Lagrangian mechanics

$$Q_i = \frac{d}{dt}\left(\frac{\partial KE}{\partial \dot{r_i}}\right) - \frac{\partial KE}{\partial r_i} = \frac{d}{dt}\left(\frac{\partial PE}{\partial \dot{r_i}}\right) - \frac{\partial PE}{\partial r_i}$$

For an object with mass m, position $P + X$ and velocity $v = \sum_i \frac{\partial X}{\partial r_i}\dot{r_i} = \frac{\partial X}{\partial r}\dot{r}$

X is the position relative to point P, and it can be written as a function of parameters. However, P depends on the trajectory, but the velocity of P is constrained such that there is no slipping

Assuming parameters are x_P, y_P, r

Then, velocity $v = \frac{dP}{dt} + \frac{dX}{dt} = \begin{bmatrix} \dot{x_P} \\ \dot{y_P} \\ 0 \end{bmatrix} + \sum_j \frac{\partial X}{\partial r_j}\dot{r_j}$

The constraints of point P are the following:

$$f_x = -\dot{x_P} + v_{Px}(r) = 0$$
$$f_y = -\dot{y_P} + v_{Py}(r) = 0$$

v_{Px}, v_{Py} are function of r

Therefore,

$$Q_{x_P} = \frac{d}{dt}\left(mv^T\frac{\partial v}{\partial \dot{x_P}}\right) - mv^T\frac{\partial v}{\partial x_P} + \mu_x\frac{\partial f_x}{\partial \dot{x_P}} + \mu_y\frac{\partial f_y}{\partial \dot{x_P}} = m\left(\frac{dv}{dt}\right)^T\begin{bmatrix}1\\0\\0\end{bmatrix} - 0 - \mu_x + 0$$

$$= 0 \Rightarrow \mu_x = m\left(\frac{dv}{dt}\right)^T\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$Q_{y_P} = \frac{d}{dt}\left(mv^T\frac{\partial v}{\partial \dot{y_P}}\right) - mv^T\frac{\partial v}{\partial y_P} + \mu_x\frac{\partial f_x}{\partial \dot{y_P}} + \mu_y\frac{\partial f_y}{\partial \dot{y_P}} = m\left(\frac{dv}{dt}\right)^T\begin{bmatrix}0\\1\\0\end{bmatrix} - 0 + 0 - \mu_y$$

$$= 0 \Rightarrow \mu_y = m\left(\frac{dv}{dt}\right)^T\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$Q_{r_i} = \frac{d}{dt}\left(mv^T\frac{\partial v}{\partial \dot{r_i}}\right) - mv^T\frac{\partial v}{\partial r_i} + \mu_x\frac{\partial f_x}{\partial \dot{r_i}} + \mu_y\frac{\partial f_y}{\partial \dot{r_i}}$$

$$= m\left(\frac{dv}{dt}\right)^T \frac{\partial}{\partial \dot{r}_i}\left(\frac{dX}{dt}\right) + mv^T\left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{r}_i}\left(\frac{dX}{dt}\right)\right) - \frac{\partial}{\partial r_i}\left(\frac{dX}{dt}\right)\right] + \mu_x + \mu_y$$

$$= m\left(\frac{dv}{dt}\right)^T \frac{\partial}{\partial \dot{r}_i}\left(\begin{bmatrix} v_{Px} \\ v_{Py} \\ 0 \end{bmatrix} + \frac{dX}{dt}\right) + mv^T \sum_j \left[\frac{\partial}{\partial r_j}\left(\frac{\partial X}{\partial r_i}\right)\dot{r}_j - \frac{\partial}{\partial r_i}\left(\frac{\partial X}{\partial r_j}\right)\dot{r}_j\right]$$

Because X is a function of r, $\frac{\partial}{\partial r_j}\left(\frac{\partial X}{\partial r_i}\right) = \frac{\partial}{\partial r_i}\left(\frac{\partial X}{\partial r_j}\right)$

Therefore,

$$Q_{r_i} = m\left(\frac{dv}{dt}\right)^T \frac{\partial}{\partial \dot{r}_i}\left(\begin{bmatrix} v_{Px} \\ v_{Py} \\ 0 \end{bmatrix} + \frac{dX}{dt}\right) = m\left(\frac{\partial v}{\partial \dot{r}_i}\right)^T \frac{dv}{dt}$$

For an object with moment of inertia I, rotation q

and angular velocity $\omega = 2\bar{q}\dot{q} = 2\bar{q}\sum_i \frac{\partial q}{\partial r_i}\dot{r}_i = 2\bar{q}\frac{\partial q}{\partial r}\dot{r}$

And $KE = \frac{1}{2}\omega^T I\omega$. Therefore,

$$Q_i = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{r}_i}\left(\frac{1}{2}\omega^T I\omega\right)\right) - \frac{\partial}{\partial r_i}\left(\frac{1}{2}\omega^T I\omega\right) = \frac{d}{dt}\left(\omega^T I \frac{\partial}{\partial \dot{r}_i}(\omega)\right) - \omega^T I \frac{\partial}{\partial r_i}(\omega)$$

$$= \frac{d}{dt}\left(\omega^T I \frac{\partial}{\partial \dot{r}_i}(\omega)\right) - \omega^T I \frac{\partial}{\partial r_i}(\omega)$$

$$= \frac{d}{dt}\left(\omega^T I 2\bar{q}\frac{\partial q}{\partial r_i}\right) - \omega^T I \frac{\partial}{\partial r_i}\left(2\bar{q}\frac{\partial q}{\partial r}\dot{r}\right)$$

$$= \frac{d}{dt}(\omega^T) I 2\bar{q}\frac{\partial q}{\partial r_i} + \omega^T I \frac{d}{dt}\left(2\bar{q}\frac{\partial q}{\partial r_i}\right) - \omega^T I \frac{\partial}{\partial r_i}\left(2\bar{q}\frac{\partial q}{\partial r}\dot{r}\right)$$

$$= \frac{d}{dt}(\omega^T) I 2\bar{q}\frac{\partial q}{\partial r_i} + \omega^T I \frac{\partial}{\partial r}\left(2\bar{q}\frac{\partial q}{\partial r_i}\right)\dot{r} - \omega^T I \frac{\partial}{\partial r_i}\left(2\bar{q}\frac{\partial q}{\partial r}\right)\dot{r}$$

$$= \frac{d}{dt}(\omega^T) I 2\bar{q}\frac{\partial q}{\partial r_i} + \omega^T I 2\frac{\partial \bar{q}}{\partial r}\dot{r}\frac{\partial q}{\partial r_i} - \omega^T I 2\frac{\partial \bar{q}}{\partial r_i}\frac{\partial q}{\partial r}\dot{r}$$

$$= \frac{d}{dt}(\omega^T) I 2\bar{q}\frac{\partial q}{\partial r_i} + \omega^T I 2 \frac{\partial \bar{q}}{\partial r}\dot{r} 2 \frac{\partial q}{\partial r_i}$$

$$= \left(\frac{d}{dt}\left(2\bar{q}\frac{\partial q}{\partial r}\dot{r}\right)\right)^T I 2\bar{q}\frac{\partial q}{\partial r_i} + \left(2\bar{q}\frac{\partial q}{\partial r}\dot{r}\right)^T I 2 \frac{\partial \bar{q}}{\partial r}\dot{r} 2 \frac{\partial q}{\partial r_i}$$

$$= \left(2\bar{q}\frac{\partial q}{\partial r_i}\right)^T I \frac{d}{dt}\left(2\bar{q}\frac{\partial q}{\partial r}\dot{r}\right) + \left(2\bar{q}\frac{\partial q}{\partial r}\dot{r}\right)^T I 2 \frac{\partial \bar{q}}{\partial r}\dot{r} 2 \frac{\partial q}{\partial r_i}$$

$$= \left(\bar{q} 2\frac{\partial q}{\partial r_i}\right)^T I \bar{q} 2\frac{\partial q}{\partial r}\ddot{r} + \left(\bar{q} 2\frac{\partial q}{\partial r_i}\right)^T I \bar{q} 2\frac{\partial^2 q}{\partial r^2}\dot{r}\dot{r} + \left(2\frac{\partial \bar{q}}{\partial r}\dot{r} 2\frac{\partial q}{\partial r_i}\right)^T I \bar{q} 2\frac{\partial q}{\partial r}\dot{r}$$

$$= \left(\bar{q} 2\frac{\partial q}{\partial r_i}\right)^T I \bar{q} 2\frac{\partial q}{\partial r}\ddot{r} + \left(\bar{q} 2\frac{\partial q}{\partial r_i}\right)^T I \sum_{j,k} \bar{q} 2\frac{\partial}{\partial r_k}\left(\frac{\partial q}{\partial r_j}\right)\dot{r}_k\dot{r}_j$$

$$+ \left(\sum_k \bar{q} 2\frac{\partial q}{\partial r_k}\dot{r}_k\right)^T I \sum_j 2\frac{\partial \bar{q}}{\partial r_j}\dot{r}_j 2\frac{\partial q}{\partial r_i}$$

$$Q = \left(\bar{q} 2\frac{\partial q}{\partial r}\right)^T I \bar{q} 2\frac{\partial q}{\partial r}\ddot{r} + \left(\bar{q} 2\frac{\partial q}{\partial r}\right)^T I \bar{q} 2\frac{\partial^2 q}{\partial r^2}\dot{r}\dot{r} + \left(2\frac{\partial \bar{q}}{\partial r}\dot{r} 2\frac{\partial q}{\partial r}\right)^T I \bar{q} 2\frac{\partial q}{\partial r}\dot{r}$$

$$Q = \left(\bar{q} 2\frac{\partial q}{\partial r}\right)^T I \bar{q} 2\frac{\partial q}{\partial r}\ddot{r} + \dot{r}^T\dot{r}^T\left(\bar{q} 2\frac{\partial^2 q}{\partial r^2}\right)^T I \bar{q} 2\frac{\partial q}{\partial r} + \left(2\frac{\partial \bar{q}}{\partial r}\dot{r} 2\frac{\partial q}{\partial r}\right)^T I \bar{q} 2\frac{\partial q}{\partial r}\dot{r}$$

Necessary terms are $2\frac{\partial q}{\partial r}, 2\frac{\partial \bar{q}}{\partial r}$, and $2\frac{\partial^2 q}{\partial r^2}$

Noted that $2\frac{\partial}{\partial \theta}R(\theta) = R(\theta + \pi), 2\frac{\partial}{\partial \theta}R(-\theta) = R(-\theta - \pi)$, also

$$2\frac{\partial}{\partial \theta}R(\theta) 2\frac{\partial}{\partial \theta}R(-\theta) = 0$$

If $q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{n}$, then

$$v' = qvq^* = \sin\theta\, \hat{n} \times v + \cos\theta\, v + 2 * \sin^2\frac{\theta}{2}(\hat{n}\cdot v)\hat{n}$$

$$\frac{d}{d\theta}v' = \cos\theta\, \hat{n} \times v - \sin\theta\, v + \sin\theta\, (\hat{n}\cdot v)\hat{n} = \cos\theta\, \hat{n} \times v + \sin\theta\, \hat{n} \times (\hat{n} \times v)$$

Wheel on the ground in ZXY rotation

-z faces up, +x faces forward, $\dot{\theta_R} < 0 \Rightarrow$ move forward

Mass m, moment of inertia $I = \begin{bmatrix} I_{xz} & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_{xz} \end{bmatrix}$

$$q = R_z(\psi)R_x(\phi)R_y(\theta_R)$$

$$v = \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} + \frac{d}{dt}\left(R_z(\psi)R_x(\phi) \begin{bmatrix} 0 \\ 0 \\ -r_R \end{bmatrix} R_x(-\phi)R_z(-\psi) \right)$$

$$= \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} + \frac{d}{dt}\left(\begin{bmatrix} -r_R \sin\phi \sin\psi \\ r_R \sin\phi \cos\psi \\ -r_R \cos\phi \end{bmatrix} \right)$$

$$= \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} -r_R \sin\phi \cos\psi \\ -r_R \sin\phi \sin\psi \\ 0 \end{bmatrix} \dot{\psi} + \begin{bmatrix} -r_R \cos\phi \sin\psi \\ r_R \cos\phi \cos\psi \\ r_R \sin\phi \end{bmatrix} \dot{\phi}$$

Constraints are

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} = \begin{bmatrix} -r_R \cos\psi \\ -r_R \sin\psi \\ 0 \end{bmatrix} \dot{\theta_R}$$

Let $r = [\theta_R \quad \psi \quad \phi]^T$

Translational kinetic energy

$$\frac{dv_x}{dt} = \frac{d}{dt}\left(-r_R \cos\psi\, \dot{\theta}_R - r_R \sin\phi \cos\psi\, \dot{\psi} - r_R \cos\phi \sin\psi\, \dot{\phi}\right)$$

$$= -r_R \cos\psi\, \ddot{\theta_R} + r_R \sin\psi\, \dot{\theta_R}\dot{\psi} - r_R \sin\phi \cos\psi\, \ddot{\psi} + r_R \sin\phi \sin\psi\, \dot{\psi}^2 - r_R \cos\phi \sin\psi\, \ddot{\phi} + r_R \sin\phi \sin\psi\, \dot{\phi}^2 - 2r_R \cos\phi \cos\psi\, \dot{\psi}\dot{\phi}$$

$$\frac{dv_y}{dt} = \frac{d}{dt}\left(-r_R \sin\psi\, \dot{\theta}_R - r_R \sin\phi \sin\psi\, \dot{\psi} + r_R \cos\phi \cos\psi\, \dot{\phi}\right)$$

$$= -r_R \sin\psi\, \ddot{\theta_R} - r_R \cos\psi\, \dot{\theta_R}\dot{\psi} - r_R \sin\phi \sin\psi\, \ddot{\psi} - r_R \sin\phi \cos\psi\, \dot{\psi}^2 + r_R \cos\phi \cos\psi\, \ddot{\phi} - r_R \sin\phi \cos\psi\, \dot{\phi}^2 - 2r_R \cos\phi \sin\psi\, \dot{\psi}\dot{\phi}$$

$$\frac{dv_z}{dt} = \frac{d}{dt}\left(r_R \sin\phi\, \dot{\phi}\right) = r_R \sin\phi\, \ddot{\phi} + r_R \cos\phi\, \dot{\phi}^2$$

$$\frac{d}{dt}\left(\frac{\partial KE}{\partial \dot{\theta_R}}\right) - \frac{\partial KE}{\partial \theta_R} = m \begin{bmatrix} -r_R \cos\psi \\ -r_R \sin\psi \\ 0 \end{bmatrix}^T \frac{dv}{dt} = m r_R^2 \ddot{\theta_R} + m r_R^2 \sin\phi\, \ddot{\psi} + 2 m r_R^2 \cos\phi\, \dot{\psi}\dot{\phi}$$

$$\frac{d}{dt}\left(\frac{\partial KE}{\partial \dot{\psi}}\right) - \frac{\partial KE}{\partial \psi} = m \begin{bmatrix} -r_R \sin\phi \cos\psi \\ -r_R \sin\phi \sin\psi \\ 0 \end{bmatrix}^T \frac{dv}{dt} =$$

$$= mr_R^2 \sin\phi\, \ddot{\theta}_R + mr_R^2 \sin^2\phi\, \ddot{\psi} + 2mr_R^2 \sin\phi \cos\phi\, \dot{\psi}\dot{\phi}$$

$$\frac{d}{dt}\left(\frac{\partial KE}{\partial \dot{\phi}}\right) - \frac{\partial KE}{\partial \phi} = m \begin{bmatrix} -r_R \cos\phi \sin\psi \\ r_R \cos\phi \cos\psi \\ r_R \sin\phi \end{bmatrix} \frac{dv}{dt}$$

$$= mr_R^2 \ddot{\phi} - mr_R^2 \sin\phi \cos\phi\, \dot{\psi}^2 - mr_R^2 \cos\phi\, \dot{\theta}_R \dot{\psi}$$

Rotational kinetic energy

$$q = R_z(\psi) R_x(\phi) R_y(\theta_R)$$
$$\bar{q} = R_y(-\theta_R) R_x(-\phi) R_z(-\psi)$$

Let $r = [\theta_R \quad \psi \quad \phi]^T$

$$2\frac{\partial q}{\partial r}$$
$$= [R_z(\psi)R_x(\phi)R_y(\theta_R+\pi) \quad R_z(\psi+\pi)R_x(\phi)R_y(\theta_R) \quad R_z(\psi)R_x(\phi+\pi)R_y(\theta_R)]$$

$$\left(2\frac{\partial^2 q}{\partial r^2}\right)^T = \frac{1}{2}\begin{bmatrix} 0 & R_z(\psi+\pi)R_x(\phi)R_y(\theta_R+\pi) & R_z(\psi)R_x(\phi+\pi)R_y(\theta_R+\pi) \\ & 0 & R_z(\psi+\pi)R_x(\phi+\pi)R_y(\theta_R) \\ & & 0 \end{bmatrix}$$

$$\dot{r}^T \dot{r}^T \left(\bar{q}2\frac{\partial^2 q}{\partial r^2}\right)^T = \begin{bmatrix} -\cos\phi\cos\theta_R \\ 0 \\ -\cos\phi\sin\theta_R \end{bmatrix} \dot{\theta}_R\dot{\psi} + \begin{bmatrix} -\sin\theta_R \\ 0 \\ \cos\theta_R \end{bmatrix} \dot{\theta}_R\dot{\phi} + \begin{bmatrix} \sin\phi\sin\theta_R \\ \cos\phi \\ -\sin\phi\cos\theta_R \end{bmatrix} \dot{\psi}\dot{\phi}$$

$$2\frac{\partial \bar{q}}{\partial r}$$
$$= [R_y(-\theta_R-\pi)R_x(-\phi)R_z(-\psi) \quad R_y(-\theta_R)R_x(-\phi)R_z(-\psi-\pi) \quad R_y(-\theta_R)R_x(-\phi-\pi)R_z(-\psi)]$$

$$\bar{q}2\frac{\partial q}{\partial r} = \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} -\cos\phi\sin\theta_R \\ \sin\phi \\ \cos\phi\cos\theta_R \end{bmatrix} & \begin{bmatrix} \cos\theta_R \\ 0 \\ \sin\theta_R \end{bmatrix} \end{bmatrix}$$

$$\left(2\frac{\partial \bar{q}}{\partial r}\dot{r}2\frac{\partial q}{\partial r}\right)^T = \begin{bmatrix} \begin{bmatrix} \cos\phi\cos\theta_R \\ 0 \\ \cos\phi\sin\theta_R \end{bmatrix}\dot{\psi} + \begin{bmatrix} \sin\theta_R \\ 0 \\ -\cos\theta_R \end{bmatrix}\dot{\phi} \\ \begin{bmatrix} -\cos\phi\cos\theta_R \\ 0 \\ -\cos\phi\sin\theta_R \end{bmatrix}\dot{\theta}_R + \begin{bmatrix} \sin\phi\sin\theta_R \\ \cos\phi \\ -\sin\phi\cos\theta_R \end{bmatrix}\dot{\phi} \\ \begin{bmatrix} -\sin\theta_R \\ 0 \\ \cos\theta_R \end{bmatrix}\dot{\theta}_R + \begin{bmatrix} -\sin\phi\sin\theta_R \\ -\cos\phi \\ \sin\phi\cos\theta_R \end{bmatrix}\dot{\psi} \end{bmatrix}$$

$$I = \begin{bmatrix} I_{xz} & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_{xz} \end{bmatrix}$$

$$Q = \left(\bar{q}2\frac{\partial q}{\partial r}\right)^T I\bar{q}2\frac{\partial q}{\partial r}\ddot{r} + \dot{r}^T\dot{r}^T\left(\bar{q}2\frac{\partial^2 q}{\partial r^2}\right)^T I\bar{q}2\frac{\partial q}{\partial r} + \left(2\frac{\partial\bar{q}}{\partial r}\dot{r}2\frac{\partial q}{\partial r}\right)^T I\bar{q}2\frac{\partial q}{\partial r}\dot{r}$$

$$= \begin{bmatrix} I_y & I_y\sin\phi & 0 \\ I_y\sin\phi & I_y\sin^2\phi + I_{xz}\cos^2\phi & 0 \\ 0 & 0 & I_{xz} \end{bmatrix}\begin{bmatrix} \ddot{\theta_R} \\ \ddot{\psi} \\ \ddot{\phi} \end{bmatrix}$$

$$+ \begin{bmatrix} I_y\cos\phi\,\dot{\psi}\dot{\phi} \\ I_{xz}\cos\phi\,\dot{\theta_R}\dot{\phi} - I_{xz}\sin\phi\cos\phi\,\dot{\psi}\dot{\phi} + I_y\sin\phi\cos\phi\,\dot{\psi}\dot{\phi} \\ -I_{xz}\cos\phi\,\dot{\theta_R}\dot{\psi} \end{bmatrix}$$

$$+ \begin{bmatrix} I_{xz}\cos\phi\,\dot{\psi}\dot{\phi} - I_{xz}\cos\phi\,\dot{\psi}\dot{\phi} \\ -I_{xz}\cos\phi\,\dot{\theta_R}\dot{\phi} + I_y\cos\phi\,\dot{\theta_R}\dot{\phi} + I_y\sin\phi\cos\phi\,\dot{\psi}\dot{\phi} - I_{xz}\sin\phi\cos\phi\,\dot{\psi}\dot{\phi} \\ I_{xz}\cos\phi\,\dot{\theta_R}\dot{\psi} - I_y\cos\phi\,\dot{\theta_R}\dot{\psi} - I_y\sin\phi\cos\phi\,\dot{\psi}^2 + I_{xz}\sin\phi\cos\phi\,\dot{\psi}^2 \end{bmatrix}$$

$$\omega = \bar{q}2\frac{\partial q}{\partial r}\dot{r} = \begin{bmatrix} \dot{\phi}\cos\theta_R - \dot{\psi}\cos\phi\sin\theta_R \\ \dot{\theta_R} + \dot{\psi}\sin\phi \\ \dot{\phi}\sin\theta_R + \dot{\psi}\cos\phi\cos\theta_R \end{bmatrix}$$

$$KE = \frac{1}{2}\omega^T I\omega = \frac{1}{2}I_y\left(\dot{\theta_R}^2 + 2\sin\phi\,\dot{\theta_R}\dot{\psi} + \sin^2\phi\,\dot{\psi}^2\right) + \frac{1}{2}I_{xz}\left(\dot{\phi}^2 + \cos^2\phi\,\dot{\psi}^2\right)$$

$$\frac{d}{dt}\left(\frac{\partial KE}{\partial\dot{\psi}}\right) - \frac{\partial KE}{\partial\psi} = \frac{d}{dt}\left(I_y\sin\phi\,\dot{\theta_R} + I_y\sin^2\phi\,\dot{\psi} + I_{xz}\cos^2\phi\,\dot{\psi}\right) - 0$$

$$= I_y\cos\phi\,\dot{\phi}\dot{\theta_R} + I_y 2\sin\phi\cos\phi\,\dot{\psi}\dot{\phi} + I_y\sin^2\phi\,\ddot{\psi}$$

$$- I_{xz}2\sin\phi\cos\phi\,\dot{\psi}\dot{\phi} + I_{xz}\cos^2\phi\,\ddot{\psi}$$

$$\frac{d}{dt}\left(\frac{\partial KE}{\partial\dot{\phi}}\right) - \frac{\partial KE}{\partial\phi} = I_{xz}\ddot{\phi} - \left(I_y\cos\phi\,\dot{\theta_R}\dot{\psi} + I_y\sin\phi\cos\phi\,\dot{\psi}^2 - I_{xz}\sin\phi\cos\phi\,\dot{\psi}^2\right)$$

$$= I_{xz}\ddot{\phi} - I_y\cos\phi\,\dot{\theta_R}\dot{\psi} - I_y\sin\phi\cos\phi\,\dot{\psi}^2 + I_{xz}\sin\phi\cos\phi\,\dot{\psi}^2$$

Wheel on the ground in ZXZ rotation

+z faces up, +x faces forward, when set, wheel is lay flat on the ground. When $\theta = \frac{\pi}{2}$, the wheel stays up

Wheel radius r

Mass m, moment of inertia $I = \begin{bmatrix} I_t & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & I_a \end{bmatrix}$

$$q = R_z(\psi)R_x(\theta)R_z(\phi)$$

$$\omega = \begin{bmatrix} \dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi \\ \dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi \\ \dot{\psi}\cos\theta + \dot{\phi} \end{bmatrix}$$

$$v = \begin{bmatrix} -r\cos\psi \\ -r\sin\psi \\ 0 \end{bmatrix}\dot{\phi} + \frac{d}{dt}\left(R_z(\psi)R_x(\theta)\begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix} R_x(-\theta)R_z(-\psi) \right)$$

$$= \begin{bmatrix} -r\cos\psi \\ -r\sin\psi \\ 0 \end{bmatrix}\dot{\phi} + \frac{d}{dt}\left(\begin{bmatrix} -r\cos\theta\sin\psi \\ r\cos\theta\cos\psi \\ r\sin\theta \end{bmatrix} \right)$$

$$= \begin{bmatrix} -r\cos\psi \\ -r\sin\psi \\ 0 \end{bmatrix}\dot{\phi} + \begin{bmatrix} r\sin\theta\sin\psi \\ -r\sin\theta\cos\psi \\ r\cos\theta \end{bmatrix}\dot{\theta} + \begin{bmatrix} -r\cos\theta\cos\psi \\ -r\cos\theta\sin\psi \\ 0 \end{bmatrix}\dot{\psi}$$

$$\frac{dv_x}{dt} = -r\cos\psi\,\ddot{\phi} + r\sin\psi\,\dot{\phi}\dot{\psi} + r\sin\theta\sin\psi\,\ddot{\theta} + r\cos\theta\sin\psi\,\dot{\theta}^2$$
$$- r\cos\theta\cos\psi\,\ddot{\psi} + r\cos\theta\sin\psi\,\dot{\psi}^2 + 2r\sin\theta\cos\psi\,\dot{\theta}\dot{\psi}$$

$$\frac{dv_y}{dt} = -r\sin\psi\,\ddot{\phi} - r\cos\psi\,\dot{\phi}\dot{\psi} - r\sin\theta\cos\psi\,\ddot{\theta} - r\cos\theta\cos\psi\,\dot{\theta}^2$$
$$- r\cos\theta\sin\psi\,\ddot{\psi} - r\cos\theta\cos\psi\,\dot{\psi}^2 + 2r\sin\theta\sin\psi\,\dot{\theta}\dot{\psi}$$

$$\frac{dv_z}{dt} = r\cos\theta\,\ddot{\theta} - r\sin\theta\,\dot{\theta}^2$$

$$Q_\psi = m\frac{\partial v}{\partial\dot{\psi}}\cdot\frac{dv}{dt} = mr^2\left(\cos\theta\,\ddot{\phi} + \cos^2\theta\,\ddot{\psi} - 2\sin\theta\cos\theta\,\dot{\theta}\dot{\psi}\right)$$

$$Q_\theta = m\frac{\partial v}{\partial\dot{\theta}}\cdot\frac{dv}{dt} = mr^2\left(\sin\theta\,\dot{\phi}\dot{\psi} + \ddot{\theta} + \sin\theta\cos\theta\,\dot{\psi}^2\right)$$

$$Q_\phi = m\frac{\partial v}{\partial\dot{\phi}}\cdot\frac{dv}{dt} = mr^2\left(\ddot{\phi} + \cos\theta\,\ddot{\psi} - 2\sin\theta\,\dot{\theta}\dot{\psi}\right)$$

In conclusion,

$$
\begin{bmatrix} (I_a + mr^2)\cos^2\theta + I_t\sin^2\theta & 0 & (I_a + mr^2)\cos\theta \\ 0 & I_t + mr^2 & 0 \\ (I_a + mr^2)\cos\theta & 0 & I_a + mr^2 \end{bmatrix} \begin{bmatrix} \ddot{\psi} \\ \ddot{\theta} \\ \ddot{\phi} \end{bmatrix}
= \begin{bmatrix} (I_a - I_t + mr^2) 2\sin\theta\cos\theta\,\dot{\psi}\dot{\theta} + I_a \sin\theta\,\dot{\theta}\dot{\phi} \\ -(I_a - I_t + mr^2)\sin\theta\cos\theta\,\dot{\psi}^2 - (I_a + mr^2)\sin\theta\,\dot{\psi}\dot{\phi} - mgr\cos\theta \\ (I_a + 2mr^2)\sin\theta\,\dot{\psi}\dot{\theta} \end{bmatrix}
$$

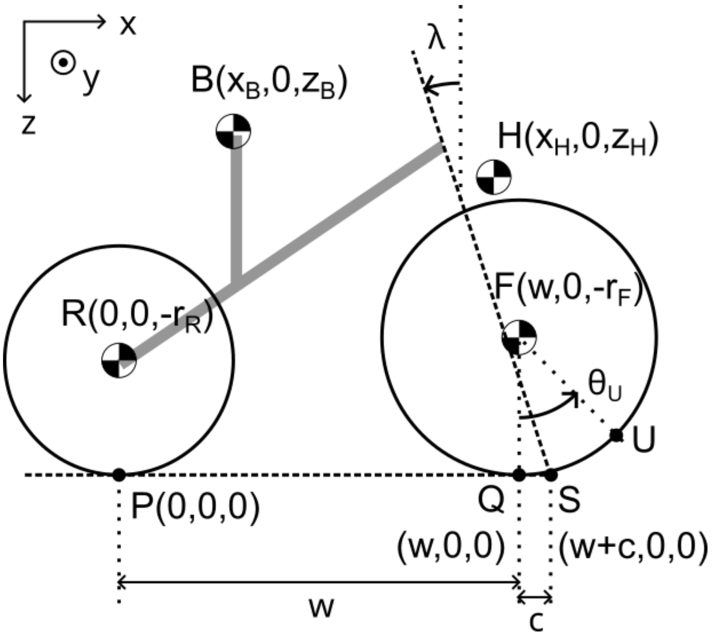
Conversion between ZXY and ZXZ

For ZXY, $q_1 = R_z(\psi_1)R_x(\phi_1)R_y(\theta_1)$

For ZXZ, $q_2 = R_z(\psi_2)R_x(\theta_2)R_z(\phi_2)$

$$
\begin{cases} \theta_1 = \phi_2 \\ \phi_1 = \theta_2 - \dfrac{\pi}{2} \\ \psi_1 = -\psi_2 \end{cases}
$$

Bike



All the parameters: $\psi, \phi, \theta_R, \theta_B, \delta, \theta_F, \theta_U$

$$q_P = R_z(\psi)R_x(\phi)$$

$$q_R = R_z(\psi)R_x(\phi)R_y(\theta_R)$$

$$q_B = R_z(\psi)R_x(\phi)R_y(\theta_B)$$

$$q_H = R_z(\psi)R_x(\phi)R_y(\theta_B)R_\lambda(\delta)$$

$$q_F = R_z(\psi)R_x(\phi)R_y(\theta_B)R_\lambda(\delta)R_y(\theta_F)$$

$$R_\lambda(\delta) = \begin{bmatrix} \cos\frac{1}{2}\delta \\ \sin\frac{1}{2}\delta\sin\lambda \\ 0 \\ \sin\frac{1}{2}\delta\cos\lambda \end{bmatrix}$$

$$PR = q_P \begin{bmatrix} 0 \\ 0 \\ -r_R \end{bmatrix} q_P^* = \begin{bmatrix} -r_R\sin\phi\sin\psi \\ r_R\sin\phi\cos\psi \\ -r_R\cos\phi \end{bmatrix}$$

$$RB = q_B \begin{bmatrix} x_B \\ 0 \\ z_B + r_R \end{bmatrix} q_B^*$$

$$RS = q_B \begin{bmatrix} w + c \\ 0 \\ r_R \end{bmatrix} q_B^*$$

$$SH = q_H \begin{bmatrix} x_H - w - c \\ 0 \\ z_H \end{bmatrix} q_H^*$$

$$SF = q_H \begin{bmatrix} -c \\ 0 \\ -r_F \end{bmatrix} q_H^*$$

$$FQ = q_H \begin{bmatrix} 0 \\ 0 \\ r_F \end{bmatrix} q_H^*$$

$$FU = q_H \begin{bmatrix} r_F \sin\theta_U \\ 0 \\ r_F \cos\theta_U \end{bmatrix} q_H^*$$

To make sure that the front wheel is contact with ground, 2 constrains are required:

$$\begin{cases} U_z = 0 \\ \left(\dfrac{\partial U_z}{\partial \theta_U}\right)_z = 0 \end{cases}$$

The first constraint means that point U is contact with ground, but U is just a point on front wheel, so the second constraint makes sure that U is the lowest point of front wheel.

Based on definition:

$$R_z(-\psi) U R_z(\psi)$$

$$= R_x(\phi)\begin{bmatrix} 0 \\ 0 \\ -r_R \end{bmatrix} R_x(-\phi) + R_x(\phi) R_y(\theta_B) \begin{bmatrix} w+c \\ 0 \\ r_R \end{bmatrix} R_y(-\theta_B) R_x(-\phi)$$

$$+ R_x(\phi) R_y(\theta_B) R_\lambda(\delta) \begin{bmatrix} r_F \sin\theta_U - c \\ 0 \\ r_F \cos\theta_U - r_F \end{bmatrix} R_\lambda(-\delta) R_y(-\theta_B) R_x(-\phi)$$

$$= \begin{bmatrix} r_R \sin\theta_B + (w+c)\cos\theta_B \\ r_R \sin\phi - \sin\phi\,(r_R \cos\theta_B - (w+c)\sin\theta_B) \\ -r_R \cos\phi + \cos\phi\,(r_R\cos\theta_B - (w+c)\sin\theta_B) \end{bmatrix}$$

$$+ R_x(\phi) R_y(\theta_B) v_H R_y(-\theta_B) R_x(-\phi)$$

$$= \begin{bmatrix} r_R \sin\theta_B + (w+c)\cos\theta_B \\ r_R \sin\phi - \sin\phi\,(r_R \cos\theta_B - (w+c)\sin\theta_B) \\ -r_R \cos\phi + \cos\phi\,(r_R\cos\theta_B - (w+c)\sin\theta_B) \end{bmatrix}$$

$$+ \begin{bmatrix} \cos\theta_B\, v_{Hx} + \sin\theta_B\, v_{Hz} \\ \sin\phi \sin\theta_B\, v_{Hx} + \cos\phi\, v_{Hy} - \sin\phi\cos\theta_B\, v_{Hz} \\ -\cos\phi\sin\theta_B\, v_{Hx} + \sin\phi\, v_{Hy} + \cos\phi\cos\theta_B\, v_{Hz} \end{bmatrix}$$

where

$$v_H = \begin{bmatrix} \cos\delta\,(r_F\sin\theta_U - c) + 2\sin^2\frac{\delta}{2}\sin\lambda\left(\sin\lambda\,(r_F\sin\theta_U - c) + \cos\lambda\,(r_F\cos\theta_U - r_F)\right) \\ \sin\delta\left(\cos\lambda\,(r_F\sin\theta_U - c) - \sin\lambda\,(r_F\cos\theta_U - r_F)\right) \\ \cos\delta\,(r_F\cos\theta_U - r_F) + 2\sin^2\frac{\delta}{2}\cos\lambda\left(\sin\lambda\,(r_F\sin\theta_U - c) + \cos\lambda\,(r_F\cos\theta_U - r_F)\right) \end{bmatrix}$$

For the first constraint, let $Zero = U_z$

$$\begin{aligned} \Rightarrow Zero = U_z = 0 = &-r_R\cos\phi + \cos\phi\,(r_R\cos\theta_B - (w+c)\sin\theta_B) \\ &- \cos\phi\sin\theta_B\, v_{Hx} + \cos\phi\cos\theta_B\, v_{Hz} + \sin\phi\, v_{Hy} \end{aligned}$$

$$\begin{aligned} \Rightarrow 0 = &-r_R\cos\phi - c\cos\lambda\sin\phi\sin\delta + r_R\cos\phi\cos\theta_B - (w+c)\cos\phi\sin\theta_B \\ &+ r_F\cos\lambda\sin\phi\sin\delta\sin\theta_U + r_F\sin\lambda\sin\phi\sin\delta \\ &- r_F\sin\lambda\sin\phi\sin\delta\cos\theta_U \\ &-2r_F\sin^2\lambda\cos\phi\sin^2\frac{\delta}{2}\sin\theta_B\sin\theta_U - 2r_F\sin\lambda\cos\lambda\cos\phi\sin^2\frac{\delta}{2}\sin\theta_B\cos\theta_U \\ &+2r_F\sin\lambda\cos\lambda\cos\phi\sin^2\frac{\delta}{2}\cos\theta_B\sin\theta_U + 2r_F\cos^2\lambda\cos\phi\sin^2\frac{\delta}{2}\cos\theta_B\cos\theta_U \\ &- \cos\phi\left(-c\cos\delta - 2c\sin^2\lambda\sin^2\frac{\delta}{2} - 2r_F\sin\lambda\cos\lambda\sin^2\frac{\delta}{2}\right)\sin\theta_B \\ &+ \cos\phi\left(-r_F\cos\delta - 2c\sin\lambda\cos\lambda\sin^2\frac{\delta}{2} - 2r_F\cos^2\lambda\sin^2\frac{\delta}{2}\right)\cos\theta_B \\ &+r_F\cos\phi\cos\delta\cos\theta_B\cos\theta_U - r_F\cos\phi\cos\delta\sin\theta_B\sin\theta_U \end{aligned}$$

$$\begin{aligned} \frac{\partial Zero}{\partial\theta_B} = &-r_R\cos\phi\sin\theta_B - (w+c)\cos\phi\cos\theta_B \\ &- 2r_F\sin^2\lambda\cos\phi\sin^2\frac{\delta}{2}\cos\theta_B\sin\theta_U \\ &- 2r_F\sin\lambda\cos\lambda\cos\phi\sin^2\frac{\delta}{2}\cos\theta_B\cos\theta_U \\ &- 2r_F\sin\lambda\cos\lambda\cos\phi\sin^2\frac{\delta}{2}\sin\theta_B\sin\theta_U \\ &- 2r_F\cos^2\lambda\cos\phi\sin^2\frac{\delta}{2}\sin\theta_B\cos\theta_U \\ &- \cos\phi\left(-c\cos\delta - 2c\sin^2\lambda\sin^2\frac{\delta}{2} - 2r_F\sin\lambda\cos\lambda\sin^2\frac{\delta}{2}\right)\cos\theta_B \\ &- \cos\phi\left(-r_F\cos\delta - 2c\sin\lambda\cos\lambda\sin^2\frac{\delta}{2} - 2r_F\cos^2\lambda\sin^2\frac{\delta}{2}\right)\sin\theta_B \\ &-r_F\cos\phi\cos\delta\sin\theta_B\cos\theta_U - r_F\cos\phi\cos\delta\cos\theta_B\sin\theta_U \end{aligned}$$

$$\frac{\partial Zero}{\partial \theta_U} = r_F \cos\lambda \sin\phi \sin\delta \cos\theta_U + r_F \sin\lambda \sin\phi \sin\delta \sin\theta_U$$

$$- 2r_F \sin^2\lambda \cos\phi \sin^2\frac{\delta}{2} \sin\theta_B \cos\theta_U$$

$$+ 2r_F \sin\lambda \cos\lambda \cos\phi \sin^2\frac{\delta}{2} \sin\theta_B \sin\theta_U$$

$$+ 2r_F \sin\lambda \cos\lambda \cos\phi \sin^2\frac{\delta}{2} \cos\theta_B \cos\theta_U$$

$$- 2r_F \cos^2\lambda \cos\phi \sin^2\frac{\delta}{2} \cos\theta_B \sin\theta_U$$

$$-r_F \cos\phi \cos\delta \cos\theta_B \sin\theta_U - r_F \cos\phi \cos\delta \sin\theta_B \cos\theta_U$$

$$\frac{\partial Zero}{\partial \phi} = r_R \sin\phi - c\cos\lambda \cos\phi \sin\delta - r_R \sin\phi \cos\theta_B + (w + c)\sin\phi \sin\theta_B$$

$$+ r_F \cos\lambda \cos\phi \sin\delta \sin\theta_U + r_F \sin\lambda \cos\phi \sin\delta$$

$$- r_F \sin\lambda \cos\phi \sin\delta \cos\theta_U$$

$$+2r_F \sin^2\lambda \sin\phi \sin^2\frac{\delta}{2} \sin\theta_B \sin\theta_U + 2r_F \sin\lambda \cos\lambda \sin\phi \sin^2\frac{\delta}{2} \sin\theta_B \cos\theta_U$$

$$-2r_F \sin\lambda \cos\lambda \sin\phi \sin^2\frac{\delta}{2} \cos\theta_B \sin\theta_U - 2r_F \cos^2\lambda \sin\phi \sin^2\frac{\delta}{2} \cos\theta_B \cos\theta_U$$

$$+ \sin\phi \left(-c\cos\delta - 2c\sin^2\lambda \sin^2\frac{\delta}{2} - 2r_F \sin\lambda \cos\lambda \sin^2\frac{\delta}{2}\right) \sin\theta_B$$

$$- \sin\phi \left(-r_F \cos\delta - 2c\sin\lambda \cos\lambda \sin^2\frac{\delta}{2} - 2r_F \cos^2\lambda \sin^2\frac{\delta}{2}\right) \cos\theta_B$$

$$-r_F \sin\phi \cos\delta \cos\theta_B \cos\theta_U + r_F \sin\phi \cos\delta \sin\theta_B \sin\theta_U$$

$$\frac{\partial Zero}{\partial \delta} = -c\cos\lambda \sin\phi \cos\delta + r_F \cos\lambda \sin\phi \cos\delta \sin\theta_U + r_F \sin\lambda \sin\phi \cos\delta$$

$$- r_F \sin\lambda \sin\phi \cos\delta \cos\theta_U$$

$$-r_F \sin^2\lambda \cos\phi \sin\delta \sin\theta_B \sin\theta_U - r_F \sin\lambda \cos\lambda \cos\phi \sin\delta \sin\theta_B \cos\theta_U$$

$$+r_F \sin\lambda \cos\lambda \cos\phi \sin\delta \cos\theta_B \sin\theta_U + r_F \cos^2\lambda \cos\phi \sin\delta \cos\theta_B \cos\theta_U$$

$$- \cos\phi \,(c\sin\delta - c\sin^2\lambda \sin\delta - r_F \sin\lambda \cos\lambda \sin\delta) \sin\theta_B$$

$$+ \cos\phi \,(r_F \sin\delta - c\sin\lambda \cos\lambda \sin\delta - r_F \cos^2\lambda \sin\delta) \cos\theta_B$$

$$-r_F \cos\phi \sin\delta \cos\theta_B \cos\theta_U + r_F \cos\phi \sin\delta \sin\theta_B \sin\theta_U$$

For the second constraint, let $Low = \left(\frac{\partial U}{\partial \theta_U}\right)_z$

$$\frac{\partial U}{\partial \theta_U} = q_H \begin{bmatrix} r_F \cos\theta_U \\ 0 \\ -r_F \sin\theta_U \end{bmatrix} q_H^* = R_z(\psi) R_x(\phi) R_y(\theta_B) \frac{\partial v_H}{\partial \theta_U} R_y(-\theta_B) R_x(-\phi) R_z(-\psi)$$

$$\frac{\partial v_H}{\partial \theta_U} = \begin{bmatrix} r_F \cos\delta \cos\theta_U + \left(2r_F \sin^2\lambda \sin^2\frac{\delta}{2} \cos\theta_U - 2r_F \sin\lambda \cos\lambda \sin^2\frac{\delta}{2} \sin\theta_U \right) \\ r_F \cos\lambda \sin\delta \cos\theta_U + r_F \sin\lambda \sin\delta \sin\theta_U \\ -r_F \cos\delta \sin\theta_U + 2r_F \sin\lambda \cos\lambda \sin^2\frac{\delta}{2} \cos\theta_U - 2r_F \cos^2\lambda \sin^2\frac{\delta}{2} \sin\theta_U \end{bmatrix}$$

$$\Rightarrow 0 = Low = \left(\frac{\partial U}{\partial \theta_U}\right)_z$$

$$= \cos\lambda \sin\phi \sin\delta \cos\theta_U + \sin\lambda \sin\phi \sin\delta \sin\theta_U$$

$$- \left(\cos\phi \cos\delta + 2\cos^2\lambda \cos\phi \sin^2\frac{\delta}{2} \right) \cos\theta_B \sin\theta_U$$

$$- \left(\cos\phi \cos\delta + 2\sin^2\lambda \cos\phi \sin^2\frac{\delta}{2} \right) \sin\theta_B \cos\theta_U$$

$$+ 2\sin\lambda \cos\lambda \cos\phi \sin^2\frac{\delta}{2} \cos\theta_B \cos\theta_U$$

$$+ 2\sin\lambda \cos\lambda \cos\phi \sin^2\frac{\delta}{2} \sin\theta_B \sin\theta_U$$

$$\frac{\partial Low}{\partial \theta_B} = \left(\cos\phi \cos\delta + 2\cos^2\lambda \cos\phi \sin^2\frac{\delta}{2} \right) \sin\theta_B \sin\theta_U$$

$$- \left(\cos\phi \cos\delta + 2\sin^2\lambda \cos\phi \sin^2\frac{\delta}{2} \right) \cos\theta_B \cos\theta_U$$

$$- 2\sin\lambda \cos\lambda \cos\phi \sin^2\frac{\delta}{2} \sin\theta_B \cos\theta_U$$

$$+ 2\sin\lambda \cos\lambda \cos\phi \sin^2\frac{\delta}{2} \cos\theta_B \sin\theta_U$$

$$\frac{\partial Low}{\partial \theta_U} = -\cos\lambda \sin\phi \sin\delta \sin\theta_U + \sin\lambda \sin\phi \sin\delta \cos\theta_U$$

$$- \left(\cos\phi \cos\delta + 2\cos^2\lambda \cos\phi \sin^2\frac{\delta}{2} \right) \cos\theta_B \cos\theta_U$$

$$+ \left(\cos\phi \cos\delta + 2\sin^2\lambda \cos\phi \sin^2\frac{\delta}{2} \right) \sin\theta_B \sin\theta_U$$

$$- 2\sin\lambda \cos\lambda \cos\phi \sin^2\frac{\delta}{2} \cos\theta_B \sin\theta_U$$

$$+ 2\sin\lambda \cos\lambda \cos\phi \sin^2\frac{\delta}{2} \sin\theta_B \cos\theta_U$$

$$\frac{\partial Low}{\partial \phi} = \cos\lambda\cos\phi\sin\delta\cos\theta_U + \sin\lambda\cos\phi\sin\delta\sin\theta_U$$
$$- \left(-\sin\phi\cos\delta - 2\cos^2\lambda\sin\phi\sin^2\frac{\delta}{2}\right)\cos\theta_B\sin\theta_U$$
$$- \left(-\sin\phi\cos\delta - 2\sin^2\lambda\sin\phi\sin^2\frac{\delta}{2}\right)\sin\theta_B\cos\theta_U$$
$$- 2\sin\lambda\cos\lambda\sin\phi\sin^2\frac{\delta}{2}\cos\theta_B\cos\theta_U$$
$$- 2\sin\lambda\cos\lambda\sin\phi\sin^2\frac{\delta}{2}\sin\theta_B\sin\theta_U$$

$$\frac{\partial Low}{\partial \delta} = \cos\lambda\sin\phi\cos\delta\cos\theta_U + \sin\lambda\sin\phi\cos\delta\sin\theta_U$$
$$- (-\cos\phi\sin\delta + 2\cos^2\lambda\cos\phi\sin\delta)\cos\theta_B\sin\theta_U$$
$$- (-\cos\phi\sin\delta + 2\sin^2\lambda\cos\phi\sin\delta)\sin\theta_B\cos\theta_U$$
$$+ 2\sin\lambda\cos\lambda\cos\phi\sin\delta\cos\theta_B\cos\theta_U$$
$$+ 2\sin\lambda\cos\lambda\cos\phi\sin\delta\sin\theta_B\sin\theta_U$$

Combine the two constraints, let $\theta_B = \theta_B(\phi,\delta), \theta_U = \theta_U(\phi,\delta)$, because $Zero = Low = 0 = constant$,

$$\begin{cases} 0 = \dfrac{dZero}{dt} = \dfrac{\partial Zero}{\partial \theta_B}\dot{\theta}_B + \dfrac{\partial Zero}{\partial \theta_U}\dot{\theta}_U + \dfrac{\partial Zero}{\partial \phi}\dot{\phi} + \dfrac{\partial Zero}{\partial \delta}\dot{\delta} \\ 0 = \dfrac{dLow}{dt} = \dfrac{\partial Low}{\partial \theta_B}\dot{\theta}_B + \dfrac{\partial Low}{\partial \theta_U}\dot{\theta}_U + \dfrac{\partial Low}{\partial \phi}\dot{\phi} + \dfrac{\partial Low}{\partial \delta}\dot{\delta} \end{cases}$$

$$\Rightarrow \begin{bmatrix}\dot{\theta}_B \\ \dot{\theta}_U\end{bmatrix} = \begin{bmatrix} \dfrac{\partial Zero}{\partial \theta_B} & \dfrac{\partial Zero}{\partial \theta_U} \\ \dfrac{\partial Low}{\partial \theta_B} & \dfrac{\partial Low}{\partial \theta_U} \end{bmatrix}^{-1} \begin{bmatrix} -\dfrac{\partial Zero}{\partial \phi}\dot{\phi} - \dfrac{\partial Zero}{\partial \delta}\dot{\delta} \\ -\dfrac{\partial Low}{\partial \phi}\dot{\phi} - \dfrac{\partial Low}{\partial \delta}\dot{\delta} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial Zero}{\partial \theta_B} & \dfrac{\partial Zero}{\partial \theta_U} \\ \dfrac{\partial Low}{\partial \theta_B} & \dfrac{\partial Low}{\partial \theta_U} \end{bmatrix}^{-1} \begin{bmatrix} -\dfrac{\partial Zero}{\partial \phi} \\ -\dfrac{\partial Low}{\partial \phi} \end{bmatrix}\dot{\phi}$$

$$+ \begin{bmatrix} \dfrac{\partial Zero}{\partial \theta_B} & \dfrac{\partial Zero}{\partial \theta_U} \\ \dfrac{\partial Low}{\partial \theta_B} & \dfrac{\partial Low}{\partial \theta_U} \end{bmatrix}^{-1} \begin{bmatrix} -\dfrac{\partial Zero}{\partial \delta} \\ -\dfrac{\partial Low}{\partial \delta} \end{bmatrix}\dot{\delta}$$

$$= \begin{bmatrix} \dfrac{\partial \theta_B}{\partial \phi} \\ \dfrac{\partial \theta_U}{\partial \phi} \end{bmatrix}\dot{\phi} + \begin{bmatrix} \dfrac{\partial \theta_B}{\partial \delta} \\ \dfrac{\partial \theta_U}{\partial \delta} \end{bmatrix}\dot{\delta}$$

$$\Rightarrow \dot{\theta_B} = \frac{\partial \theta_B}{\partial \phi}\dot{\phi} + \frac{\partial \theta_B}{\partial \delta}\dot{\delta}, \dot{\theta_U} = \frac{\partial \theta_U}{\partial \phi}\dot{\phi} + \frac{\partial \theta_U}{\partial \delta}\dot{\delta}$$

At set point

$$\frac{\partial Zero}{\partial \phi} = \frac{\partial Zero}{\partial \delta} = \frac{\partial Low}{\partial \phi} = \frac{\partial Low}{\partial \delta} = 0, \frac{\partial \theta_B}{\partial \phi} = \frac{\partial \theta_B}{\partial \delta} = \frac{\partial \theta_U}{\partial \phi} = \frac{\partial \theta_U}{\partial \delta} = 0$$

$$\begin{bmatrix} \frac{\partial Zero}{\partial \theta_B} & \frac{\partial Zero}{\partial \theta_U} \\ \frac{\partial Low}{\partial \theta_B} & \frac{\partial Low}{\partial \theta_U} \end{bmatrix}^{-1} = \begin{bmatrix} -w & 0 \\ -1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{w} & 0 \\ \frac{1}{w} & -1 \end{bmatrix}$$

Therefore,

$$\frac{\partial \theta_B}{\partial \phi} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial Zero}{\partial \theta_B} & \frac{\partial Zero}{\partial \theta_U} \\ \frac{\partial Low}{\partial \theta_B} & \frac{\partial Low}{\partial \theta_U} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{\partial Zero}{\partial \phi} \\ -\frac{\partial Low}{\partial \phi} \end{bmatrix} = \begin{bmatrix} -\frac{1}{w} & 0 \end{bmatrix} \begin{bmatrix} -\frac{\partial Zero}{\partial \phi} \\ -\frac{\partial Low}{\partial \phi} \end{bmatrix}$$

$$\frac{\partial \theta_B}{\partial \delta} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial Zero}{\partial \theta_B} & \frac{\partial Zero}{\partial \theta_U} \\ \frac{\partial Low}{\partial \theta_B} & \frac{\partial Low}{\partial \theta_U} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{\partial Zero}{\partial \delta} \\ -\frac{\partial Low}{\partial \delta} \end{bmatrix} = \begin{bmatrix} -\frac{1}{w} & 0 \end{bmatrix} \begin{bmatrix} -\frac{\partial Zero}{\partial \delta} \\ -\frac{\partial Low}{\partial \delta} \end{bmatrix}$$

then

$$\frac{\partial}{\partial \phi}\left(\frac{\partial \theta_B}{\partial \phi}\right) = \begin{bmatrix} -\frac{1}{w} & 0 \end{bmatrix} \begin{bmatrix} -\frac{\partial}{\partial \phi}\left(\frac{\partial Zero}{\partial \phi}\right) \\ -\frac{\partial}{\partial \phi}\left(\frac{\partial Low}{\partial \phi}\right) \end{bmatrix} = \frac{1}{w}\frac{\partial}{\partial \phi}\left(\frac{\partial Zero}{\partial \phi}\right)$$

$$\frac{\partial}{\partial \delta}\left(\frac{\partial \theta_B}{\partial \phi}\right) = \frac{\partial}{\partial \phi}\left(\frac{\partial \theta_B}{\partial \delta}\right) = \frac{1}{w}\frac{\partial}{\partial \delta}\left(\frac{\partial Zero}{\partial \phi}\right)$$

$$\frac{\partial}{\partial \delta}\left(\frac{\partial \theta_B}{\partial \delta}\right) = \frac{1}{w}\frac{\partial}{\partial \delta}\left(\frac{\partial Zero}{\partial \delta}\right)$$

$$\begin{aligned} \frac{\partial}{\partial \phi}\left(\frac{\partial Zero}{\partial \phi}\right) = {} & r_R \cos\phi + c\cos\lambda\sin\phi\sin\delta - r_R\cos\phi\cos\theta_B \\ & + (w+c)\cos\phi\sin\theta_B - r_F\cos\lambda\sin\phi\sin\delta\sin\theta_U \\ & - r_F\sin\lambda\sin\phi\sin\delta + r_F\sin\lambda\sin\phi\sin\delta\cos\theta_U \end{aligned}$$

$$+2r_F\sin^2\lambda\cos\phi\sin^2\frac{\delta}{2}\sin\theta_B\sin\theta_U + 2r_F\sin\lambda\cos\lambda\cos\phi\sin^2\frac{\delta}{2}\sin\theta_B\cos\theta_U$$

$$-2r_F \sin\lambda \cos\lambda \cos\phi \sin^2\frac{\delta}{2}\cos\theta_B \sin\theta_U - 2r_F \cos^2\lambda \cos\phi \sin^2\frac{\delta}{2}\cos\theta_B \cos\theta_U$$

$$+\cos\phi\left(-c\cos\delta - 2c\sin^2\lambda \sin^2\frac{\delta}{2} - 2r_F \sin\lambda\cos\lambda\sin^2\frac{\delta}{2}\right)\sin\theta_B$$

$$-\cos\phi\left(-r_F\cos\delta - 2c\sin\lambda\cos\lambda\sin^2\frac{\delta}{2} - 2r_F\cos^2\lambda\sin^2\frac{\delta}{2}\right)\cos\theta_B$$

$$-r_F\cos\phi\cos\delta\cos\theta_B\cos\theta_U + r_F\cos\phi\cos\delta\sin\theta_B\sin\theta_U$$

$$= r_R - r_R + r_F - r_F = 0$$

$$\frac{\partial}{\partial\delta}\left(\frac{\partial Zero}{\partial\phi}\right) = -c\cos\lambda\cos\phi\cos\delta + r_F\cos\lambda\cos\phi\cos\delta\sin\theta_U$$

$$+ r_F\sin\lambda\cos\phi\cos\delta - r_F\sin\lambda\cos\phi\cos\delta\cos\theta_U$$

$$+r_F\sin^2\lambda\sin\phi\sin\delta\sin\theta_B\sin\theta_U + r_F\sin\lambda\cos\lambda\sin\phi\sin\delta\sin\theta_B\cos\theta_U$$

$$-r_F\sin\lambda\cos\lambda\sin\phi\sin\delta\cos\theta_B\sin\theta_U - r_F\cos^2\lambda\sin\phi\sin\delta\cos\theta_B\cos\theta_U$$

$$+\sin\phi\,(c\sin\delta - c\sin^2\lambda\sin\delta - r_F\sin\lambda\cos\lambda\sin\delta)\sin\theta_B$$

$$-\sin\phi\,(r_F\sin\delta - c\sin\lambda\cos\lambda\sin\delta - r_F\cos^2\lambda\sin\delta)\cos\theta_B$$

$$+r_F\sin\phi\sin\delta\cos\theta_B\cos\theta_U - r_F\sin\phi\sin\delta\sin\theta_B\sin\theta_U$$

$$= -c\cos\lambda + r_F\sin\lambda - r_F\sin\lambda = -c\cos\lambda$$

$$\frac{\partial}{\partial\delta}\left(\frac{\partial Zero}{\partial\delta}\right) = c\cos\lambda\sin\phi\sin\delta - r_F\cos\lambda\sin\phi\sin\delta\sin\theta_U$$

$$-r_F\sin\lambda\sin\phi\sin\delta + r_F\sin\lambda\sin\phi\sin\delta\cos\theta_U$$

$$-r_F\sin^2\lambda\cos\phi\cos\delta\sin\theta_B\sin\theta_U - r_F\sin\lambda\cos\lambda\cos\phi\cos\delta\sin\theta_B\cos\theta_U$$

$$+r_F\sin\lambda\cos\lambda\cos\phi\cos\delta\cos\theta_B\sin\theta_U + r_F\cos^2\lambda\cos\phi\cos\delta\cos\theta_B\cos\theta_U$$

$$-\cos\phi\,(c\cos\delta - c\sin^2\lambda\cos\delta - r_F\sin\lambda\cos\lambda\cos\delta)\sin\theta_B$$

$$+\cos\phi\,(r_F\cos\delta - c\sin\lambda\cos\lambda\cos\delta - r_F\cos^2\lambda\cos\delta)\cos\theta_B$$

$$-r_F\cos\phi\cos\delta\cos\theta_B\cos\theta_U + r_F\cos\phi\cos\delta\sin\theta_B\sin\theta_U$$

$$= -c\sin\lambda\, cos\,\lambda$$

This equation below is used in potential energy

$$\begin{bmatrix} \frac{\partial}{\partial\phi}\left(\frac{\partial\theta_B}{\partial\phi}\right) & \frac{\partial}{\partial\delta}\left(\frac{\partial\theta_B}{\partial\phi}\right) \\ \frac{\partial}{\partial\phi}\left(\frac{\partial\theta_B}{\partial\delta}\right) & \frac{\partial}{\partial\delta}\left(\frac{\partial\theta_B}{\partial\delta}\right) \end{bmatrix} = \frac{1}{w}\begin{bmatrix} 0 & -c\cos\lambda \\ -c\cos\lambda & -c\sin\lambda\, cos\,\lambda \end{bmatrix}$$

Tires don't slip.

Because $U_z = 0 \Rightarrow \left(\frac{dU}{dt}\right)_z = 0$, we only need to constrain the xy component of $\frac{dU}{dt}$

$$\frac{dP}{dt} = q_P \begin{bmatrix} -r_R \\ 0 \\ 0 \end{bmatrix} q_P^* \dot{\theta_R} = \begin{bmatrix} -r_R \cos\psi \\ -r_R \sin\psi \\ 0 \end{bmatrix} \dot{\theta_R}$$

$$\frac{\partial U}{\partial \theta_F} \dot{\theta_F} = q_H \begin{bmatrix} -r_F \cos\theta_U \\ 0 \\ r_F \sin\theta_U \end{bmatrix} q_H^* \dot{\theta_F} = \frac{dU}{dt}$$

$$= \frac{\partial U}{\partial \psi}\dot{\psi} + \frac{\partial U}{\partial \theta_R}\dot{\theta_R} + \frac{\partial U}{\partial \phi}\dot{\phi} + \frac{\partial U}{\partial \delta}\dot{\delta} + \frac{\partial U}{\partial \theta_B}\dot{\theta_B} + \frac{\partial U}{\partial \theta_U}\dot{\theta_U}$$

$$\begin{bmatrix} \dot{\theta_F} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial U}{\partial \theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial \psi}\right)_{xy} \end{bmatrix}^{-1} \left(\frac{\partial U}{\partial \theta_R}\dot{\theta_R} + \frac{\partial U}{\partial \phi}\dot{\phi} + \frac{\partial U}{\partial \delta}\dot{\delta} + \frac{\partial U}{\partial \theta_B}\dot{\theta_B} + \frac{\partial U}{\partial \theta_U}\dot{\theta_U}\right)_{xy}$$

$$= \begin{bmatrix} \left(\frac{\partial U}{\partial \theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial \psi}\right)_{xy} \end{bmatrix}^{-1} \left(\frac{\partial U}{\partial \theta_R}\dot{\theta_R} + \left(\frac{\partial U}{\partial \phi} + \frac{\partial U}{\partial \theta_B}\frac{\partial \theta_B}{\partial \phi} + \frac{\partial U}{\partial \theta_U}\frac{\partial \theta_U}{\partial \phi}\right)\dot{\phi} + \left(\frac{\partial U}{\partial \delta} + \frac{\partial U}{\partial \theta_B}\frac{\partial \theta_B}{\partial \delta} + \frac{\partial U}{\partial \theta_U}\frac{\partial \theta_U}{\partial \delta}\right)\dot{\delta}\right)_{xy}$$

$$\frac{\partial U}{\partial \theta_F} = q_H \begin{bmatrix} -r_F \cos\theta_U \\ 0 \\ r_F \sin\theta_U \end{bmatrix} q_H^* = q_B v_B q_B^*$$

$$v_B = \begin{bmatrix} -r_F \cos\delta \cos\theta_U + 2\sin^2\frac{\delta}{2}\sin\lambda\,(-r_F \sin\lambda\cos\theta_U + r_F\cos\lambda\sin\theta_U) \\ \sin\delta\,(-r_F\cos\lambda\cos\theta_U - r_F\sin\lambda\sin\theta_U) \\ r_F\cos\delta\sin\theta_U + 2\sin^2\frac{\delta}{2}\cos\lambda\,(-r_F\sin\lambda\cos\theta_U + r_F\cos\lambda\sin\theta_U) \end{bmatrix}$$

$$(q_B v_B q_B^*)_{xy}$$
$$= \begin{bmatrix} \cos\psi\,(\cos\theta_B\, v_{Bx} + \sin\theta_B\, v_{Bz}) - \sin\psi\,(\sin\phi\sin\theta_B\, v_{Bx} + \cos\phi\, v_{By} - \sin\phi\cos\theta_B\, v_{Bz}) \\ \cos\psi\,(\sin\phi\sin\theta_B\, v_{Bx} + \cos\phi\, v_{By} - \sin\phi\cos\theta_B\, v_{Bz}) + \sin\psi\,(\cos\theta_B\, v_{Bx} + \sin\theta_B\, v_{Bz}) \end{bmatrix}$$
$$= \begin{bmatrix} \cos\psi\cos\theta_B - \sin\psi\sin\phi\sin\theta_B & -\sin\psi\cos\phi & \cos\psi\sin\theta_B + \sin\psi\sin\phi\cos\theta_B \\ \sin\psi\cos\theta_B + \cos\psi\sin\phi\sin\theta_B & \cos\psi\cos\phi & \sin\psi\sin\theta_B - \cos\psi\sin\phi\cos\theta_B \end{bmatrix} v_B$$

$$\frac{\partial U}{\partial \psi} = \cos\psi\left(\begin{bmatrix} -r_R \sin\phi + \sin\phi\,(r_R\cos\theta_B - (w+c)\sin\theta_B) \\ r_R\sin\theta_B + (w+c)\cos\theta_B \\ 0 \end{bmatrix} + \begin{bmatrix} -\sin\phi\sin\theta_B\, v_{Hx} - \cos\phi\, v_{Hy} + \sin\phi\cos\theta_B\, v_{Hz} \\ \cos\theta_B\, v_{Hx} + \sin\theta_B\, v_{Hz} \\ 0 \end{bmatrix}\right)$$

$$- \sin\psi\left(\begin{bmatrix} r_R\sin\theta_B + (w+c)\cos\theta_B \\ r_R\sin\phi - \sin\phi\,(r_R\cos\theta_B - (w+c)\sin\theta_B) \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\theta_B\, v_{Hx} + \sin\theta_B\, v_{Hz} \\ \sin\phi\sin\theta_B\, v_{Hx} + \cos\phi\, v_{Hy} - \sin\phi\cos\theta_B\, v_{Hz} \\ 0 \end{bmatrix}\right)$$

$$-\left(\frac{\partial U}{\partial \psi}\right)_{xy} = \begin{bmatrix} \sin\psi\cos\theta_B + \cos\psi\sin\phi\sin\theta_B & \cos\psi\cos\phi & \sin\psi\sin\theta_B - \cos\psi\sin\phi\cos\theta_B \\ -\cos\psi\cos\theta_B + \sin\psi\sin\phi\sin\theta_B & \sin\psi\cos\phi & -\cos\psi\sin\theta_B - \sin\psi\sin\phi\cos\theta_B \end{bmatrix} v_H + \begin{bmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{bmatrix}\begin{bmatrix} r_R\sin\phi - \sin\phi\,(r_R\cos\theta_B - (w+c)\sin\theta_B) \\ r_R\sin\theta_B + (w+c)\cos\theta_B \end{bmatrix}$$

Such that

$$\dot{\theta_F} = \frac{\partial \theta_F}{\partial \theta_R}\dot{\theta_R} + \frac{\partial \theta_F}{\partial \phi}\dot{\phi} + \frac{\partial \theta_F}{\partial \delta}\dot{\delta}, \dot{\psi} = \frac{\partial \psi}{\partial \theta_R}\dot{\theta_R} + \frac{\partial \psi}{\partial \phi}\dot{\phi} + \frac{\partial \psi}{\partial \delta}\dot{\delta}$$

At set point

$$\left(\frac{\partial U}{\partial \theta_F}\right)_{xy} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \end{bmatrix}\begin{bmatrix} -r_F \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -r_F\cos\psi \\ -r_F\sin\psi \end{bmatrix}$$

$$-\left(\frac{\partial U}{\partial \psi}\right)_{xy} = \begin{bmatrix} \sin\psi & \cos\psi & 0 \\ -\cos\psi & \sin\psi & 0 \end{bmatrix}\begin{bmatrix} -c \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{bmatrix}\begin{bmatrix} 0 \\ w+c \end{bmatrix} = \begin{bmatrix} w\sin\psi \\ -w\cos\psi \end{bmatrix}$$

$$\left[\left(\frac{\partial U}{\partial \theta_F}\right)_{xy} \quad -\left(\frac{\partial U}{\partial \psi}\right)_{xy}\right]^{-1} = \begin{bmatrix} -r_F\cos\psi & w\sin\psi \\ -r_F\sin\psi & -w\cos\psi \end{bmatrix}^{-1}$$

$$= \frac{1}{r_F w}\begin{bmatrix} -w\cos\psi & -w\sin\psi \\ r_F\sin\psi & -r_F\cos\psi \end{bmatrix} = \begin{bmatrix} -\frac{1}{r_F}\cos\psi & -\frac{1}{r_F}\sin\psi \\ \frac{1}{w}\sin\psi & -\frac{1}{w}\cos\psi \end{bmatrix}$$

$$U = R_z(\psi)\begin{bmatrix} r_R\sin\theta_B + (w+c)\cos\theta_B \\ r_R\sin\phi - \sin\phi\,(r_R\cos\theta_B - (w+c)\sin\theta_B) \\ -r_R\cos\phi + \cos\phi\,(r_R\cos\theta_B - (w+c)\sin\theta_B) \end{bmatrix} R_z(-\psi)$$

$$+R_z(\psi)\begin{bmatrix}\cos\theta_B\, v_{Hx}+\sin\theta_B\, v_{Hz}\\ \sin\phi\sin\theta_B\, v_{Hx}+\cos\phi\, v_{Hy}-\sin\phi\cos\theta_B\, v_{Hz}\\ -\cos\phi\sin\theta_B\, v_{Hx}+\sin\phi\, v_{Hy}+\cos\phi\cos\theta_B\, v_{Hz}\end{bmatrix}R_z(-\psi)$$

$$v_H$$

$$=\begin{bmatrix}\cos\delta\,(r_F\sin\theta_U-c)+2\sin^2\frac{\delta}{2}\sin\lambda\left(\sin\lambda\,(r_F\sin\theta_U-c)+\cos\lambda\,(r_F\cos\theta_U-r_F)\right)\\ \sin\delta\left(\cos\lambda\,(r_F\sin\theta_U-c)-\sin\lambda\,(r_F\cos\theta_U-r_F)\right)\\ \cos\delta\,(r_F\cos\theta_U-r_F)+2\sin^2\frac{\delta}{2}\cos\lambda\left(\sin\lambda\,(r_F\sin\theta_U-c)+\cos\lambda\,(r_F\cos\theta_U-r_F)\right)\end{bmatrix}$$

$$=\begin{bmatrix}-c\\0\\0\end{bmatrix}$$

$$\frac{\partial v_H}{\partial\delta}$$

$$=\begin{bmatrix}-\sin\delta\,(r_F\sin\theta_U-c)+\sin\delta\sin\lambda\left(\sin\lambda\,(r_F\sin\theta_U-c)+\cos\lambda\,(r_F\cos\theta_U-r_F)\right)\\ \cos\delta\left(\cos\lambda\,(r_F\sin\theta_U-c)-\sin\lambda\,(r_F\cos\theta_U-r_F)\right)\\ -\sin\delta\,(r_F\cos\theta_U-r_F)+\sin\delta\cos\lambda\left(\sin\lambda\,(r_F\sin\theta_U-c)+\cos\lambda\,(r_F\cos\theta_U-r_F)\right)\end{bmatrix}$$

$$=\begin{bmatrix}0\\-c\cos\lambda\\0\end{bmatrix}$$

$$\frac{\partial^2 v_H}{\partial\delta^2}=\begin{bmatrix}c\cos^2\lambda\\0\\-c\sin\lambda\cos\lambda\end{bmatrix}$$

Because $\frac{dP}{dt}=\begin{bmatrix}-r_R\cos\psi\\-r_R\sin\psi\\0\end{bmatrix}\dot{\theta}_R=\frac{\partial U}{\partial\theta_R}\dot{\theta}_R$, therefore

$$\frac{\partial U}{\partial\theta_R}=\begin{bmatrix}-r_R\cos\psi\\-r_R\sin\psi\end{bmatrix}$$

Others

$$\frac{\partial U}{\partial\phi}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\frac{\partial U}{\partial\delta}=R_z(\psi)\frac{\partial v_H}{\partial\delta}R_z(-\psi)=R_z(\psi)\begin{bmatrix}0\\-c\cos\lambda\\0\end{bmatrix}R_z(-\psi)=\begin{bmatrix}c\cos\lambda\sin\psi\\-c\cos\lambda\cos\psi\\0\end{bmatrix}$$

$$\frac{\partial U}{\partial \theta_B} = R_z(\psi)\left(\begin{bmatrix} r_R \\ 0 \\ -(w+c) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}\right) R_z(-\psi) = R_z(\psi)\begin{bmatrix} r_R \\ 0 \\ -w \end{bmatrix} R_z(-\psi) = \begin{bmatrix} r_R\cos\psi \\ r_R\sin\psi \\ -w \end{bmatrix}$$

$$\frac{\partial U}{\partial \theta_U} = R_z(\psi)\frac{\partial v_H}{\partial \theta_U} R_z(-\psi) = R_z(\psi)\begin{bmatrix} r_F \\ 0 \\ 0 \end{bmatrix} R_z(-\psi) = \begin{bmatrix} r_F\cos\psi \\ r_F\sin\psi \\ 0 \end{bmatrix}$$

Therefore,

$$\frac{\partial \psi}{\partial \theta_R} = \begin{bmatrix} \frac{1}{w}\sin\psi & -\frac{1}{w}\cos\psi \end{bmatrix} \frac{\partial U}{\partial \theta_R} = \begin{bmatrix} \frac{1}{w}\sin\psi & -\frac{1}{w}\cos\psi \end{bmatrix} \begin{bmatrix} -r_R\cos\psi \\ -r_R\sin\psi \end{bmatrix} = 0$$

$$\frac{\partial \psi}{\partial \phi} = \begin{bmatrix} \frac{1}{w}\sin\psi & -\frac{1}{w}\cos\psi \end{bmatrix} \left(\frac{\partial U}{\partial \phi} + \frac{\partial U}{\partial \theta_B}\frac{\partial \theta_B}{\partial \phi} + \frac{\partial U}{\partial \theta_U}\frac{\partial \theta_U}{\partial \phi}\right) = 0$$

$$\frac{\partial \psi}{\partial \delta} = \begin{bmatrix} \frac{1}{w}\sin\psi & -\frac{1}{w}\cos\psi \end{bmatrix} \left(\frac{\partial U}{\partial \delta} + \frac{\partial U}{\partial \theta_B}\frac{\partial \theta_B}{\partial \delta} + \frac{\partial U}{\partial \theta_U}\frac{\partial \theta_U}{\partial \delta}\right)$$

$$= \begin{bmatrix} \frac{1}{w}\sin\psi & -\frac{1}{w}\cos\psi \end{bmatrix} \left(\begin{bmatrix} c\cos\lambda\sin\psi \\ -c\cos\lambda\cos\psi \end{bmatrix}\right) = \frac{c}{w}\cos\lambda = \mu$$

$$\frac{\partial \theta_F}{\partial \theta_R} = \begin{bmatrix} -\frac{1}{r_F}\cos\psi & -\frac{1}{r_F}\sin\psi \end{bmatrix} \frac{\partial U}{\partial \theta_R} = \begin{bmatrix} -\frac{1}{r_F}\cos\psi & -\frac{1}{r_F}\sin\psi \end{bmatrix} \begin{bmatrix} -r_R\cos\psi \\ -r_R\sin\psi \end{bmatrix}$$

$$= r_R/r_F$$

$$\frac{\partial \theta_F}{\partial \phi} = \begin{bmatrix} -\frac{1}{r_F}\cos\psi & -\frac{1}{r_F}\sin\psi \end{bmatrix} \left(\frac{\partial U}{\partial \phi} + \frac{\partial U}{\partial \theta_B}\frac{\partial \theta_B}{\partial \phi} + \frac{\partial U}{\partial \theta_U}\frac{\partial \theta_U}{\partial \phi}\right) = 0$$

$$\frac{\partial \theta_F}{\partial \delta} = \begin{bmatrix} -\frac{1}{r_F}\cos\psi & -\frac{1}{r_F}\sin\psi \end{bmatrix} \left(\frac{\partial U}{\partial \delta} + \frac{\partial U}{\partial \theta_B}\frac{\partial \theta_B}{\partial \delta} + \frac{\partial U}{\partial \theta_U}\frac{\partial \theta_U}{\partial \delta}\right)$$

$$= \begin{bmatrix} -\frac{1}{r_F}\cos\psi & -\frac{1}{r_F}\sin\psi \end{bmatrix} \begin{bmatrix} c\cos\lambda\sin\psi \\ -c\cos\lambda\cos\psi \end{bmatrix} = 0$$

$$\left(\frac{\partial}{\partial \phi}\left(\frac{\partial U}{\partial \theta_F}\right)\right)_{xy} = 0$$

$$\left(\frac{\partial}{\partial \delta}\left(\frac{\partial U}{\partial \theta_F}\right)\right)_{xy} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -r_F\cos\lambda \\ 0 \end{bmatrix} = r_F\cos\lambda \begin{bmatrix} \sin\psi \\ -\cos\psi \end{bmatrix}$$

$$\left(\frac{\partial}{\partial \phi}\left(-\frac{\partial U}{\partial \psi}\right)\right)_{xy} = 0$$

$$\left(\frac{\partial}{\partial \delta}\left(-\frac{\partial U}{\partial \psi}\right)\right)_{xy} = c\cos\lambda \begin{bmatrix} -\cos\psi \\ -\sin\psi \end{bmatrix}$$

$$\frac{\partial}{\partial\phi}\left(\frac{\partial U}{\partial\theta_R}\right)=\frac{\partial}{\partial\phi}\left(\begin{bmatrix}-r_R\cos\psi\\-r_R\sin\psi\end{bmatrix}\right)=\begin{bmatrix}r_R\sin\psi\\-r_R\operatorname{coss}\psi\end{bmatrix}\frac{\partial\psi}{\partial\phi}=0$$

$$\frac{\partial}{\partial\delta}\left(\frac{\partial U}{\partial\theta_R}\right)=\frac{\partial}{\partial\delta}\left(\begin{bmatrix}-r_R\cos\psi\\-r_R\sin\psi\end{bmatrix}\right)=\begin{bmatrix}r_R\sin\psi\\-r_R\cos\psi\end{bmatrix}\frac{\partial\psi}{\partial\phi}=\mu\begin{bmatrix}r_R\sin\psi\\-r_R\cos\psi\end{bmatrix}$$

$$\frac{\partial}{\partial\phi}\left(\frac{\partial U}{\partial\phi}\right)=0$$

$$\frac{\partial}{\partial\delta}\left(\frac{\partial U}{\partial\phi}\right)=0$$

$$\frac{\partial}{\partial\phi}\left(\frac{\partial U}{\partial\delta}\right)=0$$

$$\frac{\partial}{\partial\delta}\left(\frac{\partial U}{\partial\delta}\right)=R_z(\psi)\begin{bmatrix}c\cos^2\lambda\\0\\-c\sin\lambda\cos\lambda\end{bmatrix}R_z(-\psi)=\begin{bmatrix}c\cos^2\lambda\cos\psi\\c\cos^2\lambda\sin\psi\\-c\sin\lambda\cos\lambda\end{bmatrix}$$

$$\frac{\partial}{\partial\phi}\left(\frac{\partial\psi}{\partial\theta_R}\right)=\frac{\partial}{\partial\phi}\left(\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\end{bmatrix}^{-1}\frac{\partial U}{\partial\theta_R}\right)$$

$$=\begin{bmatrix}0 & 1\end{bmatrix}\frac{\partial}{\partial\phi}\left(\begin{bmatrix}\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\end{bmatrix}^{-1}\right)\begin{bmatrix}-r_R\cos\psi\\-r_R\sin\psi\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{1}{w}\sin\psi & \frac{1}{w}\cos\psi\end{bmatrix}\frac{\partial}{\partial\phi}\left(\begin{bmatrix}\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\end{bmatrix}\right)\begin{bmatrix}\frac{r_R}{r_F}\\0\end{bmatrix}$$

$$=\frac{r_R}{r_F}\begin{bmatrix}-\frac{1}{w}\sin\psi & \frac{1}{w}\cos\psi\end{bmatrix}\left(\frac{\partial}{\partial\phi}\left(\frac{\partial U}{\partial\theta_F}\right)\right)_{xy}=0$$

$$\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right)=\frac{\partial}{\partial\delta}\left(\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\end{bmatrix}^{-1}\frac{\partial U}{\partial\theta_R}\right)$$

$$=\begin{bmatrix}0 & 1\end{bmatrix}\frac{\partial}{\partial\delta}\left(\begin{bmatrix}\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\end{bmatrix}^{-1}\right)\begin{bmatrix}-r_R\cos\psi\\-r_R\sin\psi\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{1}{w}\sin\psi & \frac{1}{w}\cos\psi\end{bmatrix}\frac{\partial}{\partial\delta}\left(\begin{bmatrix}\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\end{bmatrix}\right)\begin{bmatrix}\frac{r_R}{r_F}\\0\end{bmatrix}$$

$$= \frac{r_R}{r_F}\begin{bmatrix}-\frac{1}{w}\sin\psi & \frac{1}{w}\cos\psi\end{bmatrix}\left(\frac{\partial}{\partial\delta}\left(\frac{\partial U}{\partial\theta_F}\right)\right)_{xy} = -\frac{r_R}{w}\cos\lambda$$

$$\frac{\partial}{\partial\phi}\left(\frac{\partial\psi}{\partial\phi}\right) = \frac{\partial}{\partial\phi}\left(\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\end{bmatrix}^{-1}\frac{\partial U}{\partial\phi}\right)$$

$$= \begin{bmatrix}0 & 1\end{bmatrix}\frac{\partial}{\partial\phi}\left(\begin{bmatrix}\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\end{bmatrix}^{-1}\right)\frac{\partial U}{\partial\phi} = 0$$

$$\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\phi}\right) = \frac{\partial}{\partial\delta}\left(\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\end{bmatrix}^{-1}\frac{\partial U}{\partial\phi}\right)$$

$$= \begin{bmatrix}0 & 1\end{bmatrix}\frac{\partial}{\partial\delta}\left(\begin{bmatrix}\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\end{bmatrix}^{-1}\right)\frac{\partial U}{\partial\phi} = 0$$

$$\frac{\partial}{\partial\phi}\left(\frac{\partial\psi}{\partial\delta}\right) = \frac{\partial}{\partial\phi}\left(\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\end{bmatrix}^{-1}\frac{\partial U}{\partial\delta}\right)$$

$$= \begin{bmatrix}0 & 1\end{bmatrix}\frac{\partial}{\partial\phi}\left(\begin{bmatrix}\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\end{bmatrix}^{-1}\right)\begin{bmatrix}c\cos\lambda\sin\psi \\ -c\cos\lambda\cos\psi\end{bmatrix}$$

$$= \begin{bmatrix}-\frac{1}{w}\sin\psi & \frac{1}{w}\cos\psi\end{bmatrix}\frac{\partial}{\partial\phi}\left(\begin{bmatrix}\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\end{bmatrix}\right)\begin{bmatrix}0 \\ \frac{c}{w}\cos\lambda\end{bmatrix}$$

$$= \frac{c}{w}\cos\lambda\begin{bmatrix}-\frac{1}{w}\sin\psi & \frac{1}{w}\cos\psi\end{bmatrix}\left(\frac{\partial}{\partial\phi}\left(-\frac{\partial U}{\partial\psi}\right)\right)_{xy} = 0$$

$$\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\delta}\right) = \frac{\partial}{\partial\delta}\left(\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\end{bmatrix}^{-1}\frac{\partial U}{\partial\delta}\right)$$

$$= \begin{bmatrix}0 & 1\end{bmatrix}\frac{\partial}{\partial\delta}\left(\begin{bmatrix}\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\end{bmatrix}^{-1}\right)\begin{bmatrix}c\cos\lambda\sin\psi \\ -c\cos\lambda\cos\psi\end{bmatrix}$$

$$+ \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\end{bmatrix}^{-1}\frac{\partial}{\partial\delta}\left(\frac{\partial U}{\partial\delta}\right)$$

$$= \begin{bmatrix}-\frac{1}{w}\sin\psi & \frac{1}{w}\cos\psi\end{bmatrix}\frac{\partial}{\partial\delta}\left(\begin{bmatrix}\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\end{bmatrix}\right)\begin{bmatrix}0 \\ \frac{c}{w}\cos\lambda\end{bmatrix}$$

$$= \frac{c}{w}\cos\lambda\left[-\frac{1}{w}\sin\psi \quad \frac{1}{w}\cos\psi\right]\left(\frac{\partial}{\partial\delta}\left(-\frac{\partial U}{\partial\psi}\right)\right)_{xy} = 0$$

$$\frac{\partial}{\partial\phi}\left(\frac{\partial\theta_F}{\partial\delta}\right) = \frac{\partial}{\partial\phi}\left([1 \quad 0]\left[\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} \quad -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\right]^{-1}\frac{\partial U}{\partial\delta}\right)$$

$$= [1 \quad 0]\frac{\partial}{\partial\phi}\left(\left[\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} \quad -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\right]^{-1}\right)\frac{\partial U}{\partial\delta}$$

$$+ [1 \quad 0]\left[\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} \quad -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\right]^{-1}\frac{\partial}{\partial\phi}\left(\frac{\partial U}{\partial\delta}\right)$$

$$= \left[\frac{1}{r_F}\cos\psi \quad \frac{1}{r_F}\sin\psi\right]\frac{\partial}{\partial\phi}\left(\left[\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} \quad -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\right]\right)\begin{bmatrix}\frac{c}{r_F}\cos\lambda \\ 0\end{bmatrix}$$

$$+ \left[-\frac{1}{r_F}\cos\psi \quad -\frac{1}{r_F}\sin\psi\right]\frac{\partial}{\partial\phi}\left(\frac{\partial U}{\partial\delta}\right)$$

$$= \left[-\frac{1}{r_F}\cos\psi \quad -\frac{1}{r_F}\sin\psi\right]\frac{\partial}{\partial\phi}\left(\frac{\partial U}{\partial\delta}\right)$$

$$\frac{\partial}{\partial\delta}\left(\frac{\partial\theta_F}{\partial\phi}\right) = \frac{\partial}{\partial\delta}\left([1 \quad 0]\left[\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} \quad -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\right]^{-1}\frac{\partial U}{\partial\phi}\right)$$

$$= [1 \quad 0]\frac{\partial}{\partial\delta}\left(\left[\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} \quad -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\right]^{-1}\right)\frac{\partial U}{\partial\phi}$$

$$+ [1 \quad 0]\left[\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} \quad -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\right]^{-1}\frac{\partial}{\partial\delta}\left(\frac{\partial U}{\partial\phi}\right)$$

$$= \left[-\frac{1}{r_F}\cos\psi \quad -\frac{1}{r_F}\sin\psi\right]\frac{\partial}{\partial\delta}\left(\frac{\partial U}{\partial\phi}\right) = 0?$$

$$\frac{\partial}{\partial\delta}\left(\frac{\partial\theta_F}{\partial\delta}\right) = \frac{\partial}{\partial\delta}\left([1 \quad 0]\left[\left(\frac{\partial U}{\partial\theta_F}\right)_{xy} \quad -\left(\frac{\partial U}{\partial\psi}\right)_{xy}\right]^{-1}\frac{\partial U}{\partial\delta}\right)$$

$$= \begin{bmatrix}1 & 0\end{bmatrix} \frac{\partial}{\partial \delta} \left(\begin{bmatrix} \left(\frac{\partial U}{\partial \theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial \psi}\right)_{xy} \end{bmatrix}^{-1} \right) \begin{bmatrix} c\cos\lambda\sin\psi \\ -c\cos\lambda\cos\psi \end{bmatrix}$$

$$+ \begin{bmatrix}1 & 0\end{bmatrix} \begin{bmatrix} \left(\frac{\partial U}{\partial \theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial \psi}\right)_{xy} \end{bmatrix}^{-1} \frac{\partial}{\partial \delta} \left(\frac{\partial U}{\partial \delta}\right)$$

$$= \begin{bmatrix} \frac{1}{r_F}\cos\psi & \frac{1}{r_F}\sin\psi \end{bmatrix} \frac{\partial}{\partial \delta} \left(\begin{bmatrix} \left(\frac{\partial U}{\partial \theta_F}\right)_{xy} & -\left(\frac{\partial U}{\partial \psi}\right)_{xy} \end{bmatrix} \right) \begin{bmatrix} 0 \\ \frac{c}{w}\cos\lambda \end{bmatrix}$$

$$+ \begin{bmatrix} -\frac{1}{r_F}\cos\psi & -\frac{1}{r_F}\sin\psi \end{bmatrix} \begin{bmatrix} c\cos^2\lambda\cos\psi \\ c\cos^2\lambda\sin\psi \end{bmatrix}$$

$$= \frac{c}{w}\cos\lambda \begin{bmatrix} \frac{1}{r_F}\cos\psi & \frac{1}{r_F}\sin\psi \end{bmatrix} \frac{\partial}{\partial \delta} \left(-\frac{\partial U}{\partial \psi} \right)_{xy} - \frac{c}{r_F}\cos^2\lambda$$

$$= -\frac{w+c}{w}\frac{c}{r_F}\cos^2\lambda$$

In conclusion, the only free parameters are θ_R, ϕ, δ. If we set θ_R to constant, there are only two parameters ϕ and δ.

$$\frac{dX}{dt} = \frac{\partial X}{\partial \theta_R}\dot{\theta}_R + \frac{\partial X}{\partial \theta_B}\dot{\theta}_B + \frac{\partial X}{\partial \theta_U}\dot{\theta}_U + \frac{\partial X}{\partial \psi}\dot{\psi} + \frac{\partial X}{\partial \phi}\dot{\phi} + \frac{\partial X}{\partial \delta}\dot{\delta}$$

$$= \frac{\partial X}{\partial \theta_R}\dot{\theta}_R + \frac{\partial X}{\partial \phi}\dot{\phi} + \frac{\partial X}{\partial \delta}\dot{\delta} + \frac{\partial X}{\partial \theta_B}\frac{\partial \theta_B}{\partial \phi}\dot{\phi} + \frac{\partial X}{\partial \theta_B}\frac{\partial \theta_B}{\partial \delta}\dot{\delta} + \frac{\partial X}{\partial \theta_U}\frac{\partial \theta_U}{\partial \phi}\dot{\phi} + \frac{\partial X}{\partial \theta_U}\frac{\partial \theta_U}{\partial \delta}\dot{\delta}$$
$$+ \frac{\partial X}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\dot{\theta}_R + \frac{\partial X}{\partial \psi}\frac{\partial \psi}{\partial \phi}\dot{\phi} + \frac{\partial X}{\partial \psi}\frac{\partial \psi}{\partial \delta}\dot{\delta}$$

$$\frac{dX}{dt} = \left(\frac{\partial X}{\partial \theta_R} + \frac{\partial X}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\right)\dot{\theta}_R + \left(\frac{\partial X}{\partial \phi} + \frac{\partial X}{\partial \theta_B}\frac{\partial \theta_B}{\partial \phi} + \frac{\partial X}{\partial \theta_U}\frac{\partial \theta_U}{\partial \phi} + \frac{\partial X}{\partial \psi}\frac{\partial \psi}{\partial \phi}\right)\dot{\phi}$$
$$+ \left(\frac{\partial X}{\partial \delta} + \frac{\partial X}{\partial \theta_B}\frac{\partial \theta_B}{\partial \delta} + \frac{\partial X}{\partial \theta_U}\frac{\partial \theta_U}{\partial \delta} + \frac{\partial X}{\partial \psi}\frac{\partial \psi}{\partial \delta}\right)\dot{\delta}$$

$$\omega = \bar{q}2\left(\frac{\partial q}{\partial \theta_R} + \frac{\partial q}{\partial \theta_F}\frac{\partial \theta_F}{\partial \theta_R} + \frac{\partial q}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\right)\dot{\theta}_R$$
$$+ \bar{q}2\left(\frac{\partial q}{\partial \phi} + \frac{\partial q}{\partial \theta_B}\frac{\partial \theta_B}{\partial \phi} + \frac{\partial q}{\partial \theta_U}\frac{\partial \theta_U}{\partial \phi} + \frac{\partial q}{\partial \theta_F}\frac{\partial \theta_F}{\partial \phi} + \frac{\partial q}{\partial \psi}\frac{\partial \psi}{\partial \phi}\right)\dot{\phi}$$
$$+ \bar{q}2\left(\frac{\partial q}{\partial \delta} + \frac{\partial q}{\partial \theta_B}\frac{\partial \theta_B}{\partial \delta} + \frac{\partial q}{\partial \theta_U}\frac{\partial \theta_U}{\partial \delta} + \frac{\partial q}{\partial \theta_F}\frac{\partial \theta_F}{\partial \delta} + \frac{\partial q}{\partial \psi}\frac{\partial \psi}{\partial \delta}\right)\dot{\delta}$$

Translational kinetic energy

$$\frac{Q_\phi}{m} = \left(\frac{\partial X}{\partial \phi}\right)^T \frac{\partial X}{\partial r}\ddot{r}$$

$$+ \left(\frac{\partial X}{\partial \phi}\right)^T \frac{\partial}{\partial \theta_R}\left(\frac{\partial X}{\partial \theta_R}\right)\dot{\theta}_R\dot{\theta}_R$$

$$+ \left(\frac{\partial X}{\partial \phi}\right)^T \frac{\partial}{\partial \phi}\left(\frac{\partial X}{\partial \theta_R}\right)\dot{\theta}_R\dot{\phi}$$

$$+ \left(\frac{\partial X}{\partial \phi}\right)^T \frac{\partial}{\partial \delta}\left(\frac{\partial X}{\partial \theta_R}\right)\dot{\theta}_R\dot{\delta}$$

$$\frac{Q_\delta}{m} = \left(\frac{\partial X}{\partial \delta}\right)^T \frac{\partial X}{\partial r}\ddot{r}$$

$$+ \left(\frac{\partial X}{\partial \delta}\right)^T \frac{\partial}{\partial \theta_R}\left(\frac{\partial X}{\partial \theta_R}\right)\dot{\theta}_R\dot{\theta}_R$$

$$+\left(\frac{\partial X}{\partial \delta}\right)^T \frac{\partial}{\partial \phi}\left(\frac{\partial X}{\partial \theta_R}\right)\dot{\theta_R}\dot{\phi}$$

$$+\left(\frac{\partial X}{\partial \delta}\right)^T \frac{\partial}{\partial \delta}\left(\frac{\partial X}{\partial \theta_R}\right)\dot{\theta_R}\dot{\delta}$$

Because $\frac{\partial X}{\partial \theta_R} = R_z(\psi)\begin{bmatrix}-r_R\\0\\0\end{bmatrix} + R_z(\psi)V_\psi\frac{\partial \psi}{\partial \theta_R}$, also the only parameter depending on θ_R and the derivative might not be zero is ψ, therefore

$$\frac{\partial}{\partial \theta_R}\left(\frac{\partial X}{\partial \theta_R}\right) = \frac{\partial R_z(\psi)}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\begin{bmatrix}-r_R\\0\\0\end{bmatrix} + \frac{\partial R_z(\psi)}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}V_\psi\frac{\partial \psi}{\partial \theta_R} = \frac{\partial R_z(\psi)}{\partial \psi}\begin{bmatrix}-r_R\\0\\0\end{bmatrix}\frac{\partial}{\partial \delta}\left(\frac{\partial \psi}{\partial \theta_R}\right)\delta$$

$$= R_z(\psi)\begin{bmatrix}0\\-r_R\\0\end{bmatrix}\frac{\partial}{\partial \delta}\left(\frac{\partial \psi}{\partial \theta_R}\right)\delta$$

Because any parameter derivative of ϕ is zero,

$$\frac{\partial}{\partial \phi}\left(\frac{\partial X}{\partial \theta_R}\right) = 0$$

$$\frac{\partial}{\partial \delta}\left(\frac{\partial X}{\partial \theta_R}\right) = \frac{\partial R_z(\psi)}{\partial \psi}\frac{\partial \psi}{\partial \delta}\begin{bmatrix}-r_R\\0\\0\end{bmatrix} + R_z(\psi)V_\psi\frac{\partial}{\partial \delta}\left(\frac{\partial \psi}{\partial \theta_R}\right)$$

$$= R_z(\psi)\left(\begin{bmatrix}0\\-r_R\mu\\0\end{bmatrix} + V_\psi\frac{\partial}{\partial \delta}\left(\frac{\partial \psi}{\partial \theta_R}\right)\right)$$

Rotational kinetic energy

$$Q_\phi = \left(\bar{q}2\frac{\partial q}{\partial \phi}\right)^T I\bar{q}2\frac{\partial q}{\partial r}\ddot{r} + \left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \phi}\dot{\theta_R}^2$$

$$+\left(\bar{q}2\frac{\partial q}{\partial \phi}\right)^T I\bar{q}2\frac{\partial}{\partial \theta_R}\left(\frac{\partial q}{\partial \theta_R}\right)\dot{\theta_R}^2$$

$$+\left(\bar{q}2\frac{\partial q}{\partial \phi}\right)^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \phi}\dot{\phi}\dot{\theta_R} + \left(\bar{q}2\frac{\partial q}{\partial \phi}\right)^T I\bar{q}2\frac{\partial}{\partial \phi}\left(\frac{\partial q}{\partial \theta_R}\right)\dot{\phi}\dot{\theta_R}$$

$$+\left(\bar{q}2\frac{\partial q}{\partial \phi}\right)^T I\bar{q}2\frac{\partial}{\partial \theta_R}\left(\frac{\partial q}{\partial \phi}\right)\dot{\phi}\dot{\theta_R}$$

$$+\left(\bar{q}2\frac{\partial q}{\partial \delta}\right)^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \phi}\dot{\delta}\dot{\theta_R} + \left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I2\frac{\partial \bar{q}}{\partial \delta}2\frac{\partial q}{\partial \phi}\dot{\delta}\dot{\theta_R}$$

$$+\left(\bar{q}2\frac{\partial q}{\partial \phi}\right)^T I\bar{q}2\frac{\partial}{\partial \delta}\left(\frac{\partial q}{\partial \theta_R}\right)\dot{\delta}\dot{\theta_R} + \left(\bar{q}2\frac{\partial q}{\partial \phi}\right)^T I\bar{q}2\frac{\partial}{\partial \theta_R}\left(\frac{\partial q}{\partial \delta}\right)\dot{\delta}\dot{\theta_R}$$

$$Q_\delta = \left(\bar{q}2\frac{\partial q}{\partial \delta}\right)^T I\bar{q}2\frac{\partial q}{\partial r}\ddot{r} + \left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \delta}\dot{\theta_R}^2 + \left(\bar{q}2\frac{\partial q}{\partial \delta}\right)^T I\bar{q}2\frac{\partial}{\partial \theta_R}\left(\frac{\partial q}{\partial \theta_R}\right)\dot{\theta_R}^2$$

$$+ \left(\bar{q}2\frac{\partial q}{\partial \phi}\right)^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \delta}\dot{\phi}\dot{\theta_R} + \left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I2\frac{\partial \bar{q}}{\partial \phi}2\frac{\partial q}{\partial \delta}\dot{\phi}\dot{\theta_R} + \left(\bar{q}2\frac{\partial q}{\partial \delta}\right)^T I\bar{q}2\frac{\partial}{\partial \phi}\left(\frac{\partial q}{\partial \theta_R}\right)\dot{\phi}\dot{\theta_R} + \left(\bar{q}2\frac{\partial q}{\partial \delta}\right)^T I\bar{q}2\frac{\partial}{\partial \theta_R}\left(\frac{\partial q}{\partial \phi}\right)\dot{\phi}\dot{\theta_R}$$

$$+ \left(\bar{q}2\frac{\partial q}{\partial \delta}\right)^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \delta}\dot{\delta}\dot{\theta_R} + \left(\bar{q}2\frac{\partial q}{\partial \delta}\right)^T I\bar{q}2\frac{\partial}{\partial \delta}\left(\frac{\partial q}{\partial \theta_R}\right)\dot{\delta}\dot{\theta_R} + \left(\bar{q}2\frac{\partial q}{\partial \delta}\right)^T I\bar{q}2\frac{\partial}{\partial \theta_R}\left(\frac{\partial q}{\partial \delta}\right)\dot{\delta}\dot{\theta_R}$$

Cheat sheet of derivative

$$\frac{\partial\theta_B}{\partial\phi} = \frac{\partial\theta_B}{\partial\delta} = \frac{\partial\theta_U}{\partial\phi} = \frac{\partial\theta_U}{\partial\delta} = 0$$

$$\frac{\partial}{\partial\phi}\left(\frac{\partial\theta_B}{\partial\phi}\right) = 0, \frac{\partial}{\partial\delta}\left(\frac{\partial\theta_B}{\partial\phi}\right) = \frac{\partial}{\partial\phi}\left(\frac{\partial\theta_B}{\partial\delta}\right) = -\mu, \frac{\partial}{\partial\delta}\left(\frac{\partial\theta_B}{\partial\delta}\right) = -\sin\lambda\,\mu$$

$$\frac{\partial\theta_F}{\partial\theta_R} = \frac{r_R}{r_F}, \frac{\partial\theta_F}{\partial\phi} = 0, \frac{\partial\theta_F}{\partial\delta} = 0$$

$$\frac{\partial}{\partial\phi}\left(\frac{\partial\theta_F}{\partial\theta_R}\right) = 0, \frac{\partial}{\partial\phi}\left(\frac{\partial\theta_F}{\partial\phi}\right) = 0, \frac{\partial}{\partial\phi}\left(\frac{\partial\theta_F}{\partial\delta}\right) \neq 0$$

$$\frac{\partial}{\partial\delta}\left(\frac{\partial\theta_F}{\partial\theta_R}\right) = 0, \frac{\partial}{\partial\delta}\left(\frac{\partial\theta_F}{\partial\phi}\right) \neq 0, \frac{\partial}{\partial\delta}\left(\frac{\partial\theta_F}{\partial\delta}\right) \neq 0$$

$$\frac{\partial\psi}{\partial\theta_R} = 0, \frac{\partial\psi}{\partial\phi} = 0, \frac{\partial\psi}{\partial\delta} = \mu$$

$$\frac{\partial}{\partial\phi}\left(\frac{\partial\psi}{\partial\theta_R}\right) = 0, \frac{\partial}{\partial\phi}\left(\frac{\partial\psi}{\partial\phi}\right) = 0, \frac{\partial}{\partial\phi}\left(\frac{\partial\psi}{\partial\delta}\right) = 0$$

$$\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right) = -\frac{r_R}{w}\cos\lambda, \frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\phi}\right) = 0, \frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\delta}\right) = 0$$

Translational kinetic energy

For RB, $\begin{bmatrix} x_{BS} \\ 0 \\ z_{BS} \end{bmatrix} = \begin{bmatrix} x_B \\ 0 \\ z_B + r_R \end{bmatrix}$, for RS, $\begin{bmatrix} x_{BS} \\ 0 \\ z_{BS} \end{bmatrix} = \begin{bmatrix} w + c \\ 0 \\ r_R \end{bmatrix}$

For SH, $\begin{bmatrix} x_{HF} \\ 0 \\ z_{HF} \end{bmatrix} = \begin{bmatrix} x_H - w - c \\ 0 \\ z_H \end{bmatrix}$, for SF, $\begin{bmatrix} x_{HF} \\ 0 \\ z_{HF} \end{bmatrix} = \begin{bmatrix} -c \\ 0 \\ -r_F \end{bmatrix}$

$$\frac{\partial P}{\partial t} = \frac{\partial P}{\partial \theta_R}\dot{\theta}_R = \begin{bmatrix} -r_R \cos\psi \\ -r_R \sin\psi \\ 0 \end{bmatrix}\dot{\theta}_R = R_z(\psi)\begin{bmatrix} -r_R \\ 0 \\ 0 \end{bmatrix}\dot{\theta}_R$$

$$\frac{\partial PR}{\partial t} = \frac{\partial PR}{\partial \psi}\dot{\psi} + \frac{\partial PR}{\partial \phi}\dot{\phi} = \begin{bmatrix} -r_R \sin\phi\cos\psi \\ -r_R \sin\phi\sin\psi \\ 0 \end{bmatrix}\dot{\psi} + \begin{bmatrix} -r_R \cos\phi\sin\psi \\ r_R \cos\phi\cos\psi \\ r_R \sin\phi \end{bmatrix}\dot{\phi}$$

$$= R_z(\psi)\begin{bmatrix} -r_R \sin\phi \\ 0 \\ 0 \end{bmatrix}\dot{\psi} + R_z(\psi)\begin{bmatrix} 0 \\ r_R \cos\phi \\ r_R \sin\phi \end{bmatrix}\dot{\phi}$$

$$\frac{\partial RBS}{\partial t} = \frac{\partial RBS}{\partial \psi}\dot{\psi} + \frac{\partial RBS}{\partial \phi}\dot{\phi} + \frac{\partial RBS}{\partial \theta_B}\dot{\theta}_B$$

$$= R_z(\psi)\begin{bmatrix} \sin\phi\,(-x_{BS}\sin\theta_B + z_{BS}\cos\theta_B) \\ x_{BS}\cos\theta_B + z_{BS}\sin\theta_B \\ 0 \end{bmatrix}\dot{\psi}$$

$$+R_z(\psi)\begin{bmatrix} 0 \\ -\cos\phi\,(-x_{BS}\sin\theta_B + z_{BS}\cos\theta_B) \\ -\sin\phi\,(-x_{BS}\sin\theta_B + z_{BS}\cos\theta_B) \end{bmatrix}\dot{\phi}$$

$$+R_z(\psi)\begin{bmatrix} -x_{BS}\sin\theta_B + z_{BS}\cos\theta_B \\ -\sin\phi\,(-x_{BS}\cos\theta_B - z_{BS}\sin\theta_B) \\ \cos\phi\,(-x_{BS}\cos\theta_B - z_{BS}\sin\theta_B) \end{bmatrix}\dot{\theta}_B$$

$$R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$v_{HF} = \begin{bmatrix} \cos\delta\, x_{HF} + 2\sin^2\frac{\delta}{2}\sin\lambda\,(\sin\lambda\, x_{HF} + \cos\lambda\, z_{HF}) \\ \sin\delta\,(\cos\lambda\, x_{HF} - \sin\lambda\, z_{HF}) \\ \cos\delta\, z_{HF} + 2\sin^2\frac{\delta}{2}\cos\lambda\,(\sin\lambda\, x_{HF} + \cos\lambda\, z_{HF}) \end{bmatrix}$$

$$SHF = R_z(\psi)\begin{bmatrix} \cos\theta_B & 0 & \sin\theta_B \\ \sin\phi\sin\theta_B & \cos\phi & -\sin\phi\cos\theta_B \\ -\cos\phi\sin\theta_B & \sin\phi & \cos\phi\cos\theta_B \end{bmatrix} v_{HF}$$

$$\frac{\partial SHF}{\partial t} = \frac{\partial SHF}{\partial\psi}\dot{\psi} + \frac{\partial SHF}{\partial\phi}\dot{\phi} + \frac{\partial SHF}{\partial\theta_B}\dot{\theta}_B + \frac{\partial SHF}{\partial\delta}\dot{\delta}$$

$$\frac{\partial SHF}{\partial\psi} = \begin{bmatrix} -\sin\psi & -\cos\psi & 0 \\ \cos\psi & -\sin\psi & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta_B & 0 & \sin\theta_B \\ \sin\phi\sin\theta_B & \cos\phi & -\sin\phi\cos\theta_B \\ -\cos\phi\sin\theta_B & \sin\phi & \cos\phi\cos\theta_B \end{bmatrix} v_{HF}$$

$$= R_z(\psi)\begin{bmatrix} -\sin\phi\sin\theta_B & -\cos\phi & \sin\phi\cos\theta_B \\ \cos\theta_B & 0 & \sin\theta_B \\ 0 & 0 & 0 \end{bmatrix} v_{HF} = R_z(\psi)\begin{bmatrix} 0 \\ x_{HF} \\ 0 \end{bmatrix}$$

$$\frac{\partial}{\partial\phi}\left(R_z(-\psi)\frac{\partial SH}{\partial\psi}\right) = R_z(\psi)\begin{bmatrix} z_{HF} \\ x_{HF} \\ 0 \end{bmatrix}$$

$$\frac{\partial}{\partial\delta}\left(R_z(-\psi)\frac{\partial SH}{\partial\psi}\right) = R_z(\psi)\begin{bmatrix} -\cos\lambda\, x_{HF} + \sin\lambda\, z_{HF} \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial SHF}{\partial\phi} = R_z(\psi)\begin{bmatrix} 0 & 0 & 0 \\ \cos\phi\sin\theta_B & -\sin\phi & -\cos\phi\cos\theta_B \\ \sin\phi\sin\theta_B & \cos\phi & -\sin\phi\cos\theta_B \end{bmatrix} v_{HF} = R_z(\psi)\begin{bmatrix} 0 \\ -z_{HF} \\ 0 \end{bmatrix}$$

$$\frac{\partial SHF}{\partial\theta_B} = R_z(\psi)\begin{bmatrix} -\sin\theta_B & 0 & \cos\theta_B \\ \sin\phi\cos\theta_B & 0 & \sin\phi\sin\theta_B \\ -\cos\phi\cos\theta_B & 0 & -\cos\phi\sin\theta_B \end{bmatrix} v_{HF} = R_z(\psi)\begin{bmatrix} z_{HF} \\ 0 \\ -x_{HF} \end{bmatrix}$$

$$\frac{\partial SHF}{\partial\delta} = R_z(\psi)\begin{bmatrix} \cos\theta_B & 0 & \sin\theta_B \\ \sin\phi\sin\theta_B & \cos\phi & -\sin\phi\cos\theta_B \\ -\cos\phi\sin\theta_B & \sin\phi & \cos\phi\cos\theta_B \end{bmatrix}\frac{\partial v_{HF}}{\partial\delta}$$

$$= R_z(\psi)\begin{bmatrix} 0 \\ \cos\lambda\, x_{HF} - \sin\lambda\, z_{HF} \\ 0 \end{bmatrix}$$

$$\frac{\partial}{\partial\phi}\left(R_z(-\psi)\frac{\partial SH}{\partial\delta}\right) = \begin{bmatrix} 0 \\ 0 \\ \cos\lambda\, x_{HF} - \sin\lambda\, z_{HF} \end{bmatrix}$$

$$v_{HF} = \begin{bmatrix} \cos\delta\, x_{HF} + 2\sin^2\frac{\delta}{2}\sin\lambda\,(\sin\lambda\, x_{HF} + \cos\lambda\, z_{HF}) \\ \sin\delta\,(\cos\lambda\, x_{HF} - \sin\lambda\, z_{HF}) \\ \cos\delta\, z_{HF} + 2\sin^2\frac{\delta}{2}\cos\lambda\,(\sin\lambda\, x_{HF} + \cos\lambda\, z_{HF}) \end{bmatrix} = \begin{bmatrix} x_{HF} \\ 0 \\ z_{HF} \end{bmatrix}$$

$$\frac{\partial v_{HF}}{\partial \delta} = \begin{bmatrix} -\sin\delta\, x_{HF} + \sin\delta\sin\lambda\,(\sin\lambda\, x_{HF} + \cos\lambda\, z_{HF}) \\ \cos\delta\,(\cos\lambda\, x_{HF} - \sin\lambda\, z_{HF}) \\ -\sin\delta\, z_{HF} + \sin\delta\cos\lambda\,(\sin\lambda\, x_{HF} + \cos\lambda\, z_{HF}) \end{bmatrix}$$

$$= \begin{bmatrix} \sin\delta\,(\sin\lambda\,(\sin\lambda\, x_{HF} + \cos\lambda\, z_{HF}) - x_{HF}) \\ \cos\delta\,(\cos\lambda\, x_{HF} - \sin\lambda\, z_{HF}) \\ \sin\delta\,(\cos\lambda\,(\sin\lambda\, x_{HF} + \cos\lambda\, z_{HF}) - z_{HF}) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \cos\lambda\, x_{HF} - \sin\lambda\, z_{HF} \\ 0 \end{bmatrix}$$

$$R_z(\psi)^T \frac{d}{d\psi} R_z(\psi)$$

$$= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \frac{d}{d\psi} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin\psi & -\cos\psi & 0 \\ \cos\psi & -\sin\psi & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{d}{d\psi}(R_z(\psi)^T) R_z(\psi) = \begin{bmatrix} -\sin\psi & \cos\psi & 0 \\ -\cos\psi & -\sin\psi & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{d}{d\psi}(R_z(\psi)^T) \frac{d}{d\psi} R_z(\psi) = \begin{bmatrix} -\sin\psi & \cos\psi & 0 \\ -\cos\psi & -\sin\psi & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\sin\psi & -\cos\psi & 0 \\ \cos\psi & -\sin\psi & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_z(\psi)^T \frac{d^2}{d\psi^2} R_z(\psi) = R_z(\psi)^T \frac{d}{d\psi} \begin{bmatrix} -\sin\psi & -\cos\psi & 0 \\ \cos\psi & -\sin\psi & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\cos\psi & \sin\psi & 0 \\ -\sin\psi & -\cos\psi & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{d}{d\psi}(R_z(\psi)^T) \frac{d}{d\psi} R_z(\psi) + R_z(\psi)^T \frac{d^2}{d\psi^2} R_z(\psi) = 0$$

Therefore

$$(R_z(\psi)Y)^T \frac{d}{dr}(R_z(\psi)Z) = Y^T R_z(\psi)^T \frac{d}{dr}\big(R_z(\psi)\big)Z + Y^T R_z(\psi)^T R_z(\psi) \frac{d}{dr}(Z)$$

$$= Y^T \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Z \frac{\partial\psi}{\partial r} + Y^T \frac{d}{dr}(Z)$$

If Y=z or $\frac{\partial\psi}{\partial r} = \frac{\partial\psi}{\partial\phi}$, $Y^T \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Z \frac{\partial\psi}{\partial r} = 0$. Therefore,

$$(R_z(\psi)Y)^T \frac{d}{dr}(R_z(\psi)Z) = Y^T \frac{d}{dr}(Z)$$

For point R

$$\frac{\partial R}{\partial t} = \frac{\partial P}{\partial t} + \frac{\partial PR}{\partial t} = \frac{\partial P}{\partial \theta_R}\dot{\theta_R} + \frac{\partial PR}{\partial \psi}\dot{\psi} + \frac{\partial PR}{\partial \phi}\dot{\phi}$$

$$= \frac{\partial P}{\partial \theta_R}\dot{\theta_R} + \frac{\partial PR}{\partial \psi}\left(\frac{\partial \psi}{\partial \theta_R}\dot{\theta_R} + \frac{\partial \psi}{\partial \phi}\dot{\phi} + \frac{\partial \psi}{\partial \delta}\dot{\delta}\right) + \frac{\partial PR}{\partial \phi}\dot{\phi}$$

$$= \left(\frac{\partial P}{\partial \theta_R} + \frac{\partial PR}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\right)\dot{\theta_R} + \left(\frac{\partial PR}{\partial \psi}\frac{\partial \psi}{\partial \phi} + \frac{\partial PR}{\partial \phi}\right)\dot{\phi} + \frac{\partial PR}{\partial \psi}\frac{\partial \psi}{\partial \delta}\dot{\delta}$$

$$= R_z(\psi)\left(\begin{bmatrix} -r_R \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -r_R \sin\phi \\ 0 \\ 0 \end{bmatrix}\frac{\partial \psi}{\partial \theta_R}\right)\dot{\theta_R}$$

$$+R_z(\psi)\left(\begin{bmatrix} -r_R \sin\phi \\ 0 \\ 0 \end{bmatrix}\frac{\partial \psi}{\partial \phi} + \begin{bmatrix} 0 \\ r_R \cos\phi \\ r_R \sin\phi \end{bmatrix}\right)\dot{\phi}$$

$$+R_z(\psi)\left(\begin{bmatrix} -r_R \sin\phi \\ 0 \\ 0 \end{bmatrix}\frac{\partial \psi}{\partial \delta}\right)\dot{\delta}$$

$$\frac{\partial X}{\partial \theta_R} = R_z(\psi)\begin{bmatrix} -r_R \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial X}{\partial \phi} = R_z(\psi)\begin{bmatrix} 0 \\ r_R \\ 0 \end{bmatrix}$$

$$\frac{\partial X}{\partial \delta} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\frac{\partial X}{\partial r}\right)^T \frac{\partial X}{\partial r} = \begin{bmatrix} r_R^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\frac{\partial}{\partial\theta_R}\left(\frac{\partial X}{\partial\theta_R}\right) = \frac{\partial R_z(\psi)}{\partial\psi}\frac{\partial\psi}{\partial\theta_R}\left(\begin{bmatrix}-r_R\\0\\0\end{bmatrix} + \begin{bmatrix}-r_R\sin\phi\\0\\0\end{bmatrix}\frac{\partial\psi}{\partial\theta_R}\right)$$

$$= R_z(\psi)\frac{\partial\psi}{\partial\theta_R}\left(\begin{bmatrix}0\\-r_R\\0\end{bmatrix} + \begin{bmatrix}0\\-r_R\sin\phi\\0\end{bmatrix}\frac{\partial\psi}{\partial\theta_R}\right)$$

$$\left(\frac{\partial X}{\partial\phi}\right)^T\frac{\partial}{\partial\theta_R}\left(\frac{\partial X}{\partial\theta_R}\right) = \begin{bmatrix}0\\r_R\\0\end{bmatrix}\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right)\left(\begin{bmatrix}0\\-r_R\\0\end{bmatrix} + \begin{bmatrix}0\\-r_R\sin\phi\\0\end{bmatrix}\frac{\partial\psi}{\partial\theta_R}\right)\delta = -r_R^2\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right)\delta$$

$$\left(\frac{\partial X}{\partial\delta}\right)^T\frac{\partial}{\partial\theta_R}\left(\frac{\partial X}{\partial\theta_R}\right) = 0$$

$$\frac{\partial}{\partial\phi}\left(\frac{\partial X}{\partial\theta_R}\right) = \frac{\partial R_z(\psi)}{\partial\psi}\begin{bmatrix}-r_R\\0\\0\end{bmatrix}\frac{\partial\psi}{\partial\phi} + \frac{\partial R_z(\psi)}{\partial\psi}\begin{bmatrix}-r_R\sin\phi\\0\\0\end{bmatrix}\frac{\partial\psi}{\partial\phi}\frac{\partial\psi}{\partial\theta_R}$$

$$+ R_z(\psi)\begin{bmatrix}-r_R\cos\phi\\0\\0\end{bmatrix}\frac{\partial\psi}{\partial\theta_R} + R_z(\psi)\begin{bmatrix}-r_R\sin\phi\\0\\0\end{bmatrix}\frac{\partial}{\partial\phi}\left(\frac{\partial\psi}{\partial\theta_R}\right) = 0$$

$$\left(\frac{\partial X}{\partial\phi}\right)^T\frac{\partial}{\partial\phi}\left(\frac{\partial X}{\partial\theta_R}\right) = 0$$

$$\left(\frac{\partial X}{\partial\delta}\right)^T\frac{\partial}{\partial\phi}\left(\frac{\partial X}{\partial\theta_R}\right) = 0$$

$$\frac{\partial}{\partial\delta}\left(\frac{\partial X}{\partial\theta_R}\right) = \frac{\partial R_z(\psi)}{\partial\psi}\begin{bmatrix}-r_R\\0\\0\end{bmatrix}\frac{\partial\psi}{\partial\delta} + \frac{\partial R_z(\psi)}{\partial\psi}\begin{bmatrix}-r_R\sin\phi\\0\\0\end{bmatrix}\frac{\partial\psi}{\partial\delta}\frac{\partial\psi}{\partial\theta_R}$$

$$+ R_z(\psi)\begin{bmatrix}-r_R\sin\phi\\0\\0\end{bmatrix}\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right) = \frac{\partial R_z(\psi)}{\partial\psi}\begin{bmatrix}-r_R\mu\\0\\0\end{bmatrix} = R_z(\psi)\begin{bmatrix}0\\-r_R\mu\\0\end{bmatrix}$$

$$\left(\frac{\partial X}{\partial\phi}\right)^T\frac{\partial}{\partial\delta}\left(\frac{\partial X}{\partial\theta_R}\right) = \begin{bmatrix}0\\r_R\\0\end{bmatrix}^T\begin{bmatrix}0\\-r_R\mu\\0\end{bmatrix} = -r_R^2\mu$$

$$\left(\frac{\partial X}{\partial\delta}\right)^T\frac{\partial}{\partial\delta}\left(\frac{\partial X}{\partial\theta_R}\right) = 0$$

$$\frac{Q_\phi}{m} = \left(\frac{\partial X}{\partial \phi}\right)^T \frac{\partial X}{\partial r}\ddot{r} - r_R^2 \frac{\partial}{\partial \delta}\left(\frac{\partial \psi}{\partial \theta_R}\right)\delta \dot{\theta_R}^2 - r_R^2 \mu \dot{\delta}\dot{\theta_R}$$

$$\frac{Q_\delta}{m} = \left(\frac{\partial X}{\partial \delta}\right)^T \frac{\partial X}{\partial r}\ddot{r}$$

In conclusion,

$$\begin{bmatrix} Q_\phi \\ Q_\delta \end{bmatrix} = m_R \begin{bmatrix} r_R^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\phi} \\ \ddot{\delta} \end{bmatrix} + r_R^2 \dot{\theta_R}^2 m_R \begin{bmatrix} 0 & \frac{\cos \lambda}{w} r_R \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \delta \end{bmatrix} - r_R \dot{\theta_R} m_R \begin{bmatrix} 0 & \mu r_R \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\delta} \end{bmatrix}$$

For point B

$$\frac{\partial B}{\partial t}=\frac{\partial P}{\partial t}+\frac{\partial PR}{\partial t}+\frac{\partial RB}{\partial t}=\frac{\partial P}{\partial \theta_R}\dot{\theta_R}+\frac{\partial PR}{\partial \psi}\dot{\psi}+\frac{\partial PR}{\partial \phi}\dot{\phi}+\frac{\partial RB}{\partial \psi}\dot{\psi}+\frac{\partial RB}{\partial \phi}\dot{\phi}+\frac{\partial RB}{\partial \theta_B}\dot{\theta_B}$$

$$=\frac{\partial P}{\partial \theta_R}\dot{\theta_R}+\frac{\partial PR}{\partial \psi}\left(\frac{\partial \psi}{\partial \theta_R}\dot{\theta_R}+\frac{\partial \psi}{\partial \phi}\dot{\phi}+\frac{\partial \psi}{\partial \delta}\dot{\delta}\right)+\frac{\partial PR}{\partial \phi}\dot{\phi}$$

$$+\frac{\partial RB}{\partial \psi}\left(\frac{\partial \psi}{\partial \theta_R}\dot{\theta_R}+\frac{\partial \psi}{\partial \phi}\dot{\phi}+\frac{\partial \psi}{\partial \delta}\dot{\delta}\right)+\frac{\partial RB}{\partial \phi}\dot{\phi}+\frac{\partial RB}{\partial \theta_B}\left(\frac{\partial \theta_B}{\partial \phi}\dot{\phi}+\frac{\partial \theta_B}{\partial \delta}\dot{\delta}\right)$$

$$=\left(\frac{\partial P}{\partial \theta_R}+\frac{\partial PR}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}+\frac{\partial RB}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\right)\dot{\theta_R}$$

$$+\left(\frac{\partial PR}{\partial \psi}\frac{\partial \psi}{\partial \phi}+\frac{\partial PR}{\partial \phi}+\frac{\partial RB}{\partial \psi}\frac{\partial \psi}{\partial \phi}+\frac{\partial RB}{\partial \phi}+\frac{\partial RB}{\partial \theta_B}\frac{\partial \theta_B}{\partial \phi}\right)\dot{\phi}$$

$$+\left(\frac{\partial PR}{\partial \psi}\frac{\partial \psi}{\partial \delta}+\frac{\partial RB}{\partial \psi}\frac{\partial \psi}{\partial \delta}+\frac{\partial RB}{\partial \theta_B}\frac{\partial \theta_B}{\partial \delta}\right)\dot{\delta}$$

$$=R_z(\psi)\left(\begin{bmatrix}-r_R\\0\\0\end{bmatrix}+\begin{bmatrix}\sin\phi\,(-r_R-x_{BS}\sin\theta_B+z_{BS}\cos\theta_B)\\x_{BS}\cos\theta_B+z_{BS}\sin\theta_B\\0\end{bmatrix}\frac{\partial \psi}{\partial \theta_R}\right)\dot{\theta_R}$$

$$+R_z(\psi)\left(\begin{bmatrix}-r_R\sin\phi\\0\\0\end{bmatrix}\frac{\partial \psi}{\partial \phi}+\begin{bmatrix}0\\r_R\cos\phi\\r_R\sin\phi\end{bmatrix}+\begin{bmatrix}\sin\phi\,(-x_{BS}\sin\theta_B+z_{BS}\cos\theta_B)\\x_{BS}\cos\theta_B+z_{BS}\sin\theta_B\\0\end{bmatrix}\frac{\partial \psi}{\partial \phi}\right.$$

$$+\begin{bmatrix}0\\-\cos\phi\,(-x_{BS}\sin\theta_B+z_{BS}\cos\theta_B)\\-\sin\phi\,(-x_{BS}\sin\theta_B+z_{BS}\cos\theta_B)\end{bmatrix}$$

$$\left.+\begin{bmatrix}-x_{BS}\sin\theta_B+z_{BS}\cos\theta_B\\-\sin\phi\,(-x_{BS}\cos\theta_B-z_{BS}\sin\theta_B)\\\cos\phi\,(-x_{BS}\cos\theta_B-z_{BS}\sin\theta_B)\end{bmatrix}\frac{\partial \theta_B}{\partial \phi}\right)\dot{\phi}$$

$$+R_z(\psi)\left(\begin{bmatrix}-r_R\sin\phi\\0\\0\end{bmatrix}\frac{\partial \psi}{\partial \delta}+\begin{bmatrix}\sin\phi\,(-x_{BS}\sin\theta_B+z_{BS}\cos\theta_B)\\x_{BS}\cos\theta_B+z_{BS}\sin\theta_B\\0\end{bmatrix}\frac{\partial \psi}{\partial \delta}\right.$$

$$\left.+\begin{bmatrix}-x_{BS}\sin\theta_B+z_{BS}\cos\theta_B\\-\sin\phi\,(-x_{BS}\cos\theta_B-z_{BS}\sin\theta_B)\\\cos\phi\,(-x_{BS}\cos\theta_B-z_{BS}\sin\theta_B)\end{bmatrix}\frac{\partial \theta_B}{\partial \delta}\right)\dot{\delta}$$

$$\frac{\partial X}{\partial \theta_R}=R_z(\psi)\begin{bmatrix}-r_R\\0\\0\end{bmatrix}$$

$$\frac{\partial X}{\partial \phi} = R_z(\psi)\begin{bmatrix} 0 \\ r_R - z_{BS} \\ 0 \end{bmatrix}$$

$$\frac{\partial X}{\partial \delta} = R_z(\psi)\begin{bmatrix} 0 \\ \mu x_{BS} \\ 0 \end{bmatrix}$$

$$\left(\frac{\partial X}{\partial r}\right)^T \frac{\partial X}{\partial r} = \begin{bmatrix} (r_R - z_{BS})^2 & \mu x_{BS}(r_R - z_{BS}) \\ \mu x_{BS}(r_R - z_{BS}) & \mu^2 x_{BS}^2 \end{bmatrix}$$

$$\frac{\partial}{\partial \theta_R}\left(\frac{\partial X}{\partial \theta_R}\right) = \frac{\partial R_z(\psi)}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\left(\begin{bmatrix} -r_R \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \sin\phi\,(-r_R - x_{BS}\sin\theta_B + z_{BS}\cos\theta_B) \\ x_{BS}\cos\theta_B + z_{BS}\sin\theta_B \\ 0 \end{bmatrix}\frac{\partial \psi}{\partial \theta_R}\right)$$

$$= R_z(\psi)\frac{\partial \psi}{\partial \theta_R}\left(\begin{bmatrix} 0 \\ -r_R \\ 0 \end{bmatrix} + \begin{bmatrix} -x_{BS}\cos\theta_B - z_{BS}\sin\theta_B \\ \sin\phi\,(-r_R - x_{BS}\sin\theta_B + z_{BS}\cos\theta_B) \\ 0 \end{bmatrix}\frac{\partial \psi}{\partial \theta_R}\right)$$

$$\left(\frac{\partial X}{\partial \phi}\right)^T \frac{\partial}{\partial \theta_R}\left(\frac{\partial X}{\partial \theta_R}\right) = -r_R(r_R - z_{BS})\frac{\partial}{\partial \delta}\left(\frac{\partial \psi}{\partial \theta_R}\right)\delta$$

$$\left(\frac{\partial X}{\partial \delta}\right)^T \frac{\partial}{\partial \theta_R}\left(\frac{\partial X}{\partial \theta_R}\right) = -r_R \mu x_{BS}\frac{\partial}{\partial \delta}\left(\frac{\partial \psi}{\partial \theta_R}\right)\delta$$

$$\frac{\partial}{\partial \phi}\left(\frac{\partial X}{\partial \theta_R}\right) = \frac{\partial R_z(\psi)}{\partial \psi}\frac{\partial \psi}{\partial \phi}\left(\begin{bmatrix} -r_R \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \sin\phi\,(-r_R - x_{BS}\sin\theta_B + z_{BS}\cos\theta_B) \\ x_{BS}\cos\theta_B + z_{BS}\sin\theta_B \\ 0 \end{bmatrix}\frac{\partial \psi}{\partial \theta_R}\right)$$

$$+ R_z(\psi)\frac{\partial}{\partial \phi}\left(\begin{bmatrix} \sin\phi\,(-r_R - x_{BS}\sin\theta_B + z_{BS}\cos\theta_B) \\ x_{BS}\cos\theta_B + z_{BS}\sin\theta_B \\ 0 \end{bmatrix}\frac{\partial \psi}{\partial \theta_R}\right) = 0$$

$$\left(\frac{\partial X}{\partial \phi}\right)^T \frac{\partial}{\partial \phi}\left(\frac{\partial X}{\partial \theta_R}\right) = 0$$

$$\left(\frac{\partial X}{\partial \delta}\right)^T \frac{\partial}{\partial \phi}\left(\frac{\partial X}{\partial \theta_R}\right) = 0$$

$$\frac{\partial}{\partial\delta}\left(\frac{\partial X}{\partial\theta_R}\right) = \frac{\partial R_z(\psi)}{\partial\psi}\frac{\partial\psi}{\partial\delta}\left(\begin{bmatrix}-r_R\\0\\0\end{bmatrix}\right) + R_z(\psi)\begin{bmatrix}0\\x_{BS}\\0\end{bmatrix}\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right)$$

$$= R_z(\psi)\begin{bmatrix}0\\1\\0\end{bmatrix}\left(-r_R\mu + x_{BS}\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right)\right)$$

$$\left(\frac{\partial X}{\partial\phi}\right)^T\frac{\partial}{\partial\delta}\left(\frac{\partial X}{\partial\theta_R}\right) = (r_R - z_{BS})\left(-r_R\mu + x_{BS}\left(-\frac{r_R}{w}\cos\lambda\right)\right)$$

$$\left(\frac{\partial X}{\partial\delta}\right)^T\frac{\partial}{\partial\delta}\left(\frac{\partial X}{\partial\theta_R}\right) = \mu x_{BS}\left(-r_R\mu + x_{BS}\left(-\frac{r_R}{w}\cos\lambda\right)\right)$$

$$\frac{Q_\phi}{m} = \left(\frac{\partial X}{\partial\phi}\right)^T\frac{\partial X}{\partial r}\ddot{r} - r_R(r_R - z_{BS})\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right)\delta\dot{\theta_R}^{\,2}$$
$$+ (r_R - z_{BS})\left(-r_R\mu + x_{BS}\left(-\frac{r_R}{w}\cos\lambda\right)\right)\dot{\delta}\dot{\theta_R}$$

$$\frac{Q_\delta}{m} = \left(\frac{\partial X}{\partial\delta}\right)^T\frac{\partial X}{\partial r}\ddot{r} - r_R\mu x_{BS}\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right)\delta\dot{\theta_R}^{\,2} + \mu x_{BS}\left(-r_R\mu + x_{BS}\left(-\frac{r_R}{w}\cos\lambda\right)\right)\dot{\delta}\dot{\theta_R}$$

For B, $\begin{bmatrix}x_{BS}\\0\\z_{BS}\end{bmatrix} = \begin{bmatrix}x_B\\0\\z_B + r_R\end{bmatrix}$

In conclusion,

$$\begin{bmatrix}Q_\phi\\Q_\delta\end{bmatrix} = m_B\begin{bmatrix}z_B^2 & -\mu x_B z_B\\-\mu x_B z_B & \mu^2 x_B^2\end{bmatrix}\begin{bmatrix}\ddot{\phi}\\\ddot{\delta}\end{bmatrix} + r_R^2\dot{\theta_R}^{\,2} m_B\begin{bmatrix}0 & -\frac{\cos\lambda}{w}z_B\\0 & \frac{\cos\lambda}{w}\mu x_B\end{bmatrix}\begin{bmatrix}\phi\\\delta\end{bmatrix}$$

$$- r_R\dot{\theta_R} m_B\begin{bmatrix}0 & -z_B\left(\mu + \frac{\cos\lambda}{w}x_B\right)\\0 & \mu x_B\left(\mu + \frac{\cos\lambda}{w}x_B\right)\end{bmatrix}\begin{bmatrix}\dot{\phi}\\\dot{\delta}\end{bmatrix}$$

For point H and F

$$\frac{\partial H}{\partial t}=\frac{\partial P}{\partial t}+\frac{\partial PR}{\partial t}+\frac{\partial RS}{\partial t}+\frac{\partial SH}{\partial t}$$

$$=\frac{\partial P}{\partial \theta_R}\dot{\theta_R}+\frac{\partial PR}{\partial \psi}\dot{\psi}+\frac{\partial PR}{\partial \phi}\dot{\phi}+\frac{\partial RS}{\partial \psi}\dot{\psi}+\frac{\partial RS}{\partial \phi}\dot{\phi}+\frac{\partial RS}{\partial \theta_B}\dot{\theta}_B+\frac{\partial SH}{\partial \psi}\dot{\psi}+\frac{\partial SH}{\partial \phi}\dot{\phi}$$
$$+\frac{\partial SH}{\partial \theta_B}\dot{\theta}_B+\frac{\partial SH}{\partial \delta}\dot{\delta}$$
$$=\frac{\partial P}{\partial \theta_R}\dot{\theta_R}+\frac{\partial PR}{\partial \psi}\left(\frac{\partial \psi}{\partial \theta_R}\dot{\theta_R}+\frac{\partial \psi}{\partial \phi}\dot{\phi}+\frac{\partial \psi}{\partial \delta}\dot{\delta}\right)+\frac{\partial PR}{\partial \phi}\dot{\phi}$$
$$+\frac{\partial RS}{\partial \psi}\left(\frac{\partial \psi}{\partial \theta_R}\dot{\theta_R}+\frac{\partial \psi}{\partial \phi}\dot{\phi}+\frac{\partial \psi}{\partial \delta}\dot{\delta}\right)+\frac{\partial RS}{\partial \phi}\dot{\phi}+\frac{\partial RS}{\partial \theta_B}\left(\frac{\partial \theta_B}{\partial \phi}\dot{\phi}+\frac{\partial \theta_B}{\partial \delta}\dot{\delta}\right)$$
$$+\frac{\partial SH}{\partial \psi}\left(\frac{\partial \psi}{\partial \theta_R}\dot{\theta_R}+\frac{\partial \psi}{\partial \phi}\dot{\phi}+\frac{\partial \psi}{\partial \delta}\dot{\delta}\right)+\frac{\partial SH}{\partial \phi}\dot{\phi}+\frac{\partial SH}{\partial \theta_B}\left(\frac{\partial \theta_B}{\partial \phi}\dot{\phi}+\frac{\partial \theta_B}{\partial \delta}\dot{\delta}\right)$$
$$+\frac{\partial SH}{\partial \delta}\dot{\delta}$$

$$=\left(\frac{\partial P}{\partial \theta_R}+\frac{\partial PR}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}+\frac{\partial RS}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}+\frac{\partial SH}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\right)\dot{\theta_R}$$

$$+\left(\frac{\partial PR}{\partial \psi}\frac{\partial \psi}{\partial \phi}+\frac{\partial PR}{\partial \phi}+\frac{\partial RS}{\partial \psi}\frac{\partial \psi}{\partial \phi}+\frac{\partial RS}{\partial \phi}+\frac{\partial RS}{\partial \theta_B}\frac{\partial \theta_B}{\partial \phi}+\frac{\partial SH}{\partial \psi}\frac{\partial \psi}{\partial \phi}+\frac{\partial SH}{\partial \phi}+\frac{\partial SH}{\partial \theta_B}\frac{\partial \theta_B}{\partial \phi}\right)\dot{\phi}$$

$$+\left(\frac{\partial PR}{\partial \psi}\frac{\partial \psi}{\partial \delta}+\frac{\partial RS}{\partial \psi}\frac{\partial \psi}{\partial \delta}+\frac{\partial RS}{\partial \theta_B}\frac{\partial \theta_B}{\partial \delta}+\frac{\partial SH}{\partial \psi}\frac{\partial \psi}{\partial \delta}+\frac{\partial SH}{\partial \theta_B}\frac{\partial \theta_B}{\partial \delta}+\frac{\partial SH}{\partial \delta}\right)\dot{\delta}$$

$$=R_z(\psi)\left(\begin{bmatrix}-r_R\\0\\0\end{bmatrix}+\begin{bmatrix}\sin\phi\,(-r_R-x_{BS}\sin\theta_B+z_{BS}\cos\theta_B)\\x_{BS}\cos\theta_B+z_{BS}\sin\theta_B\\0\end{bmatrix}\frac{\partial \psi}{\partial \theta_R}+\frac{\partial SH}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\right)\dot{\theta_R}$$

$$+R_z(\psi)\left(\begin{bmatrix}-r_R\sin\phi\\0\\0\end{bmatrix}\frac{\partial \psi}{\partial \phi}+\begin{bmatrix}0\\r_R\cos\phi\\r_R\sin\phi\end{bmatrix}+\begin{bmatrix}\sin\phi\,(-x_{BS}\sin\theta_B+z_{BS}\cos\theta_B)\\x_{BS}\cos\theta_B+z_{BS}\sin\theta_B\\0\end{bmatrix}\frac{\partial \psi}{\partial \phi}\right.$$
$$+\begin{bmatrix}0\\-\cos\phi\,(-x_{BS}\sin\theta_B+z_{BS}\cos\theta_B)\\-\sin\phi\,(-x_{BS}\sin\theta_B+z_{BS}\cos\theta_B)\end{bmatrix}$$
$$+\begin{bmatrix}-x_{BS}\sin\theta_B+z_{BS}\cos\theta_B\\-\sin\phi\,(-x_{BS}\cos\theta_B-z_{BS}\sin\theta_B)\\\cos\phi\,(-x_{BS}\cos\theta_B-z_{BS}\sin\theta_B)\end{bmatrix}\frac{\partial \theta_B}{\partial \phi}+\frac{\partial SH}{\partial \psi}\frac{\partial \psi}{\partial \phi}+\frac{\partial SH}{\partial \phi}$$
$$\left.+\frac{\partial SH}{\partial \theta_B}\frac{\partial \theta_B}{\partial \phi}\right)\dot{\phi}$$

$$+R_z(\psi)\left(\begin{bmatrix}-r_R\sin\phi\\0\\0\end{bmatrix}\frac{\partial\psi}{\partial\delta}+\begin{bmatrix}\sin\phi\,(-x_{BS}\sin\theta_B+z_{BS}\cos\theta_B)\\x_{BS}\cos\theta_B+z_{BS}\sin\theta_B\\0\end{bmatrix}\frac{\partial\psi}{\partial\delta}\right.$$

$$+\begin{bmatrix}-x_{BS}\sin\theta_B+z_{BS}\cos\theta_B\\-\sin\phi\,(-x_{BS}\cos\theta_B-z_{BS}\sin\theta_B)\\\cos\phi\,(-x_{BS}\cos\theta_B-z_{BS}\sin\theta_B)\end{bmatrix}\frac{\partial\theta_B}{\partial\delta}+\frac{\partial SH}{\partial\psi}\frac{\partial\psi}{\partial\delta}+\frac{\partial SH}{\partial\theta_B}\frac{\partial\theta_B}{\partial\delta}$$

$$\left.+\frac{\partial SH}{\partial\delta}\right)\dot{\delta}$$

$$\frac{\partial X}{\partial\theta_R}=R_z(\psi)\begin{bmatrix}-r_R\\0\\0\end{bmatrix}$$

$$\frac{\partial X}{\partial\phi}=R_z(\psi)\left(\begin{bmatrix}0\\r_R-z_{BS}-z_{HF}\\0\end{bmatrix}\right)$$

$$\frac{\partial X}{\partial\delta}=R_z(\psi)\begin{bmatrix}0\\\mu x_{BS}+\mu x_{HF}+\cos\lambda\,x_{HF}-\sin\lambda\,z_{HF}\\0\end{bmatrix}=R_z(\psi)\begin{bmatrix}0\\\mu x_{SHF}\\0\end{bmatrix}$$

$$\left(\frac{\partial X}{\partial r}\right)^T\frac{\partial X}{\partial r}=\begin{bmatrix}(r_R-z_{BS}-z_{HF})^2&\mu x_{SHF}(r_R-z_{BS}-z_{HF})\\\mu x_{SHF}(r_R-z_{BS}-z_{HF})&\mu^2x_{SHF}^2\end{bmatrix}$$

$$\frac{\partial}{\partial\theta_R}\left(\frac{\partial X}{\partial\theta_R}\right)=\frac{\partial R_z(\psi)}{\partial\psi}\frac{\partial\psi}{\partial\theta_R}\left(\begin{bmatrix}-r_R\\0\\0\end{bmatrix}+\begin{bmatrix}\sin\phi\,(-r_R-x_{BS}\sin\theta_B+z_{BS}\cos\theta_B)\\x_{BS}\cos\theta_B+z_{BS}\sin\theta_B\\0\end{bmatrix}\frac{\partial\psi}{\partial\theta_R}\right.$$

$$\left.+\frac{\partial SH}{\partial\psi}\frac{\partial\psi}{\partial\theta_R}\right)$$

$$\left(\frac{\partial X}{\partial\phi}\right)^T\frac{\partial}{\partial\theta_R}\left(\frac{\partial X}{\partial\theta_R}\right)=-r_R(r_R-z_{BS}-z_{HF})\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right)\delta$$

$$\left(\frac{\partial X}{\partial\delta}\right)^T\frac{\partial}{\partial\theta_R}\left(\frac{\partial X}{\partial\theta_R}\right)=-r_R\mu x_{SHF}\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right)\delta$$

$$\frac{\partial}{\partial \phi}\left(\frac{\partial X}{\partial \theta_R}\right) = 0$$

$$\left(\frac{\partial X}{\partial \phi}\right)^T \frac{\partial}{\partial \phi}\left(\frac{\partial X}{\partial \theta_R}\right) = 0$$

$$\left(\frac{\partial X}{\partial \delta}\right)^T \frac{\partial}{\partial \phi}\left(\frac{\partial X}{\partial \theta_R}\right) = 0$$

$$\frac{\partial}{\partial \delta}\left(\frac{\partial X}{\partial \theta_R}\right) = \frac{\partial R_z(\psi)}{\partial \psi}\frac{\partial \psi}{\partial \delta}\left(\begin{bmatrix} -r_R \\ 0 \\ 0 \end{bmatrix}\right) + R_z(\psi)\begin{bmatrix} 0 \\ x_{BS} \\ 0 \end{bmatrix}\frac{\partial}{\partial \delta}\left(\frac{\partial \psi}{\partial \theta_R}\right) + R_z(\psi)\begin{bmatrix} 0 \\ x_{HF} \\ 0 \end{bmatrix}\frac{\partial}{\partial \delta}\left(\frac{\partial \psi}{\partial \theta_R}\right)$$

$$= R_z(\psi)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\left(-r_R\mu + (x_{BS} + x_{HF})\frac{\partial}{\partial \delta}\left(\frac{\partial \psi}{\partial \theta_R}\right)\right)$$

$$\left(\frac{\partial X}{\partial \phi}\right)^T \frac{\partial}{\partial \delta}\left(\frac{\partial X}{\partial \theta_R}\right) = (r_R - z_{BS} - z_{HF})\left(-r_R\mu + (x_{BS} + x_{HF})\left(-\frac{r_R}{w}\cos\lambda\right)\right)$$

$$\left(\frac{\partial X}{\partial \delta}\right)^T \frac{\partial}{\partial \delta}\left(\frac{\partial X}{\partial \theta_R}\right) = \mu x_{SHF}\left(-r_R\mu + (x_{BS} + x_{HF})\left(-\frac{r_R}{w}\cos\lambda\right)\right)$$

$$\frac{Q_\phi}{m} = \left(\frac{\partial X}{\partial \phi}\right)^T \frac{\partial X}{\partial r}\ddot{r} - r_R(r_R - z_{BS} - z_{HF})\frac{\partial}{\partial \delta}\left(\frac{\partial \psi}{\partial \theta_R}\right)\delta\dot{\theta}_R^{\ 2}$$

$$+ (r_R - z_{BS} - z_{HF})\left(-r_R\mu + (x_{BS} + x_{HF})\left(-\frac{r_R}{w}\cos\lambda\right)\right)\dot{\delta}\dot{\theta}_R$$

$$\frac{Q_\delta}{m} = \left(\frac{\partial X}{\partial \delta}\right)^T \frac{\partial X}{\partial r}\ddot{r} - r_R\mu x_{SHF}\frac{\partial}{\partial \delta}\left(\frac{\partial \psi}{\partial \theta_R}\right)\delta\dot{\theta}_R^{\ 2}$$

$$+ \mu x_{SHF}\left(-r_R\mu + (x_{BS} + x_{HF})\left(-\frac{r_R}{w}\cos\lambda\right)\right)\dot{\delta}\dot{\theta}_R$$

$$\mu x_{SHF} = \mu x_{BS} + \mu x_{HF} + \cos\lambda\, x_{HF} - \sin\lambda\, z_{HF}$$

$$-r_R(r_R - z_{BS} - z_{HF})\left(\mu + \frac{\cos\lambda}{w}(x_{BS} + x_{HF})\right)$$

$$-r_R(\mu x_{BS} + \mu x_{HF} + \cos\lambda\, x_{HF} - \sin\lambda\, z_{HF})\left(\mu + \frac{\cos\lambda}{w}(x_{BS} + x_{HF})\right)$$

for RS, $\begin{bmatrix} x_{BS} \\ 0 \\ z_{BS} \end{bmatrix} = \begin{bmatrix} w+c \\ 0 \\ r_R \end{bmatrix}$

For SH, $\begin{bmatrix} x_{HF} \\ 0 \\ z_{HF} \end{bmatrix} = \begin{bmatrix} x_H - w - c \\ 0 \\ z_H \end{bmatrix}$, for SF, $\begin{bmatrix} x_{HF} \\ 0 \\ z_{HF} \end{bmatrix} = \begin{bmatrix} -c \\ 0 \\ -r_F \end{bmatrix}$

$$\begin{bmatrix} Q_\phi \\ Q_\delta \end{bmatrix} = m_H \begin{bmatrix} {z_{HF}}^2 & -\mu x_{SHF} z_{HF} \\ -\mu x_{SHF} z_{HF} & \mu^2 x_{SHF}^2 \end{bmatrix} \begin{bmatrix} \ddot{\phi} \\ \ddot{\delta} \end{bmatrix}$$
$$+ {\dot{\theta}_R}^2 r_R^2 m_H \begin{bmatrix} 0 & \frac{\cos\lambda}{w}(r_R - z_{BS} - z_{HF}) \\ 0 & b_{HF} \end{bmatrix} \begin{bmatrix} \phi \\ \delta \end{bmatrix} - \dot{\theta}_R r_R m_H \begin{bmatrix} 0 & a_{HF} \\ 0 & c_{HF} \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\delta} \end{bmatrix}$$

$$a_{HF} = (r_R - z_{BS} - z_{HF}) \left(\mu + \frac{\cos\lambda}{w}(x_{BS} + x_{HF}) \right)$$

$$b_{HF} = \frac{\cos\lambda}{w}(\mu x_{BS} + \mu x_{HF} + \cos\lambda\, x_{HF} - \sin\lambda\, z_{HF})$$

$$c_{HF} = (\mu x_{BS} + \mu x_{HF} + \cos\lambda\, x_{HF} - \sin\lambda\, z_{HF}) \left(\mu + \frac{\cos\lambda}{w}(x_{BS} + x_{HF}) \right)$$

In conclusion, for H

$$\begin{bmatrix} Q_\phi \\ Q_\delta \end{bmatrix} = m_H \begin{bmatrix} {z_H}^2 & -\mu x_{SH} z_H \\ -\mu x_{SH} z_H & \mu^2 x_{SH}^2 \end{bmatrix} \begin{bmatrix} \ddot{\phi} \\ \ddot{\delta} \end{bmatrix} + {\dot{\theta}_R}^2 r_R^2 m_H \begin{bmatrix} 0 & -\frac{\cos\lambda}{w} z_H \\ 0 & b_H \end{bmatrix} \begin{bmatrix} \phi \\ \delta \end{bmatrix}$$
$$- \dot{\theta}_R r_R m_H \begin{bmatrix} 0 & -z_H \left(\mu + \frac{\cos\lambda}{w} x_H \right) \\ 0 & c_H \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\delta} \end{bmatrix}$$

$$b_H = \frac{\cos\lambda}{w}(\mu x_H + \cos\lambda\,(x_H - w - c) - \sin\lambda\, z_H)$$

$$c_H = (\mu x_H + \cos\lambda\,(x_H - w - c) - \sin\lambda\, z_H) \left(\mu + \frac{\cos\lambda}{w} x_H \right)$$

In conclusion, for F

$$\begin{bmatrix} Q_\phi \\ Q_\delta \end{bmatrix} = m_F \begin{bmatrix} r_F^2 & \mu x_{SF} r_F \\ \mu x_{SF} r_F & \mu^2 x_{SF}^2 \end{bmatrix} \begin{bmatrix} \ddot{\phi} \\ \ddot{\delta} \end{bmatrix} + \dot{\theta}_R{}^2 r_R^2 m_F \begin{bmatrix} 0 & \dfrac{\cos\lambda}{w} r_F \\ 0 & b_F \end{bmatrix} \begin{bmatrix} \phi \\ \delta \end{bmatrix}$$

$$- \dot{\theta}_R r_R m_F \begin{bmatrix} 0 & r_F(\mu + \cos\lambda) \\ 0 & c_F \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\delta} \end{bmatrix}$$

$$b_F = \frac{\cos\lambda}{w}(\mu w - \cos\lambda\, c + \sin\lambda\, r_F)$$

$$c_F = (\mu w - \cos\lambda\, c + \sin\lambda\, r_F)(\mu + \cos\lambda)$$

Rotational kinetic energy

$$\omega = \bar{q}2\dot{q}$$

$$R_\lambda(\delta) = \begin{bmatrix} \cos\frac{\delta}{2} \\ \sin\frac{\delta}{2}\sin\lambda \\ 0 \\ \sin\frac{\delta}{2}\cos\lambda \end{bmatrix} = \cos\frac{\delta}{2} + \sin\frac{\delta}{2}\begin{bmatrix} \sin\lambda \\ 0 \\ \cos\lambda \end{bmatrix}$$

$$R_\lambda(\delta) = \begin{bmatrix} 1 - 2\sin^2\frac{\delta}{2}\cos^2\lambda & -\sin\delta\cos\lambda & 2\sin^2\frac{\delta}{2}\sin\lambda\cos\lambda \\ \sin\delta\cos\lambda & \cos\delta & -\sin\delta\sin\lambda \\ 2\sin^2\frac{\delta}{2}\sin\lambda\cos\lambda & \sin\delta\sin\lambda & 1 - 2\sin^2\frac{\delta}{2}\sin^2\lambda \end{bmatrix}$$

$$R_\lambda(-\delta) = \begin{bmatrix} 1 - 2\sin^2\frac{\delta}{2}\cos^2\lambda & \sin\delta\cos\lambda & 2\sin^2\frac{\delta}{2}\sin\lambda\cos\lambda \\ -\sin\delta\cos\lambda & \cos\delta & \sin\delta\sin\lambda \\ 2\sin^2\frac{\delta}{2}\sin\lambda\cos\lambda & -\sin\delta\sin\lambda & 1 - 2\sin^2\frac{\delta}{2}\sin^2\lambda \end{bmatrix}$$

$$qvq^* = \sin\theta\,\hat{n} \times v + \cos\theta\, v + 2 * \sin^2\frac{\theta}{2}(\hat{n} \cdot v)\hat{n}$$

$$I_R = \begin{bmatrix} I_{Rxx} & 0 & 0 \\ 0 & I_{Ryy} & 0 \\ 0 & 0 & I_{Rxx} \end{bmatrix}$$

$$q_R = R_z(\psi)R_x(\phi)R_y(\theta_R)$$

$$\omega_R = \dot{\psi}\begin{bmatrix} -\cos\phi\sin\theta_R \\ \sin\phi \\ \cos\phi\cos\theta_R \end{bmatrix} + \dot{\phi}\begin{bmatrix} \cos\theta_R \\ 0 \\ \sin\theta_R \end{bmatrix} + \dot{\theta_R}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$I_B = \begin{bmatrix} I_{Bxx} & 0 & I_{Bxz} \\ 0 & I_{Byy} & 0 \\ I_{Bxz} & 0 & I_{Bzz} \end{bmatrix}$$

$$q_B = R_z(\psi)R_x(\phi)R_y(\theta_B)$$

$$\omega_B = \dot{\psi}\begin{bmatrix} -\cos\phi\sin\theta_B \\ \sin\phi \\ \cos\phi\cos\theta_B \end{bmatrix} + \dot{\phi}\begin{bmatrix} \cos\theta_B \\ 0 \\ \sin\theta_B \end{bmatrix} + \dot{\theta_B}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$I_H = \begin{bmatrix} I_{Hxx} & 0 & I_{Hxz} \\ 0 & I_{Hyy} & 0 \\ I_{Hxz} & 0 & I_{Hzz} \end{bmatrix}$$

$$q_H = R_z(\psi)R_x(\phi)R_y(\theta_B)R_\lambda(\delta)$$

$$\omega_H = R_\lambda(-\delta)R_y(-\theta_B)R_x(-\phi)R_z(-\psi)2\left(\dot{\psi}\frac{\partial}{\partial\psi}q_H + \dot{\phi}\frac{\partial}{\partial\phi}q_H + \dot{\theta_B}\frac{\partial}{\partial\theta_B}q_H + \dot{\delta}\frac{\partial}{\partial\delta}q_H\right)$$

$$= \dot{\psi}R_\lambda(-\delta)\begin{bmatrix}-\cos\phi\sin\theta_B\\ \sin\phi\\ \cos\phi\cos\theta_B\end{bmatrix}R_\lambda(\delta) + \dot{\phi}R_\lambda(-\delta)\begin{bmatrix}\cos\theta_B\\ 0\\ \sin\theta_B\end{bmatrix}R_\lambda(\delta) + \dot{\theta_B}R_\lambda(-\delta)\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}R_\lambda(\delta) + \dot{\delta}R_\lambda(\pi)$$

$$= \dot{\psi}\begin{bmatrix}1-2\sin^2\frac{\delta}{2}\cos^2\lambda & \sin\delta\cos\lambda & 2\sin^2\frac{\delta}{2}\sin\lambda\cos\lambda\\ -\sin\delta\cos\lambda & \cos\delta & \sin\delta\sin\lambda\\ 2\sin^2\frac{\delta}{2}\sin\lambda\cos\lambda & -\sin\delta\sin\lambda & 1-2\sin^2\frac{\delta}{2}\sin^2\lambda\end{bmatrix}\begin{bmatrix}-\cos\phi\sin\theta_B\\ \sin\phi\\ \cos\phi\cos\theta_B\end{bmatrix}$$
$$+ \dot{\phi}\begin{bmatrix}\cos\theta_B - \cos\theta_B\,2\sin^2\frac{\delta}{2}\cos^2\lambda + \sin\theta_B\,2\sin^2\frac{\delta}{2}\sin\lambda\cos\lambda\\ -\cos\theta_B\sin\delta\cos\lambda + \sin\theta_B\sin\delta\sin\lambda\\ \cos\theta_B\,2\sin^2\frac{\delta}{2}\sin\lambda\cos\lambda + \sin\theta_B - \sin\theta_B\,2\sin^2\frac{\delta}{2}\sin^2\lambda\end{bmatrix}$$
$$+ \dot{\theta_B}\begin{bmatrix}\sin\delta\cos\lambda\\ \cos\delta\\ -\sin\delta\sin\lambda\end{bmatrix} + \dot{\delta}\begin{bmatrix}\sin\lambda\\ 0\\ \cos\lambda\end{bmatrix}$$

$$I_F = \begin{bmatrix}I_{Fxx} & 0 & 0\\ 0 & I_{Fyy} & 0\\ 0 & 0 & I_{Fxx}\end{bmatrix}$$

$$q_F = R_z(\psi)R_x(\phi)R_y(\theta_B)R_\lambda(\delta)R_y(\theta_F)$$

For point R

$$I_R = \begin{bmatrix} I_{Rxx} & 0 & 0 \\ 0 & I_{Ryy} & 0 \\ 0 & 0 & I_{Rxx} \end{bmatrix}$$

$$q_R = R_z(\psi)R_x(\phi)R_y(\theta_R)$$

$$\overline{q_R} = R_y(-\theta_R)R_x(-\phi)R_z(-\psi)$$

$$\frac{dq_R}{dt} = \frac{\partial q_R}{\partial \psi}\dot{\psi} + \frac{\partial q_R}{\partial \phi}\dot{\phi} + \frac{\partial q_R}{\partial \theta_R}\dot{\theta_R} = \frac{\partial q_R}{\partial \psi}\left(\frac{\partial \psi}{\partial \theta_R}\dot{\theta_R} + \frac{\partial \psi}{\partial \phi}\dot{\phi} + \frac{\partial \psi}{\partial \delta}\dot{\delta}\right) + \frac{\partial q_R}{\partial \phi}\dot{\phi} + \frac{\partial q_R}{\partial \theta_R}\dot{\theta_R}$$
$$= \left(\frac{\partial q_R}{\partial \theta_R} + \frac{\partial q_R}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\right)\dot{\theta_R} + \left(\frac{\partial q_R}{\partial \phi} + \frac{\partial q_R}{\partial \psi}\frac{\partial \psi}{\partial \phi}\right)\dot{\phi} + \frac{\partial q_R}{\partial \psi}\frac{\partial \psi}{\partial \delta}\dot{\delta}$$

$$\bar{q}2\frac{\partial q}{\partial \theta_R} = \overline{q_R}2\frac{\partial q_R}{\partial \theta_R} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\bar{q}2\frac{\partial q}{\partial \phi} = \overline{q_R}2\frac{\partial q_R}{\partial \phi} + \overline{q_R}2\frac{\partial q_R}{\partial \psi}\frac{\partial \psi}{\partial \phi} = \begin{bmatrix} \cos\theta_R \\ 0 \\ \sin\theta_R \end{bmatrix} + \begin{bmatrix} -\cos\phi\sin\theta_R \\ \sin\phi \\ \cos\phi\cos\theta_R \end{bmatrix}\frac{\partial \psi}{\partial \phi} = \begin{bmatrix} \cos\theta_R \\ 0 \\ \sin\theta_R \end{bmatrix}$$

$$\bar{q}2\frac{\partial q}{\partial \delta} = \bar{q}2\frac{\partial q_R}{\partial \psi}\frac{\partial \psi}{\partial \delta} = \frac{\partial \psi}{\partial \delta}\begin{bmatrix} -\cos\phi\sin\theta_R \\ \sin\phi \\ \cos\phi\cos\theta_R \end{bmatrix} = \mu\begin{bmatrix} -\sin\theta_R \\ 0 \\ \cos\theta_R \end{bmatrix}$$

$$\left(\bar{q}2\frac{\partial q}{\partial r}\right)^T I\bar{q}2\frac{\partial q}{\partial r} = \begin{bmatrix} I_{Rxx} & 0 \\ 0 & \mu^2 I_{Rxx} \end{bmatrix}$$

$$\frac{\partial}{\partial \phi}\left[\left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \phi}\right]$$
$$= \frac{\partial}{\partial \phi}\left[\bar{q}2\frac{\partial q}{\partial \theta_R}\right]^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \phi} + \left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I\frac{\partial}{\partial \phi}\left[2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \phi}\right] =$$
$$= \left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I\frac{\partial}{\partial \phi}\left[2\left(\frac{\partial \overline{q_R}}{\partial \theta_R} + \frac{\partial \overline{q_R}}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\right)2\left(\frac{\partial q_R}{\partial \phi} + \frac{\partial q_R}{\partial \psi}\frac{\partial \psi}{\partial \phi}\right)\right]$$

$$= \left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I\begin{bmatrix} -\cos\theta_R \\ 0 \\ \sin\theta_R \end{bmatrix}\frac{\partial}{\partial \phi}\left[\frac{\partial \psi}{\partial \phi}\right] = 0$$

$$\frac{\partial}{\partial\delta}\left[\left(\bar{q}2\frac{\partial q}{\partial\theta_R}\right)^T I2\frac{\partial\bar{q}}{\partial\theta_R}2\frac{\partial q}{\partial\phi}\right]$$

$$= \frac{\partial}{\partial\delta}\left[\bar{q}2\frac{\partial q}{\partial\theta_R}\right]^T I2\frac{\partial\bar{q}}{\partial\theta_R}2\frac{\partial q}{\partial\phi} + \left(\bar{q}2\frac{\partial q}{\partial\theta_R}\right)^T I\frac{\partial}{\partial\delta}\left[2\frac{\partial\bar{q}}{\partial\theta_R}2\frac{\partial q}{\partial\phi}\right]$$

$$= \left(-\frac{r_R}{w}\cos\lambda\right)\begin{bmatrix}-\sin\theta_R\\0\\\cos\theta_R\end{bmatrix}^T I\begin{bmatrix}-\sin\theta_R\\0\\\cos\theta_R\end{bmatrix} + \left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)^T I\begin{bmatrix}0\\-1\\0\end{bmatrix}\left(-\frac{r_R}{w}\cos\lambda\right)$$

$$= \left(-\frac{r_R}{w}\cos\lambda\right)\left(I_{Rxx} - I_{Ryy}\right)$$

$$2\frac{\partial\bar{q}}{\partial\theta_R}2\frac{\partial q}{\partial\phi} = 2\left(\frac{\partial\overline{q_R}}{\partial\theta_R} + \frac{\partial\overline{q_R}}{\partial\psi}\frac{\partial\psi}{\partial\theta_R}\right)2\left(\frac{\partial q_R}{\partial\phi} + \frac{\partial q_R}{\partial\psi}\frac{\partial\psi}{\partial\phi}\right) = 2\frac{\partial\overline{q_R}}{\partial\theta_R}2\frac{\partial q_R}{\partial\phi} = \begin{bmatrix}-\sin\theta_R\\0\\\cos\theta_R\end{bmatrix}$$

$$\bar{q}2\frac{\partial}{\partial\phi}\left(\frac{\partial q}{\partial\theta_R}\right) = \overline{q_R}2\frac{\partial}{\partial\phi}\left(\frac{\partial q_R}{\partial\theta_R} + \frac{\partial q_R}{\partial\psi}\frac{\partial\psi}{\partial\theta_R}\right) = \frac{1}{2}\begin{bmatrix}-\sin\theta_R\\0\\\cos\theta_R\end{bmatrix}$$

$$\bar{q}2\frac{\partial}{\partial\theta_R}\left(\frac{\partial q}{\partial\phi}\right) = \frac{1}{2}\begin{bmatrix}-\sin\theta_R\\0\\\cos\theta_R\end{bmatrix}$$

$$2\frac{\partial\bar{q}}{\partial\delta}2\frac{\partial q}{\partial\phi} = 2\left(\frac{\partial\overline{q_R}}{\partial\psi}\frac{\partial\psi}{\partial\delta}\right)2\left(\frac{\partial q_R}{\partial\phi} + \frac{\partial q_R}{\partial\psi}\frac{\partial\psi}{\partial\phi}\right) = \mu 2\frac{\partial\overline{q_R}}{\partial\psi}2\frac{\partial q_R}{\partial\phi} = \begin{bmatrix}0\\-\mu\\0\end{bmatrix}$$

$$\bar{q}2\frac{\partial}{\partial\delta}\left(\frac{\partial q}{\partial\theta_R}\right) = \overline{q_R}2\frac{\partial}{\partial\delta}\left(\frac{\partial q_R}{\partial\theta_R} + \frac{\partial q_R}{\partial\psi}\frac{\partial\psi}{\partial\theta_R}\right) = \frac{1}{2}\mu\begin{bmatrix}-\cos\theta_R\\0\\-\sin\theta_R\end{bmatrix} + \left(-\frac{r_R}{w}\cos\lambda\right)\begin{bmatrix}-\sin\theta_R\\0\\\cos\theta_R\end{bmatrix}$$

$$\bar{q}2\frac{\partial}{\partial\theta_R}\left(\frac{\partial q}{\partial\delta}\right) = \frac{1}{2}\mu\begin{bmatrix}-\cos\theta_R\\0\\-\sin\theta_R\end{bmatrix}$$

$$\frac{\partial}{\partial \phi}\left[\left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \delta}\right]$$

$$=\frac{\partial}{\partial \phi}\left[\bar{q}2\frac{\partial q}{\partial \theta_R}\right]^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \delta}+\left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I\frac{\partial}{\partial \phi}\left[2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \delta}\right]$$

$$=\left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I\frac{\partial}{\partial \phi}\left[R_y(-\theta_R)\begin{bmatrix}-\cos\phi\\0\\0\end{bmatrix}R_y(\theta_R)\frac{\partial \psi}{\partial \delta}\right]$$

$$=\left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I\begin{bmatrix}-\cos\theta_R\\0\\-\sin\theta_R\end{bmatrix}\frac{\partial}{\partial \phi}\left(\frac{\partial \psi}{\partial \delta}\right)=0$$

$$\frac{\partial}{\partial \delta}\left[\left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \delta}\right]$$

$$=\frac{\partial}{\partial \delta}\left[\bar{q}2\frac{\partial q}{\partial \theta_R}\right]^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \delta}+\left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I\frac{\partial}{\partial \delta}\left[2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \delta}\right]$$

$$=\frac{\partial}{\partial \delta}\left(\frac{\partial \psi}{\partial \theta_R}\right)\begin{bmatrix}-\sin\theta_R\\0\\\cos\theta_R\end{bmatrix}^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \delta}+\left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I\frac{\partial}{\partial \delta}\left[2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \delta}\right]$$

$$=\begin{bmatrix}-\sin\theta_R\\0\\\cos\theta_R\end{bmatrix}^T I\begin{bmatrix}-\cos\theta_R\\0\\-\sin\theta_R\end{bmatrix}\frac{\partial \psi}{\partial \delta}\frac{\partial}{\partial \delta}\left(\frac{\partial \psi}{\partial \theta_R}\right)+\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)^T I\begin{bmatrix}-\cos\theta_R\\0\\-\sin\theta_R\end{bmatrix}\frac{\partial}{\partial \delta}\left[\frac{\partial \psi}{\partial \delta}\right]$$

$$=\begin{bmatrix}-\sin\theta_R\\0\\\cos\theta_R\end{bmatrix}^T I\begin{bmatrix}-\cos\theta_R\\0\\-\sin\theta_R\end{bmatrix}\mu\left(-r_R\cos\lambda-r_R\frac{\mu}{w}\right)=0$$

$$2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \delta}=2\left(\frac{\partial \overline{q_R}}{\partial \theta_R}+\frac{\partial \overline{q_R}}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\right)2\left(\frac{\partial q_R}{\partial \psi}\frac{\partial \psi}{\partial \delta}\right)=2\frac{\partial \overline{q_R}}{\partial \theta_R}2\frac{\partial q_R}{\partial \psi}\frac{\partial \psi}{\partial \delta}=\mu\begin{bmatrix}-\cos\theta_R\\0\\-\sin\theta_R\end{bmatrix}$$

$$2\frac{\partial \bar{q}}{\partial \phi}2\frac{\partial q}{\partial \delta}=2\left(\frac{\partial \overline{q_R}}{\partial \phi}+\frac{\partial \overline{q_R}}{\partial \psi}\frac{\partial \psi}{\partial \phi}\right)2\left(\frac{\partial q_R}{\partial \psi}\frac{\partial \psi}{\partial \delta}\right)=2\frac{\partial \overline{q_R}}{\partial \phi}2\frac{\partial q_R}{\partial \psi}\frac{\partial \psi}{\partial \delta}=\begin{bmatrix}0\\\mu\\0\end{bmatrix}$$

$$\bar{q}2\frac{\partial}{\partial \theta_R}\left(\frac{\partial q}{\partial \theta_R}\right)=\begin{bmatrix}-\cos\phi\cos\theta_R\\0\\-\cos\phi\sin\theta_R\end{bmatrix}\frac{\partial \psi}{\partial \theta_R}$$

$$\left(\bar{q}2\frac{\partial q}{\partial \phi}\right)^T I\bar{q}2\frac{\partial}{\partial \theta_R}\left(\frac{\partial q}{\partial \theta_R}\right)\dot{\theta_R}^2=-\cos\phi\, I_{Rxx}\frac{\partial \psi}{\partial \theta_R}\dot{\theta_R}^2=-I_{Rxx}\frac{\partial}{\partial \delta}\left(\frac{\partial \psi}{\partial \theta_R}\right)\delta\dot{\theta_R}^2$$

$$\left(\bar{q}2\frac{\partial q}{\partial \delta}\right)^T I\bar{q}2\frac{\partial}{\partial \theta_R}\left(\frac{\partial q}{\partial \theta_R}\right)\dot{\theta_R}^2 = 0$$

$$Q_\phi = \left(\bar{q}2\frac{\partial q}{\partial \phi}\right)^T I\bar{q}2\frac{\partial q}{\partial r}\ddot{r} + \left(-\frac{r_R}{w}\cos\lambda\right)\left(I_{Rxx} - I_{Ryy}\right)\delta\dot{\theta_R}^2 - I_{Rxx}\left(-\frac{r_R}{w}\cos\lambda\right)\delta\dot{\theta_R}^2$$

$$-I_{Ryy}\mu\dot{\delta}\dot{\theta}_R$$

$$Q_\delta = \left(\bar{q}2\frac{\partial q}{\partial \delta}\right)^T I\bar{q}2\frac{\partial q}{\partial r}\ddot{r}$$

$$+I_{Ryy}\mu\dot{\phi}\dot{\theta}_R - \mu I_{Rxx}\frac{r_R}{w}\cos\lambda\,\dot{\delta}\dot{\theta}_R$$

$$\begin{bmatrix} Q_\phi \\ Q_\delta \end{bmatrix} = \begin{bmatrix} I_{Rxx} & 0 \\ 0 & \mu^2 I_{Rxx} \end{bmatrix}\begin{bmatrix} \ddot{\phi} \\ \ddot{\delta} \end{bmatrix} + \dot{\theta_R}^2 r_R^2 \begin{bmatrix} 0 & \frac{1}{r_R}\frac{\cos\lambda}{w} I_{Ryy} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \phi \\ \delta \end{bmatrix}$$

$$-\dot{\theta}_R r_R \begin{bmatrix} 0 & \frac{\mu}{r_R} I_{Ryy} \\ -\frac{\mu}{r_R} I_{Ryy} & \mu\frac{\cos\lambda}{w} I_{Rxx} \end{bmatrix}\begin{bmatrix} \dot{\phi} \\ \dot{\delta} \end{bmatrix}$$

For point B

$$I_B = \begin{bmatrix} I_{Bxx} & 0 & I_{Bxz} \\ 0 & I_{Byy} & 0 \\ I_{Bxz} & 0 & I_{Bzz} \end{bmatrix}$$

$$q_B = R_z(\psi)R_x(\phi)R_y(\theta_B)$$
$$\overline{q_B} = R_y(-\theta_B)R_x(-\phi)R_z(-\psi)$$

$$\frac{dq_B}{dt} = \frac{\partial q_B}{\partial \psi}\dot{\psi} + \frac{\partial q_B}{\partial \phi}\dot{\phi} + \frac{\partial q_B}{\partial \theta_B}\dot{\theta_B}$$
$$= \frac{\partial q_B}{\partial \psi}\left(\frac{\partial \psi}{\partial \theta_R}\dot{\theta_R} + \frac{\partial \psi}{\partial \phi}\dot{\phi} + \frac{\partial \psi}{\partial \delta}\dot{\delta}\right) + \frac{\partial q_B}{\partial \phi}\dot{\phi} + \frac{\partial q_B}{\partial \theta_B}\left(\frac{\partial \theta_B}{\partial \phi}\dot{\phi} + \frac{\partial \theta_B}{\partial \delta}\dot{\delta}\right)$$
$$= \left(\frac{\partial q_B}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\right)\dot{\theta_R} + \left(\frac{\partial q_B}{\partial \phi} + \frac{\partial q_B}{\partial \psi}\frac{\partial \psi}{\partial \phi} + \frac{\partial q_B}{\partial \theta_B}\frac{\partial \theta_B}{\partial \phi}\right)\dot{\phi}$$
$$+ \left(\frac{\partial q_B}{\partial \psi}\frac{\partial \psi}{\partial \delta} + \frac{\partial q_B}{\partial \theta_B}\frac{\partial \theta_B}{\partial \delta}\right)\dot{\delta}$$

$$\bar{q}2\frac{\partial q}{\partial \theta_R} = \overline{q_B}2\left(\frac{\partial q_B}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\right) = 0$$

$$\bar{q}2\frac{\partial q}{\partial \phi} = \overline{q_B}2\left(\frac{\partial q_B}{\partial \phi} + \frac{\partial q_B}{\partial \psi}\frac{\partial \psi}{\partial \phi} + \frac{\partial q_B}{\partial \theta_B}\frac{\partial \theta_B}{\partial \phi}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{q}2\frac{\partial q}{\partial \delta} = \overline{q_B}2\frac{\partial q_B}{\partial \psi}\frac{\partial \psi}{\partial \delta} = \begin{bmatrix} 0 \\ 0 \\ \mu \end{bmatrix}$$

$$\left(\bar{q}2\frac{\partial q}{\partial r}\right)^T I\bar{q}2\frac{\partial q}{\partial r} = \begin{bmatrix} I_{Bxx} & \mu I_{Bxz} \\ \mu I_{Bxz} & \mu^2 I_{Bzz} \end{bmatrix}$$

$$\frac{\partial}{\partial \phi}\left(\left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \phi}\right) = 0$$

$$\frac{\partial}{\partial \delta}\left(\left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \phi}\right) = 0$$

$$\frac{\partial}{\partial \phi}\left(\left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \delta}\right) = 0$$

$$\frac{\partial}{\partial \delta}\left(\left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \delta}\right) = 0$$

$$2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \phi} = 2\left(\frac{\partial q_B}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\right)2\frac{\partial q}{\partial \phi} = 0$$

$$2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \delta} = 2\left(\frac{\partial q_B}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\right)2\frac{\partial q}{\partial \delta} = 0$$

$$2\frac{\partial \bar{q}}{\partial \phi}2\frac{\partial q}{\partial \delta} = 2\left(\frac{\partial \overline{q_B}}{\partial \phi}\right)2\left(\frac{\partial q_B}{\partial \psi}\mu\right) = \begin{bmatrix}0\\ \mu\\ 0\end{bmatrix}$$

$$2\frac{\partial \bar{q}}{\partial \delta}2\frac{\partial q}{\partial \phi} = -2\frac{\partial \bar{q}}{\partial \phi}2\frac{\partial q}{\partial \delta} = \begin{bmatrix}0\\ -\mu\\ 0\end{bmatrix}$$

$$\bar{q}2\frac{\partial}{\partial \phi}\left(\frac{\partial q}{\partial \theta_R}\right) = \bar{q}2\frac{\partial}{\partial \phi}\left(2\frac{\partial q_B}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\right) = 0$$

$$\bar{q}2\frac{\partial}{\partial \theta_R}\left(\frac{\partial q}{\partial \phi}\right) = 0$$

$$\bar{q}2\frac{\partial}{\partial \delta}\left(\frac{\partial q}{\partial \theta_R}\right) = \bar{q}2\frac{\partial q_B}{\partial \psi}\frac{\partial}{\partial \delta}\left(\frac{\partial \psi}{\partial \theta_R}\right) = \begin{bmatrix}0\\ 0\\ -\frac{r_R}{w}\cos\lambda\end{bmatrix}$$

$$\bar{q}2\frac{\partial}{\partial \theta_R}\left(\frac{\partial q}{\partial \delta}\right) = \bar{q}2\frac{\partial}{\partial \psi}\left(\frac{\partial q_B}{\partial \psi}\right)\frac{\partial \psi}{\partial \theta_R}\frac{\partial \psi}{\partial \delta} = 0$$

$$\bar{q}2\frac{\partial}{\partial \theta_R}\left(\frac{\partial q}{\partial \theta_R}\right) = \bar{q}2\frac{\partial}{\partial \theta_R}\left(\frac{\partial q_B}{\partial \psi}\right)\frac{\partial \psi}{\partial \theta_R} = 0$$

$$\left(\bar{q}2\frac{\partial q}{\partial \phi}\right)^T I\bar{q}2\frac{\partial}{\partial \theta_R}\left(\frac{\partial q}{\partial \theta_R}\right)\dot{\theta_R}^2 = 0$$

$$\left(\bar{q}2\frac{\partial q}{\partial \delta}\right)^T I\bar{q}2\frac{\partial}{\partial \theta_R}\left(\frac{\partial q}{\partial \theta_R}\right)\dot{\theta_R}^2 = 0$$

$$Q_\phi = \left(\bar{q}2\frac{\partial q}{\partial \phi}\right)^T I\bar{q}2\frac{\partial q}{\partial r}\ddot{r} - I_{Bxz}\frac{r_R}{w}\cos\lambda\,\dot{\delta}\dot{\theta}_R$$

$$Q_\delta = \left(\bar{q}2\frac{\partial q}{\partial \delta}\right)^T I\bar{q}2\frac{\partial q}{\partial r}\ddot{r} - I_{Bzz}\mu\frac{r_R}{w}\cos\lambda\,\dot{\delta}\dot{\theta}_R$$

$$\begin{bmatrix} Q_\phi \\ Q_\delta \end{bmatrix} = \begin{bmatrix} I_{Bxx} & \mu I_{Bxz} \\ \mu I_{Bxz} & \mu^2 I_{Bzz} \end{bmatrix} \begin{bmatrix} \ddot{\phi} \\ \ddot{\delta} \end{bmatrix} - \dot{\theta}_R r_R \begin{bmatrix} 0 & \dfrac{\cos\lambda}{w} I_{Bxz} \\ 0 & \mu \dfrac{\cos\lambda}{w} I_{Bzz} \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\delta} \end{bmatrix}$$

For point H

$$I_H = \begin{bmatrix} I_{Hxx} & 0 & I_{Hxz} \\ 0 & I_{Hyy} & 0 \\ I_{Hxz} & 0 & I_{Hzz} \end{bmatrix}$$

$$q_H = R_z(\psi)R_x(\phi)R_y(\theta_B)R_\lambda(\delta)$$
$$\overline{q_H} = R_\lambda(-\delta)R_y(-\theta_B)R_x(-\phi)R_z(-\psi)$$
$$\frac{dq_H}{dt} = \frac{\partial q_H}{\partial \psi}\dot{\psi} + \frac{\partial q_H}{\partial \phi}\dot{\phi} + \frac{\partial q_H}{\partial \theta_B}\dot{\theta_B} + \frac{\partial q_H}{\partial \delta}\dot{\delta}$$

$$= \frac{\partial q_H}{\partial \psi}\left(\frac{\partial \psi}{\partial \theta_R}\dot{\theta_R} + \frac{\partial \psi}{\partial \phi}\dot{\phi} + \frac{\partial \psi}{\partial \delta}\dot{\delta}\right) + \frac{\partial q_H}{\partial \phi}\dot{\phi} + \frac{\partial q_H}{\partial \theta_B}\left(\frac{\partial \theta_B}{\partial \phi}\dot{\phi} + \frac{\partial \theta_B}{\partial \delta}\dot{\delta}\right) + \frac{\partial q_H}{\partial \delta}\dot{\delta}$$
$$= \left(\frac{\partial q_H}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\right)\dot{\theta_R} + \left(\frac{\partial q_H}{\partial \phi} + \frac{\partial q_H}{\partial \psi}\frac{\partial \psi}{\partial \phi} + \frac{\partial q_H}{\partial \theta_B}\frac{\partial \theta_B}{\partial \phi}\right)\dot{\phi}$$
$$+ \left(\frac{\partial q_H}{\partial \delta} + \frac{\partial q_H}{\partial \psi}\frac{\partial \psi}{\partial \delta} + \frac{\partial q_H}{\partial \theta_B}\frac{\partial \theta_B}{\partial \delta}\right)\dot{\delta}$$

$$\bar{q}2\frac{\partial q}{\partial \theta_R} = \overline{q_H}2\left(\frac{\partial q_H}{\partial \psi}\frac{\partial \psi}{\partial \theta_R}\right) = 0$$

$$\bar{q}2\frac{\partial q}{\partial \phi} = \overline{q_H}2\left(\frac{\partial q_H}{\partial \phi} + \frac{\partial q_H}{\partial \psi}\frac{\partial \psi}{\partial \phi} + \frac{\partial q_H}{\partial \theta_B}\frac{\partial \theta_B}{\partial \phi}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{q}2\frac{\partial q}{\partial \delta} = \overline{q_H}2\left(\frac{\partial q_H}{\partial \delta} + \frac{\partial q_H}{\partial \psi}\frac{\partial \psi}{\partial \delta}\right) = \overline{q_H}2\frac{\partial q_H}{\partial \delta} + \mu\overline{q_H}2\frac{\partial q_H}{\partial \psi} = \begin{bmatrix} \sin\lambda \\ 0 \\ \mu + \cos\lambda \end{bmatrix}$$

$$\left(\bar{q}2\frac{\partial q}{\partial r}\right)^T I\bar{q}2\frac{\partial q}{\partial r}$$
$$= \begin{bmatrix} I_{Hxx} & \sin\lambda\, I_{Hxx} + (\mu + \cos\lambda)I_{Hxz} \\ \sin\lambda\, I_{Hxx} + (\cos\lambda + \mu)I_{Hxz} & (\mu + \cos\lambda)^2 I_{Hzz} + 2\sin\lambda\,(\mu + \cos\lambda)I_{Hxz} + \sin^2\lambda\, I_{Hxx} \end{bmatrix}$$

$$\frac{\partial}{\partial \phi}\left(\left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \phi}\right) = 0$$

$$\frac{\partial}{\partial \delta}\left(\left(\bar{q}2\frac{\partial q}{\partial \theta_R}\right)^T I2\frac{\partial \bar{q}}{\partial \theta_R}2\frac{\partial q}{\partial \phi}\right) = 0$$

$$\frac{\partial}{\partial\phi}\left(\left(\bar{q}2\frac{\partial q}{\partial\theta_R}\right)^T I2\frac{\partial\bar{q}}{\partial\theta_R}2\frac{\partial q}{\partial\delta}\right)=0$$

$$\frac{\partial}{\partial\delta}\left(\left(\bar{q}2\frac{\partial q}{\partial\theta_R}\right)^T I2\frac{\partial\bar{q}}{\partial\theta_R}2\frac{\partial q}{\partial\delta}\right)=0$$

$$2\frac{\partial\bar{q}}{\partial\theta_R}2\frac{\partial q}{\partial\phi}=0$$

$$2\frac{\partial\bar{q}}{\partial\theta_R}2\frac{\partial q}{\partial\delta}=0$$

$$2\frac{\partial\bar{q}}{\partial\phi}2\frac{\partial q}{\partial\delta}=2\frac{\partial\overline{q_H}}{\partial\phi}2\frac{\partial q_H}{\partial\delta}+2\frac{\partial\overline{q_H}}{\partial\phi}2\frac{\partial q_H}{\partial\psi}\frac{\partial\psi}{\partial\delta}=\begin{bmatrix}0\\ \mu+\cos\lambda\\ 0\end{bmatrix}$$

$$2\frac{\partial\bar{q}}{\partial\delta}2\frac{\partial q}{\partial\phi}=-2\frac{\partial\bar{q}}{\partial\phi}2\frac{\partial q}{\partial\delta}=\begin{bmatrix}0\\ -\mu-\cos\lambda\\ 0\end{bmatrix}$$

$$\bar{q}2\frac{\partial}{\partial\phi}\left(\frac{\partial q}{\partial\theta_R}\right)=0$$

$$\bar{q}2\frac{\partial}{\partial\theta_R}\left(\frac{\partial q}{\partial\phi}\right)=0$$

$$\bar{q}2\frac{\partial}{\partial\delta}\left(\frac{\partial q}{\partial\theta_R}\right)=\bar{q}2\frac{\partial q_H}{\partial\psi}\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right)=\begin{bmatrix}0\\ 0\\ -\frac{r_R}{w}\cos\lambda\end{bmatrix}$$

$$\bar{q}2\frac{\partial}{\partial\theta_R}\left(\frac{\partial q}{\partial\delta}\right)=0$$

$$\bar{q}2\frac{\partial}{\partial\theta_R}\left(\frac{\partial q}{\partial\theta_R}\right)=0$$

$$\left(\bar{q}2\frac{\partial q}{\partial\phi}\right)^T I\bar{q}2\frac{\partial}{\partial\theta_R}\left(\frac{\partial q}{\partial\theta_R}\right)\dot{\theta_R}^2=0$$

$$\left(\bar{q}2\frac{\partial q}{\partial\delta}\right)^T I\bar{q}2\frac{\partial}{\partial\theta_R}\left(\frac{\partial q}{\partial\theta_R}\right)\dot{\theta_R}^2=0$$

$$Q_\phi=\left(\bar{q}2\frac{\partial q}{\partial\phi}\right)^T I\bar{q}2\frac{\partial q}{\partial r}\ddot{r}-I_{Hxz}\frac{r_R}{w}\cos\lambda\,\dot{\delta}\dot{\theta}_R$$

$$Q_\delta = \left(\bar{q}2\frac{\partial q}{\partial \delta}\right)^T I\bar{q}2\frac{\partial q}{\partial r}\ddot{r} - (\sin\lambda\, I_{Hxz} + (\mu + \cos\lambda)I_{Hzz})\frac{r_R}{w}\cos\lambda\,\dot{\delta}\dot{\theta}_R$$

$$\begin{bmatrix} Q_\phi \\ Q_\delta \end{bmatrix}$$
$$= \begin{bmatrix} I_{Hxx} & \sin\lambda\, I_{Hxx} + (\mu + \cos\lambda)I_{Hxz} \\ \sin\lambda\, I_{Hxx} + (\cos\lambda + \mu)I_{Hxz} & (\mu + \cos\lambda)^2 I_{Hzz} + 2\sin\lambda\,(\mu + \cos\lambda)I_{Hxz} + \sin^2\lambda\, I_{Hxx} \end{bmatrix} \begin{bmatrix} \ddot{\phi} \\ \ddot{\delta} \end{bmatrix}$$
$$- \dot{\theta}_R r_R \begin{bmatrix} 0 & \frac{\cos\lambda}{w} I_{Hxz} \\ 0 & \frac{\cos\lambda}{w}(\sin\lambda\, I_{Hxz} + (\mu + \cos\lambda)I_{Hzz}) \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\delta} \end{bmatrix}$$

For point F

$$I_F = \begin{bmatrix} I_{Fxx} & 0 & 0 \\ 0 & I_{Fyy} & 0 \\ 0 & 0 & I_{Fxx} \end{bmatrix}$$

$$q_F = R_z(\psi)R_x(\phi)R_y(\theta_B)R_\lambda(\delta)R_y(\theta_F)$$
$$\overline{q_F} = R_y(-\theta_F)R_\lambda(-\delta)R_y(-\theta_B)R_x(-\phi)R_z(-\psi)$$

$$\begin{aligned}
\frac{dq_F}{dt} = \frac{\partial q_F}{\partial \psi}\dot{\psi} + \frac{\partial q_F}{\partial \phi}\dot{\phi} + \frac{\partial q_F}{\partial \theta_B}\dot{\theta}_B + \frac{\partial q_F}{\partial \delta}\dot{\delta} + \frac{\partial q_F}{\partial \theta_F}\dot{\theta}_F \\
&= \frac{\partial q_F}{\partial \psi}\left(\frac{\partial \psi}{\partial \theta_R}\dot{\theta}_R + \frac{\partial \psi}{\partial \phi}\dot{\phi} + \frac{\partial \psi}{\partial \delta}\dot{\delta}\right) + \frac{\partial q_F}{\partial \phi}\dot{\phi} + \frac{\partial q_F}{\partial \theta_B}\left(\frac{\partial \theta_B}{\partial \phi}\dot{\phi} + \frac{\partial \theta_B}{\partial \delta}\dot{\delta}\right) \\
&+ \frac{\partial q_F}{\partial \delta}\dot{\delta} + \frac{\partial q_F}{\partial \theta_F}\left(\frac{\partial \theta_F}{\partial \theta_R}\dot{\theta}_R + \frac{\partial \theta_F}{\partial \phi}\dot{\phi} + \frac{\partial \theta_F}{\partial \delta}\dot{\delta}\right) \\
&= \left(\frac{\partial q_F}{\partial \psi}\frac{\partial \psi}{\partial \theta_R} + \frac{\partial q_F}{\partial \theta_F}\frac{\partial \theta_F}{\partial \theta_R}\right)\dot{\theta}_R \\
&+ \left(\frac{\partial q_F}{\partial \phi} + \frac{\partial q_F}{\partial \psi}\frac{\partial \psi}{\partial \phi} + \frac{\partial q_F}{\partial \theta_B}\frac{\partial \theta_B}{\partial \phi} + \frac{\partial q_F}{\partial \theta_F}\frac{\partial \theta_F}{\partial \phi}\right)\dot{\phi} \\
&+ \left(\frac{\partial q_F}{\partial \delta} + \frac{\partial q_F}{\partial \psi}\frac{\partial \psi}{\partial \delta} + \frac{\partial q_F}{\partial \theta_B}\frac{\partial \theta_B}{\partial \delta} + \frac{\partial q_F}{\partial \theta_F}\frac{\partial \theta_F}{\partial \delta}\right)\dot{\delta}
\end{aligned}$$

$$\bar{q}2\frac{\partial q}{\partial \theta_R} = \overline{q_F}2\left(\frac{\partial q_F}{\partial \psi}\frac{\partial \psi}{\partial \theta_R} + \frac{\partial q_F}{\partial \theta_F}\frac{\partial \theta_F}{\partial \theta_R}\right) = \begin{bmatrix} 0 \\ \frac{r_R}{r_F} \\ 0 \end{bmatrix}$$

$$\bar{q}2\frac{\partial q}{\partial \phi} = \overline{q_F}2\left(\frac{\partial q_B}{\partial \phi} + \frac{\partial q_B}{\partial \psi}\frac{\partial \psi}{\partial \phi} + \frac{\partial q_B}{\partial \theta_R}\frac{\partial \theta_B}{\partial \phi}\right) = \begin{bmatrix} \cos\theta_F \\ 0 \\ \sin\theta_F \end{bmatrix}$$

$$\bar{q}2\frac{\partial q}{\partial \delta} = \overline{q_F}2\left(\frac{\partial q_F}{\partial \delta} + \frac{\partial q_F}{\partial \psi}\frac{\partial \psi}{\partial \delta} + \frac{\partial q_F}{\partial \theta_R}\frac{\partial \theta_B}{\partial \delta} + \frac{\partial q_F}{\partial \theta_F}\frac{\partial \theta_F}{\partial \delta}\right) = \overline{q_F}2\frac{\partial q_F}{\partial \delta} + \overline{q_F}2\frac{\partial q_F}{\partial \psi}\frac{\partial \psi}{\partial \delta}$$

$$= \begin{bmatrix} -(\mu + \cos\lambda)\sin\theta_F + \sin\lambda\cos\theta_F \\ 0 \\ (\mu + \cos\lambda)\cos\theta_F + \sin\lambda\sin\theta_F \end{bmatrix}$$

$$\left(\bar{q}2\frac{\partial q}{\partial r}\right)^T I\bar{q}2\frac{\partial q}{\partial r} = \begin{bmatrix} I_{Fxx} & \sin\lambda\, I_{Fxx} \\ \sin\lambda\, I_{Fxx} & I_{Fxx}((\mu + \cos\lambda)^2 + \sin^2\lambda) \end{bmatrix}$$

$$\frac{\partial}{\partial\phi}\left(\left(\bar{q}2\frac{\partial q}{\partial\theta_R}\right)^T I2\frac{\partial\bar{q}}{\partial\theta_R}2\frac{\partial q}{\partial\phi}\right)=0$$

$$\frac{\partial}{\partial\delta}\left(\left(\bar{q}2\frac{\partial q}{\partial\theta_R}\right)^T I2\frac{\partial\bar{q}}{\partial\theta_R}2\frac{\partial q}{\partial\phi}\right)=-\frac{r_R}{r_F}I_{Fyy}\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right)+\frac{r_R}{r_F}I_{Fxx}\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right)$$

$$\frac{\partial}{\partial\phi}\left(\left(\bar{q}2\frac{\partial q}{\partial\theta_R}\right)^T I2\frac{\partial\bar{q}}{\partial\theta_R}2\frac{\partial q}{\partial\delta}\right)=0$$

$$\frac{\partial}{\partial\delta}\left(\left(\bar{q}2\frac{\partial q}{\partial\theta_R}\right)^T I2\frac{\partial\bar{q}}{\partial\theta_R}2\frac{\partial q}{\partial\delta}\right)=\frac{r_R}{r_F}\sin\lambda\, I_{Fxx}\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right)-\sin\lambda\frac{r_R}{r_F}I_{Fyy}\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right)$$

$$2\frac{\partial\bar{q}}{\partial\theta_R}2\frac{\partial q}{\partial\phi}=2\left(\frac{\partial q_F}{\partial\theta_F}\frac{\partial\theta_F}{\partial\theta_R}\right)2\left(\frac{\partial q_F}{\partial\phi}\right)=\frac{r_R}{r_F}\begin{bmatrix}-\sin\theta_F\\0\\\cos\theta_F\end{bmatrix}$$

$$\begin{aligned}2\frac{\partial\bar{q}}{\partial\theta_R}2\frac{\partial q}{\partial\delta}&=2\left(\frac{\partial q_F}{\partial\theta_F}\frac{\partial\theta_F}{\partial\theta_R}\right)2\left(\frac{\partial q_F}{\partial\delta}+\frac{\partial q_F}{\partial\psi}\frac{\partial\psi}{\partial\delta}\right)\\&=\frac{r_R}{r_F}\begin{bmatrix}-(\mu+\cos\lambda)\cos\theta_F-\sin\lambda\sin\theta_F\\0\\-(\mu+\cos\lambda)\sin\theta_F+\sin\lambda\cos\theta_F\end{bmatrix}\end{aligned}$$

$$2\frac{\partial\bar{q}}{\partial\phi}2\frac{\partial q}{\partial\delta}=2\left(\frac{\partial q_F}{\partial\phi}\right)2\left(\frac{\partial q_F}{\partial\delta}+\frac{\partial q_F}{\partial\psi}\frac{\partial\psi}{\partial\delta}\right)=\begin{bmatrix}0\\\mu+\cos\lambda\\0\end{bmatrix}$$

$$2\frac{\partial\bar{q}}{\partial\delta}2\frac{\partial q}{\partial\phi}=-2\frac{\partial\bar{q}}{\partial\phi}2\frac{\partial q}{\partial\delta}=\begin{bmatrix}0\\-\mu-\cos\lambda\\0\end{bmatrix}$$

$$\bar{q}2\frac{\partial}{\partial\phi}\left(\frac{\partial q}{\partial\theta_R}\right)=\bar{q}2\frac{\partial}{\partial\phi}\left(\frac{\partial q_F}{\partial\psi}\frac{\partial\psi}{\partial\theta_R}+\frac{\partial q_F}{\partial\theta_F}\frac{\partial\theta_F}{\partial\theta_R}\right)=\bar{q}2\frac{\partial}{\partial\phi}\left(\frac{\partial q_F}{\partial\theta_F}\right)\frac{\partial\theta_F}{\partial\theta_R}=\frac{1}{2}\frac{r_R}{r_F}\begin{bmatrix}-\sin\theta_F\\0\\\cos\theta_F\end{bmatrix}$$

$$\bar{q}2\frac{\partial}{\partial\theta_R}\left(\frac{\partial q}{\partial\phi}\right)=\bar{q}2\frac{\partial}{\partial\theta_R}\left(\frac{\partial q_F}{\partial\phi}\right)=\bar{q}2\frac{\partial}{\partial\theta_F}\left(\frac{\partial q_F}{\partial\phi}\right)\frac{\partial\theta_F}{\partial\theta_R}=\frac{1}{2}\frac{r_R}{r_F}\begin{bmatrix}-\sin\theta_F\\0\\\cos\theta_F\end{bmatrix}$$

$$\begin{aligned}\bar{q}2\frac{\partial}{\partial\delta}\left(\frac{\partial q}{\partial\theta_R}\right)&=\bar{q}2\frac{\partial}{\partial\delta}\left(\frac{\partial q_F}{\partial\psi}\frac{\partial\psi}{\partial\theta_R}+\frac{\partial q_F}{\partial\theta_F}\frac{\partial\theta_F}{\partial\theta_R}\right)=\bar{q}2\frac{\partial q_F}{\partial\psi}\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right)+\bar{q}2\frac{\partial}{\partial\delta}\left(\frac{\partial q_F}{\partial\theta_F}\right)\frac{\partial\theta_F}{\partial\theta_R}\\&=\begin{bmatrix}-\sin\theta_F\\0\\\cos\theta_F\end{bmatrix}\left(-\frac{r_R}{w}\cos\lambda\right)+\frac{1}{2}\begin{bmatrix}-(\mu+\cos\lambda)\cos\theta_F-\sin\lambda\sin\theta_F\\0\\-(\mu+\cos\lambda)\sin\theta_F+\sin\lambda\cos\theta_F\end{bmatrix}\frac{r_R}{r_F}\end{aligned}$$

$$\bar{q}2\frac{\partial}{\partial\theta_R}\left(\frac{\partial q}{\partial\delta}\right)=\bar{q}2\frac{\partial}{\partial\theta_R}\left(\frac{\partial q_F}{\partial\delta}\right)+\bar{q}2\frac{\partial}{\partial\theta_R}\left(\frac{\partial q_F}{\partial\psi}\right)\frac{\partial\psi}{\partial\delta}$$

$$=\frac{1}{2}\begin{bmatrix}-(\mu+\cos\lambda)\cos\theta_F-\sin\lambda\sin\theta_F\\0\\-(\mu+\cos\lambda)\sin\theta_F+\sin\lambda\cos\theta_F\end{bmatrix}\frac{r_R}{r_F}$$

$$\bar{q}2\frac{\partial}{\partial\theta_R}\left(\frac{\partial q}{\partial\theta_R}\right)=R_y(-\theta_F)R_\lambda(-\delta)\begin{bmatrix}-\cos\phi\sin\theta_B\\\sin\phi\\\cos\phi\cos\theta_B\end{bmatrix}R_\lambda(\delta)R_y(\theta_F+\pi)\frac{\partial\psi}{\partial\theta_R}\frac{\partial\theta_F}{\partial\theta_R}=$$

$$=\begin{bmatrix}-\cos\theta_F\\0\\-\sin\theta_F\end{bmatrix}\frac{\partial\psi}{\partial\theta_R}\frac{\partial\theta_F}{\partial\theta_R}$$

$$\left(\bar{q}2\frac{\partial q}{\partial\phi}\right)^T I\bar{q}2\frac{\partial}{\partial\theta_R}\left(\frac{\partial q}{\partial\theta_R}\right)\dot{\theta_R}^{\,2}=\left(\bar{q}2\frac{\partial q}{\partial\phi}\right)^T I\frac{\partial}{\partial\delta}\left(\bar{q}2\frac{\partial}{\partial\theta_R}\left(\frac{\partial q}{\partial\theta_R}\right)\right)\delta\dot{\theta_R}^{\,2}$$

$$=\left(\bar{q}2\frac{\partial q}{\partial\phi}\right)^T I\begin{bmatrix}-\cos\theta_F\\0\\-\sin\theta_F\end{bmatrix}\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right)\frac{\partial\theta_F}{\partial\theta_R}\delta\dot{\theta_R}^{\,2}$$

$$=-I_{Fxx}\left(-\frac{r_R}{w}\cos\lambda\right)\frac{r_R}{r_F}\delta\dot{\theta_R}^{\,2}$$

$$\left(\bar{q}2\frac{\partial q}{\partial\delta}\right)^T I\bar{q}2\frac{\partial}{\partial\theta_R}\left(\frac{\partial q}{\partial\theta_R}\right)\dot{\theta_R}^{\,2}=\left(\bar{q}2\frac{\partial q}{\partial\delta}\right)^T I\frac{\partial}{\partial\delta}\left(\bar{q}2\frac{\partial}{\partial\theta_R}\left(\frac{\partial q}{\partial\theta_R}\right)\right)\delta\dot{\theta_R}^{\,2}$$

$$=\left(\bar{q}2\frac{\partial q}{\partial\delta}\right)^T I\begin{bmatrix}-\cos\theta_F\\0\\-\sin\theta_F\end{bmatrix}\frac{\partial}{\partial\delta}\left(\frac{\partial\psi}{\partial\theta_R}\right)\frac{\partial\theta_F}{\partial\theta_R}\delta\dot{\theta_R}^{\,2}$$

$$=-\sin\lambda\, I_{Fxx}\left(-\frac{r_R}{w}\cos\lambda\right)\frac{r_R}{r_F}\delta\dot{\theta_R}^{\,2}$$

$$Q_\phi=\left(\bar{q}2\frac{\partial q}{\partial\phi}\right)^T I\bar{q}2\frac{\partial q}{\partial r}\ddot{r}+\frac{r_R}{r_F}I_{Fyy}\frac{r_R}{w}\cos\lambda\,\delta\dot{\theta_R}^{\,2}-I_{Fyy}(\mu+\cos\lambda)\frac{r_R}{r_F}\dot{\delta}\dot{\theta_R}$$

$$Q_\delta=\left(\bar{q}2\frac{\partial q}{\partial\delta}\right)^T I\bar{q}2\frac{\partial q}{\partial r}\ddot{r}+\sin\lambda\frac{r_R}{r_F}I_{Fyy}\frac{r_R}{w}\cos\lambda\,\delta\dot{\theta_R}^{\,2}+I_{Fyy}(\mu+\cos\lambda)\frac{r_R}{r_F}\dot{\phi}\dot{\theta_R}$$

$$+(\mu+\cos\lambda)I_{Fxx}\left(-\frac{r_R}{w}\cos\lambda\right)\dot{\delta}\dot{\theta_R}$$

$$\begin{bmatrix} Q_\phi \\ Q_\delta \end{bmatrix} = \begin{bmatrix} I_{Fxx} & \sin\lambda\, I_{Fxx} \\ \sin\lambda\, I_{Fxx} & I_{Fxx}((\mu + \cos\lambda)^2 + \sin^2\lambda) \end{bmatrix} \begin{bmatrix} \ddot{\phi} \\ \ddot{\delta} \end{bmatrix}$$

$$+ \dot{\theta}_R{}^2 r_R^2 \begin{bmatrix} 0 & \dfrac{1}{r_F} \dfrac{\cos\lambda}{w} I_{Fyy} \\ 0 & \dfrac{\sin\lambda}{r_F} \dfrac{\cos\lambda}{w} I_{Fyy} \end{bmatrix} \begin{bmatrix} \phi \\ \delta \end{bmatrix}$$

$$- \dot{\theta}_R r_R \begin{bmatrix} 0 & \dfrac{(\mu + \cos\lambda)}{r_F} I_{Fyy} \\ -\dfrac{(\mu + \cos\lambda)}{r_F} I_{Fyy} & \cos\lambda \dfrac{(\mu + \cos\lambda)}{w} I_{Fxx} \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\delta} \end{bmatrix}$$

Gravitational potential energy:

$$m_4 = m_R + m_B + m_H + m_F$$

$$m_3 = m_B + m_H + m_F$$

$$m_2 = m_H + m_F$$

Because gravity is facing toward $+z$ direction instead of $-z$ direction

$$PE = -gm_R R_z - gm_B B_z - gm_H H_z - gm_F F_z$$

$$= -gm_4 (PR)_z - gm_3 (RBS)_z - gm_2 (SHF)_z$$

$$= gm_4 r_R \cos\phi - gm_3 \cos\phi\,(-x_{BS}\sin\theta_B + z_{BS}\cos\theta_B) - gm_2\big(-\cos\phi\sin\theta_B\, v_{HFx} + \sin\phi\, v_{HFy} + \cos\phi\cos\theta_B\, v_{HFz}\big)$$

$$= gm_4 r_R \cos\phi - gm_3 \cos\phi\,(-x_{BS}\sin\theta_B + z_{BS}\cos\theta_B) - gm_2[-\cos\phi\sin\theta_B \quad \sin\phi \quad \cos\phi\cos\theta_B] v_{HF}$$

$$v_{HF} = \begin{bmatrix} \cos\delta\, x_{HF} + 2\sin^2\frac{\delta}{2}\sin\lambda\,(\sin\lambda\, x_{HF} + \cos\lambda\, z_{HF}) \\ \sin\delta\,(\cos\lambda\, x_{HF} - \sin\lambda\, z_{HF}) \\ \cos\delta\, z_{HF} + 2\sin^2\frac{\delta}{2}\cos\lambda\,(\sin\lambda\, x_{HF} + \cos\lambda\, z_{HF}) \end{bmatrix}$$

$$\frac{\partial v_{HF}}{\partial\delta} = \begin{bmatrix} -\sin\delta\, x_{HF} + \sin\delta\sin\lambda\,(\sin\lambda\, x_{HF} + \cos\lambda\, z_{HF}) \\ \cos\delta\,(\cos\lambda\, x_{HF} - \sin\lambda\, z_{HF}) \\ -\sin\delta\, z_{HF} + \sin\delta\cos\lambda\,(\sin\lambda\, x_{HF} + \cos\lambda\, z_{HF}) \end{bmatrix}$$

$$\frac{\partial^2 v_{HF}}{\partial\delta^2} = \begin{bmatrix} -\cos\delta\, x_{HF} + \cos\delta\sin\lambda\,(\sin\lambda\, x_{HF} + \cos\lambda\, z_{HF}) \\ -\sin\delta\,(\cos\lambda\, x_{HF} - \sin\lambda\, z_{HF}) \\ -\cos\delta\, z_{HF} + \cos\delta\cos\lambda\,(\sin\lambda\, x_{HF} + \cos\lambda\, z_{HF}) \end{bmatrix}$$

$$\frac{\partial PE}{\partial\phi} = -gm_4 r_R \sin\phi + gm_3 \sin\phi\,(-x_{BS}\sin\theta_B + z_{BS}\cos\theta_B) - gm_2[\sin\phi\sin\theta_B \quad \cos\phi \quad -\sin\phi\cos\theta_B] v_{HF} = 0$$

$$\frac{\partial PE}{\partial\delta} = -gm_2[-\cos\phi\sin\theta_B \quad \sin\phi \quad \cos\phi\cos\theta_B]\frac{\partial v_{HF}}{\partial\delta} = 0$$

$$\frac{\partial PE}{\partial\theta_B} = gm_3\cos\phi\,(x_{BS}\cos\theta_B + z_{BS}\sin\theta_B) + gm_2[\cos\phi\cos\theta_B \quad 0 \quad \cos\phi\sin\theta_B] v_{HF}$$

$$= gm_3 x_{BS} + gm_2 x_{HF}$$

$$\frac{\partial}{\partial \phi}\left(\frac{\partial PE}{\partial \theta_B}\right) = -gm_3 \sin\phi\,(x_{BS}\cos\theta_B + z_{BS}\sin\theta_B)$$

$$+ gm_2[\sin\phi\cos\theta_B \quad 0 \quad \sin\phi\sin\theta_B]v_{HF} = 0$$

$$\frac{\partial}{\partial \delta}\left(\frac{\partial PE}{\partial \theta_B}\right) = gm_2[\cos\phi\cos\theta_B \quad 0 \quad \cos\phi\sin\theta_B]\frac{\partial v_{HF}}{\partial \delta} = 0$$

$$\frac{\partial}{\partial \phi}\left(\frac{\partial PE}{\partial \phi}\right) = -gm_4 r_R \cos\phi + gm_3\cos\phi\,(-x_{BS}\sin\theta_B + z_{BS}\cos\theta_B)$$

$$- gm_2[\cos\phi\sin\theta_B \quad -\sin\phi \quad -\cos\phi\cos\theta_B]v_{HF}$$

$$= -gm_4 r_R + gm_3 z_{BS} + gm_2 z_{HF}$$

$$\frac{\partial}{\partial \delta}\left(\frac{\partial PE}{\partial \phi}\right) = \frac{\partial}{\partial \phi}\left(\frac{\partial PE}{\partial \delta}\right) = -gm_2[\sin\phi\sin\theta_B \quad \cos\phi \quad -\sin\phi\cos\theta_B]\frac{\partial v_{HF}}{\partial \delta}$$

$$= gm_2(\sin\lambda\, z_{HF} - \cos\lambda\, x_{HF})$$

$$\frac{\partial}{\partial \delta}\left(\frac{\partial PE}{\partial \delta}\right) = -gm_2[-\cos\phi\sin\theta_B \quad \sin\phi \quad \cos\phi\cos\theta_B]\frac{\partial^2 v_{HF}}{\partial \delta^2}$$

$$= gm_2\sin\lambda\,(sin\,\lambda\, z_{HF} - \cos\lambda\, x_{HF})$$

$$-Q_\phi = \frac{\partial PE}{\partial \phi} = \frac{\partial}{\partial \phi}\left(\frac{\partial PE}{\partial \phi}\right)\phi + \frac{\partial}{\partial \delta}\left(\frac{\partial PE}{\partial \phi}\right)\delta$$

$$= \frac{\partial}{\partial \phi}\left(\frac{\partial PE}{\partial \phi} + \frac{\partial PE}{\partial \theta_B}\frac{\partial \theta_B}{\partial \phi}\right)\phi + \frac{\partial}{\partial \delta}\left(\frac{\partial PE}{\partial \phi} + \frac{\partial PE}{\partial \theta_B}\frac{\partial \theta_B}{\partial \phi}\right)\delta$$

$$= \left(\frac{\partial}{\partial \phi}\left(\frac{\partial PE}{\partial \phi}\right) + \frac{\partial}{\partial \phi}\left(\frac{\partial PE}{\partial \theta_B}\right)\frac{\partial \theta_B}{\partial \phi} + \frac{\partial PE}{\partial \theta_B}\frac{\partial}{\partial \phi}\left(\frac{\partial \theta_B}{\partial \phi}\right)\right)\phi$$

$$+ \left(\frac{\partial}{\partial \delta}\left(\frac{\partial PE}{\partial \phi}\right) + \frac{\partial}{\partial \delta}\left(\frac{\partial PE}{\partial \theta_B}\right)\frac{\partial \theta_B}{\partial \phi} + \frac{\partial PE}{\partial \theta_B}\frac{\partial}{\partial \delta}\left(\frac{\partial \theta_B}{\partial \phi}\right)\right)\delta$$

$$= \left(-gm_4 r_R + gm_3 z_{BS} + gm_2 z_{HF} + (gm_3 x_{BS} + gm_2 x_{HF})\frac{\partial}{\partial \phi}\left(\frac{\partial \theta_B}{\partial \phi}\right)\right)\phi$$

$$+\left(gm_2(\sin\lambda\, z_{HF}-\cos\lambda\, x_{HF})+(gm_3x_{BS}+gm_2x_{HF})\frac{\partial}{\partial\delta}\left(\frac{\partial\theta_B}{\partial\phi}\right)\right)\delta$$

$$-Q_\delta=\frac{\partial PE}{\partial\delta}=\frac{\partial}{\partial\phi}\left(\frac{\partial PE}{\partial\delta}\right)\phi+\frac{\partial}{\partial\delta}\left(\frac{\partial PE}{\partial\delta}\right)\delta$$

$$=\frac{\partial}{\partial\phi}\left(\frac{\partial PE}{\partial\delta}+\frac{\partial PE}{\partial\theta_B}\frac{\partial\theta_B}{\partial\delta}\right)\phi+\frac{\partial}{\partial\delta}\left(\frac{\partial PE}{\partial\delta}+\frac{\partial PE}{\partial\theta_B}\frac{\partial\theta_B}{\partial\delta}\right)\delta$$

$$=\left(\frac{\partial}{\partial\phi}\left(\frac{\partial PE}{\partial\delta}\right)+\frac{\partial}{\partial\phi}\left(\frac{\partial PE}{\partial\theta_B}\right)\frac{\partial\theta_B}{\partial\delta}+\frac{\partial PE}{\partial\theta_B}\frac{\partial}{\partial\phi}\left(\frac{\partial\theta_B}{\partial\delta}\right)\right)\phi$$

$$+\left(\frac{\partial}{\partial\delta}\left(\frac{\partial PE}{\partial\delta}\right)+\frac{\partial}{\partial\delta}\left(\frac{\partial PE}{\partial\theta_B}\right)\frac{\partial\theta_B}{\partial\delta}+\frac{\partial PE}{\partial\theta_B}\frac{\partial}{\partial\delta}\left(\frac{\partial\theta_B}{\partial\delta}\right)\right)\delta$$

$$=\left(gm_2(\sin\lambda\, z_{HF}-\cos\lambda\, x_{HF})+(gm_3x_{BS}+gm_2x_{HF})\frac{\partial}{\partial\phi}\left(\frac{\partial\theta_B}{\partial\delta}\right)\right)\phi$$

$$+\left(gm_2\sin\lambda\,(sin\,\lambda\, z_{HF}-\cos\lambda\, x_{HF})+(gm_3x_{BS}+gm_2x_{HF})\frac{\partial}{\partial\delta}\left(\frac{\partial\theta_B}{\partial\delta}\right)\right)\delta$$

$$-\begin{bmatrix}Q_\phi\\Q_\delta\end{bmatrix}=\begin{bmatrix}-gm_4r_R+gm_3z_{BS}+gm_2z_{HF} & gm_2(\sin\lambda\, z_{HF}-\cos\lambda\, x_{HF})\\ gm_2(\sin\lambda\, z_{HF}-\cos\lambda\, x_{HF}) & gm_2\sin\lambda\,(sin\,\lambda\, z_{HF}-\cos\lambda\, x_{HF})\end{bmatrix}\begin{bmatrix}\phi\\\delta\end{bmatrix}$$

$$+(gm_3x_{BS}+gm_2x_{HF})\begin{bmatrix}\frac{\partial}{\partial\phi}\left(\frac{\partial\theta_B}{\partial\phi}\right) & \frac{\partial}{\partial\delta}\left(\frac{\partial\theta_B}{\partial\phi}\right)\\ \frac{\partial}{\partial\phi}\left(\frac{\partial\theta_B}{\partial\delta}\right) & \frac{\partial}{\partial\delta}\left(\frac{\partial\theta_B}{\partial\delta}\right)\end{bmatrix}\begin{bmatrix}\phi\\\delta\end{bmatrix}$$

$$-\begin{bmatrix}Q_\phi\\Q_\delta\end{bmatrix}=g\begin{bmatrix}-m_4r_R+m_3z_{BS}+m_2z_{HF} & m_2(\sin\lambda\, z_{HF}-\cos\lambda\, x_{HF})\\ m_2(\sin\lambda\, z_{HF}-\cos\lambda\, x_{HF}) & m_2\sin\lambda\,(sin\,\lambda\, z_{HF}-\cos\lambda\, x_{HF})\end{bmatrix}\begin{bmatrix}\phi\\\delta\end{bmatrix}$$

$$+g\frac{(m_3x_{BS}+m_2x_{HF})}{w}\begin{bmatrix}0 & -c\cos\lambda\\ -c\cos\lambda & -c\sin\lambda\, cos\ \lambda\end{bmatrix}\begin{bmatrix}\phi\\ \delta\end{bmatrix}$$

$$-\begin{bmatrix}Q_\phi\\ Q_\delta\end{bmatrix}=g\begin{bmatrix}-m_Rr_R+m_Bz_B+m_Hz_H-m_Fr_F & -S_A\\ -S_A & -S_A\sin\lambda\end{bmatrix}\begin{bmatrix}\phi\\ \delta\end{bmatrix}$$

$$S_A=-(m_Hz_H-m_Fr_F)\sin\lambda$$

$$+(m_H(x_H-w-c)-m_Fc)\cos\lambda$$

$$+\frac{c}{w}\cos\lambda\,(m_Bx_B+m_Hx_H+m_Fw)$$

Combine everything

$$v = -r_R\dot{\theta}_R, q = \begin{bmatrix}\phi\\ \delta\end{bmatrix}$$

$$M\begin{bmatrix}\ddot{\phi}\\ \ddot{\delta}\end{bmatrix} + vC_1\begin{bmatrix}\dot{\phi}\\ \dot{\delta}\end{bmatrix} + (v^2K_2 + gK_0)\begin{bmatrix}\phi\\ \delta\end{bmatrix} = \tau\begin{bmatrix}0\\ 1\end{bmatrix}$$

$$\begin{bmatrix}\dot{\phi}\\ \dot{\delta}\\ \ddot{\phi}\\ \ddot{\delta}\end{bmatrix} = \begin{bmatrix}\begin{matrix}0 & 0\\ 0 & 0\end{matrix} & \begin{matrix}1 & 0\\ 0 & 1\end{matrix}\\ -M^{-1}(v^2K_2 + gK_0) & -M^{-1}vC_1\end{bmatrix}\begin{bmatrix}\phi\\ \delta\\ \dot{\phi}\\ \dot{\delta}\end{bmatrix} + \tau\begin{bmatrix}0\\ 0\\ M^{-1}\begin{bmatrix}0\\ 1\end{bmatrix}\end{bmatrix}$$

From literature

$$v = -r_R \dot{\theta}_R, q = \begin{bmatrix} \phi \\ \delta \end{bmatrix}$$

$$M\ddot{q} + vC_1\dot{q} + v^2 K_2 q + gK_0 q = 0$$

$$\mu = \frac{c}{w}\cos\lambda$$

$$m_T x_T = m_B x_B + m_H x_H + m_F w$$

$$m_T z_T = -m_R r_R + m_B z_B + m_H z_H - m_F r_F$$

$$m_A = m_H + m_F, x_A = \frac{m_H x_H + m_F w}{m_A}, z_A = \frac{m_H z_H - m_F r_F}{m_A}$$

$$\mu_A = (x_A - w - c)\cos\lambda - z_A \sin\lambda$$

$$S_A = (m_H + m_F)\mu_A + \mu m_T x_T$$

$$S_T = S_R + S_F = \frac{I_{Ryy}}{r_R} + \frac{I_{Fyy}}{r_F}$$

$$K_0 = \begin{bmatrix} m_T z_T & -S_A \\ -S_A & -S_A \sin\lambda \end{bmatrix}$$

$$K_2 = \frac{\cos\lambda}{w}\begin{bmatrix} 0 & S_T - m_T z_T \\ 0 & S_A + S_F \sin\lambda \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & \mu S_T + S_F \cos\lambda + \frac{\cos\lambda}{w} I_{Txz} - \mu m_T z_T \\ -\mu S_T - S_F \cos\lambda & \frac{\cos\lambda}{w} I_{A\lambda z} + \mu S_A + \mu \frac{\cos\lambda}{w} I_{Tzz} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \mu S_T + S_F \cos\lambda \\ -\mu S_T - S_F \cos\lambda & 0 \end{bmatrix}$$

$$+ \frac{\cos\lambda}{w}\begin{bmatrix} 0 & I_{Txz} - c m_T z_T \\ 0 & I_{A\lambda z} + \mu I_{Tzz} + c S_A \end{bmatrix}$$

$$M = \begin{bmatrix} I_{Txx} & I_{A\lambda x} + \mu I_{Txz} \\ I_{A\lambda x} + \mu I_{Txz} & I_{A\lambda\lambda} + 2\mu I_{A\lambda z} + \mu^2 I_{Tzz} \end{bmatrix}$$

$$I_{Txx} = m_R r_R^2 + m_B z_B^2 + m_H z_H^2 + m_F r_F^2 + I_{Rxx} + I_{Bxx} + I_{Hxx} + I_{Fxx}$$

$$I_{Tzz} = m_B x_B^2 + m_H x_H^2 + m_F w^2 + I_{Rzz} + I_{Bzz} + I_{Hzz} + I_{Fzz}$$

$$I_{Txz} = -m_B x_B z_B - m_H x_H z_H + m_F r_F w + I_{Bxz} + I_{Hxz}$$

$$I_{Axx} = m_H (z_H - z_A)^2 + m_F (r_F + z_A)^2 + I_{Hxx} + I_{Fxx}$$

$$I_{Azz} = m_H (x_H - x_A)^2 + m_F (w - x_A)^2 + I_{Hzz} + I_{Fzz}$$

$$I_{Axz} = -m_H (x_H - x_A)(z_H - z_A) + m_F (w - x_A)(r_F + z_A) + I_{Hxz}$$

$$I_{A\lambda\lambda} = m_A \mu_A^2 + I_{Axx} \sin^2 \lambda + 2I_{Axz} \sin \lambda \cos \lambda + I_{Azz} \cos^2 \lambda$$

$$I_{A\lambda x} = -m_A \mu_A z_A + I_{Axx} \sin \lambda + I_{Axz} \cos \lambda$$

$$I_{A\lambda z} = m_A \mu_A x_A + I_{Axz} \sin \lambda + I_{Azz} \cos \lambda$$

B4:

$$\dot{y_P} = v\psi$$

B6:

$$\dot{\psi} = \frac{\cos \lambda}{w} v\delta + \mu\dot{\delta}, \ddot{\psi} = \frac{\cos \lambda}{w} v\dot{\delta} + \mu\ddot{\delta}$$

B7:

$$\ddot{y_P} = \frac{\cos \lambda}{w} v^2 \delta + \mu v \dot{\delta}$$

B1:

$$\begin{gathered} -m_T \ddot{y_P} z_T + I_{Txx} \ddot{\phi} + I_{Txz} \ddot{\psi} + I_{A\lambda x} \ddot{\delta} + \dot{\psi} v S_T + \dot{\delta} v S_F \cos \lambda \\ = T_{B\phi} - g m_T z_T \phi + g S_A \delta \end{gathered}$$

$$\begin{gathered} \Rightarrow T_{B\phi} = I_{Txx} \ddot{\phi} + (\mu I_{Txz} + I_{A\lambda x}) \ddot{\delta} + v \left(\mu S_T + S_F \cos \lambda - m_T z_T \mu + \frac{\cos \lambda}{w} I_{Txz} \right) \dot{\delta} \\ + v^2 \frac{\cos \lambda}{w} (S_T - m_T z_T) \delta + g m_T z_T \phi - g S_A \delta \end{gathered}$$

B2:

$$m_T \ddot{y_P} x_T + I_{Txz} \ddot{\phi} + I_{Tzz} \ddot{\psi} + I_{A\lambda z} \ddot{\delta} - \dot{\phi} v S_T - \dot{\delta} v S_F \sin \lambda = w F_{Fy}$$

B3:

$$m_A \ddot{y}_P \mu_A + I_{A\lambda x}\ddot{\phi} + I_{A\lambda z}\ddot{\psi} + I_{A\lambda\lambda}\ddot{\delta} + vS_F\left(-\dot{\phi}\cos\lambda + \dot{\psi}\sin\lambda\right)$$
$$= T_{H\delta} - cF_{Fy}\cos\lambda + g(\phi + \delta\sin\lambda)S_A$$

B2+B3:

$$T_{H\delta} = (I_{A\lambda x} + \mu I_{Txz})\ddot{\phi} + (I_{A\lambda\lambda} + \mu I_{A\lambda z} + \mu^2 I_{Tzz} + \mu I_{A\lambda z})\ddot{\delta} + v(-S_F\cos\lambda - \mu S_T)\dot{\phi}$$
$$+ v\left(\mu^2 m_T x_T + \mu m_A \mu_A + \mu\frac{\cos\lambda}{w}I_{Tzz} + \frac{\cos\lambda}{w}I_{A\lambda z}\right)\dot{\delta}$$
$$+ v^2\frac{\cos\lambda}{w}(\mu m_T x_T + m_A\mu_A + S_F\sin\lambda)\delta - g(\phi + \delta\sin\lambda)S_A$$

The parameters provided in the article are

For point R

$$r_R = 0.3, m_R = 2$$

$$I_R = \begin{bmatrix} I_{Rxx} & 0 & 0 \\ 0 & I_{Ryy} & 0 \\ 0 & 0 & I_{Rxx} \end{bmatrix} = \begin{bmatrix} 0.0603 & 0 & 0 \\ 0 & 0.12 & 0 \\ 0 & 0 & 0.603 \end{bmatrix}$$

I_R can be model as an annulus, $I_{Ryy} = \frac{1}{2}m * (r_R^2 + r_{R2}^2), I_{Rxx} = \frac{1}{4}m * (r_R^2 + r_{R2}^2)$

The inner radius of annulus $r_{R2} = 0.1732$

For point B

$$(x_B, z_B) = (0.3, -0.9), m_B = 85$$

$$I_B = \begin{bmatrix} I_{Bxx} & 0 & I_{Bxz} \\ 0 & I_{Byy} & 0 \\ I_{Bxz} & 0 & I_{Bzz} \end{bmatrix} = \begin{bmatrix} 9.2 & 0 & 2.4 \\ 0 & 11 & 0 \\ 2.4 & 0 & 2.8 \end{bmatrix}$$

For point H

$$(x_H, z_H) = (0.9, -0.7), m_H = 4$$

$$I_H = \begin{bmatrix} I_{Hxx} & 0 & I_{Hxz} \\ 0 & I_{Hyy} & 0 \\ I_{Hxz} & 0 & I_{Hzz} \end{bmatrix} = \begin{bmatrix} 0.05892 & 0 & -0.00756 \\ 0 & 0.06 & 0 \\ -0.00756 & 0 & 0.00708 \end{bmatrix}$$

$$= R_y \begin{bmatrix} 0.06 & 0 & 0 \\ 0 & 0.06 & 0 \\ 0 & 0 & 0.006 \end{bmatrix} R_y^{-1}$$

$$R_y = \begin{bmatrix} \frac{7}{\sqrt{50}} & 0 & \frac{1}{\sqrt{50}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{50}} & 0 & \frac{7}{\sqrt{50}} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \theta = \sin^{-1}\frac{1}{\sqrt{50}} = 8.1301°$$

The principal moment of inertia can be modeled as a solid cylinder,

$$0.006 = \frac{1}{2}mr^2, 0.06 = \frac{1}{12}m(3r^2 + h^2) = \frac{1}{4}mr^2 + \frac{1}{12}mh^2$$

$$r = 0.0547, h = 0.4135$$

The equivalent height is 41cm which is a little bit short.

For point F

$$r_F = 0.35, m_F = 3$$

$$I_F = \begin{bmatrix} I_{Fxx} & 0 & 0 \\ 0 & I_{Fyy} & 0 \\ 0 & 0 & I_{Fxx} \end{bmatrix} = \begin{bmatrix} 0.1405 & 0 & 0 \\ 0 & 0.28 & 0 \\ 0 & 0 & 0.1405 \end{bmatrix}$$

I_F can be model as an annulus, $I_{Fyy} = \frac{1}{2}m * (r_F^2 + r_{F2}^2), I_{Fxx} = \frac{1}{4}m * (r_F^2 + r_{F2}^2)$

The inner radius of annulus $r_{F2} = 0.2533$