

第三节

高阶导数

一、高阶导数的概念

二、高阶导数的运算法则



一、高阶导数的概念

引例：变速直线运动 $s = s(t)$

速度 $v = \frac{ds}{dt}$, 即 $v = s'$

加速度 $a = \frac{dv}{dt} = \frac{d}{dt} \left(\frac{ds}{dt} \right)$

即 $a = (s')'$



定义. 若函数 $y = f(x)$ 的导数 $y' = f'(x)$ 可导, 则称 $f'(x)$ 的导数为 $f(x)$ 的**二阶导数**, 记作 y'' 或 $\frac{d^2 y}{dx^2}$, 即

$$y'' = (y')' \quad \text{或} \quad \frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

类似地, 二阶导数的导数称为三阶导数, 依次类推, $n-1$ 阶导数的导数称为 n 阶导数, 分别记作

$$y''', \quad y^{(4)}, \quad \dots, \quad y^{(n)}$$

或

$$\frac{d^3 y}{dx^3}, \quad \frac{d^4 y}{dx^4}, \quad \dots, \quad \frac{d^n y}{dx^n}$$



例1. 设 $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, 求 $y^{(n)}$.

解: $y' = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}$

$$y'' = 2 \cdot 1a_2 + 3 \cdot 2a_3x + \cdots + n(n-1)a_nx^{n-2}$$

依次类推, 可得

$$y^{(n)} = n!a_n$$

思考: 设 $y = x^\mu$ (μ 为任意常数), 问 $y^{(n)} = ?$

$$(x^\mu)^{(n)} = \mu(\mu-1)(\mu-2)\cdots(\mu-n+1)x^{\mu-n}$$



例2. 设 $y = e^{ax}$, 求 $y^{(n)}$.

解: $y' = ae^{ax}$, $y'' = a^2 e^{ax}$, $y''' = a^3 e^{ax}$, \dots ,

$$y^{(n)} = a^n e^{ax}$$

特别有: $(e^x)^{(n)} = e^x$

$$y' = -\frac{1}{1-x}$$
$$y'' = -\frac{1}{(1-x)^2}$$

例3. 设 $y = \ln(1+x)$, 求 $y^{(n)}$.

解: $y' = \frac{1}{1+x}$, $y'' = -\frac{1}{(1+x)^2}$, $y''' = (-1)^2 \frac{1 \cdot 2}{(1+x)^3}$,

$$\dots, \quad y^{(n)} = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n}$$

规定 $0! = 1$

思考: $y = \ln(1-x)$, $y^{(n)} = -\frac{(n-1)!}{(1-x)^n}$



例4. 设 $y = \sin x$, 求 $y^{(n)}$.

解: $y' = \cos x = \sin(x + \frac{\pi}{2})$

$$y'' = \cos(x + \frac{\pi}{2}) = \sin(x + \frac{\pi}{2} + \frac{\pi}{2})$$

$$= \sin(x + 2 \cdot \frac{\pi}{2})$$

$$y''' = \cos(x + 2 \cdot \frac{\pi}{2}) = \sin(x + 3 \cdot \frac{\pi}{2})$$

一般地, $(\sin x)^{(n)} = \sin(x + n \cdot \frac{\pi}{2})$

类似可证:

$$(\cos x)^{(n)} = \cos(x + n \cdot \frac{\pi}{2})$$



例5. 设 $y = e^{ax} \sin bx$ (a, b 为常数), 求 $y^{(n)}$.

解: $y' = ae^{ax} \sin bx + be^{ax} \cos bx$

$$= e^{ax} (a \sin bx + b \cos bx)$$

$$= e^{ax} \sqrt{a^2 + b^2} \sin(bx + \varphi) \quad (\varphi = \arctan \frac{b}{a})$$

$$y'' = \sqrt{a^2 + b^2} [ae^{ax} \sin(bx + \varphi) + be^{ax} \cos(bx + \varphi)]$$

$$= \sqrt{a^2 + b^2} e^{ax} \sqrt{a^2 + b^2} \sin(bx + 2\varphi)$$

.....

$$y^{(n)} = (a^2 + b^2)^{\frac{n}{2}} e^{ax} \sin(bx + n\varphi) \quad (\varphi = \arctan \frac{b}{a})$$



例6. 设 $f(x) = 3x^3 + x^2|x|$, 求使 $f^{(n)}(0)$ 存在的最高阶数 $n = \underline{2}$.

分析:
$$f(x) = \begin{cases} 4x^3, & x \geq 0 \\ 2x^3, & x < 0 \end{cases}$$

$$\therefore f'_-(0) = \lim_{x \rightarrow 0^-} \frac{2x^3 - 0}{x} = 0$$

$$f'_+(0) = \lim_{x \rightarrow 0^+} \frac{4x^3 - 0}{x} = 0$$

$$\text{又 } f''_-(0) = \lim_{x \rightarrow 0^-} \frac{6x^2}{x} = 0$$

$$f''_+(0) = \lim_{x \rightarrow 0^+} \frac{12x^2}{x} = 0$$

$$\therefore f'(x) = \begin{cases} 12x^2, & x \geq 0 \\ 6x^2, & x < 0 \end{cases}$$

$$\therefore f''(x) = \begin{cases} 24x, & x \geq 0 \\ 12x, & x < 0 \end{cases}$$

但是 $f''_-(0) = 12$, $f''_+(0) = 24$, $\therefore f'''(0)$ 不存在.



二、高阶导数的运算法则

设函数 $u = u(x)$ 及 $v = v(x)$ 都有 n 阶导数, 则

$$1. (u \pm v)^{(n)} = u^{(n)} \pm v^{(n)}$$

$$2. (Cu)^{(n)} = Cu^{(n)} \quad (C \text{ 为常数})$$

$$\begin{aligned} 3. (uv)^{(n)} = & u^{(n)}v + nu^{(n-1)}v' + \frac{n(n-1)}{2!}u^{(n-2)}v'' + \\ & + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}u^{(n-k)}v^{(k)} \\ & + \cdots + uv^{(n)} \end{aligned}$$

莱布尼兹(Leibniz) 公式



例7. $y = x^2 e^{2x}$, 求 $y^{(20)}$.

解: 设 $u = e^{2x}$, $v = x^2$, 则

$$u^{(k)} = 2^k e^{2x} \quad (k = 1, 2, \cdots, 20)$$

$$v' = 2x, \quad v'' = 2,$$

$$v^{(k)} = 0 \quad (k = 3, \cdots, 20)$$

代入莱布尼兹公式, 得

$$\begin{aligned} y^{(20)} &= 2^{20} e^{2x} \cdot x^2 + 20 \cdot 2^{19} e^{2x} \cdot 2x + \frac{20 \cdot 19}{2!} 2^{18} e^{2x} \cdot 2 \\ &= 2^{20} e^{2x} (x^2 + 20x + 95) \end{aligned}$$



例8. 设 $y = \arctan x$, 求 $y^{(n)}(0)$.

解: $y' = \frac{1}{1+x^2}$, 即 $(1+x^2)y' = 1$

↓ 用莱布尼兹公式求 n 阶导数

$$(1+x^2) y^{(n+1)} + n \cdot 2x y^{(n)} + \frac{n(n-1)}{2!} \cdot 2 y^{(n-1)} = 0$$

令 $x = 0$, 得 $y^{(n+1)}(0) = -n(n-1)y^{(n-1)}(0) \quad (n = 1, 2, \dots)$

由 $y(0) = 0$, 得 $y''(0) = 0, y^{(4)}(0) = 0, \dots, y^{(2m)}(0) = 0$

由 $y'(0) = 1$, 得 $y^{(2m+1)}(0) = (-1)^m (2m)! y'(0)$

$$\text{即 } y^{(n)}(0) = \begin{cases} 0, & n = 2m \\ (-1)^m (2m)!, & n = 2m + 1 \end{cases} \quad (m = 0, 1, 2, \dots)$$



内容小结

高阶导数的求法

- (1) 逐阶求导法
- (2) 利用归纳法
- (3) 间接法 —— 利用已知的高阶导数公式

$$\text{如, } \left(\frac{1}{a+x} \right)^{(n)} = (-1)^n \frac{n!}{(a+x)^{n+1}}$$

$$\left(\frac{1}{a-x} \right)^{(n)} = \frac{n!}{(a-x)^{n+1}}$$

- (4) 利用莱布尼兹公式



思考与练习

1. 如何求下列函数的 n 阶导数?

$$(1) \quad y = \frac{1-x}{1+x}$$

$$\text{解: } y = -1 + \frac{2}{1+x}$$

$$y^{(n)} = 2(-1)^n \frac{n!}{(1+x)^{n+1}}$$

$$(2) \quad y = \frac{x^3}{1-x}$$

$$\text{解: } y = -x^2 - x - 1 + \frac{1}{1-x}$$

$$y^{(n)} = \frac{n!}{(1-x)^{n+1}}, \quad n \geq 3$$



$$(3) \quad y = \frac{1}{x^2 - 3x + 2}$$

提示: 令 $\frac{1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$

$$A = (x-2) \cdot \text{原式} \Big|_{x=2} = 1$$

$$B = (x-1) \cdot \text{原式} \Big|_{x=1} = -1$$

$$\therefore y = \frac{1}{x-2} - \frac{1}{x-1}$$

$$y^{(n)} = (-1)^n n! \left[\frac{1}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}} \right]$$



$$(4) \quad y = \sin^6 x + \cos^6 x$$

解: $y = (\sin^2 x)^3 + (\cos^2 x)^3$

$$= \sin^4 x - \sin^2 x \cos^2 x + \cos^4 x$$

$$= (\sin^2 x + \cos^2 x)^2 - 3\sin^2 x \cos^2 x$$

$$= 1 - \frac{3}{4} \sin^2 2x$$

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$$

$$= \frac{5}{8} + \frac{3}{8} \cos 4x$$

$$y^{(n)} = \frac{3}{8} \cdot 4^n \cos(4x + n\frac{\pi}{2})$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$



2. (填空题) (1) 设 $f(x) = (x^2 - 3x + 2)^n \cos \frac{\pi x^2}{16}$, 则

$$f^{(n)}(2) = \underline{n! \frac{\sqrt{2}}{2}}$$

提示: $f(x) = (x-2)^n (x-1)^n \cos \frac{\pi x^2}{16}$

各项均含因子 $(x-2)$

$$f^{(n)}(x) = n! (x-1)^n \cos \frac{\pi x^2}{16} + \dots$$

(2) 已知 $f(x)$ 任意阶可导, 且 $f'(x) = [f(x)]^2$, 则当

$$n \geq 2 \text{ 时 } f^{(n)}(x) = \underline{n! [f(x)]^{n+1}}$$

提示: $f''(x) = 2f(x)f'(x) = 2![f(x)]^3$

$$f'''(x) = 2! \cdot 3[f(x)]^2 f'(x) = 3![f(x)]^4$$

.....



3. 试从 $\frac{dx}{dy} = \frac{1}{y'}$ 导出 $\frac{d^2 x}{d y^2} = -\frac{y''}{(y')^3}$.

解:
$$\begin{aligned}\frac{d^2 x}{d y^2} &= \frac{d}{d y} \left(\frac{dx}{dy} \right) = \frac{d}{d x} \left(\frac{1}{y'} \right) \cdot \frac{dx}{dy} \\ &= -\frac{y''}{(y')^2} \cdot \frac{1}{y'} = -\frac{y''}{(y')^3}\end{aligned}$$

同样可求 $\frac{d^3 x}{d y^3}$

作业

P97 97,101,104,110,113,115, 118



备用题

设 $y = x^2 f(\sin x)$ 求 y'' , 其中 f 二阶可导.

解: $y' = 2x \cdot f(\sin x) + x^2 \cdot f'(\sin x) \cdot \cos x$

$$y'' = (2x f(\sin x))' + (x^2 f'(\sin x) \cos x)'$$

$$= 2f(\sin x) + 2x \cdot f'(\sin x) \cdot \cos x$$

$$+ 2x f'(\sin x) \cos x + x^2 f''(\sin x) \cos^2 x$$

$$+ x^2 f'(\sin x)(-\sin x)$$

$$= 2f(\sin x) + (4x \cos x - x^2 \sin x) f'(\sin x)$$

$$+ x^2 \cos^2 x f''(\sin x)$$

