Abstract

This short article aims to explain the behavior of motion under linear drag, originating from the study of rigidbody dynamics in Unity. In unity, setting drag to zero is not always practical, making it worthwhile to derive the relevant equations. I begin by analyzing Unity's rigidbody algorithm to understand how speed and movement are calculated. Next, I will derive the equations of trajectory based on an initial velocity. Finally, I will address the inverse problem: calculating the initial velocity required to reach a given final position.

Rigidbody dynamics in Unity

In Unity, the air resistance force acting on a rigidbody is proportional to its velocity, whereas in reality, it is proportional to the square of the velocity. This is why it is referred to as "linear drag." Consequently, the derivative of velocity can be expressed as follows:

$$\frac{dv}{dt} = \frac{F}{m} - drag * v$$

v is the velocity of the object. F is the applied force on the object. m is the mass of the object. $drag$ is the coefficient of drag force which is greater than or equal to zero.

We can solve the differential equation:

$$\frac{dv}{dt} = \frac{F}{m} - drag * v = -drag * \left(v - \frac{F}{m * drag}\right)$$

$$\Rightarrow \mathrm{v} = \mathrm{C_1 e^{-drag*t}} + \frac{F}{m * drag}$$

$$\Rightarrow \mathrm{v} = \left(v_0 - \frac{F}{m * drag}\right)\mathrm{e^{-drag*t}} + \frac{F}{m * drag} \tag{1}$$

Interestingly, in Unity, the calculation of applied force and drag is not handled in a straightforward manner.

The part of applied force is straightforward: $\frac{dv}{dt} = \frac{F}{m} \Rightarrow v_{t+\Delta t} = v_t + \frac{F}{m}\Delta \mathrm{t}$

The part of linear drag:

$$\frac{dv}{dt} = -drag * v \Rightarrow v_{t+\Delta t} = v_t - drag * v_t * \Delta t = v_t(1 - drag * \Delta t)$$

Combine these two parts:

$$v_{t+\Delta t} = \left(v_t + \frac{F}{m}\Delta \mathrm{t}\right)(1 - drag * \Delta t)$$

It can be confirmed by calculating the terminal velocity

$$\mathrm{v_{Terminal}} = \left(\mathrm{v_{Terminal}} + \frac{F}{m}\Delta t\right)(1 - drag * \Delta t)$$

$$\mathrm{v_{Terminal}} = \frac{\frac{F}{m}\Delta t(1 - drag * \Delta t)}{1 - (1 - \mathrm{drag} * \Delta \mathrm{t})} = \frac{F}{m}\left(\frac{1}{\mathrm{drag}} - \Delta \mathrm{t}\right) = \frac{F}{m}\frac{1}{drag} - \frac{F}{m}\Delta t$$

I check the value in unity. It is the same as the above equation.

Trajectory of rigidbody

Based on eq1 derived from last paragraph,

$$v = \left(v_0 - \frac{F}{m * drag}\right)e^{-drag*t} + \frac{F}{m * drag} \tag{1}$$

We can integrate both to get the equation of trajectory. Let d be the position and $d(0) = 0$

$$d = \frac{v_0 - \frac{F}{m * drag}}{drag}\left(1 - e^{-drag*t}\right) + \frac{F}{m * drag}t \tag{2}$$

In the x-direction, no force is applied. Therefore, $F = 0$. The position in the x-direction d_x is the following:

$$d_x = \frac{v_{x0}}{drag}\left(1 - e^{-drag*t}\right) \tag{3}$$

Furthermore, we can calculate the time required to travel horizontally for d_x:

$$t = \frac{1}{drag}\ln\left(\frac{v_{x0}}{v_{x0} - \mathrm{d_x} * drag}\right) \tag{4}$$

In the y-direction, gravity force is applied. Let $v_T = \frac{F}{m*drag} = -\frac{g}{drag}$. The position in the y-direction d_y is the following:

$$d_y = \frac{v_{y0} - v_T}{drag}\left(1 - e^{-drag*t}\right) + v_T t \tag{5}$$

Furthermore, we can calculate the time required to travel vertically for d_y.

Let $d_f = \frac{v_{y0} - v_T}{drag}, t_f = \frac{-d_y + d_f}{v_T}$, then

$$d_y = \frac{v_{y0} - v_T}{drag}\left(1 - e^{-drag*t}\right) + v_T t = d_f\left(1 - e^{-drag*t}\right) + v_T t$$

$$\Rightarrow e^{-drag*t} = 1 + \frac{-d_y}{d_f} + \frac{v_T}{d_f}t = \frac{v_T}{d_f}\left(t + \left(\frac{-d_y}{v_T} + \frac{d_f}{v_T}\right)\right) = \frac{v_T}{d_f}\left(t + t_f\right)$$

$$\Rightarrow t = -t_f + \frac{1}{drag}W\left(\frac{drag * d_f}{v_T}e^{drag*t_f}\right) \tag{6}$$

Calculation of initial velocity for a given final position

Introduce eq4 into eq5

$$1-\frac{\mathrm{drag}*\mathrm{d_x}}{\mathrm{v_{x0}}}=\exp\left(\frac{drag*d_x}{v_T}\frac{v_{y0}}{v_{x0}}-\frac{drag*d_x}{v_{x0}}-\frac{drag*d_y}{v_T}\right)$$

Let

$$\mathrm{v}_i=\frac{\mathrm{drag}*\mathrm{d_x}}{\mathrm{v_{x0}}} \tag{7}$$

then

$$1-\mathrm{v_i}=\exp\left(v_i\frac{v_{y0}}{v_T}-v_i-\frac{drag*d_y}{v_T}\right)$$

$$\Rightarrow 1-\mathrm{v_i}=\exp\left(\left(\frac{v_{y0}}{v_T}-1\right)v_i-\frac{drag*d_y}{v_T}\right) \tag{8}$$

Solving this equation is challenging, so rather than constraining the velocity magnitude to a specific value, I opted to set the sum of the velocity magnitudes in the x and y directions to a positive value $v_0=|v_{x0}|+|\mathrm{v_{y0}}|$.

Assuming the final position in x-direction is greater than zero, so we can guarantee that $v_{x0}\geq 0$. Then $v_0=v_{x0}+|\mathrm{v_{y0}}|$ which suggest that $v_{x0}<v_0$

Now, I'll discuss the problem in two different cases, one is where $\mathrm{v_{y0}}>0$ and the other one is where $\mathrm{v_{y0}}\leq 0$

Case 1: $\mathrm{v_{y0}}=v_0-v_{x0}>0$, where object goes up initially

Introducing $\mathrm{v_{y0}}=v_0-v_{x0}$ into eq8:

$$1-\mathrm{v_i}=\exp\left(\left(\frac{v_0}{v_T}-1\right)\mathrm{v_i}+\frac{drag*\left(-d_y-d_x\right)}{v_T}\right)$$

$$\Rightarrow \mathrm{v_i}=1-\exp\left(\frac{drag*\left(-d_y-d_x\right)}{v_T}\right)\exp\left(\left(\frac{v_0}{v_T}-1\right)\mathrm{v_i}\right)$$

let $\mathrm{r_{ie}}=\exp\left(-\frac{drag*(d_y+d_x)}{v_T}\right), r_{iv}=\frac{v_0}{v_T}-1, r_{ic}=\frac{-drag*(d_y+d_x)+v_0-v_T}{v_T}$, then

$$\mathrm{v_i}=1-r_{ie}\exp(r_{iv}v_i)$$

$$\Rightarrow \mathrm{v_i}=1-\frac{1}{r_{iv}}W(r_{iv}\mathrm{r_{ie}}\,e^{r_{iv}})=1-\frac{1}{r_{iv}}W(r_{iv}e^{r_{ic}})$$

$$\Rightarrow v_{x0}=\frac{\mathrm{drag}*\mathrm{d_x}}{\mathrm{v_i}}\ and\ \mathrm{v_{y0}}=v_0-v_{x0}$$

Case 2: $\mathrm{v_{y0}}=v_{x0}-v_0\leq 0$, where object goes down initially

Introducing $\mathrm{v_{y0}}=v_{x0}-v_0$ into eq8:

$$1-\mathrm{v_i}=\exp\left(\left(-1-\frac{v_0}{v_T}\right)v_i+\frac{drag*(-d_y+\mathrm{d_x})}{v_T}\right)$$

$$\Rightarrow \mathrm{v_i}=1-\exp\left(\frac{drag*\left(-d_y+\mathrm{d_x}\right)}{v_T}\right)\exp\left(\left(-1-\frac{v_0}{v_T}\right)v_i\right)$$

let $\mathrm{r_{ie2}}=\exp\left(-\frac{drag*(d_y-d_x)}{v_T}\right), r_{iv2}=-\frac{v_0}{v_T}-1, r_{ic2}=\frac{-drag*(d_y-d_x)-v_0-v_T}{v_T}$, then

$$\mathrm{v_i}=1-r_{ie2}\exp(r_{iv2}v_i)$$

$$\Rightarrow \mathrm{v_i}=1-\frac{1}{r_{iv2}}W(\mathrm{r_{ie2}}r_{iv2}\,e^{r_{iv2}})=1-\frac{1}{r_{iv2}}W(r_{iv2}e^{r_{ic2}})$$

$$\Rightarrow v_{x0}=\frac{\mathrm{drag}*\mathrm{d_x}}{\mathrm{v_i}}\ and\ \mathrm{v_{y0}}=v_{x0}-v_0$$

Now we have derived the equations of initial velocity for a given final position. Furthermore, we must find the correct branch in Lambert W function and the domain of the equations and. I'll start with case 2 because it is easier.

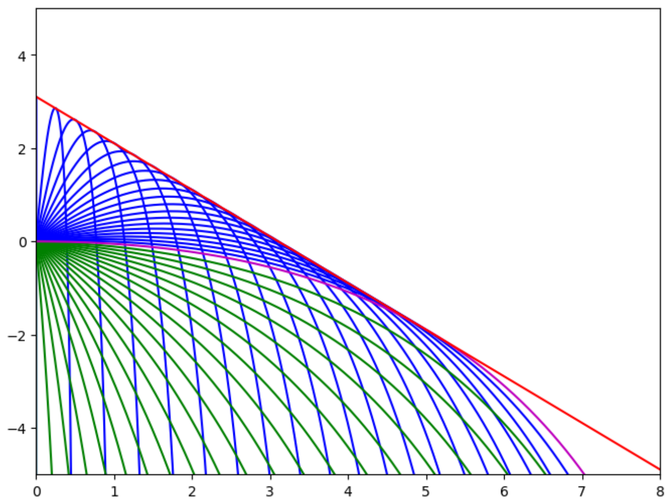

Figure 1. Simulation of trajectory where $v_0=10, drag=1$

Green curve is the trajectory in case 2, blue curve is the trajectory in case 1, magenta curve is the horizontal projectile trajectory, and the red line is the upper bound of the all the trajectories

Case 2: where object goes down initially

If $v_0>-v_T=\frac{g}{drag}$, then $r_{iv2}>0\Rightarrow r_{iv2}e^{r_{ic2}}>0$. Because the input of Lambert W function is greater than zero, we choose W_0 branch.

On the other hand, if $v_0<-v_T=\frac{g}{drag}$, the input of Lambert W function is smaller than zero. Therefore, we must choose a branch. By definition, $v_{x0} > 0$,

$$\Rightarrow v_i = \frac{\mathrm{drag} * \mathrm{d_x}}{\mathrm{v_{x0}}} > 0 \Rightarrow W(r_{iv2}e^{r_{ic2}}) > r_{iv2}$$

Because $r_{iv2} = -\frac{v_0}{v_T} - 1 < -1$ and the final position to initial velocity is bijection function, we choose W_0 branch.

From the discussion above, no matter the magnitude of v_0, we always choose W_0 branch.

All the possible trajectories must be below the horizontal projectile trajectory where $v_{x0} = v_0$, and $v_{y0} = 0$. Introducing the condition into eq7 and eq8:

$$\begin{cases} v_i = \dfrac{drag * d_x}{v_0} \\ 1 - v_i = exp\left(-v_i - \dfrac{drag * d_y}{v_T}\right) \end{cases}$$

$$\Rightarrow v_i = 1 - exp\left(\frac{-drag * d_y}{v_T}\right) exp(-v_i)$$

$$\Rightarrow v_i = 1 + W\left(- exp\left(\frac{-drag * d_y - v_T}{v_T}\right)\right)$$

$$\Rightarrow d_x = \frac{v_0}{drag}\left(1 + W\left(- exp\left(\frac{-drag * d_y - v_T}{v_T}\right)\right)\right) = d_x(v_0, d_y) \qquad (9)$$

This is the equation of magenta line in fig 1, the domain in case1 is the area under the curve of eq9.

Case 1 where object goes up initially

Because $v_T = -\frac{g}{drag} < 0$, $r_{iv} < 0 \Rightarrow r_{iv}e^{r_{ic}} < 0$. Because the input of Lambert W function is smaller than zero, there are always two possible branches.

From fig 1 we can see that if the final position is below the curve of eq9, there should be only one solution. By definition, $v_{x0} > 0$,

$$\Rightarrow v_i = \frac{\mathrm{drag} * \mathrm{d_x}}{\mathrm{v_{x0}}} > 0 \Rightarrow W(r_{iv}e^{r_{ic}}) > r_{iv}$$

Because $r_{iv} = \frac{v_0}{v_T} - 1 < -1$, so we choose W_0 branch.

Above the curve of eq9, both W_0 and W_{-1} branches are reasonable.

W_0 branch represents the higher trajectory and W_{-1} branch represents the lower trajectory.

The upper bound of final position happen when the input of W function equals to $-\frac{1}{e}$. In this situation, both branches approach the same state.

$$r_{iv}e^{r_{ic}} = -\frac{1}{e} \Rightarrow e^{r_{ic}+1} = -\frac{1}{r_{iv}} = \frac{v_T}{v_T - v_0}$$

$$\Rightarrow \exp\left(\frac{-drag * \left(d_y + d_x\right) + v_0}{v_T}\right) = \frac{v_T}{v_T - v_0}$$

$$\Rightarrow \exp\left(\frac{drag * \left(d_y + d_x\right) - v_0}{v_T}\right) = \frac{v_T - v_0}{v_T} = 1 - \frac{v_0}{v_T}$$

$$\Rightarrow \frac{drag * \left(d_y + d_x\right) - v_0}{v_T} = \ln\left(1 - \frac{v_0}{v_T}\right)$$

$$\Rightarrow d_x = -d_y + \frac{v_0 + v_T \ln\left(1 - \frac{v_0}{v_T}\right)}{drag} \tag{10}$$

This is the equation of the red line in fig 1.

Noted that the eq9 and eq10 are tangent to one another at their point of intersection in fig 1, and the domain composed of the area enclosed by eq9, eq10 and y axis, and the area under curve of eq9.

It is possible to calculate the point of intersection of eq9 and 10, but it is way too complicated.

Instead, I use a different approach. The derivative of curve of eq 9 is the following:

$$d_x = \frac{v_0}{drag}\left(1 + W\left(-\,exp\left(\frac{-drag * d_y - v_T}{v_T}\right)\right)\right) = d_x\left(v_0, d_y\right) \tag{9}$$

$$\Rightarrow \frac{\mathrm{drag}}{\mathrm{v}_0}d_x - 1 = W\left(-\exp\left(\frac{-drag * d_y - v_T}{v_T}\right)\right)$$

$$\Rightarrow \exp\left(\frac{-drag}{v_T}d_y - 1\right) = \left(1 - \frac{\mathrm{drag}}{\mathrm{v}_0}d_x\right)\exp\left(\frac{\mathrm{drag}}{\mathrm{v}_0}d_x - 1\right)$$

$$\Rightarrow \frac{dd_y}{dd_x}\left(\frac{-drag}{v_T}\right)\exp\left(\frac{-drag}{v_T}d_y - 1\right)$$

$$= \left(-\frac{\mathrm{drag}}{\mathrm{v}_0}\right)\exp\left(\frac{\mathrm{drag}}{\mathrm{v}_0}d_x - 1\right)$$

$$+ \left(1 - \frac{\mathrm{drag}}{\mathrm{v}_0}d_x\right)\frac{\mathrm{drag}}{\mathrm{v}_0}\exp\left(\frac{\mathrm{drag}}{\mathrm{v}_0}d_x - 1\right)$$

$$= -\frac{\mathrm{drag}}{\mathrm{v_0}} d_x \frac{\mathrm{drag}}{\mathrm{v_0}} \exp\left(\frac{\mathrm{drag}}{\mathrm{v_0}} d_x - 1\right)$$

$$\Rightarrow \frac{dd_y}{dd_x} = d_x \frac{\mathrm{drag}}{\mathrm{v_0}} \frac{v_T}{\mathrm{v_0}} \exp\left(\frac{\mathrm{drag}}{\mathrm{v_0}} d_x + \frac{drag}{v_T} d_y\right)$$

Within the area enclosed by eq9, eq10 and y axis, $\frac{dd_y}{dd_x} \geq -1$

Appendix1: lambert W function

1.General solution $\mathrm{x} = \mathrm{a} + \mathrm{be}^{-\mathrm{cx}} \Rightarrow \mathrm{x} = \mathrm{a} + \frac{1}{\mathrm{c}} W(bc\, e^{-ac})$

2.Approximation:

Initial guess

If (z > 0) $\mathrm{w}_0 = 0.75 * \ln(z+1)$

If (-1/e < z < 0) $w_0 = (e * z + 1)^{0.42} - 1,\ \mathrm{w}_{-1} = \ln(-z)$

And then 3 Iteration of $\mathrm{w}_{\mathrm{new}} = w - \frac{we^w - z}{e^w(w+1)} = \frac{e^w * w * w * + z}{e^w(w+1)}$, (Newton's method)

Alternative method:

Halley's method: $\mathrm{w}_{\mathrm{new}} = w - \frac{we^w - z}{e^w(w+1) - \frac{(w+2)(we^w - z)}{2w+2}}$

Appendix2: Ideal situation without drag

$$\mathrm{d_x} = v_{x0}t, d_y = v_{y0}t - \frac{1}{2}gt^2$$

1. the time required to reach d_y

$$\mathrm{t} = \frac{\mathrm{v_{y0}}}{g} + \sqrt{\left(\frac{v_{y0}}{g}\right)^2 - \frac{2d_y}{g}}$$

2. angle of velocity based on final position d_x and d_y

let $\mathrm{a} = \frac{\mathrm{g}}{2}\left(\frac{\mathrm{d_x}}{v_0}\right)^2$

$$\frac{v_y}{v_x} = \tan\theta = \frac{\mathrm{d_x} \pm \sqrt{\mathrm{d_x^2} - 4a(d_y + a)}}{2a} = \frac{\mathrm{v_0^2}}{gd_x}\left(1 \pm \sqrt{1 - \frac{2gd_y}{v_0^2} - \frac{g^2 d_x^2}{v_0^4}}\right)$$

Let $\mathrm{r} = \frac{gd_x}{\mathrm{v_0^2}}, \mathrm{Delta} = -r^2 - \frac{2d_y}{d_x}\mathrm{r} + 1$

$$\tan\theta = \frac{1}{r}\left(1 \pm \sqrt{1 - \frac{2d_y}{d_x}\mathrm{r} - r^2}\right) = \frac{1}{r}\left(1 \pm \sqrt{Delta}\right)$$