Дегтралов

10.2. Лапласиан векторной ф-ции

$div\, grad\, \bar{a} = \Delta\bar{a} = \nabla^2 \bar{a} \sim g^{ij}\nabla_i\nabla_j a^k$ (10.2.1) для ортог. корд. $div\, grad\, \bar{a} = \nabla^2\bar{a} \sim g^{ii}\nabla_i\nabla_i a^k$, в декартовых $g^{ii}=1$; $\Delta\bar{a} = \nabla^2\bar{a} \sim \nabla_i\nabla_i a^k \sim \begin{bmatrix}\Delta a_x\\ \Delta a_y\\ \Delta a_z\end{bmatrix}$.

$$\boxed{\frac{\partial\sqrt{g}}{\partial x^k} = \frac{1}{2\sqrt{g}}\frac{\partial g}{\partial x^k} = \sqrt{g}\,\Gamma^o_{ok}} \quad (5.13)$$

11. Формулы

Тождества с операторами 1-го порядка.

1. $grad(\varphi\psi) = \nabla(\varphi\psi) = \psi\frac{\partial\varphi}{\partial x^i}e^i + \varphi\frac{\partial\psi}{\partial x^i}e^i = \psi\nabla\varphi + \varphi\nabla\psi$; $\quad ![\frac{\partial a^o}{\partial x^o} + \frac{a^k}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^k} = \frac{1}{\sqrt{g}}\frac{\partial(a^o\sqrt{g})}{\partial x^o}]!$

2. $div\,\bar{a} = \nabla_o a^o = \frac{\partial a^o}{\partial x^o} + a^k\Gamma^o_{ko}$ исп. ф-лу (5.13) $\Gamma^o_{ko} = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^k}$, получ. $\boxed{div\,\bar{a} = \nabla_o a^o = \frac{1}{\sqrt{g}}\frac{\partial(a^o\sqrt{g})}{\partial x^o}}$

3. $div(\varphi\bar{a}) = \nabla_o(\varphi a^o) = \varphi\nabla_o a^o + \bar{a}\cdot\nabla\varphi = \varphi\, div\,\bar{a} + \bar{a}\, grad\,\varphi$, $div(\varphi\bar{a}) = \nabla_i(\varphi a^i) = \frac{\partial\varphi a^i}{\partial x^i} + \varphi a^o\Gamma^i_{oi} = a^i\frac{\partial\varphi}{\partial x^i} + \varphi\frac{\partial a^i}{\partial x^i} + \varphi a^o\Gamma^i_{oi} = a^i\frac{\partial\varphi}{\partial x^i} + \varphi\left(\frac{\partial a^i}{\partial x^i} + a^o\Gamma^i_{oi}\right)$ $\quad [i(\nabla_y\nabla_z - \nabla_z\nabla_y) - j(\nabla_x\nabla_z - \nabla_z\nabla_x) + k(\nabla_x\nabla_y - \nabla_y\nabla_x)]$

4. $rot(\varphi\bar{a}) = \nabla\times(\varphi\bar{a}) = \varphi\nabla\times\bar{a} + \nabla\varphi\times a = \varphi\, rot\,\bar{a} + grad\,\varphi\times\bar{a}$, $[rot(\varphi\bar{a})]^k = \varepsilon^{ijk}\nabla_i(\varphi a_j) = \varepsilon^{ijk}\left(\frac{\partial\varphi a_j}{\partial x^i} + \varphi a_o\Gamma^o_{ji}\right) =$
$= \varepsilon_{ijk}\left(a_j\frac{\partial\varphi}{\partial x^i} + \varphi\frac{\partial a_j}{\partial x^i} + \varphi a_o\Gamma^o_{ji}\right) = \varphi\varepsilon^{ijk}\left(\frac{\partial a_j}{\partial x^i} + a_o\Gamma^o_{ji}\right) + \varepsilon^{ijk}\frac{\partial\varphi}{\partial x^i}a_j$

5. $grad(\bar{a}\cdot\bar{b})$, $[\bar{a}\times rot\,\bar{b}]_n = \varepsilon_{mkn}a^m(\varepsilon^{ijk}\nabla_i b_j) = -\varepsilon_{mnk}\varepsilon^{ijk}a^m\nabla_i b_j = (\delta^i_n\delta^j_m - \delta^i_m\delta^j_n)a^m\nabla_i b_j = a^j\nabla_n b_j - a^m\nabla_m b_n =$
$= a^o\nabla_n b_o - a^o\nabla_o b_n \Rightarrow a^o\nabla_i b_o = [\bar{a}\times rot\,\bar{b}]_i + a^o\nabla_o b_i$ или $a^o\nabla_i b_o e^i = \bar{a}\times rot\,\bar{b} + (a^o\nabla_o)\bar{b}$.

6. $div(\bar{a}\times\bar{b}) = \nabla_k(\varepsilon^{ijk}a_i b_j) = \varepsilon^{ijk}\nabla_k(a_i b_j) = a_i\varepsilon^{ijk}\nabla_k b_j + b_j\varepsilon^{ijk}\nabla_k a_i = -a_i\varepsilon^{ikj}\nabla_k b_j + b_j\varepsilon^{jki}\nabla_k a_i =$
$= \bar{b}\cdot rot\,\bar{a} - \bar{a}\, rot\,\bar{b}$

7. $[\nabla\times(b\times c)]_n = \varepsilon^{ijn}\nabla_i(\varepsilon_{jmn}b^m c^n) = -\varepsilon^{inj}\varepsilon_{mnj}\nabla_i b^m c^n = (\delta^i_n\delta^k_m - \delta^i_m\delta^k_n)\nabla_i b^m c^n = \nabla_n b^k c^n - \nabla_m b^m c^k =$
$= b^k\nabla_n c^n - c^k\nabla_m b^m + (\nabla_n b^k)c^n - (\nabla_m c^k)b^m = \bar{b}(\nabla\bar{c}) - \bar{c}(\nabla\bar{b}) + (\bar{c}\cdot\nabla)\bar{b} - (\bar{b}\cdot\nabla)\bar{c}$ или
$rot(\bar{b}\times\bar{c}) = \bar{b}\, div\,\bar{c} - \bar{c}\, div\,\bar{b} + (\bar{c}\cdot\nabla)\bar{b} - (\bar{b}\cdot\nabla)\bar{c}$

8. Заменим $b(5)$ $\bar{a}$ и b на $\bar{v}$: $(\bar{v}\nabla)\bar{v} = \bar{v}^o\nabla_i\bar{v}_o e^i - \bar{v}\times rot\,\bar{v} \Rightarrow (\bar{v}\cdot\nabla)\bar{v} = grad\frac{v^2}{2} + (rot\,\bar{v})\times\bar{v}$![т.к. $\bar{a}\times\bar{b} = -\bar{b}\times\bar{a}$]
$![grad\frac{\bar{v}^2}{2} = \nabla_i\frac{\bar{v}^o\bar{v}_o}{2} = [\bar{v}_o\nabla_i\bar{v}^o + \bar{v}^o\nabla_i\bar{v}_o]\cdot\frac{1}{2} = v_o\nabla_i v^o]!$ $\quad \begin{bmatrix}\nabla_2\nabla_3 - \nabla_3\nabla_2\\ \nabla_3\nabla_1 - \nabla_1\nabla_3\\ \nabla_1\nabla_2 - \nabla_2\nabla_1\end{bmatrix}$ $\quad grad\,\varphi = \frac{\partial\varphi}{\partial x}e^1 + \frac{\partial\varphi}{\partial y}e^2 + \frac{\partial\varphi}{\partial z}e^3$, $div\,\bar{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$

Тождества, вкл. операторы 2-го порядка.

1. $div\, rot\, T = \nabla\cdot(\nabla\times T) = \nabla_k(\varepsilon^{ijk}\nabla_i T^m_j) = \varepsilon^{kij}\nabla_k\nabla_i T^m_j = (\nabla_2\nabla_3 - \nabla_3\nabla_2)T^m_1 + (\nabla_3\nabla_1 - \nabla_1\nabla_3)T^m_2 + (\nabla_1\nabla_2 - \nabla_2\nabla_1)T^m_3 = 0$ – справедливо в евклидовом пр-ве, т.к. в нём м. операторы ков. пр-ой местами.

2. $rot\, grad\, T = \nabla\times(\nabla T) = \varepsilon^{ijk}\nabla_i\nabla_j T^m_n = \sqrt{g}^{-1}((\nabla_2\nabla_3 - \nabla_3\nabla_2)^1 + (\nabla_3\nabla_1 - \nabla_1\nabla_3)^2 + (\nabla_1\nabla_2 - \nabla_2\nabla_1)^3)T^m_n = 0$ как и в предыдущем.

3. $rot\, rot\,\bar{a} = \nabla\times(\nabla\times\bar{a}) = \varepsilon_{nkp}g^{nm}\nabla_m(\varepsilon^{ijk}\nabla_i a_j)e^p = -g^{nm}\varepsilon_{npk}\varepsilon^{ijk}\nabla_m\nabla_i a_j e^p = -g^{nm}(\delta^i_n\delta^j_p - \delta^i_p\delta^j_n)\nabla_m\nabla_i a_j e^p =$
$= (g^{nm}\nabla_m\nabla_p a_n - g^{nm}\nabla_m\nabla_n a_p)e^p = (\nabla_p(\nabla_m a^m) - g^{nm}\nabla_m\nabla_n a_p)e^p = grad(div\,\bar{a}) - \Delta\bar{a}$
![$g^{in}\nabla_i\nabla_n$ – скалярный квадрат оператора Гамильтона]!

4. Из (3), получаем $\Delta\bar{a} = grad(div\,\bar{a}) - rot\, rot\,\bar{a}$

5. $rot((\bar{a}\cdot\nabla)\bar{a})$, восп. (8) из предыдущего: $(\bar{a}\cdot\nabla)\bar{a} = grad\frac{a^2}{2} + rot\,\bar{a}\times\bar{a}$, $rot((\bar{a}\cdot\nabla)\bar{a}) = rot\left(grad\frac{a^2}{2}\right) + rot(rot\,\bar{a}\times\bar{a})$
ротор градиента скалярной ф-ции равен нулю $\Rightarrow rot((\bar{a}\cdot\nabla)\bar{a}) = rot(rot\,\bar{a}\times a)$,
$rot(rot\,\bar{a}\times\bar{a}) = \varepsilon^{pnr}\nabla_p(\varepsilon_{kmn}(\varepsilon^{ijk}\nabla_i a_j)a^m) = -\varepsilon^{prn}\varepsilon_{kmn}\nabla_p(a^m(\varepsilon^{ijk}\nabla_i a_j)) = (\delta^p_m\delta^r_k - \delta^p_k\delta^r_m)\nabla_p(a^m(\varepsilon^{ijk}\nabla_i a_j)) =$
$= \nabla_m(a^m(\varepsilon^{ijr}\nabla_i a_j)) - \nabla_k(a^r(\varepsilon^{ijk}\nabla_i a_j)) = \nabla_m(a^m(\varepsilon^{ijr}\nabla_i a_j)) - \nabla_k(a^r(\varepsilon^{ijk}\nabla_i a_j))$, $\nabla_m(a^m(\varepsilon^{ijr}\nabla_i a_j)) = (\nabla_m a^m)(\varepsilon^{ijr}\nabla_i a_j) +$
$+ (a^m\nabla_m)(\varepsilon^{ijr}\nabla_i a_j) = \underline{(\nabla_o a^o)(\varepsilon^{ijr}\nabla_i a_j)} + \underline{(a^o\nabla_o)(\varepsilon^{ijr}\nabla_i a_j)}$, $-\nabla_k(a^r(\varepsilon^{ijk}\nabla_i a_j)) = -(\nabla_k a^r)(\varepsilon^{ijk}\nabla_i a_j) - a^r(\varepsilon^{ijk}\nabla_k\nabla_i a_j) =$
$= \underline{-(\varepsilon^{ijk}\nabla_i a_j)(\nabla_k a^r)}$, т.к. $\varepsilon^{ijk}\nabla_k\nabla_i a_j$ в евкл. пр-ве равно нулю.
Т.о. $rot((\bar{a}\cdot\nabla)\bar{a}) = (\nabla_o a^o)(\varepsilon^{ijr}\nabla_i a_j) + (a^o\nabla_o)(\varepsilon^{ijk}\nabla_i a_j) - (\varepsilon^{ijk}\nabla_i a_j)(\nabla_k a^r) = div\,\bar{a}\, rot\,\bar{a} + (\bar{a}\cdot\nabla)rot\,\bar{a} - (rot\,\bar{a})\cdot\nabla\bar{a}$
$\underline{rot((\bar{a}\cdot\nabla)\bar{a}) = div\,\bar{a}\, rot\,\bar{a} + (\bar{a}\cdot\nabla)rot\,\bar{a} - (rot\,\bar{a})\cdot\nabla\bar{a}}$

12. Криволинейные, поверхностные и объёмные интегралы

Криволинейн. интеграл $A_{12} = \int_1^2 \bar{F}\cdot d\bar{r}$

Поверхн. интеграл $\Phi_S = \int_S \bar{v}\cdot d\bar{S} = \int_S \bar{v}\cdot n\, ds$

Объемн. интеграл $M = \int_V \rho\, dV$

Теорема Стокса.

$\int_S rot\,\bar{v}\cdot n\, ds = \oint_L \bar{v}\cdot d\bar{r}$ (12.1) $\quad rot_n\bar{v} = \lim_{\Delta S\to 0}\frac{1}{\Delta S}\oint_L \bar{v}\cdot d\bar{r}$, $\Gamma_L = \oint_L \bar{v}\cdot d\bar{r} \approx rot_n\bar{v}\Delta S = rot\,\bar{v}\cdot n\Delta S$ (12.2)
$[rot\,\bar{v}]_i\cdot n\Delta S = \Gamma_i$; $[rot\,\bar{v}_i]$ – ротор в опред. точке эл-та ΔS_i; $\sum\Gamma_i = \Gamma_L$, тогда

$\sum [\mathrm{rot}\,\bar{v}]_i \cdot n \Delta S = \text{Ц}_L \to \oint\limits_L \bar{v} \cdot d\bar{r}$ (12.3) $\quad \int\limits_S \mathrm{rot}\,\bar{v} \cdot n\, dS = \oint\limits_L \bar{v} \cdot d\bar{r}$ (12.4)

Теорема Гаусса-Остроградского

$\int\limits_S \bar{v} \cdot n\, d\bar{s} = \int\limits_V \mathrm{div}\,\bar{v}\, dv$ (12.5) $\quad \Big| * \Delta t \Rightarrow \int\limits_S v_n \Delta t\, dS = \int\limits_V \mathrm{div}\,\bar{v}\, dV dt, \quad \int\limits_S v_n \Delta t\, dS = V' - V$