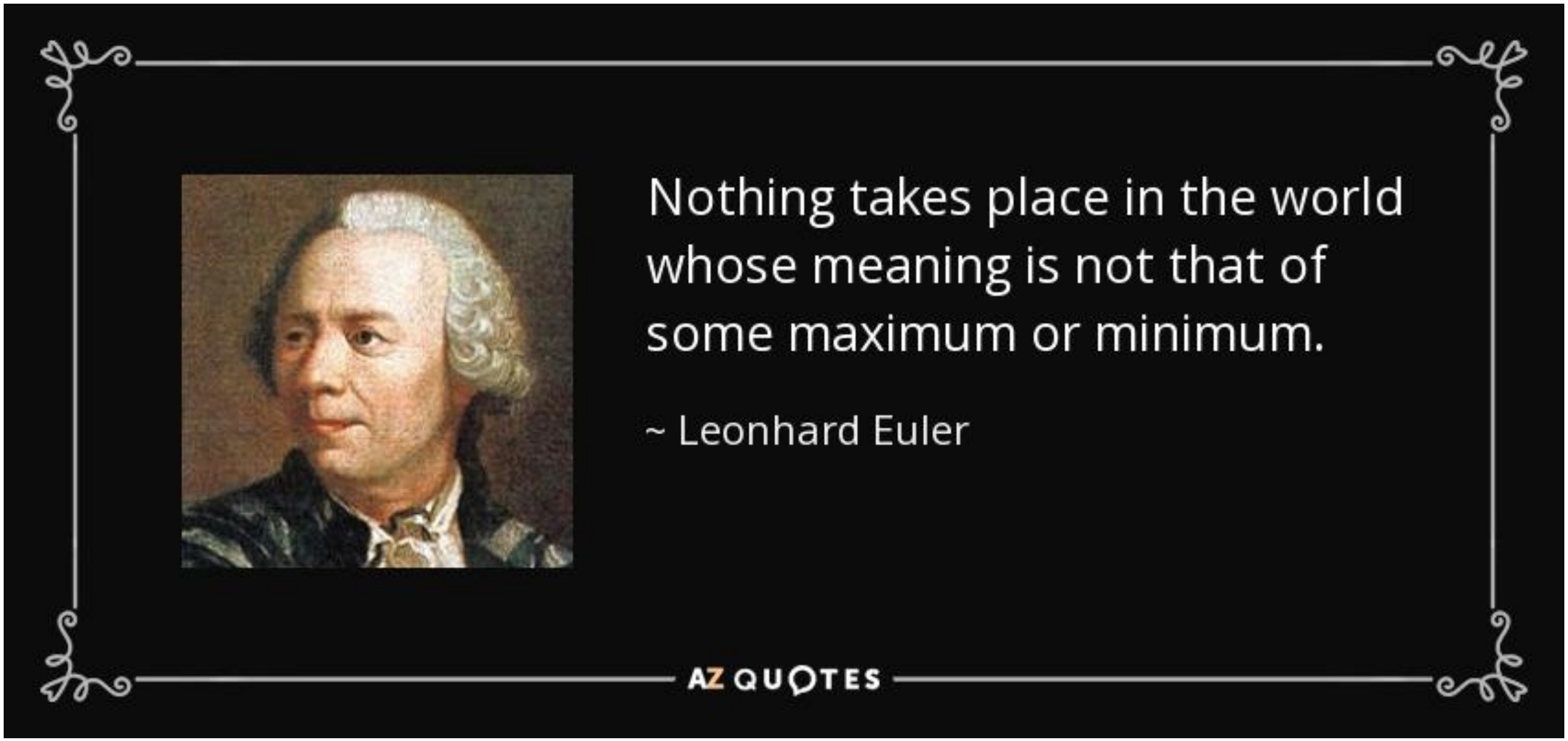
Maximum and Minimum Values

Section 4.1

Outline


Nothing takes place in the world whose meaning is not that of some maximum or minimum.
~ Leonhard Euler
AZ QUOTES

Definition

1 Definition Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geqslant f(x)$ for all x in D.
- **absolute minimum** value of f on D if $f(c) \leqslant f(x)$ for all x in D.

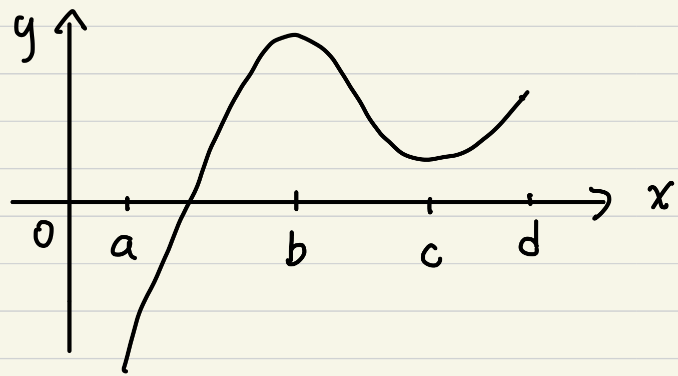
2 Definition The number $f(c)$ is a

- **local maximum** value of f if $f(c) \geqslant f(x)$ when x is near c.
- **local minimum** value of f if $f(c) \leqslant f(x)$ when x is near c.

Definition

- An absolute maximum or minimum is sometimes called a **global** maximum or minimum.
- The maximum and minimum values of f are called **extreme values** of f.
- In Definition 2, if we say that something is true **near** c, we mean that it is true on some open interval containing c. Hence, the local extreme values always occur in the **interior** of the domain of f.

Ex: The graph of $f(x)$ is given as below. Find the local extreme values and global extreme values of $f(x)$.



Local extreme values

Global extreme values

Theorem

$\boxed{3}$ **The Extreme Value Theorem** If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Remark:
Be careful about the requirements of the Extreme Value Theorem: f is *continuous* on a *closed* interval.

Remark: If requirements are not satisfied, the conclusion is not true.

① Continuity of $f(x)$

Counterexample:

② The interval is closed and bounded

counterexample:

Theorem

$\boxed{4}$ **Fermat's Theorem** If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

- Note:
- Be careful of applying Fermat's Theorem. The inverse of the theorem may not be true.

Ex: Prove Fermat's Theorem.

Warning: Common mistakes of applying Fermat's Theorem

① If $f(x)$ has local extreme value at c, then $f'(c) = 0$

counterexample:

② If $f'(c) = 0$, then $f(x)$ attains local extreme value at c.

counterexample:

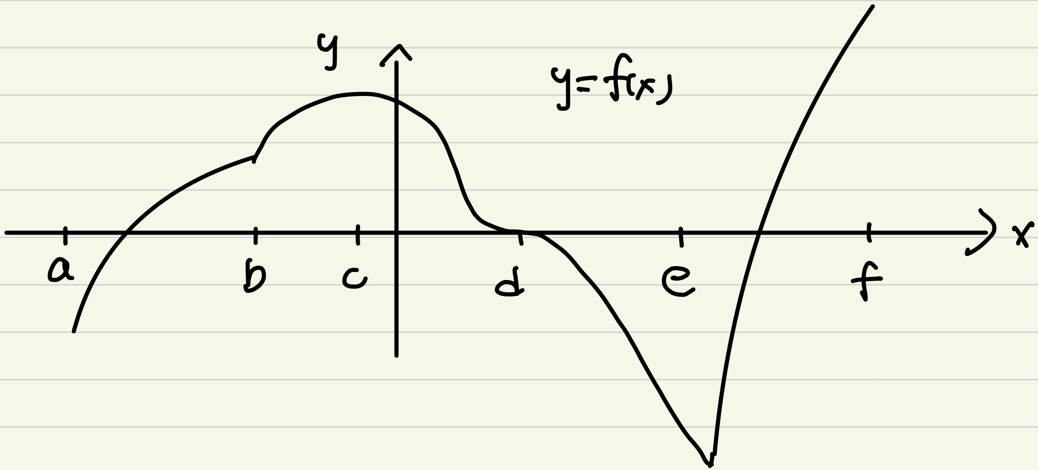
Theorem

- Fermat's Theorem does suggest that we should at least start looking for extreme values of f at the numbers c where $f'(c) = 0$ or where $f'(c)$ does not exist. Such numbers are given a special name.

6 **Definition** A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

7 If f has a local maximum or minimum at c, then c is a critical number of f.

Ex: The graph of $f(x)$ is given as below. Find critical numbers of $f(x)$ and local / global extreme values.



Summary

- To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local or it occurs at an endpoint of the interval.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Ex: Find absolute extreme values of $f(x) = x^2 - 3|x+1|$ on $[-2, 3]$.

Ex: Find absolute extreme values of $f(x) = x^{\frac{1}{3}}(x-2)$ on $[-1, 2]$.

Ex: Find absolute extreme values of $f(x) = 2\sin x - \sin 2x$ on the interval $[-\frac{\pi}{2}, \pi]$

Review

- State the definition of the absolute / local maximum and minimum of a function. What is a critical number of a function?
- State the Extreme Value Theorem.
- State Fermat's theorem.
- Describe the method of finding extreme values of a function on a closed interval.