computational mathematics

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# Numerik I – Homework 4

Deadline: 22.3.2019, 13:00

**Exercise 1** (8 P.) (Theoretical task)

Given a matrix $\mathbf{A}_\varepsilon = \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix}$ for $0 \le \varepsilon \le \frac{1}{2}$.

(a) Compute the QR-decomposition $\mathbf{A}_\varepsilon = \mathbf{Q}_\varepsilon \mathbf{R}_\varepsilon$ where $\mathbf{Q}_\varepsilon \mathbf{Q}_\varepsilon^T = \mathbf{I}$ and $\mathbf{R}_\varepsilon = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$.

(b) Compute the LU-decomposition $\mathbf{A}_\varepsilon = \mathbf{L}_\varepsilon \mathbf{U}_\varepsilon$ where $\mathbf{L}_\varepsilon = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}$ and $\mathbf{U}_\varepsilon = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$.

(c) The condition number of an $n \times n$ matrix $\mathbf{M}$ is defined as $\kappa_*(\mathbf{M}) := \|\mathbf{M}\|_* \|\mathbf{M}^{-1}\|_*$ where $\|\mathbf{M}\|_* := \max\limits_{1 \le i,j \le n} |m_{ij}|$. For a linear system $\mathbf{M}\mathbf{u} = \mathbf{r}$, it indicates the influence of the perturbation in $\tilde{\mathbf{r}} \approx \mathbf{r}$ on the perturbed solution of the system $\mathbf{M}\tilde{\mathbf{u}} = \tilde{\mathbf{r}}$.

(i) Calculate the condition numbers of $\mathbf{A}_\varepsilon$, $\mathbf{L}_\varepsilon$ and $\mathbf{U}_\varepsilon$. Investigate how the condition numbers change as $\varepsilon$ approaches 0.

(ii) Determine the condition numbers of $\mathbf{Q}_\varepsilon$ and $\mathbf{R}_\varepsilon$ for $\varepsilon \to 0$. (The norm is continuous, and therefore the norm and the limit are exchangeable.)

(iii) Comment on your results in (i) and (ii).

**Exercise 2** (6 P.) (Programming task)

Note: Do not use implemented Matlab functions to solve the exercises.

a) Implement the QR-decomposition to solve a system of linear equations. Write a function that takes the matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ as input and returns the vectors $\mathbf{u}$, $\beta$ as defined in (3.5) of the lecture notes.

b) Write a program for solving $\mathbf{A}\mathbf{x} = \mathbf{b}$ with input $\mathbf{A}$ and the right-hand side vector $\mathbf{b} \in \mathbb{C}^{n\times 1}$ using your program from a) to get $\mathbf{u}$ and $\beta$. Use the algorithm in proof of Satz 3.6 to efficiently solve the system. Adapt the algorithm on page 30 to solve a system with upper-triangular matrix. Use the following $\mathbf{A}$ and $\mathbf{b}$ to test your program:

$$\begin{aligned} \mathbf{A} =& (a_{ij})_{i,j=1}^{100} \in \mathbb{R}^{100\times 100} \quad \text{where } a_{ij} := \begin{cases} 100 & \text{if } i = j, \\ e^{-\frac{|i-j|}{100}} & \text{if } i \ne j. \end{cases} \\ \mathbf{b} =& [1, 0, 1, 0, \cdots, 1, 0]^T \in \mathbb{R}^{100\times 1}. \end{aligned}$$

Check the result by calculating the residual $\|\mathbf{r}\|_\infty = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_\infty$ where $\|\mathbf{r}\|_\infty = \max\limits_{1\le i\le n} |r_i|$.

**Exercise 3** (4 P.)(Theoretical task)

a) Prove **Lemma 3.5** of the lecture notes for the case $a_{11} = 0$.

b) Assume $dim(ker(\mathbf{A})) = m > 0$. What is the consequence on the matrix $\mathbf{R}$ in (3.2) of the lecture notes?