Machine learning from scratch

Lecture 1: Mathematical background

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Before we start

IT STEP will be organizing a Tech night on **February 16th** (Thursday) from 7pm. I will (probably) be giving a talk. The course will most likely be postponed.

Motivation, vocabulary and notations

In **supervised learning** tasks, we are given a *data set* of the form:

$$D = \left\{ \left(\boldsymbol{x}^{(i)}, y^{(i)} \right), \boldsymbol{x}^{(i)} \in \mathcal{X}, y^{(i)} \in \mathcal{Y}, i \in \{1, \ldots, n\} \right\}$$

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- n is the size of the data set (number of *instances*/*samples*)
- In most applications:
  - $\mathcal{X} = \mathbb{R}^d$ (d is the *dimensionality*)
  - $\mathcal{Y} = \mathbb{R}$ (*regression*) or $\mathcal{Y} \subset \mathbb{N}$ (*classification*)
- $\boldsymbol{x} \in \mathcal{X}$ is the *feature vector* and $y \in \mathcal{Y}$ is the *label*

Motivation, vocabulary and notations

Solving a **supervised learning problem** is finding (or *learning*) a function (or *hypothesis*) $h : \mathcal{X} \mapsto \mathcal{Y}$ such that for $(\boldsymbol{x}, y) \in D$, $h(\boldsymbol{x})$ is a *good* estimation (or approximation) of y.

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We often write $h(\boldsymbol{x}) = \hat{y}$ ($\hat{y}$ is the *prediction* of $\boldsymbol{x}$ by h).

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This raises **2 questions**:

- How to define h?
- How to assess whether $\hat{y}$ is a good approximation of y?

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Several ways to parametrize h exist:

- **Linear model**: $h(\boldsymbol{x}) = \theta^T \boldsymbol{x}$
- **Polynomial kernel** (degree k): $h(\boldsymbol{x}) = \left(1 + \theta^T \boldsymbol{x}\right)^k$
- Other kernels exist, more on this when we talk about duality
- With a **neural net**, more on this later as well
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How you define h highly depends on the application, for example:

- Sometimes a lot of data preprocessing has been made and a simple model (e.g. linear) would work well
- You might have **time/hardware constraints**: In this case going for a too complex model might be crippling
- For neural net, the architecture depends a lot on the type of data you have

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Illustration of the introduced notations:

- $\mathcal{X} = \mathbb{R}^2$ ($d = 2$ dimensions: living area and $\#$ bedrooms)
- $\mathcal{Y} = \mathbb{R}$ (regression task)
- $\boldsymbol{x}^{(1)} = [50, 1]^T$ and $y^{(1)} = 30000$

Linear algebra concepts (2)

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Using a **linear regression model** gives

$$h_\theta(\boldsymbol{x}) = \theta_0 + \theta_1 x_1 + \theta_2 x_2$$

Generalization to any dimensionality d:

$$h(\boldsymbol{x}) = \sum_{j=0}^{d} \theta_j x_j = \theta^T \boldsymbol{x}$$

Here, we set $\boldsymbol{x}_0 = 1$ so that θ_0 is included in θ. $\theta^T \boldsymbol{x}$ is called the dot product (or inner product) between *theta* and $\boldsymbol{x}$ and is sometimes noted $\langle \theta, \boldsymbol{x} \rangle$.

Ordinary least squares

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Suppose we chose the following loss function:

$$\ell\left(y, \hat{y}\right) = \frac{1}{2}\left(y - \hat{y}\right)^2$$

This leads to the following least squares *cost function*:

$$J(\theta) = \frac{1}{2} \sum_{i=1}^{n} \left(h\left(\boldsymbol{x}^{(i)}\right) - y^{(i)} \right)^2$$

This problem the **ordinary least squares** (OLS) regression model.

Least Mean Squares (LMS) update rule

We want to minimize the following cost function:

$$J(\theta) = \frac{1}{2} \sum_{i=1}^{n} \left(h\left(\boldsymbol{x}^{(i)}\right) - y^{(i)} \right)^2$$

One way to do it is by using the **gradient descent** algorithm:

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta)$$

for all $j \in \{0, \ldots, d\}$. α is called the **step size** or the **learning rate**. This update rule can be rewritten in a more compact way:

$$\theta := \theta - \alpha \nabla J(\theta)$$

where $\nabla J(\theta)$ is the **gradient** of J in θ. We have, by definition:

$$\nabla J(\theta) = \left[\frac{\partial}{\partial \theta_1} J(\theta), \ldots, \frac{\partial}{\partial \theta_d} J(\theta) \right]^T$$

Least Mean Squares (LMS) update rule

To apply the LMS update rule, we need to compute the gradient of J. Let's compute it for a single $(\boldsymbol{x}, y)$ sample:

$$\begin{aligned} \frac{\partial}{\partial \theta_j} J(\theta) &= \frac{\partial}{\partial \theta_j} \frac{1}{2} \left(h\left(\boldsymbol{x}^{(i)} \right) - y^{(i)} \right)^2 \\ &= \end{aligned}$$

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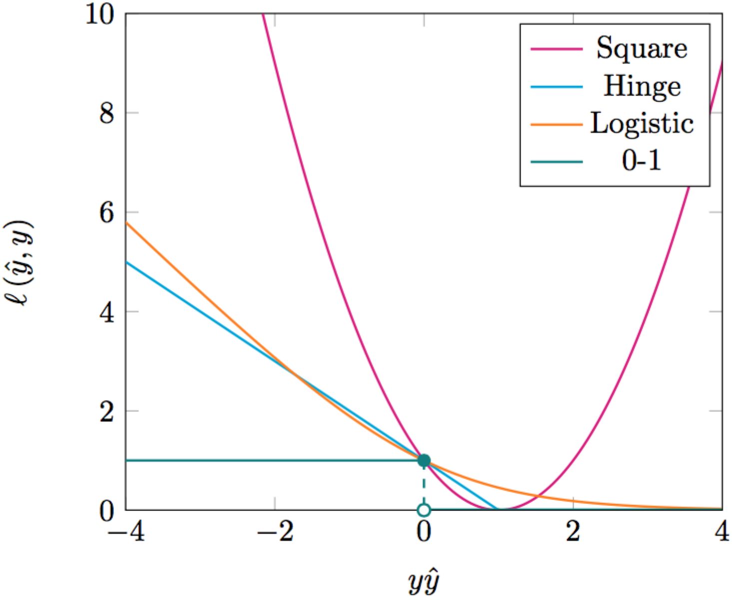
Possibility to add a weight to the loss!

Loss functions

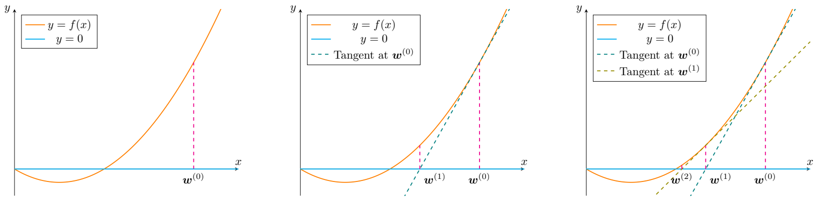


Regularization

Optimization

Now, we want to minimize a function of the form

$$J(\theta) = \sum_{i=1}^{n} \ell\left(y^{(i)}, \hat{y}^{(i)}\right) + \lambda R(\theta) \tag{1}$$



y
$y = f(x)$
$y = 0$
x
$\boldsymbol{w}^{(0)}$
y
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Tangent at $\boldsymbol{w}^{(0)}$
x
$\boldsymbol{w}^{(1)}$
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Conclusion

Next week we will implement some of the concepts we've seen today.

Thank you! Questions?

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https://github.com/azubiolo/itstep