computational mathematics

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Numerik I – Homework 7

Deadline: 12.4.2019 13:00

Exercise 1 (6 P.) (Mixed taks)
We want to approximate

$$\int_{-1}^{1} f(x)dx.$$

(a) Find nodes $x_0, x_1, x_2 \in [-1, 1]$ and weights w_0, w_1, w_2 such that the quadrature

$$Q(f) = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2)$$

has maximal degrees of exactness (Definition 6.5).

(b) Implement the composed quadrature method derived in (a). Input: number of subintervals n, values of f in the nodes.

(c) Compare the convergence of the method with the Simpson's quadrature w.r.t. the same number of subintervals.

(d) What is the advantage of each of the two methods?

Exercise 2 (4 P.) (Theoretical task)
Construct a quadrature formula for the approximation of

$$I(f) = \int_0^1 \omega(x) f(x)\, dx \quad \text{with} \quad \omega(x) = x^{1/3},$$

having maximum degree of exactness, which uses the information $f(0)$ and $\int_0^1 f(x)dx$.
Derive an error estimate using the Peano kernels.

Exercise 3 (4 Pkt.) (Theoretical task)
We define the monic Legendre polynomials $L_n(x)$ as the monic polynomials of increasing degree which are orthogonal with respect to the L^2-inner product $\int_{-1}^{1} uv\, dx$. Show that the following 3-term recursion gives the Legendre polynomials.

$$\begin{aligned} P_0 &\equiv 1, \quad P_1 = x, \\ P_{i+1} &= xP_i - \left(\frac{i^2}{4i^2-1}\right) P_{i-1} \quad \text{for } i = 1, 2, \ldots \end{aligned} \tag{1}$$

Hint: We know from the lectures that the Legendre polynomials satisfy the 3-term recursion

$$\begin{aligned} P_0 &\equiv 1, \quad P_{-1} \equiv 0 \\ P_{k+1} &= (x - \alpha_k)P_k - \beta_k P_{k-1} \quad \text{for } k = 1, 2, \ldots \end{aligned} \tag{2}$$

with $\alpha_k = \frac{(xP_k(x), P_k(x))}{(P_k(x), P_k(x))}$ and $\beta_k = \frac{(P_k(x), P_k(x))}{(P_{k-1}(x), P_{k-1}(x))}$.

Use symmetry with respect to 0 to compute α_k. For β_k use the equivalent formula for P_k given by

$$P_k(x) = (-1)^k \frac{k!}{(2k)!} \left(\frac{d}{dx}\right)^k \left(\left(1 - x^2\right)^k\right). \tag{3}$$

Use integration by parts (k times) and a trigonometric change of variables to compute (P_k, P_k) and, consequently, to find the formula for β_k.