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Numerik I – Homework 10

Deadline: 10.5.2019, 13:00

Exercise 1 (3 Points, Computational task)
Given a function f, we want to approximate its zeros by a combined bisection and Newton's method. Write a function that takes as input a starting interval and:

- iterates with the bisection method to determine an interval $I_\delta(a) := [a-\delta, a+\delta]$, where the Newton's method converges to the zero of f for all start values in this interval;
- uses Newton's method on $f(x)$ using the starting value a found before.

Test your code with the function

$$f(x) = x - \cos(x) + \sin(x).$$

Hint: For the bisection method, find δ using Corollary 7.11 from the script. Plot the given function to guess the starting interval to use as the input of your code.

Exercise 2 (8 Points)
Let x_* be a zero of f, i.e. $f(x_*) = 0$. We assume: $f'(x_*) > 0$, $f''(x_*) \neq 0$.

(a) (Theoretical task) Find a function φ such that x_* is a solution of the fixed-point equation $x = \varphi(x)$, and the iteration formula based on φ satisfies

$$x^{(i+1)} = \varphi(x^{(i)}), \quad x^{(i)} \xrightarrow[i\to\infty]{} x_*,$$

with the order of convergence 3. To do this, replace the function f by its Taylor series expansion of order 2 around $x^{(i)}$ and determine $x^{(i+1)}$ as the zero of this parabola. Think about which zero point should be chosen as $x^{(i+1)}$ if the parabola has two zeros.

(b) (Computational task) Implement this formula and also the Newton's method to determine $2^{1/3}$. Use the starting value $x_0 = 2$. Compare their convergence rate along with the bisection method using starting interval $[1, 2]$. In all the three methods, approximate up to an accuracy of 10^{-8} or maximum of 30 iterations.
What are the disadvantages of the method of order 3?

Exercise 3 (5 Points)
The function $f(x) = \sin(x)$ is only available in a perturbed (or noisy) form:

$$\tilde{f}(x) = f(x) + \delta\,\frac{\sin\frac{1}{x}}{x+1}$$

(a) (Computational task) Approximate $f'(0)$ using symmetric differences (i.e. $f'(x) \approx \frac{\tilde{f}(x+h)-\tilde{f}(x-h)}{2h}$).

(b) (Theoretical task) How should h be choosen optimally in terms of δ?

(c) (Computational task) Plot the error of the approximation $e_\delta(h)$ for $h = 10^0, 10^{-0.5}, \ldots, 10^{-3}$ and for $\delta = 10^{-12}, 10^{-10}, 10^{-8}, 10^{-6}$. Check if $h = h_{opt}$ from the script is a good choice.