Linear Approximations and Differentials

Section 3.10

Outline

- Linear Approximations
- Differentials

Linear Approximations

- We use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a.
- Definition: $f(x) \approx f(a) + f'(a)(x-a)$ is called the **linear approximation** or **tangent line approximation** of f at a.
- $L(x) = f(a) + f'(a)(x-a)$ is called the **linearization** of f at a.

Ex: Estimate $(0.999)^{\frac{1}{5}}$ by linear approximation

Ex: Estimate $\sin(0.002)$

Ex: Estimate $\cos(58^\circ)$

Ex: Estimate $\tan^{-1}(1.03) - \frac{\pi}{4}$.

Linear Approximation

- Applications:
- Estimate the following values by linear approximation.

$(0.999)^{\frac{1}{5}}$ = 0.9997999199519664...

$\sqrt{82}$ = 9.055385138137417...

$\sin(0.002)$ = 0.0019999986666669333...

$\tan^{-1}(1.03) - \frac{\pi}{4}$ = 0.014777249407492...

Ex: $e^{\sin x} y^3 + xy = 1$ defines y implicitly as a function of x near $(0, 1)$, say $y = f(x)$. Use the linear approximation to estimate $f(-0.05)$.

5. (14%) Consider the equation $y^5 + 1.009y^3 + y = 3$.

(a) (6%) Show that the equation has exactly one real solution.

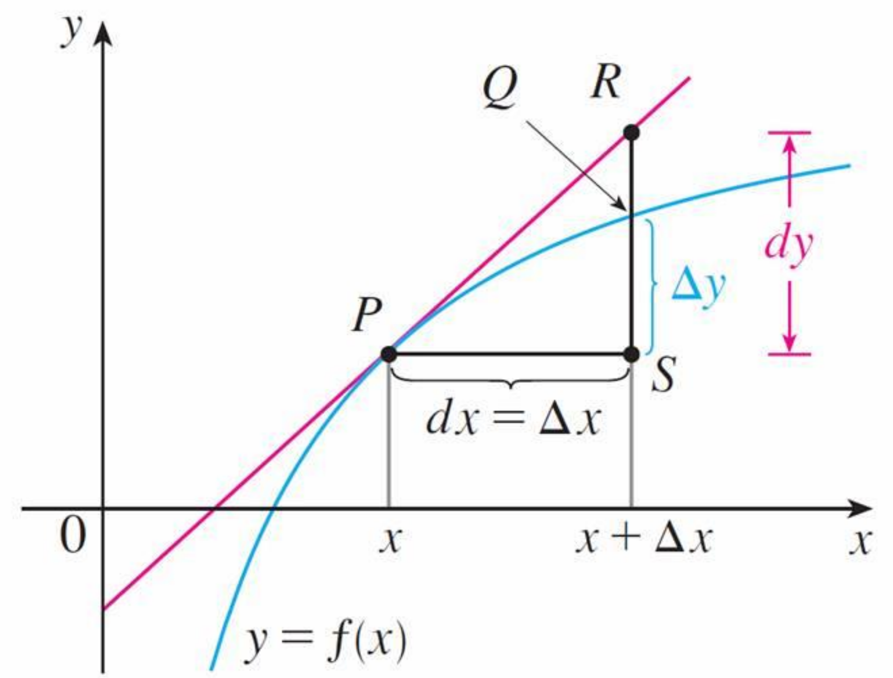
(b) (4%) Given $y^5 + xy^3 + y = 3$, find $\frac{dy}{dx}$ at $(1, 1)$.

(c) (4%) Use a linear approximation to estimate the real root of $y^5 + 1.009y^3 + y = 3$.

Differentials

- If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an **independent variable**; that is, dx can any real number. The **differential** dy is then defined in terms of dx and x by the equation $dy = f'(x)dx$.
- So dy is a **dependent variable**; it depends on the values of x and dx . We can use dy to estimate the change of f due to the change of the variable x by an amount dx.

Differentials



dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx.

Ex: The radius of a sphere is measured and found to be 20 cm with possible error at most 0.05 cm. Approximate the maximum error of the volume.

How good is the linear approximation?

Suppose that $f(x)$ is differentiable at $x=a$.

The linearization of $f(x)$ at $x=a$ is $L(x) = f(a) + f'(a)(x-a)$

① $\lim_{x \to a} f(x) - L(x)$

② $\lim_{x \to a} \dfrac{f(x) - L(x)}{x-a}$

Ex: Prove the chain Rule .

Review

- Write down the linearization of a function at a point.
- For a function $y = f(x)$, how do we define the differential dy in terms of dx and x?
- What is the geometrical meaning of dy?