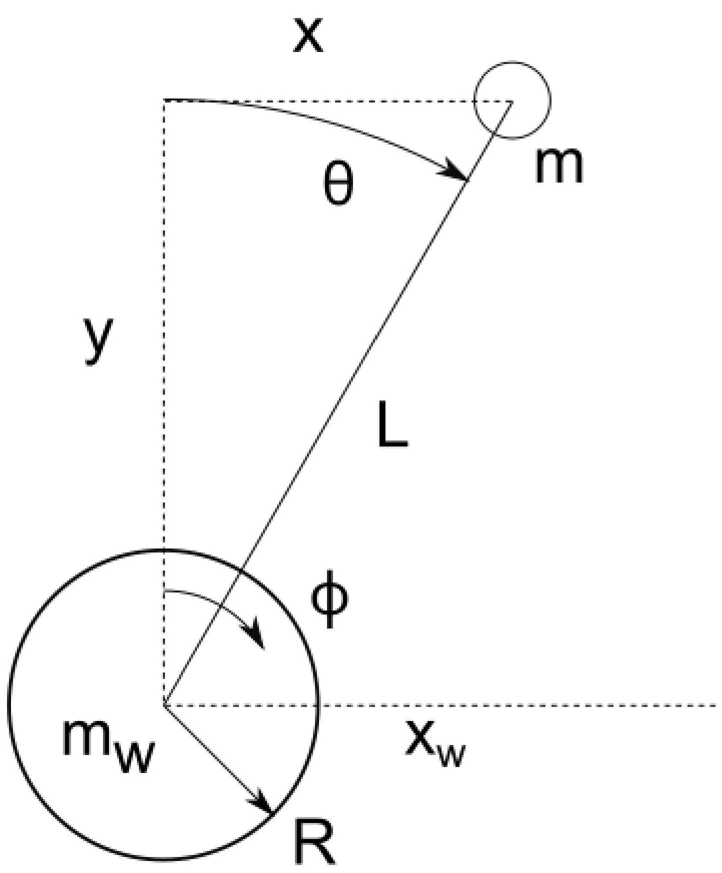
Inverted pendulum on wheel



x_w: horizontal position of the center of wheel relative to a defined origin
x: horizontal position of the center of pendulum relative to a defined origin
y: vertical position of the center of pendulum relative to a defined origin
phi: clockwise rotational angle of the wheel from +y axis
theta: clockwise rotational angle of the pendulum from +y axis
tau: clockwise torque applied to wheel from pendulum
m: mass of the pendulum part
m_w: mass of the wheel part
R: radius of the wheel
L: length between center of pendulum and center of wheel

I: inertia of the pendulum

I_w: intertia of wheel

Br: friction between wheel and floor

Bm: friction between wheel and pendulum

$$x_w = R\phi \ \Rightarrow \dot{x}_w = R\dot{\phi}$$

$$x = R\phi + L\sin\theta \ \Rightarrow \ \dot{x} = R\dot{\phi} + L\dot{\theta}\cos\theta$$

$$y = L\cos\theta \ \Rightarrow \ \dot{y} = -L\dot{\theta}\sin\theta$$

$$EK = \frac{1}{2}I_w\dot{\phi}^2 + \frac{1}{2}m_w\dot{x}_w^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2$$

$$= \frac{1}{2}(I_w + (m_w + m)R^2)\dot{\phi}^2 + mRL\cos\theta\,\dot{\theta}\dot{\phi} + \frac{1}{2}(I + mL^2)\dot{\theta}^2$$

$$PK = mgL\cos\theta$$

$$L = \frac{1}{2}(I_w + (m_w + m)R^2)\dot{\phi}^2 + mRL\cos\theta\,\dot{\theta}\dot{\phi} + \frac{1}{2}(I + mL^2)\dot{\theta}^2 - mgL\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = (I_w + (m_w + m)R^2)\ddot{\phi} + mRL\cos\theta\,\ddot{\theta} - mRL\sin\theta\,\dot{\theta}^2 = \mu$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = (I + mL^2)\ddot{\theta} + mRL\cos\theta\,\ddot{\phi} - mgL\sin\theta = \chi$$

$$\mu = \tau - \beta_m(\dot{\phi} - \dot{\theta}) - \beta_r\dot{\phi}$$

$$\chi = -\tau + \beta_m(\dot{\phi} - \dot{\theta})$$

$$\begin{bmatrix} I_w + (m_w + m)R^2 & mRL\cos\theta \\ mRL\cos\theta & I + mL^2 \end{bmatrix}\begin{bmatrix}\ddot{\phi} \\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix} \tau - \beta_m(\dot{\phi} - \dot{\theta}) - \beta_r\dot{\phi} + mRL\sin\theta\,\dot{\theta}^2 \\ -\tau + \beta_m(\dot{\phi} - \dot{\theta}) + mgL\sin\theta \end{bmatrix}$$

$$G\begin{bmatrix}\ddot{\phi} \\ \ddot{\theta}\end{bmatrix} = H$$

$$\begin{bmatrix}\ddot{\phi} \\ \ddot{\theta}\end{bmatrix} = G^{-1}H$$

linearization at $\theta = 0, \dot{\theta} = 0, \dot{\phi} = 0, \tau = 0, H = 0$

$$\frac{\partial}{\partial\theta}\begin{bmatrix}\ddot{\phi} \\ \ddot{\theta}\end{bmatrix} = G^{-1}\frac{\partial}{\partial\theta}H = G^{-1}\begin{bmatrix}0 \\ mgL\end{bmatrix}$$

$$\frac{\partial}{\partial\dot{\theta}}\begin{bmatrix}\ddot{\phi} \\ \ddot{\theta}\end{bmatrix} = G^{-1}\frac{\partial}{\partial\dot{\theta}}H = G^{-1}\begin{bmatrix}\beta_m \\ -\beta_m\end{bmatrix}$$

$$\frac{\partial}{\partial\phi}\begin{bmatrix}\ddot{\phi} \\ \ddot{\theta}\end{bmatrix} = G^{-1}\frac{\partial}{\partial\phi}H = G^{-1}\begin{bmatrix}0 \\ 0\end{bmatrix}$$

$$\frac{\partial}{\partial\dot{\phi}}\begin{bmatrix}\ddot{\phi} \\ \ddot{\theta}\end{bmatrix} = G^{-1}\frac{\partial}{\partial\dot{\phi}}H = G^{-1}\begin{bmatrix}-\beta_m - \beta_r \\ \beta_m\end{bmatrix}$$

$$\frac{\partial}{\partial\tau}\begin{bmatrix}\ddot{\phi}\\ \ddot{\theta}\end{bmatrix} = G^{-1}\frac{\partial}{\partial\tau}H = G^{-1}\begin{bmatrix}1\\ -1\end{bmatrix}$$

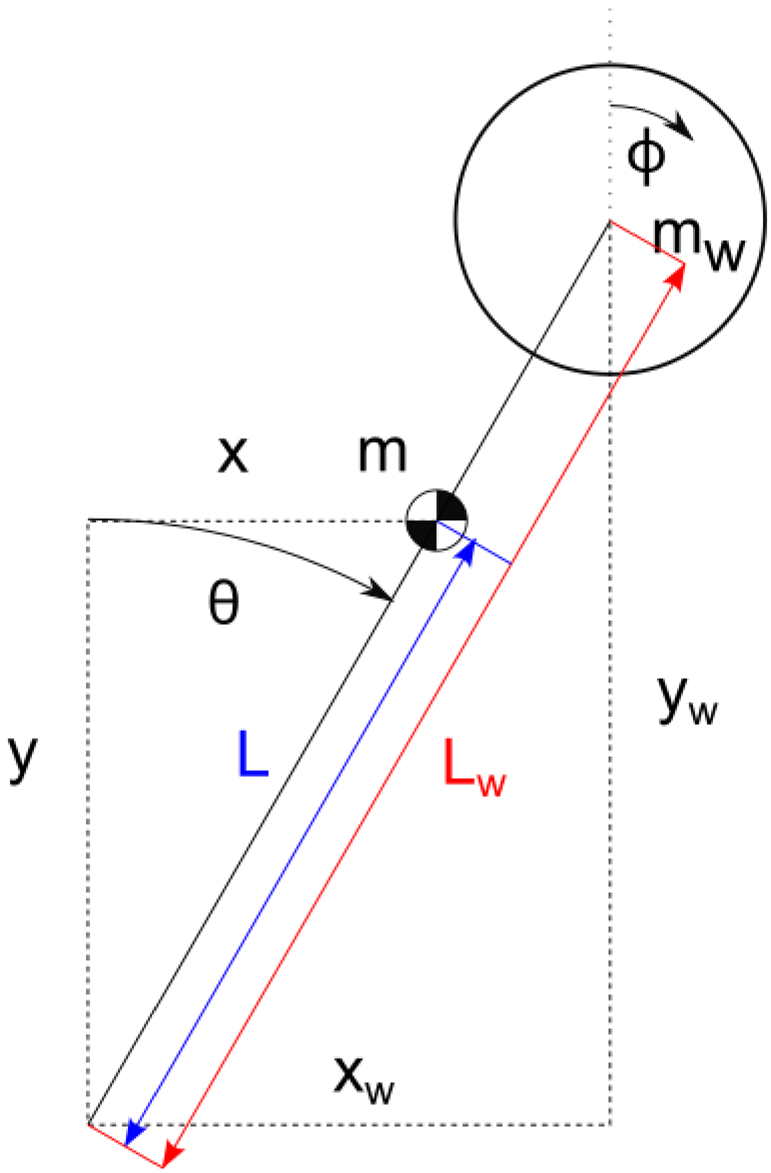
$$\begin{bmatrix}\ddot{\phi}\\ \ddot{\theta}\end{bmatrix} = G^{-1}\begin{bmatrix}0 & 0 & -\mathrm{\beta_m} - \beta_r & \mathrm{\beta_m}\\ 0 & mgL & \mathrm{\beta_m} & -\mathrm{\beta_m}\end{bmatrix}\begin{bmatrix}\phi\\ \theta\\ \dot{\phi}\\ \dot{\theta}\end{bmatrix} + G^{-1}\begin{bmatrix}1\\ -1\end{bmatrix}\tau$$

$$= G^{-1}\begin{bmatrix}0 & -\mathrm{\beta_m} - \beta_r & \mathrm{\beta_m}\\ mgL & \mathrm{\beta_m} & -\mathrm{\beta_m}\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\phi}\\ \dot{\theta}\end{bmatrix} + G^{-1}\begin{bmatrix}1\\ -1\end{bmatrix}\tau$$

$$= G^{-1}F\begin{bmatrix}\theta\\ \dot{\phi}\\ \dot{\theta}\end{bmatrix} + G^{-1}\begin{bmatrix}1\\ -1\end{bmatrix}\tau$$

$$\begin{bmatrix}\dot{\theta}\\ \ddot{\phi}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}0 \quad 0 \quad 1\\ G^{-1}F\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\phi}\\ \dot{\theta}\end{bmatrix} + \begin{bmatrix}0\\ G^{-1}\begin{bmatrix}1\\ -1\end{bmatrix}\end{bmatrix}\tau$$

Reaction wheel on inverted pendulum



x_w: horizontal position of the center of wheel relative to a defined origin

y_w: vertical position of the center of wheel relative to a defined origin

x: horizontal position of the center of pendulum relative to a defined origin

y: vertical position of the center of pendulum relative to a defined origin

phi: clockwise rotational angle of the wheel from +y axis

theta: clockwise rotational angle of the pendulum from +y axis

tau: clockwise torque applied to wheel from pendulum

m: mass of the pendulum part

m_w: mass of the wheel part

L: length between center of pendulum and pivot point

L_w: length between center of wheel and pivot point

I: inertia of the pendulum

I_w: intertia of wheel

Ba: friction between pendulum and floor

Bm: friction between wheel and pendulum

$$\mathrm{x_w} = \mathrm{L_w}\sin\theta \quad \Rightarrow \dot{x}_W \ = \mathrm{L}_w\dot{\theta}\cos\theta$$
$$\mathrm{y_w} = \mathrm{L_w}\cos\theta \quad \Rightarrow \dot{y}_W \ = -\mathrm{L_w}\dot{\theta}\sin\theta$$
$$\mathrm{x} = \mathrm{L}\sin\theta \ \Rightarrow \ \dot{x} = \mathrm{L}\dot{\theta}\cos\theta$$
$$\mathrm{y} = \mathrm{L}\cos\theta \ \Rightarrow \ \dot{y} = -\mathrm{L}\dot{\theta}\sin\theta$$
$$KE = \frac{1}{2}I_w\dot{\phi}^2 + \frac{1}{2}m_w(\dot{x}_w^2 + \dot{y}_w^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$= \frac{1}{2}I_w\dot{\phi}^2 + \frac{1}{2}(I + m_w\mathrm{L_w^2} + mL^2)\dot{\theta}^2$$
$$\mathrm{PE} = (\mathrm{mL} + \mathrm{m_w L_w})\mathrm{g}\cos\theta$$
$$\mathrm{L} = \frac{1}{2}I_w\dot{\phi}^2 + \frac{1}{2}(I + m_w\mathrm{L_w^2} + mL^2)\dot{\theta}^2 - (\mathrm{mL} + \mathrm{m_w L_w})\mathrm{g}\cos\theta$$
$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{L}}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = \mathrm{I_w}\ddot{\phi} = \mu$$
$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{L}}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = (I + m_w\mathrm{L_w^2} + mL^2)\ddot{\theta} - (\mathrm{mL} + \mathrm{m_w L_w})\mathrm{g}\sin\theta = \chi$$

$$\mu = \tau - \beta_\mathrm{m}(\dot{\phi} - \dot{\theta})$$
$$\chi = -\tau + \beta_\mathrm{m}(\dot{\phi} - \dot{\theta}) - \beta_a\dot{\theta}$$
$$\begin{bmatrix}\mu \\ \chi\end{bmatrix} = \begin{bmatrix}1 \\ -1\end{bmatrix}\tau - \begin{bmatrix}\beta_m & -\beta_m \\ -\beta_m & \beta_m + \beta_a\end{bmatrix}\begin{bmatrix}\dot{\phi} \\ \dot{\theta}\end{bmatrix}$$

linearization at $\theta = 0, \dot{\theta} = 0, \dot{\phi} = 0$

$$\begin{bmatrix}I_w & 0 \\ 0 & I + m_w\mathrm{L_w^2} + mL^2\end{bmatrix}\begin{bmatrix}\ddot{\phi} \\ \ddot{\theta}\end{bmatrix} + \begin{bmatrix}\beta_m & -\beta_m \\ -\beta_m & \beta_m + \beta_a\end{bmatrix}\begin{bmatrix}\dot{\phi} \\ \dot{\theta}\end{bmatrix} + \begin{bmatrix}0 \\ -(\mathrm{mL} + \mathrm{m_w L_w})\mathrm{g}\end{bmatrix}\theta$$
$$= \begin{bmatrix}1 \\ -1\end{bmatrix}\tau$$

$$\ddot{\phi} = \frac{-\beta_\mathrm{m}(\dot{\phi} - \dot{\theta}) + \tau}{\mathrm{I_w}}$$
$$\ddot{\theta} = \frac{(\mathrm{mL} + \mathrm{m_w L_w})\mathrm{g}\sin\theta + \beta_\mathrm{m}(\dot{\phi} - \dot{\theta}) - \beta_a\dot{\theta} - \tau}{I + m_w\mathrm{L_w^2} + mL^2}$$

Let $f = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \ddot{\phi} \\ \ddot{\theta} \end{bmatrix}, x = \begin{bmatrix} \phi \\ \theta \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix}$

$$\frac{\partial}{\partial x} f = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{\beta_m}{I_w} & \frac{\beta_m}{I_w} \\ 0 & \frac{(mL + m_w L_w) g \cos\theta}{I + m_w L_w^2 + mL^2} & \frac{\beta_m}{I + m_w L_w^2 + mL^2} & \frac{-\beta_m - \beta_a}{I + m_w L_w^2 + mL^2} \end{bmatrix}$$

$$\frac{\partial}{\partial \tau_0} f = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{I_w} \\ \frac{-1}{I + m_w L_w^2 + mL^2} \end{bmatrix}$$

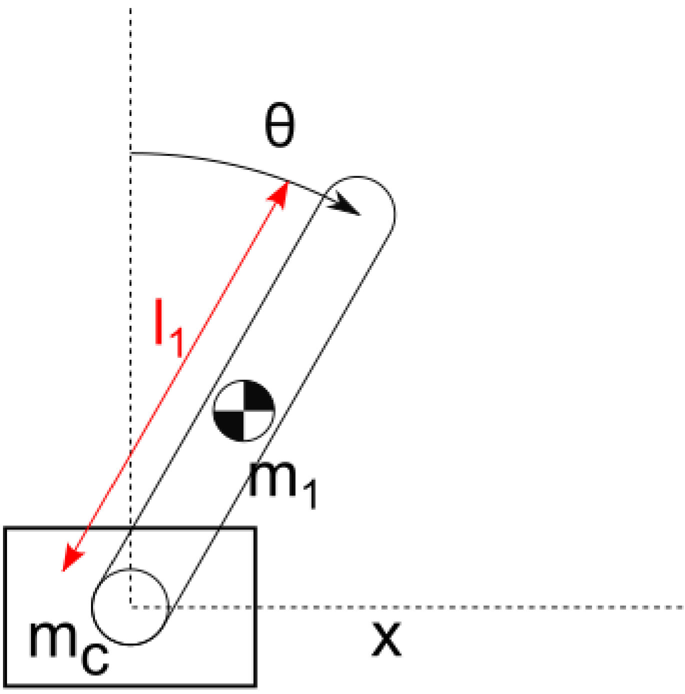
Let $x_d = \begin{bmatrix} \phi \\ \theta \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, then $u_d = 0$ such that $f(x_d, u_d) = 0$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \ddot{\phi} \\ \ddot{\theta} \end{bmatrix} = \frac{\partial}{\partial x} f(x_d, u_d) \begin{bmatrix} \phi \\ \theta \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} + \frac{\partial}{\partial \tau_0} f(x_d, u_d) * u$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \ddot{\phi} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{\beta_m}{I_w} & \frac{\beta_m}{I_w} \\ 0 & \frac{(mL + m_w L_w) g}{I + m_w L_w^2 + mL^2} & \frac{\beta_m}{I + m_w L_w^2 + mL^2} & \frac{-\beta_m - \beta_a}{I + m_w L_w^2 + mL^2} \end{bmatrix} \begin{bmatrix} \phi \\ \theta \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{I_w} \\ \frac{-1}{I + m_w L_w^2 + mL^2} \end{bmatrix} u$$

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\phi} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{\mathrm{\beta_m}}{\mathrm{I_w}} & \frac{\mathrm{\beta_m}}{\mathrm{I_w}} \\ \frac{(\mathrm{mL} + \mathrm{m_w L_w})\mathrm{g}}{I + m_w \mathrm{L_w^2} + mL^2} & \frac{\mathrm{\beta_m}}{I + m_w \mathrm{L_w^2} + mL^2} & \frac{-\mathrm{\beta_m} - \mathrm{\beta_a}}{I + m_w \mathrm{L_w^2} + mL^2} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I_w} \\ \frac{-1}{I + m_w \mathrm{L_w^2} + mL^2} \end{bmatrix} u$$

Inverted pendulum on cart



Cart has mass m_c

Pendulum1 mass m_1, length l1, moment of inertia I1, angle from y to x (CW) is theta

$$m_c \ @ \ (x, 0) \Rightarrow v_c = (\dot{x}, 0)$$

$$m_1 \ @ \ \left(x + \frac{1}{2} l_1 \sin\theta \, , \frac{1}{2} l_1 \cos\theta\right)$$

$$\Rightarrow v_1 = \left(\dot{x} + \frac{1}{2} l_1 \dot{\theta} \cos\theta \, , -\frac{1}{2} l_1 \dot{\theta} \sin\theta\right)$$

$$KE = \frac{1}{2} m_c v_c^2 + \frac{1}{2} m_1 v_1^2 + \frac{1}{2} I_1 \dot{\theta}^2$$

$$PE = \frac{1}{2} m_1 g l_1 \cos\theta$$

$$L = KE - PE = \frac{1}{2} m_1 \left(\left(\dot{x} + \frac{1}{2} l_1 \dot{\theta} \cos\theta \right)^2 + \left(-\frac{1}{2} l_1 \dot{\theta} \sin\theta \right)^2 \right) + \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} m_c \dot{x}^2$$

$$- \frac{1}{2} m_1 g l_1 \cos\theta$$

$$= \frac{1}{2}(m_1 + m_c)\dot{x}^2 + \frac{1}{2} m_1 l_1 \dot{x}\dot{\theta}\cos\theta + \frac{1}{2}\left(\frac{1}{4} m_1 l_1^2 + I_1\right)\dot{\theta}^2 - \frac{1}{2} m_1 g l_1 \cos\theta$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{d}{dt}\left((m_1 + m_c)\dot{x} + \frac{1}{2} m_1 l_1 \dot{\theta} \cos\theta \right) - 0$$

$$= (m_1 + m_c)\ddot{x} + \frac{1}{2} m_1 l_1 \cos\theta\, \ddot{\theta} - \frac{1}{2} m_1 l_1 \dot{\theta}^2 \sin\theta$$

$$(m_1 + m_c)\ddot{x} + \frac{1}{2} m_1 l_1 \cos\theta\, \ddot{\theta} = F + \frac{1}{2} m_1 l_1 \dot{\theta}^2 \sin\theta$$

$$-b_1 \dot{\theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta}$$

$$= \frac{d}{dt}\left(\frac{1}{2} m_1 l_1 \dot{x} \cos\theta + \left(\frac{1}{4} m_1 l_1^2 + I_1\right)\dot{\theta}\right)$$

$$- \left(-\frac{1}{2} m_1 \dot{x} l_1 \dot{\theta} \sin\theta + \frac{1}{2} m_1 g l_1 \sin\theta \right)$$

$$= \frac{1}{2} m_1 l_1 \cos\theta\, \ddot{x} + \left(\frac{1}{4} m_1 l_1^2 + I_1\right)\ddot{\theta} - \frac{1}{2} m_1 g l_1 \sin\theta$$

$$\frac{1}{2} m_1 l_1 \cos\theta\, \ddot{x} + \left(\frac{1}{4} m_1 l_1^2 + I_1\right)\ddot{\theta} = \frac{1}{2} m_1 g l_1 \sin\theta - b_1 \dot{\theta}$$

$$\begin{bmatrix} m_1 + m_c & \frac{1}{2} m_1 l_1 \cos\theta \\ \frac{1}{2} m_1 l_1 \cos\theta & \frac{1}{4} m_1 l_1^2 + I_1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} F + \frac{1}{2} m_1 l_1 \dot{\theta}^2 \sin\theta \\ \frac{1}{2} m_1 g l_1 \sin\theta - b_1 \dot{\theta} \end{bmatrix}$$

$$G \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = H \Rightarrow \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = G^{-1} H$$

Set point is $\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} = H = F = 0$

$$\frac{\partial}{\partial x}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \frac{\partial}{\partial \dot{x}}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = 0$$

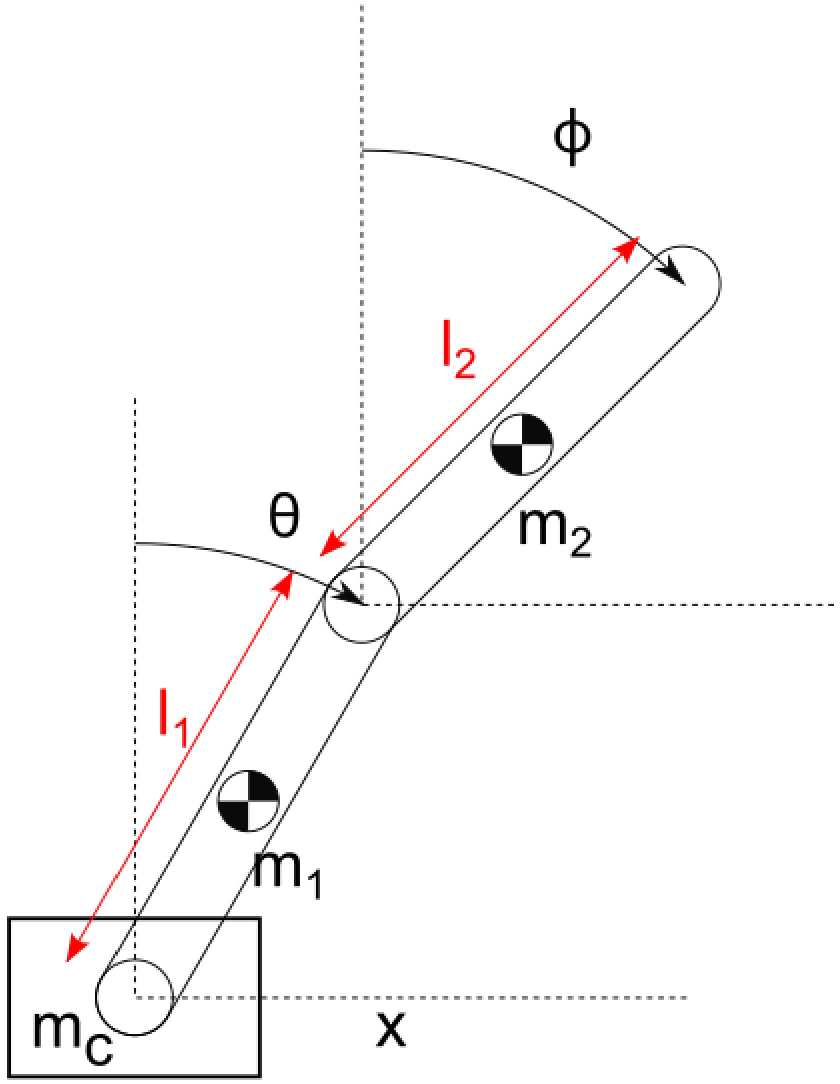
$$\frac{\partial}{\partial \theta}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = G^{-1} \frac{\partial}{\partial \theta} H = G^{-1} \begin{bmatrix} 0 \\ \frac{1}{2} m_1 g l_1 \end{bmatrix}$$

$$\frac{\partial}{\partial\dot{\theta}}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} = G^{-1}\frac{\partial}{\partial\dot{\theta}}H = G^{-1}\begin{bmatrix}0\\ -b_1\end{bmatrix}$$

$$\frac{\partial}{\partial F}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} = G^{-1}\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} = G^{-1}\begin{bmatrix}0 & 0 & 0\\ \frac{1}{2}m_1 g l_1 & 0 & -b_1\end{bmatrix}\begin{bmatrix}\theta\\ \dot{x}\\ \dot{\theta}\end{bmatrix} + G^{-1}\begin{bmatrix}1\\ 0\end{bmatrix}F$$

Double Inverted pendulum on cart



Cart has mass m_c

Pendulum1 mass m_1, length l_1, MoI I_1, angle from y to x (CW) is theta

Pendulum2 mass m_2, length l_2, MoI I_2 angle from y to x (CW) is phi

$$m_c \ @ \ (x, 0) \Rightarrow v_c = (\dot{x}, 0)$$

$$m_1 \ @ \ \left(x + \frac{1}{2} l_1 \sin\theta \, , \frac{1}{2} l_1 \cos\theta\right)$$

$$\Rightarrow v_1 = \left(\dot{x} + \frac{1}{2} l_1 \dot{\theta} \cos\theta \, , -\frac{1}{2} l_1 \dot{\theta} \sin\theta\right)$$

$$m_2 \ @ \ \left(x + l_1 \sin\theta + \frac{1}{2} l_2 \sin\phi \, , l_1 \cos\theta + \frac{1}{2} l_2 \cos\phi\right)$$

$$\Rightarrow v_2 = \left(\dot{x} + l_1\dot{\theta}\cos\theta + \frac{1}{2}l_2\dot{\phi}\cos\phi\,, -l_1\dot{\theta}\sin\theta - \frac{1}{2}l_2\dot{\phi}\sin\phi\right)$$

$$KE = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_cv_c^2 + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2\dot{\phi}^2$$

$$PE = m_1g * \frac{1}{2}l_1\cos\theta + m_2g * \left(l_1\cos\theta + \frac{1}{2}l_2\cos\phi\right)$$

$$= (\frac{1}{2}m_1 + m_2)gl_1\cos\theta + \frac{1}{2}m_2gl_2\cos\phi$$

$$L = KE - PE = \frac{1}{2}m_1\left(v_{1x}^2 + v_{1y}^2\right) + \frac{1}{2}m_2\left(v_{2x}^2 + v_{2y}^2\right) + \frac{1}{2}m_c\dot{x}^2 + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2\dot{\phi}^2$$

$$- \left(\frac{1}{2}m_1 + m_2\right)gl_1\cos\theta - \frac{1}{2}m_2gl_2\cos\phi$$

X part

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left(\frac{1}{2}m_1 2v_{1x} + \frac{1}{2}m_2 2v_{2x} + \frac{1}{2}m_c 2\dot{x}\right)$$

$$= \frac{d}{dt}\left(m_1\left(\dot{x} + \frac{1}{2}l_1\dot{\theta}\cos\theta\right) + m_2\left(\dot{x} + l_1\dot{\theta}\cos\theta + \frac{1}{2}l_2\dot{\phi}\cos\phi\right) + m_c\dot{x}\right)$$

$$= m_1(\ddot{x} + \frac{1}{2}l_1\ddot{\theta}\cos\theta - \frac{1}{2}l_1\dot{\theta}^2\sin\theta)$$

$$+ m_2\left(\ddot{x} + l_1\ddot{\theta}\cos\theta - l_1\dot{\theta}^2\sin\theta + \frac{1}{2}l_2\ddot{\phi}\cos\phi - \frac{1}{2}l_2\dot{\phi}^2\sin\phi\right)$$

$$+ m_c\ddot{x}$$

$$= (m_c + m_1 + m_2)\ddot{x} + \left(\frac{1}{2}m_1l_1\cos\theta + m_2l_1\cos\theta\right)\ddot{\theta} - \left(\frac{1}{2}m_1 + m_2\right)l_1\sin\theta\,\dot{\theta}^2$$

$$+ \frac{1}{2}m_2l_2\cos\phi\,\ddot{\phi} - \frac{1}{2}m_2l_2\sin\phi\,\dot{\phi}^2$$

$$F = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - 0$$

$$= (m_c + m_1 + m_2)\ddot{x} + \left(\frac{1}{2}m_1l_1\cos\theta + m_2l_1\cos\theta\right)\ddot{\theta}$$

$$- \left(\frac{1}{2}m_1 + m_2\right)l_1\sin\theta\,\dot{\theta}^2 + \frac{1}{2}m_2l_2\cos\phi\,\ddot{\phi} - \frac{1}{2}m_2l_2\sin\phi\,\dot{\phi}^2$$

Therefore,

$$(m_c + m_1 + m_2)\ddot{x} + \left(\frac{1}{2}m_1 + m_2\right)l_1\cos\theta\,\ddot{\theta} + \frac{1}{2}m_2l_2\cos\phi\,\ddot{\phi}$$

$$= F + \left(\frac{1}{2}m_1 + m_2\right)l_1\sin\theta\,\dot{\theta}^2 + \frac{1}{2}m_2l_2\sin\phi\,\dot{\phi}^2$$

Theta part

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)=\frac{d}{dt}\left(m_1\left(v_{1x}\left(\frac{1}{2}l_1\cos\theta\right)+v_{1y}\left(-\frac{1}{2}l_1\sin\theta\right)\right)\right.$$
$$\left.+m_2\left(v_{2x}(l_1\cos\theta)+v_{2y}(-l_1\sin\theta)\right)+I_1\dot{\theta}\right)$$

$$=m_1\dot{v_{1x}}\left(\frac{1}{2}l_1\cos\theta\right)+m_1\dot{v_{1y}}\left(-\frac{1}{2}l_1\sin\theta\right)+m_2\dot{v_{2x}}(l_1\cos\theta)+m_2\dot{v_{2y}}(-l_1\sin\theta)$$
$$+m_1v_{1x}\left(-\frac{1}{2}l_1\sin\theta\,\dot{\theta}\right)+m_1v_{1y}\left(-\frac{1}{2}l_1\cos\theta\,\dot{\theta}\right)$$
$$+m_2v_{2x}(-l_1\sin\theta\,\dot{\theta})+m_2v_{2y}(-l_1\cos\theta\,\dot{\theta})+I_1\ddot{\theta}$$

$$\frac{\partial L}{\partial\theta}=m_1\left(v_{1x}\left(-\frac{1}{2}l_1\dot{\theta}\sin\theta\right)+v_{1y}\left(-\frac{1}{2}l_1\dot{\theta}\cos\theta\right)\right)$$
$$+m_2\left(v_{2x}(-l_1\dot{\theta}\sin\theta)+v_{2y}(-l_1\dot{\theta}\cos\theta)\right)+\left(\frac{1}{2}m_1+m_2\right)gl_1\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\frac{\partial L}{\partial\theta}=-b_1\dot{\theta}$$

$$=m_1\dot{v_{1x}}\left(\frac{1}{2}l_1\cos\theta\right)+m_1\dot{v_{1y}}\left(-\frac{1}{2}l_1\sin\theta\right)+m_2\dot{v_{2x}}(l_1\cos\theta)+m_2\dot{v_{2y}}(-l_1\sin\theta)$$
$$-\left(\frac{1}{2}m_1+m_2\right)gl_1\sin\theta+I_1\ddot{\theta}$$
$$=m_1\left(\ddot{x}+\frac{1}{2}l_1\ddot{\theta}\cos\theta-\frac{1}{2}l_1\dot{\theta}^2\sin\theta\right)\left(\frac{1}{2}l_1\cos\theta\right)$$
$$+m_1\left(-\frac{1}{2}l_1\ddot{\theta}\sin\theta-\frac{1}{2}l_1\dot{\theta}^2\cos\theta\right)\left(-\frac{1}{2}l_1\sin\theta\right)$$
$$+m_2\left(\ddot{x}+l_1\ddot{\theta}\cos\theta-l_1\dot{\theta}^2\sin\theta+\frac{1}{2}l_2\ddot{\phi}\cos\phi-\frac{1}{2}l_2\dot{\phi}^2\sin\phi\right)(l_1\cos\theta)$$
$$+m_2\left(-l_1\ddot{\theta}\sin\theta-l_1\dot{\theta}^2\cos\theta-\frac{1}{2}l_2\ddot{\phi}\sin\phi-\frac{1}{2}l_2\dot{\phi}^2\cos\phi\right)(-l_1\sin\theta)$$
$$-\left(\frac{1}{2}m_1+m_2\right)gl_1\sin\theta+I_1\ddot{\theta}$$
$$=(\frac{1}{2}m_1+m_2)l_1\cos\theta\,\ddot{x}+(\frac{1}{4}m_1+m_2)l_1^2\ddot{\theta}+I_1\ddot{\theta}$$
$$+\frac{1}{2}m_2l_1l_2(\sin\theta\sin\phi+\cos\theta\cos\phi)\ddot{\phi}+\frac{1}{2}m_2l_1l_2(\sin\theta\cos\phi-\cos\theta\sin\phi)\dot{\phi}^2$$
$$-\left(\frac{1}{2}m_1+m_2\right)gl_1\sin\theta$$

Therefore,

$$\left(\frac{1}{2}m_1+m_2\right)l_1\cos\theta\,\ddot{x}+\left(\frac{1}{4}m_1l_1^2+m_2l_1^2+I_1\right)\ddot{\theta}+\frac{1}{2}m_2l_1l_2\cos(\phi-\theta)\,\ddot{\phi}$$
$$=\frac{1}{2}m_2l_1l_2\sin(\phi-\theta)\,\dot{\phi}^2+\left(\frac{1}{2}m_1+m_2\right)gl_1\sin\theta-b_1\dot{\theta}$$

Phi part

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right)=\frac{d}{dt}\left(m_2\left(v_{2x}\left(\frac{1}{2}l_2\cos\phi\right)+v_{2y}\left(-\frac{1}{2}l_2\sin\phi\right)\right)+I_2\dot{\phi}\right)$$
$$=m_2\dot{v_{2x}}\left(\frac{1}{2}l_2\cos\phi\right)+m_2\dot{v_{2y}}\left(-\frac{1}{2}l_2\sin\phi\right)+m_2v_{2x}\left(-\frac{1}{2}l_2\sin\phi\,\dot{\phi}\right)$$
$$+m_2v_{2y}\left(-\frac{1}{2}l_2\cos\phi\,\dot{\phi}\right)+I_2\ddot{\phi}$$
$$\frac{\partial L}{\partial\phi}=m_2\left(v_{2x}\left(-\frac{1}{2}l_2\dot{\phi}\sin\phi\right)+v_{2y}\left(-\frac{1}{2}l_2\dot{\phi}\cos\phi\right)\right)+\frac{1}{2}m_2gl_2\sin\phi$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right)-\frac{\partial L}{\partial\phi}=-b_2(\dot{\phi}-\dot{\theta})$$
$$=m_2\dot{v_{2x}}\left(\frac{1}{2}l_2\cos\phi\right)+m_2\dot{v_{2y}}\left(-\frac{1}{2}l_2\sin\phi\right)-\frac{1}{2}m_2gl_2\sin\phi+I_2\ddot{\phi}$$
$$=m_2\left(\ddot{x}+l_1\ddot{\theta}\cos\theta-l_1\dot{\theta}^2\sin\theta+\frac{1}{2}l_2\ddot{\phi}\cos\phi-\frac{1}{2}l_2\dot{\phi}^2\sin\phi\right)\left(\frac{1}{2}l_2\cos\phi\right)$$
$$+m_2\left(-l_1\ddot{\theta}\sin\theta-l_1\dot{\theta}^2\cos\theta-\frac{1}{2}l_2\ddot{\phi}\sin\phi-\frac{1}{2}l_2\dot{\phi}^2\cos\phi\right)\left(-\frac{1}{2}l_2\sin\phi\right)$$
$$-\frac{1}{2}m_2gl_2\sin\phi+I_2\ddot{\phi}$$
$$=\frac{1}{2}m_2l_2\cos\phi\,\ddot{x}+\frac{1}{2}m_2l_1l_2(\sin\theta\sin\phi+\cos\theta\cos\phi)\ddot{\theta}+\frac{1}{4}m_2l_2^2\ddot{\phi}$$
$$+\frac{1}{2}m_2l_1l_2(\cos\theta\sin\phi-\sin\theta\cos\phi)\dot{\theta}^2-\frac{1}{2}m_2gl_2\sin\phi+I_2\ddot{\phi}$$

Therefore,

$$\frac{1}{2}m_2l_2\cos\phi\,\ddot{x}+\frac{1}{2}m_2l_1l_2\cos(\phi-\theta)\,\ddot{\theta}+\frac{1}{4}m_2l_2^2\ddot{\phi}+I_2\ddot{\phi}$$
$$=-\frac{1}{2}m_2l_1l_2\sin(\phi-\theta)\,\dot{\theta}^2+\frac{1}{2}m_2gl_2\sin\phi-b_2(\dot{\phi}-\dot{\theta})$$

Combine all three equations

$$\begin{bmatrix} m_c+m_1+m_2 & \left(\frac{1}{2}m_1+m_2\right)l_1\cos\theta & \frac{1}{2}m_2l_2\cos\phi \\ \left(\frac{1}{2}m_1+m_2\right)l_1\cos\theta & \left(\frac{1}{4}m_1+m_2\right)l_1^2+I_1 & \frac{1}{2}m_2l_1l_2\cos(\phi-\theta) \\ \frac{1}{2}m_2l_2\cos\phi & \frac{1}{2}m_2l_1l_2\cos(\phi-\theta) & \frac{1}{4}m_2l_2^2+I_2 \end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\\ \ddot{\phi}\end{bmatrix}$$

$$= \begin{bmatrix} F+\left(\frac{1}{2}m_1+m_2\right)l_1\sin\theta\,\dot{\theta}^2+\frac{1}{2}m_2l_2\sin\phi\,\dot{\phi}^2 \\ \frac{1}{2}m_2l_1l_2\sin(\phi-\theta)\,\dot{\phi}^2+\left(\frac{1}{2}m_1+m_2\right)gl_1\sin\theta-b_1\dot{\theta} \\ -\frac{1}{2}m_2l_1l_2\sin(\phi-\theta)\,\dot{\theta}^2+\frac{1}{2}m_2gl_2\sin\phi-b_2\left(\dot{\phi}-\dot{\theta}\right) \end{bmatrix}$$

$$G\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\\ \ddot{\phi}\end{bmatrix}=H\Rightarrow\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\\ \ddot{\phi}\end{bmatrix}=G^{-1}H$$

If $m_2=0$, then

$$\begin{bmatrix} m_c+m_1 & \frac{1}{2}m_1l_1\cos\theta & 0 \\ \frac{1}{2}m_1l_1\cos\theta & \frac{1}{4}m_1l_1^2+I_1 & 0 \\ 0 & 0 & I_2 \end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\\ \ddot{\phi}\end{bmatrix}=\begin{bmatrix} F+\frac{1}{2}m_1l_1\sin\theta\,\dot{\theta}^2 \\ \frac{1}{2}m_1gl_1\sin\theta-b_1\dot{\theta} \\ -b_2\left(\dot{\phi}-\dot{\theta}\right) \end{bmatrix}$$

If $l_1=0, m_1=0$, then

$$\begin{bmatrix} m_c+m_2 & 0 & \frac{1}{2}m_2l_2\cos\phi \\ 0 & I_1 & 0 \\ \frac{1}{2}m_2l_2\cos\phi & 0 & \frac{1}{4}m_2l_2^2+I_2 \end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\\ \ddot{\phi}\end{bmatrix}=\begin{bmatrix} F+\frac{1}{2}m_2l_2\sin\phi\,\dot{\phi}^2 \\ -b_1\dot{\theta} \\ \frac{1}{2}m_2gl_2\sin\phi-b_2\left(\dot{\phi}-\dot{\theta}\right) \end{bmatrix}$$

They reduce to the single pendulum case

$$\begin{bmatrix} m_1+m_c & \frac{1}{2}m_1l_1\cos\theta \\ \frac{1}{2}m_1l_1\cos\theta & \frac{1}{4}m_1l_1^2+I_1 \end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix}=\begin{bmatrix} F+\frac{1}{2}m_1l_1\dot{\theta}^2\sin\theta \\ \frac{1}{2}m_1gl_1\sin\theta-b_1\dot{\theta} \end{bmatrix}$$

Set point is $\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\\ \ddot{\phi}\end{bmatrix}=\begin{bmatrix}x\\ \theta\\ \phi\end{bmatrix}=H=0, F=0$

$$\frac{\partial}{\partial x}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\\ \ddot{\phi}\end{bmatrix}=G^{-1}\left(\frac{\partial}{\partial x}H\right)=0$$

$$\frac{\partial}{\partial \dot{x}}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\\ \ddot{\phi}\end{bmatrix} = G^{-1}\left(\frac{\partial}{\partial \dot{x}}H\right) = 0$$

$$\frac{\partial}{\partial \theta}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\\ \ddot{\phi}\end{bmatrix} = G^{-1}\left(\frac{\partial}{\partial \theta}H\right) = G^{-1}\left(\begin{bmatrix}0\\ \left(\frac{1}{2}m_1 + m_2\right)gl_1\\ 0\end{bmatrix}\right)$$

$$\frac{\partial}{\partial \dot{\theta}}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\\ \ddot{\phi}\end{bmatrix} = G^{-1}\left(\frac{\partial}{\partial \dot{\theta}}H\right) = G^{-1}\left(\begin{bmatrix}0\\ -b_1\\ b_2\end{bmatrix}\right)$$

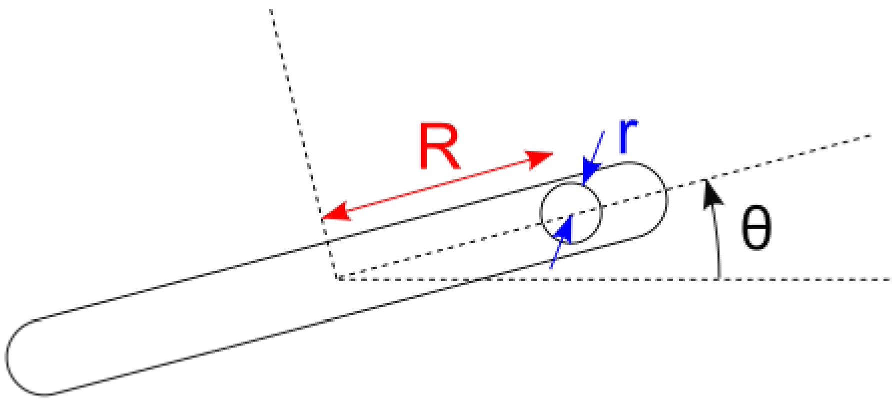
$$\frac{\partial}{\partial \phi}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\\ \ddot{\phi}\end{bmatrix} = G^{-1}\left(\frac{\partial}{\partial \phi}H\right) = G^{-1}\left(\begin{bmatrix}0\\ 0\\ \frac{1}{2}m_2gl_2\end{bmatrix}\right)$$

$$\frac{\partial}{\partial \dot{\phi}}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\\ \ddot{\phi}\end{bmatrix} = G^{-1}\left(\frac{\partial}{\partial \dot{\phi}}H\right) = G^{-1}\left(\begin{bmatrix}0\\ 0\\ -b_2\end{bmatrix}\right)$$

$$\frac{\partial}{\partial F}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\\ \ddot{\phi}\end{bmatrix} = G^{-1}\left(\frac{\partial}{\partial F}H\right) = G^{-1}\left(\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\right)$$

$$\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\\ \ddot{\phi}\end{bmatrix} = G^{-1}\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ \left(\frac{1}{2}m_1 + m_2\right)gl_1 & 0 & 0 & -b_1 & 0\\ 0 & \frac{1}{2}m_2gl_2 & 0 & b_2 & -b_2\end{bmatrix}\begin{bmatrix}\theta\\ \phi\\ \dot{x}\\ \dot{\theta}\\ \dot{\phi}\end{bmatrix} + G^{-1}\left(\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\right)F$$

Ball rolling on a rotated linear slide (assume no slip)



theta: counterclockwise rotational angle of the linear slide from +x axis

R: distance between mass and origin

r: radius of the ball

phi: clockwise rotated angle of the ball

tau: counterclockwise torque applied to linear slide

m: mass of the pendulum part

I: inertia of the linear slide

b: friction at R direction

$$\mathrm{KE} = \frac{1}{2}m\left(\dot{R}^2 + R^2\dot{\theta}^2\right) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}\frac{2}{5}mr^2\dot{\phi}^2$$

Note that $\dot{\phi} \approx \frac{\dot{R}}{r}$

$$\mathrm{KE} = \frac{1}{2}m\dot{R}^2 + \frac{1}{2}(mR^2 + I)\dot{\theta}^2 + \frac{1}{2}\frac{2}{5}m\dot{R}^2$$

$$\mathrm{KE} = \frac{1}{2}\frac{7}{5}m\dot{R}^2 + \frac{1}{2}(mR^2 + I)\dot{\theta}^2$$

$$\mathrm{PK} = \mathrm{mgR}\sin\theta$$

$$L = \frac{1}{2}\frac{7}{5}m\dot{R}^2 + \frac{1}{2}(mR^2 + I)\dot{\theta}^2 - \mathrm{mgR}\sin\theta$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{L}}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 2mR\dot{R}\dot{\theta} + (mR^2 + I)\ddot{\theta} + mgR\cos\theta = \tau_0$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{L}}{\partial \dot{R}}\right)-\frac{\partial L}{\partial R}=\frac{7}{5}m\ddot{R}-mR\dot{\theta}^2+mg\sin\theta=-b\dot{R}$$

$$\ddot{\theta}=\frac{-2mR\dot{R}\dot{\theta}-mgR\cos\theta+\tau}{mR^2+I}$$

$$\ddot{R}=\frac{5}{7}\left(R\dot{\theta}^2-g\sin\theta-\frac{b}{m}\dot{R}\right)$$

Let $\mathrm{f}=\begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\\ \dot{R}\\ \ddot{R}\end{bmatrix},\mathrm{x}=\begin{bmatrix}\theta\\ \dot{\theta}\\ R\\ \dot{R}\end{bmatrix}$

$$\frac{\partial}{\partial \mathrm{x}}\mathrm{f}=\begin{bmatrix}0 & 1 & 0 & 0\\ \dfrac{mgR\sin\theta}{mR^2+I} & \dfrac{-2mR\dot{R}}{mR^2+I} & f_{\ddot{\theta},R} & 0\\ 0 & 0 & 0 & 1\\ -\dfrac{5}{7}g\cos\theta & \dfrac{5}{7}2R\dot{\theta} & \dfrac{5}{7}\dot{\theta}^2 & -\dfrac{5}{7}\dfrac{b}{m}\end{bmatrix}$$

$$f_{\ddot{\theta},R}=\frac{\partial}{\partial R}\frac{-2mR\dot{R}\dot{\theta}-mgR\cos\theta+\tau}{mR^2+I}$$
$$=\frac{\left(-2m\dot{R}\dot{\theta}-mg\cos\theta\right)(mR^2+I)-2mR\left(-2mR\dot{R}\dot{\theta}-mgR\cos\theta+\tau\right)}{(mR^2+I)^2}$$

$$\frac{\partial}{\partial \tau_0}\mathrm{f}=\begin{bmatrix}0\\ \dfrac{1}{mR^2+I}\\ 0\\ 0\end{bmatrix}$$

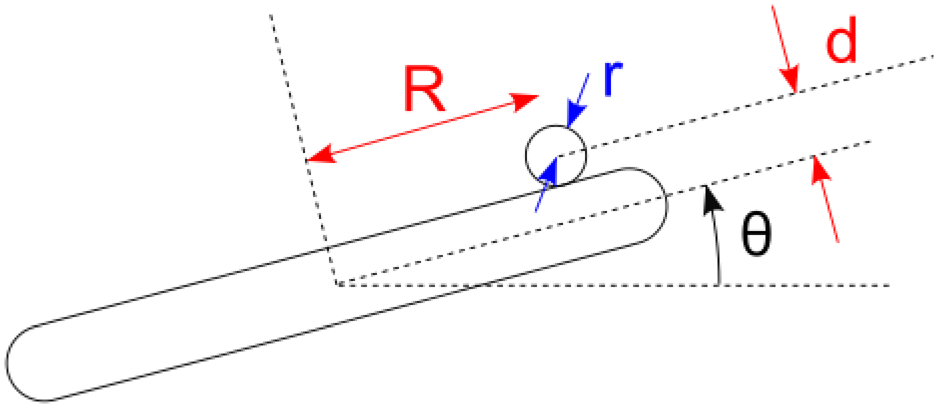
Let $\mathrm{x_d}=\begin{bmatrix}\theta\\ \dot{\theta}\\ R\\ \dot{R}\end{bmatrix}=\begin{bmatrix}0\\ 0\\ R_d\\ 0\end{bmatrix}$, then $\mathrm{u}_d=mgR$ such that $\mathrm{f(x_d,u_d)}=0$

$$\begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\\ \dot{R}\\ \ddot{R}\end{bmatrix}=\frac{\partial}{\partial \mathrm{x}}\mathrm{f(x_d,u_d)}\begin{bmatrix}\theta\\ \dot{\theta}\\ R\\ \dot{R}\end{bmatrix}+\frac{\partial}{\partial \tau_0}\mathrm{f}(\mathrm{x_d},u_d)*u$$

$$\begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\\ \dot{R}\\ \ddot{R}\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & \dfrac{-mg}{mR_d^2+I} & 0\\ 0 & 0 & 0 & 1\\ -\dfrac{5}{7}g & 0 & 0 & -\dfrac{5}{7}\dfrac{b}{m}\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\\ R\\ \dot{R}\end{bmatrix}+\begin{bmatrix}0\\ \dfrac{1}{mR_d^2+I}\\ 0\\ 0\end{bmatrix}u$$

$$\begin{bmatrix} \dot{\theta} \\ \dot{R} \\ \ddot{R} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{5}{7}g & 0 & -\frac{5}{7}\frac{b}{m} \end{bmatrix} \begin{bmatrix} \theta \\ R \\ \dot{R} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \dot{\theta}$$

Ball rolling on a rotated linear slide with offset (assume no slip)



theta: counterclockwise rotational angle of the linear slide from +x axis

phi: counterclockwise rotated angle of the ball

R: distance between mass and origin

tau: counterclockwise torque applied to linear slide

m: mass of the pendulum part

r: radius of the ball

d: offset between center of ball and origin

I: inertia of the linear slide

b: friction at R direction

$$x = R\cos\theta - d\sin\theta$$
$$y = R\sin\theta + d\cos\theta$$

$$\dot{x} = \dot{R}\cos\theta - R\dot{\theta}\sin\theta - d\dot{\theta}\cos\theta$$
$$\dot{y} = \dot{R}\sin\theta + R\dot{\theta}\cos\theta - d\dot{\theta}\sin\theta$$
$$\dot{\phi} \approx \dot{\theta} - \frac{\dot{R}}{r}$$
$$PE = mgy = mgR\sin\theta + mgd\cos\theta$$
$$KE = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}\frac{2}{5}mr^2\dot{\phi}^2$$
$$= \frac{1}{2}m\left(\dot{R}^2 + R^2\dot{\theta}^2 + d^2\dot{\theta}^2 - 2d\dot{R}\dot{\theta}\right) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}\frac{2}{5}mr^2(\dot{\theta}^2 - \frac{2}{r}\dot{\theta}\dot{R} + \frac{1}{r^2}\dot{R}^2)$$

$$= \frac{1}{2} m \left(\frac{7}{5} \dot{R}^2 + R^2 \dot{\theta}^2 + \left(d^2 + \frac{I}{m} + \frac{2}{5} r^2 \right) \dot{\theta}^2 + \left(-2d - \frac{4}{5} r \right) \dot{\mathrm{R}} \dot{\theta} \right)$$

$$L = \frac{1}{2} m \left(\frac{7}{5} \dot{R}^2 + R^2 \dot{\theta}^2 + \left(d^2 + \frac{I}{m} + \frac{2}{5} r^2 \right) \dot{\theta}^2 + \left(-2d - \frac{4}{5} r \right) \dot{\mathrm{R}} \dot{\theta} \right) - \mathrm{mgR} \sin\theta - \mathrm{mgd} \cos\theta$$

$$\frac{\partial \mathrm{L}}{\partial \dot{\theta}} = \frac{1}{2} m \left(2R^2 \dot{\theta} + 2 \left(d^2 + \frac{I}{m} + \frac{2}{5} r^2 \right) \dot{\theta} + \left(-2d - \frac{4}{5} r \right) \dot{\mathrm{R}} \right)$$

$$\frac{\mathrm{d}}{\mathrm{dt}} \left(\frac{\partial \mathrm{L}}{\partial \dot{\theta}} \right) = \frac{1}{2} m \left(4R \dot{\mathrm{R}} \dot{\theta} + 2R^2 \ddot{\theta} + 2 \left(d^2 + \frac{I}{m} + \frac{2}{5} r^2 \right) \ddot{\theta} + \left(-2d - \frac{4}{5} r \right) \ddot{\mathrm{R}} \right)$$

$$\frac{\partial L}{\partial \theta} = -\mathrm{mgR} \cos\theta + \mathrm{mgd} \sin\theta$$

$$\frac{\mathrm{d}}{\mathrm{dt}} \left(\frac{\partial \mathrm{L}}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = \frac{1}{2} m \left(4R \dot{\mathrm{R}} \dot{\theta} + 2R^2 \ddot{\theta} + 2 \left(d^2 + \frac{I}{m} + \frac{2}{5} r^2 \right) \ddot{\theta} + \left(-2d - \frac{4}{5} r \right) \ddot{\mathrm{R}} \right) + \mathrm{mgR} \cos\theta - \mathrm{mgd} \sin\theta = \tau$$

$$\frac{\partial \mathrm{L}}{\partial \dot{R}} = \frac{1}{2} m \left(\frac{14}{5} \dot{\mathrm{R}} + \left(-2d - \frac{4}{5} r \right) \dot{\theta} \right)$$

$$\frac{\mathrm{d}}{\mathrm{dt}} \left(\frac{\partial \mathrm{L}}{\partial \dot{R}} \right) = \frac{1}{2} m \left(\frac{14}{5} \ddot{\mathrm{R}} + \left(-2d - \frac{4}{5} r \right) \ddot{\theta} \right)$$

$$\frac{\partial L}{\partial R} = mR\dot{\theta}^2 - \mathrm{mg} \sin\theta$$

$$\frac{\mathrm{d}}{\mathrm{dt}} \left(\frac{\partial \mathrm{L}}{\partial \dot{R}} \right) - \frac{\partial L}{\partial R} = \frac{1}{2} m \left(\frac{14}{5} \ddot{\mathrm{R}} + \left(-2d - \frac{4}{5} r \right) \ddot{\theta} \right) - mR\dot{\theta}^2 + \mathrm{mg} \sin\theta = -b\dot{R}$$

$$\left(-d - \frac{2}{5} r \right) \ddot{\mathrm{R}} + \left(R^2 + d^2 + \frac{I}{m} + \frac{2}{5} r^2 \right) \ddot{\theta} = -2R \dot{\mathrm{R}} \dot{\theta} - \mathrm{gR} \cos\theta + \mathrm{gd} \sin\theta + \frac{1}{m} \tau$$

$$\frac{7}{5} \ddot{\mathrm{R}} + \left(-d - \frac{2}{5} r \right) \ddot{\theta} = R\dot{\theta}^2 - \frac{b}{m} \dot{R} - \mathrm{g} \sin\theta$$

$$\begin{bmatrix} d + \frac{2}{5} r & -\left(R^2 + d^2 + \frac{I}{m} + \frac{2}{5} r^2 \right) \\ -\frac{7}{5} & d + \frac{2}{5} r \end{bmatrix} \begin{bmatrix} \ddot{\mathrm{R}} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 2R \dot{\mathrm{R}} \dot{\theta} + \mathrm{gR} \cos\theta - \mathrm{gd} \sin\theta - \frac{1}{m} \tau \\ -R\dot{\theta}^2 + \frac{b}{m} \dot{R} + \mathrm{g} \sin\theta \end{bmatrix}$$

$$\begin{bmatrix} \ddot{\mathrm{R}} \\ \ddot{\theta} \end{bmatrix} = \frac{1}{det} \begin{bmatrix} d + \frac{2}{5} r & R^2 + d^2 + \frac{I}{m} + \frac{2}{5} r^2 \\ \frac{7}{5} & d + \frac{2}{5} r \end{bmatrix} \begin{bmatrix} 2R \dot{\mathrm{R}} \dot{\theta} + \mathrm{gR} \cos\theta - \mathrm{gd} \sin\theta - \frac{1}{m} \tau \\ -R\dot{\theta}^2 + \frac{b}{m} \dot{R} + \mathrm{g} \sin\theta \end{bmatrix}$$

$$\det = \left(d + \frac{2}{5}r\right)^2 - \frac{7}{5}\left(R^2 + d^2 + \frac{I}{m} + \frac{2}{5}r^2\right)$$

$$= d^2 + \frac{4}{5}dr + \frac{4}{25}r^2 - \frac{7}{5}\left(R^2 + d^2 + \frac{I}{m} + \frac{2}{5}r^2\right)$$

$$= -\frac{7}{5}R^2 + \left(-\frac{2}{5}d^2 - \frac{2}{5}r^2 + \frac{4}{5}dr - \frac{7}{5}\frac{I}{m}\right)$$

$$\frac{\partial}{\partial R}\frac{1}{det} = \frac{\frac{14}{5}R}{\det * \det}$$

$$\ddot{\mathrm{R}} * det(R) = \left(d + \frac{2}{5}r\right)\left(2R\dot{\mathrm{R}}\dot{\theta} + \mathrm{gR}\cos\theta - \mathrm{gd}\sin\theta - \frac{1}{m}\tau\right)$$
$$+ \left(R^2 + d^2 + \frac{I}{m} + \frac{2}{5}r^2\right)\left(-R\dot{\theta}^2 + \frac{b}{m}\dot{R} + \mathrm{g}\sin\theta\right)$$

$$\ddot{\theta} * det(R) = \frac{7}{5}\left(2R\dot{\mathrm{R}}\dot{\theta} + \mathrm{gR}\cos\theta - \mathrm{gd}\sin\theta - \frac{1}{m}\tau\right)$$
$$+ \left(d + \frac{2}{5}r\right)\left(-R\dot{\theta}^2 + \frac{b}{m}\dot{R} + \mathrm{g}\sin\theta\right)$$

$$\tau = m\left(2R\dot{\mathrm{R}}\dot{\theta} + \mathrm{gR}\cos\theta - \mathrm{gd}\sin\theta\right)$$
$$- \frac{5}{7}m\left(\ddot{\theta} * det(R) - \left(d + \frac{2}{5}r\right)\left(-R\dot{\theta}^2 + \frac{b}{m}\dot{R} + \mathrm{g}\sin\theta\right)\right)$$

Let $\mathrm{f} = \begin{bmatrix}\dot{R}\\ \ddot{\mathrm{R}}\\ \dot{\theta}\\ \ddot{\theta}\end{bmatrix}, \mathrm{x} = \begin{bmatrix}R\\ \dot{R}\\ \theta\\ \dot{\theta}\end{bmatrix}, \mathrm{x_d} = \begin{bmatrix}R_d\\ 0\\ 0\\ 0\end{bmatrix}$

$$\frac{\partial}{\partial R}\frac{1}{\det(R)} = \frac{\frac{14}{5}R_d}{\det(\mathrm{R_d}) * \det(\mathrm{R_d})}$$

$$\frac{\partial}{\partial \mathrm{x}}\mathrm{f} = \begin{bmatrix}0 & 1 & 0 & 0\\ A_{21} & A_{22} & A_{23} & A_{24}\\ 0 & 0 & 0 & 1\\ A_{41} & A_{42} & A_{43} & A_{44}\end{bmatrix}$$

$$\mathrm{A}_{21} = \frac{\partial}{\partial R}\ddot{R} = \frac{\partial}{\partial R}\frac{1}{\det(R)} * \ddot{\mathrm{R}} * det(R)$$

$$+ \frac{1}{det}\left(\left(d + \frac{2}{5}r\right)\left(2\dot{\mathrm{R}}\dot{\theta} + \mathrm{g}\cos\theta\right) + 2R\left(-R\dot{\theta}^2 + \frac{b}{m}\dot{R} + \mathrm{g}\sin\theta\right)\right.$$

$$\left. + \left(R^2 + d^2 + \frac{I}{m} + \frac{2}{5}r^2\right)\left(-\dot{\theta}^2\right)\right)$$

$$= \frac{\frac{14}{5}R_d}{\det * \det} * \left(d + \frac{2}{5}r\right)\left(\mathrm{gR_d} - \frac{1}{m}\tau\right) + \frac{1}{det}\left(d + \frac{2}{5}r\right)g$$

$$\mathrm{A}_{22} = \frac{\partial}{\partial \dot{R}}\ddot{R} = \frac{1}{\det}\left(\left(d + \frac{2}{5}r\right)2R\dot{\theta} + \left(R^2 + d^2 + \frac{I}{m} + \frac{2}{5}r^2\right)\frac{b}{m}\right)$$

$$= \frac{1}{\det}\left(R_d^2 + d^2 + \frac{I}{m} + \frac{2}{5}r^2\right)\frac{b}{m}$$

$$\mathrm{A}_{23} = \frac{\partial}{\partial \theta}\ddot{R} = \frac{1}{det}\left(\left(d + \frac{2}{5}r\right)\left(-\mathrm{gR}\sin\theta - \mathrm{gd}\cos\theta\right)\right.$$

$$\left. + \left(R^2 + d^2 + \frac{I}{m} + \frac{2}{5}r^2\right)\mathrm{g}\cos\theta\right)$$

$$= \frac{1}{det}\left(\left(d + \frac{2}{5}r\right)(-\mathrm{gd}) + \left(R_d^2 + d^2 + \frac{I}{m} + \frac{2}{5}r^2\right)\mathrm{g}\right)$$

$$= \frac{g}{det}\left(R_d^2 + \frac{2}{5}r^2 - \frac{2}{5}rd + \frac{I}{m}\right)$$

$$\mathrm{A}_{24} = \frac{\partial}{\partial \dot{\theta}}\ddot{R} = \frac{1}{det}\left(\left(d + \frac{2}{5}r\right)2R\dot{\mathrm{R}} + \left(R^2 + d^2 + \frac{I}{m} + \frac{2}{5}r^2\right)\left(-2R\dot{\theta}\right)\right) = 0$$

$$\mathrm{A}_{41} = \frac{\partial}{\partial R}\ddot{\theta} = \frac{\partial}{\partial R}\frac{1}{\det(R)} * \ddot{\theta} * det(R)$$

$$+ \frac{1}{det}\left(\frac{7}{5}\left(2\dot{\mathrm{R}}\dot{\theta} + \mathrm{g}\cos\theta\right) + \left(d + \frac{2}{5}r\right)\left(-\dot{\theta}^2\right)\right)$$

$$= \frac{\frac{14}{5}R_d}{\det * \det} * \frac{7}{5}\left(\mathrm{gR_d} - \frac{1}{m}\tau\right) + \frac{1}{det}\frac{7}{5}g$$

$$\mathrm{A}_{42} = \frac{\partial}{\partial \dot{R}}\ddot{\theta} = \frac{1}{det}\left(\frac{7}{5}2R\dot{\theta} + \left(d + \frac{2}{5}r\right)\frac{b}{m}\right) = \frac{1}{det}\left(d + \frac{2}{5}r\right)\frac{b}{m}$$

$$\mathrm{A}_{43} = \frac{\partial}{\partial \theta}\ddot{\theta} = \frac{1}{det}\left(\frac{7}{5}\left(-\mathrm{gR}\sin\theta - \mathrm{gd}\cos\theta\right) + \left(d + \frac{2}{5}r\right)\mathrm{g}\cos\theta\right)$$

$$= \frac{1}{det}\left(\frac{7}{5}(-\mathrm{gd}) + \left(d + \frac{2}{5}r\right)\mathrm{g}\right) = \frac{g}{det}\left(-\frac{2}{5}d + \frac{2}{5}r\right)$$

$$\mathrm{A}_{44} = \frac{\partial}{\partial\dot{\theta}}\ddot{\theta} = \frac{1}{det}\left(\frac{14}{5}R\dot{\mathrm{R}} + \left(d + \frac{2}{5}r\right)(-2R\dot{\theta})\right) = 0$$

$$\frac{\partial}{\partial\tau}f = -\frac{1}{m}\frac{1}{det}\begin{bmatrix} 0 \\ d + \frac{2}{5}r \\ 0 \\ \frac{7}{5} \end{bmatrix}$$

$$\begin{bmatrix} \dot{R} \\ \ddot{\mathrm{R}} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix}$$

$$= \frac{1}{det}\begin{bmatrix} 0 & 1 & 0 & 0 \\ H_{21} + g\left(d + \frac{2}{5}r\right) & \frac{b}{m}\left(R_d^2 + d^2 + \frac{I}{m} + \frac{2}{5}r^2\right) & g\left(\left(R_d^2 + \frac{2}{5}r^2 - \frac{2}{5}rd + \frac{I}{m}\right)\right) & 0 \\ 0 & 0 & 0 & 1 \\ H_{41} + \frac{7}{5}g & \frac{b}{m}\left(d + \frac{2}{5}r\right) & g\left(-\frac{2}{5}d + \frac{2}{5}r\right) & 0 \end{bmatrix}\begin{bmatrix} R \\ \dot{R} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

$$-\frac{1}{m}\frac{1}{det}\begin{bmatrix} 0 \\ d + \frac{2}{5}r \\ 0 \\ \frac{7}{5} \end{bmatrix}\tau - \frac{14\mathrm{R_d}}{5\mathrm{m}}\frac{1}{\det * \det}\begin{bmatrix} 0 \\ \left(d + \frac{2}{5}r\right)\dot{R} \\ 0 \\ \frac{7}{5}\dot{\theta} \end{bmatrix}\tau$$

$$\det = -\frac{7}{5}R_d^2 + \left(-\frac{2}{5}d^2 - \frac{2}{5}r^2 + \frac{4}{5}dr - \frac{7}{5}\frac{I}{m}\right)$$

$$H_{21} = \frac{1}{\det}\frac{14}{5}g\left(d + \frac{2}{5}r\right)\mathrm{R_d^2}$$

$$H_{41} = \frac{1}{\det}\frac{14}{5}g\frac{7}{5}\mathrm{R_d^2}$$

If $\mathrm{d} = \mathrm{r}$

$$\begin{bmatrix} \dot{R} \\ \ddot{\mathrm{R}} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \frac{1}{det}\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{7}{5}\frac{gr}{det}\left(R_d^2 - \frac{I}{m}\right) & \frac{b}{m}\left(\frac{5}{7}R_d^2 + \frac{5}{7}\frac{I}{m} + r^2\right) & g\left(\frac{5}{7}R_d^2 + \frac{5}{7}\frac{I}{m}\right) & 0 \\ 0 & 0 & 0 & 1 \\ \frac{7}{5}\frac{g}{\det}(R_d^2 - \frac{I}{m}) & \frac{b}{m}r & 0 & 0 \end{bmatrix}\begin{bmatrix} R \\ \dot{R} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

$$-\frac{1}{m}\frac{1}{det}\begin{bmatrix} 0 \\ r \\ 0 \\ 1 \end{bmatrix}\tau - \frac{2\mathrm{R_d}}{\mathrm{m}}\frac{1}{\det * \det}\begin{bmatrix} 0 \\ r\dot{R} \\ 0 \\ \dot{\theta} \end{bmatrix}\tau$$

$$\det = -\frac{7}{5}R_d^2 - \frac{7}{5}\frac{I}{m}$$

if $\mathrm{I} \gg \mathrm{mR_d^2}, \mathrm{I} \gg \frac{7}{5}mr^2$

$$\begin{bmatrix} \dot{R} \\ \ddot{\mathrm{R}} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{5}{7}\frac{mgr}{I} & -\frac{25}{49}\frac{b}{m} & -\frac{25}{49}g & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{5}{7}\frac{mg}{I} & -\frac{5}{7}\frac{br}{I} & 0 & 0 \end{bmatrix} \begin{bmatrix} R \\ \dot{R} \\ \theta \\ \dot{\theta} \end{bmatrix} + \frac{5}{7}\frac{1}{I}\begin{bmatrix} 0 \\ r \\ 0 \\ 1 \end{bmatrix}\tau + \frac{50\mathrm{m}}{49\mathrm{I}^2}\mathrm{R_d}\begin{bmatrix} 0 \\ r\dot{R} \\ 0 \\ \dot{\theta} \end{bmatrix}\tau$$

At the same time, if $\mathrm{I} \gg r, \mathrm{I}^2 \gg mrR_d$

$$\begin{bmatrix} \dot{R} \\ \ddot{\mathrm{R}} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{5}{7}\frac{mgr}{I} & -\frac{25}{49}\frac{b}{m} & -\frac{25}{49}g & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{5}{7}\frac{mg}{I} & -\frac{5}{7}\frac{br}{I} & 0 & 0 \end{bmatrix} \begin{bmatrix} R \\ \dot{R} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{5}{7}\frac{1}{I} \end{bmatrix}\tau$$

$$\begin{bmatrix} \dot{R} \\ \ddot{\mathrm{R}} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{25}{49}\frac{b}{m} & -\frac{25}{49}g \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R \\ \dot{R} \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\dot{\theta}$$

$$\begin{bmatrix} \dot{\theta} \\ \dot{R} \\ \ddot{R} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{25}{49}g & 0 & -\frac{25}{49}\frac{b}{m} \end{bmatrix} \begin{bmatrix} \theta \\ R \\ \dot{R} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\dot{\theta}$$

On the other hand, if $\mathrm{R_d} = 0, \tau_\mathrm{d} = 0$

$$\begin{bmatrix} \dot{R} \\ \ddot{\mathrm{R}} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \frac{1}{det}\begin{bmatrix} 0 & 1 & 0 & 0 \\ g\left(d + \frac{2}{5}r\right) & \frac{b}{m}\left(d^2 + \frac{I}{m} + \frac{2}{5}r^2\right) & g\left(\left(\frac{2}{5}r^2 - \frac{2}{5}rd + \frac{I}{m}\right)\right) & 0 \\ 0 & 0 & 0 & 1 \\ \frac{7}{5}g & \frac{b}{m}\left(d + \frac{2}{5}r\right) & g\left(-\frac{2}{5}d + \frac{2}{5}r\right) & 0 \end{bmatrix} \begin{bmatrix} R \\ \dot{R} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

$$-\frac{1}{m}\frac{1}{det}\begin{bmatrix} 0 \\ d + \frac{2}{5}r \\ 0 \\ \frac{7}{5} \end{bmatrix}\tau$$

$$\det = \left(-\frac{2}{5}d^2 - \frac{2}{5}r^2 + \frac{4}{5}dr - \frac{7}{5}\frac{I}{m}\right)$$

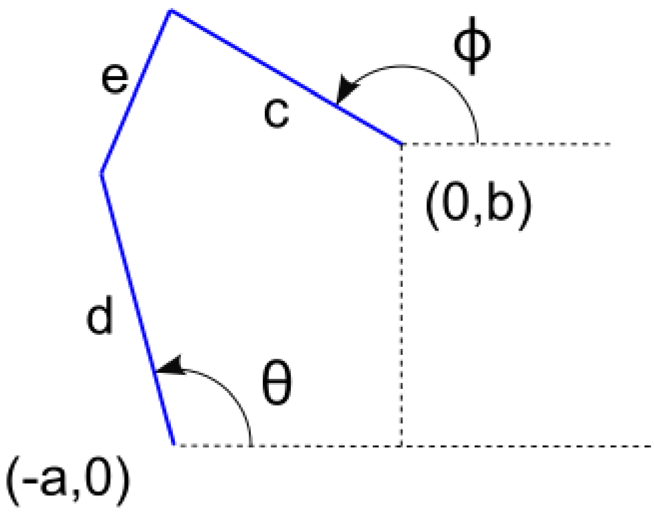
if $\mathrm{I} \gg d + \frac{2}{5}r$

$$\begin{bmatrix}\dot{R}\\ \ddot{\mathrm{R}}\\ \dot{\theta}\end{bmatrix}=\frac{1}{det}\begin{bmatrix}0 & 1 & 0\\ g\left(d+\frac{2}{5}r\right) & \frac{b}{m}\left(d^2+\frac{I}{m}+\frac{2}{5}r^2\right) & g\left(\left(\frac{2}{5}r^2-\frac{2}{5}rd+\frac{I}{m}\right)\right)\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}R\\ \dot{R}\\ \theta\end{bmatrix}$$

$$+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\dot{\theta}$$

$$\begin{bmatrix}\dot{\theta}\\ \dot{R}\\ \ddot{R}\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ -g & 0 & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}\theta\\ R\\ \dot{R}\end{bmatrix}+\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\dot{\theta}$$

Ball rolling on a platform inverse kinetic



Motor angle is theta from x axis

Platform angle is phi from x axis

Motor at (-a,0)

Platform pivot at (0,b)

Platform hinge is c away from pivot

Motor arm length is d

Fourth linkage length is e, which is the distance between $(\mathrm{d}\cos\theta - \mathrm{a}, \mathrm{d}\sin\theta)$ and $(\mathrm{c}\cos\phi, \mathrm{c}\sin\phi + b)$

Therefore,

$$(\mathrm{d}\cos\theta - \mathrm{c}\cos\phi - \mathrm{a})^2 + (\mathrm{d}\sin\theta - \mathrm{c}\sin\phi - b)^2 = \mathrm{e}^2$$

$$2ad\cos\theta + 2bd\sin\theta = \mathrm{a}^2 + b^2 + c^2 + d^2 - e^2 - 2cd + 2ac\cos\phi + 2bc\sin\phi$$

$$= f + 2ac\cos\phi + 2bc\sin\phi$$

$$\mathrm{f} = \mathrm{a}^2 + b^2 - e^2 + (c-d)^2$$

$$a\cos\theta + b\sin\theta = \frac{ac}{d}\cos\phi + \frac{bc}{d}\sin\phi + \frac{f}{2d} = g$$

$$\cos\theta = \frac{ag \pm b\sqrt{a^2 + b^2 - g^2}}{a^2 + b^2}$$

$$\sin\theta = \frac{bg \mp a\sqrt{a^2 + b^2 - g^2}}{a^2 + b^2}$$

$$\dot{\theta} = -\sqrt{\left(\frac{d}{dt}\cos\theta\right)^2 + \left(\frac{d}{dt}\sin\theta\right)^2} = -\frac{\mathrm{c}}{\mathrm{d}}\frac{-a\sin\phi + b\cos\phi}{\sqrt{a^2 + b^2 - g^2}}\dot{\phi}$$

The other way around

$$a\cos\phi + b\sin\phi = \frac{ad}{c}\cos\theta + \frac{bd}{c}\sin\theta - \frac{f}{2c} = h$$

$$\cos\phi = \frac{ag \pm b\sqrt{a^2+b^2-h^2}}{a^2+b^2}$$

$$\sin\phi = \frac{bg \mp a\sqrt{a^2+b^2-h^2}}{a^2+b^2}$$

$$\dot{\phi} = -\frac{d}{c}\frac{-a\sin\theta + b\cos\theta}{\sqrt{a^2+b^2-h^2}}\dot{\theta}$$

If we want to simplify the equation

Let $\mathrm{f} = \mathrm{a}^2 + \mathrm{b}^2 - \mathrm{e}^2 + (\mathrm{c}-\mathrm{d})^2 = 0, \mathrm{c} = \mathrm{d}, \mathrm{a} = \mathrm{b}, e = \sqrt{2}b$

such that $\mathrm{h} = \mathrm{b}\cos\theta + b\sin\theta, \mathrm{g} = \mathrm{b}\cos\phi + b\sin\phi$

$$\dot{\theta} = \frac{\sin\phi - \cos\phi}{\sqrt{1 - 2\cos\phi\sin\phi}}\dot{\phi} = \pm\dot{\phi}$$

Angular velocity and quaternion

let $\mathrm{q} = \mathrm{R_y}(\phi)R_z(\psi)$

then $\mathrm{v} = \mathrm{q}^*\begin{bmatrix}0\\0\\0\\-1\end{bmatrix}q = \begin{bmatrix}0\\ \sin\phi\cos\psi\\ -\sin\phi\sin\psi\\ -\cos\phi\end{bmatrix}$

let $\mathrm{q} = \mathrm{R_x}(\theta)\mathrm{R_y}(\phi)R_z(\psi) = \begin{bmatrix}\cos\theta\cos\phi\cos\psi - \sin\theta\sin\phi\sin\psi\\ \cos\theta\sin\phi\sin\psi + \sin\theta\cos\phi\cos\psi\\ \cos\theta\sin\phi\cos\psi - \sin\theta\cos\phi\sin\psi\\ \cos\theta\cos\phi\sin\psi + \sin\theta\sin\phi\cos\psi\end{bmatrix}$

then $\mathrm{v} = \mathrm{q}^*\begin{bmatrix}0\\0\\0\\-1\end{bmatrix}q = \cos\theta\begin{bmatrix}0\\ \sin\phi\cos\psi\\ -\sin\phi\sin\psi\\ -\cos\phi\end{bmatrix} + \sin\theta\begin{bmatrix}0\\ -\sin\psi\\ -\cos\psi\\ 0\end{bmatrix}$

body ref:

$$\omega = 2\mathrm{q}^*\dot{\mathrm{q}}$$

$$= 2\dot{\theta}\mathrm{q}^*\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{q}\right) + 2\dot{\phi}\mathrm{q}^*\left(\frac{\mathrm{d}}{\mathrm{d}\phi}\mathrm{q}\right) + 2\dot{\psi}\mathrm{q}^*\left(\frac{\mathrm{d}}{\mathrm{d}\psi}\mathrm{q}\right)$$

$$2\mathrm{q}^*\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{q}\right) = R_z(-\psi)\mathrm{R_y}(-\phi)\mathrm{R_x}(-\theta)\left(2\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{R_x}(\theta)\right)\mathrm{R_y}(\phi)R_z(\psi)$$

$$= \begin{bmatrix}\cos\frac{\psi}{2}\\0\\0\\-\sin\frac{\psi}{2}\end{bmatrix}\begin{bmatrix}\cos\frac{\phi}{2}\\0\\-\sin\frac{\phi}{2}\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\begin{bmatrix}\cos\frac{\phi}{2}\\0\\ \sin\frac{\phi}{2}\\0\end{bmatrix}\begin{bmatrix}\cos\frac{\psi}{2}\\0\\0\\ \sin\frac{\psi}{2}\end{bmatrix}$$

$$= \begin{bmatrix}\cos\frac{\psi}{2}\\0\\0\\-\sin\frac{\psi}{2}\end{bmatrix}\begin{bmatrix}0\\ \cos\phi\\0\\ \sin\phi\end{bmatrix}\begin{bmatrix}\cos\frac{\psi}{2}\\0\\0\\ \sin\frac{\psi}{2}\end{bmatrix} = \begin{bmatrix}\cos\frac{\psi}{2}\\0\\0\\-\sin\frac{\psi}{2}\end{bmatrix}\begin{bmatrix}-\sin\phi\sin\frac{\psi}{2}\\ \cos\phi\cos\frac{\psi}{2}\\ -\cos\phi\sin\frac{\psi}{2}\\ \sin\phi\cos\frac{\psi}{2}\end{bmatrix}$$

$$= \cos\frac{\psi}{2}\begin{bmatrix} -\sin\phi\sin\frac{\psi}{2} \\ \cos\phi\cos\frac{\psi}{2} \\ -\cos\phi\sin\frac{\psi}{2} \\ \sin\phi\cos\frac{\psi}{2} \end{bmatrix} + \sin\frac{\psi}{2}\begin{bmatrix} \sin\phi\cos\frac{\psi}{2} \\ -\cos\phi\sin\frac{\psi}{2} \\ -\cos\phi\cos\frac{\psi}{2} \\ \sin\phi\sin\frac{\psi}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \cos\phi\cos\psi \\ -\cos\phi\sin\psi \\ \sin\phi \end{bmatrix}$$

$$2\left(\frac{\mathrm{d}}{\mathrm{d}\phi}\mathrm{q}\right)\mathrm{q}^* = R_z(-\psi)\mathrm{R_y}(-\phi)\mathrm{R_x}(-\theta)\mathrm{R_x}(\theta)\left(2\frac{\mathrm{d}}{\mathrm{d}\phi}\mathrm{R_y}(\phi)\right)R_z(\psi)$$

$$= \begin{bmatrix} \cos\frac{\psi}{2} \\ 0 \\ 0 \\ -\sin\frac{\psi}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} \cos\frac{\psi}{2} \\ 0 \\ 0 \\ \sin\frac{\psi}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \sin\psi \\ \cos\psi \\ 0 \end{bmatrix}$$

$$2\left(\frac{\mathrm{d}}{\mathrm{d}\psi}\mathrm{q}\right)\mathrm{q}^* = R_z(-\psi)\mathrm{R_y}(-\phi)\mathrm{R_x}(-\theta)\mathrm{R_x}(\theta)\mathrm{R_y}(\phi)\left(2\frac{\mathrm{d}}{\mathrm{d}\phi}R_z(\psi)\right)$$

$$= R_z(-\psi)\left(2\frac{\mathrm{d}}{\mathrm{d}\psi}R_z(\psi)\right)$$

$$\begin{bmatrix} \cos\frac{\psi}{2} \\ 0 \\ 0 \\ -\sin\frac{\psi}{2} \end{bmatrix}\begin{bmatrix} -\sin\frac{\psi}{2} \\ 0 \\ 0 \\ \cos\frac{\psi}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\omega = \begin{bmatrix} \dot\theta\cos\phi\cos\psi + \dot\phi\sin\psi \\ -\dot\theta\cos\phi\sin\psi + \dot\phi\cos\psi \\ \dot\theta\sin\phi + \dot\psi \end{bmatrix}$$

extra, if $\mathrm{q} = \mathrm{R_z}(\phi)R_x(\theta)R_z(\psi)$, then

$$\omega = \begin{bmatrix} \dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi \\ \dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi \\ \dot\phi\cos\theta + \dot\psi \end{bmatrix}$$

fixed ref:

$$\omega = 2\dot{\mathrm{q}}\mathrm{q}^*$$

$$= 2\dot{\theta}\left(\frac{\mathrm{d}}{\mathrm{d\theta}}\mathrm{q}\right)\mathrm{q}^* + 2\dot{\phi}\left(\frac{\mathrm{d}}{\mathrm{d\phi}}\mathrm{q}\right)\mathrm{q}^* + 2\dot{\psi}\left(\frac{\mathrm{d}}{\mathrm{d\psi}}\mathrm{q}\right)\mathrm{q}^*$$

$$2\left(\frac{\mathrm{d}}{\mathrm{d\theta}}\mathrm{q}\right)\mathrm{q}^* = \left(2\frac{\mathrm{d}}{\mathrm{d\theta}}\mathrm{R_x}(\theta)\right)\mathrm{R_y}(\phi)R_z(\psi)R_z(-\psi)\mathrm{R_y}(-\phi)\mathrm{R_x}(-\theta)$$

$$= \left(2\frac{\mathrm{d}}{\mathrm{d\theta}}\mathrm{R_x}(\theta)\right)\mathrm{R_x}(-\theta) = \begin{bmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$2\left(\frac{\mathrm{d}}{\mathrm{d\phi}}\mathrm{q}\right)\mathrm{q}^* = \mathrm{R_x}(\theta)\left(2\frac{\mathrm{d}}{\mathrm{d\phi}}\mathrm{R_y}(\phi)\right)R_z(\psi)R_z(-\psi)\mathrm{R_y}(-\phi)\mathrm{R_x}(-\theta)$$

$$= \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}$$

$$2\left(\frac{\mathrm{d}}{\mathrm{d\psi}}\mathrm{q}\right)\mathrm{q}^* = \mathrm{R_x}(\theta)\mathrm{R_y}(\phi)\left(2\frac{\mathrm{d}}{\mathrm{d\psi}}R_z(\psi)\right)R_z(-\psi)\mathrm{R_y}(-\phi)\mathrm{R_x}(-\theta)$$

$$= \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ \sin\phi \\ 0 \\ \cos\phi \end{bmatrix}\begin{bmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} \sin\phi\sin\frac{\theta}{2} \\ \sin\phi\cos\frac{\theta}{2} \\ -\cos\phi\sin\frac{\theta}{2} \\ \cos\phi\cos\frac{\theta}{2} \end{bmatrix}$$

$$= \cos\frac{\theta}{2}\begin{bmatrix} \sin\phi\sin\frac{\theta}{2} \\ \sin\phi\cos\frac{\theta}{2} \\ -\cos\phi\sin\frac{\theta}{2} \\ \cos\phi\cos\frac{\theta}{2} \end{bmatrix} + \sin\frac{\theta}{2}\begin{bmatrix} -\sin\phi\cos\frac{\theta}{2} \\ \sin\phi\sin\frac{\theta}{2} \\ -\cos\phi\cos\frac{\theta}{2} \\ -\cos\phi\sin\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \sin\phi \\ -\cos\phi\sin\theta \\ \cos\phi\cos\theta \end{bmatrix}$$

$$\omega = \begin{bmatrix} \dot{\theta} + \dot{\psi} \sin \phi \\ \dot{\phi} \cos \theta - \dot{\psi} \cos \phi \sin \theta \\ \dot{\phi} \sin \theta + \dot{\psi} \cos \phi \cos \theta \end{bmatrix}$$

$$\omega^2 = \dot{\theta}^2 + \dot{\psi}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\psi} \sin \phi$$

$$\mathrm{I} = \mathrm{RI}_0 R^T$$

Gyroscope zxz

$$q = R_z(\phi)R_x(\theta)R_z(\psi)$$

wheel $\omega = \begin{bmatrix} \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi \\ \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi \\ \dot\phi \cos\theta + \dot\psi \end{bmatrix}$

Wheel moment of inertia is $I_w = \begin{bmatrix} I_{xy} + mR^2 & 0 & 0 \\ 0 & I_{xy} + mR^2 & 0 \\ 0 & 0 & I_z \end{bmatrix}$

Center of mass radius is R

Wheel mass is m

P=mgR

Center of mass at $R_z(\phi)R_x(\theta)\begin{bmatrix} 0 \\ 0 \\ R \end{bmatrix} R_x(-\theta)R_z(-\phi) = \begin{bmatrix} 0 \\ R\sin\phi\sin\theta \\ -R\cos\phi\sin\theta \\ R\cos\theta \end{bmatrix}$

$$L = KE - PE = \frac{1}{2}\omega^T I_w \omega - p\cos\theta$$

$$= \frac{1}{2}I_{xy}\left(\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi\right)^2 + \frac{1}{2}I_{xy}\left(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi\right)^2$$

$$+\frac{1}{2}I_z\left(\dot\phi\cos\theta + \dot\psi\right)^2 - p\cos\theta$$

$$L = \frac{1}{2}I_{xy}\left(\dot\phi^2\sin^2\theta + \dot\theta^2\right) + \frac{1}{2}I_z\left(\dot\phi^2\cos^2\theta + 2\dot\psi\dot\phi\cos\theta + \dot\psi^2\right) - p\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$= I_{xy}\ddot\theta - \left(\frac{1}{2}I_{xy}\dot\phi^2\sin 2\theta + \frac{1}{2}I_z\left(-\dot\phi^2\sin 2\theta - 2\dot\psi\dot\phi\sin\theta\right) + p\sin\theta\right)$$

$$\ddot\theta = \frac{1}{2}\dot\phi^2\sin 2\theta + \frac{I_z}{2I_{xy}}\left(-\dot\phi^2\sin 2\theta - 2\dot\psi\dot\phi\sin\theta\right) + \frac{1}{I_{xy}}p\sin\theta$$

$$I_c\dot\phi^2\cos\theta + I_z\dot\psi\dot\phi = p$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi}\right) - \frac{\partial L}{\partial\phi} = 0 = \frac{d}{dt}\left(I_{xy}\dot\phi\sin^2\theta + I_z\dot\phi\cos^2\theta + I_z\dot\psi\cos\theta\right) - 0$$

$$= I_{xy}\ddot\phi\sin^2\theta + I_{xy}\dot\theta\dot\phi\sin 2\theta + I_z\ddot\phi\cos^2\theta - I_z\dot\theta\dot\phi\sin 2\theta - I_z\dot\psi\dot\theta\sin\theta + I_z\ddot\psi\cos\theta$$

Let $I_c = I_z - I_{xy}$

$$\left(I_{xy}\sin^2\theta + I_z\cos^2\theta\right)\ddot\phi + I_z\ddot\psi\cos\theta = I_c\dot\theta\dot\phi\sin 2\theta + I_z\dot\psi\dot\theta\sin\theta$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{L}}{\partial \dot{\psi}}\right)-\frac{\partial L}{\partial \psi}=0=\frac{\mathrm{d}}{\mathrm{dt}}\left(I_z\left(\dot{\phi}\cos\theta+\dot{\psi}\right)\right)=\ddot{\phi}\cos\theta-\dot{\theta}\dot{\phi}\sin\theta+\ddot{\psi}$$

$$\ddot{\psi}+\ddot{\phi}\cos\theta=\dot{\theta}\dot{\phi}\sin\theta$$

$$\left(I_{xy}\sin^2\theta+I_z\cos^2\theta\right)\dot{\phi}+I_z\dot{\psi}\cos\theta=p_\phi$$

$$\dot{\phi}\cos\theta+\dot{\psi}=p_\psi$$

$$\begin{bmatrix} I_{xy}\sin^2\theta+I_z\cos^2\theta & I_z\cos\theta \\ \cos\theta & 1 \end{bmatrix}\begin{bmatrix}\dot{\phi}\\ \dot{\psi}\end{bmatrix}=\begin{bmatrix}p_\phi\\ p_\psi\end{bmatrix}$$

$$\begin{bmatrix}\dot{\phi}\\ \dot{\psi}\end{bmatrix}=\frac{1}{I_{xy}\sin^2\theta}\begin{bmatrix} 1 & -I_z\cos\theta \\ -\cos\theta & I_{xy}\sin^2\theta+I_z\cos^2\theta \end{bmatrix}\begin{bmatrix}p_\phi\\ p_\psi\end{bmatrix}$$

$$\begin{bmatrix} I_{xy}\sin^2\theta+I_z\cos^2\theta & I_z\cos\theta \\ \cos\theta & 1 \end{bmatrix}\begin{bmatrix}\ddot{\phi}\\ \ddot{\psi}\end{bmatrix}=\begin{bmatrix} I_c\dot{\theta}\dot{\phi}\sin 2\theta+I_z\dot{\psi}\dot{\theta}\sin\theta \\ \dot{\theta}\dot{\phi}\sin\theta \end{bmatrix}$$

$$\begin{bmatrix}\ddot{\phi}\\ \ddot{\psi}\end{bmatrix}=\frac{1}{I_{xy}\sin^2\theta}\begin{bmatrix} 1 & -I_z\cos\theta \\ -\cos\theta & I_{xy}\sin^2\theta+I_z\cos^2\theta \end{bmatrix}\begin{bmatrix} I_c\dot{\theta}\dot{\phi}\sin 2\theta+I_z\dot{\psi}\dot{\theta}\sin\theta \\ \dot{\theta}\dot{\phi}\sin\theta \end{bmatrix}$$

$$=\frac{1}{I_{xy}\sin^2\theta}\begin{bmatrix} I_c\dot{\theta}\dot{\phi}\sin 2\theta+I_z\dot{\psi}\dot{\theta}\sin\theta-\frac{1}{2}I_z\dot{\theta}\dot{\phi}\sin 2\theta \\ -\cos\theta\left(I_c\dot{\theta}\dot{\phi}\sin 2\theta+I_z\dot{\psi}\dot{\theta}\sin\theta\right)+\dot{\theta}\dot{\phi}\sin\theta\left(I_{xy}\sin^2\theta+I_z\cos^2\theta\right) \end{bmatrix}$$

$$=\frac{1}{I_{xy}\sin\theta}\begin{bmatrix} I_z\dot{\theta}\dot{\phi}\cos\theta-2I_{xy}\dot{\theta}\dot{\phi}\cos\theta+I_z\dot{\psi}\dot{\theta} \\ -I_z\dot{\theta}\dot{\phi}\cos^2\theta+2I_{xy}\dot{\theta}\dot{\phi}\cos^2\theta-I_z\dot{\psi}\dot{\theta}\cos\theta+I_{xy}\dot{\theta}\dot{\phi}\sin^2\theta \end{bmatrix}$$

$$=\frac{1}{I_{xy}\sin\theta}\begin{bmatrix} I_z\dot{\theta}\dot{\phi}\cos\theta-2I_{xy}\dot{\theta}\dot{\phi}\cos\theta+I_z\dot{\psi}\dot{\theta} \\ -I_c\dot{\theta}\dot{\phi}\cos^2\theta+I_{xy}\dot{\theta}\dot{\phi}-I_z\dot{\psi}\dot{\theta}\cos\theta \end{bmatrix}$$

Gyroscope xyz

$$\mathrm{q} = \mathrm{R_x}(\theta)\mathrm{R_y}(\phi)R_z(\psi)$$

wheel $\omega = \begin{bmatrix} \dot{\theta}\cos\phi\cos\psi + \dot{\phi}\sin\psi \\ -\dot{\theta}\cos\phi\sin\psi + \dot{\phi}\cos\psi \\ \dot{\theta}\sin\phi + \dot{\psi} \end{bmatrix}$

Wheel moment of inertia is $\mathrm{I_w} = \begin{bmatrix} I_{xy} + mR^2 & 0 & 0 \\ 0 & I_{xy} + mR^2 & 0 \\ 0 & 0 & I_z \end{bmatrix}$

Center of mass radius is R

Wheel mass is m

P=mgR

Center of mass at $\mathrm{R_x}(\theta)\mathrm{R_y}(\phi)\begin{bmatrix} 0 \\ 0 \\ R \end{bmatrix}\mathrm{R_y}(-\phi)\mathrm{R_x}(-\theta)$

$$= \mathrm{R_x}(\theta)\begin{bmatrix} \cos\frac{\phi}{2} \\ 0 \\ \sin\frac{\phi}{2} \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ R \end{bmatrix}\begin{bmatrix} \cos\frac{\phi}{2} \\ 0 \\ -\sin\frac{\phi}{2} \\ 0 \end{bmatrix}\mathrm{R_x}(-\theta) = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ R\sin\phi \\ 0 \\ R\cos\phi \end{bmatrix}\begin{bmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ R\sin\phi \\ -R\cos\phi\sin\theta \\ R\cos\phi\cos\theta \end{bmatrix}$$

$$\mathrm{L} = \mathrm{KE} - \mathrm{PE} = \frac{1}{2}\omega^T I_w \omega - p\cos\theta\cos\phi$$

$$= \frac{1}{2}I_{xy}\left(\dot{\theta}\cos\phi\cos\psi + \dot{\phi}\sin\psi\right)^2 + \frac{1}{2}I_{xy}\left(-\dot{\theta}\cos\phi\sin\psi + \dot{\phi}\cos\psi\right)^2$$

$$+ \frac{1}{2}I_z\left(\dot{\theta}\sin\phi + \dot{\psi}\right)^2 - p\cos\theta\cos\phi$$

$$\mathrm{L} = \frac{1}{2}I_{xy}\left(\dot{\theta}^2\cos^2\phi + \dot{\phi}^2\right) + \frac{1}{2}I_z\left(\dot{\theta}^2\sin^2\phi + 2\dot{\psi}\dot{\theta}\sin\phi + \dot{\psi}^2\right) - p\cos\theta\cos\phi$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{L}}{\partial \dot{\phi}}\right)-\frac{\partial L}{\partial \phi}=0$$

$$=I_{xy}\ddot{\phi}$$

$$-\left(-\frac{1}{2}I_{xy}\dot{\theta}^2\sin 2\phi+\frac{1}{2}I_z\left(\dot{\theta}^2\sin 2\phi+2\dot{\psi}\dot{\theta}\cos\phi\right)+p\cos\theta\sin\phi\right)$$

$$\ddot{\phi}=-\frac{1}{2}\dot{\theta}^2\sin 2\phi+\frac{I_z}{2I_{xy}}\left(\dot{\theta}^2\sin 2\phi+2\dot{\psi}\dot{\theta}\cos\phi\right)+\frac{1}{I_{xy}}p\cos\theta\sin\phi$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{L}}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0=\frac{\mathrm{d}}{\mathrm{dt}}\left(I_{xy}\dot{\theta}\cos^2\phi+I_z\dot{\theta}\sin^2\phi+I_z\dot{\psi}\sin\phi\right)-(\mathrm{p}\sin\theta\cos\phi)$$

$$=I_{xy}\ddot{\theta}\cos^2\phi-I_{xy}\dot{\theta}\dot{\phi}\sin 2\phi+I_z\ddot{\theta}\sin^2\phi+I_z\dot{\theta}\dot{\phi}\sin 2\phi+I_z\dot{\psi}\dot{\phi}\cos\phi+I_z\ddot{\psi}\sin\phi$$

$$-\mathrm{p}\sin\theta\cos\phi$$

Let $\mathrm{I_c}=I_z-I_{xy}$

$$\left(I_{xy}\cos^2\phi+I_z\sin^2\phi\right)\ddot{\theta}=-I_c\dot{\theta}\dot{\phi}\sin 2\phi-I_z\dot{\psi}\dot{\phi}\cos\phi+\mathrm{p}\sin\theta\cos\phi-I_z\ddot{\psi}\sin\phi$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{L}}{\partial \dot{\psi}}\right)-\frac{\partial L}{\partial \psi}=0=\frac{\mathrm{d}}{\mathrm{dt}}\left(\mathrm{I_z}\left(\dot{\theta}\sin\phi+\dot{\psi}\right)\right)-0=\ddot{\theta}\sin\phi+\dot{\theta}\dot{\phi}\cos\phi+\ddot{\psi}$$

$$\ddot{\psi}=-\ddot{\theta}\sin\phi-\dot{\theta}\dot{\phi}\cos\phi$$

$$\begin{bmatrix} I_{xy}\cos^2\phi+I_z\sin^2\phi & I_z\sin\phi \\ \sin\phi & 1\end{bmatrix}\begin{bmatrix}\ddot{\theta}\\ \ddot{\psi}\end{bmatrix}$$

$$=\begin{bmatrix}-I_c\dot{\theta}\dot{\phi}\sin 2\phi-I_z\dot{\psi}\dot{\phi}\cos\phi+\mathrm{p}\sin\theta\cos\phi\\ -\dot{\theta}\dot{\phi}\cos\phi\end{bmatrix}$$

$$\begin{bmatrix}\ddot{\theta}\\ \ddot{\psi}\end{bmatrix}$$

$$=\frac{1}{I_{xy}\cos^2\phi}\begin{bmatrix}1 & -I_z\sin\phi\\ -\sin\phi & I_{xy}\cos^2\phi+I_z\sin^2\phi\end{bmatrix}\begin{bmatrix}-I_c\dot{\theta}\dot{\phi}\sin 2\phi-I_z\dot{\psi}\dot{\phi}\cos\phi+\mathrm{p}\sin\theta\cos\phi\\ -\dot{\theta}\dot{\phi}\cos\phi\end{bmatrix}$$

$$=\frac{1}{I_{xy}\,cos\,\phi}\begin{bmatrix}1 & -I_z\sin\phi\\ -\sin\phi & I_{xy}\cos^2\phi+I_z\sin^2\phi\end{bmatrix}\begin{bmatrix}-2I_c\dot{\theta}\dot{\phi}\sin\phi-I_z\dot{\psi}\dot{\phi}+\mathrm{p}\sin\theta\\ -\dot{\theta}\dot{\phi}\end{bmatrix}$$

$$=\frac{1}{I_{xy}\,cos\,\phi}\begin{bmatrix}-I_z\dot{\theta}\dot{\phi}\sin\phi+2I_{xy}\dot{\theta}\dot{\phi}\sin\phi-I_z\dot{\psi}\dot{\phi}+\mathrm{p}\sin\theta\\ I_z\dot{\theta}\dot{\phi}\sin^2\phi-I_{xy}\dot{\theta}\dot{\phi}\sin^2\phi-I_{xy}\dot{\theta}\dot{\phi}+I_z\dot{\psi}\dot{\phi}\sin\phi-\mathrm{p}\sin\theta\sin\phi\end{bmatrix}$$

Control moment gyroscope

Robot forward is $+x\ direction$, it can tilt along x axis for θ,

The wheel is fixed on the robot, spinning along z axis for ψ,

Wheel rotation axis can be tilted along y axis for ϕ by applying torque

For the wheel, the rotation is $\mathrm{q} = \mathrm{R_x}(\theta)\mathrm{R_y}(\phi)R_z(\psi)$

Wheel angular velocity $\omega = \begin{bmatrix} \dot\theta\cos\phi\cos\psi + \dot\phi\sin\psi \\ -\dot\theta\cos\phi\sin\psi + \dot\phi\cos\psi \\ \dot\theta\sin\phi + \dot\psi \end{bmatrix}$

Wheel moment of inertia is $\mathrm{I_w} = \begin{bmatrix} I_{xy} & 0 & 0 \\ 0 & I_{xy} & 0 \\ 0 & 0 & I_z \end{bmatrix}$

(2 Wheel mass * wheel height^2+ Body mass * body height^2) is I_b

g*(2 Wheel mass * wheel height+ Body mass * body height) is p

$$\mathrm{L} = \mathrm{KE} - \mathrm{PE} = 2 * \frac{1}{2}\omega^T I_w \omega + \frac{1}{2} I_b \dot\theta^2 - p\cos\theta$$

$$= I_{xy}\left(\dot\theta\cos\phi\cos\psi + \dot\phi\sin\psi\right)^2 + I_{xy}\left(-\dot\theta\cos\phi\sin\psi + \dot\phi\cos\psi\right)^2$$

$$+ I_z\left(\dot\theta\sin\phi + \dot\psi\right)^2 + \frac{1}{2} I_b \dot\theta^2 - p\cos\theta$$

$$\mathrm{L} = I_{xy}\left(\dot\theta^2\cos^2\phi + \dot\phi^2\right) + I_z\left(\dot\theta^2\sin^2\phi + 2\dot\psi\dot\theta\sin\phi + \dot\psi^2\right) + \frac{1}{2} I_b\dot\theta^2 - p\cos\theta$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{L}}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$= \frac{\mathrm{d}}{\mathrm{dt}}\left(2I_{xy}\dot\theta\cos^2\phi + 2I_z\dot\theta\sin^2\phi + 2I_z\dot\psi\sin\phi + I_b\dot\theta\right) - (\mathrm{p}\sin\theta)$$

$$= 2I_{xy}\ddot\theta\cos^2\phi - 2I_{xy}\dot\theta\dot\phi\sin 2\phi + 2I_z\ddot\theta\sin^2\phi + 2I_z\dot\theta\dot\phi\sin 2\phi + 2I_z\dot\psi\dot\phi\cos\phi$$
$$+ I_b\ddot\theta - p\sin\theta$$

$$\left(2I_{xy}\cos^2\phi + 2I_z\sin^2\phi + I_b\right)\ddot\theta$$
$$= 2I_{xy}\dot\theta\dot\phi\sin 2\phi - 2I_z\dot\theta\dot\phi\sin 2\phi - 2I_z\dot\psi\dot\phi\cos\phi + p\sin\theta$$

Let $\mathrm{I_c} = I_z - I_{xy}$

$$\left(2I_{xy}\cos^2\phi + 2I_z\sin^2\phi + I_b\right)\ddot\theta = -2I_c\dot\theta\dot\phi\sin 2\phi - 2I_z\dot\psi\dot\phi\cos\phi + p\sin\theta$$

$$\mathrm{L} = I_{xy}\left(\dot\theta^2\cos^2\phi + \dot\phi^2\right) + I_z\left(\dot\theta^2\sin^2\phi + 2\dot\psi\dot\theta\sin\phi + \dot\psi^2\right) + \frac{1}{2} I_b\dot\theta^2 - p\cos\theta$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{L}}{\partial\dot\phi}\right) - \frac{\partial L}{\partial\phi} = \tau - \mathrm{B}_w\dot\phi$$

$$= 2I_{xy}\ddot\phi - \left(-I_{xy}\dot\theta^2\sin 2\phi + I_z\left(\dot\theta^2\sin 2\phi + 2\dot\psi\dot\theta\cos\phi\right)\right)$$

$$2I_{xy}\ddot{\phi} = -I_{xy}\dot{\theta}^2 \sin 2\phi + I_z\dot{\theta}^2 \sin 2\phi + 2I_z\dot{\psi}\dot{\theta}\cos\phi - \mathrm{B}_w\dot{\phi} + \tau$$
$$2I_{xy}\ddot{\phi} = I_c\dot{\theta}^2 \sin 2\phi + 2I_z\dot{\psi}\dot{\theta}\cos\phi - \mathrm{B}_w\dot{\phi} + \tau$$

Linearization at $\begin{bmatrix}\theta\\ \dot{\theta}\\ \phi\\ \dot{\phi}\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

$$\left(2I_{xy}\cos^2\phi + 2I_z\sin^2\phi + I_b\right)\ddot{\theta} = -2I_c\dot{\theta}\dot{\phi}\sin 2\phi - 2I_z\dot{\psi}\dot{\phi}\cos\phi + p\sin\theta$$

$$\left(2I_{xy}\cos^2\phi + 2I_z\sin^2\phi + I_b\right)\frac{\partial}{\partial\theta}\ddot{\theta} = p\cos\theta$$

$$\frac{\partial}{\partial\theta}\ddot{\theta} = \frac{p}{2I_{xy} + I_b}$$

$$\left(2I_{xy}\cos^2\phi + 2I_z\sin^2\phi + I_b\right)\frac{\partial}{\partial\dot{\theta}}\ddot{\theta} = -2I_c\dot{\phi}\sin 2\phi$$

$$\frac{\partial}{\partial\dot{\theta}}\ddot{\theta} = 0$$

$$\left(2I_{xy}\cos^2\phi + 2I_z\sin^2\phi + I_b\right)\frac{\partial}{\partial\phi}\ddot{\theta} = -4I_c\dot{\theta}\dot{\phi}\cos 2\phi + 4I_z\dot{\psi}\dot{\phi}\sin\phi$$

$$\frac{\partial}{\partial\phi}\ddot{\theta} = 0$$

$$\left(2I_{xy}\cos^2\phi + 2I_z\sin^2\phi + I_b\right)\frac{\partial}{\partial\dot{\phi}}\ddot{\theta} = -2I_c\dot{\theta}\sin 2\phi - 2I_z\dot{\psi}\cos\phi$$

$$\frac{\partial}{\partial\dot{\phi}}\ddot{\theta} = \frac{-2I_z\dot{\psi}}{2I_{xy} + I_b}$$

$$\frac{\partial}{\partial\theta}\ddot{\phi} = 0$$

$$2I_{xy}\frac{\partial}{\partial\dot{\theta}}\ddot{\phi} = 2I_c\dot{\theta}\sin 2\phi + 2I_z\dot{\psi}\cos\phi = 2I_z\dot{\psi}$$

$$\frac{\partial}{\partial\dot{\theta}}\ddot{\phi} = \frac{I_z}{I_{xy}}\dot{\psi}$$

$$2I_{xy}\frac{\partial}{\partial\phi}\ddot{\phi} = 2I_c\dot{\theta}^2\cos 2\phi - 2I_z\dot{\psi}\dot{\theta}\sin\phi$$

$$\frac{\partial}{\partial\phi}\ddot{\phi} = 0$$

$$2I_{xy}\frac{\partial}{\partial\dot{\phi}}\ddot{\phi} = -\mathrm{B}_w$$

$$\frac{\partial}{\partial \dot{\phi}} \ddot{\phi} = -\frac{\mathrm{B}_w}{2I_{xy}}$$

$$\frac{\partial}{\partial \tau} \ddot{\phi} = \frac{1}{2I_{xy}}$$

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{p}{2I_{xy} + I_b} & 0 & 0 & \frac{-2I_z \dot{\psi}}{2I_{xy} + I_b} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{I_z}{I_{xy}} \dot{\psi} & 0 & -\frac{\mathrm{B}_w}{2I_{xy}} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix} + \tau \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2I_{xy}} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{p}{2I_{xy} + I_b} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \end{bmatrix} + \dot{\phi} \begin{bmatrix} 0 \\ \frac{-2I_z \dot{\psi}}{2I_{xy} + I_b} \\ 1 \end{bmatrix}$$

Control theory

Plant without noise: input=u, output=y

$$\frac{\mathrm{d}}{\mathrm{dt}}x = Ax + Bu$$

$$\mathrm{y} = \mathrm{Cx}$$

Plant with noise: input=(u,wd,wn), output=y,

in this case, wd has length of x

$$\frac{\mathrm{d}}{\mathrm{dt}}x = Ax + \begin{bmatrix} B & 1 & 0 \end{bmatrix}\begin{bmatrix} u \\ wd \\ wn \end{bmatrix}$$

$$\mathrm{y} = \mathrm{Cx} + \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} u \\ wd \\ wn \end{bmatrix}$$

LQG feedback: input=(u,y,r), output=u

$$\frac{\mathrm{d}}{\mathrm{dt}}\hat{x} = \left(A - K_f C\right)\hat{x} + \begin{bmatrix} B & K_f & 0 \end{bmatrix}\begin{bmatrix} u \\ y \\ r \end{bmatrix}$$

$$\mathrm{u} = -K_r\hat{x} + \begin{bmatrix} 0 & 0 & K_c \end{bmatrix}\begin{bmatrix} u \\ y \\ r \end{bmatrix}$$

QN is the covariance of wd, RN is the covariance of wn

$$K_r, S, E = lrq(A, B, QW, RW)$$
$$K_f, P, E = lre(A, eye, C, QN, RN)$$

Control with desired point

Let r is the reference value, K_c is a scaling coefficient

$$\mathrm{u} = K_c r - K_r x$$

$$\dot{x} = 0 = Ax_d + Bu = Ax_d - BK_r x_d + BK_c r$$

$$-(A - BK_r)x_d = BK_c r$$

$$x_d = -(A - BK_r)^{-1}BK_c r$$

If $r = Cx_d$, then

$$r = Cx_d = -C(A - BK_r)^{-1}BK_c r$$

$$1 = -C(A - BK_r)^{-1}BK_c$$

$$K_c = -\frac{1}{C(A - BK_r)^{-1}B}$$

Discrete Kalman filter

Xk is the real state at time k

uk is the input at time k

yk is the real measurement at time k

Xhatk is the estimated state at time k

$$\dot{\mathrm{x}} = Ax + Bu + w$$

$$\mathrm{x}_{\mathrm{t+\Delta t}} = x_t + A\Delta \mathrm{t} x_t + B\Delta \mathrm{t} u_t + \Delta \mathrm{t} w = (I + A\Delta \mathrm{t})x_t + B\Delta \mathrm{t} u_t + \Delta \mathrm{t} w$$

$$x_k = Fx_{k-1} + Bu_{k-1} + w$$

$$\mathrm{y_k} = Hx_k + v$$

$$P_k = \mathrm{cov}(x_k - \hat{x}_k)$$

From initial state $\hat{x}_0, P_0$

Prediction based on previous $\hat{x}_{k-1}, P_{k-1}$:

$$\hat{x}_{k|k-1} = F\hat{x}_{k-1} + Bu_{k-1}$$

$$\begin{aligned} P_{k|k-1} &= \mathrm{cov}\left(\mathrm{x}_k - \hat{x}_{k|k-1}\right) = \mathrm{cov}(Fx_{k-1} + Bu_{k-1} + w - F\hat{x}_{k-1} - Bu_{k-1}) \\ &= \mathrm{cov}(F(x_{k-1} - \hat{x}_{k-1}) + w) = F\mathrm{cov}(x_{k-1} - \hat{x}_{k-1})F^T + cov(w) \\ &= FP_{k-1}F^T + Q \end{aligned}$$

Update:

$$\hat{x}_k = \hat{x}_{k|k-1} + \mathrm{K}\left(y_k - H\hat{x}_{k|k-1}\right)$$

$$\hat{x}_k = (I - KH)\hat{x}_{k|k-1} + \mathrm{K}y_k$$

$$\begin{aligned} P_k &= \mathrm{cov}(\mathrm{x}_k - \hat{x}_k) = \mathrm{cov}\left(\mathrm{x}_k - (I - KH)\hat{x}_{k|k-1} - KHx_k - Kv\right) \\ &= \mathrm{cov}\left((I - KH)\left(\mathrm{x}_k - \hat{x}_{k|k-1}\right) - Kv\right) \\ &= (I - KH)cov\left(\mathrm{x}_k - \hat{x}_{k|k-1}\right)(I - KH)^T + Kcov(v)K^T \\ &= (I - KH)P_{k|k-1}(I - KH)^T + KRK^T \\ &= P_{k|k-1} - KHP_{k|k-1} - P_{k|k-1}H^TK^T + KHP_{k|k-1}H^TK^T + KRK^T \\ &= P_{k|k-1} - KHP_{k|k-1} - P_{k|k-1}H^TK^T + K\left(\mathrm{H}P_{k|k-1}H^T + R\right)K^T \\ &= P_{k|k-1} - KHP_{k|k-1} - P_{k|k-1}H^TK^T + KS_kK^T \end{aligned}$$

To minimize the trace of P_k, the kalman gain is derived as following:

$$\frac{\partial \mathrm{tr}(\mathrm{P_k})}{\partial \mathrm{K}} = -2P_{k|k-1}H^T + 2KS_k = 0$$

$$\mathrm{K} = P_{k|k-1}H^TS_k^{-1}$$

Therefore

$$KS_kK^T = P_{k|k-1}H^TK^T$$

$$\begin{aligned} P_k &= P_{k|k-1} - KHP_{k|k-1} - P_{k|k-1}H^TK^T + P_{k|k-1}H^TK^T \\ &= P_{k|k-1} - KHP_{k|k-1} = (I - KH)P_{k|k-1} \end{aligned}$$