Vector Fields and Line Integrals

Section 16.1-16.3

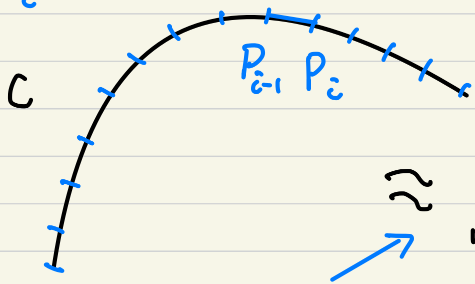
Outline

- Vector Field, Gradient Vector Fields
- Line Integrals
  - With Respect to Arc Length
  - With Respect to Variables
  - Integrate Vector Fields Along a Curve
- The Fundamental Theorem for Line Integrals
  - Independence of Path

Suppose that $f(x,y)$ is the density function of the curve $C: \vec{r}(t) = (x(t), y(t))$, $a \le t \le b$. We want to find the mass of C. Divide C with partition $t_0 = a < t_1 < t_2 < \cdots < t_n = b$

$$M \equiv \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \overline{P_{i-1}P_i}$$

$P_i = (x(t_i), y(t_i))$



$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}$$

$$\approx \lim_{n\to\infty} \sum_{i=1}^{n} f(x(t_i^*), y(t_i^*)) \sqrt{(x_i'(t_i^*))^2 + (y_i'(t_i^*))^2}\, \Delta t$$

FTC

$x(t_i) - x(t_{i-1})$

$= x'(t_i^*)\, \Delta t$

Line Integrals

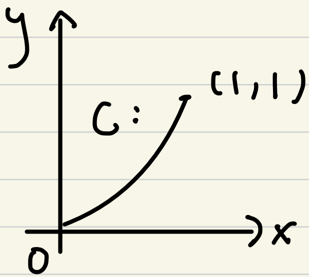
- **Line Integrals with respect to arc length**
- If f is a continuous scale function whose domain contains a smooth curve C : $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, $a \leq t \leq b$, then we

define the line integral of f along the curve C as

$$\boxed{\int_C f(x, y)\, ds} = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

- The value of the line integral does not depend on the parametrization of the curve.

Ex: Evaluate $\int_C 2x\, ds$, where C is the parabola $y=x^2$ from $(0,0)$ to $(1,1)$.



sol :

Ex: Compute $\int_C y \, ds$, where C is $\vec{r}(t) = (t - \sin t, 1 - \cos t)$, $0 \le t \le 2\pi$

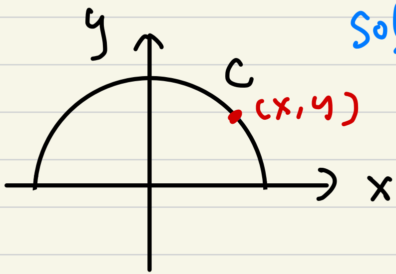
Sol:

Line Integrals

- Any physical interpretation of a line integral $\int_C f(x,y)ds$ depends on the physical interpretation of the function f.
- Suppose that $\rho(x,y)$ represents the linear density at a point (x,y) of a thin wire shaped like a curve C. Then the line integral of $\rho(x,y)$ along C is the mass of C.
- The **center of mass** of the wire is

$$\overline{x} = \frac{1}{m}\int_C x\rho(x,y)ds \qquad \overline{y} = \frac{1}{m}\int_C y\rho(x,y)ds$$

Ex: C is the upper half circle $x^2+y^2=1$, $y \geq 0$. Find the center of mass if the density at (x,y) is proportional to its distance from the y-axis.



Sol:

Line Integrals

- **Line integrals with respect to x and y :**
- On a curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, $a \le t \le b$.

$$\boxed{\int_C f(x,y)\,dx} = \int_a^b f(x(t),y(t))x'(t)\,dt$$

$$\boxed{\int_C f(x,y)\,dy} = \int_a^b f(x(t),y(t))y'(t)\,dt$$

Line Integrals

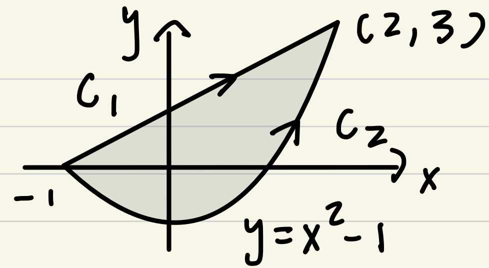
- It frequently happens that line integrals with respect to x and y occur together.
- When this happens, it's customary to abbreviate by writing

$$\int_C P(x,y)dx + \int_C Q(x,y)dy = \int_C Pdx + Qdy$$

Line Integrals

- If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation, then we have
- $\int_{-C} f(x,y)dx = -\int_C f(x,y)dx$
- $\int_{-C} f(x,y)dy = -\int_C f(x,y)dy$
- $\int_{-C} f(x,y)ds = \int_C f(x,y)ds$
- This is because Δs_i is always positive, whereas Δx_i and Δy_i change sign when we reverse the orientation of C.

Ex: Find $\int_{C_i} y\,dx + x\,dy$, $i=1,2$, where



Sol:

Line Integrals

- We can extend the line integrals along a space curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $a \leq t \leq b$.

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\underbrace{\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2+(\frac{dz}{dt})^2}dt}_{ds}$$

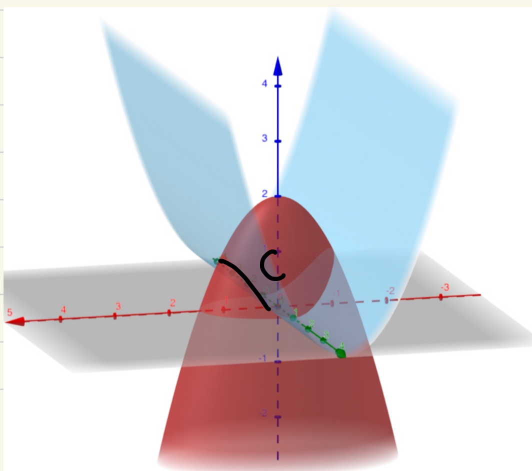
$$\int_C f(x,y,z)\,dz = \int_a^b f(x(t),y(t),z(t))\underbrace{z'(t)\,dt}_{dz}$$

Ex: C is the smaller part of the intersection of two surfaces

$z = 2 - x^2 - 2y^2$, $z = x^2$ between $(1, 0, 1)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

Compute $\int_C (x^2 - y^2)\, ds$.

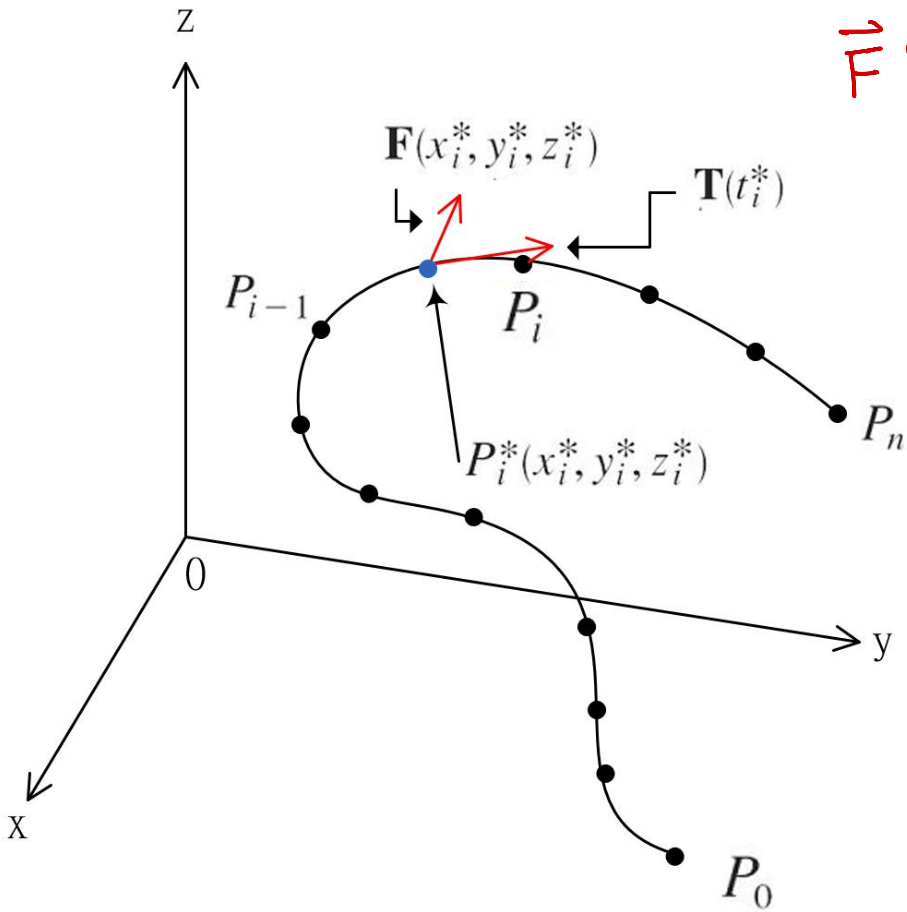
sol:



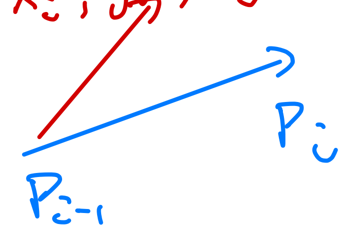
Line Integrals of Vector Fields

- Now suppose that $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a continuous force field on R^3. We wish to compute the work done by this force in moving a particle along a smooth curve C.
- We divide C into subarcs $P_{i-1}P_i$ with lengths Δs_i by dividing the parameter interval $[a, b]$ into subintervals of equal width. Choose a point $P_i^*(x_i^*, y_i^*, z_i^*)$ on the ith subarc corresponding to the parameter value t_i^*.

Line Integrals of Vector Fields



$\vec{F}(x_i^*, y_i^*, z_i^*)$



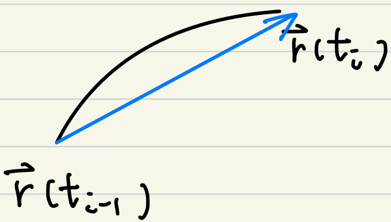
$W_i \approx \vec{F} \cdot \overrightarrow{P_{i-1}P_i}$

$C: \vec{r}(t) = (x(t), y(t), z(t))$, $a \le t \le b$.

Divide C with partition $a = t_0 < t_1 < \cdots < t_n = b$.

$$W \equiv \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(\vec{r}(t_i^*)) \cdot (\vec{r}(t_i) - \vec{r}(t_{i-1}))$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(\vec{r}(t_i^*)) \cdot \left(\frac{x(t_i)-x(t_{i-1})}{\Delta t}, \frac{y(t_i)-y(t_{i-1})}{\Delta t}, \frac{z(t_i)-z(t_{i-1})}{\Delta t} \right) \Delta t$$



$$\approx \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(\vec{r}(t_i^*)) \cdot (x'(t_i^*), y'(t_i^*), z'(t_i^*)) \Delta t$$

Line Integrals of Vector Fields

- The work done by the force $\vec{F}$ in moving the particle from P_{i-1} to P_i is approximately $\vec{F}(x_i^*, y_i^*, z_i^*) \cdot \vec{T}(t_i^*)\Delta s_i$, where $\vec{T}(x, y, z)$ is the unit tangent vector of C at (x, y, z).
- Therefore we define the **work** W done by the force field $\vec{F}$ as the limit of the Riemann sums

$$W = \int_C \vec{F}(x, y, z) \cdot \vec{T}(x, y, z) ds = \int_C \vec{F} \cdot \vec{T} ds$$

Line Integrals of Vector Fields

- If the curve C is given by the vector equation $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, then
$$\boxed{W = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt}$$
- This integral is often abbreviated as $\boxed{\int_C \vec{F} \cdot d\vec{r}}$ and occurs in other areas of physics as well.

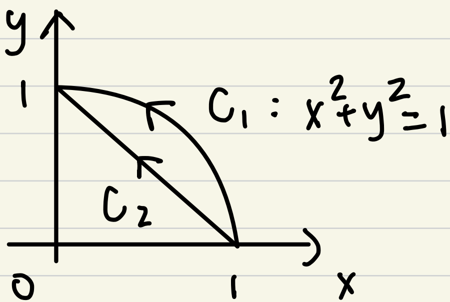
Line Integrals of Vector Fields

- Definition:
- Let $\vec{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\vec{r}(t)$, $a \le t \le b$. Then **the line integral of $\vec{F}$ along C** is

$$\boxed{\int_C \vec{F} \cdot d\vec{r}} = \int_C \vec{F} \cdot \vec{T} ds = \boxed{\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt}$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \qquad ds = |\vec{r}'(t)|\, dt$$

Ex: The force field is $\vec{F}(x,y) = y\vec{i} + x\vec{j}$. Find work done by $\vec{F}$ along C_i, $i=1,2$.



sol:

Ex: $\vec{F}(x,y) = y\,\vec{i} + 2x^2\,\vec{j}$. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the curve $y=x^2$ from $(0,0)$ to $(1,1)$.

Sol:

Ex: The position of a particle with mass m at time t is C:

$\vec{r}(t) = 2t^2\vec{i} - t^3\vec{j} + t\,\vec{k}$. Find the work done on the particle for $0 \le t \le 1$.

sol:

Line Integrals of Vector Fields

- Note that line integrals of vector fields are related to line integrals of scalar functions in the following way.
- Property: $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$

  where $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$.

Proof: $\vec{F} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$. Let $\vec{r}(t) = (x(t), y(t), z(t))$,

$a \le t \le b$ be a parametrization of C.

Then

$$\int_C \vec{F} \cdot d\vec{r} =$$