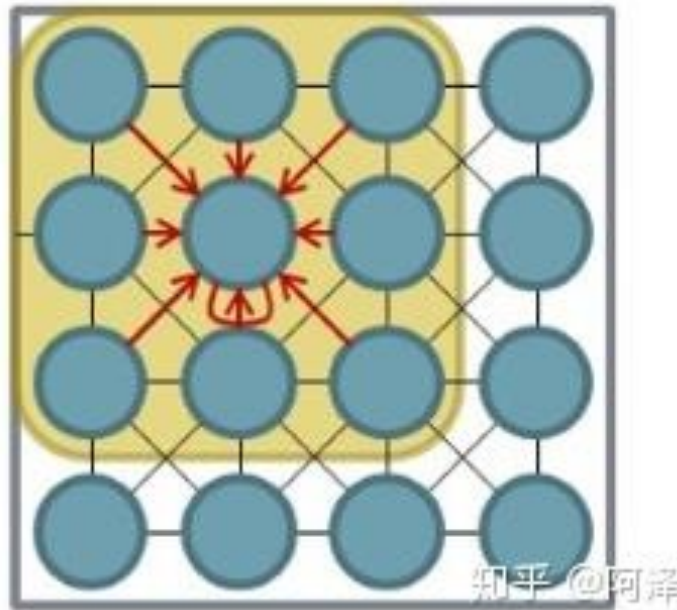

学术分享

GCN入门

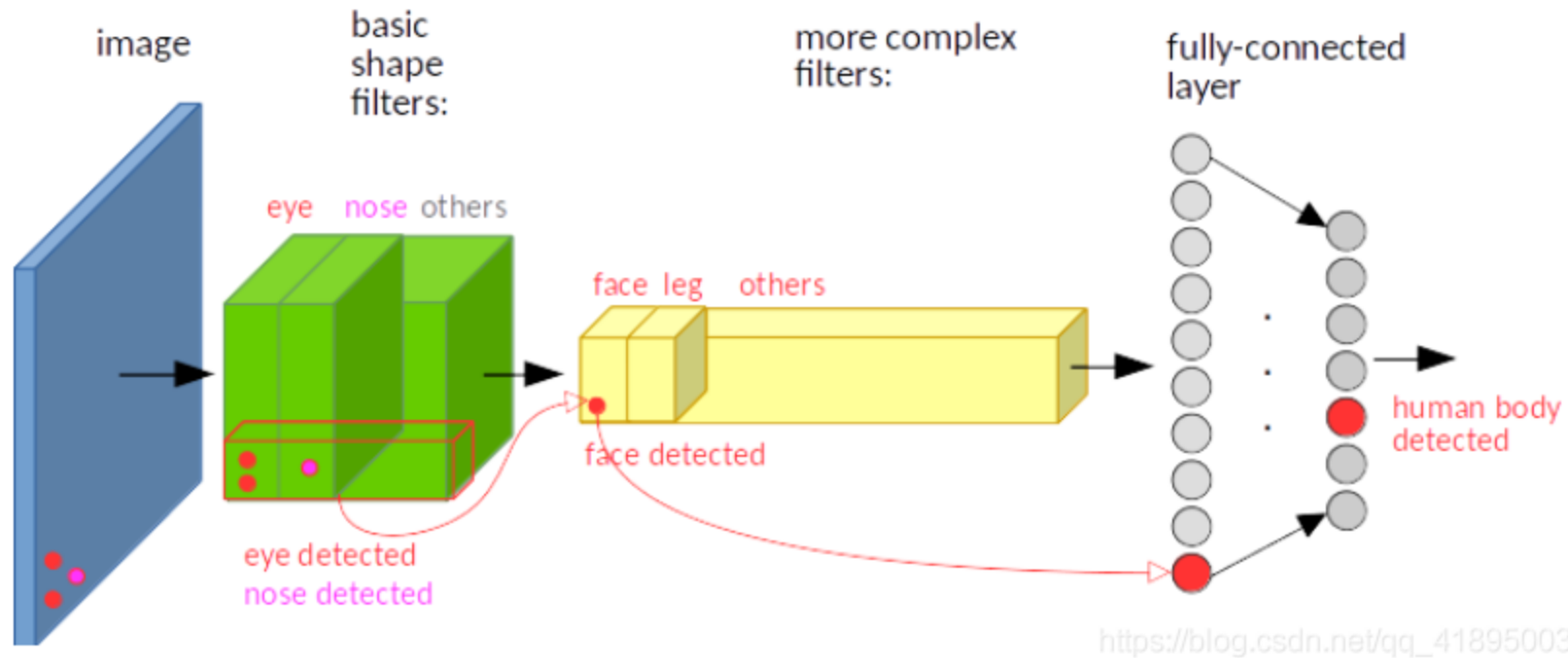
北京理工大学宇航学院

李嘉鑫

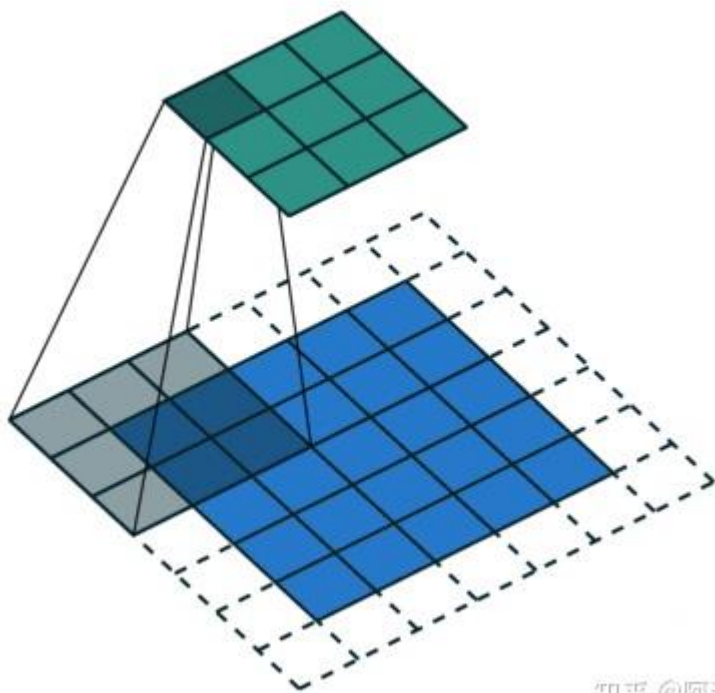
◆ Convolutional Neural Network



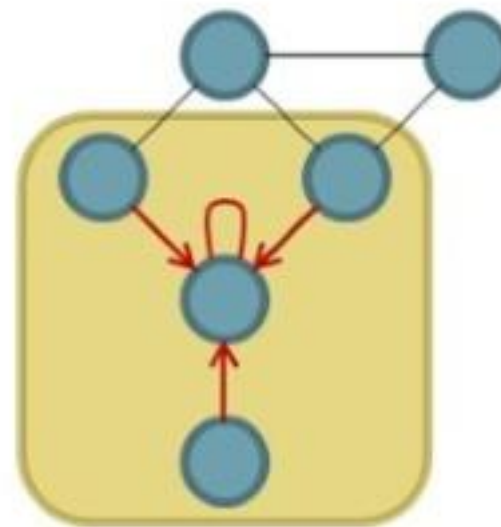
◆ Local Translational Invariance



◆ Local Translational Invariance

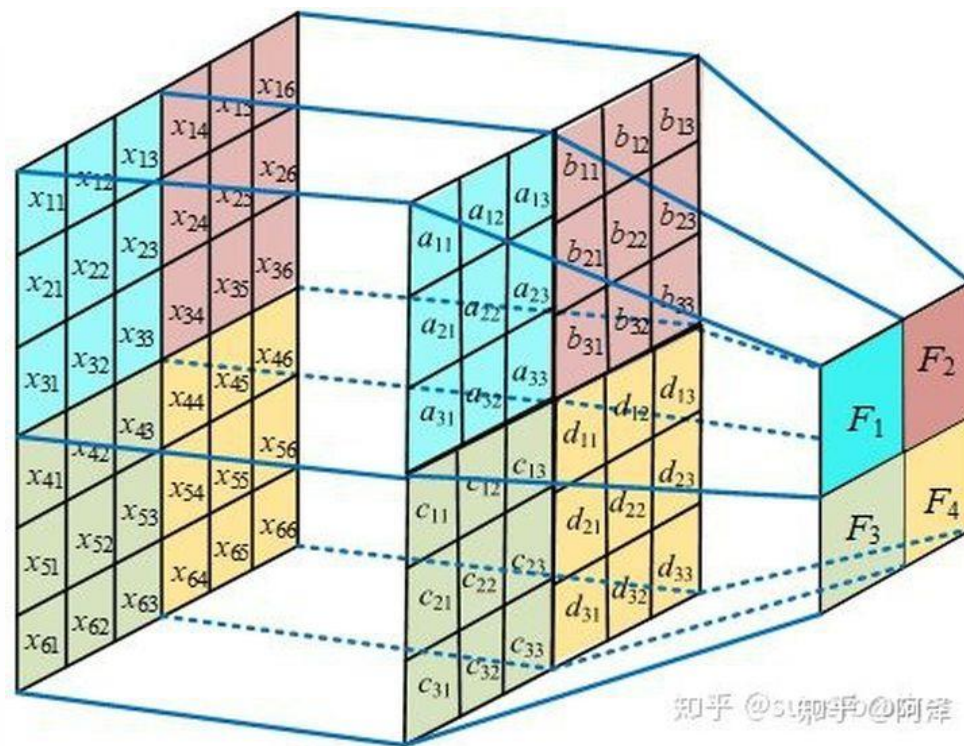
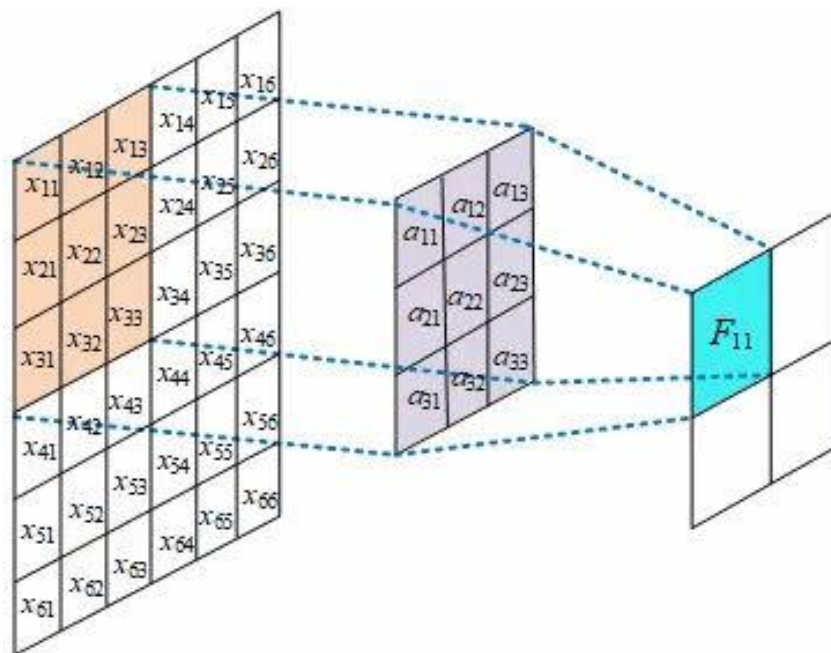


知乎 @阿泽



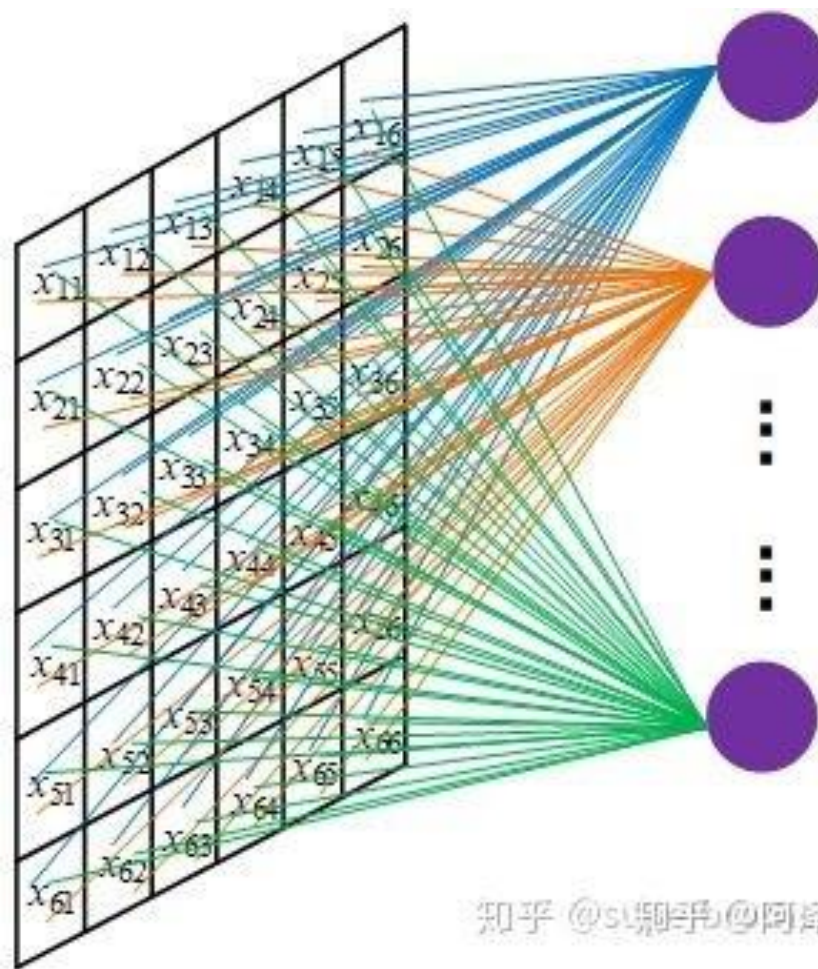
知乎 @阿泽

◆ Convolution Kernels



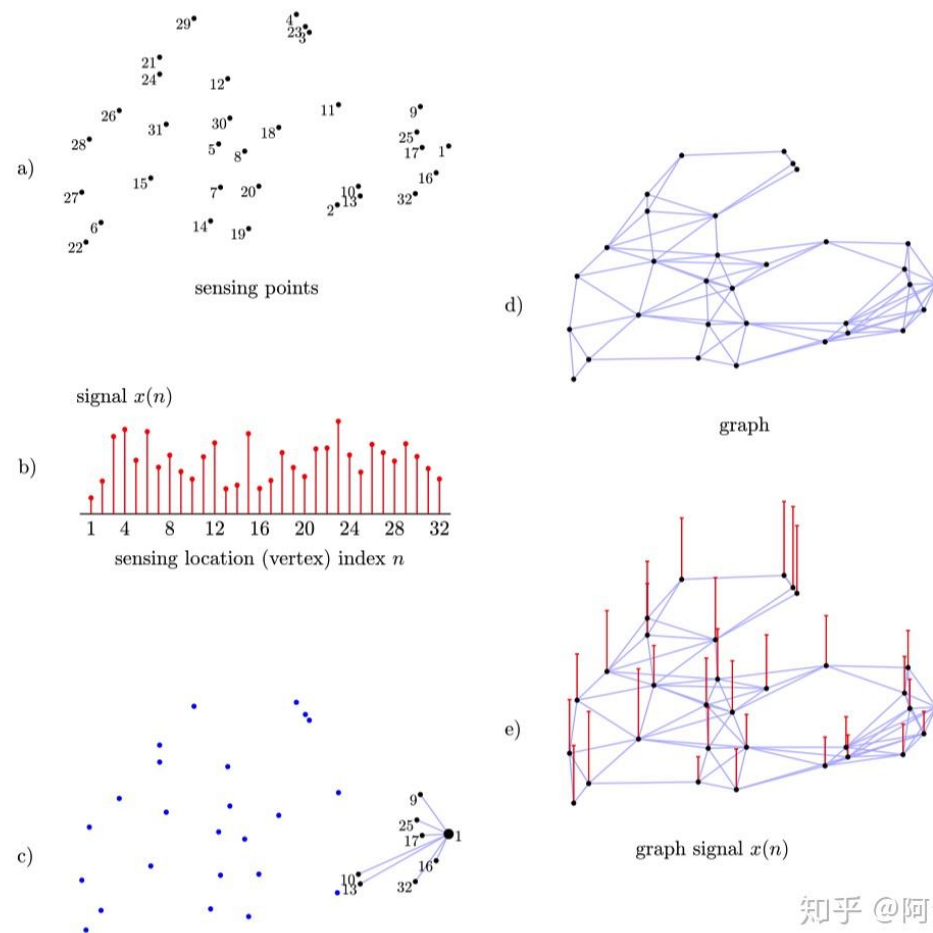
知乎 @SL知乎@阿洋

◆ Convolution Kernels

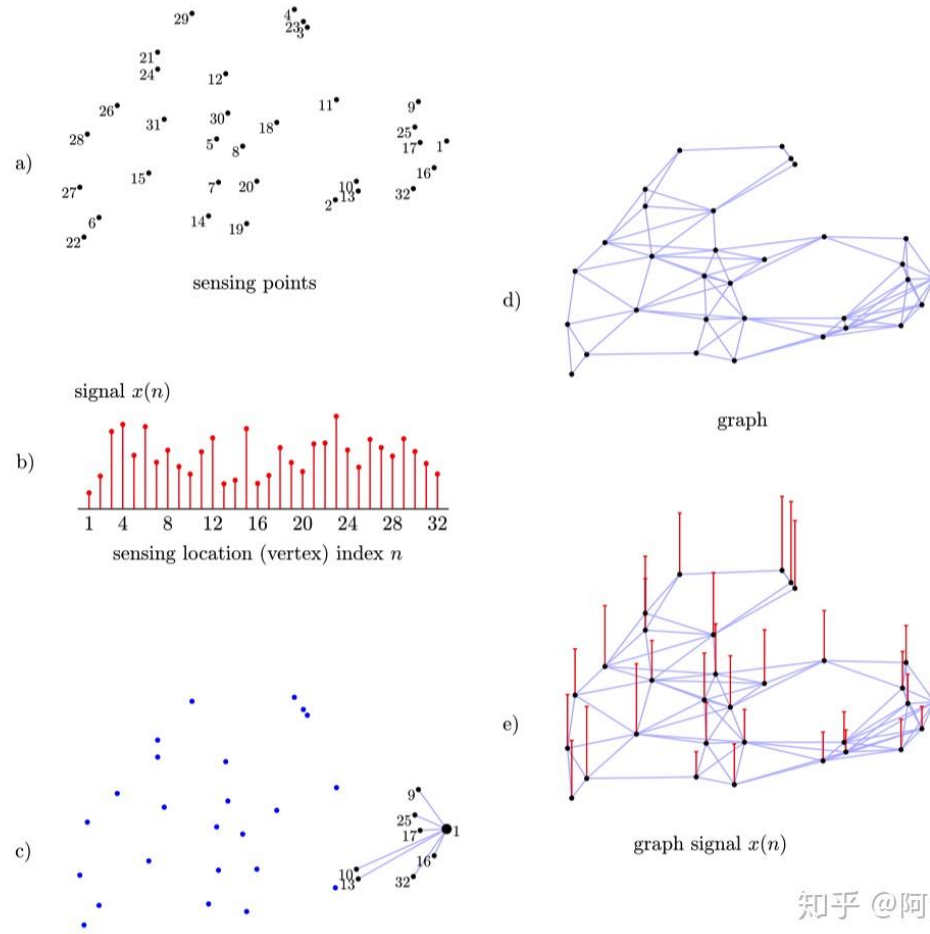


知乎 @知乎@阿峰

◆图信号处理(Graph Signal Processing)



◆图信号处理(Graph Signal Processing)



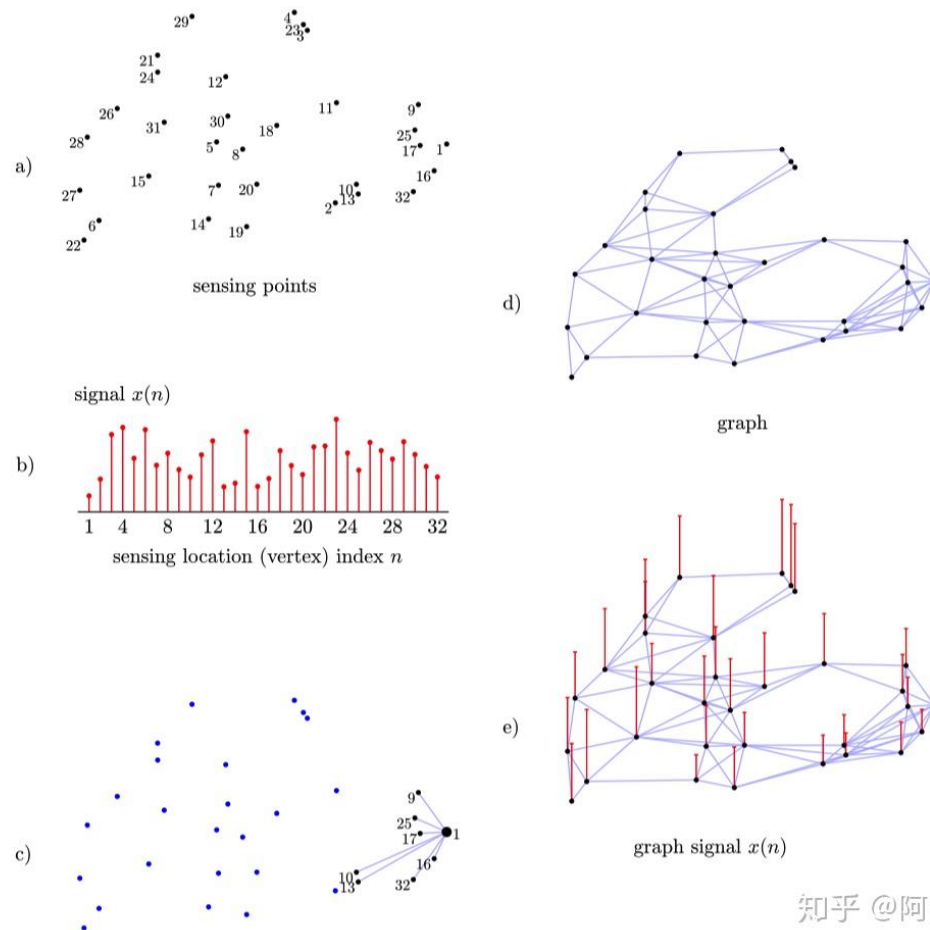
$$y(n) = x(n) + \sum_{m \in N(n)} x(m)$$

$$\mathbf{y} = \mathbf{x} + \mathbf{A}\mathbf{x}$$

$$\mathbf{y} = \mathbf{x} + \mathbf{W}\mathbf{x}$$

$$\mathbf{y} = \frac{1}{2}(\mathbf{x} + \mathbf{D}^{-1}\mathbf{W}\mathbf{x})$$

◆图信号处理(Graph Signal Processing)



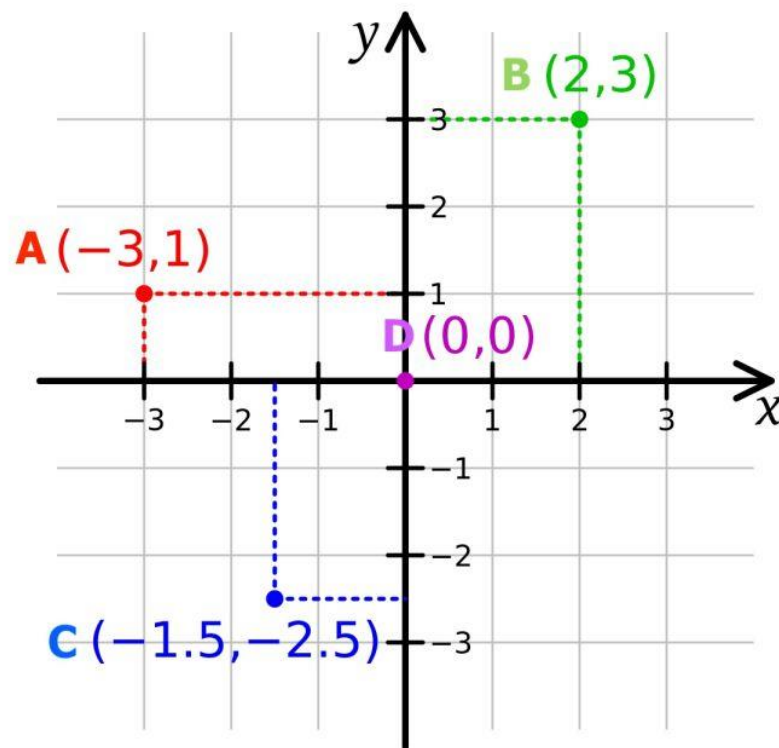
知乎 @阿泽

1. 测量点构成节点（图 a），节点间的连通性和相关性构成边；
2. 节点和边构成图（图 b），该图是信号域，表示测量信号的点以及它们之间的关系，并使用该图进行分析和处理；
3. 测量温度是图的信号（图 e），这里的信号由真实温度和测量噪声所组成；
4. 考虑测量位置，我们提出了局部平均和加权平均，这是最简单的图信号处理方式（Linear first-order）。

◆傅里叶变换 (Fourier Transformer)

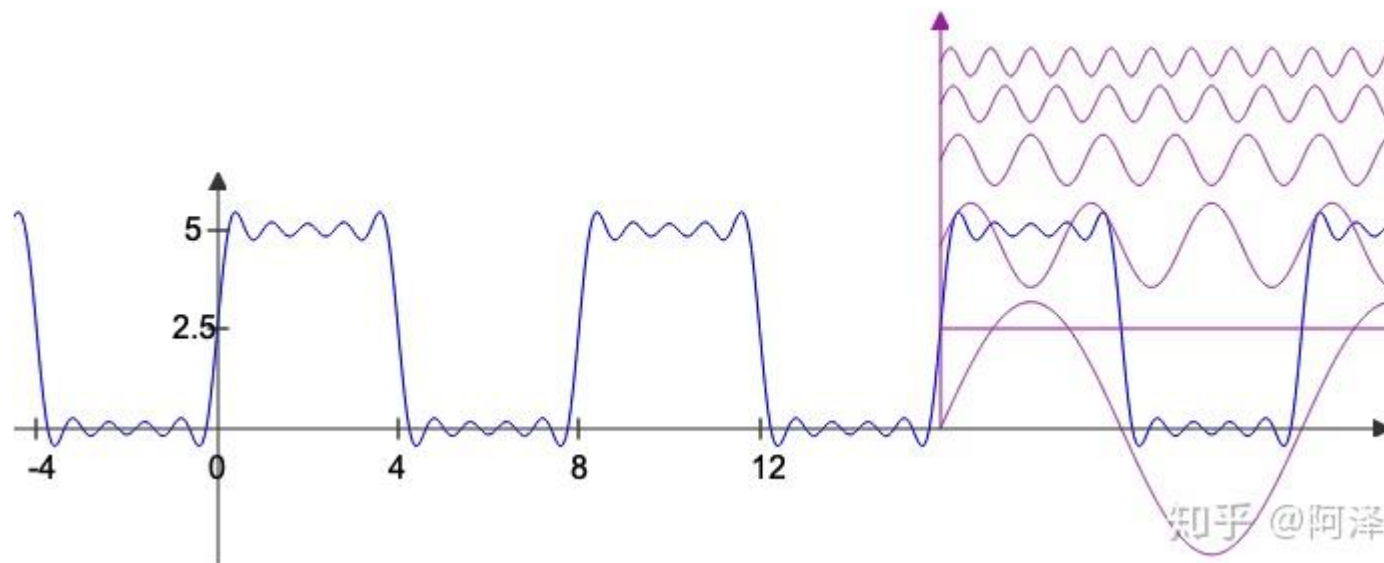
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2i\pi x\xi} dx$$

◆ 傅里叶变换 (Fourier Transformer)



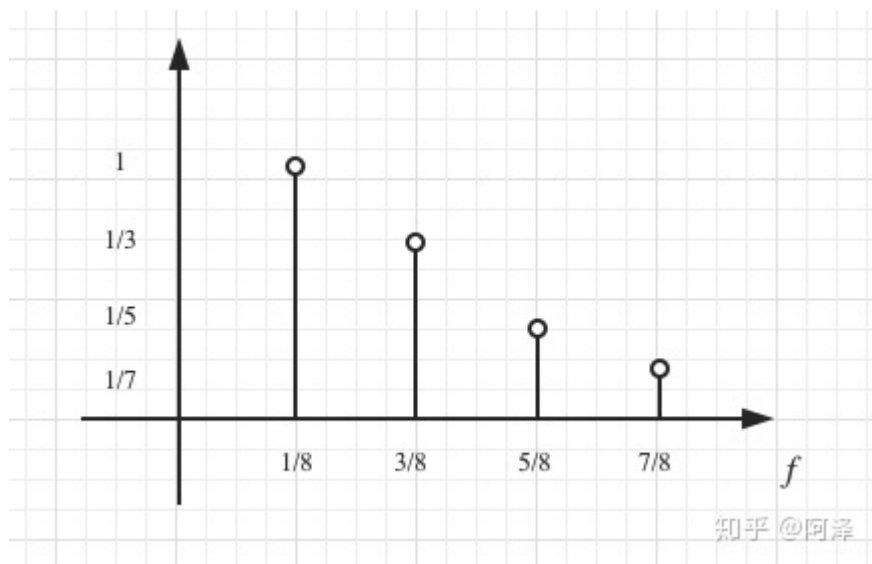
知乎 @阿泽

◆傅里叶变换 (Fourier Transformer)

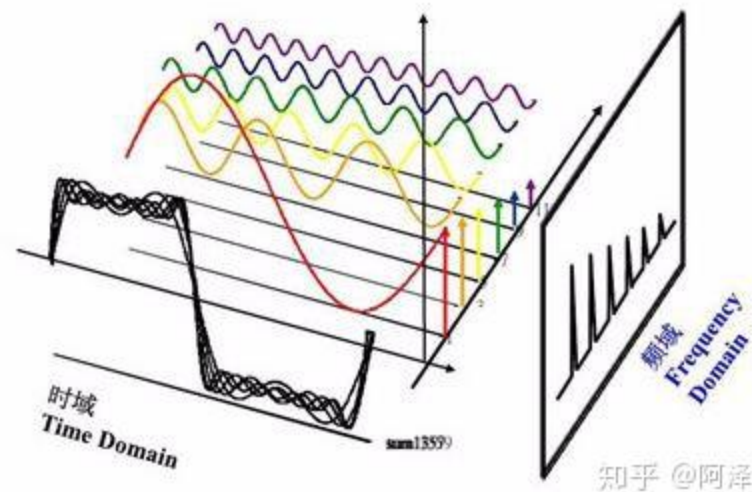


$$f(t) \approx 2.5 + \frac{10}{\pi} \left(\sin \frac{\pi t}{4} + \frac{1}{3} \sin \frac{3\pi t}{4} + \frac{1}{5} \sin \frac{5\pi t}{4} + \frac{1}{7} \sin \frac{7\pi t}{4} \right)$$

◆傅里叶变换 (Fourier Transformer)



频域与时域



$$f(t) \approx 2.5 + \frac{10}{\pi} \left(\sin \frac{\pi t}{4} + \frac{1}{3} \sin \frac{3\pi t}{4} + \frac{1}{5} \sin \frac{5\pi t}{4} + \frac{1}{7} \sin \frac{7\pi t}{4} \right)$$

◆傅里叶变换 (Fourier Transformer)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin(n\omega x + \varphi_n)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin(n\omega x) + \sum_{n=1}^{\infty} b_n \cos(n\omega x)$$

◆傅里叶变换 (Fourier Transformer)

$$\int_0^{2\pi} \cos(mx) \sin(nx) dx = 0$$

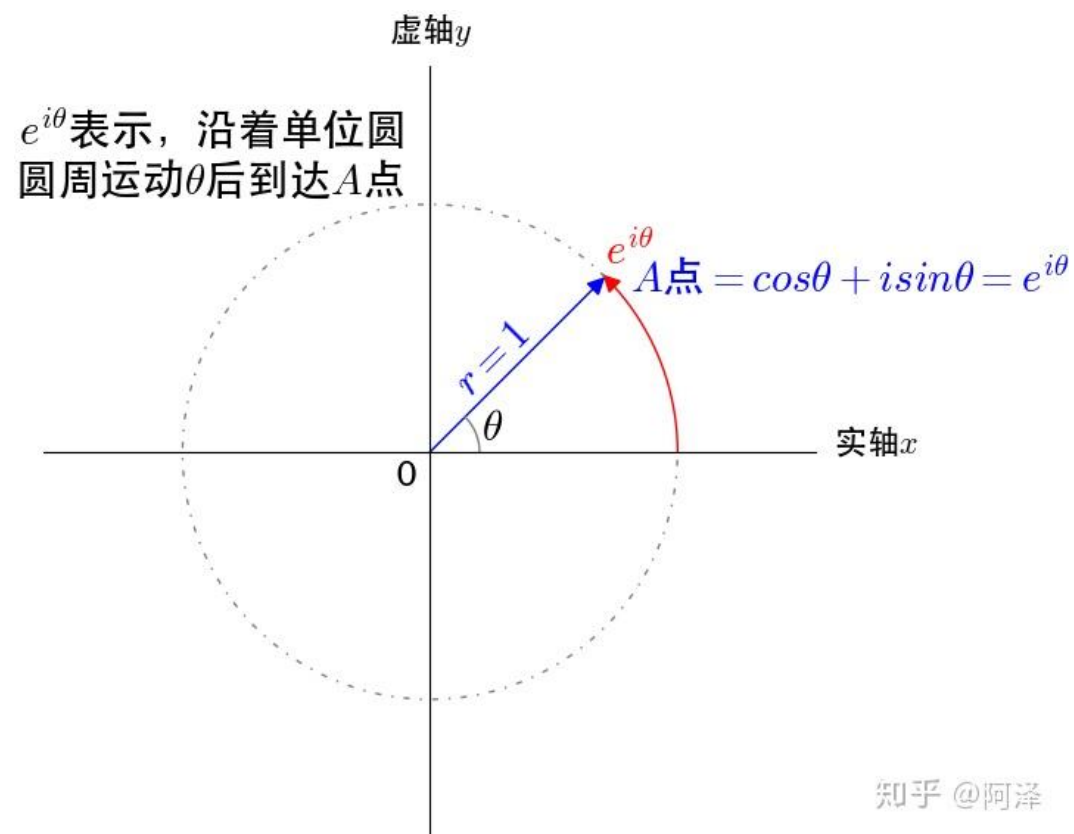
$$\int_0^{2\pi} \sin(mx) \sin(nx) dx = \pi \delta_{mn} \quad m, n > 1$$

$$\int_0^{2\pi} \cos(mx) \cos(nx) dx = \pi \delta_{mn} \quad m, n > 1$$

$$\text{where } \delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

◆傅里叶变换 (Fourier Transformer)

$$\cos\theta + i \sin\theta = e^{i\theta}$$



◆傅里叶变换 (Fourier Transformer)

$$\mathcal{F}_T(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t} dt$$

◆图拉普拉斯

$$\mathbf{L} = \mathbf{D} - \mathbf{W}$$

$$\mathbf{L}_N = \mathbf{D}^{-1/2} (\mathbf{D} - \mathbf{W}) \mathbf{D}^{-1/2}$$

◆图拉普拉斯

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f$$

$$\frac{\partial^2 f}{\partial x_i^2} = f''(x)$$

$$\approx f'(x) - f'(x-1)$$

$$\approx f(x+1) - f(x) - (f(x) - f(x-1))$$

$$= f(x+1) + f(x-1) - 2f(x)$$

◆图拉普拉斯

$$\begin{aligned}
 \Delta f_i &= \sum_{j \in N_i} \frac{\partial f_i}{\partial j^2} \\
 &\approx \sum_j w_{ij} (f_i - f_j) \\
 &= \sum_j w_{ij} (f_i - f_j) \\
 &= \left(\sum_j w_{ij} \right) f_i - \sum_j w_{ij} f_j \\
 &= d_i f_i - w_{i:} f_i
 \end{aligned}
 \qquad
 \begin{aligned}
 \Delta f &= \begin{pmatrix} \Delta f_1 \\ \vdots \\ \Delta f_N \end{pmatrix} = \begin{pmatrix} d_1 f_1 - w_{1:} f \\ \vdots \\ d_N f_N - w_{N:} f \end{pmatrix} \\
 &= \begin{pmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_N \end{pmatrix} f - \begin{pmatrix} w_{1:} \\ \vdots \\ w_{N:} \end{pmatrix} f \\
 &= \text{diag}(d_i) f - \mathbf{W} f \\
 &= (\mathbf{D} - \mathbf{W}) f \\
 &= \mathbf{L} f
 \end{aligned}$$

◆图拉普拉斯谱分解

$$\mathbf{L} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{-1} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$$

$$\begin{aligned} f^T \mathbf{L} f &= f^T D f - f^T W f \\ &= \sum_i d_i f_i^2 - \sum_{i,j} f_i f_j w_{ij} \\ &= \frac{1}{2} \left(\sum_i d_i f_i^2 - 2 \sum_{ij} f_i f_j w_{ij} + \sum_i d_i f_i^2 \right) \\ &= \frac{1}{2} \sum_{i,j} w_{ij} (f_i - f_j)^2 \end{aligned}$$

◆图傅里叶变换

首先考虑亥姆霍兹方程

$$\nabla^2 f = -k^2 f$$

$$\mathcal{F}_T(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

$$\nabla^2 e^{-i\omega t} = \frac{\partial^2 e^{-i\omega t}}{\partial t^2} = -\omega^2 e^{-i\omega t}$$

◆图傅里叶变换

$$\mathcal{F}_T(\lambda_k) = \hat{f}_k = \sum_{i=1}^N f(i)u_k(i)$$

其中, $f_k = (f_k(i), \dots, f_k(n))$ 为网络图上的 n 维向量, $f_k(i)$ 表示网络中的节点 i 的第 k 个分量, $u_k(i)$ 表示特征向量 k 的第 i 个分量。做个类比解释: 特征值 (频率) λ_k 下, f 的图傅立叶变换 (振幅) 等于 f 与 λ_x 对应的特征向量 \mathbf{u}_k 的内积。

考虑矩阵乘法:

$$\hat{f} = \begin{pmatrix} \hat{f}_1 \\ \vdots \\ \hat{f}_N \end{pmatrix} = \begin{pmatrix} u_1(1) & \cdots & u_1(n) \\ \vdots & \ddots & \vdots \\ u_n(1) & \cdots & u_n(n) \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} = \mathbf{U}^T f$$

◆ Graph Convolutional Network

$$\begin{aligned}(f * h)_G &= \mathcal{F}^{-1}[\mathcal{F}\{f\} \cdot \mathcal{F}\{h\}] \\ &= \mathcal{F}^{-1}[\mathbf{U}^T f \cdot \hat{h}]\end{aligned}$$

其中，向量 \hat{f} 与向量 \hat{h} 的元素点积，等价于将 \hat{h} 组织成对角矩阵的形式进行矩阵乘法，所以我们有：

$$\begin{aligned}(f * h)_G &= \mathcal{F}^{-1}[\mathbf{U}^T f \cdot \hat{h}] \\ &= \mathcal{F}^{-1}[\text{diag}[\hat{h}_1, \dots, \hat{h}_n] \mathbf{U}^T f]\end{aligned}$$

最后我们再左乘 \mathbf{U} 进行逆变换：

$$(f * h)_G = \mathbf{U} \text{diag}[\hat{h}_1, \dots, \hat{h}_n] \mathbf{U}^T f$$

◆ GCN-1

$$y = \sigma(\mathbf{U}g_{\theta}\mathbf{U}^T x) = \sigma(\mathbf{U}diag[\theta_1, \dots, \theta_n]\mathbf{U}^T x)$$

$$x_{k+1,j} = h\left(V \sum_{i=1}^{f_{k-1}} F_{k,i,j} V^T x_{k,i}\right)$$

$$x_{k+1,j} = L_k h\left(\sum_{i=1}^{f_{k-1}} F_{k,i,j} x_{k,i}\right) \quad j = 1, \dots, f_k$$

◆GCN-2 ChbeyNet

$$\mathcal{F}_T(\lambda_k) = \hat{g}_k = \sum_{i=1}^N g(i) u_k(i)$$

$$y = \sigma(\mathbf{U} g_{\theta} \mathbf{U}^T x) = \sigma(\mathbf{U} g_{\theta}(\Lambda) \mathbf{U}^T x)$$

$$g_{\theta}(\Lambda) \approx \sum_{k=0}^{K-1} \theta_k \Lambda^k$$

◆GCN-2 ChbeyNet

$$y = \sigma(\mathbf{U}g_{\theta}(\Lambda)\mathbf{U}^T x) = \sigma(\mathbf{U} \sum_{k=0}^{K-1} \theta_k \Lambda^k \mathbf{U}x) = \sigma(\sum_{k=0}^{K-1} \theta_k L^k x)$$

设 $T_k(x)$ 为切比雪夫多项式的第 k 阶式子，切比雪夫多项式的递归式为：

$T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x)$, $T_0(x) = 1$, $T_1(x) = x$ 。所以我们有：

$$g_{\theta}(\Lambda) \approx \sum_{k=0}^{K-1} \theta_k T_k(\tilde{\Lambda})$$

其中, $\tilde{\Lambda} = \frac{2}{\lambda_{max}} \Lambda - I_N$; λ_{max} 是指拉普拉斯矩阵 L 的最大值。

◆ GCN-3

$$g_{\theta} * x \approx \sum_{k=0}^{K-1} \theta_k T_k(\tilde{L})x$$

$$g_{\theta} * x \approx \theta_0 x + \theta_1 (L - I_N)x = \theta_0 x - \theta_1 D^{-\frac{1}{2}} A D^{-\frac{1}{2}} x$$

$$g_{\theta} * x \approx \theta (I_N + D^{-\frac{1}{2}} A D^{-\frac{1}{2}})x$$

$$I_N + D^{-\frac{1}{2}} A D^{-\frac{1}{2}} \rightarrow \tilde{D}^{-\frac{1}{2}} \tilde{A} \tilde{D}^{-\frac{1}{2}} \quad \text{where } \tilde{D}_{ii} = \sum_j \tilde{A}_{ij} \quad \tilde{A} = A + I_N$$

◆GCN-3

$$H^{(l+1)} = \sigma(\widetilde{D}^{-\frac{1}{2}} \widetilde{A} \widetilde{D}^{-\frac{1}{2}} H^{(l)} W^{(l)})$$

其中, $\widetilde{A} = A + I_N$, A 为邻接矩阵, I_N 为单位矩阵, 所以 \widetilde{A} 为添加自连接的邻接矩阵;

$\widetilde{D}_{ii} = \sum_j \widetilde{A}_{ij}$, \widetilde{D} 为节点的度数矩阵; $W^{(l)}$ 为神经网络第 l 层的权重矩阵; $\sigma(\cdot)$ 是激

活函数; $H^{(l)} \in R^{N \times D}$ 是第 l 层的激活矩阵, 并且 $H^{(0)} = X$, X 是由节点 x_i 的特征向量组成矩阵。

学术分享

预告：Winograd算法（加速卷积运算）

北京理工大学宇航学院

李嘉鑫