An Introduction to Higher Global Symmetries

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Abstract

Much in the structure of modern physical theories relies on the concept of symmetry. Ordinary symmetries act on local operators. Extending the domain to non-local objects leads to the formulation of generalised global symmetries. The dissertation will give an introduction to higher-form symmetries and higher-group symmetries. The physical, topological origin and construction will be discussed. In the presentation, applications of higher-form symmetries to pure Maxwell theory and BF theory will be analysed along with their behaviour under spontaneous symmetry breaking. A separate discussion about the role of 1-form symmetries in distinguishing between $SU(N)$ and $PSU(N)$ gauge theories is considered. 2-group symmetries are introduced using the formalism of topological networks of defects. The emergence of $U(1)^{(0)} \times U(1)^{(1)}$ 2-group symmetry will be justified to come as a requirement of 't Hooft anomaly cancellation when gauging one of the components of a $U(1) \times U(1)$ global symmetry.

Contents

1. Introduction

Symmetries lie at the core of modern theoretical developments in physics. While standard field theoretical approaches consider symmetries acting on local fields, the emergence of extended objects suggests that a complete treatment should include non-local objects in the spectrum of operators. Historically, charged, string-like objects have been extensively studied, leading to formulations of higher gauge theories [1]. A classic example is the Kalb-Ramond $U(1)$ field [2]. However, gauge invariance is merely an unphysical redundancy and has no direct observable consequences.

Instead of focusing on higher gauge theories, one can proceed by describing higher global symmetries, subsequently studying the possibility of gauging. In contrast to gauge symmetries, global symmetries are relevant because they imply relations between physical quantities. They can lead to degeneracies in the states and can be used to classify the spectrum of operators. Having control over higher global symmetries can lead to a study of the dynamics of operators even in the absence of an action description.

The current literature lacks introductory reviews of higher global symmetries. As different approaches to the field can be adopted, this dissertation will offer a pedagogical introduction to two types of higher symmetries, namely higher-form and higher-group symmetry, with the focus of the latter on 2-group symmetry. The examples discussed have been present in standard QFTs all along. The topics have not been chosen for their complexity, but to illustrate the new framework, while also making it easier to address questions about the implications of the novel symmetries. The structure of the thesis will be as follows. In Chapter 2.1 ordinary symmetries are reviewed using the language of topological operators which makes the transition to higher-form symmetries manifest. Chapter 2.2 explicitly lays down the building blocks of higher-form symmetries, their objects and constraints. Chapter 2.3 complements the general exposition by a discussion of the two 1-form symmetries present in a four dimensional $U(1)$ gauge theory. Ordinary global symmetries can be either explicitly or spontaneously broken, the latter leading to the production of Goldstone scalar modes. Higher-form symmetries undergo the same processes, the result being the emergence of higher-spin Goldstone modes. The mechanism is discussed in Chapter 2.5.

If the higher symmetry is broken to a discrete subgroup it can give rise to a gauge theory described by a discrete symmetry group. An explicit example is discussed in Chapter 3.1. Global differences between gauge groups cannot be distinguished by local operators. However, they can appear as discrepancies in the number of electric and magnetic extended operators as considered in Chapter 3.2 for the gauge groups $SU(N)$ and $PSU(N)$.

In the later part of the thesis, the emphasis is shifted to an alternative description of topological operators using a discrete network of defects. The construction provides a framework for studying the mixing of a 0-form and 1-form symmetry. More concretely, modifying the associativity of the 0-form symmetry multiplication by an element of $H^3(BG, \mathcal{A})$ leads to the emergence of 1-form symmetry defects. The novel relation is best described by a 2-group, a categorical structure. However, a formal description of the embedding lies beyond the scope of the dissertation. In Chapters 4.1 and 4.2 the discussion provides the essential elements that will allow an understanding of the mixed symmetry transformation, with further references where necessary. Finally, an application of 2-group symmetry to 't Hooft anomaly cancellation for a $U(1) \times U(1)$ global symmetry and the obstructions encountered when gauging the 2-group symmetry are offered in Chapters 4.3 and 4.4.

2. Higher-Form Symmetries

2.1 Topological Approach to Ordinary Symmetries

In this section we briefly review core aspects of ordinary symmetries in quantum field theories that will become useful in transitioning to higher symmetries. The discussion in the next two sections follows [3]. The setting will be a theory with a generic field Φ, charged under some global symmetry, characterized by the group G. We will assume that the theory is specified by an action S and the manifolds under consideration have Euclidean signature. The discussion can be generalized straightforwardly to more fields. Under the transformation:

$$\Phi \to \Phi + \varepsilon_a \delta_a \Phi, \tag{2.1}$$

with ε_a a constant global parameter, labelled by an internal or space-time index a, the action is invariant. From Noether's theorem, to this symmetry there is an associated conserved 1-form current that can be identified by promoting the parameter $\varepsilon_a \to \varepsilon_a(x)$ to a position dependent function and varying the action:

$$\delta S = \int d^d x j_a^\mu \partial_\mu \varepsilon_a(x). \tag{2.2}$$

The variation vanishes for a symmetry and integrating by parts leads to the conservation equation $\partial_\mu j_a^\mu = 0$. We can define the charge associated to the symmetry as:

$$Q_a = \int d^{d-1} x j_a^0 \tag{2.3}$$

and the above conservation of j translates into $\frac{dQ_a}{dt} = 0$. In the canonical formalism, the Noether charges (2.3) generate the symmetry transformations (2.1) through the Poisson bracket:

$$\delta_a \Phi = \{\Phi, Q_a\}. \tag{2.4}$$

The discussion so far has been classical. Moving to the quantum realm, the expression of interest is (2.4). It becomes the well-known Ward identity for x and y, different points in space:

$$i\langle \partial_\mu j_a^\mu(x)\Phi(y)\rangle = \delta^d(x-y)\langle \delta_a \Phi(y)\rangle. \tag{2.5}$$

The usual QFT approach is to integrate both sides over the volume $\mathcal{V} = [y^0 - \epsilon, y^0 + \epsilon] \times \mathbb{R}^{d-1}$ as in Fig. 2.1a. Because the expectation value in Eq. (2.5) is time ordered, we get, in the limit $\epsilon \to 0$ the commutator between the charge $Q_a(y^0)$ and the field Φ:

$$i\langle [Q_a(y^0), \Phi(y)]\rangle = \langle \delta_a \Phi(y)\rangle. \tag{2.6}$$

The objects of a higher symmetry, however, are not local operators, but have support on higher-dimensional manifolds. It is not then obvious how to use (2.6) and define time ordering for objects extended on a timelike direction. We aim instead at a more general relation that captures the configurations that allow an extended object to be transformed. From now on, it will prove to be more transparent to work with explicit differential forms, where the conserved current becomes: $j_a = j_{\mu,a} dx^\mu$. The current conservation law becomes $d * j_a = \partial_\mu j_a^\mu dx^0 \wedge ... \wedge dx^{d-1} = 0$. The Noether charge (2.3) can be defined as the integral of the Hodge

(a) Volume of integration $\mathcal{V}$ for charge in (2.6) is the whole space between times $y^0+\varepsilon$ and $y^0-\varepsilon$

(b) Closed volume of integration $\mathcal{V}$ for charge in (2.7)

Figure 2.1. Volumes of integration for charges inside expectation values

dual to the current over a closed $(d-1)$-dimensional volume $\mathcal{V}$:

$$Q_a(\mathcal{V}) = \int_{\mathcal{V}} \star j_a = \int_{\mathcal{V}} \frac{1}{(d-1)!} j_{\mu,a} \epsilon^{\mu}_{\mu_1 \dots \mu_{d-1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{d-1}}. \tag{2.7}$$

Then we can integrate the Ward identity (2.5) over a d-dimensional volume $\Omega_{\mathcal{V}}$ such that its boundary is the $(d-1)$-dimensional submanifold $\mathcal{V}$ as in Fig. 2.1b. The left hand side of (2.5) becomes $i\int_{\Omega_{\mathcal{V}}} \langle d\star j_a \Phi(y)\rangle = i\int_{\mathcal{V}} \langle \star j_a \Phi(y)\rangle = i\langle Q_a(\mathcal{V})\Phi(y)\rangle$. On the right we have $\int_{\Omega_{\mathcal{V}}} d^d x \delta^{(d)}(x-y)\langle \delta_a \Phi(y)\rangle$. The delta function integral is 0 or 1, depending on whether the field Φ is inside the volume $\mathcal{V}$. It is the integral representation of the topological *linking number* between the two manifolds $\mathcal{V}$ and y:

$$\mathrm{Link}(\mathcal{V}, y) = \int_{\Omega_{\mathcal{V}}} d^d x \delta^{(d)}(x-y). \tag{2.8}$$

Finally the Ward identity is written as:

$$i\langle Q_a(\mathcal{V})\Phi(y)\rangle = \mathrm{Link}(\mathcal{V}, y)\langle \delta_a \Phi(y)\rangle. \tag{2.9}$$

What makes this relationship powerful is its topological nature. The right hand side is invariant under deformations of $\mathcal{V}$ as long as they do not cross y. We expect the same from the charge $Q_a(\mathcal{V})$ appearing on the right. This can be seen by considering an extension of $\Omega_{\mathcal{V}}$ as $\Omega'_{\mathcal{V}} = \Omega_{\mathcal{V}} \cup \Omega_0$, where $y \notin \Omega_0$:

$$Q_a(\mathcal{V}+\partial\Omega_0) = \int_{\Omega_{\mathcal{V}}\cup\Omega_0} \langle d\star j_a \Phi(y)\rangle = \int_{\Omega_{\mathcal{V}}} \langle d\star j_a \Phi(y)\rangle + \int_{\Omega_0} \langle d\star j_a \Phi(y)\rangle = \int_{\Omega_{\mathcal{V}}} \langle d\star j_a \Phi(y)\rangle = Q_a(\mathcal{V}).$$

The extra term vanishes because we can set $d\star j_a = 0$ inside the correlator as $y \notin \Omega_0$. The origin of the invariance in the case of a continuous symmetry is, thus, the conservation of j_a.

For continuous symmetries, the unitary operator generating the symmetry is given by $U = e^{i\varepsilon_a Q_a}$. We can exponentiate (2.9) to obtain the finite transformation of a charged field:

$$\langle U_g(\mathcal{V})\Phi(y)\rangle = R(g)\langle\Phi(y)\rangle, \tag{2.10}$$

where $R(g) = \exp(i\varepsilon_a t_a)$ is the representation in which Φ transforms, with t_a the group generators in the same representation.

The particular case of a charged scalar (or other local field) is referred to as 0-form symmetry because the support of the charged field is a 0-dimensional manifold, a point. As it is the case with global symmetries, the symmetry parameter is a constant ε_a, which can be interpreted as a closed 0-form.

2.2 Introducing Higher-Form Symmetries

From the above discussion we see that an ordinary global symmetry can be characterized by the support of the charged field, here a 0-dimensional manifold, a topological operator $U(\mathcal{V})$ defined on a $(d-1)$-dimensional manifold and a closed 0-form, the parameter of the symmetry. We can generalize the transformation laws and topological operators directly to symmetries acting on extended objects by changing the dimensions of operators above in a consistent manner. It is instructive to start from the symmetry parameter and argue that the charged objects are higher dimensional. For simplicity of notation, we will drop the index a in what follows.

As such, consider a closed q-form ε with $d\varepsilon = 0$. Under a symmetry transformation, by promoting ε to a local operator as before, the change in action (2.2) becomes:

$$\delta S = \int_{\mathcal{M}^{(d)}} \star j \wedge d\varepsilon \tag{2.11}$$

where j is now a $(q+1)$-form current. We can further define the charge as:

$$Q(\mathcal{V}_{d-q-1}) = \int_{\mathcal{V}_{d-q-1}} \star j. \tag{2.12}$$

So far we have not explicitly mentioned the extended charged objects. To argue for their dimensionality, we return to the case of a 0-form symmetry and remark that from the perspective of the Hilbert space, there is an equivalence between the action of the charge (2.7), defined on a $(d-1)$-dimensional, closed manifold $\mathcal{V}$, and the operator transformation (2.1). The Poincaré duality can be used to associate to $\mathcal{V}$ a form. Generally, to a p-dimensional manifold $\mathcal{V}^{(p)}$ we can associate a $(d-p-1)$-form using the Thom class [4]. In components, the dual form reads:

$$\varepsilon_{i_{p+2}\ldots i_d} = \frac{1}{p!}\int_{\mathcal{V}^{(p)}} \epsilon_{i_1\ldots i_{d-1}}\delta^{(d-1)}(x-y)dy^{i_1}\wedge\ldots\wedge dy^{i_p}. \tag{2.13}$$

For an ordinary symmetry, $p=d-1$ and the dual is a constant scalar:

$$\varepsilon = \frac{1}{(d-1)!}\int_{\mathcal{V}^{(d-1)}} \epsilon_{i_1\ldots i_{d-1}}\delta^{(d-1)}(x-y)dy^{i_1}\wedge\ldots\wedge dy^{i_{d-1}} = 1.$$

Hence the transformation parameter is a closed 0-dimensional form, supported on 0-dimensional regions of the base manifold. They can be associated with the transformation of similar objects defined on 0-dimensional spaces. These are the local operators.

For higher symmetries, the charge (2.12) has support on a $(d-q-1)$-region, implying that the symmetry factor ε is a q-form. By integrating it over a q-dimensional surface $\Gamma^{(q)}$ we obtain the parametrization of the transformation. The space over which we integrate ε has to be characteristic to the extended operator, and hence the charged objects under a q-form symmetry are supported on q-dimensional spaces. Consider such an object $W(\Gamma^{(q)})$. Its transformation under a global q-form symmetry is:

$$W(\Gamma^{(q)}) \to W(\Gamma^{(q)}) + \int_{\Gamma^{(q)}} \varepsilon(\mathcal{V}^{(d-q-1)})\delta W(\Gamma^{(q)}). \tag{2.14}$$

It is important to remark that if $\Gamma^{(q)}$ is a closed manifold, this does not imply that $\int_{\Gamma^{(q)}}\varepsilon = \int_{\Omega^{(q+1)}} d\varepsilon = 0$ with $\partial\Omega^{(q+1)} = \Gamma^{(q)}$. We cannot simply apply Stokes' theorem because $\Gamma^{(q)}$ and $\mathcal{V}^{(d-q-1)}$ may intersect.

The Ward identity can be derived by considering the variation of the partition function under the symmetry:

$$\int D[\Phi]e^{iS[\Phi]}W = \int D[\Phi]e^{iS[\Phi]}\Big(1 + i\delta S W + \int_{\Gamma^{(q)}} \varepsilon\delta W\Big).$$

Using (2.11) and introducing a delta function as $1 = \int_{\mathcal{M}^{(d)}}\delta^{(d)}$, where $\mathcal{M}^{(d)}$ is the whole d-dimensional space, we obtain:

$$i\langle d{\star}j\ W\rangle = \int_{\Gamma^{(q)}} \delta^{(d)}\langle\delta W\rangle. \tag{2.15}$$

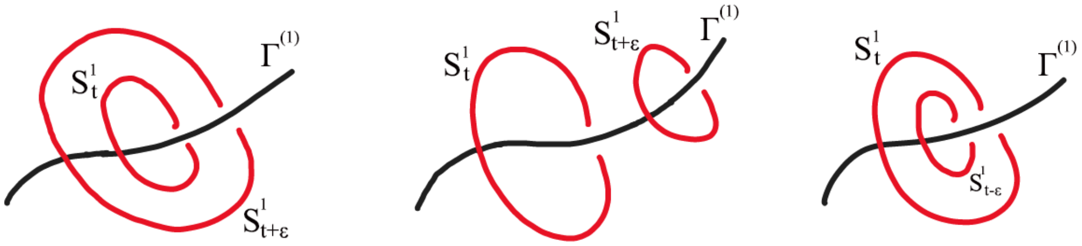

Figure 2.2. An extended operator with support on a one dimensional manifold $\Gamma^{(1)}$ in 3 dimensions. The figures depict a series of deformations applied to two closed manifolds S^1, surrounding $\Gamma^{(1)}$. They are defined at different times, as described at the end of Section 2.2. The circle $S^1_{t+\epsilon}$ (left figure) can be slid along $\Gamma^{(1)}$ (middle figure), contracted and brought to $S^1_{t-\epsilon}$ (right figure) without intersecting S^1_t.

Integrating the above over a closed surface $\mathcal{V}^{(d-q)}$ gives the charge on the left hand side:

$$i\langle Q(\mathcal{V}_{d-q-1})\ W\rangle = \mathrm{Link}(\Gamma,\mathcal{V})\langle \delta W\rangle, \quad \mathrm{Link}(\Gamma,\mathcal{V}) = \int_{\Gamma^{(q)}}\int_{\mathcal{V}^{(d-q)}} \delta^{(d)} \tag{2.16}$$

The exponential of (2.16) is recast in the form of:

$$U_g(\mathcal{V}^{(d-q-1)})W(\Gamma^{(q)}) = R(g)^{\mathrm{Link}(\Gamma,\mathcal{V})}W(\Gamma^{(q)}) \tag{2.17}$$

where U_g is the topological operator associated to the group element g. There is a caveat to the manifold $\mathcal{V}^{(d-q-1)}$ over which U_g is defined, as an exact treatment uses the small sphere linking $\Gamma^{(q)}$. We recommend [4, 5] for a detailed discussion.

One aspect that has been ignored so far is related to the nature of the group G describing the global symmetries. We want our topological operators (2.17), as representations, to respect the multiplication law of the group:

$$U_g(\mathcal{V}^{(d-q-1)})U_{g'}(\mathcal{V}^{(d-q-1)}) = U_{gg'}(\mathcal{V}^{(d-q-1)}). \tag{2.18}$$

The law does not bring constraints on G for the case of ordinary symmetries, i.e $q = 0$, but when $q > 0$, the topological nature of U_g comes into play. In order to see this, we can interpret (2.18) as the action of two operators at times t and $t+\epsilon$ along $\mathcal{V}^{(d-q-1)}$. For $q > 0$, because of the higher codimensionality, $\mathcal{V}^{(d-q-1)}$ at $t+\epsilon$ can be deformed to $\mathcal{V}^{(d-q-1)}$ at $t-\epsilon$ as in Fig. 2.2. Hence there is no definite order in which the transformations may act and the topological operators are commutative. Because this property has to be reflected at the group level, we are forced to use only Abelian groups for higher-form symmetries.

2.3 Simple Example: Pure Maxwell Theory

We can study a simple, prototypical, example of a theory which displays global higher-form symmetries. This is pure Maxwell theory in $\mathbb{R}^4$. The well known action $S[A] = -\frac{1}{2e^2}\int F \wedge \star F$ is written in terms of the gauge field A from which we build the field strength $F = dA$. The equations of motion and Bianchi identity are:

$$d\star F = 0 \quad \text{and} \quad dF = 0. \tag{2.19}$$

Each of these can be associated to a conserved current as follows. We can first construct $j^E = \frac{1}{e^2}\star F$ with the symmetry operator:

$$U^E_\alpha(\mathcal{V}^{(2)}) = e^{\frac{i\alpha}{e^2}\int_{\mathcal{V}^{(2)}}\star F}. \tag{2.20}$$

The integral of $\star F$ over a closed 2-dimensional surface is the electric flux, which is quantized. The charged objects are Wilson lines $W(\Gamma^{(1)}) = e^{i\int_{\Gamma^{(1)}} A}$, where we choose unit charge for the probe particle integrated along $\Gamma^{(1)}$, but this can be lifted to any integer charge. To see the explicit action of the symmetry on W we insert the operator (2.20) into the path integral:

$$\langle U^E_\alpha(\mathcal{V}^{(2)}) W(\Gamma^{(1)})\rangle = \int \mathcal{D}[A] e^{i\frac{\alpha}{e^2}\int_{\mathcal{V}^{(2)}}\star F + iS + i\int_{\Gamma^{(1)}} A}. \tag{2.21}$$

We want to perform a change of variables such that the extra exponential factor of the symmetry operator is absorbed in the action. The first step is to rewrite the charge of the symmetry as an integral over $\Omega_\mathcal{V}$, a 3-dimensional surface with $\partial\Omega_\mathcal{V} = \mathcal{V}^{(2)}$:

$$\int_{\mathcal{V}^{(2)}} \star F = \int_{\Omega_\mathcal{V}} d\star F. \tag{2.22}$$

We can further massage the above by inserting a delta function $1 = \int_{\mathbb{R}^4}\delta^4$ and swapping the order of integrals such that (2.22) becomes $\int_{\mathbb{R}^4} d\star F \wedge J$, with $J = \int_{\Omega_\mathcal{V}}\delta^4$. This equation can be inserted back in (2.21) and by redefining $A \to A - \alpha J$, the change in the action can absorb the symmetry charge such that we end up with:

$$\langle U^E_\alpha(\mathcal{V}^{(2)}) W(\Gamma^{(1)})\rangle = e^{i\alpha\int_{\Gamma^{(1)}}\int_{\Omega_\mathcal{V}}\delta^4}\langle W(\Gamma^{(1)})\rangle.$$

This is exactly what we would expect from a 1-form symmetry with the factor multiplying the phase α being the Linking number.

So far only one of the conserved current has been discussed, but from the Bianchi identity (2.19), $j^M = \frac{1}{2\pi}F$ is also a closed current. As before, the symmetry operator is:

$$U_\alpha^M(\mathcal{V}^{(2)}) = e^{i\frac{\alpha}{2\pi}\int_{\mathcal{V}^{(2)}} F}. \tag{2.23}$$

Because the surface integral gives the magnetic flux, this operator corresponds to magnetically charged line objects. However, in contrast with Wilson lines, we cannot construct a magnetically charged object by integrating a magnetic charge along a worldline. We can instead impose the constraint that if there is a magnetic monopole of charge 1 then for any 2-surface surrounding its worldline, the magnetic flux is given by:

$$\frac{1}{2\pi}\int_{\mathcal{V}^{(2)}} F = 1. \tag{2.24}$$

This can be achieved by adding $dF = 0$ as a Lagrange multiplier in the action and then dualizing the theory. This procedure leads to the following dual action:

$$S = -\frac{e^2}{16\pi^2}\int G \wedge \star G, \tag{2.25}$$

with $G = d\tilde{A}$, $\tilde{A}$ the dual gauge field. We are able to construct a line operator now which is referred to as *'t Hooft line*: $T(\Gamma^{(1)}) = e^{i\int_{\Gamma^{(1)}}\tilde{A}}$. From the dualization procedure, $F = \frac{e^2}{4\pi}\star G$, and the symmetry operator (2.23) can be rewritten in terms of $\star G$ as for the electric symmetry.

2.4 Adding charged matter

For a pure $U(1)$ gauge theory the topological nature of $U_\alpha^E(\mathcal{V}^{(2)})$ is inherently linked to the conservation of the current j^E, which can be translated into the invariance of the charge $Q(\mathcal{V}^{(2)})$ under deformations of the volume $\mathcal{V}^{(2)}$. If the charge changes under such deformations, then such an operator cannot be defined and the $U(1)$ 1-form symmetry is broken. This type of breaking can be accessed explicitly by adding electrically charged matter to the pure Maxwell theory. It amounts to adding a scalar field action S_ϕ and a coupling between the electromagnetic potential and scalar current J_ϕ:

$$S = S_\phi + S_{EM} + \int A \wedge \star J_\phi. \tag{2.26}$$

The 2-form electric current $\star F$ is not conserved anymore, but:

$$d\star F = \frac{e^2}{2}\star J_\phi. \tag{2.27}$$

The integral of $\star F$ on closed 2-surfaces is $\frac{e^2}{2}\mathbb{Z}$. Then $\int J_\phi$ is an integer. If we consider a charge Q defined on a two dimensional surface $\mathcal{V}^{(2)}$ and deform this surface to $\mathcal{V}'^{(2)}$, with the difference between the two being $\partial\Omega_\mathcal{V}$, the charge transforms as:

$$Q(\mathcal{V}^{(2)}) \to Q(\mathcal{V}^{(2)}) + \int_{\Omega_\mathcal{V}} dj^E = Q(\mathcal{V}^{(2)}) + \mathbb{Z}. \tag{2.28}$$

The operator U_α transforms as: $U_\alpha(\mathcal{V}^{(2)}) \to U_\alpha(\mathcal{V}^{(2)})e^{i2\pi\alpha\mathbb{Z}}$. The extra phase has to be equal to unity for it to be topological. This forces alpha to 0 or 1, breaking the $U(1)$ electric symmetry completely. In contrast, the magnetic 1-form symmetry remains intact by adding electric charges. It could be broken as well if magnetic charges were present.

Breaking $U(1)$ completely as above is not the only possibility as the conservation of j^E can be modified in such a way that a subgroup still acts as a global symmetry. One such example can be obtained by adding electrically charged $n \in \mathbb{Z}$ matter. The action is modified to:

$$S = S_\phi + S_{EM} + \int nA \wedge \star J_\phi \tag{2.29}$$

and the previous equation of motion (2.27) to: $d\star F = \frac{ne^2}{2}\star J_\phi$. This time the charge picked up by deforming the support surface is not any integer, but $n\mathbb{Z}$. The change in the U_α operator becomes accordingly: $U_\alpha(\mathcal{V}^{(2)}) \to U_\alpha(\mathcal{V}^{(2)})e^{i2n\pi\alpha\mathbb{Z}}$. If invariance of U_α is desired, then α has to be a ratio $\frac{l}{n}$, with $l = 0, 1, ..., n-1$. The initial $U(1)$ symmetry is thus broken to $\mathbb{Z}_n$ by adding n charged matter. An explicit example of breaking to $\mathbb{Z}_n$ will be studied in a further section along with a discussion of charged operators.

2.5 Spontaneous Symmetry Breaking

Besides explicit breaking, an ordinary global symmetry can be spontaneously broken if the charged scalar field acquires a non-zero vev at infinity. Similarly, a higher-form global symmetry is spontaneously broken if the vev of the charged extended operator does not vanish for support manifolds extended to infinity. This leads to the appearance of higher-spin Goldstone modes that will be discussed below. In this section we will prove the generalization of Goldstone's theorem to q-form symmetries, following [6].

Let us write the charge on a closed manifold in yet a different way:

$$Q(\mathcal{V}^{(d-q-1)}) = \int_{\mathcal{V}^{(d-q-1)}} \star j = \int_{M^{(d-1)}} \star j \wedge \tilde{\mathcal{V}}, \tag{2.30}$$

where $M^{(d-1)}$ is a $(d-1)$-dimensional manifold, which can be interpreted as a constant time slice, and $\tilde{\mathcal{V}}$ is the Poincaré dual of $\mathcal{V}^{(d-q-1)}$ with respect to $M^{(d-1)}$. For an operator W, charged under the symmetry, we can define $\mathcal{C} = \langle 0|[Q(\mathcal{V}^{(d-q-1)}), V(\Gamma^{(q)})]|0\rangle$, where $|0\rangle$ is a reference symmetry breaking ground state. In this case, $\langle 0|W(\Gamma^{(q)})|0\rangle \neq 0$. Inserting a complete set of states inside $\mathcal{C}$ we obtain:

$$\mathcal{C} = \sum_n \int_{M^{(d-1)}} (\langle 0|\star j \wedge \tilde{\mathcal{V}}|n\rangle\langle n|V(\Gamma^{(q)})|0\rangle - \langle 0|V(\Gamma^{(q)})|n\rangle\langle n|\star j \wedge \tilde{\mathcal{V}}|0\rangle). \tag{2.31}$$

We can treat the differential forms inside the integral as functions of x by trivialization. Then we know $\langle 0|\star j \wedge \tilde{\mathcal{V}}(x)|n\rangle = e^{i\mathbf{p}_n\cdot\mathbf{x}-i\omega_n t}\langle 0|\star j \wedge \tilde{\mathcal{V}}(0)|n\rangle$ and inserting this in $\mathcal{C}$ we can perform the integral over x:

$$\mathcal{C} = \sum_n (2\pi)^{d-1}\delta^{d-1}(\mathbf{p}_n)(\langle 0|\star j \wedge \tilde{\mathcal{V}}(0)|n\rangle\langle n|W(\Gamma^{(q)})|0\rangle e^{-i\omega_n t} - \langle 0|W(\Gamma^{(q)})|n\rangle\langle n|\star j \wedge \tilde{\mathcal{V}}|0\rangle e^{-i\omega_n t}). \tag{2.32}$$

On the other hand, starting from the definition of $\mathcal{C}$, this quantity turns out to be time independent. In order to see this, differentiate with respect to time:

$$\partial_0 \mathcal{C} = \int_{\mathcal{V}^{(d-q-1)}} \langle 0|[\partial_0 \star j, W(\Gamma^{(q)})]|0\rangle. \tag{2.33}$$

We can use $d\star j = 0$, written as $\partial_0 j^{0\mu_1\cdots\mu_{q+1}} = -\partial_i j^{i\mu_1\cdots\mu_{q+1}}$, and Stokes' theorem to bring the integral to:

$$\partial_0 \mathcal{C} = -\int_{\partial\mathcal{V}^{(d-q-1)}} \langle 0|[\star j, W(\Gamma^{(q)})]|0\rangle. \tag{2.34}$$

The relevant observation here is that W has codimension $d-q-1$ with respect to $\partial M^{(d-1)}$, while $\partial\mathcal{V}^{(d-q-1)}$ has codimension $q+1$. Hence, we can use a previous argument that higher codimension surfaces can be deformed such that they do not intersect, to conclude that for a generic $\Gamma^{(q)}$ we can choose a $\mathcal{V}^{(d-q-1)}$ such that $\Gamma^{(q)}$ is not contained in $\partial\mathcal{V}^{(d-q-1)}$. As a consequence, the commutator in (2.34) vanishes, and $\mathcal{C}$ is time independent.

However, the expression in terms of modes (2.32) has an explicit time dependence. When $\mathbf{p}_n$ is non zero, this is not a problem, but when the 0 momentum case is approached, time independent modes are needed, hence ω_n also needs to approach 0. Then there is a linear relationship between $\mathbf{p}$ and ω as we expect for massless modes.

Since Q is generated by the integral of $\star j$, which is a $(d-q-1)$-form, these massless modes are not just spin 0 bosons, but can have higher integer spin. We can apply this to $U(1)$ gauge theory in $\mathbb{R}^4$ by choosing $M^{(3)}$ as the xy plane. Then $\star j = j^{03} dx \wedge dy$. But $j \propto \star F$ and the current will describe a component of the E field. Since it transforms as a vector, the resulting boson will also be vector-like and we can associate it to the photon.

From a quantum perspective, the presence of a massless mode is associated to a pole in amplitudes. We can illustrate this by considering again the case of a 1-form symmetry and computing the correlator between the current and a Wilson line. The Ward identity (2.15) can be written in components as [3]:

$$\langle \partial_\mu j^{\mu\nu}(x) W(\Gamma^{(1)})\rangle = -\int_{\Gamma^{(1)}} dy^\nu \delta^d(x-y)\langle W(\Gamma^{(1)})\rangle. \tag{2.35}$$

Changing to momentum space, it becomes $ip_\mu\langle j^{\mu\nu}(p)W(\Gamma^{(1)})\rangle = \epsilon^\nu(p)\langle W(\Gamma^{(1)})\rangle$, with $\epsilon^\nu(p) = \int_{\Gamma^{(1)}} dx^\nu e^{ipx}$. We note that $\epsilon^\nu(0) \neq 0$ and $p_\nu \epsilon^\nu(p) = 0$. Then in the limit $p \to 0$, the correlator needs to behave as:

$$\langle j^{\mu\nu}(p) W(\Gamma^{(1)})\rangle \sim \frac{p^\mu \epsilon^\nu(p) - p^\nu \epsilon^\mu(p)}{p^2}. \tag{2.36}$$

The pole at $p^2 = 0$ signals the presence of a massless excitation in the spectrum.

3. Discrete Higher Symmetries

3.1 $\mathbb{Z}_N$ gauge symmetry

The example of a $U(1)$ gauge theory and its two 1-form symmetries is simplified by the existence of conserved currents. Their origin is in the continuous nature of the symmetry group. It is natural to ask whether discrete versions of higher symmetries exist and if there are ways to translate the previous formalism to them. One approach is to embed the discrete symmetry in a larger continuous group, such configurations being described by discrete gauge theories with gauge fields measuring discrete holonomies [5, 7]. However, the discrete group does not need to have a continuous counterpart and in this case there are no continuous degrees of freedom. An alternative description of the gauge bundle can be obtained through the insertion of topological defects that represent the transition functions of the bundle. This approach will be discussed in the next chapter. Other sophisticated methods such as *differential characters* can be used that do not make explicit use of the differential forms involved. It offers an alternative to parallel transport and holonomies on topologically non-trivial bundles [8]. Most importantly, a rigorous treatment can lead to an explicit construction of a TQFT involving a q-form and $(d-q-1)$-form, where the latter is not topologically trivial in general [9]. Such a theory is referred to as a BF theory.

We choose instead to follow [10, 11] and obtain the BF theory by higgsing a $U(1)$ gauge theory in $d=4$ dimensions. The setup will be a $U(1)$ gauge theory with complex scalar field of charge N, $\phi = \rho e^{i\varphi}$ with $\varphi \in \mathbb{S}^1$. Considering only the angular variable and the vev, $v \in \mathbb{R}$, the Lagrangian for the scalar field becomes:

$$\mathcal{L} = v^2(d\varphi - NA_1) \wedge \star(d\varphi - NA_1). \tag{3.1}$$

We have the identifications $\varphi \sim \varphi + 2\pi$ and under a gauge transformation $\varphi \to \varphi + N\alpha$ and $A_1 \to A_1 + d\alpha$, with α also periodic. We can access the low energy limit of this theory by sending $v \to \infty$. Then we are forced to have $A_1 = \frac{1}{N}d\varphi$. This means that, locally, A_1 is flat and the dynamics is trivial. However, the holonomy of A_1 around non-trivial loops is non zero as $\frac{1}{2\pi}\oint A_1 = \frac{1}{N}\mathbb{Z}$. As before, the $U(1)$ symmetry is broken to $\mathbb{Z}_N$ by the presence of N charged matter.

A more insightful understanding comes from dualizing φ, by introducing a 3-form H_3 with integer integrals over 3-cycles. We can then write:

$$\mathcal{L} = \frac{1}{(4\pi)^2 v^2} H_3 \wedge \star H_3 + \frac{i}{2\pi} H_3 \wedge (d\phi - N A_1). \tag{3.2}$$

Integrating out H_3 through its equation of motion brings us back to the original theory, but we can instead send $v \to \infty$ here to obtain the constraint as before $A_1 = \frac{1}{N} d\varphi$, but with H_3 acting as a Lagrange multiplier. Integrating out φ from the Lagrangian above imposes the flatness condition $dH_3 = 0$. Using the Poincaré lemma [12], $H_3 = dB_2$, locally. We are then left with the following Lagrangian in the $v \to \infty$ limit:

$$\mathcal{L} = \frac{iN}{2\pi} B_2 \wedge dA_1. \tag{3.3}$$

The resulting theory has no propagating degrees of freedom as, locally, both differential forms are closed. The gauge transformations are $A_1 \to A_1 + d\alpha$ and $B_2 \to B_2 + d\alpha_1$, with α_1 a 1-form. Moreover, the gauge symmetry of A_1 is $\mathbb{Z}_N$. Whether the 1-form gauge symmetry of B_2 is $U(1)$ or is broken down to a subgroup is not obvious in this formulation, but we can further dualize A_1 to make it explicit. We first add a Maxwell term for $F_2 = dA_1$ and a Lagrange multiplier $d\hat{A}_1$ in the form:

$$\mathcal{L} = -\frac{1}{2e^2} F_2 \wedge \star F_2 - \frac{i}{2\pi} F_2 \wedge (d\hat{A}_1 - N B_2). \tag{3.4}$$

The field $\hat{A}_1$ has gauge symmetry $\hat{A}_1 \to \hat{A}_1 - N\alpha_1$. Integrating out F_2 using its equation of motion leads to:

$$\mathcal{L} = \frac{e^2}{8\pi^2} (d\hat{A}_1 - N B_2) \wedge \star (d\hat{A}_1 - N B_2). \tag{3.5}$$

We can compare the final Lagrangian with the initial one (3.1) and interpret the gauge field $\hat{A}_1$ as being charged under the B_2 gauge symmetry with charge N. Thus the gauge redundancy for B_2 is broken as well to $U(1) \to \mathbb{Z}_N$.

We further want to study the operator content of this theory. The local, gauge invariant, operators are:

$$d\varphi - NA_1 \sim \star H_3, \quad d\hat{A}_1 - NB_2 \sim \star F_2. \tag{3.6}$$

However, these are trivial, from the equations of motion. Turning our attention to extended operators, there are two electric Wilson-like operators $W_A[\mathcal{C}] = e^{in_1 \int_{\mathcal{C}} A_1}$ and $W_B[\mathcal{S}] = e^{in_2 \int_{\mathcal{S}} B_2}$, where $\mathcal{C}$ is a curve and $\mathcal{S}$ is a surface. The Wilson line describes the worldline of an electric probe of charge n_1, while the Wilson surface generalizes this to the case of an insertion of a vortex string with flux n_2/N. We know from the Aharonov-Bohm effect that by moving a charged particle around a magnetic flux tube, it picks up a holonomy of $\frac{2\pi n_1 n_2}{N}$ and thus we can write the correlation function between these two operators as:

$$\langle W_A[\mathcal{C}] W_B[\mathcal{S}] \rangle \sim \exp\Big(i \frac{2\pi n_1 n_2 \mathrm{Link}(\mathcal{C}, \mathcal{S})}{N}\Big). \tag{3.7}$$

This relation can be read in two distinct ways: either the Wilson surface induces a flux in the Wilson line, or the Wilson surface is charged under the action of the Wilson line, with both playing the role of topological operators. These operators describe two $\mathbb{Z}_N$ electric symmetries. From the above relation we see that if an operator with has a charge equal to a multiple of N, the commutation relation is trivial. The operators are labelled by charges modulo N, justifying the $\mathbb{Z}_N$ symmetry assignment.

Constructing magnetic operators is not as simple as in the unbroken $U(1)$ case. We can proceed in a similar manner and use the dual of A_1, $\hat{A}_1$, and B_2 and construct an operator from them. Since we are looking for a gauge invariant operators, a good candidate would be:

$$\exp\Big(i \int_{\mathcal{C}} \hat{A}_1 - iN \int_{\mathcal{S}} B_2\Big). \tag{3.8}$$

with the surface $\mathcal{S}$ ending on the line $\mathcal{C}$. However, this operator is also trivial in our theory after using the second equation in (3.6). In order to obtain non-trivial magnetically charged operators, we need to integrate the above fields over *torsion cycles*. Simply described, instead of having $\partial\mathcal{S} = \mathcal{C}$ as above, we would instead have $\partial\mathcal{S} = P\mathcal{C}$, where the multiplication by $P \in \mathbb{Z}$ here is seen as composition in the singular homology of our base manifold.

A full treatment of these issues requires the use of differential characters and can be found in [9]. The main result is that the gauge invariant observables are:

$$\exp\left(i\frac{P}{\gcd(P,N)}\int_{\mathcal{C}}\hat{A}_1 - i\frac{N}{\gcd(P,N)}\int_{\mathcal{S}}B_2\right). \tag{3.9}$$

3.2 $SU(N)$ and $PSU(N)$ Theories

In the previous section the higher-form symmetries of the $U(1)$ theory and its $\mathbb{Z}_N$ subgroup have been discussed. However, there are different embeddings of $\mathbb{Z}_N$ and one of them is in a $SU(N)$ gauge theory. They also give rise to 1-form symmetries as we will describe below.

To clarify notation, our starting point will be the pure Yang-Mills action in 4 dimensions $S_{YM}[A] = -\frac{1}{g^2}\int Tr[F \wedge \star F]$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ and $A_\mu = T^a A^a_\mu$, with T^a being the Hermition generators satisfying $[T^a, T^b] = if^{abc}T^c$. Finally, gauge transformations are generated by $U(x) \in SU(N)$:

$$A_\mu \to U(x)A_\mu U^\dagger(x) + iU(x)\partial_\mu U^\dagger(x). \tag{3.10}$$

The discussion in this section follows that in [3]. The $\mathbb{Z}_N$ subgroup of $SU(N)$ is generated by the element $U = e^{i2\pi T^{N^2-1}} = e^{\frac{i2\pi}{N}}\mathbb{1}$, where $T^{N^2-1} = \text{diag}(\frac{1}{N}, \frac{1}{N}, ..., -1+\frac{1}{N})$. To make the global center symmetry explicit, it is useful to consider our base manifold with periodic time, identifying $x^0 \sim x^0 + \tau$. Let us consider the operator:

$$\tilde{U}(x^0) = \exp\left(i2\pi k\frac{x^0}{\tau}T^{N^2-1}\right), \quad k = 0, 1, ..., N-1. \tag{3.11}$$

It satisfies $\tilde{U}(x^0+\tau) = \tilde{U}(x^0)e^{\frac{i2\pi k}{N}}$, so it cannot correspond to a local or large gauge transformation which would have $U(x^0+\tau) = U(x^0)$. However, under (3.10) the gauge field changes as $A_0 \to \tilde{U}(x^0)A_0\tilde{U}^\dagger(x^0) + \frac{2\pi k}{\tau}T^{N^2-1}$. While this shift cannot be accounted by any gauge transformation, the field strength tensor $F_{\mu\nu}$ remains invariant, implying that this is a true global symmetry of our action.

We can look at the action of this operator (3.11) on a Wilson line:

$$\begin{aligned} \mathcal{P}e^{i\int_{x^0}^{x^0+\tau}A_0dx'^0} &\to \tilde{U}(x^0+\tau)\mathcal{P}e^{i\int_{x^0}^{x^0+\tau}A_0dx'^0}\tilde{U}(x^0) \\ &= e^{\frac{i2\pi k}{N}}\tilde{U}(x^0)\mathcal{P}e^{i\int_{x^0}^{x^0+\tau}A_0dx'^0}\tilde{U}(x^0). \end{aligned} \tag{3.12}$$

Taking the trace brings us to $\operatorname{Tr}\mathcal{P}e^{i\int_{x^0}^{x^0+\tau}A_0dx'^0} \to e^{\frac{i2\pi k}{N}}\operatorname{Tr}\mathcal{P}e^{i\int_{x^0}^{x^0+\tau}A_0dx'^0}$. Hence the Wilson line acquires a $\mathbb{Z}_N$ phase. The operator (3.11) generates the electric $\mathbb{Z}_N$ center symmetry of an $SU(N)$ theory.

The discussion of magnetically charged operators needs a preamble on the possible magnetic charges in Yang-Mills. Recall that in the case of a $U(1)$ theory, the existence of magnetic monopoles was imposed through the constraint (2.24). We can apply the same reasoning and require that for a monopole at the origin, the magnetic field takes the form:

$$B^i \to \frac{x^i}{4\pi r^3}Q(x), \quad r \to 0. \tag{3.13}$$

The magnetic charge is specified by the Lie algebra valued object $Q(x)$. We can patch the $\mathbf{S}^2$ sphere surrounding the origin by two charts and in each pick $Q(x)$ to be constant, and, through a gauge transformation on B we can pick Q to be in the Cartan algebra. It can thus be written as [10] $Q = \mathbf{m}\cdot\mathbf{H}$, where $\mathbf{H}$ denotes the Cartan algebra of rank r and $\mathbf{m}$ is a vector of length r that specifies the magnetic charge in this basis.

We recall that having both electric and magnetic charges leads to the Dirac quantization as a consistency condition (see [13] for a review). The relation can be generalized in our case to $\exp(i\mathbf{m}\cdot\boldsymbol{\mu}) = 1$, where $\boldsymbol{\mu}$ corresponds to the weight of the representation of the electric Wilson line inserted around the $\mathbf{S}^2$. The important remark here is that this relation needs to hold for any representation of the Wilson line. To solve for this constraint, remember that for $\boldsymbol{\alpha}$ in the root lattice, $2\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\mu}}{\boldsymbol{\alpha}^2} \in \mathbb{Z}$. Hence the Dirac quantization is satisfied if $\mathbf{m} = 2\pi\frac{2\boldsymbol{\alpha}}{\boldsymbol{\alpha}^2}$. The magnetic charges, thus, turn out to be associated with the roots. Hence if we want to build 't Hooft lines, they have to lie in the adjoint representation. However, it turns out that adjoint lines can be written formally as the modulus squared of lines in the fundamental representation: $\operatorname{Tr}_{Adj}\mathcal{P}e^{i\oint A} \sim |\operatorname{Tr}\mathcal{P}e^{i\oint A}|^2$. Since the charged lines in the fundamental representation shift by a phase under the center symmetry, adjoint lines are invariant under the global $\mathbb{Z}_N$ symmetry. Summarizing, while Wilson lines can be labelled by any tensor product of fundamental representations and are charged under $\mathbb{Z}_N$, the 't Hooft lines have to lie in the adjoint representation and are not charged under a magnetic 1-form symmetry.

In a $SU(N)$ theory as above we observe there is an imbalance between the number of electrically and magnetically charged objects under the center symmetry. This difference becomes important when comparing this theory with that of $PSU(N)$. From the perspective of local operators the two gauge theories are the same as local fields are not affected by the global difference between the two gauge groups. However, the spectrum of extended operators differs. To see this, we have to notice that since we are modding out by $\mathbb{Z}_N$, the electric representations that transform non-trivially under the centre of $SU(N)$ are not allowed to appear. This restricts the labelling of the Wilson lines, from tensor products of fundamentals, to adjoint represenations. The Dirac quantization can be applied again, this time allowing the 't Hooft lines to be labelled by elements of the weight lattice, i.e. fundamental representations. We see that the role of the Wilson and 't Hooft lines changes between the two gauge theories, even if locally they act similarly.

4. Global 2-group symmetries

4.1 Discretized Higher Symmetries and Symmetry Defects

The previous chapter dealt with discrete symmetries that benefit from being able to write down explicitly the charged operators or the symmetry generators. However, as noted before, this is not always the case and one has to find alternative ways to implement the transformations. The idea exposed in this chapter will be to use discrete networks of topological defects. In order to use them, we couple the topological operators and charged objects to a background gauge field, which becomes the connection of the bundle describing the symmetry group. The transformation of the operators is equivalent in this way to parallel transport along the bundle. Furthermore, the parallel transport will turn out to be equivalent to a transformation of the discretized gauge field. It is useful to use geometrical properties of the base manifold, and as such the gauge field will be represented in terms of simplicical complexes. The symmetry operators will appear as topological defects. An introductory account of simplicical and singular homology/cohomology and the notions used can be found in [12, 14]. A caveat will be that throughout the chapter, the notions of singular and simplicical complexes will be used interchangeably as this will suffice for the discussion, but in general they should be distinguished. We follow [15, 16] as main references.

We proceed by first recalling that a 0-form symmetry can be described by unitary operators $U_g(M^{(d-1)})$ supported on codimension-1 manifolds $M^{(d-1)}$. We saw that the action of the symmetry is implemented by passing $M^{(d-1)}$ through the charged operator. Since we are going to couple the charged fields to a background gauge connection, we need to find a suitable description for the bundle. We cover the manifold $M^{(d)}$ with open, contractible patches V_i. We denote intersections by ordered indices $V_{ij} = V_i \cap V_j$. Transition functions between patches for the gauge field are A_{ij} and they satisfy the usual cocycle condition:

$$A_{ij}A_{jk} = A_{ik} \text{ on intersections } V_{ijk}. \tag{4.1}$$

The multiplicative convention has been used here. Focusing on the indices labelling the transition functions, the above condition can be translated at the level of the geometrical patches (in the additive convention) as $\langle ij\rangle + \langle jk\rangle = \langle ik\rangle$. In an abstract language, this is the cocycle condition in the simplicical formalism for ordered $\langle ijk\rangle$ and we will make use of it in the next paragraphs.

If we want to describe the parallel transport and action on extended objects, we need to embed higher connections in a similar manner. The next step is triangulating the manifold in simplices. We use the following convention for denoting simplices: vertices or 0-simplices are $\langle i\rangle$, edges or 1-simplices are $\langle ij\rangle$, faces are $\langle ijk\rangle$ etc. There are two distinct coverings of $M^{(d)}$ and we can connect the open cover in terms of V_i's to the simplicical triangulation by associating a vertex $\langle i\rangle$ to an open subset V_i, an edge $\langle ij\rangle$ to a double intersection V_{ij}, a face $\langle ijk\rangle$ to a triple intersection etc. We can now represent a q-form on the manifold as a q-cochain $A \in C^q(M^{(d)}, G)$, where $C^q(M^{(d)}, G)$ is dual to the group $\mathrm{Hom}(C_q(M^{(d)}), G)$ and a cochain A assigns to each q-simplex a value in G. With this notion of cochains, the previous A_{ij} is a cocycle in the singular cohomology and satisfies $dA = 0$. The constraint condition (4.1) can be expressed through the geometrical patching and complexes introduced.

In this formalism we can finally describe the action of topological operators by associating group factors to simplices. The identifications will be described below. Let us focus again on ordinary symmetries. We associate an element g to a transition function A_{ij} if the edge $\langle ij\rangle$ crosses $M^{(d-1)}$. Depending on the orientation of the edge we associate either g or g^{-1}. Otherwise $A_{ij} = \mathbb{1}$. We have to check that this construction satisfies the cocycle condition. We note that a triple intersection is described by a 2-cycle, or triangle. The triangle sides could either not cross the manifold at all so the cocycle condition is satisfied trivially or the triangle could cut the manifold $M^{(d-1)}$ with 2 of its sides (if all 3 sides intersect it, we have to choose a refinement of the open cover). But in this case, the 2 edges would cross the manifold in opposite directions, so the cocycle condition is satisfied.

We insert a local charged object at one of the vertices of the triangulation, surrounded by the support $M^{(d-1)}$. If we want to remove the topological operator and act on the charged field we can perform a transformation on A_{ij}:

$$A_{ij} \to A^f_{ij} = A_{ij} + f_i - f_j = A_{ij} + (df)_{ij}, \tag{4.2}$$

where $f_i = g^{-1}$ if the vertex $\langle i\rangle$ is inside the manifold and identity otherwise. This amounts to changing A by a flat connection. The path integral defined on these refined simplices picks up a phase from this transformation and implements the transformation on charged objects.

We can perform a similar analysis for 1-form symmetries. Now the symmetry group is abelian, $\mathcal{A}$, and the topological operators are supported on codimension-2 manifolds $M^{(d-2)}$. We assign an element of $\mathcal{A}$ to any 2-chain, or triple intersection, B_{ijk} such that the cocycle condition is satisfied:

$$(dB)_{ijkl} = B_{jkl} - B_{ikl} + B_{ijl} - B_{ijk} = 0. \tag{4.3}$$

A value $a \in \mathcal{A}$ is assigned to B_{ijk} if the triangle $\langle ijk\rangle$ crosses the manifold $M^{(d-2)}$ with positive orientation and $-a$ for negative orientation. The cocycle condition is again satisfied as for a tetrahedron, or 3-simplex, the defect enters one face and exits on another one. We remark here that we are not defining the actual structure underpinning the equivalent notion of a principal bundle for 1-form gauge transformations. The actual construction is referred to as a *gerbe* and we direct the reader to [17] for a review.

As before, a transformation by a flat connection induces the transformation on a line operator that lies along some edges of the triangulation inside $M^{(d-2)}$ as $B_{ijk} \to B_{ijk} + (d\gamma)_{ijk}$. Geometrically, we can draw a codimension-1 surface $\Sigma^{(d-1)}$ whose boundary is $M^{(d-2)}$. Then we assign $\gamma_{ij} = a$ if the edge $\langle ij \rangle$ crosses $\Sigma^{(d-1)}$ with positive orientation and the negative for the inverse orientation. The extended object here is an edge itself, so it pierces some face of a triangle, hence it gains a phase in the path integral.

4.2 Emergence of 2-group Symmetry

The description of higher symmetries in terms of networks of defects unveils an intricate behaviour that arises when a 0-form and a 1-form symmetry are brought together. The resulting symmetry will not merely be a product of the two, but the transformation of connections will mix the components. The resulting symmetry has the name of *2-group* symmetry. The exact structure of a 2-group has categorical origins and we will not delve into the topic of categorical groups here, but quote the properties when needed. We refer the reader to [1] for a comprehensive introduction to 2-groups and their applications to higher gauge theory.

The first type of mixing arises if we consider 2 symmetries, a 0-form one generated by a group G and a 1-form one associated to a group A. If the charged operators are isolated and do not interact with one another, then the symmetry group is simply $G \times A$. However, suppose we have an action of G on A via a homomorphism $\rho : G \to \mathrm{Aut}(A)$. At the level of the defects, when an operator $U_a(M^{(d-2)})$ of type $a \in A$ crosses the symmetry operator $U_g(M^{(d-1)})$, it appears on the other side as an operator $U_{\rho_g(a)}(M^{(d-2)})$ corresponding to the group element $\rho_g(a)$. The action on A also influences the way line objects behave. If a line operator W with charge $\alpha : A \to \mathbb{Z}$ crosses U_g, then its charge becomes $\alpha \circ \rho_g^{-1}$. This assignment makes sure that moving both a topological operator corresponding to the 1-form symmetry and its associated object past a U_g operator will not change the action of the A symmetry on it, namely the charge, $\alpha(a)$, is invariant.

In the previous section we asserted that the connections, there cochains, needed to satisfy a cocycle condition. Once we allow for mixing, the cocycle condition for A_{ij} is unchanged, but a modification is induced in the cocycle condition for B_{ijk}. Geometrically, we would consider a tetrahedron $\langle ijkl \rangle$ and starting from one edge, say $\langle ij \rangle$ we want to move through adjacent

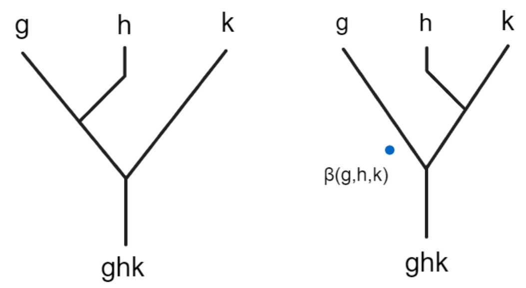

Figure 4.1. Representation of 1-form symmetry defect (the blue dot) arising as a result of changing the multiplication order between the 0-form symmetry operators at a junction. Figure inspired from [15].

edges to $\langle ik\rangle$. We can either do this directly through the face $\langle ijk\rangle$ or pass through the other faces as: $ij \to jl \to lk \to ik$. The presence of a gauge field A_{ij} in the network leads to a change in the gauge field B_{jkl} as we pass past that edge. As a result, the B field satisfies a *twisted* cocycle condition:

$$\rho(A_{ij})B_{jkl} - B_{ikl} + B_{ijl} - B_{ijk} \equiv (d_A B)_{ijkl} = 0, \quad \text{on } V_{ijkl}. \tag{4.4}$$

A further action of the group G on A involves a Postnikov class [12] $[\beta] \in H^3(BG, \mathcal{A})$, where BG is the classifying bundle for G. For our purposes, we can treat β as a function $\beta : G \times G \times G \to \mathcal{A}$. To identify the modification such an element brings, let us look at the intersection of three defects U_g, U_h, U_k with $g, h, k \in G$ (see Fig. 4.1). On the left, we have $(U_g U_h)U_k = U_{ghk}$. Under a local transformation of the network, we can make the configuration look like that on the right. This is equivalent to topologically moving the defect U_h to the right (technically, this corresponds to an F move [18]), or a gauge transformation. If we want the correlation functions to remain invariant under this move, we need [15, 16]: $U_g(U_h U_k) = U_{ghk} + \beta(g, h, k)$. In this case, at the junction of the support of the three defects, which is a codimension-2 manifold, a 1-form symmetry defect $\beta(g, h, k)$ emerges. As this has to be reflected at the level of the gauge field describing 1-form symmetries, the field B is not co-closed anymore but satisfies:

$$(d_A B)_{ijkl} = \beta(A_{ij}, A_{jk}, A_{kl}). \tag{4.5}$$

The scope of this section is to emphasize how the presence of a non-trivial associativity law is linked with the emergence of a 1-form symmetry, generated by the connection of the

0-form counterpart. This becomes the clearest at the level of gauge transformations. We will assume $\rho = \mathbb{1}$ for simplicity. Then A_{ij} transforms as in (4.2) and its cocycle condition remains invariant. But the transformation of A needs to be accompanied by a transformation of B as:

$$B_{ijk} \to B_{ijk} + \zeta_{ijk}(A, f). \tag{4.6}$$

At this point ζ is arbitrary, but we want B to satisfy (4.5) after the transformation, as $[\beta]$ is invariant under a gauge transformation [15]. This leads to a consistency relation satisfied by ζ:

$$(d_{A^f}\zeta)_{ijkl} = \beta(A^f_{ij}, A^f_{jk}, A^f_{kl}) - \beta(A_{ij}, A_{jk}, A_{kl}). \tag{4.7}$$

Explicit examples of the emergence of a 1-form symmetry gauge field for simple configurations such as that in Fig. (4.1) are discussed in [15, 16]. We instead choose to contextualize this new symmetry with an application in the next section.

4.3 Application of 2-group Symmetry

One motivation for studying 2-group symmetries comes from the phenomenon of mixed 't Hooft anomaly when gauging one member of a global $U(1) \times U(1)$ symmetry. In this section, we will see how imposing a 2-group symmetry structure prevents the theory from suffering from an anomaly. The discussion and notation is based on [19].

The starting point is a theory with global symmetry $G = U(1)^{(0)}_A \times U(1)^{(0)}_C$, where the superscript (0) labels the higher-form degree of the symmetry. Let $A^{(1)}$ and $C^{(1)}$ be the background fields that couple to the conserved currents. We would like to gauge the $U(1)^{(0)}_C$ symmetry and promote the background fields to dynamical variables on their own. It is only possible to gauge a symmetry if it is non-anomalous, so we have to study the anomaly associated to the global symmetry. A general study of anomalies can be found in [12]. We will quote the relevant expressions throughout this section.

In order to exhibit a possible anomaly, we couple the conserved current to the background field as mentioned above. We can perform a gauge transformation on the generic background field $\mathcal{B} \to \mathcal{B} + \delta\mathcal{B}$. The variation δ is the Becchi-Rouet-Stora (BRS) operator [12, 20]. For our purposes, the exact details of the transformation will not interfere with the final result, hence we will keep the variation general. If the partition function is invariant under this gauge transformation, the theory is non-anomalous and we can gauge the symmetry. However, the effective action can have non-trivial variation under the transformation $W[\mathcal{B} + \delta\mathcal{B}] = W[\mathcal{B}] + \mathcal{A}[\mathcal{B}]$.

The anomaly $\mathcal{A}[B]$ is a c-number functional of the background fields, which satisfies the Wess-Zumino consistency conditions, namely $\delta\mathcal{A}[\mathcal{B}] = 0$. Another notable property is that the anomaly $\mathcal{A}[\mathcal{B}]$ vanishes when the background field is turned off.

Following Wess-Zumino [19, 21], the anomalous c-number in d dimensions can be obtained from an invariant polynomial in $(d+2)$ dimensions $\mathcal{I}^{(d+2)}[\mathcal{B}]$ through the descent equations. The polynomial $\mathcal{I}$ being invariant, can be written in terms of various characteristic classes for the field strength of the background field $\mathcal{B}$. Through the descent equation the polynomial can be reduced to a different one in d dimensions, such that the final form depends not only on the gauge field but also on its variation. The descent equations read:

$$d\mathcal{I}^{(d+1)}[\mathcal{B}] = \mathcal{I}^{(d+2)}[\mathcal{B}], \quad d\mathcal{I}^{(d)}[\mathcal{B}, \delta\mathcal{B}] = \delta\mathcal{I}^{(d+1)}[\mathcal{B}]. \tag{4.8}$$

Then the anomalous coefficient can be expressed in terms of the d-dimensional polynomial as:

$$\mathcal{A}[\mathcal{B}] = 2\pi i \int_{\mathcal{M}^{(d)}} \mathcal{I}^{(d)}[\mathcal{B}, \delta\mathcal{B}]. \tag{4.9}$$

For this procedure to work, one property we are seeking is that of reducibility. If the anomaly polynomial is reducible, it factorizes into a product of closed, gauge-invariant lower degree polynomials:

$$\mathcal{I}^{(d+2)}_{red} = \mathcal{J}^{(p)} \wedge \mathcal{K}^{(d+2-p)}, \quad d\mathcal{I}^{(p)} = d\mathcal{K}^{(d+2-p)} = 0. \tag{4.10}$$

The anomaly polynomials that will be encountered benefit from this decomposition, so we can generically apply the descent procedure (4.8) to the reducible polynomial (4.10). The first step would be to remove a derivative from the $d+2$ dimensional polynomial. This step is, however, ambiguous as for a general reducible polynomial we can remove the derivative from any of the constituent terms. The ambiguity leads to a new factor, $s \in \mathbb{R}$, that parametrizes the reduction procedure:

$$\mathcal{I}^{(d+1)} = \mathcal{J}^{(p-1)} \wedge \mathcal{K}^{d+2-p} + sd(\mathcal{J}^{(p-1)} \wedge \mathcal{K}^{d+1-p}). \tag{4.11}$$

With this nomenclature, we can return to our gauge group G in 4 dimensions. The most general polynomial we can write contains all possible combinations of wedge products between the field strengths of the two $U(1)$ groups, as the first Chern class for a $U(1)$ is $c_1(F^{(2)}) = \frac{1}{2\pi}F^{(2)}$. Each term is parametrized by an anomaly coefficient which can be extracted by computing correlation functions for the corresponding currents. We will keep these generic for now.

$$\mathcal{I}^{(6)} = \frac{1}{(2\pi)^3}\Big(\frac{\kappa_{A^3}}{3!}F_A^{(2)} \wedge F_A^{(2)} \wedge F_A^{(2)} + \frac{\kappa_{A^2C}}{2!}F_A^{(2)} \wedge F_A^{(2)} \wedge F_C^{(2)} + \\ \frac{\kappa_{AC^2}}{2!}F_A^{(2)} \wedge F_C^{(2)} \wedge F_C^{(2)} + \frac{\kappa_{C^3}}{3!}F_C^{(2)} \wedge F_C^{(2)} \wedge F_C^{(2)}\Big). \quad (4.12)$$

All the terms in the above expression are reducible and we can construct a polynomial in 5 dimensions, up to an ambiguous term. An example of such a term comes from the A^2C contribution, which in 5 dimensions reads:

$$\mathcal{I}^{(5)} \supset \frac{\kappa_{A^2C}}{2(2\pi)^3}A^{(1)} \wedge F_A^{(2)} \wedge F_C^{(2)} + sd(A^{(1)} \wedge F_A^{(2)} \wedge C^{(1)}). \quad (4.13)$$

We can shed some light on the nature of the free parameter s by describing the 't Hooft anomaly as an anomaly inflow from a higher dimension. To understand the origin of this contribution, let us return to the anomaly coefficient (4.9). We observe that it can be accounted by considering $M^{(d)} = \partial M^{(d+1)}$, for a $(d+1)$-dimensional space $M^{(d+1)}$, and the following term in the action $S_{d+1}[\mathcal{B}] = \int_{M^{(d+1)}} \mathcal{I}^{(d+1)}[\mathcal{B}]$. Varying the gauge field $\mathcal{B}$ generates the coefficient (4.9) as an anomaly inflow from the $(d+1)$-dimensional bulk to the boundary. The s term in (4.11) or (4.13) multiplies an exact term in the $(d+1)$-dimensional anomaly inflow action. Thus, in d dimensions, it appears as a local counterterm that can be adjusted, modifying the representation of the anomaly, without completely cancelling it. It will turn out that the counterterms have to vanish in order to ensure the absence of a non-zero $\mathcal{A}_C$. We choose to set them to 0 at this point as it will not make a difference and it simplifies the appearance of the coefficients.

After performing this procedure for all terms in $\mathcal{I}^{(6)}$ a 5 dimensional polynomial containing terms such as $A^{(1)} \wedge F_A^{(2)} \wedge F_A^{(2)}$ is obtained. Terms containing $A^{(1)}$ or $C^{(1)}$ can be further reduced, leading to the 4 dimensional anomaly polynomials:

$$
\begin{aligned}
\mathcal{A}_A &= \frac{i}{4\pi^2}\int_{\mathcal{M}_4} \lambda_A^{(0)}\Big(\frac{\kappa_{A^3}}{3!}F_A^{(2)}\wedge F_A^{(2)} + \frac{\kappa_{A^2C}}{2!}F_A^{(2)}\wedge F_C^{(2)} + \frac{\kappa_{AC^2}}{2!}F_C^{(2)}\wedge F_C^{(2)}\Big)\\
\mathcal{A}_C &= \frac{i}{4\pi^2}\int_{\mathcal{M}_4} \lambda_C^{(0)}\Big(\frac{\kappa_{C^3}}{3!}F_C^{(2)}\wedge F_C^{(2)}\Big).
\end{aligned}
\tag{4.14}
$$

where $\lambda_A^{(0)}$ and $\lambda_C^{(0)}$ are the gauge parameters of $A^{(1)}$ and $C^{(1)}$. We remark that even if $\mathcal{I}^{(6)}$ was symmetric in the field content of $A^{(1)}$ and $C^{(1)}$, the final coefficients (4.14) do not share this property. This is a result of our choice of counterterms and a different parametrization can restore the symmetry. However, this representation is more advantageous for the upcoming parts.

The gauging of $U(1)_C^{(0)}$ can only be performed by ensuring first that the anomalous variation $\mathcal{A}_C$ vanishes. This implies $\kappa_{C^3} = 0$. With this, the polynomial for $U(1)_A^{(0)}$ becomes:

$$
\mathcal{A}_A = \frac{i}{4\pi^2}\int_{\mathcal{M}_4} \lambda_A^{(0)}\Big(\frac{\kappa_{A^3}}{3!}F_A^{(2)}\wedge F_A^{(2)} + \frac{\kappa_{A^2C}}{2!}F_A^{(2)}\wedge F_C^{(2)} + \frac{\kappa_{AC^2}}{2!}F_C^{(2)}\wedge F_C^{(2)}\Big). \tag{4.15}
$$

When we gauge $U(1)_C^{(0)}$ the field strengths $F_C^{(2)}$ become operators $f_C^{(2)}$. Correspondingly, the terms containing $F_C^{(2)}$ in $\mathcal{A}_A$ will act as operators and not c-numbers. This behaviour is inconsistent with the way $\mathcal{A}_A$ is defined, as a background gauge field is varied, and not a dynamical one. Hence, if we want consistent gauging, we have to correct for these terms. The AC^2 term in (4.14) gives rise to an ABJ anomaly for the $U(1)_A^{(0)}$ current: $d\star j_A^{(1)} = -\frac{i\kappa_{AC^2}}{8\pi^2}f_C^{(2)}\wedge f_C^{(2)}$. The non-conservation of the current remains even when the $A^{(1)}$ field is turned off, thus breaking the symmetry by an ABJ anomaly. This is an undesired feature, not related to 2-group symmetry, and we set $\kappa_{AC^2} = 0$.

Let us focus on the A^2C term instead. Its contribution is $\frac{i\kappa_{A^2C}}{8\pi^2}F_A^{(2)}\wedge f_C^{(2)}$. To cancel it, we remember from Maxwell theory that once we gauge a $U(1)$ 0-form symmetry, we acquire two 1-form global symmetries. The magnetic symmetry has an associated current $J_B^{(2)} = \frac{i}{2\pi}\star f_c^{(2)}$. We couple the current to a background gauge field, $B^{(2)}$, for the 1-form symmetry as:

$$
S \supset \int B^{(2)}\wedge \star J_B^{(2)} = \frac{i}{2\pi}\int B^{(2)}\wedge f_c^{(2)}. \tag{4.16}
$$

The trick is to impose a non-trivial 2-group transformation on $B^{(2)}$ that mixes it with the $A^{(1)}$ gauge field:

$$B^{(2)} \to B^{(2)} + \frac{\hat{\kappa}_A}{2\pi}\lambda_A^{(0)} F_A^{(2)}, \quad \hat{\kappa}_A = -\frac{1}{2}\kappa_{A^2C}. \tag{4.17}$$

This is the continuous version of (4.6), the transformation of the 1-form gauge field considered on the discretized network. The contribution from the 2-group shift exactly cancels the undesired term in the anomaly coefficient of $A^{(1)}$, rendering the symmetry non-anomalous.

We saw previously that 2-group symmetries are characterized by an element of $H^3(BU(1), U(1))$, which in this case are labelled by integers [19]. This implies that $\hat{\kappa}_A$ has to be an integer. But from the definition above, it is only true if $\kappa_{A^2C} \in 2\mathbb{Z}$. Let us check that this is true in a theory with a set of Weyl fermions ψ_α^i with integer charges q_A^i and q_C^i. In terms of the charges, the anomaly coefficients are given by $\kappa_{A^2C} = \sum_i (q_A^i)^2 q_C^i$ and $\kappa_{AC^2} = \sum_i q_A^i (q_C^i)^2$. If any of q_A^i or q_C^i is even, then the i-th term in each sum is even, otherwise they are both odd. Hence they have the same value modulo 2. Summing over all fermions establishes $\kappa_{AC^2} \equiv \kappa_{A^2C} \pmod 2$. However, we set $\kappa_{AC^2} = 0$ earlier, thus forcing $\kappa_{A^2C} \in 2\mathbb{Z}$.

4.4 't Hooft Anomaly of 2-group Symmetry

In this section we will see how further gauging the $U(1)_A^{(0)}$ is obstructed by the presence of a 't Hooft anomaly. After cancelling the terms containing $F_C^{(2)}$ from (4.9), there is still a contribution to the anomaly coefficient purely from the $A^{(1)}$ field:

$$\mathcal{A}_A = \frac{i\kappa_{A^3}}{24\pi^2} \int_{\mathcal{M}_4} \lambda_A^{(0)} F_A^{(2)} \wedge F_A^{(2)}. \tag{4.18}$$

As before, we have to consider extra terms that can potentially cancel this factor. A term having the right structure is a Green-Schwarz counterterm:

$$S_{GS} = \frac{in}{2\pi} \int_{\mathcal{M}_4} B^{(2)} \wedge F_A^{(2)}. \tag{4.19}$$

Under the previous 2-group transformation, it shifts as $S_{GS} \to S_{GS} + \frac{in\hat{\kappa}_A}{4\pi^2} \int_{\mathcal{M}_4} \lambda_A^{(0)} F_A^{(2)} \wedge F_A^{(2)}$. The integral of the extra term matches that in (4.18). It furthermore shifts the anomaly coefficient $\kappa_{A^3} \to \kappa_{A^3} + 6n\hat{\kappa}_A$. We first note that invariance under large $U(1)_B^{(0)}$ gauge transformations requires $n \in \mathbb{Z}$ in order to preserve (4.19) modulo $2\pi\mathbb{Z}$. Then n is quantized and by adding a GS term, we can cancel the anomaly coefficient κ_{A^3} only modulo $6\hat{\kappa}_A$.

We could instead give up large gauge invariance of (4.19). Then $n \in \mathbb{R}$ and under the 2-group transformation κ_{A^3} can be made to vanish. However, under a large gauge transformation, the partition function picks up a phase proportional to κ_{A^3} modulo $6\hat{\kappa}_A$ as before and hence is anomalous.

The conclusion is that the 2-group anomaly is characterized by κ_{A^3} modulo $6\hat{\kappa}_A$ and not by the integer value of κ_{A^3}. There is a mixed 't Hooft anomaly between $U(1)_A^{(0)}$ and large $U(1)_B^{(1)}$ and we cannot further gauge $U(1)_A^{(0)}$.

5. Conclusion and Outlook

In this dissertation, we introduced higher-form global symmetries by highlighting their topological origin and described their action on operators for both continuous and discrete symmetries. Furthermore, the link between non-trivial mixing of a 0-form and 1-form symmetry and relaxing the operator associativity was discussed and we argued that it leads to an enhanced symmetry whose structure cannot be captured by groups, but finds its description in terms of 2-categories.

Higher symmetries can be used to approach areas not accessible to ordinary symmetries. One such field where higher-form symmetries bring new insights is in characterizing phase transitions between confinement and Higgs or Coulomb phases in gauge theories. Landau's paradigm that phase transitions occur when ordinary global symmetries are spontaneously broken can be extended to 1-form symmetries [22]. The Wilson or 't Hooft lines act as order parameters for the different phases and their behaviour at infinity dictates whether the symmetry is broken [23, 24]. In previous sections we saw that a broken higher-form symmetry signals a gauge theory. In the IR these can give rise to gapped TQFTs [25].

The previous generalizations can arise in relativistic theories, but higher symmetries have many applications to condensed matter systems (see [26] for a review). For instance, another class of generalized symmetries are *subsystem symmetries*. The distinction from the higher-form case is that the symmetry operators act on rigid subspaces and cannot be deformed. Thus, they are not topological anymore, but can be made to commute with the Hamiltonian. The charged objects are characterized in terms of the fracton phases [27]. Subsystem symmetries are not entirely independent from higher-form versions as the former can be obtained as p-string condensations of higher-form symmetries [28]. All of the above symmetries are described by unitary, invertible operators. When two defects are fused together on the same submanifold, they give rise to a single defect corresponding to a further element in the symmetry group. This restriction can be lifted and the fusion of two operators be expressed as a combination of other defects. Symmetries described by such fusion algebras are called *non-invertible* or *categorical symmetries*. In 2 dimensional QFTs, they can be used to constrain RG flows [29].

The language of generalized symmetries constitutes a fruitful arena for addressing questions about intricate dynamics that cannot be captured in the more narrow sense of ordinary symmetries. Rephrasing known results in terms of higher symmetries shed new light over phenomena which were assumed to be of different nature, identifying common structures that govern them. As more symmetries imply more control, the hope is that the new language of generalized symmetries will help answer fundamental questions such as classifying more phases of matter and constraining the principles of the physics beyond the Standard Model.

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