Name: ....

computational mathematics

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Numerik I – Midterm

Exercises marked with * give extra points.

Exercise 1 (Computational task)
Let a be a real positive parameter and $f_a : [-1, 1] \to \mathbb{R}$ the given function $f_a(x) := \arctan(a \sin(x))$. For $n \in \mathbb{N}$ let $p_{a,n} \in \mathbb{P}_n$ be the interpolation polynomial with the following properties

$$p_{a,n}(x_i) = f_a(x_i) \qquad \text{for nodes } x_i = -1 + \frac{2i}{n} \qquad 0 \leq i \leq n.$$

You find the function computing the divided differences and the corresponding interpolation polynomial in your folder (**Newton_Interpol.m** or **Newton_Interpol.py**)

(a) (3 points) Plot the functions f_a and $p_{a,n}$ for the parameter $a = 1$ and polynomial degrees $n = 4, 6, 8, 10, 12$. Store the figure for $n = 8$ as **fig1a.eps**). Compute the error

$$E_{a,n} := \max_{-500 \leq j \leq 500} \left| (f_a - p_{a,n}) \left(\frac{j}{500} \right) \right|$$

with respect to n into a table. Is the decay of the error exponential or polynomial with respect to n?

(b) (1 point) Estimate, through computer experiments, the approximated value of a_{max}, such that $\forall a \leq a_{max}$ the interpolation polynomial $p_{a,n}$ is converging on $[-1, 1]$.

(c) (2 points) Fix $a = a_{max}/2$ and find an upper bound for the error with respect to n, estimating β and C in the following inequality:

$$E_{a,n} \leq Ce^{-\beta n} =: B(C, \beta, n).$$

Save a figure with the error and the bound $B(C, \beta, \cdot)$, with respect to n, as **fig1c.eps**.

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Exercise 2 (Computational task)
Let

$$A_n = (a_{ij})_{i,j=1}^n, \qquad a_{ij} := \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{else} \end{cases} \qquad \text{and } b = (b_i)_{i=1}^n, \qquad b_i := \begin{cases} 1 & \text{if } i \text{ even}, \\ 0 & \text{if } i \text{ odd}. \end{cases}$$

(a) (2 points) Test the running time for the given QR decomposition (**solveLinSysQR.m** or **solveLinSysQR.py**) to solve the system $A_n x = b$ for different values of n (take $n \geq 500$) and verify the asymptotic behaviour of the computational cost with respect to the dimension n of the system.

(b*) (6 points) Fix $n = 100$. We consider a perturbed vector on the right hand side

$$b^\delta = (b_i)_{i=1}^n, \qquad b_i := \begin{cases} 1 & \text{if } i \text{ even}, \\ \delta & \text{if } i \text{ odd} \end{cases}$$

for $0 < \delta < 0.1$. Compute the error $\|x - x^\delta\|$ for different values of δ, where x is the solution of $A_n x = b$, and x^δ is the solution of $A_n x^\delta = b^\delta$. Plot the error with respect to δ. What can you say about the asymptotic behaviour of the error for $\delta \to 0$? Repeat your experiment with the matrix

$$\tilde{A}_n = (a_{ij})_{i,j=1}^n, \qquad a_{ij} := \begin{cases} 4 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{else} \end{cases}$$

and the same right hand side b and b^δ. Compare the error $\|x - x^\delta\|$ and $\|\tilde{x} - \tilde{x}^\delta\|$ in a table. How do you explain the difference of the magnitudes?

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Exercise 3 (Mixed task)
Let $I := [0, 1]$. We want to approximate the integral of a given function $f : I \to \mathbb{R}$

$$\mathcal{I}(f) := \int_0^1 \omega(x) f(x) dx$$

using a quadrature of the form

$$\mathcal{I}(f) \approx Q(f) = \alpha f(x_0)$$

(a) (2 points) Assume that we know the values $\int_0^1 \omega(x)dx$ and $\int_0^1 x\omega(x)dx$. Find a formula for x_0 and α such that the quadrature has maximal degree of exactness. Compute x_0 and α explicitly for the choice $\omega(x) = 1 + x$.

From now on, we consider $\omega(x) = 1$.

(b) (2 points) Write a program that computes the composite quadrature $Q^n(f)$ (based on $Q(f)$) on n equidistant subintervals.

(c) (2 points) Run the program for function $f(x) := \dfrac{1}{1+x^2}$. The exact value of $\mathcal{I}$ is given by $\mathcal{I}(f) = \pi/4$. Compute error $e^n(f) := |Q^n(f) - \mathcal{I}(f)|$ for different values of n. What can you say about the convergence of the error with respect to h, where h is the length of each subinterval?

(d) (2 points) (Theoretical task) Prove that for $f \in \mathcal{C}^\infty(I)$

$$e^n(f) \leq Ch^2.$$

(e) (2 points) Use the quadrature for the function $f(x) := x^{2/3}$. Compute analytically the exact value of $I(f)$. Estimate numerically the exponent r in the following inequality

$$e^n(f) \leq Ch^r.$$

Explain why $r < 2$.