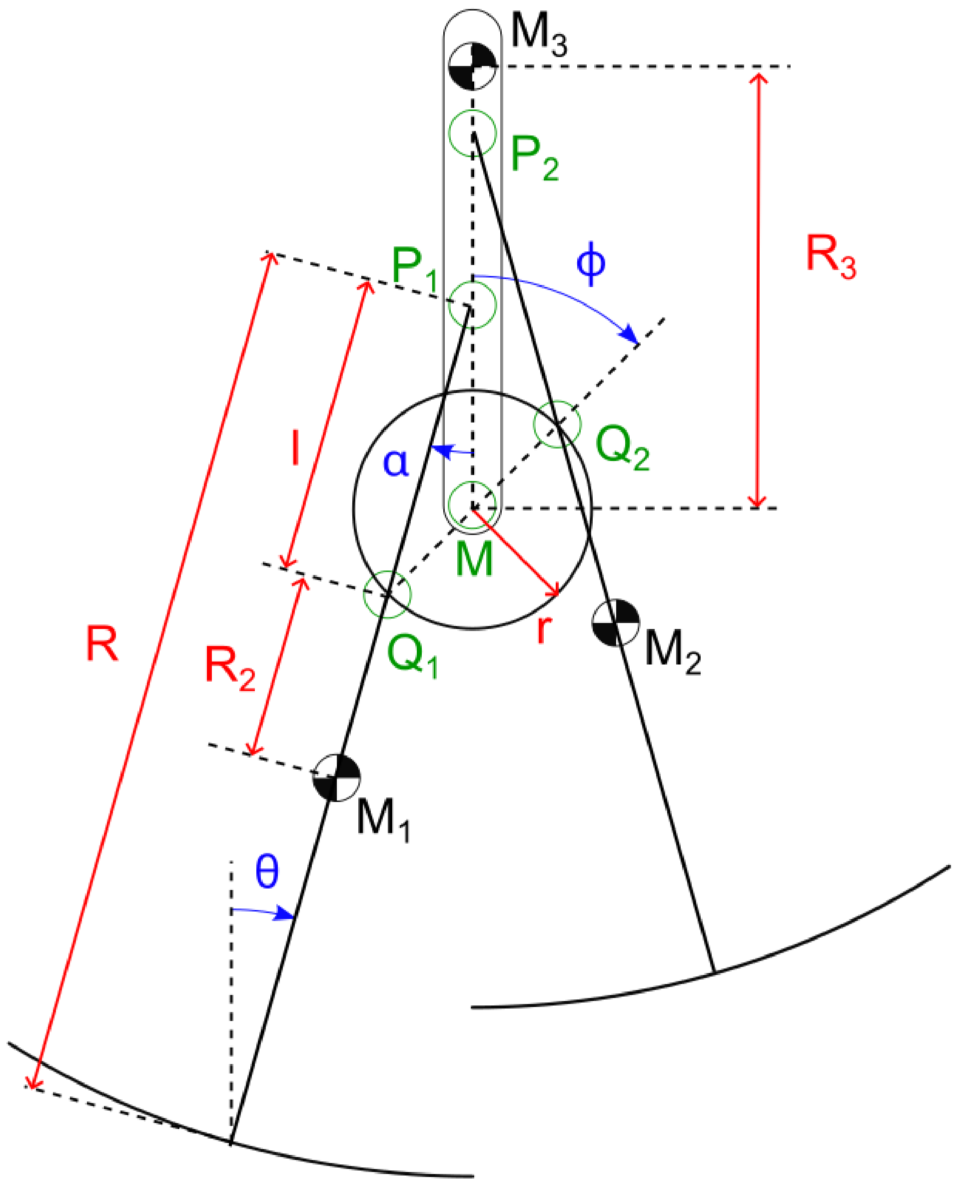

$$\beta = \arcsin\frac{r}{l}$$

$$P_1 = (R\theta, R)$$

$$Q_1 = P_1 + l(-\sin\theta, -\cos\theta)$$

$$\alpha = \sin^{-1}\left(\frac{r}{l}\sin\phi\right) \Rightarrow \sin\alpha = \frac{r}{l}\sin\phi \Rightarrow \dot{\alpha} = \frac{r\cos\phi}{l\cos\alpha}\dot{\phi} \Rightarrow \frac{\partial\alpha}{\partial\phi} = \frac{\partial\dot{\alpha}}{\partial\dot{\phi}} = \frac{r\cos\phi}{l\cos\alpha}$$

$$\Rightarrow \ddot{\alpha} = \frac{r\cos\phi}{l\cos\alpha}\ddot{\phi} - \frac{r\sin\phi}{l\cos\alpha}\dot{\phi}^2 + \tan\alpha\,\dot{\alpha}^2$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial \dot{\alpha}}{\partial \dot{\phi}}\right) = \frac{d}{dt}\left(\frac{r\cos\phi}{l\cos\alpha}\right) = -\frac{r\sin\phi}{l\cos\alpha}\dot{\phi} + \frac{r\cos\phi}{l\cos\alpha}\tan\alpha\,\dot{\alpha}$$

$$\Rightarrow \left(\frac{\partial \dot{\alpha}}{\partial \phi}\right) = -\frac{r\sin\phi}{l\cos\alpha}\dot{\phi} + \frac{r\cos\phi}{l\cos\alpha}\frac{r\cos\phi}{l\cos\alpha}\tan\alpha\,\dot{\phi}$$

$$M = Q_1 + r(\sin\theta + \phi - \alpha, \cos\theta + \phi - \alpha)$$

$$Q_2 = Q_1 + 2r(\sin\theta + \phi - \alpha, \cos\theta + \phi - \alpha)$$

$$P_2 = Q_2 + l(\sin\theta - 2\alpha, \cos\theta - 2\alpha)$$

$$M_1 = Q_1 + R_2(-\sin\theta, -\cos\theta) = (R\theta, R) + (l + R_2)(-\sin\theta, -\cos\theta)$$

$$\begin{aligned} M_2 = Q_2 + R_2(-\sin\theta - 2\alpha, -\cos\theta - 2\alpha) \\ &= (R\theta, R) + l(-\sin\theta, -\cos\theta) + 2r(\sin\theta + \phi - \alpha, \cos\theta + \phi - \alpha) \\ &+ R_2(-\sin\theta - 2\alpha, -\cos\theta - 2\alpha) \end{aligned}$$

$$\begin{aligned} M_3 = M + R_3(\sin\theta - \alpha, \cos\theta - \alpha) \\ &= (R\theta, R) + l(-\sin\theta, -\cos\theta) + r(\sin\theta + \phi - \alpha, \cos\theta + \phi - \alpha) \\ &+ R_3(\sin\theta - \alpha, \cos\theta - \alpha) \end{aligned}$$

$$M_T = \frac{m_1 M_1 + m_2 M_2 + m_3 M_3}{m_1 + m_2 + m_3}$$

$$\frac{dM_1}{dt} = (R, 0)\dot{\theta} + (l + R_2)(-\cos\theta, \sin\theta)\dot{\theta}$$

$$\frac{d^2 M_1}{dt^2} = (R, 0)\ddot{\theta} + (l + R_2)(-\cos\theta, \sin\theta)\ddot{\theta} + (l + R_2)(\sin\theta, \cos\theta)\dot{\theta}^2$$

$$\frac{\partial}{\partial \dot{\theta}}\left(\frac{dM_1}{dt}\right) = (R, 0) + (l + R_2)(-\cos\theta, \sin\theta)$$

$$\frac{\partial}{\partial \dot{\phi}}\left(\frac{dM_1}{dt}\right) = 0$$

$$\frac{\partial}{\partial\dot{\theta}}\left(\frac{dM_1}{dt}\right)\cdot\frac{d^2M_1}{dt^2}$$
$$= R^2\ddot{\theta} - R(l+R_2)\cos\theta\,\ddot{\theta} + R(l+R_2)\sin\theta\,\dot{\theta}^2 - R(l+R_2)\cos\theta\,\ddot{\theta}$$
$$+ (l+R_2)^2\ddot{\theta}$$
$$= [R^2 - 2R(l+R_2)\cos\theta + (l+R_2)^2]\ddot{\theta} + R(l+R_2)\sin\theta\,\dot{\theta}^2$$

$$\frac{\partial}{\partial\dot{\phi}}\left(\frac{dM_1}{dt}\right)\cdot\frac{d^2M_1}{dt^2} = 0$$

$$\frac{dM_2}{dt} = (R,0)\dot{\theta} + l(-\cos\theta\,,\sin\theta)\dot{\theta}$$
$$+ 2r(\cos\theta + \phi - \alpha\,, -\sin\theta + \phi - \alpha)\left(\dot{\theta} + \dot{\phi} - \dot{\alpha}\right)$$
$$+ R_2(-\cos\theta - 2\alpha\,, \sin\theta - 2\alpha)\left(\dot{\theta} - 2\dot{\alpha}\right)$$

$$\frac{d^2M_2}{dt^2} = (R,0)\ddot{\theta} + l(-\cos\theta\,,\sin\theta)\ddot{\theta}$$
$$+ 2r(\cos\theta + \phi - \alpha\,, -\sin\theta + \phi - \alpha)\left(\ddot{\theta} + \ddot{\phi} - \ddot{\alpha}\right)$$
$$+ R_2(-\cos\theta - 2\alpha\,, \sin\theta - 2\alpha)\left(\ddot{\theta} - 2\ddot{\alpha}\right) + l(\sin\theta\,,\cos\theta)\dot{\theta}^2$$
$$+ 2r(-\sin\theta + \phi - \alpha\,, -\cos\theta + \phi - \alpha)\left(\dot{\theta} + \dot{\phi} - \dot{\alpha}\right)^2$$
$$+ R_2(\sin\theta - 2\alpha\,, \cos\theta - 2\alpha)\left(\dot{\theta} - 2\dot{\alpha}\right)^2$$

$$\frac{\partial}{\partial\dot{\theta}}\left(\frac{dM_2}{dt}\right) = (R,0) + l(-\cos\theta\,,\sin\theta) + 2r(\cos\theta + \phi - \alpha\,, -\sin\theta + \phi - \alpha)$$
$$+ R_2(-\cos\theta - 2\alpha\,, \sin\theta - 2\alpha)$$

$$\frac{\partial}{\partial\dot{\phi}}\left(\frac{dM_2}{dt}\right) = 2r(\cos\theta + \phi - \alpha\,, -\sin\theta + \phi - \alpha)\left(1 - \frac{r\cos\phi}{l\cos\alpha}\right)$$
$$+ R_2(-\cos\theta - 2\alpha\,, \sin\theta - 2\alpha)\left(-2\frac{r\cos\phi}{l\cos\alpha}\right)$$

$$\frac{\partial}{\partial\dot{\theta}}\left(\frac{dM_2}{dt}\right)\cdot\frac{d^2M_2}{dt^2}$$
$$= R^2\ddot{\theta} - Rl\cos\theta\,\ddot{\theta} + 2Rr\cos\theta + \phi - \alpha\left(\ddot{\theta} + \ddot{\phi} - \ddot{\alpha}\right)$$
$$- RR_2\cos\theta - 2\alpha\left(\ddot{\theta} - 2\ddot{\alpha}\right) + Rl\sin\theta\,\dot{\theta}^2$$
$$- 2Rr\sin\theta + \phi - \alpha\left(\dot{\theta} + \dot{\phi} - \dot{\alpha}\right)^2 + RR_2\sin\theta - 2\alpha\left(\dot{\theta} - 2\dot{\alpha}\right)^2$$

$$-Rl\cos\theta\,\ddot{\theta}+l^2\ddot{\theta}-2rl\cos\phi-\alpha\left(\ddot{\theta}+\ddot{\phi}-\ddot{\alpha}\right)+R_2l\cos 2\alpha\left(\ddot{\theta}-2\ddot{\alpha}\right)$$
$$+2rl\sin\phi-\alpha\left(\dot{\theta}+\dot{\phi}-\dot{\alpha}\right)^2+R_2l\sin 2\alpha\left(\dot{\theta}-2\dot{\alpha}\right)^2$$

$$2Rr\cos\theta+\phi-\alpha\,\ddot{\theta}-2rl\cos\phi-\alpha\,\ddot{\theta}+4r^2\left(\ddot{\theta}+\ddot{\phi}-\ddot{\alpha}\right)$$
$$-2rR_2\cos\phi+\alpha\left(\ddot{\theta}-2\ddot{\alpha}\right)-2rl\sin\phi-\alpha\,\dot{\theta}^2$$
$$-2rR_2\sin\phi+\alpha\left(\dot{\theta}-2\dot{\alpha}\right)^2$$

$$-RR_2\cos\theta-2\alpha\,\ddot{\theta}+lR_2\cos 2\alpha\,\ddot{\theta}-2rR_2\cos\phi+\alpha\left(\ddot{\theta}+\ddot{\phi}-\ddot{\alpha}\right)+R_2^2\left(\ddot{\theta}-2\ddot{\alpha}\right)$$
$$-lR_2\sin 2\alpha\,\dot{\theta}^2+2rR_2\sin\phi+\alpha\left(\dot{\theta}+\dot{\phi}-\dot{\alpha}\right)^2$$

$$=(R^2+l^2+4r^2+R_2^2-2Rl\cos\theta+4Rr\cos\theta+\phi-\alpha-2RR_2\cos\theta-2\alpha$$
$$-4lr\cos\phi-\alpha+2lR_2\cos 2\alpha-4rR_2\cos\phi+\alpha)\ddot{\theta}$$
$$+(4r^2+2Rr\cos\theta+\phi-\alpha-2rl\cos\phi-\alpha-2rR_2\cos\phi+\alpha)\ddot{\phi}$$
$$+(-4r^2-2R_2^2-2Rr\cos\theta+\phi-\alpha+2RR_2\cos\theta-2\alpha+2lr\cos\phi-\alpha$$
$$-2lR_2\cos 2\alpha+6rR_2\cos\phi+\alpha)\ddot{\alpha}$$

$$+Rl\sin\theta\,\dot{\theta}^2-2Rr\sin\theta+\phi-\alpha\left(\dot{\theta}+\dot{\phi}-\dot{\alpha}\right)^2+RR_2\sin\theta-2\alpha\left(\dot{\theta}-2\dot{\alpha}\right)^2$$
$$+2lr\sin\phi-\alpha\left(2\dot{\theta}+\dot{\phi}-\dot{\alpha}\right)\left(\dot{\phi}-\dot{\alpha}\right)$$
$$+lR_2\sin 2\alpha\left(2\dot{\theta}-2\dot{\alpha}\right)(-2\dot{\alpha})$$
$$+2rR_2\sin\phi+\alpha\left(2\dot{\theta}+\dot{\phi}-3\dot{\alpha}\right)\left(\dot{\phi}+\dot{\alpha}\right)$$

$$\frac{\partial}{\partial\dot{\phi}}\left(\frac{dM_2}{dt}\right)\cdot\frac{d^2M_2}{dt^2}$$

$$=\left(1-\frac{r\cos\phi}{l\cos\alpha}\right)\Big((4r^2+2Rr\cos\theta+\phi-\alpha-2lr\cos\phi-\alpha-2rR_2\cos\phi+\alpha)\ddot{\theta}$$
$$+4r^2\ddot{\phi}+(-4r^2+4rR_2\cos\phi+\alpha)\ddot{\alpha}-2lr\sin\phi-\alpha\,\dot{\theta}^2$$
$$-2rR_2\sin\phi+\alpha\left(\dot{\theta}-2\dot{\alpha}\right)^2\Big)$$

$$+\left(-2\frac{r\cos\phi}{l\cos\alpha}\right)\Big((R_2^2 - RR_2\cos\theta - 2\alpha + lR_2\cos 2\alpha - 2rR_2\cos\phi + \alpha)\ddot{\theta}$$
$$- 2rR_2\cos\phi + \alpha\,\ddot{\phi} + (-2R_2^2 + 2rR_2\cos\phi + \alpha)\ddot{\alpha} - lR_2\sin 2\alpha\,\dot{\theta}^2$$
$$+ 2rR_2\sin\phi + \alpha\left(\dot{\theta} + \dot{\phi} - \dot{\alpha}\right)^2\Big)$$

$$\frac{dM_3}{dt} = (R, 0)\dot{\theta} + l(-\cos\theta, \sin\theta)\dot{\theta}$$
$$+ r(\cos\theta + \phi - \alpha, -\sin\theta + \phi - \alpha)\left(\dot{\theta} + \dot{\phi} - \dot{\alpha}\right)$$
$$+ R_3(\cos\theta - \alpha, -\sin\theta - \alpha)\left(\dot{\theta} - \dot{\alpha}\right)$$

$$\frac{d^2M_3}{dt^2} = (R, 0)\ddot{\theta} + l(-\cos\theta, \sin\theta)\ddot{\theta}$$
$$+ r(\cos\theta + \phi - \alpha, -\sin\theta + \phi - \alpha)\left(\ddot{\theta} + \ddot{\phi} - \ddot{\alpha}\right)$$
$$+ R_3(\cos\theta - \alpha, -\sin\theta - \alpha)\left(\ddot{\theta} - \ddot{\alpha}\right) + l(\sin\theta, \cos\theta)\dot{\theta}^2$$
$$+ r(-\sin\theta + \phi - \alpha, -\cos\theta + \phi - \alpha)\left(\dot{\theta} + \dot{\phi} - \dot{\alpha}\right)^2$$
$$+ R_3(-\sin\theta - \alpha, -\cos\theta - \alpha)\left(\dot{\theta} - \dot{\alpha}\right)^2$$

$$\frac{\partial}{\partial\dot{\theta}}\left(\frac{dM_3}{dt}\right) = (R, 0) + l(-\cos\theta, \sin\theta) + r(\cos\theta + \phi - \alpha, -\sin\theta + \phi - \alpha)$$
$$+ R_3(\cos\theta - \alpha, -\sin\theta - \alpha)$$

$$\frac{\partial}{\partial\dot{\phi}}\left(\frac{dM_3}{dt}\right) = r(\cos\theta + \phi - \alpha, -\sin\theta + \phi - \alpha)\left(1 - \frac{r\cos\phi}{l\cos\alpha}\right)$$
$$+ R_3(\cos\theta - \alpha, -\sin\theta - \alpha)\left(-\frac{r\cos\phi}{l\cos\alpha}\right)$$

$$\frac{\partial}{\partial\dot{\theta}}\left(\frac{dM_3}{dt}\right)\cdot\frac{d^2M_3}{dt^2}$$
$$= R^2\ddot{\theta} - Rl\cos\theta\,\ddot{\theta} + Rr\cos\theta + \phi - \alpha\left(\ddot{\theta} + \ddot{\phi} - \ddot{\alpha}\right)$$
$$+ RR_3\cos\theta - \alpha\left(\ddot{\theta} - \ddot{\alpha}\right) + Rl\sin\theta\,\dot{\theta}^2$$
$$- Rr\sin\theta + \phi - \alpha\left(\dot{\theta} + \dot{\phi} - \dot{\alpha}\right)^2 - RR_3\sin\theta - \alpha\left(\dot{\theta} - \dot{\alpha}\right)^2$$

$$-Rl\cos\theta\,\ddot{\theta} + l^2\ddot{\theta} - rl\cos\phi - \alpha\left(\ddot{\theta} + \ddot{\phi} - \ddot{\alpha}\right) - R_3 l\cos\alpha\left(\ddot{\theta} - \ddot{\alpha}\right)$$
$$+ rl\sin\phi - \alpha\left(\dot{\theta} + \dot{\phi} - \dot{\alpha}\right)^2 - R_3 l\sin\alpha\left(\dot{\theta} - \dot{\alpha}\right)^2$$

$$Rr\cos\theta+\phi-\alpha\,\ddot{\theta}-rl\cos\phi-\alpha\,\ddot{\theta}+r^2\left(\ddot{\theta}+\ddot{\phi}-\ddot{\alpha}\right)+rR_3\cos\phi\left(\ddot{\theta}-\ddot{\alpha}\right)$$
$$-rl\sin\phi-\alpha\,\dot{\theta}^2+rR_3\sin\phi\left(\dot{\theta}-\dot{\alpha}\right)^2$$

$$RR_3\cos\theta-\alpha\,\ddot{\theta}-lR_3\cos\alpha\,\ddot{\theta}+rR_3\cos\phi\left(\ddot{\theta}+\ddot{\phi}-\ddot{\alpha}\right)+R_3^2\left(\ddot{\theta}-\ddot{\alpha}\right)$$
$$+lR_3\sin\alpha\,\dot{\theta}^2-rR_3\sin\phi\left(\dot{\theta}+\dot{\phi}-\dot{\alpha}\right)^2$$

$$=(R^2+l^2+r^2+R_3^2-2Rl\cos\theta+2Rr\cos\theta+\phi-\alpha+2RR_3\cos\theta-\alpha$$
$$-2lr\cos\phi-\alpha-2lR_3\cos\alpha+2rR_3\cos\phi)\ddot{\theta}$$

$$+(r^2+Rr\cos\theta+\phi-\alpha-lr\cos\phi-\alpha+rR_3\cos\phi)\ddot{\phi}$$

$$(-r^2-R_3^2-Rr\cos\theta+\phi-\alpha-RR_3\cos\theta-\alpha+lr\cos\phi-\alpha+lR_3\cos\alpha$$
$$-2rR_3\cos\phi)\ddot{\alpha}$$

$$+Rl\sin\theta\,\dot{\theta}^2-Rr\sin\theta+\phi-\alpha\left(\dot{\theta}+\dot{\phi}-\dot{\alpha}\right)^2-RR_3\sin\theta-\alpha\left(\dot{\theta}-\dot{\alpha}\right)^2$$

$$+lr\sin\phi-\alpha\left(2\dot{\theta}+\dot{\phi}-\dot{\alpha}\right)\left(\dot{\phi}-\dot{\alpha}\right)$$

$$+lR_3\sin\alpha\left(2\dot{\theta}-\dot{\alpha}\right)(\dot{\alpha})$$

$$+rR_3\sin\phi\left(2\dot{\theta}+\dot{\phi}-2\dot{\alpha}\right)\left(-\dot{\phi}\right)$$

$$\frac{\partial}{\partial\dot{\phi}}\left(\frac{dM_3}{dt}\right)\cdot\frac{d^2M_3}{dt^2}$$
$$=\left(1-\frac{r\cos\phi}{l\cos\alpha}\right)\Big(Rr\cos\theta+\phi-\alpha\,\ddot{\theta}-rl\cos\phi-\alpha\,\ddot{\theta}$$
$$+r^2\left(\ddot{\theta}+\ddot{\phi}-\ddot{\alpha}\right)+rR_3\cos\phi\left(\ddot{\theta}-\ddot{\alpha}\right)-rl\sin\phi-\alpha\,\dot{\theta}^2$$
$$+rR_3\sin\phi\left(\dot{\theta}-\dot{\alpha}\right)^2\Big)$$
$$+\left(-\frac{r\cos\phi}{l\cos\alpha}\right)\Big(RR_3\cos\theta-\alpha\,\ddot{\theta}-lR_3\cos\alpha\,\ddot{\theta}$$
$$+rR_3\cos\phi\left(\ddot{\theta}+\ddot{\phi}-\ddot{\alpha}\right)+R_3^2\left(\ddot{\theta}-\ddot{\alpha}\right)+lR_3\sin\alpha\,\dot{\theta}^2$$
$$-rR_3\sin\phi\left(\dot{\theta}+\dot{\phi}-\dot{\alpha}\right)^2\Big)$$

$$= \left(1 - \frac{r\cos\phi}{l\cos\alpha}\right)\Big((r^2 + Rr\cos\theta + \phi - \alpha - lr\cos\phi - \alpha + rR_3\cos\phi)\ddot{\theta} + r^2\ddot{\phi}$$

$$+ (-r^2 - rR_3\cos\phi)\ddot{\alpha} - lr\sin\phi - \alpha\,\dot{\theta}^2 + rR_3\sin\phi\,\big(\dot{\theta} - \dot{\alpha}\big)^2\Big)$$

$$+ \left(-\frac{r\cos\phi}{l\cos\alpha}\right)\Big((R_3^2 + RR_3\cos\theta - \alpha - lR_3\cos\alpha + rR_3\cos\phi)\ddot{\theta} + rR_3\cos\phi\,\ddot{\phi}$$

$$+ (-R_3^2 - rR_3\cos\phi)\ddot{\alpha} + lR_3\sin\alpha\,\dot{\theta}^2 - rR_3\sin\phi\,\big(\dot{\theta} + \dot{\phi} - \dot{\alpha}\big)^2\Big)$$

$$Q_{r_i} = m\left(\frac{\partial v}{\partial \dot{r_i}}\right)^T \frac{dv}{dt}$$

Rotational energy

$$KE = \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2\big(\dot{\theta} - 2\dot{\alpha}\big)^2 + \frac{1}{2}I_3\big(\dot{\theta} - \dot{\alpha}\big)^2 + \frac{1}{2}I_D\big(\dot{\theta} - \dot{\alpha} + \dot{\phi}\big)^2$$

$$\frac{d}{dt}\left(\frac{\partial KE}{\partial \dot{\theta}}\right) = I_1\ddot{\theta} + I_2\big(\ddot{\theta} - 2\ddot{\alpha}\big) + I_3\big(\ddot{\theta} - \ddot{\alpha}\big) + I_D\big(\ddot{\theta} - \ddot{\alpha} + \ddot{\phi}\big)$$

$$= (I_1 + I_2 + I_3 + I_D)\ddot{\theta} + I_D\ddot{\phi} + (-2I_2 - I_3 - I_D)\ddot{\alpha}$$

$$\frac{d}{dt}\left(\frac{\partial KE}{\partial \dot{\phi}}\right) = \frac{d}{dt}\Big(I_2\big(\dot{\theta} - 2\dot{\alpha}\big)\left(-2\frac{r\cos\phi}{l\cos\alpha}\right) + I_3\big(\dot{\theta} - \dot{\alpha}\big)\left(-\frac{r\cos\phi}{l\cos\alpha}\right)$$

$$+ I_D\big(\dot{\theta} - \dot{\alpha} + \dot{\phi}\big)\left(1 - \frac{r\cos\phi}{l\cos\alpha}\right)\Big)$$

$$= \big(2I_2\ddot{\theta} - 4I_2\ddot{\alpha}\big)\left(-\frac{r\cos\phi}{l\cos\alpha}\right) + \big(I_3\ddot{\theta} - I_3\ddot{\alpha}\big)\left(-\frac{r\cos\phi}{l\cos\alpha}\right)$$

$$+ \big(I_D\ddot{\theta} - I_D\ddot{\alpha}\big)\left(1 - \frac{r\cos\phi}{l\cos\alpha}\right) + I_D\ddot{\phi}\left(1 - \frac{r\cos\phi}{l\cos\alpha}\right)$$

$$+ \big[-2I_2\big(\dot{\theta} - 2\dot{\alpha}\big) - I_3\big(\dot{\theta} - \dot{\alpha}\big) - I_D\big(\dot{\theta} - \dot{\alpha} + \dot{\phi}\big)\big]\frac{d}{dt}\left(\frac{\partial \dot{\alpha}}{\partial \dot{\phi}}\right)$$

$$= \left[I_D + (-2I_2 - I_3 - I_D)\frac{r\cos\phi}{l\cos\alpha}\right]\ddot{\theta} + I_D\left(1 - \frac{r\cos\phi}{l\cos\alpha}\right)\ddot{\phi}$$

$$+ \left[-I_D + (4I_2 + I_3 + I_D)\frac{r\cos\phi}{l\cos\alpha}\right]\ddot{\alpha}$$

$$+ \big[-2I_2\big(\dot{\theta} - 2\dot{\alpha}\big) - I_3\big(\dot{\theta} - \dot{\alpha}\big) - I_D\big(\dot{\theta} - \dot{\alpha} + \dot{\phi}\big)\big]\frac{d}{dt}\left(\frac{\partial \dot{\alpha}}{\partial \dot{\phi}}\right)$$

$$\frac{\partial KE}{\partial \phi} = I_2(\dot{\theta} - 2\dot{\alpha})\left(-2\frac{\partial \dot{\alpha}}{\partial \phi}\right) + I_3(\dot{\theta} - \dot{\alpha})\left(-\frac{\partial \dot{\alpha}}{\partial \phi}\right) + I_D(\dot{\theta} - \dot{\alpha} + \dot{\phi})\left(-\frac{\partial \dot{\alpha}}{\partial \phi}\right)$$

$$= \left[-2I_2(\dot{\theta} - 2\dot{\alpha}) - I_3(\dot{\theta} - \dot{\alpha}) - I_D(\dot{\theta} - \dot{\alpha} + \dot{\phi})\right]\left(\frac{\partial \dot{\alpha}}{\partial \phi}\right)$$

Gravitational energy

$$PE = m_1 g(R - (l + R_2)\cos\theta)$$
$$+ m_2 g(R - l\cos\theta + 2r\cos\theta + \phi - \alpha - R_2\cos\theta - 2\alpha)$$
$$+ m_3 g(R - l\cos\theta + r\cos\theta + \phi - \alpha + R_3\cos\theta - \alpha)$$

$$= (m_1 + m_2 + m_3)gR + (-m_1(l + R_2) - m_2 l - m_3 l)g\cos\theta$$
$$+ (2m_2 r + m_3 r)g\cos\theta + \phi - \alpha - m_2 R_2 g\cos\theta - 2\alpha$$
$$+ m_3 R_3 g\cos\theta - \alpha$$

$$\frac{\partial PE}{\partial \theta} = (m_1(l + R_2) + m_2 l + m_3 l)g\sin\theta + (-2m_2 r - m_3 r)g\sin\theta + \phi - \alpha$$
$$+ m_2 R_2 g\sin\theta - 2\alpha - m_3 R_3 g\sin\theta - \alpha$$

$$\frac{\partial PE}{\partial \phi} = -(2m_2 r + m_3 r)g\left(1 - \frac{r\cos\phi}{l\cos\alpha}\right)\sin\theta + \phi - \alpha$$
$$+ m_2 R_2 g\left(-2\frac{r\cos\phi}{l\cos\alpha}\right)\sin\theta - 2\alpha - m_3 R_3 g\left(-\frac{r\cos\phi}{l\cos\alpha}\right)\sin\theta - \alpha$$

Overall,

$$0 = m_1 \frac{\partial}{\partial \dot{\theta}}\left(\frac{dM_1}{dt}\right) \cdot \frac{d^2 M_1}{dt^2} + m_2 \frac{\partial}{\partial \dot{\theta}}\left(\frac{dM_2}{dt}\right) \cdot \frac{d^2 M_2}{dt^2} + m_3 \frac{\partial}{\partial \dot{\theta}}\left(\frac{dM_3}{dt}\right) \cdot \frac{d^2 M_3}{dt^2}$$
$$+ \frac{d}{dt}\left(\frac{\partial KE}{\partial \dot{\theta}}\right) + \frac{\partial PE}{\partial \theta}$$

$$\tau = m_1 \frac{\partial}{\partial \dot{\phi}}\left(\frac{dM_1}{dt}\right) \cdot \frac{d^2 M_1}{dt^2} + m_2 \frac{\partial}{\partial \dot{\phi}}\left(\frac{dM_2}{dt}\right) \cdot \frac{d^2 M_2}{dt^2} + m_3 \frac{\partial}{\partial \dot{\phi}}\left(\frac{dM_3}{dt}\right) \cdot \frac{d^2 M_3}{dt^2}$$
$$+ \frac{d}{dt}\left(\frac{\partial KE}{\partial \dot{\phi}}\right) + \frac{\partial PE}{\partial \phi}$$

If I keep the rotation of disk constant, such that $\dot{\phi} = \omega$.

Without noise and disturbance, if the initial state is $\begin{bmatrix}\theta \\ \phi \\ \dot{\theta} \\ \dot{\phi}\end{bmatrix} = \begin{bmatrix}\beta \\ \frac{\pi}{2} \\ 0 \\ 0\end{bmatrix}$, then for certain range of ω, the trajectory is smooth and stable. And the trajectory can be fitted by the function of $\theta = (\sin\phi\, p_1) \times p_2 + p_3 \times \phi$

Therefore, I added two feedback so that

$$\ddot{\phi} = -K_1(\dot{\phi} - \omega) - K_2[(\sin\phi\, p_1) \times p_2 + p_3 \times \phi - \theta]$$

K_1 term is to keep $\dot{\phi}$ constant, K_2 term is to keep the trajectory stable.