PLSC 504 – Fall 2020 Endogenous Selection and Potential Outcomes

September 16, 2020

Sample Selection In Theory

- Challenge: Inference to a Population from a Non-Random Sample
- Widespread Problem...
  - Heckman's wage equations...
  - Self-selection (e.g., into groups)
  - Surveys: "Screening" questions (<u>sometimes</u>...)
- Parallels in Missing Data, Causal/Counterfactual Inference

Observe:

$$
\begin{aligned}
Y_{1i}^* &= \mathbf{X}_i\boldsymbol{\beta} + u_{1i} \\
Y_{2i}^* &= \mathbf{Z}_i\gamma + u_{2i}
\end{aligned}
$$

$$
Y_{1i} = \begin{cases} Y_{1i}^* \text{ if } Y_{2i}^* > 0 \\ \text{missing if } Y_{2i}^* \leq 0 \end{cases}
$$

- Y_{2i}^* unobserved (except for sign);
- $\mathbf{X}_i$ observed iff Y_{1i} is observed;
- $\mathbf{Z}_i$ observed in every case.

$$
\begin{aligned}
\Pr(Y_{2i}^* \leq 0 | \mathbf{X}, \mathbf{Z}) &= \Pr(u_{2i} \leq -\mathbf{Z}_i \gamma) \\
&= 1 - \Pr(u_{2i} \geq -\mathbf{Z}_i \gamma) \\
&= 1 - \Pr(-u_{2i} \leq \mathbf{Z}_i \gamma) \\
&= 1 - \int_{-\infty}^{\mathbf{Z}_i \gamma} f(u_2) du_2 \\
&= 1 - F_{u_2}(\mathbf{Z}_i \gamma)
\end{aligned}
$$

Define:

$$D_i = \begin{cases} 1 & \text{if } Y_{1i} \text{ is observed.} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\Pr(D_i = 1) = F_{u_2}(\mathbf{Z}_i \gamma).$$

An Assumption

Assume:

$$\{u_1, u_2\} \sim \mathcal{BVN}(0, 0, \sigma_1^2, 1, \sigma_{12})$$

Means

$$\Pr(D_i = 1 | \mathbf{Z}_i, \mathbf{X}_i) = \Phi(\mathbf{Z}_i \gamma).$$

Define:

$$\rho = \text{corr}(u_1, u_2).$$

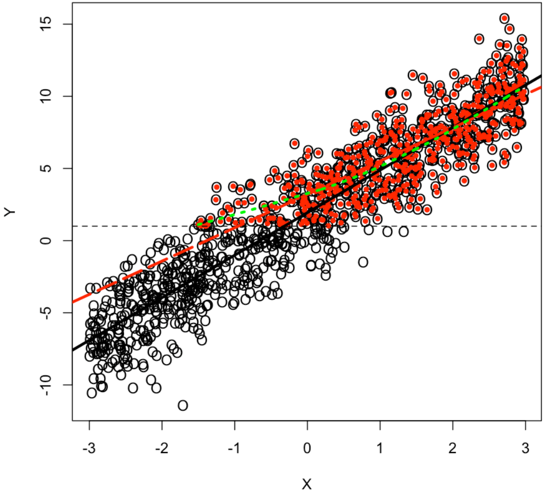
What we get:

$$\mathrm{E}(Y_{1i}|\mathbf{X}_i, \mathbf{Z}_i, D_i = 1) = \mathbf{X}_i\boldsymbol{\beta} + \rho\sigma_1 \left[\frac{\phi(\mathbf{Z}_i\gamma)}{\Phi(\mathbf{Z}_i\gamma)}\right]$$

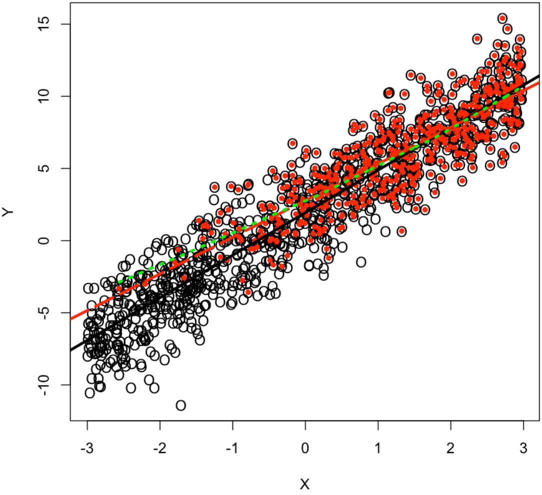
Without conditioning on $\mathbf{Z}$:

$$\mathrm{E}(Y_{1i}|\mathbf{X}_i, D_i = 1) = \mathbf{X}_i\boldsymbol{\beta} + \mathrm{E}\left\{\rho\sigma_1 \left[\frac{\phi(\mathbf{Z}_i\gamma)}{\Phi(\mathbf{Z}_i\gamma)}\right]\middle|\mathbf{X}_i\right\}$$

Truncation Bias



Sample Selection Bias



Selection Bias: Substantive Effects

- Specification Error (unless $\rho = 0$)
- Indeterminate bias in $\hat{\boldsymbol{\beta}}$
- Including $\mathbf{Z}_i$ will not generally* remove the bias
- Bias remains even if inference is limited to the "selected" group. (This point is made nicely in Berk (1983)...)

*...unless sample selection is completely deterministic (i.e., determined by $\mathbf{X}$, $\mathbf{Z}$) (Heckman & Robb 1985).

E(Y) Under Selection

Conditional Density:

$$h(Y|\mathbf{X}, \mathbf{Z}, \boldsymbol{\beta}, \gamma, \sigma_1, \rho) = \frac{\phi\left(\frac{Y_{1i} - \mathbf{X}_i\boldsymbol{\beta}}{\sigma_1}\right)}{\sigma_1 \Phi(\mathbf{Z}_i\gamma)} \cdot \Phi\left[\frac{\frac{\rho(Y_{1i} - \mathbf{X}_i\boldsymbol{\beta})}{\sigma_1} + \mathbf{Z}_i\gamma}{\sqrt{1-\rho^2}}\right]$$

Note: $\rho = 0$ yields

$$\begin{aligned} h(Y|\mathbf{X}, \mathbf{Z}, \boldsymbol{\beta}, \gamma, \sigma_1, \rho = 0) &= \frac{\phi\left(\frac{Y_{1i} - \mathbf{X}_i\boldsymbol{\beta}}{\sigma_1}\right)}{\sigma_1 \Phi(\mathbf{Z}_i\gamma)} \cdot \Phi\left[\frac{0 + \mathbf{Z}_i\gamma}{1}\right] \\ &= \frac{\phi\left(\frac{Y_{1i} - \mathbf{X}_i\boldsymbol{\beta}}{\sigma_1}\right)}{\sigma_1}. \end{aligned}$$

$$\begin{aligned}\ln L(\boldsymbol{\beta}, \gamma, \sigma_1, \rho | Y_1) &= \sum_{i=1}^{N} (1 - D_i) \ln[1 - \Phi(\mathbf{Z}_i \gamma)] \\ &+ \sum_{i=1}^{N} D_i \ln[\Phi(\mathbf{Z}_i \gamma)] \\ &+ \sum_{i=1}^{N} D_i \ln \left\{ \frac{\phi\left(\frac{Y_{1i} - \mathbf{X}_i \boldsymbol{\beta}}{\sigma_1}\right)}{\sigma_1 \Phi(\mathbf{Z}_i \gamma)} \cdot \Phi\left[\frac{\frac{\rho(Y_{1i} - \mathbf{X}_i \boldsymbol{\beta})}{\sigma_1} + \mathbf{Z}_i \gamma}{\sqrt{1 - \rho^2}}\right] \right\}\end{aligned}$$

- MLE (above)
- Or, reconsider:

$$\mathrm{E}(Y_{1i}|\mathbf{X}_i, \mathbf{Z}_i, D_i = 1) = \mathbf{X}_i\boldsymbol{\beta} + \rho\sigma_1 \left[\frac{\phi(\mathbf{Z}_i\gamma)}{\Phi(\mathbf{Z}_i\gamma)}\right]$$

- Note that $\Phi(\mathbf{Z}_i\gamma) = \Pr(D_i = 1)$
- Suggests a two-step approach...

Heckman's Two-Step Estimator

1. Estimate $\hat{\gamma}$ from

   $$\Pr(D_i = 1) = \Phi(\mathbf{Z}_i\gamma)$$

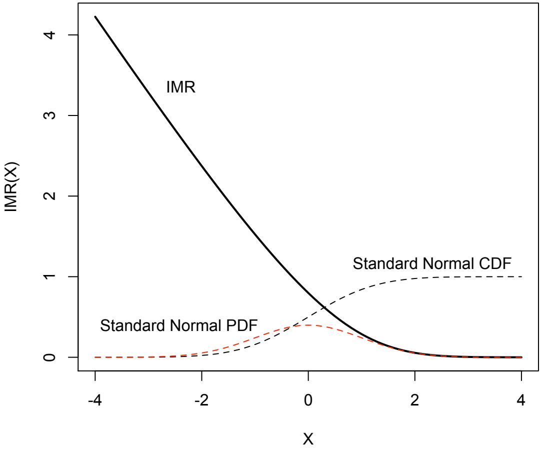
   and calculate the estimated inverse Mills' ratio:

   $$\hat{\lambda}_i = \frac{\phi(\mathbf{Z}_i\hat{\gamma})}{\Phi(-\mathbf{Z}_i\hat{\gamma})}$$

2. Estimate $\boldsymbol{\beta}, \theta(\equiv \rho\sigma_1)$ as:

   $$Y_{1i} = \mathbf{X}_i\boldsymbol{\beta} + \theta\hat{\lambda}_i + u_{1i}$$

What exactly *is* an "inverse Mills' ratio," anyway?



A Few Things...

- Since $\sigma_1 > 0$, $\hat{\theta} = 0 \implies \rho = 0$
- Two-step approach:
  - Is "LIML"...
  - Consistent for $\hat{\boldsymbol{\beta}}$, <u>but</u>
  - Inconsistent estimating $\widehat{\mathbf{V}(\boldsymbol{\beta})}$; so
  - Standard errors require correction (e.g., bootstrap)
  - *Can* yield $\hat{\rho} \notin [-1, 1]$ (because $\hat{\rho} = \hat{\theta}/\hat{\sigma}_1$)
  - Sensitive to prediction of D_i (better prediction = better precision)

- If $\mathbf{X} = \mathbf{Z}$, then $\boldsymbol{\beta}, \gamma, \rho$ (formally) identified by nonlinearity of $\Phi(\cdot)$
- (Much) better: $\geq$ one covariate in $\mathbf{Z}$ not in $\mathbf{X}$
- But...
  - Factors causing Y_1 also (often) cause D
  - $\implies \mathbf{X}, \mathbf{Z}$ highly correlated
  - ...just makes things worse (Stolzenberg and Relles 1997)

Some Practical Things

- In practice, few people use two-step anymore,
- Sensitive to joint normality of $\{u_i, u_2\}$,
- Very sensitive to model specification...
- Key issue: endogeneity of selection...

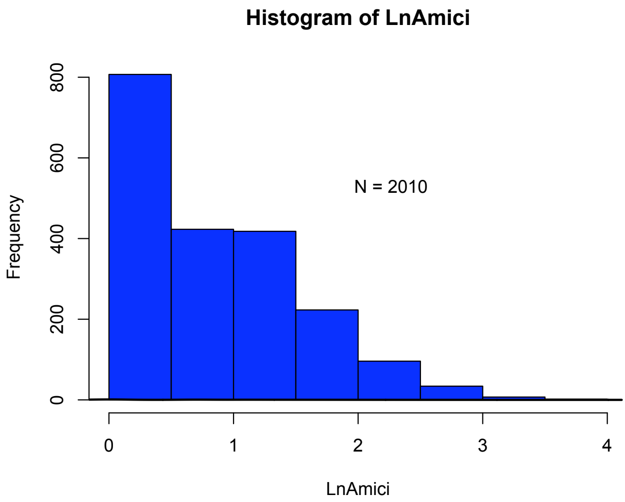
Example: SCOTUS Amicus Briefs

- `LnAmici` $= \ln(\#$ of briefs filed$)$
- For this to be defined, `Amici` > 0...
- Covariates:
  - `Year` -1900
  - `USPartic`: 1 if U.S. participated, 0 otherwise
  - `SCscore`: SCOTUS "Segal-Cover" liberalism score
  - `MultipleLegal`: 1 if multiple legal issues, 0 otherwise
  - `SGAmicus`: 1 if SG filed a brief, 0 otherwise

SCOTUS Decisions, 1953-1985

```
> summary(SCOTUS)
       ID               Docket                 Amici              LnAmici     
 Min.   : 920764   Length:7156          Min.   : 0.0000   Min.   :0.000  
 1st Qu.:3790359   Class :character     1st Qu.: 0.0000   1st Qu.:0.000  
 Median :4100519   Mode  :character     Median : 0.0000   Median :0.693  
 Mean   :4116116                        Mean   : 0.8425   Mean   :0.757  
 3rd Qu.:4460624                        3rd Qu.: 1.0000   3rd Qu.:1.386  
 Max.   :4781050                        Max.   :39.0000   Max.   :3.664  
                                                          NA's   :5146   
      Year          USPartic          FedPetit          FedResp     
 Min.   :53.00   Min.   :0.0000   Min.   :0.0000   Min.   :1.000  
 1st Qu.:65.00   1st Qu.:0.0000   1st Qu.:0.0000   1st Qu.:3.000  
 Median :73.00   Median :0.0000   Median :0.0000   Median :3.000  
 Mean   :71.93   Mean   :0.3707   Mean   :0.1722   Mean   :2.593  
 3rd Qu.:80.00   3rd Qu.:1.0000   3rd Qu.:0.0000   3rd Qu.:3.000  
 Max.   :86.00   Max.   :1.0000   Max.   :1.0000   Max.   :3.000  

    SGAmicus           SCscore          MultipleLegal        select      
 Min.   :0.00000   Min.   :-0.22444   Min.   :0.000   Min.   :0.0000  
 1st Qu.:0.00000   1st Qu.:-0.12444   1st Qu.:0.000   1st Qu.:0.0000  
 Median :0.00000   Median :-0.01778   Median :0.000   Median :0.0000  
 Mean   :0.07868   Mean   : 0.13250   Mean   :0.149   Mean   :0.2809  
 3rd Qu.:0.00000   3rd Qu.: 0.47667   3rd Qu.:0.000   3rd Qu.:1.0000  
 Max.   :1.00000   Max.   : 0.66222   Max.   :1.000   Max.   :1.0000  
```


Histogram of LnAmici
Frequency
0
200
400
600
800
N = 2010
0
1
2
3
4
LnAmici

Estimates: OLS

```
> OLS<-lm(LnAmici~Year+USPartic+MultipleLegal+SCscore,data=SCOTUS)
> summary(OLS)

Residuals:
    Min      1Q  Median      3Q     Max
-1.2328 -0.5837 -0.1223  0.4614  3.0901

Coefficients:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)   -0.737133   0.314843  -2.341   0.0193 *
Year           0.020168   0.004134   4.879 1.15e-06 ***
USPartic      -0.174420   0.034968  -4.988 6.62e-07 ***
MultipleLegal  0.199667   0.038331   5.209 2.09e-07 ***
SCscore       -0.159575   0.117648  -1.356   0.1751
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.7275 on 2005 degrees of freedom
  (5151 observations deleted due to missingness)
Multiple R-squared: 0.1003,Adjusted R-squared: 0.09854
F-statistic:  55.9 on 4 and 2005 DF,  p-value: < 2.2e-16
```

Estimates: Probit (Selection)

```
> SCOTUS$D<-SCOTUS$Amici>0
> probit<-glm(D~Year+USPartic+SCscore+MultipleLegal,data=SCOTUS,
  family=binomial(link="probit"))
> summary(probit)

Coefficients:
               Estimate Std. Error z value Pr(>|z|)
(Intercept)   -2.558970   0.273964  -9.341  < 2e-16 ***
Year           0.026875   0.003602   7.462 8.54e-14 ***
USPartic      -0.164948   0.034408  -4.794 1.64e-06 ***
SCscore       -0.089525   0.103323  -0.866    0.386
MultipleLegal  0.565585   0.043171  13.101  < 2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 8498.3  on 7155  degrees of freedom
Residual deviance: 8025.2  on 7151  degrees of freedom
  (5 observations deleted due to missingness)
AIC: 8035.2
```

Estimates: Two-Step ("By-Hand")

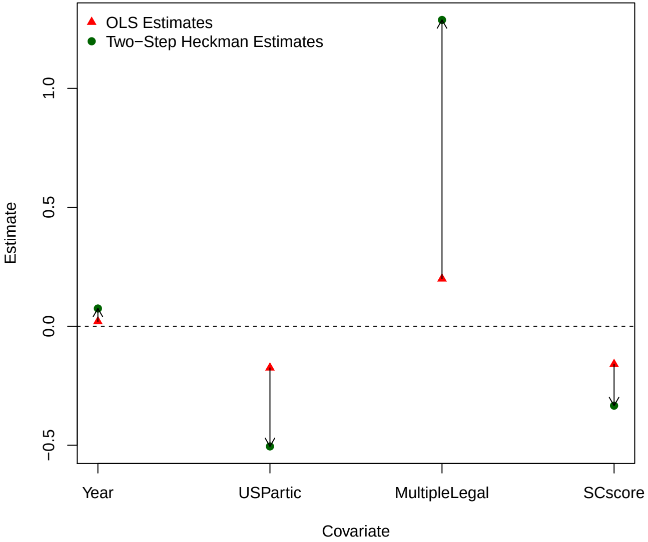
```
> SCOTUS$IMR<-((1/sqrt(2*pi))*exp(-((probit$linear.predictors)^2/2))) /
  pnorm(probit$linear.predictors)
> OLS.2step<-lm(LnAmici~Year+USPartic+MultipleLegal+SCscore+IMR,data=SCOTUS)
> summary(OLS.2step)

Call:
lm(formula = LnAmici ~ Year + USPartic + MultipleLegal + SCscore +
    IMR, data = Day17)

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)   -8.07914    3.58519  -2.253  0.02434 *
Year           0.07478    0.02688   2.782  0.00546 **
USPartic      -0.50500    0.16456  -3.069  0.00218 **
MultipleLegal  1.28738    0.53048   2.427  0.01532 *
SCscore       -0.33374    0.14490  -2.303  0.02137 *
IMR            2.75326    1.33926   2.056  0.03993 *
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.7269 on 2004 degrees of freedom
  (5146 observations deleted due to missingness)
Multiple R-squared: 0.1022,Adjusted R-squared: 0.09999
F-statistic: 45.64 on 5 and 2004 DF,  p-value: < 2.2e-16
```

OLS vs. (Two-Step) Heckman $\hat{\beta}$s

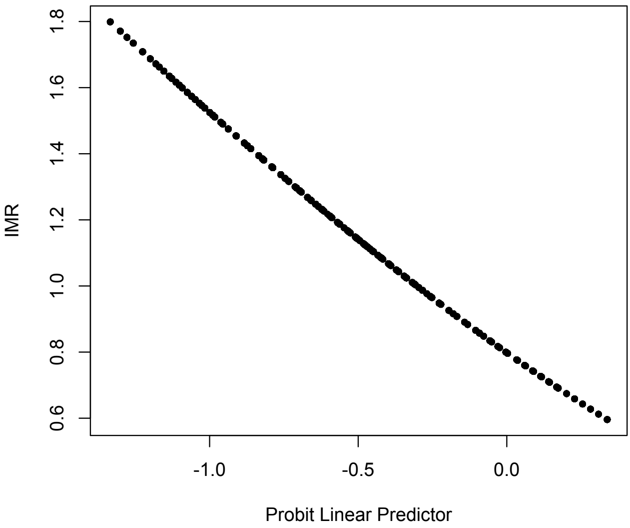


Estimates: Two-Step (Bad Specification)

```
> heckman2S<-heckit(D~Year+USPartic+SCscore+MultipleLegal, LnAmici~Year+USPartic
+SCscore+MultipleLegal,data=SCOTUS,method="2step")
> summary(heckman2S)
--------------------------------------------
Tobit 2 model (sample selection model)
2-step Heckman / heckit estimation
7156 observations (5146 censored and 2010 observed) and 13 free parameters (df = 7144)

Probit selection equation:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)   -2.558971   0.275385  -9.292  < 2e-16 ***
Year           0.026875   0.003622   7.420 1.31e-13 ***
USPartic      -0.164948   0.034366  -4.800 1.62e-06 ***
SCscore       -0.089524   0.103873  -0.862    0.389
MultipleLegal  0.565585   0.043298  13.063  < 2e-16 ***
Outcome equation:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)   -8.07914    4.56334  -1.770   0.0767 .
Year           0.07478    0.03499   2.137   0.0326 *
USPartic      -0.50500    0.21993  -2.296   0.0217 *
SCscore       -0.33374    0.25058  -1.332   0.1829
MultipleLegal  1.28738    0.67647   1.903   0.0571 .

Multiple R-Squared:0.1022,Adjusted R-Squared:0.1
Error terms:
              Estimate Std. Error t value Pr(>|t|)
invMillsRatio    2.753      1.668    1.65   0.0989 .
sigma            2.447         NA      NA       NA
rho              1.125         NA      NA       NA
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
--------------------------------------------
```



IMR
1.8
1.6
1.4
1.2
1.0
0.8
0.6
-1.0
-0.5
0.0
Probit Linear Predictor

Estimates: MLE (Bad Specification)

```
> heckmanML<-heckit(D~Year+USPartic+SCscore+MultipleLegal,
                   LnAmici~Year+USPartic+SCscore+MultipleLegal,
                   data=SCOTUS,method="ml")

> summary(heckmanML)

--------------------------------------------
Tobit 2 model (sample selection model)
Maximum Likelihood estimation
Newton-Raphson maximisation, 4 iterations
Return code 3: Last step could not find a value above the current.
Boundary of parameter space?
Consider switching to a more robust optimisation method temporarily.
Log-Likelihood: -6424.647
7156 observations (5146 censored and 2010 observed)
12 free parameters (df = 7144)

.
.
.
```

Estimates: MLE (Poor Specification)

```
Probit selection equation:
               Estimate Std. error t value  Pr(> t)
(Intercept)   -2.559549   0.331857  -7.713 1.23e-14 ***
Year           0.026862   0.004367   6.151 7.72e-10 ***
USPartic      -0.165173   0.043585  -3.790 0.000151 ***
SCscore       -0.090504   0.125536  -0.721 0.470946
MultipleLegal  0.566437   0.058852   9.625  < 2e-16 ***

Outcome equation:
              Estimate Std. error t value  Pr(> t)
(Intercept)   -8.06266    0.88402  -9.120  < 2e-16 ***
Year           0.08519    0.01182   7.205 5.80e-13 ***
USPartic      -0.49013    0.10103  -4.851 1.23e-06 ***
SCscore       -0.29510    0.34156  -0.864    0.388
MultipleLegal  1.26060    0.10607  11.885  < 2e-16 ***

Error terms:
      Estimate Std. error t value Pr(> t)
sigma  2.11218         NA      NA      NA
rho    0.99993    0.00742   134.8  <2e-16 ***
---
Signif. codes:
0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
--------------------------------------------
Warning messages:
1: In sqrt(diag(vc)) : NaNs produced
2: In sqrt(diag(vc)) : NaNs produced
```

Estimates: MLE ("Better" Specification)

```
> betterML<-heckit(D~Year+USPartic+SCscore+MultipleLegal+SGAmicus,
           LnAmici~Year+USPartic+SCscore+MultipleLegal,
           data=SCOTUS,method="ml")

> summary(betterML)

--------------------------------------------
Tobit 2 model (sample selection model)
Maximum Likelihood estimation
Newton-Raphson maximisation, 3 iterations
Return code 1: gradient close to zero
Log-Likelihood: -5689.492
7156 observations (5146 censored and 2010 observed)
13 free parameters (df = 7143)

.
.
.
```

Estimates: MLE ("Better" Specification)

```
Probit selection equation:
               Estimate Std. error t value  Pr(> t)
(Intercept)   -2.670268   0.289236  -9.232  < 2e-16 ***
Year           0.024971   0.003804   6.565 5.21e-11 ***
USPartic       0.080486   0.036022   2.234   0.0255 *
SCscore       -0.091135   0.109363  -0.833   0.4047
MultipleLegal  0.518324   0.045625  11.361  < 2e-16 ***
SGAmicus       2.167694   0.082758  26.193  < 2e-16 ***

Outcome equation:
               Estimate Std. error t value  Pr(> t)
(Intercept)   -0.177121   0.326280  -0.543 0.587233
Year           0.015413   0.004188   3.681 0.000233 ***
USPartic      -0.104100   0.036572  -2.846 0.004421 **
SCscore       -0.167759   0.117178  -1.432 0.152242
MultipleLegal  0.130377   0.039958   3.263 0.001103 **

Error terms:
      Estimate Std. error t value  Pr(> t)
sigma  0.73923    0.01270  58.199  < 2e-16 ***
rho   -0.29103    0.04419  -6.586 4.53e-11 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
--------------------------------------------
```

- Selection + binary second stage ($Y_i \in \{0,1\}$) (a/k/a "Heckit").
- Assume errors are bivariate standard Normal [so, $\{u_1, u_2 \sim \mathcal{BVN}(0,0,1,1,\rho) \equiv \Phi_2(\cdot)]$
- Log-Likelihood:

$$
\begin{aligned}
\ln L(\boldsymbol{\beta}, \gamma, \sigma_1, \rho | Y_1) \quad = \quad & \sum_{Y_{1i}=1, D_i=1} \ln[\Phi_2(\mathbf{X}_i\boldsymbol{\beta}, \mathbf{Z}_i\gamma, \rho)] \\
& + \sum_{Y_{1i}=0, D_i=1} \ln[\Phi_2(-\mathbf{X}_i\boldsymbol{\beta}, \mathbf{Z}_i\gamma, -\rho)] \\
& + \sum_{D_i=0} \ln \Phi(-\mathbf{Z}_i\gamma)
\end{aligned}
$$

- Different outcome stages:
  - Poisson (Greene 1995)
  - Durations (Boehmke et al. 2006)
  - Count/binary/ordinal (Mirand and Rabe-Hesketh 2005)

- Selection stage is <u>ordered</u> (Chiburis & Lokshin 2007)

- Multiple-stage models (not much... work in finance + Signorino and others)

Sample Selection: Software

- R (`selection` and `heckit` in `sampleSelection` package)
  - Binary selection
  - Continuous/binary outcomes
  - Also tobit, etc. models
- Stata
  - `heckman` (binary-continuous model)
  - `heckprob` (binary-binary model)
  - `oheckman` (ordered-continuous)
  - `dursel` (binary-duration model)
  - `gllamm` (various multilevel models w/selection)

Further Readings: References

Articles by Heckman (1974, 1976, 1979).

Breen, Richard. 1996. Regression Models for Censored, Sample Selected, or Truncated Data. Thousand Oaks, CA: Sage.

Stolzenberg, Ross M. and Daniel A. Relles. 1997. "Tools for Intuition about Sample Selection Bias and Its Correction." American Sociological Review 62:494-507.

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Winship, Christopher and Robert D. Mare. 1992. "Models for Sample Selection Bias." Annual Review of Sociology 18:327-350.

Further Readings: Applications

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- Blanton, Shannon Lindsey. 2000. "Promoting Human Rights and Democracy in the Developing World: U.S. Rhetoric versus U.S. Arms Exports." *American Journal of Political Science* 44:123-131.
- Hart, David M. 2001. "Why Do Some Firms Give? Why Do Some Give a Lot?: High-Tech PACs, 1977-1996." *The Journal of Politics* 63:1230-1249.
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Potential Outcomes and Counterfactual Inference

The goal: **Making causal inferences from observational data.**

- Establish and measure the *causal* relationship between variables in a non-experimental setting.
- The *fundamental problem of causal inference*:

  It is impossible to observe the causal effect of a treatment / predictor on a single unit.

- Specific challenges:
  - *Confounding*
  - *Selection bias*
  - *Heterogenous treatment effects*

Causation and Counterfactuals

Causal statements imply counterfactual reasoning.

- "If the cause(s) had been different, the outcome(s) would be different, too."
- Conditioning, probabilistic and causal:

Probabilistic conditioning	Causal conditioning
$\Pr(Y \mid X = x)$	$\Pr[Y \mid do(X = x)]$
Factual	Counterfactual
Select a sub-population	Generate a new population
Predicts passive observation	Predicts active manipulation
Calculate from full DAG*	Calculate from surgically-altered DAG*
Always identifiable when X and Y are observable	Not always identifiable even when X and Y are observable

*See below. Source: Swiped from Shalizi, "Advanced Data Analysis from an Elementary Point of View", Table 23.1.

- Causality (typically) implies / requires:
  - *Temporal ordering*
  - *Mechanism*
  - *Correlation*

Notation

- N observations indexed by i, $i \in \{1, 2, ...N\}$
- Outcome variable Y
- Interest: the effect on Y of a treatment variable W:
  - $W_i = 1 \leftrightarrow$ observation i is "treated"
  - $W_i = 0 \leftrightarrow$ observation i is "control"

Potential Outcomes

- Y_{0i} = the value of Y_i if $W_i = 0$
- Y_{1i} = the value of Y_i if $W_i = 1$
- $\delta_i = (Y_{1i} - Y_{0i})$ = the treatment effect of W

The average treatment effect (ATE) is just:

$$
\begin{aligned}
\text{ATE} \equiv \bar{\delta} &= \mathrm{E}(Y_{1i} - Y_{0i}) \\
&= \frac{1}{N} \sum_{i=1}^{N} Y_{1i} - Y_{0i}.
\end{aligned}
$$

BUT we observe only Y_i:

$$
Y_i = \begin{cases} Y_{0i} \text{ if } W_i = 0, \\ Y_{1i} \text{ if } W_i = 1. \end{cases}
$$

or (equivalently)

$$
Y_i = W_i Y_{1i} + (1 - W_i) Y_{0i}.
$$

Key to estimating treatment effects: **Assignment mechanism for** W.

Neyman/Rubin/Holland: Treat inability to observed Y_{0i} / Y_{1i} as a missing data problem.

[press "pause"]

Notation:

$$\underset{N\times k}{\mathbf{X}_i} \cup \{\mathbf{W}_i, \mathbf{Z}_i\}$$

$\mathbf{W}_i$ have some missing values,
$\mathbf{Z}_i$ are "complete"

$$R_{ik} = \begin{cases} 1 & \text{if } W_{ik} \text{ is missing,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\pi_{ik} = \Pr(R_{ik} = 1)$$

Rubin's flavors of missingness:

- Missing completely at random ("MCAR") (= "ignorable"):

$$\mathbf{R} \perp \{\mathbf{Z}, \mathbf{W}\}$$

- Missing at random ("MAR") (conditionally "ignorable"):

$$\mathbf{R} \perp \mathbf{W} | \mathbf{Z}$$

- Anything else is "informatively" (or "non-ignorably") missing.

Key to estimating treatment effects: **Assignment mechanism for W.**

Neyman/Rubin/Holland: Treat inability to observed Y_{0i} / Y_{1i} as a missing data problem.

- If the "missingness" due to the value of W_i is orthogonal to the values of Y, then it is ignorable. Formally:

$$\Pr(W_i|\mathbf{X}_i, Y_{0i}, Y_{1i}) = \Pr(W_i|\mathbf{X}_i)$$

- If that "missingness" is non-orthogonal, then it is not ignorable, and can lead to bias in estimation
- Non-ignorable assignment of W requires understanding the mechanism by which that assignment occurs

One more thing: the stable unit-treatment value assumption ("SUTVA")

- Requires that there be two and only two possible values of Y for each observation i...

- "the observation (of Y_i) on one unit should be unaffected by the particular assignment of treatments to the other units."

- $\equiv$ the "assumption of no interference between units," meaning:
  - Values of Y for any two i, j ($i \neq j$) observations do not depend on each other
  - Treatment effects are homogenous within categories defined by W

Treatment Effects Under Randomization of W

If W_i is assigned randomly, then:

$$\Pr(W_i) \perp Y_{0i}, Y_{1i}$$

and so:

$$\Pr(W_i | Y_{0i}, Y_{1i}) = \Pr(W_i) \,\forall\, Y_{0i}, Y_{1i}.$$

This means that the "missing" data on Y_0/Y_1 are ignorable (here, in the special case where the $\mathbf{X}_i$ on which W_i depends is null). This in turn means that:

$$f(Y_{0i}|W_i = 0) = f(Y_{0i}|W_i = 1) = f(Y_i|W_i = 0) = f(Y_i|W_i = 1)$$

and

$$f(Y_{1i}|W_i = 0) = f(Y_{1i}|W_i = 1) = f(Y_i|W_i = 0) = f(Y_i|W_i = 1)$$

Randomized W (continued)

Implication: Y_{0i} and Y_{1i} are (not identical but) *exchangeable*...

This in turn means that:

$$E(Y_{0i}|W_i) = E(Y_{1i}|W_i)$$

and so

$$\begin{aligned}
\widehat{\text{ATE}} &= \text{E}(Y_i|W_i = 1) - \text{E}(Y_i|W_i = 0) \\
&= \bar{Y}_{W=1} - \bar{Y}_{W=0}.
\end{aligned}$$

will be an unbiased estimate of the ATE.

Observational Data: W Depends on $\mathbf{X}$

Formally,

$$Y_{0i}, Y_{1i} \perp W_i | \mathbf{X}_i.$$

Here,

- $\mathbf{X}$ are *known confounders* that (stochastically) determine the value of W_i,
- Conditioning on $\mathbf{X}$ is necessary to achieve exchangeability.

So long as W is entirely due to $\mathbf{X}$, we can condition:

$$f(Y_{1i}|\mathbf{X}_i, W_i = 1) = f(Y_{1i}|\mathbf{X}_i, W_i = 0) = f(Y_i|\mathbf{X}_i, W_i)$$

and similarly for Y_{0i}.

W Depends on $\mathbf{X}$ (continued)

Estimands:

- the *average treatment effect for the treated* (ATT):

$$\text{ATT} = \text{E}(Y_{1i}|W_i = 1) - \text{E}(Y_{0i}|W_i = 1).$$

- the *average treatment effect for the controls* (ATC):

$$\text{ATC} = \text{E}(Y_{1i}|W_i = 0) - \text{E}(Y_{0i}|W_i = 0).$$

Corresponding estimates:

$$\widehat{\text{ATT}} = \text{E}\{[\text{E}(Y_i|\mathbf{X}_i, W_i = 1) - \text{E}(Y_i|\mathbf{X}_i, W_i = 0)]|W_i = 1\}.$$

and

$$\widehat{\text{ATC}} = \text{E}\{[\text{E}(Y_i|\mathbf{X}_i, W_i = 1) - \text{E}(Y_i|\mathbf{X}_i, W_i = 0)]|W_i = 0\}.$$

Note that in both cases **the expectation of the whole term is conditioned on W_i**.

Confounding occurs when one or more observed or unobserved factors $\mathbf{X}$ affect the causal relationship between W and Y.

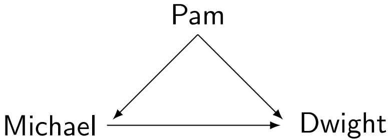
Formally, confounding requires that:

- $\text{Cov}(\mathbf{X}, W) \neq 0$ (the confounder is associated with the "treatment")
- $\text{Cov}(\mathbf{X}, Y) \neq 0$ (the confounder is associated with the outcome)
- $\mathbf{X}$ does not "lie on the path" between W and Z (that is, $\mathbf{X}$ is not affected by either W or Y).

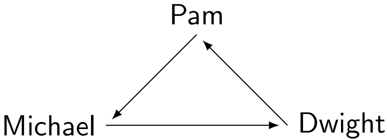
Directed acyclic graphs (DAGs) are a tool for visualizing and interpreting structural/causal phenomena.

- DAGs comprise:
  - Nodes (typically, variables / phenomena) and
  - Edges (or lines; typically, relationships/causal paths).
- Directed means each edge is *unidirectional*.
- Acyclical means exactly what it suggests: If a graph has a "feedback loop," it is not a DAG.
- Read more at the Wikipedia page, or at this useful page.

Know your DAG



A DAG



Not a DAG

DAGs and Confounding

$$W \longrightarrow Y \longleftarrow X$$

No Confounding

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ W \arrow[rr, dashed] & & Y \end{tikzcd}$$

Confounding

Confounding Bias: Some Toy Examples

Example One: Cov$(W, Y) = 0$ (ATE=2)

i	W_i	Y_{0i}	Y_{1i}	$Y_{1i} - Y_{01}$	Y_i	$(\bar{Y}\|W = 1) - (\bar{Y}\|W = 0)$
1	0	8	(10)	(2)	8	-
2	0	10	(12)	(2)	10	-
3	0	12	(14)	(2)	12	-
4	1	(8)	10	(2)	10	-
5	1	(10)	12	(2)	12	-
6	1	(12)	14	(2)	14	-
Mean_{obs}	-	10	12	-	11	2
Mean_{all}	-	(10)	(12)	(2)	-	-

$t = -1.22$, $p = 0.14$

Confounding Bias: Some Toy Examples

Example Two: Cov$(W, Y) > 0$ (ATE=2)

i	W_i	Y_{0i}	Y_{1i}	$Y_{1i} - Y_{01}$	Y_i	$(\bar{Y}\vert W=1) - (\bar{Y}\vert W=0)$
1	0	8	(10)	(2)	8	-
2	0	8	(10)	(2)	8	-
3	0	10	(12)	(2)	10	-
4	1	(10)	12	(2)	12	-
5	1	(12)	14	(2)	14	-
6	1	(12)	14	(2)	14	-
Mean$_{\text{obs}}$	-	8.67	13.33	-	11	4.67
Mean$_{\text{all}}$	-	(10)	(12)	(2)	-	-

$t = -4.95$, $p < 0.001$

Confounding Bias: Some Toy Examples

Example Three: $\text{Cov}(W, Y) < 0$ (ATE=2)

i	W_i	Y_{0i}	Y_{1i}	$Y_{1i} - Y_{01}$	Y_i	$(\bar{Y}\|W=1) - (\bar{Y}\|W=0)$
1	0	12	(14)	(2)	12	-
2	0	12	(14)	(2)	12	-
3	0	10	(12)	(2)	10	-
4	1	(10)	12	(2)	12	-
5	1	(8)	10	(2)	10	-
6	1	(8)	10	(2)	10	-
Mean_{obs}	-	11.33	10.67	-	11	-0.67
Mean_{all}	-	(10)	(12)	(2)	-	-

$t = 0.71,\ p = 0.74$

Next time: How to make causal(-ish) inferences from observational data...