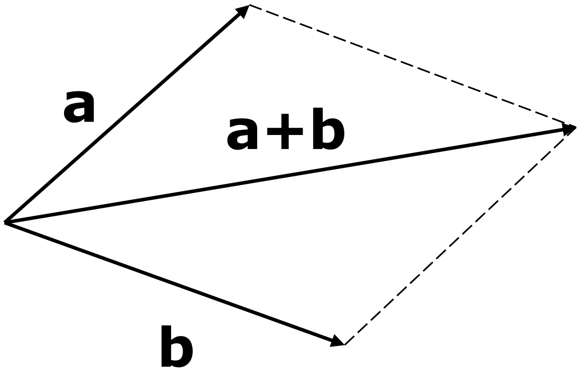
- Define vectors

$$\mathbf{a} = (a_{1,} a_2, a_3)$$

$$\mathbf{b} = (b_{1,} b_2, b_3)$$

Then vector addition:

$$\mathbf{a} + \mathbf{b} = (a_1 + b_{1,} a_2 + b_2, a_3 + b_3)$$



Vector Operations

- Define scalar, *s*
- Scaling vector by a scalar

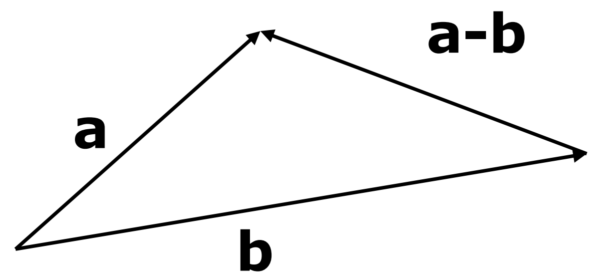
$$\mathbf{a}s = (a_1 s, a_2 s, a_3 s)$$

Note vector subtraction:

$$\mathbf{a} - \mathbf{b}$$
$$= (a_1 + (-b_1), a_2 + (-b_2), a_3 + (-b_3))$$

เพิ่ม scale vector ด้วยการคูณ

ถ้าคูณด้วย −
ทิศทางตรงกันข้าม



Vector Operations: Examples

- Scaling vector by a scalar

$$\mathbf{a}s = (a_1 s, a_2 s, a_3 s)$$

- Vector addition:

$$\mathbf{a} + \mathbf{b} = (a_1 + b_{1,} a_2 + b_2, a_3 + b_3)$$

- For example, if **a**=(2,5,6) and **b**=(-2,7,1) and *s*=6, then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_{1,} a_2 + b_2, a_3 + b_3) = (0,12,7)$$

$$\mathbf{a}s = (a_1 s, a_2 s, a_3 s) = (12,30,36)$$

Magnitude of a Vector

การคำนวณ ขนาดเวคเตอร์

- Magnitude of **a** 3D

$$|\mathbf{a}| = \sqrt{a_1^{\,2} + a_2^{\,2} \ldots\ldots\ldots + a_n^{\,2}}$$

- Normalizing a vector (unit vector)

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{vector}{magnitude}$$

จะได้เวคเตอร์ 1 หน่วย

ขนาดเวคเตอร์

- Note magnitude of normalized vector = 1. i.e

$$\sqrt{a_1^{\,2} + a_2^{\,2} \ldots\ldots\ldots + a_n^{\,2}} = 1$$

มีขนาดเป็น 1

Magnitude of a Vector

- Example: if a = (2, 5, 6)

- Magnitude of **a** $$|\mathbf{a}| = \sqrt{2^2 + 5^2 + 6^2} = \sqrt{65}$$

- Normalizing **a** หาเวคเตอร์ 1 หน่วยของ a

$$\hat{\mathbf{a}} = \left(\frac{2}{\sqrt{65}}, \frac{5}{\sqrt{65}}, \frac{6}{\sqrt{65}} \right)$$

Dot Product (Scalar product)

- Dot product, 2 vector

$$d = \mathbf{a} \cdot \mathbf{b} = a_1 \cdot b_1 + a_2 \cdot b_2 \ldots\ldots\ldots + a_3 \cdot b_3$$

- For example, if *a*=(2,3,1) and *b*=(0,4,-1) then

$$a \cdot b = (2 \times 0) + (3 \times 4) + (1 \times -1)$$
$$= 0 + 12 - 1 = 11$$

Properties of Dot Products

คุณสมบัติของ Dot Products

- Symmetry (or commutative):

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$$ ← สลับตำแหน่งได้

- Linearity:

$$(\mathbf{a}+\mathbf{c})\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{b}$$ กระจาย

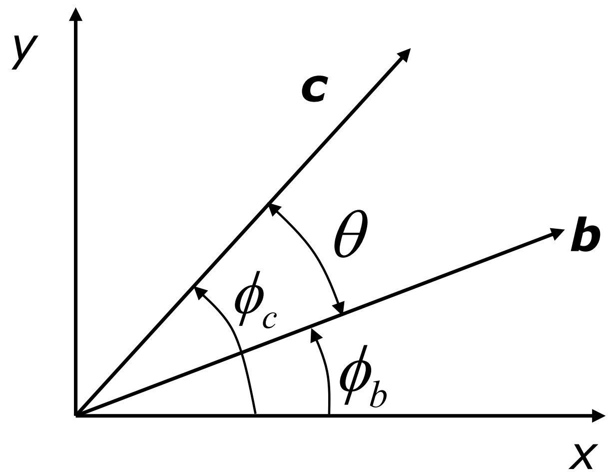
- Homogeneity:

$$(s\mathbf{a})\cdot\mathbf{b}=s(\mathbf{a}\cdot\mathbf{b})$$

- And $$|\mathbf{b}|^2=\mathbf{b}\cdot\mathbf{b}$$ หาขนาดกำลัง 2 ของ vector

Angle Between Two Vectors

หา มุม ของ 2 vectors



$$\mathbf{b} = \left(|\mathbf{b}|\cos\phi_b, |\mathbf{b}|\sin\phi_b\right)$$

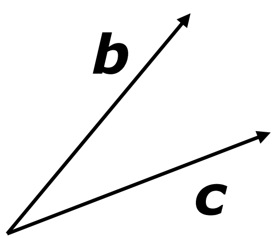
$$\mathbf{c} = \left(|\mathbf{c}|\cos\phi_c, |\mathbf{c}|\sin\phi_c\right)$$

$$\mathbf{b}\cdot\mathbf{c} = |\mathbf{b}||\mathbf{c}|\cos\theta$$

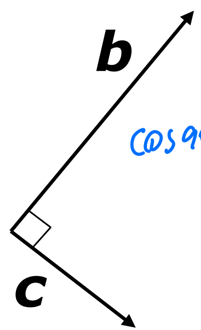
Sign of **b.c**:

ข้อดี → ตำแหน่งแรง

→ บอกการเบี่ยงเบน ทิศทางระหว่าง 2 vector

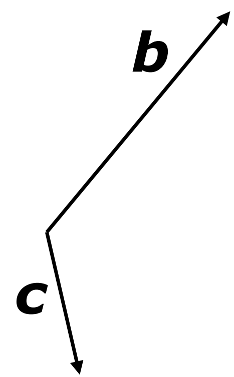


b.c > 0



cos 90° = 0

b.c = 0



b.c < 0

Angle Between Two Vectors

- **Problem:** Find angle b/w vectors **b** = (3,4) and **c = (5,2)**
- **Step 1:** Find magnitudes of vectors **b** and **c**

$$|\mathbf{b}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$|\mathbf{c}| = \sqrt{5^2 + 2^2} = \sqrt{29}$$

- **Step 2:** Normalize vectors **b** and **c**

$$\hat{\mathbf{b}} = \left(\frac{3}{5}, \frac{4}{5}\right) \qquad \hat{\mathbf{c}} = \left(\frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}}\right)$$

Angle Between Two Vectors

- **Step 3:** Find angle as dot product $\hat{\mathbf{b}} \bullet \hat{\mathbf{c}}$

$$\hat{\mathbf{b}} \bullet \hat{\mathbf{c}} = \left(\frac{3}{5}, \frac{4}{5}\right) \bullet \left(\frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}}\right)$$

$$\hat{\mathbf{b}} \bullet \hat{\mathbf{c}} = \frac{15}{5\sqrt{29}} + \frac{8}{5\sqrt{29}} = \frac{23}{5\sqrt{29}} = 0.85422$$

ใช้ func ตรีโกณ ผ่าน library
math → c, python
เวลาเรียกใช้ func เกี่ยวกับตรีโกณ ที่ส่งเข้าไป
หน่วยไม่ได้เป็นองศา หน่วยเป็น radiun

- **Step 4:** Find angle as inverse cosine

$$\theta = \text{acos}(0.85422) = 31.326^{\circ}$$

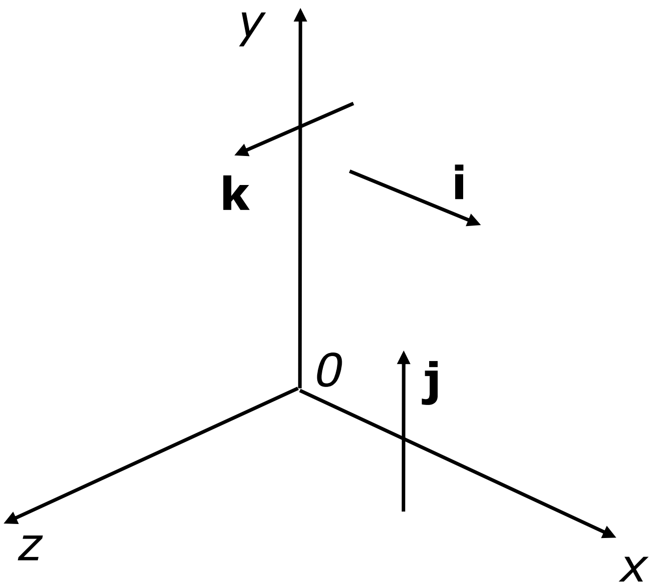
Standard Unit Vectors

Define

$$\mathbf{i} = (1,0,0)$$

$$\mathbf{j} = (0,1,0)$$

$$\mathbf{k} = (0,0,1)$$



So that any vector,

$$\mathbf{v} = (a,b,c) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

scalar

Cross Product (Vector product)

If

$$\mathbf{a} = \left(a_x, a_y, a_z\right) \qquad \mathbf{b} = \left(b_x, b_y, b_z\right)$$

คล้ายๆ การหา determinace ใน linear ↘+ ↗-

Then

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} - (a_x b_z - a_z b_x)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

Remember using determinant

$$\begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

บอกอะไรได้บ้าง

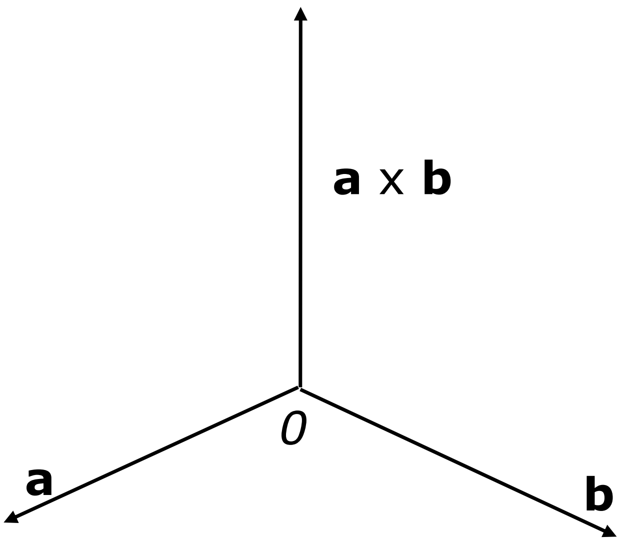
- เวกเตอร์ตั้งฉาก

เวกเตอร์ที่ตั้งฉากกับ vecter a , vecter b

Note: a x **b** is perpendicular to **a** and **b**

Cross Product

Note: a x **b** is perpendicular to both **a** and **b**



Cross Product (Vector product)

Calculate **a x b** if a = (3,0,2) and **b** = (4,1,8)

$$\mathbf{a} = (3,0,2) \qquad \mathbf{b} = (4,1,8)$$

Using determinant

$$\begin{vmatrix} i & j & k \\ 3 & 0 & 2 \\ 4 & 1 & 8 \end{vmatrix}$$

Then

$$\mathbf{a} \times \mathbf{b} = (0-2)\mathbf{i} - (24-8)\mathbf{j} + (3-0)\mathbf{k}$$

$$= -2\mathbf{i} - 16\mathbf{j} + 3\mathbf{k}$$

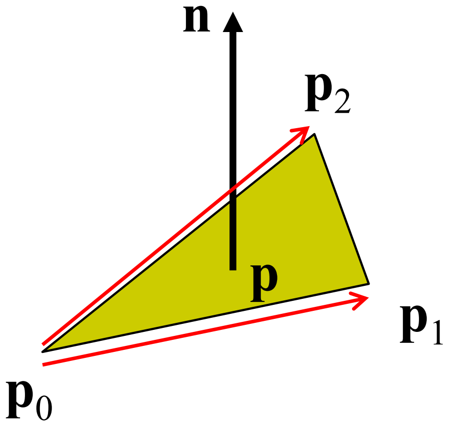
Normal for Triangle using Cross Product Method

plane $\mathbf{n}\cdot(\mathbf{p} - \mathbf{p}_0) = 0$

$\mathbf{n} = (\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0)$

normalize $\mathbf{n} \leftarrow \mathbf{n}/|\mathbf{n}|$

เวกเตอร์ 1 หน่วย

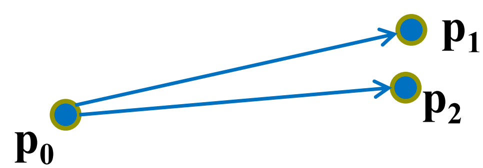


Note that right-hand rule determines outward face

เวกเตอร์ที่ตั้งฉากกับพื้นผิว

Newell Method for Normal Vectors

- Problems with cross product method:
  - calculation difficult by hand, tedious ซีเรียส, น่าเบื่อ
  - If 2 vectors almost parallel, cross product is small
  - Numerical inaccuracy may result



อักวิธีการ

- Proposed by Martin Newell at Utah (teapot guy)
  - Uses formulae, suitable for computer
  - Compute during mesh generation
  - Robust!

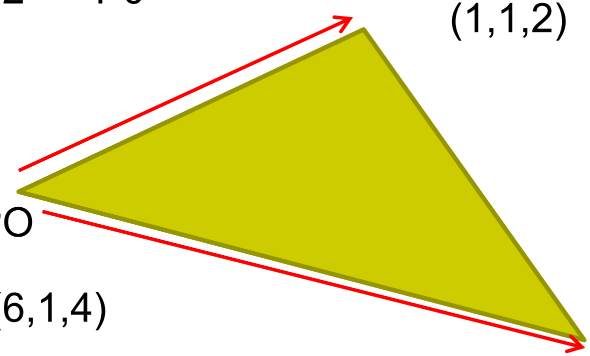
- ง่ายต่อการคำนวณ
- เสถียรกว่า

Newell Method Example

- Example: Find normal of polygon with vertices
  P0 = (6,1,4), P1=(7,0,9) and P2 = (1,1,2)

- Using simple cross product:
  ((7,0,9)-(6,1,4)) X ((1,1,2)-(6,1,4)) = (2,-23,-5)

P1 - P0 P2 - P0



Newell Method for Normal Vectors

- Formulae: Normal N = (mx, my, mz)

ตัวถัดไป

$$m_x = \sum_{i=0}^{N-1} \left(y_i - y_{next(i)}\right)\left(z_i + z_{next(i)}\right)$$

$$m_y = \sum_{i=0}^{N-1} \left(z_i - z_{next(i)}\right)\left(x_i + x_{next(i)}\right)$$

$$m_z = \sum_{i=0}^{N-1} \left(x_i - x_{next(i)}\right)\left(y_i + y_{next(i)}\right)$$

Newell Method for Normal Vectors

- Calculate x component of normal

$$m_x = \sum_{i=0}^{N-1} \left(y_i - y_{next(i)}\right)\left(z_i + z_{next(i)}\right)$$

$$m_x = \overset{1-0}{(1)}\overset{4+9}{(13)} + \overset{0-1}{(-1)}\overset{9+2}{(11)} + \overset{1-1}{(0)}\overset{2+4}{(6)}$$

$$m_x = 13 - 11 + 0$$

$$m_x = 2$$

	x	y	z
P0	6	1	4
P1	7	0	9
P2	1	1	2
P0	6	1	4

Newell Method for Normal Vectors

- Calculate y component of normal

$$m_y = \sum_{i=0}^{N-1} \left(z_i - z_{next(i)}\right)\left(x_i + x_{next(i)}\right)$$

4-9 6+7 9-2 7+1 2-4 1+6

$$m_y = (-5)(13) + (7)(8) + (-2)(7)$$

$$m_y = -65 + 56 - 14$$

$$m_y = -23$$

	x	y	z
P0	6	1	4
P1	7	0	9
P2	1	1	2
P0	6	1	4

Newell Method for Normal Vectors

- Calculate z component of normal

$$m_z = \sum_{i=0}^{N-1} \left(x_i - x_{next(i)}\right)\left(y_i + y_{next(i)}\right)$$

6-7 1+0

$$m_z = (-1)(1) + (6)(1) + (-5)(2)$$

$$m_z = -1 + 6 - 10$$

$$m_z = -5$$

	x	y	z
P0	6	1	4
P1	7	0	9
P2	1	1	2
P0	6	1	4

Note: Using Newell method yields same result as Cross product method (2,-23,-5)

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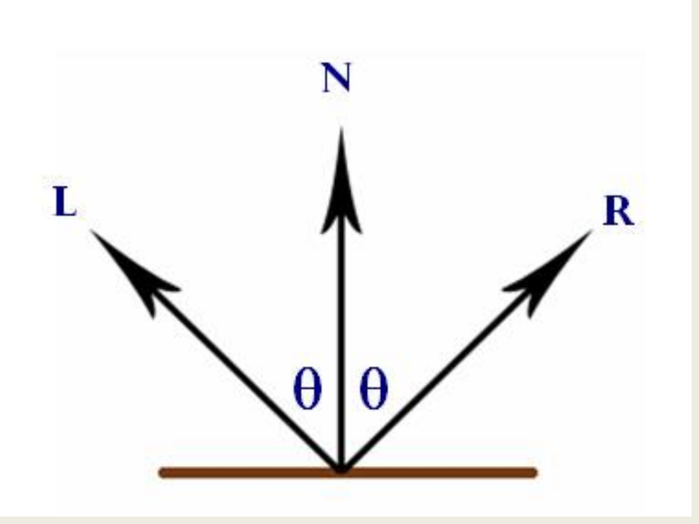
เวคเตอร์สะท้อน

- Reflected Vector R can be computed using

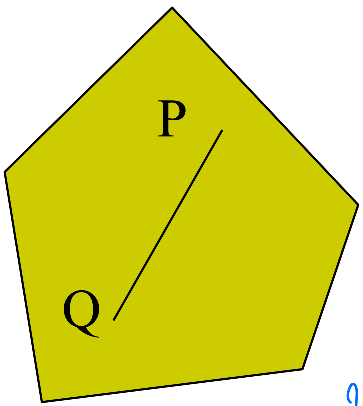
$$\underset{\text{vector}}{R} = ((\underset{\text{scalar}}{2\,(N \cdot L)})N) - L$$

ไว้ทำ
– ทำกระจก พื้นผิวกระจก



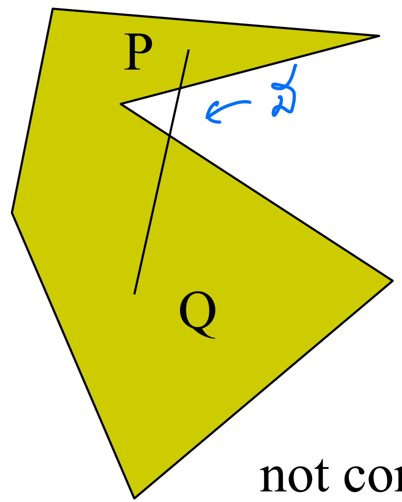
Convexity

- An object is *convex* iff for any two points in the object all points on the straight line between these points are also in the object



convex

หา 2 จุดใด บนพื้นผิว
polygon จากเส้นเชื่อม
ถ้าทั้งเส้นอยู่ในพื้นผิว
จะเป็น convex



not convex

References

- Angel and Shreiner, Interactive Computer Graphics, 6th edition, Chapter 3
- Hill and Kelley, Computer Graphics using OpenGL, 3rd edition, Sections 4.2 - 4.4