computational mathematics

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Numerik I – Homework 3

Deadline: 15.3.2019, 13:00

Exercise 1 (4 P.) (Programming task)
Implement the evaluation of the piecewise quadratic interpolation[1] on N subintervals of equal length at given points.
Input: interval $[a, b]$, the evaluation points, N, values of f at the $2N + 1$ nodes.
Output: values of the interpolation polynomial at given evaluation points.

Exercise 2 (4 P.) (Mixed task)

(a) Let $i = \sqrt{-1}$ and $\mathcal{D} := \{x + iy| - 2 \leq x \leq 2,\ -3/8 \leq y \leq 3/8\}$ with the boundary $\partial\mathcal{D}$ of the length L. Consider the function

$$f : \mathcal{D} \to \mathbb{C},\ f(z) = \frac{1}{1 + 4z^2}$$

Use you program of exercise 1 to find a piecewise quadratic interpolation p_N of $f_R(x) := \dfrac{1}{1 + 4x^2}$, $x \in [-1, 1]$, with $N \in \{1, 2, 3, 4, 5\}$ subintervals. Write the error

$$E_N := \max_{0 \leq j \leq 1000} |(f_R - p_N)(\tilde{x}_j)| \tag{1}$$

as a function of N into a table, where $\tilde{x}_j = -1 + \frac{j}{500}$.

(b) Use the estimate (derived via complex analysis)

$$|f^{(n)}(x)| \leq \frac{n!}{2\pi} L \left(\frac{8}{3}\right)^{n+1} max_{z \in \partial\mathcal{D}} |f(z)|$$

to get an explicit bound for the error $\|f - p_N\|_{\max}$. Employ the error estimate to determine N_ε such that the error $\|f - p_{N\varepsilon}\|_{\max}$ is less than $\varepsilon = 10^{-7}$.

Hint: Show that $f(x + iy)\overline{f(x + iy)}$ as a function of x is maximal at $x = 0$.

Exercise 3 (4 P.) (Theoretical task)
Let $\Theta_n := \{x_i | i = 0, 1, 2, ..., n\}$ with $-1 = x_0 < x_1 < \ldots < x_n = 1$ be the set of nodes, $\tau_i = (x_{i-1}, x_i)$ and

$$\mathcal{S}^{-1,0}_{\Theta_n} := \{\varphi : [-1, 1] \to \mathbb{R}|\ \varphi_{|\tau_i} = c_i,\ c_i \in \mathbb{R},\ \forall\ 1 \leq i \leq n\}$$

the space of piecewise constant functions. We want to approximate the linear function $f : [-1, 1] \to \mathbb{R}$, $f(x) = x$.

[1] Section 2.4 in manuscript $\mathcal{S}^{0,2}_{\mathcal{G}} := \{u \in C^0(I) | \forall \tau \in \mathcal{G} : u|_\tau \in \mathbb{P}_2\}$

(a) Choose a basis for $\mathcal{S}^{-1,0}_{\Theta_n}$ and compute the matrix

$$\mathbf{M} := \left(\int_{-1}^{1} b_i(x) b_j(x) \right)_{i,j=1}^{n}$$

and the right hand side $\mathbf{r} := \left(\int_{-1}^{1} b_i(x) f(x)) \right)_{i=1}^{n}$.

(b) Find the solution of the system $\mathbf{Mx} = \mathbf{r}$ and let $\varphi_{\text{opt}} := \sum_{i=1}^{n} x_i b_i(x)$.