1. Formulation of the problem

In order to mitigate the risk, the cedent (or primary insurance) enters into a contract with a reinsurance firm. According to this contract, at time t, the insurer assumes the responsibility to pay a fraction u_t^π of each claim, while the reinsurance firm takes care of the remaining payment $1 - u_t^\pi$ for each claim. The starting point is the Cramer-Lundberg ruin problem with controled reinsurance and dividend payments, i.e. the capital of the primary insurer, that states for the liquid assets of the company, is governed by:

$$X_t^\pi = x_0 + u_t^\pi(1+\eta)\lambda at - (1-u_t^\pi)(1+\mu)\lambda at - u_t^\pi \sum_{i=1}^{N_t^\lambda} Y_i - \sum_{i=1}^{N_t^\gamma} \xi_i^\pi \qquad \text{(initial model)}$$

Here $\{N_t^\lambda\}_{t\geq 0}$ stands for a Poisson process of rate λ, and $\{Y_j\}_{j\geq 1}$ represents i.i.d. non-negative random variables corresponding to the sizes of individual claims, which are also independent of N^λ. The premiums are paid continuously at constant rates $\lambda a[1+\eta]$ and $\lambda a[1-u_t^\pi][1+\mu]$ based on the expected value principle, where $a = \mathbb{E}[Y_1]$, $\bar{\sigma}^2 = \mathrm{Var}[Y_1]$, and safety loadings $\mu \geq \eta > 0$ are involved. The payment rate η represents the input payment rate of the insurance firm, while μ is the insurer's payment rate to the reinsurance firm as per the contract mentioned earlier. When $\mu > \eta$, we call it non-cheap reinsurance, and in the case of $\mu = \eta$, it is usually referred as a cheap reinsurance on which this paper is focused. Other unmentioned components will be formally defined later.

Notice that initial model can be rewritten in a different form as follows:

$$X_t^\pi = x_0 + u_t^\pi \, \mu\lambda at + u_t^\pi Z_t - \sum_{i=1}^{N_t^\gamma} \xi_i^\pi \qquad \text{(rewritten form)}$$

with $Z_t = \lambda at - \sum_{j=1}^{N_t^\lambda} Y_j$. Then $E[Z_t] = 0$, $\mathrm{Var}[Z_t] = \bar{\sigma}^2\lambda t$ and for $\lambda \gg 1$, $a \ll 1$ and $\lambda a \approx 1$, using Theorem 6 form Donald Iglehart paper the process Z_t is well approximated by the Brownian motion σW_t, with $\sigma = \bar{\sigma}\sqrt{a}$. This approximation is valid under the condition that the random variables $\{Y_j\}_{j\geq 1}$ are stochastically small, and it serves as the intuitive background for the following model:

$$dX_t^\pi = u_t^\pi \mu dt + u_t^\pi \sigma dW_t - \xi_t^\pi dN_t^\gamma \qquad \text{(differencial form)}$$

Then take integral of both sides of the equasion and suppose a starting point x_0:

$$X_t^\pi = x_0 + \mu \int_0^t u_s^\pi \, ds + \sigma \int_0^t u_s^\pi \, dW_s - \int_0^t \xi_s^\pi \, dN_s^\gamma \qquad \text{(integral form)}$$

Filtered probability space should be defined as $(\Omega, \mathcal{F}, \mathcal{H}, P)$ where $\{\mathcal{H}_t\}_{t\geq 0}$ is a right-continuous complete filtration generated by (W, N^γ) and the processes W and N^γ are independent. Filtration $\{\mathcal{H}_t\}_{t\geq 0}$ represents the information available at time t and any decision made is based on this information.

Where previously unmentioned parameteres are difined as:

1. $X_0 = x_0 > 0$ is the initial surplus of the insurance firm.
2. μ and σ are drift and diffusion coefficient of Black-Scholes model.
3. $u_t \in [0,1]$ is a cheap reinsurance parameter that corresponds to exposure at time t and $1 - u_t$ denotes the fraction of incoming claims that is reinsured at time t.
4. Dividend payments can only be made at arrival times
   of a Poisson process N_t^γ with intensity $\gamma > 0$ which is adapted to the filtration $\{\mathcal{H}_t\}_{t\geq 0}$.
5. W_t is a Brownian Motion which is also adapted to the filtration $\{\mathcal{H}_t\}_{t\geq 0}$.
6. ξ_t is right-continuous with left limits stochastic process (Cadlag) which is adapted to the filtration $\{\mathcal{H}_t\}_{t\geq 0}$ and represents the amount of dividend payment made at ariival times of previously defined Poisson process.

Therefore, under the strategy $\pi = (u^\pi, \xi^\pi)$, the surplus process $X_t^\pi := \{X_t^\pi : t \geq 0\}$ evolves as in (integral form), with $X_0^\pi = x_0$. Let $\mathcal{A}$ be the collection of strategies $\pi = (u^\pi, \xi^\pi)$ such that u^π and ν^π are $\{\mathcal{H}_t\}_{t\geq 0}$ adapted such that $u^\pi \in [0,1]$ and ξ^π satisfies $0 \leq \xi_t^\pi \leq X_{t-}^\pi$. We say that a strategy π is admissible if $\pi \in \mathcal{A}$.

For each $\pi \in \mathcal{A}$, the expected Net Present Value (NPV) is given by:

$$V^\pi(x) := \mathbb{E}_x\left[\int_0^{\tau^\pi} e^{-\delta t} g(\xi_t^\pi) dN_t^\gamma\right], \quad x \geq 0 \qquad \text{(Expected NPV)}$$

Where g is a function: $\mathcal{R}_+ \to \mathcal{R}$ such that $g(\xi) := -K\mathbb{I}_{\{\xi>0\}} + \xi$, where K is a constant transaction cost that is confiscated out of the economical model at arrival times when insurance company dicides to make positive dividend payments.

Also, $\mathbb{E}_x$ represents the conditional expectation given an initial surplus x, $\delta > 0$ denotes the discount rate, and $\tau^\pi := \inf\{t > 0 : X_t^\pi < 0\}$ is the first time of bankruptcy. Then, our goal is to find the value function V for the problem, which is:

$$V(x) := \sup_{\pi \in \mathcal{A}} V^\pi(x), \quad x \geq 0 \qquad \text{(Value function)}$$

Furthermore, we aim to find an optimal strategy $\pi^* \in \mathcal{A}$ for which the expected NPV V^{π^*} coincides with V, if such a strategy exists.

2. Construction of HJB equation

Let's construct our HJB equation. We have:

$$V(x) = (-K\mathbb{I}_{\{\xi>0\}} + \xi + \gamma t e^{-\delta t}\mathbb{E}_x[V(X_t - \xi)]) + (1-\gamma t)e^{-\delta t}\mathbb{E}_x[V(X_t)] \tag{2.1}$$

This equation can be derived using following heuristic argument. We have two states of our model: the first occurs with probability of γt that there occurs dividend payout of value $g(\xi)$ and the second state occurs with probability $1 - \gamma t$ when nothing happens.

Moving $V(X_n)$ to the right-hand side:

$$0 = ((-K\mathbb{I}_{\{\xi>0\}} + \xi)e^{-\delta t} + \mathbb{E}_x[V(X_t - \xi)e^{-\delta t} - V(x)])\gamma t + (1-\gamma t)\mathbb{E}_x[V(X_t)e^{-\delta t} - V(x)] \tag{2.2}$$

Assuming that $V \in C^2(0, +\infty)$, by Ito's formula we know:

(i) In the case that there is no arrival of the Poisson process N^γ:

$$V(X_t) - V(x) = \int_0^t \left[\mu u_s V'(X_s) + \frac{\sigma^2}{2}u_s^2 V''(X_s)\right] ds + \int_0^t V'(X_s)dW_s \tag{2.3}$$

(ii) In the case there is the first arrival of the Poisson process N^γ:

$$V(X_t - \xi) - V(x) = \int_0^t \left[\mu u_s V'(X_s) + \frac{\sigma^2}{2}u_s^2 V''(X_s)\right] ds + (V(X_{t-} - \xi) - V(X_{t-})) + \int_0^t V'(X_s)dW_s \tag{2.4}$$

So, now we will apply integration by parts: (i) In the case there is no arrival of Poisson process N^γ:

$$e^{-\delta t}V(X_t) - V(x) = \int_0^t e^{-\delta s}\left[\mu u_s V'(X_s) + \frac{\sigma^2}{2}u_s^2 V''(X_s) - \delta V(X_s)\right] ds + \int_0^t e^{-\delta s}V'(X_s)dW_s \tag{2.5}$$

(ii) In the case there is first arrival of Poisson process N^γ:

$$\begin{aligned} e^{-\delta t}V(X_t - \xi) - V(x) = \int_0^t e^{-\delta s}\left[\mu u_s V'(X_s) + \frac{\sigma^2}{2}u_s^2 V''(X_s) - \delta V(X_s)\right] ds + \\ e^{-\delta t}\left(V(X_{t-} - \xi) - V(X_{t-})\right) + \int_0^t e^{-\delta s}V'(X_s)dW_s \end{aligned} \tag{2.6}$$

Taking expected value in (5) and (6) and also considering $\int_0^t e^{-\delta s}V'(X_s)dW_s$ to be a local martingale, we have that:

$$\mathbb{E}_x\left[e^{-\delta t}V(X_t) - V(x)\right] = \mathbb{E}_x\left[\int_0^t e^{-\delta s}\left[\mu u_s V'(X_s) + \frac{\sigma^2}{2}u_s^2 V''(X_s) - \delta V(X_s)\right] ds\right] \tag{2.7}$$

$$\begin{aligned} \mathbb{E}_x\left[e^{-\delta t}V(X_t - \xi) - V(x)\right] = \mathbb{E}_x\left[\int_0^t e^{-\delta s}\left[\mu u_s V'(X_s) + \frac{\sigma^2}{2}u_s^2 V''(X_s) - \delta V(X_s)\right] ds\right] + \\ e^{-\delta t}\left(V(X_{t-} - \xi) - V(X_{t-})\right) \end{aligned} \tag{2.8}$$

Applying (7) and (8) into (2) we obtain:

$$0 = (\xi - K\mathbb{I}_{\{\xi>0\}})e^{-\delta t}\gamma t + \gamma t\mathbb{E}_x\left[\int_0^t e^{-\delta s}\left[\mu u_s V'(X_s) + \frac{\sigma^2}{2}u_s^2 V''(X_s) - \delta V(X_s)\right] ds\right] +$$

$$e^{-\delta t}\left[V(X_{t-}-\xi)-V(X_{t-})\right]+[1-\gamma t]\mathbb{E}_x\left[\int_0^t e^{-\delta s}\left[\mu u_s V'(X_s)+\frac{\sigma^2}{2}u_s^2 V''(X_s)-\delta V(X_s)\right]ds\right]=$$
$$(\xi-K\mathbb{I}_{\{\xi>0\}})e^{-\delta t}\gamma t+\gamma t\mathbb{E}_x\left[e^{-\delta t}\left[V(X_{t-}-\xi)-V(X_{t-})\right]\right]+$$
$$\mathbb{E}_x\left[\int_0^t e^{-\delta s}\left[\mu u_s V'(X_s)+\frac{\sigma^2}{2}u_s^2 V''(X_s)-\delta V(X_s)\right]ds\right] \tag{2.9}$$

Now, factoring t on the previous expression letting $t\to 0$ and maximizing with respect $u\in[0,1]$ and $\xi\in[0,x]$ it follows that the HJB for our problem is:

$$0=\max_{0\le\xi\le x}\gamma\left[-K\mathbb{I}_{\{\xi>0\}}+\xi+[V(x-\xi)-V(x)]\right]+\max_{0\le u\le 1}\mathcal{L}^u v(x) \tag{HJB equation}$$
$$s.t.\ \mathcal{L}^u v(x)=V'(x)\mu u+\frac{1}{2}V''(x)\sigma^2u^2-\delta V(x),\ V(0)=0$$

3. Dividend two-barrier strategy and Verification Theorem

To begin with, we should figure out, how to open the first maximum of the Hamilton-Jacobi-Bellman equation above. In order to do it, we should first talk about two-barrier dividend payment strategy that we will use in our case. Basically, we have two levels, for example b_1 and b_2. After our capital hits level b_2 there will appear a payment at dicision time, of value $X_{\tau i}-b_1$ to a shareholder. Next we will introduce lemma which is closely connected to the definition of dividend two-barrier strategy.

Lemma 3.1

Let v be a C^2-continuous, concave and increasing function on $(0,\infty)$ such that v'/v'' are decreasing on $(0,\infty)$, there exists $b_1>0$ and v has a following form:

$$v'(x)\ \ is\ \begin{cases}>1, & x\in[0,b_1)\\ =1, & x=b_1\\ <1, & x\in(b_1,\infty)\end{cases} \tag{3.1}$$

Then the first non-differential term of the HJB equation attains its maximum at:

$$\arg\max_{0\le\xi\le x}\gamma(\xi-K\mathbb{I}_{\{\xi>0\}}+v(x-\xi)-v(x))=\begin{cases}0, & x\in[0,b_2)\\ x-b_1, & x\in[b_2,\infty)\end{cases} \tag{3.2}$$

for some $b_1\le b_2$.

Now we can formulate two-barriers dividend strategy.

Definition 3.2 (Periodic Two-Barrier Dividend Strategy)

A periodic strategy (b_1,b_2) with $0\le b_1\le b_2$ is the strategy that pays $x-b_1$ whenever the surplus x is above or equal to b_2, at decision times.

$$\xi_{\tau_i}=(X_{\tau_i}-b_1)\mathbb{I}_{\{X_{\tau_i}\ge b_2\}} \tag{3.3}$$

Theorem 3.3 (Verification Theorem)

Assume that we follow a periodic dividend strategy with dividend payments $\theta=\xi_{\tau_1},\xi_{\tau_2},\ldots$ at decision times less than time of bankruptcy $t=\tau_1,\tau_2,\ldots<\tau$. Assume that there exists v be a C^2-continuous, concave and increasing function on $(0,\infty)$ such that v'/v'' are decreasing on $(0,\infty)$, which satisfies HJB equation, then we have $v^\pi(x)\ge V(x)\ge 0$. Moreover, if there exists an admissible control policy π^* satisfying:

$$\max_{0\le\xi^{\pi^*}\le x}\gamma(\xi^{\pi^*}-K\mathbb{I}_{\{\xi^{\pi^*}>0\}}+v(x-\xi^{\pi^*})-v(x))+\max_{0\le u^{\pi^*}\le 1}(\mathcal{L}_{u^{\pi^*}}(v(x)))=0 \tag{condition 1}$$

then $v^{\pi^*}(x)=V(x)\ge 0$ and π^* is an optimal dividend control strategy.

Proof.

Following ideas of Benjamin Avanzi first we apply Ito's formula to jump diffusion process:

$$
e^{-\delta(t\wedge\tau)}v(X(t\wedge\tau)) = v(x) - \int_0^{t\wedge\tau} \delta e^{-\delta s}v(X(s))ds + \mu\int_0^{t\wedge T} \delta e^{-\delta s}u(x)v'(X(s))dX^{(c)}(s) +
$$

$$
\frac{\sigma^2}{2}\int_0^{t\wedge\tau} e^{-\delta s}v''(X(s))ds + \sum_{s\le t\wedge\tau}^{\Delta X(s)\neq 0} e^{-\delta s}[v(X(s-)+\Delta X(s)) - v(X(s-))] \tag{3.4}
$$

where $X^{(c)}(t)$ is the continuous component of $X(t)$. Since the dividend process $\sum_{i=1}^{N_t^\gamma}\xi_i$ is a pure jump process and using $X(t)$ defined in rewritten form, we have:

$$
dX_t^{(c)} = \mu u_t dt + \sigma u_t dW_t \tag{3.5}
$$

In our model unlike Benjamin Avanzi 2014 paper we don't have the second Poisson jump process so the sum in Ito's formula can be rewritten as:

$$
e^{-\delta(t\wedge\tau)}v(X(t\wedge\tau)) = v(x) - \int_0^{t\wedge\tau} \delta e^{-\delta s}v(X(s))ds + \int_0^{t\wedge\tau} e^{-\delta s}v'(X(s))\mu u(s)ds +
$$

$$
\int_0^{t\wedge\tau} e^{-\delta s}v'(X(s))\sigma u(s)dW_s + \frac{\sigma^2}{2}\int_0^{t\wedge\tau} e^{-\delta s}v''(X(s))ds +
$$

$$
\int_0^{t\wedge\tau} e^{-\delta s}[g(\xi_s) + v(X(s-)) - \xi_s - v(X(s-))]dN_\gamma(s) - \int_0^{t\wedge\tau} e^{-\delta s}g(\xi_s)dN_\gamma(s) \tag{3.6}
$$

After some algebraic transformations we get:

$$
e^{-\delta(t\wedge\tau)}v(X(t\wedge\tau)) = v(x) + \int_0^{t\wedge\tau} e^{-\delta s}[\mu u(s)v'(X(s-)) + \frac{\sigma^2}{2}v''(X(s-)) - \delta v(X(s-))]ds +
$$

$$
\int_0^{t\wedge\tau} e^{-\delta s}v'(X(s-))\sigma u(s)dW_s + \int_0^{t\wedge\tau} e^{-\delta s}[g(\xi_s) + v(X(s-)) - \xi_s - v(X(s-))]dN_\gamma(s) - \int_0^{t\wedge\tau} e^{-\delta s}g(\xi_s)dN_\gamma(s) \tag{3.7}
$$

Where $-\delta v(X(s-)) + \mu u(s)v'(X(s-)) + \frac{\sigma^2}{2}v''(X(s-)) = \mathcal{L}^u(v(X(s-)))$. After taking the expectation and assuming that $M_{t\wedge\tau} = \int_0^{t\wedge\tau} e^{-\delta s}v'(X(s-))\sigma u(s)dW_s$ is a square integrable martingale and the fact that HJB equation holds for $v(x)$ we get:

$$
v(x) \ge \mathbb{E}[e^{-\delta(t\wedge T)}v(X(t\wedge T))] + \mathbb{E}\left[\int_0^{t\wedge T} e^{-\delta s}g(\xi_s)dN_\gamma(s)\right] \tag{3.8}
$$

We can actually show that $M_{t\wedge\tau}$ is a martingale. Note that M_t is the Ito's integral of the process $e^{-\delta s}v'(X(t-))\sigma u(t)$. In order to show that M_t is a uniformly integrable martingale, we need to verify that (see, for example, Theorem 8.27, 8.32 and Corollary 7.8 in Klebaner, 2005):

$$
sup_{\{t\ge 0\}}\mathbb{E}^u\langle M, M\rangle(t) = \sup_{t\ge 0}\mathbb{E}^u\left[\int_0^t \sigma u(s)e^{-\delta s}v'(X(s-))^2 d\langle W, W\rangle(s)\right] < \infty \tag{3.9}
$$

Since the sharp bracket process of a Brownian motion is $d\langle W, W\rangle(s) = ds$, it suffices to verify that:

$$
sup_{\{t\ge 0\}}\mathbb{E}^u\left[\int_0^t \sigma u(s)e^{-\delta s}v'(X(s-))^2 ds\right] < \infty \tag{3.10}
$$

Now because $v(X)$ is a concave function, consequently $v(X(s-))$ is also concave then $v'(X(s-)) \le v'(0)$ for all $s \ge 0$. Therefore we have:

$$
sup_{\{t\ge 0\}}\mathbb{E}^u\left[\int_0^t \sigma u(s)e^{-\delta s}v'(X(s-))^2 ds\right] \le sup_{\{t\ge 0\}}\mathbb{E}^u\left[\int_0^t \sigma u(s)e^{-\delta s}v'(0)^2 ds\right] \tag{3.11}
$$

which is finite because v' is finite. Hence M_t is a uniformly integrable martingale, and since τ is a stopping time, $M_{t\wedge\tau}$ is also a uniformly integrable martingale (see Theorem 7.14 in Klebaner, 2005).

Now we are interested in observing inequation (3.8) when $t \to \infty$. The first part of RHS of inequation with respect of Fatou's lemma turns to $\lim_{t\to\infty}\mathbb{E}_x[e^{-\delta(t\wedge\tau)}v(X(t\wedge\tau))] \ge 0$.

For the second part of RHS of equation (3.8) we have $dN_\gamma(s)$ and $v(s)$ greater than 0. When taking $t \to \infty$ by the monotone convergence theorem we obtain:

$$\lim_{t\to\infty}\mathbb{E}_x\left[\int_0^{t\wedge\tau} e^{-\delta s} g(\xi_s) dN_\gamma(s)\right] =$$
$$\lim_{t\to\infty}\mathbb{E}_x\left[\int_0^{t\wedge\tau} e^{-\delta s} g(\xi_s)\mathbb{I}_{\{\tau<\infty\}} dN_\gamma(s)\right] + \lim_{t\to\infty}\mathbb{E}_x\left[\int_0^{t\wedge\tau} e^{-\delta s} g(\xi_s)\mathbb{I}_{\{\tau=\infty\}} dN_\gamma(s)\right] \geq$$
$$\mathbb{E}_x\left[\int_0^{t\wedge\tau} e^{-\delta s} g(\xi_s)\mathbb{I}_{\{\tau<\infty\}} dN_\gamma(s)\right] \tag{3.12}$$

Combining all above we finally get:

$$v(x) \geq \mathbb{E}\left[\int_0^{t\wedge\tau} e^{-\delta s} g(\xi_s)\mathbb{I}_{\{\tau<\infty\}} dN_\gamma(s)\right] = V^\pi(x) \tag{3.13}$$

Since π is arbitrary:

$$v(x) \geq V^{\pi^*}(x) = V(x) \tag{3.14}$$

Now we should prove that $v(x) \leq V^{\pi^*}(x) = V(x)$. When an optimal strategy π^* is applied condition 1 implies that the sum of integrals with reference to to ds in (3.8) is zero:

$$\int_0^{t\wedge\tau} e^{-\delta s}\mathcal{L}^{u^{\pi^*}} v(X(s))ds + \int_0^{t\wedge\tau} e^{-\delta s}\gamma[g(\xi_s^{\pi^*}) + v(X(s-)) - \xi_s^{\pi^*} - v(X(s-))]ds = 0 \tag{3.15}$$

Now taking the expectation of (3.8) assuming (3.9) we will get:

$$v(x) = \mathbb{E}_x[e^{-\delta(t\wedge\tau)} v(X(t\wedge\tau))] + \mathbb{E}_x\left[\int_0^{t\wedge\tau} e^{-\delta s} g(\xi_s^*) dN_\gamma(s)\right] \tag{3.16}$$

Firstly, we should show that taking $t \to \infty$ we will have:

$$\lim_{t\to\infty}\mathbb{E}_x[e^{-\delta(t\wedge\tau)} v(X(t\wedge\tau))] = 0 \tag{3.17}$$

To verify this we can use dominated convergence theorem, which requires to identify an integrable random variable R such that $\mathbb{E}_x[e^{-\delta(t\wedge\tau)} v(X(t\wedge\tau))] < R$ with $\mathbb{E}[R] < \infty$.

Here we use the same steps as in Benjamin Avanzi 2014 paper and assuming that $\sup u(s) = 1$ as $u \in [0;1]$ we will get:

$$\mathbb{E}_x[\sup X(t)] \leq (x + \mathbb{E}_x[N_\gamma(\tau)]b^*) + (\mathbb{E}_x[N_\gamma(\tau)] + 1)(\mu + \frac{\sigma}{\sqrt{2\gamma}}) \tag{3.18}$$
$$s.t.\ X(t)\mathbb{I}_{\{t<\tau\}} \leq \sup_{0\leq t<\tau} X(t),\ \mathbb{E}_x[\sup_{0\leq t<\tau} X(t)] < \infty$$

Letting $R = \sup_{\{0\leq t<\tau\}} a_1 X(t) + a_2$ as we consider function v to be linearly bounded with reference to all above we can state that R dominates $e^{-\delta(t\wedge\tau)} v(X(t\wedge\tau))$ which means we can apply dominated convergence theorem which leads to (3.17). For the second term of RHS in (3.16) we observe that it is monotonically increasing as t increases. In addition $N_\gamma(s),\ g(\xi_s^{\pi^*}) > 0$. When taking $t \to \infty$ we need to consider case then $\tau < \infty$ and by monotone convergence theorem we have:

$$\lim_{t\to\infty}\mathbb{E}_x\left[\int_0^{t\wedge\tau} e^{-\delta s} g(\xi_s^{\pi^*}) dN_\gamma(s)\right] =$$
$$\lim_{t\to\infty}\mathbb{E}_x\left[\int_0^{t\wedge\tau} e^{-\delta s} g(\xi_s^{\pi^*})\mathbb{I}_{\{t<\tau\}} dN_\gamma(s)\right] = \mathbb{E}_x\left[\int_0^{t} e^{-\delta s} g(\xi_s^{\pi^*})\mathbb{I}_{\{t<\tau\}} dN_\gamma(s)\right] \tag{3.19}$$

Now combining (3.17) and (3.19) for (3.16) we have:

$$v(x) \leq V^{\pi^*}(x) = V(x) \tag{3.20}$$

Now the only possible solution to the system of inequations (3.14) and (3.20) is that:

$$v(x) = V(x) \geq 0 \tag{3.21}$$

This means that any admissible under periodic barrier dividend payment strategy control policy π^*, which satisfies HJB equation, make NPV equal to the Value Function $V(x)$.

4. The determination of present value function under the periodic two-barrier strategy with transaction costs

Now, after verication theorem and dividend barrier strategies were designated, it is necessary to return to the HJB equation. As said earlier, we have already opened the first maximum connected to the value paid to shareholder, now we should open the second maximum connected to the reinsured part.

$$\max_{0\le u\le 1} V'(x)\mu u(x)+\frac{1}{2}V''(x)\sigma^2u^2(x)-\delta V(x)=0,\ x<b_1 \tag{4.1}$$

Let $u^*(x)$ be the maximizer of the expression on the left side of (4.1) that is:

$$u^*(x)=-\frac{\mu V'(x)}{\sigma^2V''(x)} \tag{4.2}$$

After substituting (4.2) into (4.1) we can get the first equation of our system to find $V(x)$, because when $x<b_1:\ \xi=0$, therefore first maximum of the HJB turns to 0:

$$\frac{\mu^2[V'(x)]^2}{2\sigma^4[V''(x)]^2}\mu V''(x)-\frac{\mu V'(x)}{\sigma^2V''(x)}V'(x)-\delta V(x)=-\frac{\mu^2[V'(x)]^2}{2\sigma^2V''(x)}-\delta V(x)=0 \tag{4.3}$$

Let's define x_0 here as $x_0=arg(u^*(x)=1)$, and due to the fact that $u^*(x)$ is increasing on $(0,b_2]$ it is the first and only point where reinsurance rate reaches its maximum limit value as $u^*(x)\in[0,1]$.

Therefore, when $b_2>x>x_0$ (assuming $x_0<b_2$) we override optimal $u^*(x)=1$, which can be derived using properties of $V(x)$ ($\frac{V'}{V''}$ is decreasing), but also intuitively obvious, as bigger surpluss of insurance company leads to a smaller re-insurance needs. Therefore, we can get the second equation of our system to find $V(x)$:

$$\max_{0\le u\le 1} V'(x)\mu u(x)+\frac{1}{2}V''(x)\sigma^2u^2(x)-\delta V(x)=V'(x)\mu+\frac{1}{2}V''(x)\sigma^2-\delta V(x)=0,\ x\in[x_0;b_1] \tag{4.4}$$

For the third equation we should consider the case when $x>b_2$ when we can open both maximums and obtain the third equation of system:

$$\gamma\left(-K+\xi+(V(x-\xi)-V(x))\right)+V'(x)\mu u(x)+\frac{1}{2}V''(x)\sigma^2u^2(x)-\delta V(x)=0 \tag{4.5}$$

Overall the following system is obtained using cases (4.3), (4.4), and (4.5):

$$\begin{cases} -\frac{\mu^2[V'(x)]^2}{2\sigma^2V''(x)}-\delta V(x)=0 & ,\ x\in[0,x_0] \\ \frac{1}{2}\sigma^2V''(x)+\mu V'(x)-\delta V(x)=0 & ,\ x\in[x_0,b_2] \\ \gamma(-K+(x-b_1)+V(b_1)-V(x))+\frac{1}{2}\sigma^2V''(x)+\mu V'(x)-\delta V(x)=0 & ,\ x\in[b_2,\infty) \end{cases} \tag{4.6}$$

To find solution of the first differential equation it is necessary to multiply both terms by the $V''(x)$ and devide them by $V(x)$:

$$-\frac{\mu^2}{2\sigma^2}V'(x)^2-\delta V(x)V''(x)=0 \iff -\frac{\mu^2}{2\delta\sigma^2}\frac{V'(x)}{V(x)}=\frac{V''(x)}{V'(x)}$$

Both sides has fractions wich can be represented as derivatives of $V(x)$ and $V'(x)$ respectively:

$$-\frac{\mu^2}{2\delta\sigma^2}d\ln(V(x))=d\ln(V'(x))$$

Now, we can assuming $ln(V(x))$ and $ln(V'(x))$ as variables we integrate both parts and after some algebraic transformations we will get the solution:

$$\begin{gathered} -\frac{\mu^2}{2\delta\sigma^2}\ln(V(x))=\ln(V'(x))+C \iff \ln(V(x)^{-\frac{\mu^2}{2\delta\sigma^2}})=\ln(V'(x))+C \\ \iff V(x)^{-\frac{\mu^2}{2\delta\sigma^2}}=e^CV'(x) \iff dx=C(V(x)^{\frac{\mu^2}{2\delta\sigma^2}}dV(x)) \\ \iff x=\frac{C}{\frac{\mu^2}{2\delta\sigma^2}+1}V(x)^{\frac{\mu^2}{2\delta\sigma^2}+1} \iff V(x)=Cx^{\frac{1}{\frac{\mu^2}{2\delta\sigma^2}+1}}=Cx^{\frac{\delta}{\frac{\mu^2}{2\sigma^2}+\delta}} \end{gathered}$$

Otherwise $V(x)=Cx^{\alpha}$, where $\alpha=\frac{\delta}{\frac{\mu^2}{2\sigma^2}+\delta}$, then inserting $V(x)$ in (4.2) it follows $u^*(x)=-\frac{\mu x}{\sigma^2(\alpha-1)}$.

The solution for the second equation of the system is a solution of the homogeneous equation with corresponding characteristic equation:

$$\frac{\sigma^2}{2}\theta^2 + \mu\theta - \delta = 0$$

And it's solution is $\theta_+ = \frac{-\mu+\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2},\ \theta_- = \frac{-\mu-\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}$. The solution to the whole differential equation will be $V(x) = C^1 e^{\theta_+(x-x_0)} + C^2 e^{-\theta_+(x-x_0)}$, where C^1 and C^2 are constants.

Now using the fact that $V(x)$ and $V'(x)$ are continuous at point x_0 we can find connection between constants $C,\ C^1,\ C^2$.

$$\begin{cases} \lim\limits_{x\to x_0-} V(x) = Cx_0^\alpha \\ \lim\limits_{x\to x_0+} V(x) = C^1 + C^2 \end{cases} \Rightarrow Cx_0^\alpha = C^1 + C^2$$

$$\begin{cases} \lim\limits_{x\to x_0-} V'(x) = C\alpha x_0^{\alpha-1} \\ \lim\limits_{x\to x_0+} V'(x) = C^1\theta_+ + C^2\theta_- \end{cases} \Rightarrow C\alpha x_0^{\alpha-1} = C^1\theta_+ + C^2\theta_-$$

After trivial algebraic transformations we get:

$$C^1 = C\frac{\alpha x_0^{\alpha-1} - \theta_- x_0^\alpha}{\theta_+ - \theta_-},\ C^2 = C\frac{\theta_+ x_0^\alpha - \alpha x_0^{\alpha-1}}{\theta_+ - \theta_-}$$

To sum up, the solution of the second equation expressed through consant C from first equation is:

$$\begin{gathered} V(x) = C(a_1 e^{\theta_+(x-x_0)} + a_2 e^{\theta_-(x-x_0)}) \\ a_1 = \frac{\alpha x_0^{\alpha-1} - \theta_- x_0^\alpha}{\theta_+ - \theta_-},\ a_2 = \frac{\theta_+ x_0^\alpha - \alpha x_0^{\alpha-1}}{\theta_+ - \theta_-} \end{gathered} \tag{4.7}$$

Further we will consider this part as $V(x) = Ch(x)$, where $h(x) = a_1 e^{\theta_+(x-x_0)} + a_2 e^{\theta_-(x-x_0)}$. Also since $u^*(x_0) = 1$ it follows that $x_0 = \frac{\sigma^2(1-\alpha)}{\mu}$.

Now continue with the third equation. First, we solve the homogeneous part of the equation same way as in the second equation, where we derive two roots from characteristic equation:

$$\begin{gathered} \frac{\sigma^2}{2}\theta^2 + \mu\theta - (\delta+\gamma) = 0 \\ \alpha_\gamma = \frac{-\mu + \sqrt{\mu^2 + 2(\delta+\gamma)\sigma^2}}{\sigma^2},\ \lambda_\gamma = \frac{-\mu - \sqrt{\mu^2 + 2(\delta+\gamma)\sigma^2}}{\sigma^2} \end{gathered}$$

The solution to homogeneous part $y_1 = C^3 e^{\alpha_\gamma x} + C^4 e^{\lambda_\gamma x}$. Continuing with the solution to inhomogeneous part:

$$V''(x) + \frac{2\mu}{\sigma^2}V'(x) - \frac{2(\delta+\gamma)}{\sigma^2}V(x) = -\frac{2\gamma}{\sigma^2}\left(V(b_1) + (x - b_1) - K\right) \tag{4.8}$$

We will be looking for the solution in form of $y_0 = Ax + B,\ y_0' = A,\ y_0'' = 0$:

$$\frac{2\mu}{\sigma^2}A - \frac{2(\delta+\gamma)}{\sigma^2}(Ax + B) = -\frac{2\gamma}{\sigma^2}\left((Ab_1 + B) + (x - b_1) - K\right)$$

Let's determine the coefficient at x:

$$-\frac{2}{\sigma^2}(\delta+\gamma) = -\frac{2\gamma}{\sigma^2}(x) \Rightarrow A = \frac{\gamma}{\delta+\gamma}$$

And free term:

$$\frac{2\mu\gamma}{\sigma^2(\delta+\gamma)} - \frac{2}{\sigma^2}(\delta+\gamma)B = -\frac{2\gamma}{\sigma^2}\left(V(b_1) - b_1 - K\right)$$

$$B = \frac{\sigma^2}{2(\delta+\gamma)}\left(\frac{2\gamma}{\sigma^2}\left(V(b_1) - b_1 - K\right) + \frac{2\mu\gamma}{\sigma^2(\delta+\gamma)}\right)$$

$$B = \frac{\gamma}{\delta+\gamma}\left(V(b_1) - b_1 - K + \frac{\mu}{\delta+\gamma}\right)$$

Finally, we can obtain our generalized solution using only positive root from the homogeneous part and solution to inhomogeneous part:

$$G_R = y_0^+ + y_1 = C^4 e^{\lambda_\gamma x} + \frac{\gamma}{\delta+\gamma}\left(V(b_1) + (x - b_1) - K + \frac{\mu}{\delta+\gamma}\right) \tag{4.9}$$

So now we can construct our candidate value function:

$$v(x)=\begin{cases}Cx^{\alpha} & ,\ x\in[0,x_0]\\ Ch(x) & ,\ x\in[x_0,b_2]\\ C^4e^{\lambda_\gamma x}+\frac{\gamma}{\delta+\gamma}\left(G_R(b_1)+(x-b_1)-K+\frac{\mu}{\delta+\gamma}\right) & ,\ x\in[b_2,\infty)\end{cases}\tag{4.10}$$

Now after we figured out the shape of our candidate value function we can find the explicit value of constants. First, let's analize continuity of $V(x),\ V'(x),\ V''(x)$ in b_2:

$$\begin{cases}\lim\limits_{x\to b_2-}V(x)=Ch(b_2)\\ \lim\limits_{x\to b_2+}V(x)=C^4e^{\lambda x}+\frac{\gamma}{\delta+\gamma}\left[CA_1(b_1)+(b_2-b_1)-K+\frac{\mu}{\delta+\gamma}\right]\end{cases}\Rightarrow Ch(b_2)=C^4e^{\lambda x}+\frac{\gamma}{\delta+\gamma}\left[CA_1(b_1)+(b_2-b_1)-K+\frac{\mu}{\delta+\gamma}\right]$$

where

$$A_1(b_1)=\begin{cases}b_1^{\alpha} & ,\ b_1\in[0,x_0]\\ h(b_1) & ,\ b_1\in[x_0,b_2]\end{cases}\tag{4.11}$$

due to the fact that we do not know where, relative to the point x_0, is point b_1. Also we will asign new variable $g=b_2-b_1-K$. Now we can find C^4 as a function of C:

$$Ch(b_2)=C^4e^{\lambda_\gamma x}+\frac{\gamma}{\delta+\gamma}\left[CA_1(b_1)+g+\frac{\mu}{\delta+\gamma}\right]$$
$$C^4=e^{-\lambda_\gamma x}\left(Ch(b_2)-\frac{\gamma}{\delta+\gamma}\left[CA_1(b_1)+g+\frac{\mu}{\delta+\gamma}\right]\right)=e^{-\lambda_\gamma x}\left(C\left[h(b_2)-\frac{\gamma A_1(b_1)}{\delta+\gamma}\right]-\frac{\gamma}{\delta+\gamma}\left(g+\frac{\mu}{\delta+\gamma}\right)\right)$$

As the first derivative is continuous at point b_2, we can find right and left limits at b_2- and b_2+:

$$\begin{cases}\lim\limits_{x\to b_2-}V'(x)=Ch'(b_2)\\ \lim\limits_{x\to b_2+}V'(x)=e^{-\lambda_\gamma b_2}C^4\lambda_\gamma+\frac{\gamma}{\delta+\gamma}\end{cases}\Rightarrow Ch'(b_2)=e^{-\lambda_\gamma b_2}C^4\lambda_\gamma+\frac{\gamma}{\delta+\gamma}$$

Inserting C^4 into the equation and finally finding values of our constants C and C^4:

$$Ch'(b_2)=\lambda_\gamma e^{-2\lambda_\gamma b_2}\left[C(h(b_2)-\frac{\gamma}{\delta+\gamma}A_1(b_1))-\frac{\gamma}{\delta+\gamma}\left(g+\frac{\mu}{\delta+\gamma}\right)\right]+\frac{\gamma}{\delta+\gamma}$$

$$C=\frac{\frac{\gamma}{\delta+\gamma}\left[1-\lambda_\gamma e^{-2\lambda_\gamma b_2}\right]\left[g+\frac{\mu}{\delta+\gamma}\right]}{h'(b_2)-\lambda_\gamma e^{-2\lambda_\gamma b_2}h(b_2)+\frac{\gamma\lambda_\gamma e^{-2\lambda_\gamma b_2}A_1(b_1)}{\delta+\gamma}}\tag{4.12}$$

$$C^4=e^{-2\lambda_\gamma b_2}\left(\frac{\frac{\gamma}{\delta+\gamma}\left[1-\lambda_\gamma e^{-2\lambda_\gamma b_2}\right]\left[g+\frac{\mu}{\delta+\gamma}\right]}{h'(b_2)-\lambda_\gamma e^{-2\lambda_\gamma b_2}h(b_2)+\frac{\gamma\lambda_\gamma e^{-2\lambda_\gamma b_2}A_1(b_1)}{\delta+\gamma}}\left[h(b_2)-\frac{\gamma A_1(b_1)}{\delta+\gamma}\right]\right)-\frac{\gamma}{\delta+\gamma}\left(g+\frac{\mu}{\delta+\gamma}\right)\tag{4.13}$$