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Numerical Methods for Scientific Computing

Homework 2

Deadline: March 9, 2016

Exercise 1. (4 P)

We denote the inverse of a quadratic matrix $A \in \mathbb{R}^{m\times m}$ by $A^{-1} \in \mathbb{R}^{m\times m}$.

(a) Write a function `fweli_yourSurname.m` in Matlab which implements the forward elimination stage of the Gaussian elimination method. Write a second function `bksub_yourSurname.m` which implements the backward substitution.

(b) Compute A^{-1} using the implemented functions and compare it to the result obtained via the Matlab command `inv(A)`. Test your codes with the following matrix:

$$A = \begin{bmatrix} 50 & 1 & 3 \\ 1 & 6 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Hint. The columns of the inverse matrix $A^{-1} \in \mathbb{R}^{m\times m}$ are the solutions of the systems

$$Ax = e_i, \quad i = 1, 2, \ldots, m,$$

where $e_i = [0, 0, \ldots, 1, \ldots, 0]^T$ with 1 at the i-th position.

(c) (BONUS POINT) Implement the Gaussian elimination method with pivoting by writing a function `fweliPivot_yourSurname.m`. Test your code with the following matrix:

$$A = \begin{bmatrix} 1 & 1 + 0.5 \cdot 10^{-15} & 3 \\ 2 & 2 & 20 \\ 3 & 6 & 4 \end{bmatrix}$$

Exercise 2. (2 P)

Consider the matrix `A=rand(n);` for $n = 1000, 1100, \ldots, 5000$.

Plot the CPU time necessary to generate the LU factorization of the Vandermonde matrix as a function of the matrix dimension n . Compare the result with the theoretical computational complexity.

Hint. Use the Matlab commands `[L,U] = lu(A)` and `tic`, `toc` or `cputime`.

Exercise 3. (5 P)

Prove the following properties of the condition number $\mathrm{cond}(A) = \|A\|\|A^{-1}\|$:

(a) $\mathrm{cond}(\mathbf{I}) = 1$

(b) $\mathrm{cond}(\mathbf{A}) \geq 1$

(c) $\mathrm{cond}(\mathbf{A}) = \mathrm{cond}(\alpha\mathbf{A})$ for $\alpha \in \mathbb{R}, \alpha \neq 0$

(d) $\mathrm{cond}(\mathbf{A}) \geq \frac{|\lambda_{\max}(\mathbf{A})|}{|\lambda_{\min}(\mathbf{A})|}$ ($\lambda_{\max}, \lambda_{\min}$ are the largest and smallest eigenvalues)

(e) $\mathrm{cond}(\mathbf{AB}) \leq \mathrm{cond}(\mathbf{A})\mathrm{cond}(\mathbf{B})$

Exercise 4. (3 P)

We have seen that `cond(A)` shows both the sensibility of the **propagation of numerical errors** and the need of **computational resources** when solving the linear system $\mathbf{Ax} = \mathbf{b}$.

Consider the Vandermonde matrix `A=vander(rand(1,n));` and the random matrix `B=rand(n);` for `n=100`.

(a) Construct the vector $\mathbf{b}$ so that the system $\mathbf{Ax} = \mathbf{b}$ is solved by `x=ones(n,1)` and the vector $\mathbf{c}$ so that $\mathbf{By} = \mathbf{c}$ is solved by `y=ones(n,1)`.

(b) Compute the vectors $\mathbf{x}$ and $\mathbf{y}$ of the two systems $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{By} = \mathbf{c}$ accordingly using the Matlab command : `\`

(c) Compute the relative errors and explain the results.

(d) Plot in a semilogarithmic scale (`semilogy(N,K)`) the condition number K of the Vandermonde matrix with the dimensions varying from 2x2 to 50x50 (`N=2:1:50`). Plot analogously the condition number of the random matrix. What do you observe?