Cylinders and Quadric Surfaces

Section 12.6

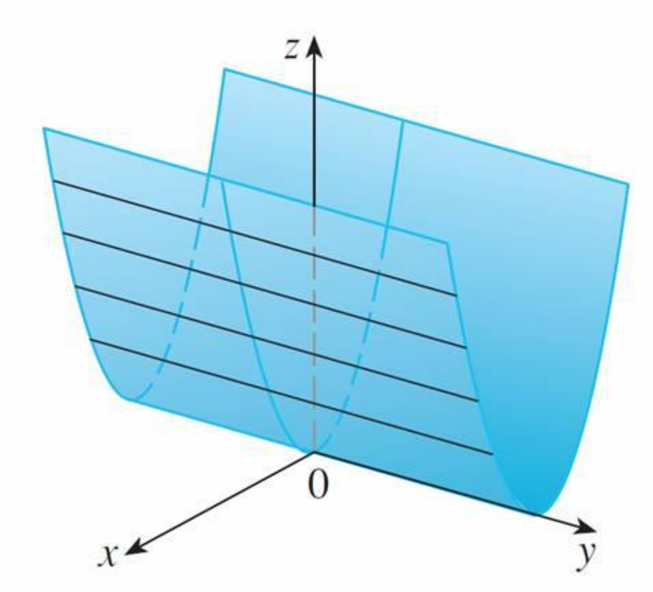
Outline

- Cylinders and Quadric Surfaces
  - Cylinders 柱面
  - Ellipsoid 橢球面
  - Hyperboloid of One Sheet 單葉双曲面
  - Hyperboloid of Two Sheets 双葉双曲面
  - Elliptic Paraboloid 橢圓拋物面
  - Hyperbolic Paraboloid 双曲拋物面

- In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called **traces** (or **cross-sections**) of the surface.
- Now we will sketch some common surfaces with the help of traces.

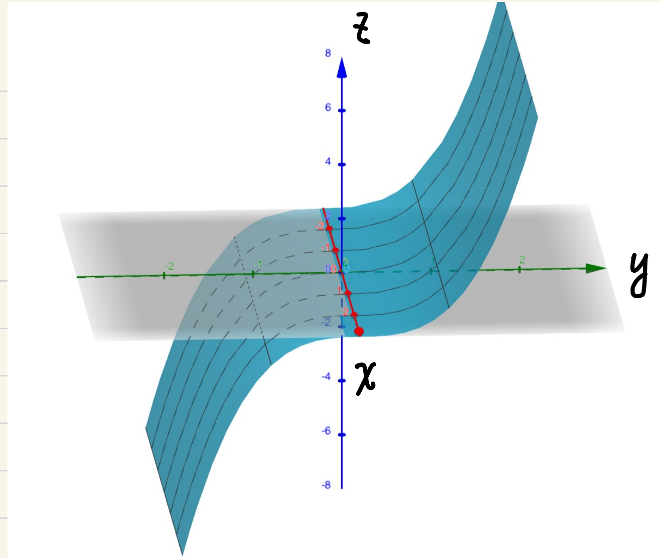
Cylinders

- Cylinders:
- A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

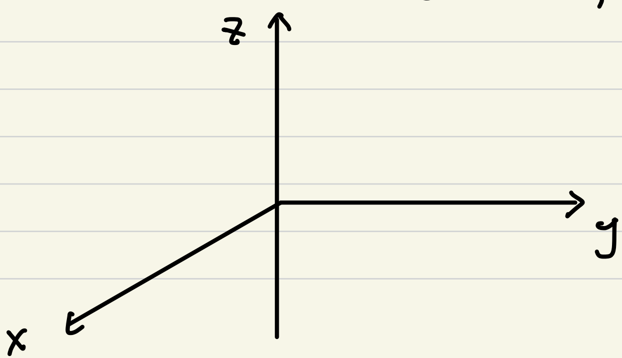


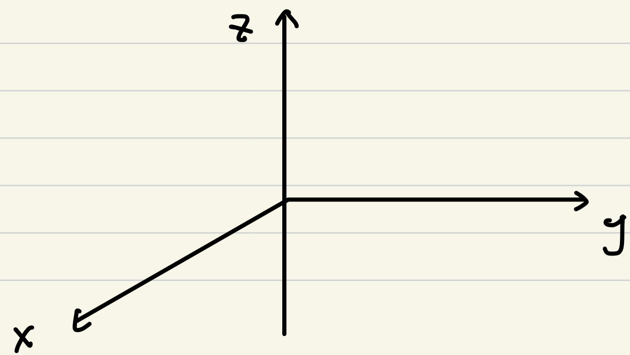
Cylinder

Ex: $z = y^3$.



Ex: Sketch $x^2 + y^2 = 4$, $4y^2 + z^2 = 1$ in $\mathbb{R}^3$





Quadric Surfaces

- Quadric Surfaces:
- A **quadric surface** is the graph of a second-degree equation in three variables *x*, *y*, and *z*. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

- By **translation** and **rotation** it can be brought into one of the two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0$$

$Ax^2 + Hy = 0$ $\qquad$ $Ax^2 + J = 0$

Ex: Reduce the equation to a standard form.

$$x^2 - \frac{y^2}{4} - 2x + y + 3z + 6 = 0.$$

Standard Forms

$$Ax^2 + By^2 + Cz^2 + J = 0$$

$$Ax^2 + By^2 + Iz = 0$$

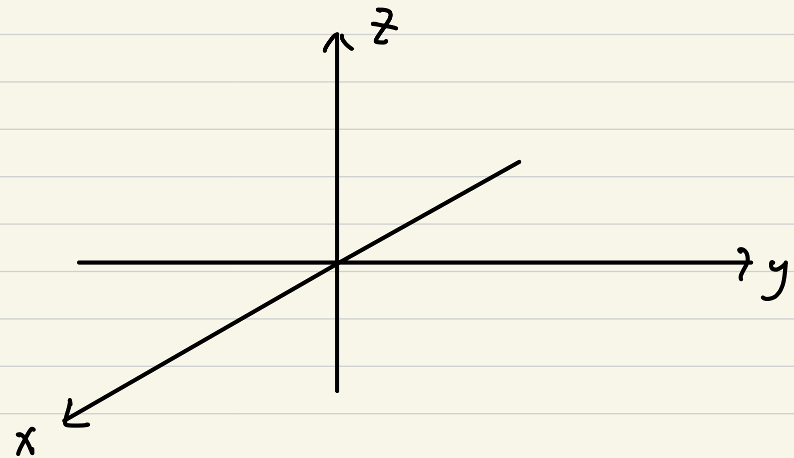
$$Ax^2 + By = 0$$

$$Ax^2 + J = 0$$

$Ax^2 + By^2 + Cz^2 = 1$ Family

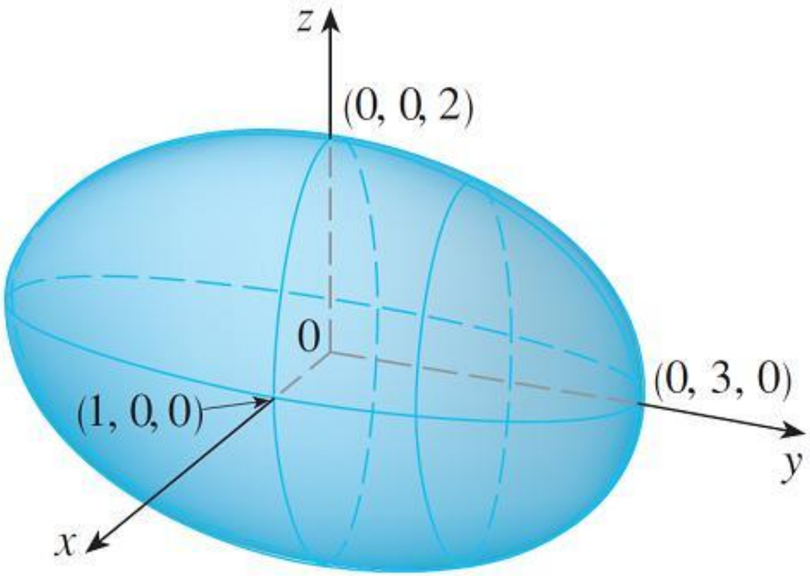
Case 1: $A, B, C > 0$.

Ex: $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$



Quadric Surfaces

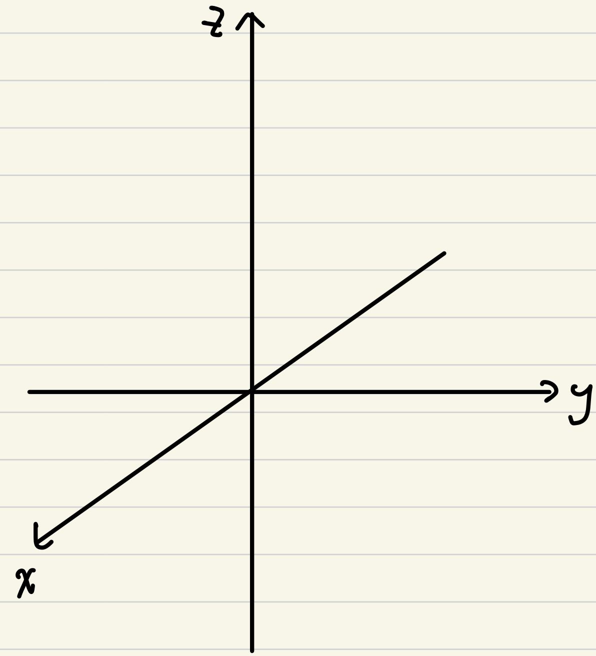
- Example: $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$
- It's called an **ellipsoid**. All of its traces are ellipses.



$Ax^2 + By^2 + Cz^2 = 1$ Family

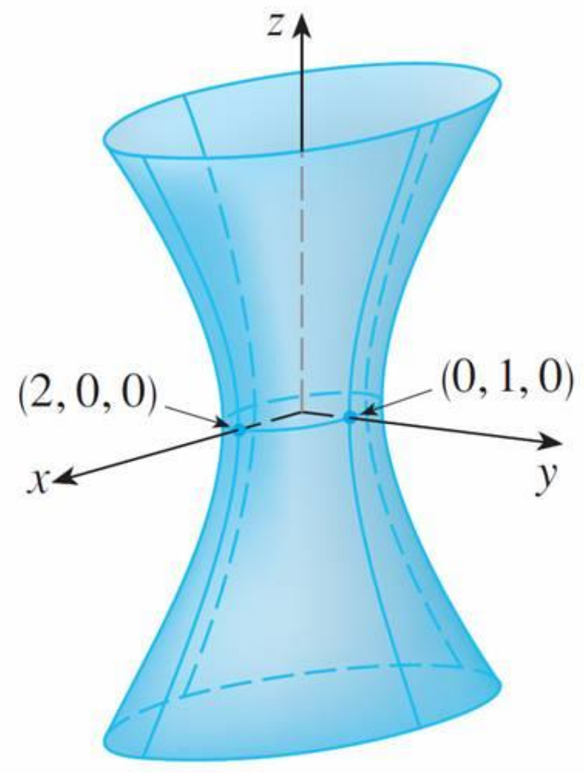
Case 2: A, B, C 2 $+$, 1 $-$.

Ex: $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$



Quadric Surfaces

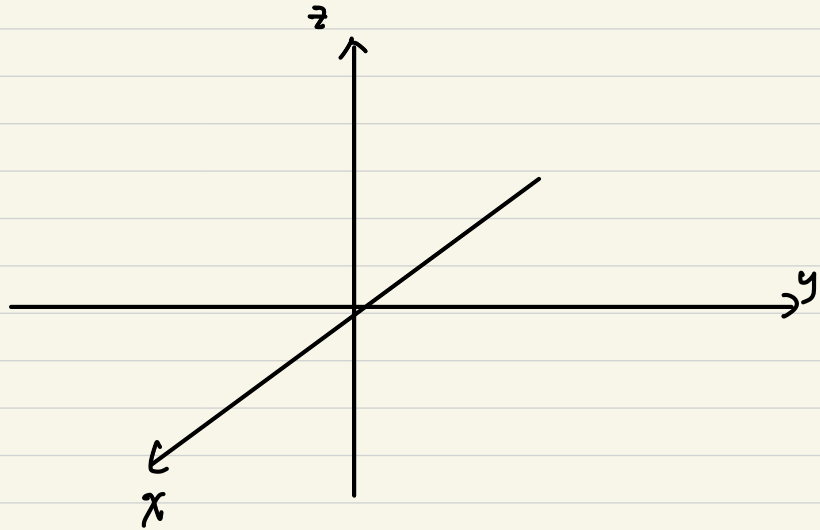
- Example: $\dfrac{x^2}{4} + y^2 - \dfrac{z^2}{4} = 1$
- This surface is called a **hyperboloid of one sheet.**



$Ax^2+By^2+Cz^2=1$ Family

Case 3: A, B, C, 1+ , 2 −

Ex: $4x^2-y^2+2z^2+4=0$



Quadric Surfaces

- Example: $4x^2 - y^2 + 2z^2 + 4 = 0$
- It is a **hyperboloid of two sheets**.

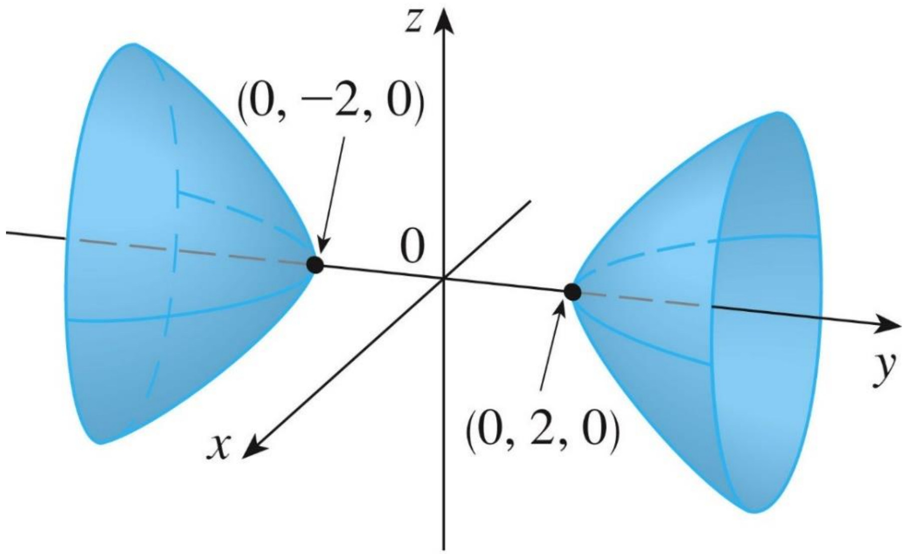
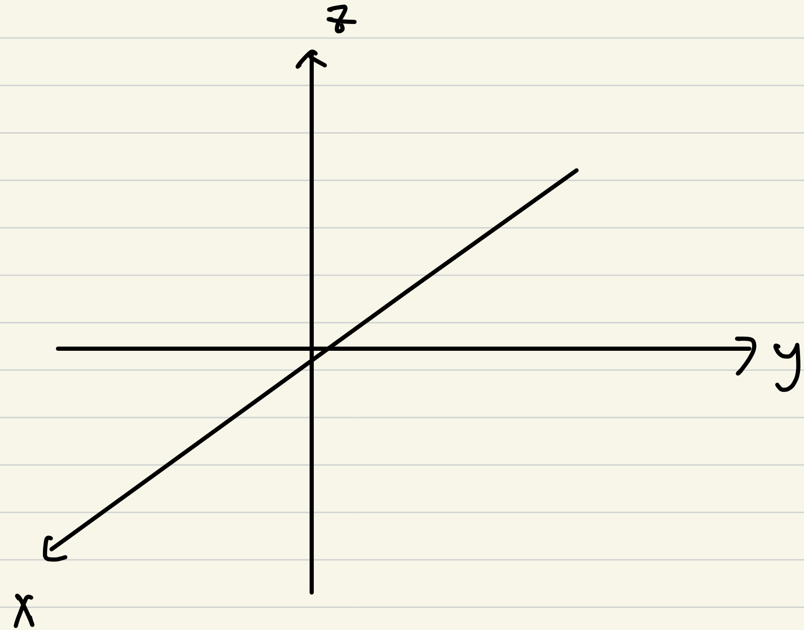


FIGURE 10

$4x^2 - y^2 + 2z^2 + 4 = 0$

$$Ax^2+By^2+Cz^2=0 \quad \text{family}$$

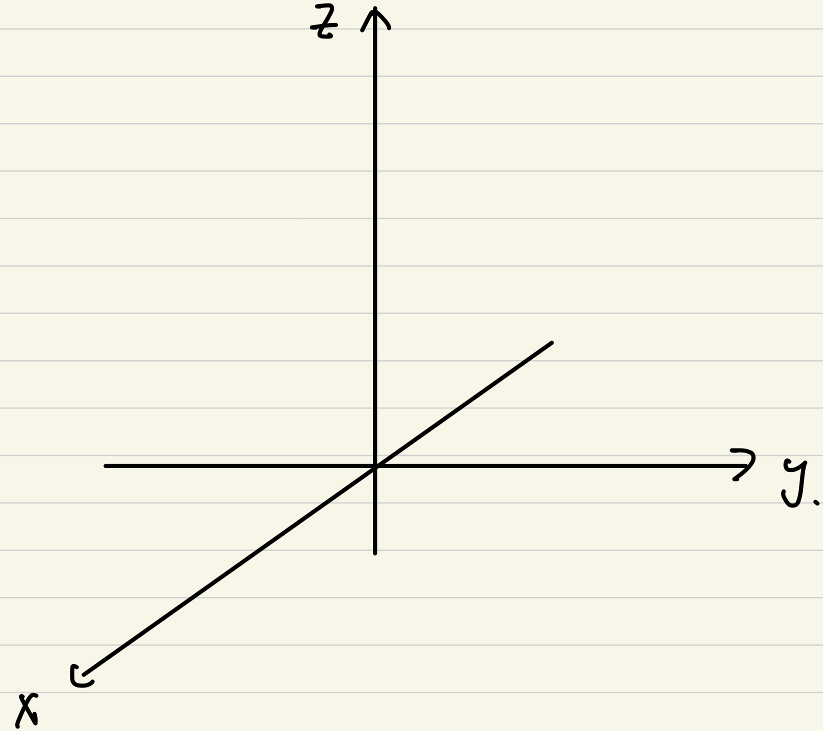
Ex: $x^2+4y^2-z^2=0$



$Ax^2 + By^2 = z$ Family

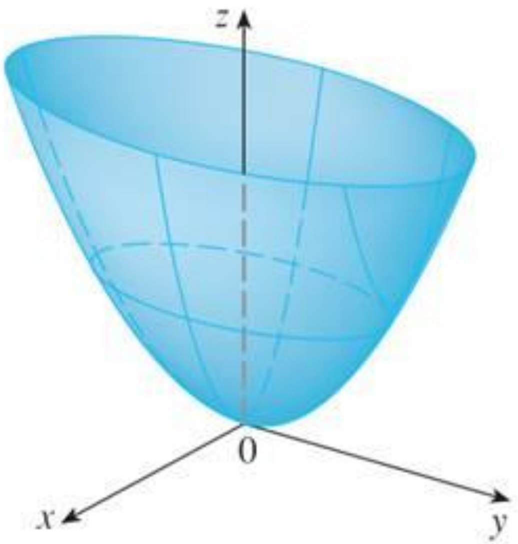
Case 1 : $A \cdot B > 0$

Ex: $4x^2 + y^2 = z$

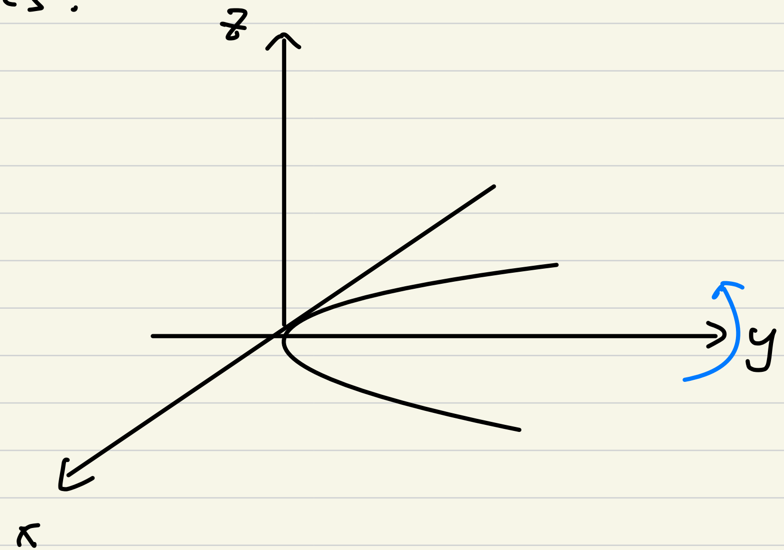


Quadric Surfaces

- Example: $z = 4x^2 + y^2$
- Because of the elliptical and parabolic traces, the quadric surface $z = 4x^2 + y^2$ is called an **elliptic paraboloid**.



Ex: Find the surface obtained by rotating the curve $y = x^2$ about the y-axis.

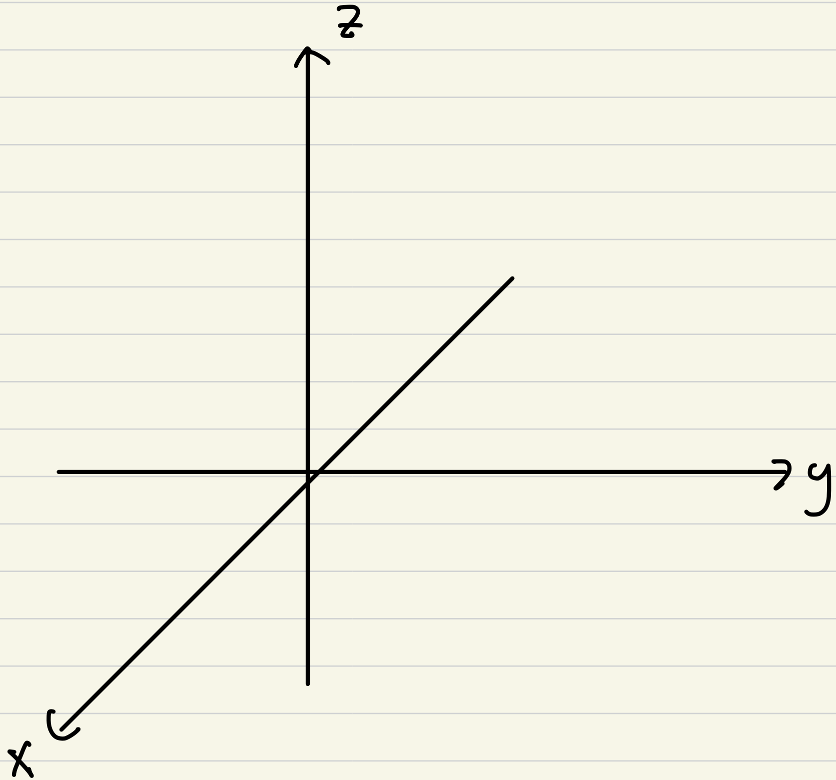


sol:

$Ax^2 + By^2 = z$ Family

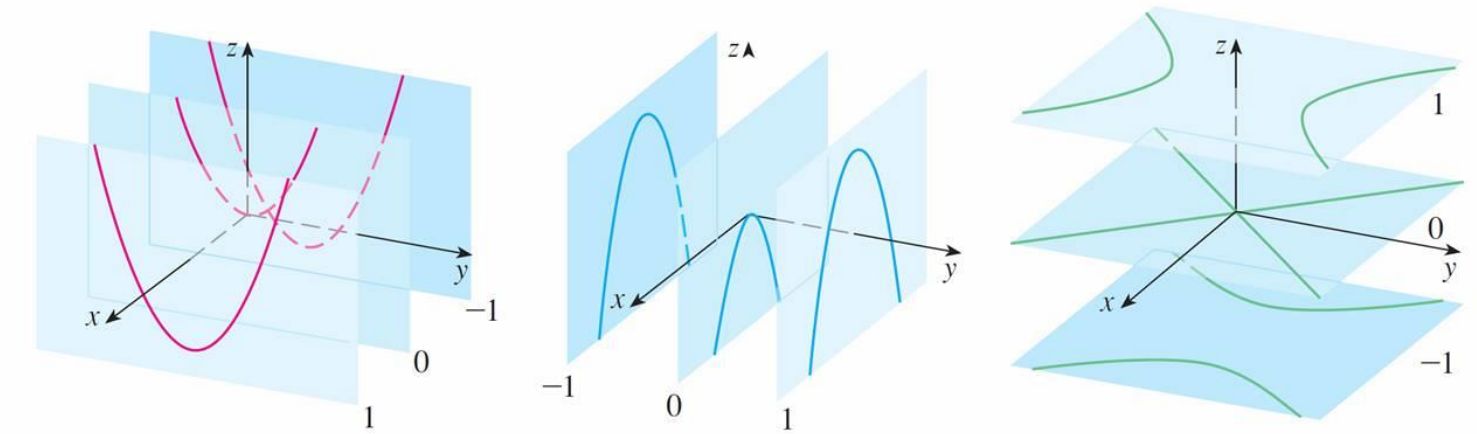
Case 2: $A \cdot B < 0$

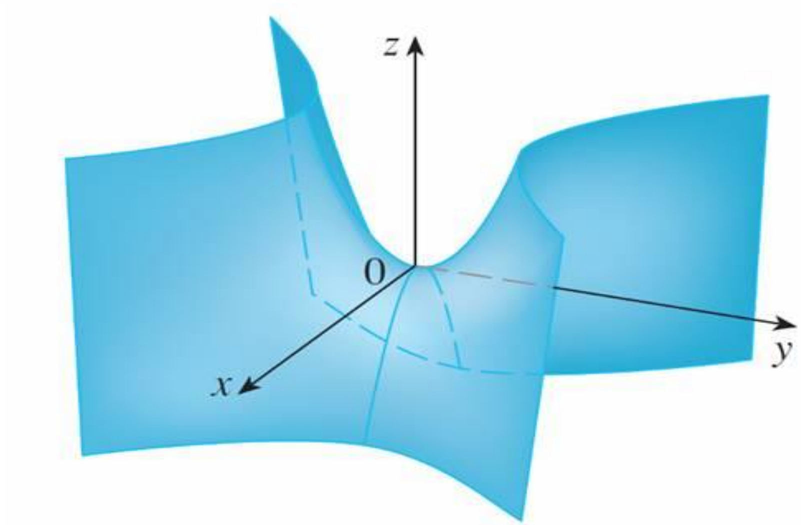
Ex: $z = y^2 - x^2$.



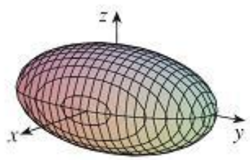
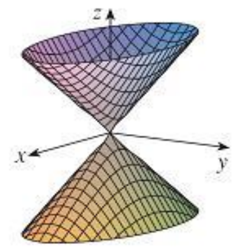
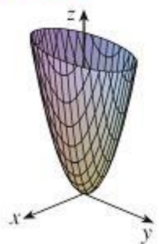
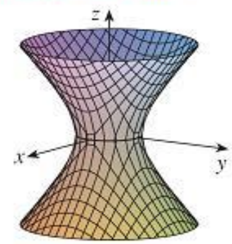
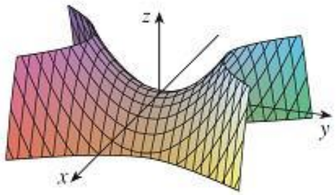
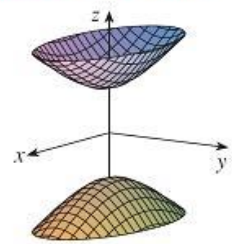
Quadric Surfaces

- Example: $z = y^2 - x^2$, a **hyperbolic paraboloid**.





Quadric Surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Ex: Classify and sketch the quadric surface

$$x^2 + \frac{y^2}{4} - z^2 - 2x + y + 6z + 1 = 0.$$

Ex: Classify the quadric surface

$2x^2 - 4y^2 + z^2 + 4y + 6z + a = 0$ according to the constant a.

Ex: Classify the surface $x^2 + ay^2 + 2x - 2ay - z = 0$ according to the constant a.

Review

- How do we sketch quadric surfaces? What are standard forms of quadric surfaces?
- Classify ellipsoid and hyerboloid of one (two) sheet(s).
- Classify elliptic paraboloid and hyperbolic paraboloid.