Computational Thinking Logic Coursework

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March 2025

1 Complete sets of logical connectives

To prove a complete set of connectives, the given set must be able to construct all the others in the complete set of logical connectives $\{\wedge, \vee, \neg, \rightarrow, \leftrightarrow\}$.

1.1 Show $\{\neg, \rightarrow\}$ is a complete set of connectives.

The logical connectives $\{\wedge, \vee, \leftrightarrow\}$ needs to be proven using propositions $\{\neg, \rightarrow\}$. This can be done by constructing a truth table of each logical connective and use a combination of propositions to prove a complete set of connectives.

1.1.1 AND ($\wedge$)

$$A \wedge B = \neg(A \rightarrow \neg B)$$

A	**B**	**A $\wedge$ B**	**$\neg$ (A $\rightarrow$ $\neg$ B)**
F	F	F	F
T	F	F	F
F	T	F	F
T	T	T	T

1.1.2 OR ($\vee$)

$$A \vee B = \neg A \rightarrow B$$

A	**B**	**A $\vee$ B**	**$\neg$ A $\rightarrow$ B**
F	F	F	F
T	F	T	T
F	T	T	T
T	T	T	T

1.1.3 IFF ($\leftrightarrow$)

$$A \leftrightarrow B = (A \wedge \neg B) \vee (\neg A \wedge B)$$

Since we have earlier proven $\wedge$ and $\vee$ can be constructed using the propositions, they can be in turn used to prove for $\leftrightarrow$

A	**B**	**A $\leftrightarrow$ B**	**A $\wedge \neg$ B**	**$\neg$ A $\wedge$ B**	**(A $\wedge \neg$ B) $\vee$ ($\neg$ A $\wedge$ B)**
F	F	F	F	F	F
T	F	T	T	F	T
F	T	T	F	T	T
T	T	T	F	F	F

Using these truth tables, we can see that $\{\neg, \rightarrow\}$ is a complete set of connectives.

1.2 Show $\{\rightarrow$, 0$\}$ is a complete set of connectives (where 0 is the constant false).

The logical connectives $\{\wedge, \vee, \leftrightarrow\ \neg\}$ needs to be proven using propositions $\{\rightarrow$, 0 (constant false)$\}$. This can be done by constructing a truth table of each logical connective and use a combination of propositions to prove a complete set of connectives.

1.2.1 NOT ($\neg$)

$$\neg A = A \rightarrow 0$$

A	**$\neg$ A**	**A $\rightarrow$ 0**
T	F	F
F	T	T

$\neg$ is now proven to be part of the set and can be similarly used in **1.1** to prove the rest.

1.2.2 AND ($\wedge$)

$$A \wedge B = \neg(A \to \neg B)$$

A	**B**	**A $\wedge$ B**	**$\neg$ (A $\to$ $\neg$ B)**
F	F	F	F
T	F	F	F
F	T	F	F
T	T	T	T

1.2.3 OR ($\vee$)

$$A \vee B = \neg A \to B$$

A	**B**	**A $\vee$ B**	**$\neg$ A $\to$ B**
F	F	F	F
T	F	T	T
F	T	T	T
T	T	T	T

1.2.4 IFF ($\leftrightarrow$)

$$A \leftrightarrow B = (A \wedge \neg B) \vee (\neg A \wedge B)$$

Since we have earlier proven $\wedge$ and $\vee$ can be constructed using the propositions, they can be in turn used to prove for $\leftrightarrow$

A	**B**	**A $\leftrightarrow$ B**	**A $\wedge$ $\neg$ B**	**$\neg$ A $\wedge$ B**	**(A $\wedge$ $\neg$ B) $\vee$ ($\neg$ A $\wedge$ B)**
F	F	F	F	F	F
T	F	T	T	F	T
F	T	T	F	T	T
T	T	T	F	F	F

1.3 Is $\{\uparrow,\wedge\}$ a complete set of connectives?

NAND $\{\uparrow\}$ can be used to construct any logical connective.

1.3.1 NOT ($\neg$)

$$\neg A = A \uparrow A$$

A	**$\neg$ A**	**A $\uparrow$ A**
T	F	F
F	T	T

1.3.2 IMPLIES ($\rightarrow$)

$$A \rightarrow B = (A \uparrow A) \uparrow B$$

A	**B**	**A $\rightarrow$ B**	**A $\uparrow$ A**	**(A $\uparrow$ A) $\uparrow$ B**
F	F	T	F	T
T	F	F	F	F
F	T	T	T	T
T	T	T	T	T

Now that the set contains $\{\neg, \rightarrow\}$, the rest of the logical connectives can be constructed similarly to **1.1** and **1.2** .

1.4 Is $\{\wedge, \vee\}$ a complete set of connectives?

This is not a complete set due to the inability to construct NOT ($\neg$).

2 Conversions of (((p $\rightarrow$ q) $\rightarrow$ r) $\rightarrow$ (s $\rightarrow$ t))

2.1 Conjunctive normal form (CNF)

Remove implications except one:

$$(((\text{p} \rightarrow \text{q}) \rightarrow \text{r}) \rightarrow (\text{s} \rightarrow \text{t}))$$

$$((\neg(\neg \text{p} \vee \text{q}) \vee \text{r}) \rightarrow (\neg \text{s} \vee \text{t}))$$

Removing one Internal NOT:

$$((\text{p} \wedge \neg \text{q}) \vee \text{r}) \rightarrow (\neg \text{s} \vee \text{t})$$

Remove last implication and NOT:

$$\neg((\text{p} \wedge \neg \text{q}) \vee \text{r}) \vee (\neg \text{s} \neg \text{t})$$
$$(\neg(\text{p} \wedge \neg \text{q}) \wedge \neg \text{r}) \vee (\neg \text{s} \vee \text{t})$$

Distribute:

$$((\neg \text{p} \vee \text{q}) \wedge \neg \text{r}) \vee (\neg \text{s} \vee \text{t})$$
$$((\neg \text{p} \vee \text{q}) \vee (\neg \text{s} \vee \text{t})) \wedge (\neg \text{r} \vee (\neg \text{s} \vee \text{t}))$$
$$(\neg \text{p} \vee \text{q} \vee \neg \text{s} \vee \text{t}) \wedge (\neg r \vee \neg \text{s} \vee \text{t})$$

2.2 Disjunctive normal form (DNF)

Remove implications except one:

$$(((\mathrm{p} \rightarrow \mathrm{q}) \rightarrow \mathrm{r}) \rightarrow (\mathrm{s} \rightarrow \mathrm{t}))$$

$$((\neg(\neg \mathrm{p} \vee \mathrm{q}) \vee \mathrm{r}) \rightarrow (\neg \mathrm{s} \vee \mathrm{t}))$$

Removing one Internal NOT:

$$((\mathrm{p} \wedge \neg \mathrm{q}) \vee \mathrm{r}) \rightarrow (\neg \mathrm{s} \vee \mathrm{t})$$

Remove last implication and NOT:

$$\neg((\mathrm{p} \wedge \neg \mathrm{q}) \vee \mathrm{r}) \vee (\neg \mathrm{s} \neg \mathrm{t})$$
$$(\neg(\mathrm{p} \wedge \neg \mathrm{q}) \wedge \neg \mathrm{r}) \vee (\neg \mathrm{s} \vee \mathrm{t})$$

Distribute:

$$((\neg \mathrm{p} \vee \mathrm{q}) \wedge \neg \mathrm{r}) \vee (\neg \mathrm{s} \vee \mathrm{t})$$
$$((\neg \mathrm{p} \vee \mathrm{q}) \wedge (\neg \mathrm{p} \vee \neg \mathrm{r})) \vee (\neg \mathrm{s} \vee \mathrm{t})$$
$$(\neg \mathrm{p} \vee \mathrm{q} \vee \neg s \vee \mathrm{t}) \wedge (\neg \mathrm{p} \vee \neg \mathrm{r} \vee \neg \mathrm{s} \vee \mathrm{t})$$

3 Tseitin's Algorithm

Tseitin's Algorithm takes in a formulation of logical connectives and converts it to CNF. This is helpful as this form is used for applications such as SAT solvers.

3.1 $(((x_1 \wedge x_2 \wedge x_3) \rightarrow (y_1 \wedge y_2 \wedge y_3)) \vee \mathbf{z})$ to CNF

Let v_1 represent $(x_1 \wedge x_2 \wedge x_3)$ and let v_2 represent $(y_1 \wedge y_2 \wedge y_3)$:

$$(v_1 \rightarrow v_2) \vee z$$
$$(\neg v_1 \vee v_2) \vee z \text{ (Clause 1 labelled } c_1\text{)}$$

Make clause from v_1:

$$v_1 \leftrightarrow (x_1 \wedge x_2 \wedge x_3)$$
$$(\neg v_1 \vee x_1),\ (\neg v_1 \vee x_2),\ (\neg v_1 \vee x_3),\ (v_1 \vee x_1 \vee x_2 \vee x_3) \text{ (Clause 2 labelled } c_2\text{)}$$

Make clause from v_2:

$$v_2 \leftrightarrow (y_1 \wedge y_2 \wedge y_3)$$
$$(\neg v_2 \vee y_1),\ (\neg v_2 \vee y_2),\ (\neg v_2 \vee y_3),\ (v_2 \vee y_1 \vee y_2 \vee y_3) \text{ (Clause 3 labelled } c_3\text{)}$$

Create conjunctions between all the clauses:

$$c_1 \wedge c_2 \wedge c_3$$
$$\equiv ((\neg v_1 \vee v_2) \vee z) \wedge((\neg v_1 \vee x_1),\ (\neg v_1 \vee x_2),\ (\neg v_1 \vee x_3),\ (v_1 \vee x_1 \vee x_2 \vee x_3))$$
$$\wedge((\neg v_2 \vee y_1),\ (\neg v_2 \vee y_2),\ (\neg v_2 \vee y_3),\ (v_2 \vee y_1 \vee y_2 \vee y_3))$$

4 Logically valid predicate logic

4.1 $(\forall x \exists y \forall z \ E(x,y) \wedge E(y,z)) \rightarrow (\forall x \forall z \exists y \ E(x,y) \wedge E(y,z))$

The left hand side means 'For every x, there exists a y such that for all z, $E(x,y)$ and $E(y,z)$ holds.' Comparatively, the right hand side means 'For every x and for every z, there exists a y such that $E(x,y)$ and $E(y,z)$ holds.'
This is logically valid as the left hand side requires a single y value for each x while saying that it works for all z values while the right hand side says that for each pair (x,y) there is a y value that works. This makes the right hand side have more possible accepted values while having all values on the left hand side still be valid.
To put this into a real world example, imagine there are students (x) that attend lectures (z) in classrooms (y). The left hand side means that for each student there is a classroom where they can attend regardless of the specific lecture. Meanwhile, the right hand side means that each student has a specific classroom and lecture (pairing) that they can attend.

4.2 $(\forall x \, \exists y \, \exists u \, \forall v \ \ E(x,y) \wedge E(u,v)) \rightarrow (\exists u \, \forall v \, \forall x \, \exists y \ \ E(x,y) \wedge E(u,v))$

The left hand side means 'For every x, there exists a y and u such that for all v, $E(x,y)$ and $E(u,v)$ holds.' Comparatively, the right hand side means 'There exists a u such that for all v and for all x, there exists a y such that $E(x,y)$ and $E(u,v)$ holds.'
This is not logically valid as the left hand side says that for every x there are specific y and u. However, the right hand side says that u is the same for every value x.
To put this into a real world example, imagine there are students (x) working together (u) on a task (v) for a overarching project (y). The left hand side says that for all students there is a corresponding project and department that is working on a task. Meanwhile, the right hand side says that there is a single department that where all students are working on a project.

4.3 $((\forall x \, \exists y \, \forall z \, R(x,y,z)) \rightarrow (\exists x \, \forall y \, \exists z \, R(x,y,z))$

The left hand side means 'For every x, there exists a y such that for all z, $R(x,y,z)$ holds.' Comparatively, the right hand side means 'There exists an x such that for all y, there exists a z such that $R(x,y,z)$ holds.'
This is not logically valid and can be proven with a real world example.
Imagine there are students (x) that study using a study method (y) to pass an exam (z). The left hand side says that each student has a study method that guarantees them to pass all the exams. This does not sound logical as there exist cases such as a student (x_1) passing all exams while using a specific method (y_1) but failing when they use another (y_2) while another student (x_2) can use y_2 but can't use y_1 effectively (this would pass under the right hand side).

4.4 $((\forall x\, \forall y\, \exists z\ (E(x,y) \wedge E(y,z))) \rightarrow (\forall x\, \forall y\, \forall z\ (E(x,y) \vee E(y,z))))$

The left hand side means 'For every x and y there exists a z such that $E(x,y)$ and $E(y,z)$ holds.' Comparatively, the right hand side means that 'For every x, y, and z, one of or both of $E(x,y)$, $E(y,z)$ holds.'
This is not logically valid as the left hand side says that for each pairing of x and y there is a z that satisfies. However, the right hand side can be satisfied by any possible triplet of x, y, and z. This contains cases that may not appear in the left hand side as it has less possible values.

5 Evaluating given sentences on the relation E over the domain $\{0, 1, 2\}$

$E := \{(0,1),(1,0),(1,2),(2,1),(2,0),(0,2)\}$

5.1 $\forall x\, \forall y\, \forall z\, \exists w\ (E(x,w) \wedge E(y,w) \wedge E(z,w))$

For every x, y, and z, there exists at least one w that satisfies E when paired with x, y, or z.
For this to be true, for some value of $w, w \in \{0,1,2\}$, all of the following must be true:

$$\begin{gathered}(0,w) \in E \\ (1,w) \in E \\ (2,w) \in E\end{gathered}$$

If $w = 0$:
  $(1,0)$ and $(2,0)$ are in E but not $(0,0)$
If $w = 1$:
  $(0,1)$ and $(2,1)$ are in E but not $(1,1)$
If $w = 2$:
  $(0,2)$ and $(1,2)$ are in E but not $(2,2)$

Thus, the sentence is false.

5.2 $\exists x\, \forall y\, \forall z\, \exists w\ (E(x,w) \wedge E(y,w) \wedge E(z,w))$

There exists an x such that for every y and z there is a w that satisfies E when paired with x, y, and z.
If $x = 0$:
  When $y = 1$, $E(1,1)$ does not hold.
If $x = 1$:
  All are satisfied.

Thus, the sentence is true.

5.3 $\forall y \, \exists x \, \forall z \, \exists w \ (E(x,w) \wedge E(y,w) \wedge E(z,w))$

For all y, there exists an x such that for all z there is a w that satisfies E when paired with x, y, and z.
If $y = 0$:
  When $w = 0$ and $z = 0$, the case does not hold.
If $y = 1$:
  When $z = 0$, the case does not hold.

Thus, the sentence is true.

5.4 $\exists x \, \exists y \, \exists z \, \forall w \ (E(x,w) \wedge E(y,w) \wedge E(z,w))$

There exists an x, y, and z such that for all w there is a w that satisfies E when paired with x, y, and z.
If $x = 1$, $y = 1$, and $z = 2$:
  When $w = 0$, the case does not hold.
If $x = 0$, $y = 1$, and $z = 0$:
  When $w = 0$, the case does not hold.
If $x = 1$, $y = 0$, and $z = 2$:
  When $w = 0$, the case does not hold.
If $x = 1$, $y = 1$, and $z = 2$:
  When $w = 1$, the case does not hold.

There is no combination of the triplets where w holds, thus the sentence is false.

5.5 $\forall x_1 \, \exists x_2 \, \forall y_1 \, \exists y_2 \, \forall z_1 \, \exists z_2 \forall z \, \exists w$

$E(x_1, x_2) \wedge E(x_2, w) \wedge E(y_1, y_2) \wedge E(y_2, w) \wedge E(z_1, z_2) \wedge E(z_2, w) \wedge E(z, w)$
For every x_1, there exists an x_2 such that for all y_1 there is a y_2 such that for all z_1 there is a z_2 for all z there exists a w that satisfies all of the following:

$E(x_1, x_2)$
$E(x_2, w)$
$E(y_1, y_2)$
$E(y_2, w)$
$E(z_1, z_2)$
$E(z_2, w)$
$E(z, w)$

If $x_1 = 0$:
The $E(0, 0)$ case does not hold.
If $x_1 = 1$:
The $E(1, 1)$ case does not hold.
If $x_1 = 2$:
The $E(1, 1)$ case does not hold.

Thus, the sentence is false.

5.6 $\forall x_1 \, \exists x_2 \, \forall y_1 \, \exists y_2 \, \forall z_1 \, \forall z \, \exists z_2 \, \exists w$

E($\mathrm{x}_1, x_2) \wedge E(x_2, w) \wedge E(y_1, y_2) \wedge E(y_2, w) \wedge E(z_1, z_2) \wedge E(z_2, w) \wedge E(z, w)$
For every x_1, there exists an x_2 such that for all y_1 there is a y_2 such that for all z and z_1 there is a z_2 and a w that satisfies all of the following:

$E(x_1, x_2)$
$E(x_2, w)$
$E(y_1, y_2)$
$E(y_2, w)$
$E(z_1, z_2)$
$E(z_2, w)$
$E(z, w)$

If $x_1 = 0$:
All cases hold.

Thus, the sentence is true.