Techniques of Integration

Section 7.1-7.3

Outline

- Integration by Parts
- Trigonometric Integrals
- Trigonometric Substitution

Trigonometric Integrals

- In section 7.2 we use trigonometric identities to integrate certain combinations of trigonometric functions.
- Powers of $\sin x$ and $\cos x$:

$$\boxed{\int \sin^m x \cos^n x \, dx}$$

Identity	
Differentials	

Ex: $\int \cos^5 x \, dx$

Ex: $\int \sin^3 x \, (\cos x)^{\frac{1}{3}} \, dx$

Ex: $\displaystyle\int \frac{\sin^3 x}{2-\cos x}\,dx$.

Ex: $\displaystyle\int x\cos^3 x\,dx$

Ex: $\int \cos^2 x \, dx$, $\int \sin^2 x \, dx$

Ex: $\int \sin^2 x \cos^4 x \, dx$

Recall that

$$\int \sin^n x \, dx = \frac{-1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx, \quad \text{for } n \geq 2.$$

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx, \quad \text{for } n \geq 2.$$

Ex: $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{1+\cos 2x} \, dx$

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$$

$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \, \sin x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \, \sin x \, dx$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$

Trigonometric Integrals

- Finally, we can make use of another set of trigonometric identities:

2 To evaluate the integrals (a) $\int \sin mx \cos nx\,dx$, (b) $\int \sin mx \sin nx\,dx$, or (c) $\int \cos mx \cos nx\,dx$, use the corresponding identity:

$$\text{(a)}\quad \sin A \cos B = \tfrac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$\text{(b)}\quad \sin A \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\text{(c)}\quad \cos A \cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]$$

Ex: Compute $\int \sin 4x \cos x \, dx$.

Ex: Compute $\int \cos 2x \cos 4x \, dx$.

Trigonometric Integrals

- Powers of $\tan x$ and $\sec x$:
- Since $(d/dx)\tan x = \sec^2 x$, we can separate a $\sec^2 x$ factor and convert the remaining even power of $\sec x$ to an expression involving $\tan x$ using the identity $\sec^2 x = 1 + \tan^2 x$.
- Or, since $(d/dx)\sec x = \sec x \tan x$, we can separate a $\sec x \tan x$ factor and convert the remaining even power of $\tan x$ to $\sec x$.

Ex: $\int \tan^2 x \sec^4 x \, dx$

Ex: $\int \tan^3 x \sec^3 x \, dx$

Ex: $\int \tan^5 x \,(\sec x)^{-3}\, dx$

Ex: $\int \tan^{e-1} x \ \sec^{-e-2} x \ dx$

Strategy for Evaluating $\int \tan^m x \sec^n x\, dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x\, dx = \int \tan^m x\, (\sec^2 x)^{k-1} \sec^2 x\, dx$$

$$= \int \tan^m x\, (1 + \tan^2 x)^{k-1} \sec^2 x\, dx$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x\, dx = \int (\tan^2 x)^k \sec^{n-1} x \;\sec x\; \tan x\, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \;\sec x\; \tan x\, dx$$

Then substitute $u = \sec x$.

Ex: $\int \tan x \, dx$

Ex: $\int \tan^2 x \, dx$

Ex: $\int \tan^3 x \, dx$

Reduction Formula

$$\int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx .$$

Ex: $\int \sec x \, dx$

Ex: $\int \sec^2 x \, dx$

Ex: $\int \sec^3 x \, dx$

Reduction Formula

$$\int \sec^n x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

Other Trigonometric Integrals

Ex: $\int \csc x \, dx$

Ex: $\int \cot^3 x \csc^4 x \, dx$

Ex: $\int \frac{1+\sin x}{1-\sin x}\,dx$, $\int \frac{\cos x}{1+\cos x}\,dx$

Ex: $\displaystyle \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$

Ex: $\int x \tan^2 x \, dx$