Separable Equations

Section 9.3

Outline

- 1. Separable Equations and Their Solutions
- 2. Applications:
  - Orthogonal Trajectories
  - Mixing Problems
  - Models for Population Growth

Separable Equations

- A **separable equation** is a first-order differential equation in which the expression for dy/dx can be factored as a function of x times a function of y. In other words, it can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

Separable Equations

- Equivalently, if $f(y) \neq 0$, we could write

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \quad , \quad h(y) = \frac{1}{f(y)}$$

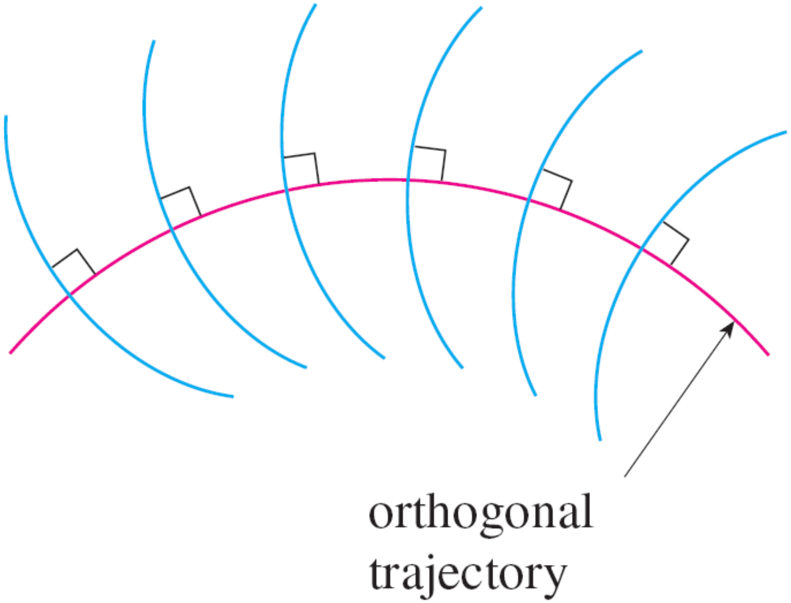
where $h(y)dy = g(x)dx$.

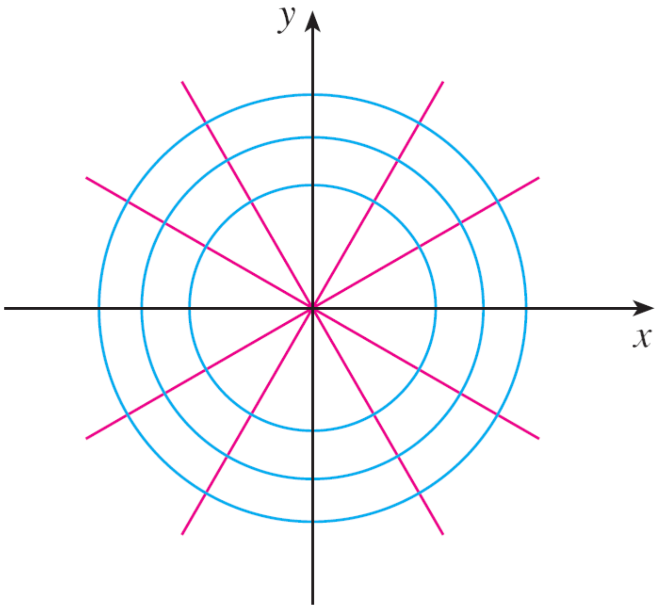
- Thus, $\int h(y)dy = \int g(x)dx$. Then we integrate both sides of the equation:
- This defines y implicitly as a function of x.

Ex: Solve the equation $\begin{cases} xy^2 \cdot y' = x + \ln x \\ y(1) = k \end{cases}$, where k is a constant.

Applications: Orthogonal Trajectories

- An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family orthogonally





Orthogonal Trajectories

Strategy for finding orthogonal trajectories of a family of curves $F(x, y, k) = 0$.

1. Find a single differential equation that is satisfied by all members of the family, $\frac{dy}{dx} = g(x, y)$ for all k.

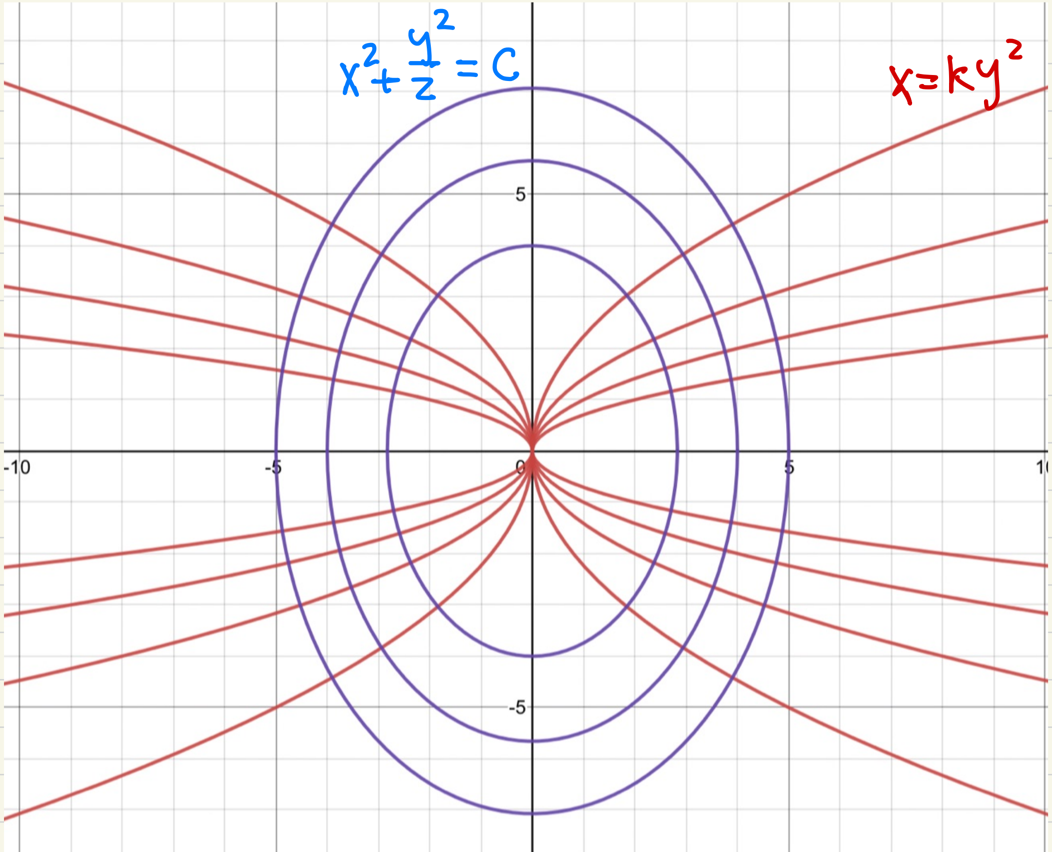
2. Then its orthogonal trajectories must satisfy the differential equation $$\frac{dy}{dx} = \frac{-1}{g(x,y)} \quad \cdots\cdots (*) .$$

3. Solve $(*)$ and find the orthogonal trajectories.

Applications: Orthogonal Trajectories

- To find the orthogonal trajectories of a family of curves, we need to solve a differential equation.
- Examples: Find the orthogonal trajectories of the family of curves
  - $xy = k$, where k is an arbitrary constant.
  - $x = ky^2$, where k is an arbitrary constant.

Ex: Find the orthogonal trajectories of $x = ky^2$, where k is a constant.



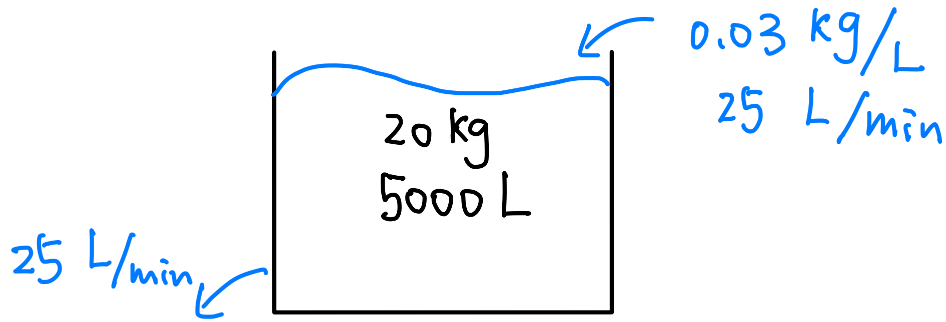
$x^2 + \frac{y^2}{2} = C$
$x = ky^2$
5
-5
-10
0
10

Applications: Mixing Problems

- A solution of a given concentration enters the tank at a fixed rate and the mixture, thoroughly stirred, leaves at a fixed rate, which may differ from the entering rate.
- If $y(t)$ denotes the amount of substance in the tank at time t , then $y'(t)$ is the rate at which the substance is being added minus the rate at which it is being removed.

Applications: Mixing Problems

- Example: A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half an hour?



Sol: Let $x(t)$ be the amount of salt (kg) in the tank at time t.

Applications: Models for Population Growth

- The Law of Natural Growth:

$$\boxed{\frac{dP}{dt} = kP}$$

2 The solution of the initial-value problem

$$\frac{dP}{dt} = kP \qquad P(0) = P_0$$

is

$$P(t) = P_0 e^{kt}$$

Applications: Models for Population Growth

- The Logistic Model:
- With carrying capacity M, the population growth satisfies the following differential equation:

$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)}$$

- The solution for the equation is of the form

$$\boxed{P(t) = \frac{M}{1 + Ae^{-kt}} \qquad \text{where } A = \frac{M - P_0}{P_0}}$$

Ex: Solve $\begin{cases} \dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{M}\right) \\ P(0) = P_0 \end{cases}$, where $k > 0$, $P_0 > 0$ are constants.

Ex :

Newton's law of cooling says that the rate of cooling of an object is proportional to the difference between the temperature of the object and that of its surroundings (provided the difference is not too large). If

$T = T(t)$ represents the temperature of a (warm) object at time

$t,$

A represents the ambient (cool) temperature, and

k is a negative constant of proportionality, which equation(s) accurately characterize Newton's law?

Review

- What is a separable equation? How do we solve it?
- How do we find an orthogonal trajectory for a family of curves?
- What is a mixing problem?
- State some models for population growth.