Differential Rules (Part 2)

Section 3.4-3.5

Outline

The Chain Rule

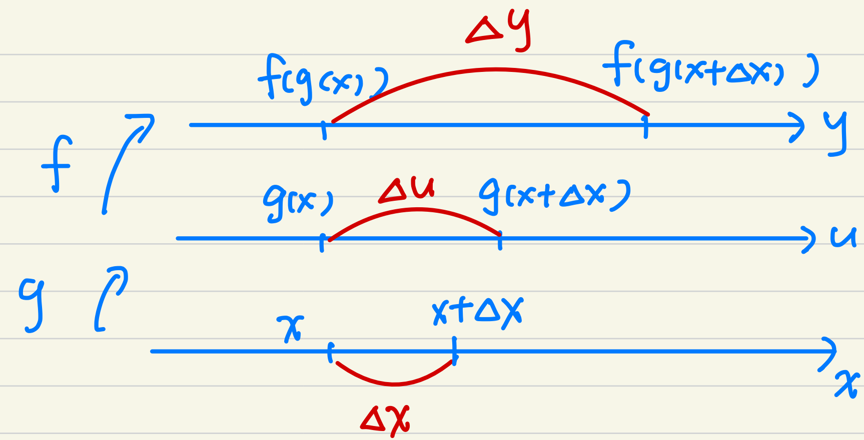
The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Idea of the Chain Rule



Let $u = g(x)$,

$y = f(u) = f(g(x))$.

If the variable x changes to $x+\Delta x$, then the change of u and y are $\Delta u = g(x+\Delta x) - g(x)$,

$\Delta y = f(g(x+\Delta x)) - f(g(x))$.

Hence $$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

$$\frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} = \frac{f(g(x+\Delta x)) - f(g(x))}{g(x+\Delta x) - g(x)} \cdot \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

Ex: A table of values for f, g, f', and g' are given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	2	2	6	3
2	3	1	7	5
3	1	5	4	-1

(a) $F(x) = f(g(x))$.

Find $F'(1)$.

(b) $G(x) = g(f(x))$.

Find $G'(3)$.

Ex: Suppose that $L(x)$ is a function such that $L'(x) = \frac{1}{x}$ for $x > 0$. Find a) $\frac{d}{dx} L(c \cdot x)$, where $c \in \mathbb{R}$.

b) $\frac{d}{dx} L(x^n)$, where $n \in \mathbb{R}$.

c) $\frac{d}{dx} L(f(x) \cdot g(x))$, where $f(x), g(x)$ are differentiable functions.

Ex: $f(x) = \sin(x^2+1)$. Compute $f'(x)$.

Sol:

Ex: $f(x) = (\sin(x))^{\sqrt{2}}$. Compute $f'(x)$.

Sol:

Ex: Find $\frac{d}{dx}(f(x))^n$ where $n \in \mathbb{R}$.

Ex. Find $\frac{d}{dx}(x^2 + e^x)^{\frac{1}{3}}$

Ex: Find $\frac{d}{dx} \frac{(3x-2)^4}{(x^2+x+1)^3}$

Ex: $\frac{d}{dx} f(g(h(x)))$

Ex: $\frac{d}{dx} (\sec(\sin x))^2$

Ex: $F(x) = f(g(x))$. Find $F''(x)$.

Ex: $\frac{d}{dx}(f(ax))$

Ex: $\frac{d}{dx}(a^x)$

Ex: $\frac{d}{dx}\left(3^{\sqrt{x}}\right)$

Ex : $\frac{d}{dx}\left(e^{2^x} + 2^{x^e} + x^{2^e}\right)$

Ex: Find $\lim\limits_{x \to 0} \dfrac{5^{\sin(2x)} - 1}{x}$

Ex: $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & \text{for } x \neq 0 \\ 0, & \text{for } x = 0. \end{cases}$

a) Is f continuous at $x=0$?

b) Compute $f'(0)$ and $f'(x)$ for $x \neq 0$.

c) Is $f'(x)$ continuous at $x=0$?

Q: Let $f(x) = \begin{cases} |x|^{\alpha} \sin(\frac{1}{x}), & \text{for } x \neq 0. \\ 0, & \text{for } x = 0. \end{cases}$

a) Find values of α such that $f(x)$ is continuous at $x=0$.

b) Find values of α such that $f(x)$ is differentiable at $x=0$.

c) Find values of α such that $f'(x)$ is continuous at $x=0$.

Applications of the Chain Rule

$\boxed{4}$ **The Power Rule Combined with the Chain Rule** If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively, $$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

$$\frac{d}{dx}(a^x) = a^x \ln a$$