§ 3 基本应用一

§ 3.1 一维谐振子

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

$\Downarrow$ 正则量子化

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{q}^2 , \quad [\hat{q}, \hat{p}] = i\hbar$$

坐标表象下定态 schrodinger 方程，

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2\right)\psi(x) = E\psi(x)$$

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(i\hat{p} + m\omega\hat{q})$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(-i\hat{p} + m\omega\hat{q})$$

$$[\hat{a}, \hat{a}^\dagger] = \frac{1}{2m\hbar\omega} 2im\omega [\hat{p}, \hat{q}] = \frac{-2im\omega i\hbar}{2m\hbar\omega} = 1 \rightarrow \hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$$

$$\rightarrow \quad \hat{p} = \frac{\sqrt{2m\hbar\omega}}{2i}(\hat{a} - \hat{a}^\dagger) = i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a}) \qquad \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a})$$

$$\hat{q} = \frac{\sqrt{2m\hbar\omega}}{2m\omega}(\hat{a} + \hat{a}^\dagger) = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}) \qquad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a})$$

$$\begin{aligned}
\hat{H} &= \frac{1}{2m}(-1)\frac{m\hbar\omega}{2}(\hat{a}^\dagger - \hat{a})^2 + \frac{1}{2} m\omega^2 \frac{\hbar}{2m\omega}(\hat{a}^\dagger + \hat{a})^2 \\
&= -\frac{1}{4}\hbar\omega(\hat{a}^\dagger - \hat{a})^2 + \frac{1}{4}\hbar\omega(\hat{a}^\dagger + \hat{a})^2 \\
&= \frac{1}{2}\hbar\omega(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) \qquad \hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1 \\
&= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)
\end{aligned}$$

$[\hat{a}, \hat{H}]$

$$= \hbar\omega\left[\hat{a}, \hat{a}^\dagger\hat{a} + \frac{1}{2}\right] = \hbar\omega\,\hat{a}[\hat{a}, \hat{a}^\dagger] = \hbar\omega\hat{a}$$

$[\hat{a}^\dagger, \hat{H}]$

$$= \hbar\omega\left[\hat{a}^\dagger, \hat{a}^\dagger\hat{a} + \frac{1}{2}\right] = \hbar\omega\,\hat{a}^\dagger[\hat{a}^\dagger, \hat{a}] = -\hbar\omega\hat{a}^\dagger$$

求 $\hat{H}$ 的本征态（设为 $|\lambda\rangle$）

$\hat{H}|\lambda\rangle = \lambda\hbar\omega|\lambda\rangle$

$\hookrightarrow \langle\lambda|\hat{H}|\lambda\rangle = \hbar\omega\langle\lambda|\hat{a}^{+}\hat{a}+\frac{1}{2}|\lambda\rangle = \lambda\hbar\omega\langle\lambda|\lambda\rangle$

$0 \le |a|\lambda\rangle|^2 = \langle\lambda|a^{+}a|\lambda\rangle = (\lambda-\frac{1}{2})\hbar\omega \quad \rightarrow \lambda \geq \frac{1}{2}$

$$\begin{cases} \hat{H}\hat{a}|\lambda\rangle = (\hat{a}\hat{H}-\hbar\omega\hat{a})|\lambda\rangle = (\lambda-1)\hbar\omega\hat{a}|\lambda\rangle \\ \hat{H}\hat{a}^{+}|\lambda\rangle = (\hat{a}^{+}\hat{H}+\hbar\omega\hat{a}^{+})|\lambda\rangle = (\lambda+1)\hbar\omega\hat{a}|\lambda\rangle \end{cases}$$

$\Rightarrow$ $\hat{a}|\lambda\rangle$，$\hat{a}^{+}|\lambda\rangle$ 也是 $\hat{H}$ 的本征态。 $(\hat{a}|\lambda\rangle, \hat{a}^{+}|\lambda\rangle \neq 0)$

$\exists\, |\lambda_{min}\rangle$ s.t. $\hat{a}|\lambda_{min}\rangle = 0$ （不存在 $|\lambda_{max}\rangle$ s.t. $\hat{a}^{+}|\lambda\rangle \neq 0$）

本征值序列 $\hbar\omega\{\lambda, \lambda-1 \cdots \frac{1}{2}\}$

$E_n = (n+\frac{1}{2})\hbar\omega$，$\hat{a}^{+}$上升算符，$\hat{a}$下降算符。

记 $\hat{H}$ 特征值 $n+\frac{1}{2}$ 对应的本征态为 $|n\rangle$

设 $\hat{a}|n\rangle = c|n-1\rangle$，求 c 的值。

归一化：$\langle n|\hat{a}^{+}\hat{a}|n\rangle = c^2\langle n-1|n-1\rangle = c^2$

$\langle n|\hat{H}|n\rangle = \langle n|\hbar\omega(\hat{a}^{+}\hat{a}+\frac{1}{2})|n\rangle = (n+\frac{1}{2})\hbar\omega$

$\hookrightarrow c^2 = n$

$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$，

设 $\hat{a}^{+}|n\rangle = c'|n+1\rangle$，求 c' 的值。

$\langle n|\hat{a}\hat{a}^{+}|n\rangle = c'^2\langle n+1|n+1\rangle = c^2$

$\langle n|\hat{H}|n\rangle = \hbar\omega\langle n|\hat{a}^{+}\hat{a}+\frac{1}{2}|n\rangle = \hbar\omega\langle n|\hat{a}\hat{a}^{+}-\frac{1}{2}|n\rangle$

$= (n+\frac{1}{2})\hbar\omega$

$\hookrightarrow c'^2 = n+1$，$c' = \sqrt{n+1}$

$\hat{a}^{+}|n\rangle = \sqrt{n+1}|n+1\rangle$

$\Rightarrow \hat{a}^{+}\hat{a}|n\rangle = \hat{a}^{+}\sqrt{n}|n-1\rangle = n|n\rangle$ 　$|n\rangle$ 是数算符 $\hat{N} = \hat{a}^{+}\hat{a}$ 的本征态，本征值为 n

$(\hat{a}^{+})^n|0\rangle = \sqrt{n!}|n\rangle$，$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^{+})^n|0\rangle$

在坐标表象下：H的本征态的形式

$$\hat a|0\rangle = 0,$$

$$\langle x|\hat a|0\rangle = \frac{1}{\sqrt{2m\hbar\omega}}\langle x|i\hat p + m\omega\hat x|0\rangle = 0$$

$\beta = \sqrt{\frac{m\omega}{\hbar}}$

$$= \frac{1}{\sqrt{2m\hbar\omega}}\left(\hbar\frac{\partial}{\partial x} + m\omega x\right)\langle x|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{\partial}{\partial x} + \beta^2 x\right)\langle x|0\rangle$$

解得 $\langle x|0\rangle = C e^{-\frac{1}{2}\beta^2x^2}$ 本征态 $|0\rangle$ 在 x 表象下的形式.

归一化：$\int_{-\infty}^{\infty}|\langle x|0\rangle|^2 dx = C^2\int_{-\infty}^{\infty} e^{-\beta^2x^2}dx$

$$= C^2\sqrt{\frac{\pi}{\beta^2}} = 1 \rightarrow C = \sqrt{\frac{\beta}{\sqrt{\pi}}}$$

$$\psi_0(x) = \sqrt{\frac{\beta}{\sqrt{\pi}}}\, e^{-\frac{1}{2}\beta^2x^2}$$

$$\langle\hat x|\hat a^\dagger|\hat x\rangle = \langle\hat x|\frac{1}{\sqrt{2m\hbar\omega}}(-i\hat p + m\omega\hat x)|\hat x\rangle$$

$$= \frac{1}{\sqrt{2m\hbar\omega}}\left(-\hbar\frac{\partial}{\partial x} + m\omega x\right) = \underline{\frac{1}{\sqrt{2}}\left(-\frac{1}{\beta}\frac{\partial}{\partial x} + \beta x\right)}$$

$$\langle x|n\rangle = \langle x|\frac{1}{\sqrt{n!}}(\hat a^\dagger)^2|0\rangle$$

$$= \langle x|\frac{1}{\sqrt{n!}}(\hat a^\dagger)^n|x\rangle\langle x|0\rangle$$

$\langle x|\hat A|x\rangle = K$

$\Updownarrow$

$\langle x|\hat A^n|x\rangle = K^n$

$$= \frac{1}{\sqrt{n!}}\underline{\left(\frac{1}{\sqrt{2}}\right)^n\left(-\frac{1}{\beta}\frac{d}{dx} + \beta x\right)^n}\langle x|0\rangle$$

$$= \left(\frac{\beta}{\sqrt{\pi}}\right)^{1/2}\frac{1}{\sqrt{2^n n!}}\underline{H_n(\beta x)}\, e^{-\frac{1}{2}\beta^2x^2}$$

$H_n(x) = \left(-\frac{d}{dx} + x\right)^n e^{-\frac{1}{2}x^2}$

例：在 $t=0$ 时刻，一个谐振子处的状态为 $|\psi\rangle = |1\rangle - i|2\rangle$

求坐标的不确定度.

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}|1\rangle e^{-i\frac{3}{2}\omega t} - \frac{1}{\sqrt{2}}i|2\rangle e^{-i\frac{5}{2}\omega t}$$

$$\langle\psi^*(t)|\hat x|\psi(t)\rangle = \langle\psi^*(t)|\sqrt{\frac{\hbar}{2m\omega}}(\hat a^\dagger + \hat a)|\psi(t)\rangle$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\Big((\langle 1|e^{i\frac{3}{2}\omega t} + i\langle 2|e^{i\frac{5}{2}\omega t})\hat a^\dagger(|1\rangle e^{-i\frac{3}{2}\omega t} - i|2\rangle e^{-i\frac{5}{2}\omega t})$$

$$+ (\langle 1|e^{i\frac{3}{2}\omega t} + i\langle 2|e^{i\frac{5}{2}\omega t})\hat a(|1\rangle e^{-i\frac{3}{2}\omega t} - i|2\rangle e^{-i\frac{5}{2}\omega t})\Big)$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\Big((\langle 1|e^{i\frac{3}{2}\omega t} + i\langle 2|e^{i\frac{5}{2}\omega t})(\sqrt{2}|2\rangle e^{-i\frac{3}{2}\omega t} - i\sqrt{3}|3\rangle e^{-i\frac{5}{2}\omega t})$$

$$+ (\langle 1|e^{i\frac{3}{2}\omega t} + i\langle 2|e^{i\frac{5}{2}\omega t})(|0\rangle e^{-i\frac{3}{2}\omega t} - \sqrt{2}i|1\rangle e^{-i\frac{5}{2}\omega t})\Big)$$

$$= \frac{1}{\sqrt{2}}\sqrt{\frac{\hbar}{2m\omega}}(ie^{i\omega t} - ie^{-i\omega t}) = \sqrt{2}\sqrt{\frac{\hbar}{2m\omega}}\sin\omega t$$

$$\langle \psi^*(t) | \hat{x}^2 | \psi(t) \rangle$$

$$= \langle \psi^*(t) | \frac{\hbar}{2m\omega} (\hat{a}^{+2} + \hat{a}^+\hat{a} + \hat{a}\hat{a}^+ + \hat{a}^2) | \psi(t) \rangle$$

$$= \frac{\hbar}{4m\omega} \Big((\langle 1| e^{i\frac{3}{2}\omega t} + i\langle 2| e^{i\frac{5}{2}\omega t}, \hat{a}^{+2} (|1\rangle e^{-i\frac{3}{2}\omega t} - i|2\rangle e^{-i\frac{5}{2}\omega t})$$

$$+ (\langle 1| e^{i\frac{3}{2}\omega t} + i\langle 2| e^{i\frac{5}{2}\omega t}, \hat{a}^+\hat{a} (|1\rangle e^{-i\frac{3}{2}\omega t} - i|2\rangle e^{-i\frac{5}{2}\omega t})$$

$$+ (\langle 1| e^{i\frac{3}{2}\omega t} + i\langle 2| e^{i\frac{5}{2}\omega t}, \hat{a}\hat{a}^+ (|1\rangle e^{-i\frac{3}{2}\omega t} - i|2\rangle e^{-i\frac{5}{2}\omega t})$$

$$+ (\langle 1| e^{i\frac{3}{2}\omega t} + i\langle 2| e^{i\frac{5}{2}\omega t}, \hat{a}^2 (|1\rangle e^{-i\frac{3}{2}\omega t} - i|2\rangle e^{-i\frac{5}{2}\omega t}) \Big)$$

$$= \frac{\hbar}{4m\omega} \Big((\langle 1| e^{i\frac{3}{2}\omega t} + i\langle 2| e^{i\frac{5}{2}\omega t}, (\sqrt{6}|3\rangle e^{-i\frac{3}{2}\omega t} - i\sqrt{12}|4\rangle e^{-i\frac{5}{2}\omega t})$$

$$+ (\langle 1| e^{i\frac{3}{2}\omega t} + i\langle 2| e^{i\frac{5}{2}\omega t}, \hat{a}^+\hat{a} (|1\rangle e^{-i\frac{3}{2}\omega t} - i|2\rangle e^{-i\frac{5}{2}\omega t})$$

$$+ (\langle 1| e^{i\frac{3}{2}\omega t} + i\langle 2| e^{i\frac{5}{2}\omega t}, \hat{a}\hat{a}^+ (|1\rangle e^{-i\frac{3}{2}\omega t} - i|2\rangle e^{-i\frac{5}{2}\omega t}) \qquad \hat{a}\hat{a}^+ = \hat{a}^+\hat{a} + 1$$

$$+ (\langle 1| e^{i\frac{3}{2}\omega t} + i\langle 2| e^{i\frac{5}{2}\omega t}, (-i|0\rangle e^{-i\frac{5}{2}\omega t}) \Big) \qquad = \hat{N} + 1$$

$$= \frac{\hbar}{4m\omega} \Big[2\cdot(\langle 1| e^{i\frac{3}{2}\omega t} + i\langle 2| e^{i\frac{5}{2}\omega t}, \hat{a}^+\hat{a} (|1\rangle e^{-i\frac{3}{2}\omega t} - i|2\rangle e^{-i\frac{5}{2}\omega t})$$

$$+ (1+1) \Big]$$

$$= \frac{\hbar}{4m\omega} \Big[2\cdot(1+2) + 2 \Big] = \frac{\hbar}{2m\omega} \cdot 4$$

$$\langle \Delta x \rangle = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2}$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \sqrt{4 - 2\sin^2 \omega t}$$

$$= \sqrt{\frac{\hbar}{m\omega}} \sqrt{2 - \sin^2 \omega t}$$

§ 3.2 角动量

$\hat{L} = \hat{r} \times \hat{p}$

$[\hat{L}_i, \hat{L}_j] = \varepsilon_{ijk} i\hbar L_k e_k$

$\hat{L}^2 = \hat{L}\cdot\hat{L} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$

$[\hat{L}^2, \hat{L}_i] = 0$

↳ 选取 $(\hat{L}^2, \hat{L}_z)$ 为力学量完全集

共同本征态，设为 $|\lambda\rangle$

定义 $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$

$[\hat{L}_z, \hat{L}_\pm] = [\hat{L}_z, \hat{L}_x] \pm i[\hat{L}_z, \hat{L}_y]$

$= i\hbar(L_y \mp iL_x)$

$= \pm\hbar\hat{L}_\pm$

$[\hat{L}^2, \hat{L}_\pm] = [\hat{L}^2, L_x] \pm i[\hat{L}^2, \hat{L}_y] = 0$

$\hat{L}^2(L_\pm|\lambda\rangle) = \hat{L}_\pm(\hat{L}^2|\lambda\rangle) = \mu\hat{L}_\pm|\lambda\rangle$

$\hat{L}_z(\hat{L}_\pm|\lambda\rangle) = \pm\hbar\hat{L}_\pm|\lambda\rangle + L_zL_\pm|\lambda\rangle = (\nu\pm\hbar)(\hat{L}_\pm|\lambda\rangle)$

$[L_z, L_\pm] = \hat{L}_z\hat{L}_\pm - \hat{L}_\pm\hat{L}_z = \pm\hbar\hat{L}_\pm$

设 $l\hbar$ 为 L_z 的最大本征值，对应本征态为 $|\lambda_t\rangle$

此时 $\hat{L}_+|\lambda_t\rangle = 0$

$\hat{L}_z|\lambda\rangle = l\hbar|\lambda\rangle$ ， $\hat{L}^2|\lambda_t\rangle = \mu|\lambda_t\rangle$

$\hat{L}^2 = \hat{L}_\pm\hat{L}_\mp + \hat{L}_z^2 \mp \hbar\hat{L}_z$

$\hat{L}^2|\lambda_t\rangle = \hat{L}_-\hat{L}_+|\lambda_t\rangle + \hat{L}_z^2|\lambda_t\rangle + \hbar\hat{L}_z|\lambda_t\rangle$

$= l^2\hbar^2|\lambda_t\rangle + l\hbar^2|\lambda_t\rangle$

$= l(l+1)\hbar^2|\lambda_t\rangle$

同理：存在 $\bar{l}\hbar$ 为 L_z 的最小本征值，对应本征态为 $|\lambda_b\rangle$

此时 $\hat{L}_-|\lambda_b\rangle = 0$

$\hat{L}_z|\lambda_b\rangle = \tilde{l}\hbar|\lambda_b\rangle$, $\hat{L}^2|\lambda_b\rangle = \mu|\lambda_b\rangle$

$$\hat{L}^2|\lambda_b\rangle = \hat{L}_+\hat{L}_-|\lambda_b\rangle + \hat{L}_z^2|\lambda_b\rangle - \hbar\hat{L}_z|\lambda_b\rangle$$
$$= \tilde{l}(\tilde{l}-1)\hbar^2|\lambda_b\rangle$$

$|\lambda_t\rangle$, $|\lambda_b\rangle$ 都是 $\hat{L}^2$ 的本征态。

$\hookrightarrow$ $L^2 = l(l+1) = \tilde{l}(\tilde{l}-1)$

解得 $\tilde{l} = -l$ 或 $l+1$ (舍去)

由 $\hat{L}_\pm$ 性质: $l - \tilde{l} = 2l = N$ (整数)

$\Rightarrow$ 共同本征态 $|l,m\rangle$ $(m = -l, -l+1, \cdots, l)$

$\hat{L}^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle$, $\hat{L}_z|l,m\rangle = m\hbar|l,m\rangle$

$\hat{L}_-|l,m\rangle = c\hbar|l,m-1\rangle$

$\langle l,m|\hat{L}_+\hat{L}_-|l,m\rangle = |c|^2\hbar^2\langle l,m-1|l,m-1\rangle$

$\hat{L}_+\hat{L}_- = \hat{L}^2 - \hat{L}_z^2 + \hbar\hat{L}_z$

$= \langle l,m|\hat{L}^2|l,m\rangle - \langle l,m|\hat{L}_z^2|l,m\rangle + \hbar\langle l,m|\hat{L}_z|l,m\rangle$

$= l(l+1)\hbar^2 - m^2\hbar^2 - m\hbar = [l(l+1) - m(m-1)]\hbar$

$\hookrightarrow$ $c = \sqrt{l(l+1) - m(m-1)}$

$\hat{L}_-|l,m\rangle = \sqrt{l(l+1) - m(m-1)}$

设 $\hat{L}_+|l,m\rangle = c\hbar|l,m+1\rangle$

$\langle l,m|\hat{L}_-\hat{L}_+|l,m\rangle$ $\quad \hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z$

$= [l(l+1) - m^2 - m]\hbar^2$

$\hookrightarrow$ $c = \sqrt{l(l+1) - m(m+1)}\ \hbar$

$\hat{L}_+|l,m\rangle = \sqrt{l(l+1) - m(m+1)}\ \hbar|l,m+1\rangle$

轨道角动量在球坐标下的具体形式：

$$Y_{lm}(\theta,\varphi) = \langle \theta,\varphi | l,m \rangle$$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\begin{cases} \hat{L}_x = -i\hbar\left(-\sin\phi \frac{\partial}{\partial\theta} - \cos\phi \cot\theta \frac{\partial}{\partial\phi}\right) \\ \hat{L}_y = -i\hbar\left(\cos\phi \frac{\partial}{\partial\theta} - \sin\phi \cot\theta \frac{\partial}{\partial\phi}\right) \\ \hat{L}_z = -i\hbar \frac{\partial}{\partial\phi} \end{cases}$$

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y = -i\hbar\left[(-\sin\phi \pm i\cos\phi)\frac{\partial}{\partial\theta} + (-\cos\phi \mp i\sin\phi)\cot\theta\frac{\partial}{\partial\phi}\right]$$

$$= i\hbar\left[(\sin\phi \mp i\cos\phi)\frac{\partial}{\partial\theta} + (\cos\phi \pm i\sin\phi)\cot\theta\frac{\partial}{\partial\phi}\right]$$

$$= \hbar\left[(\pm\cos\phi + i\sin\phi)\frac{\partial}{\partial\theta} + i e^{\pm i\phi}\cot\theta\frac{\partial}{\partial\phi}\right]$$

$$= \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) \qquad + = \pm \cdot \pm$$

$$= \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

$$= -\hbar^2\Big[\sin^2\phi\frac{\partial^2}{\partial\theta^2} + \sin\phi\cos\phi\frac{\partial}{\partial\theta}\left(\cot\theta\frac{\partial}{\partial\phi}\right) + \cos^2\phi\cot\theta\frac{\partial^2}{\partial\phi^2} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\theta}\right)$$

$$+ \cos^2\phi\frac{\partial^2}{\partial\theta^2} - \cos\phi\sin\phi\frac{\partial}{\partial\theta}\left(\cot\theta\frac{\partial}{\partial\phi}\right) + \sin^2\phi\cot^2\theta\frac{\partial^2}{\partial\phi^2} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\theta}\right) + \frac{\partial^2}{\partial\phi^2}$$

$$= -\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \cot\theta(\cos^2\phi + \sin^2\phi)\frac{\partial}{\partial\theta}\right]$$

$$= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

由 $\hat{L}_+|l,l\rangle = 0$ $\qquad$ $\hat{L}_+|\lambda_t\rangle = 0$

$$\hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)Y_{ll}(\theta,\varphi) = 0$$

分离变量法 $Y_{ll}(\theta,\varphi) = f(\theta)g(\varphi)$ $\qquad$ $\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$

$\Rightarrow$ $g\frac{\partial f}{\partial\theta} + i\cot\theta f\frac{\partial g}{\partial\phi} = 0$, $\quad \frac{f'}{f}\tan\theta = -i\frac{g'}{g} = C$ $\qquad$ $\hat{L}_z Y_{ll}(\theta,\varphi) = l\hbar fg$

$\frac{f'}{f} = l\cot\theta$, $\quad \frac{g'}{g} = il$ $\qquad$ $= -i\hbar fg' \rightarrow C = l$

解得 $f(\theta) = e^{C_1}\sin^l\theta$, $\quad g(\varphi) = e^{i(l+C_2)\varphi}$

$$Y_{ll}(\theta,\varphi) \propto \sin^l\theta\, e^{il\varphi}$$

在用下降算符：

$$Y_{lm}(\theta,\varphi) = \langle\theta,\varphi|l,m\rangle = \langle\theta,\varphi|\hat{L}_-^{l-m}|l,l\rangle$$

$$= \alpha_m(\hat{L}_-)^{l-m}\sin^l\theta\, e^{il\varphi}$$

例：在($\hat L^2$, $\hat L_z$)表象下，粒子运动状态的角度部分可用如下函数描写：

$$\psi(\theta,\varphi) = Y_{00}(\theta,\varphi) + Y_{10}(\theta,\varphi) + \sqrt{2}\,Y_{11}(\theta,\varphi)$$

(1) 求 $\hat L^2$ 的期待值.

归一化：$\psi^*\psi = 1+1+2 = 4 \rightarrow \psi = \frac{1}{2}\psi$

$$\langle \psi^*, \hat L^2, \psi\rangle = \frac{1}{4}(0 + 1\cdot 2 + 2\cdot 1\cdot 2)\hbar^2 = \frac{3}{2}\hbar^2$$

(2) 求 $L_n = \hat L\cdot\vec n$，$\vec n = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\beta)$

的可能测量值和概率

$l=0$：0. $P = |\langle 0,0|\psi\rangle|^2 = \frac{1}{4}$

$l=1$：$|1,1\rangle$ $|1,0\rangle$，$|1,-1\rangle$ 为基底.

此时 $\hat L_+|1,1\rangle = 0$，$\hat L_+|1,0\rangle = \sqrt{1(1+1) - 0(0+1)}\,\hbar|1,1\rangle = \sqrt{2}\hbar|1,1\rangle$

$\hat L_+|1,-1\rangle = \sqrt{1\cdot(1+1) - (-1)(-1+1)}\,\hbar|1,0\rangle = \sqrt{2}\hbar|1,0\rangle$

$$\hat L_+ : \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}\hbar$$

$\hat L_-|1,1\rangle = \sqrt{1(1+1) - 1(1-1)}\,\hbar|1,0\rangle = \sqrt{2}\hbar|1,0\rangle$

$\hat L_-|1,0\rangle = \sqrt{1(1+1) - 0(0-1)}\,\hbar|1,-1\rangle = \sqrt{2}\hbar|1,-1\rangle$，$\hat L_-|1,-1\rangle = 0$

$$\hat L_- : \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\hbar$$

$$\hat L_x = \frac{1}{2}(\hat L_+ + \hat L_-) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\hbar$$

$$\hat L_y = \frac{1}{2i}(\hat L_+ - \hat L_-) = \frac{1}{\sqrt{2}i}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\hbar$$

$$\hat L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\hbar$$

$$L_n = (\hat L_x, \hat L_y, \hat L_z)\cdot(\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$$

$$= \frac{\hbar}{\sqrt{2}}\begin{pmatrix} \sqrt{2}\cos\alpha & \sin\alpha\, e^{i\beta} & 0 \\ \sin\alpha\, e^{-i\beta} & 0 & \sin\alpha\, e^{i\beta} \\ 0 & \sin\alpha\, e^{-i\beta} & -\sqrt{2}\cos\alpha \end{pmatrix}$$

$$|L_n - \lambda\frac{\hbar}{\sqrt{2}} I|$$

$$= \left(\frac{\hbar}{\sqrt{2}}\right)^3\left|(\sqrt{2}\cos\alpha-\lambda)(-\lambda)(-\sqrt{2}\cos\alpha-\lambda) \quad {}^2\alpha(\sqrt{2}\cos\alpha-\lambda) - \sin^2\alpha(-\sqrt{2}\cos\alpha-\lambda)\right|$$

$$= \left(\frac{\hbar}{\sqrt{2}}\right)^3\left|(2\cos^2\alpha - \lambda^2)\lambda + 2\sin^2\alpha\,\lambda\right| = \left(\frac{\hbar}{\sqrt{2}}\right)^3\lambda(\lambda^2-2) = 0$$

$\rightarrow \lambda = \pm\sqrt{2}, 0$，对应本征值. $\pm\hbar, 0$

$$\begin{pmatrix} \sqrt{2}(\cos\alpha - 1) & \sin\alpha \, e^{i\beta} & 0 \\ \sin\alpha \, e^{-i\beta} & -\sqrt{2} & \sin\alpha \, e^{i\beta} \\ 0 & \sin\alpha \, e^{-i\beta} & -\sqrt{2}(\cos\alpha + 1) \end{pmatrix} |\lambda_1\rangle = 0$$

$\hookrightarrow$