Second-Order Linear Differential Equations

Section 17.1-17.2

Outline

- Definitions and Basic Properties
- Solve Homogeneous Equations
  - Two Distinct Real Roots
  - One Real Root
  - Two Complex Roots
- Initial-Value and Boundary-Value Problems
- Solve Nonhomogeneous Equations (Find a Particular Solution)
  - The Method of Undetermined Coefficients
  - The Method of Variation of Parameters

Solve Nonhomogeneous Equations

- Now we try to solve second-order nonhomogeneous linear differential equations with constant coefficients, that is, equations of the form $ay'' + by' + cy = G(x)$.

  The **related homogeneous equation**

$$ay'' + by' + cy = 0$$

  is called the **complementary equation.**

Solve Nonhomogeneous Equations

3 **Theorem** The general solution of the nonhomogeneous differential equation $\boxed{1}$ can be written as

$$y(x) = y_p(x) + y_c(x)$$

where y_p is a particular solution of Equation 1 and y_c is the general solution of the complementary Equation 2.

- Thus, to solve the nonhomogeneous equation, we only need to find a *particular solution*. There are two methods for finding a particular solution.

The Method of Undetermined

- $ay'' + by' + cy = G(x)$
- Case I: $G(x)$ is a polynomial.
- It is reasonable to guess that there is a particular solution y_p that is a polynomial of the same degree as G .
- Case II: $G(x)$ is $\sin(ax)$, $\cos(ax)$, or e^{kx}
- We could guess the particular solution as $c_1 \cos(ax) + c_2 \sin(ax)$, or ce^{kx} .

Ex: Find a particular solution and general solutions of

$$y'' + 2y' + y = x^2.$$

Ex: Find a particular solution and general solutions of

$$y'' + y = 3e^{-2x}$$

Ex: Find a particular solution and general solutions of

$$y'' + y' - 2y = \cos 2x .$$

The Method of Undetermined

- Case III: If $G(x)$ is a product of functions of the preceding types, then we take the trial solution to be a product of functions of the same type.
- Case IV: If $G(x)$ is a solution of the complementary equation, we choose y_p as multiply by $x\,G(x)$ (or x^2 if necessary).

Ex: Solve $y'' - 2y' + y = 2e^x$

The Method of Undetermined

Summary of the Method of Undetermined Coefficients

1. If $G(x) = e^{kx}P(x)$, where P is a polynomial of degree n, then try $y_p(x) = e^{kx}Q(x)$, where $Q(x)$ is an nth-degree polynomial (whose coefficients are determined by substituting in the differential equation).

2. If $G(x) = e^{kx}P(x) \cos mx$ or $G(x) = e^{kx}P(x) \sin mx$, where P is an nth-degree polynomial, then try

$$y_p(x) = e^{kx}Q(x) \cos mx + e^{kx}R(x) \sin mx$$

where Q and R are nth-degree polynomials.

Modification: If any term of y_p is a solution of the complementary equation, multiply y_p by x (or by x^2 if necessary).

The Method of Undetermined

- If $G(x)$ is a sum of functions of these types, we use the easily verified *principle of superposition*, which says that if y_{p_1} and y_{p_2} are solutions of
$$ay'' + by' + cy = G_1(x)$$
$$ay'' + by' + cy = G_2(x)$$
respectively, then $y_{p_1} + y_{p_2}$ is a solution of
$$ay'' + by' + cy = G_1(x) + G_2(x)$$

The Method of Variation

- Suppose we have already solved the homogeneous equation $ay'' + by' + cy = 0$ and written the solution as $y = c_1 y_1 + c_2 y_2$ where y_1 and y_2 are linearly independent solutions.
- Let's replace the constants (or parameters) c_1 and c_2 by arbitrary functions $u_1(x)$ and $u_2(x)$. Try it as a solution of the nonhomogeneous equation.

Ex: Let $y_p = u_1 y_1 + u_2 y_2$, where y_1, y_2 are solutions of $ay'' + by' + cy = 0$ and $u_1' y_1 + u_2' y_2 = 0$.

Then y_p is a solution of $ay'' + by' + cy = G(x)$ iff $a(u_1' y_1' + u_2' y_2') = G(x)$.

The Method of Variation

- We look for a particular solution of the nonhomogeneous equation
  $ay'' + by' + cy = G(x)$
  of the form $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$
- This method is called **variation of parameters.**

The Method of Variation

- Differentiating y_p, we get
  $$y_p' = (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2')$$
- Since u_1 and u_2 are arbitrary functions, we can impose two conditions on them.
- One condition is that y_p is a solution of the differential equation. Let's impose the second condition $u_1'y_1 + u_2'y_2 = 0$.

The Method of Variation

- Then $y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''$
- Substituting in the differential equation, we get $a(u_1'y_1' + u_2'y_2') = G$.
- Together with $u_1'y_1 + u_2'y_2 = 0$, we can solve u_1' and u_2' . Then, we may be able to integrate to find u_1 , u_2 and the particular solution is solved.

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = \frac{1}{a}G(x) \end{cases}$$

Ex: Solve $y'' + y = \tan x \quad , 0 < x < \frac{\pi}{2}$.

Ex: Solve $y'' - 2y' + y = \dfrac{e^x}{1+x^2}$.

Review

- What is a linear homogeneous (non – homogeneous) second order differential equation?
- How do we solve a homogeneous linear differential equation with constant coefficients?
- How do we solve a non-homogeneous differential equation?
- What are the method of undetermined coefficients and method of variation of parameters?