Infinite Sequences and Series

Section 11.1-11.2

Outline

- Sequences:
  - Definition
  - The Limit of a Sequence
  - Properties of Limits
  - Monotonic Sequence Theorem
- Series:
  - Definition
  - Examples
  - Properties

Sequences

- Definition: A **sequence** is a list of infinite numbers written in a definite order:

$$a_1, a_2, a_3, \cdots, a_n, \cdots$$

- Notation: The sequence $\{a_1, a_2, a_3, \cdots\}$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Subsequences

- Given a sequence $\{a_n\}$, we say that $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ and denote it by $\{a_{n_k}\} \subset \{a_n\}$ where $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$

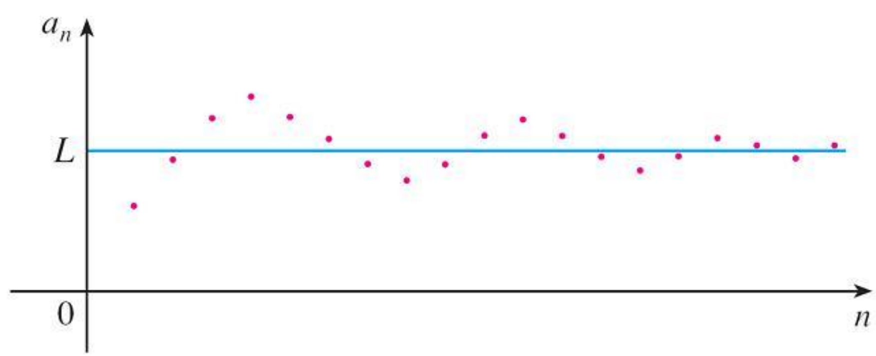
- $\{a_{2n}\}$ are the even terms of $\{a_n\}$.
- $\{a_{2n+1}\}$ are the odd terms of $\{a_n\}$.

The Limit of a Sequence

1 **Definition** A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).



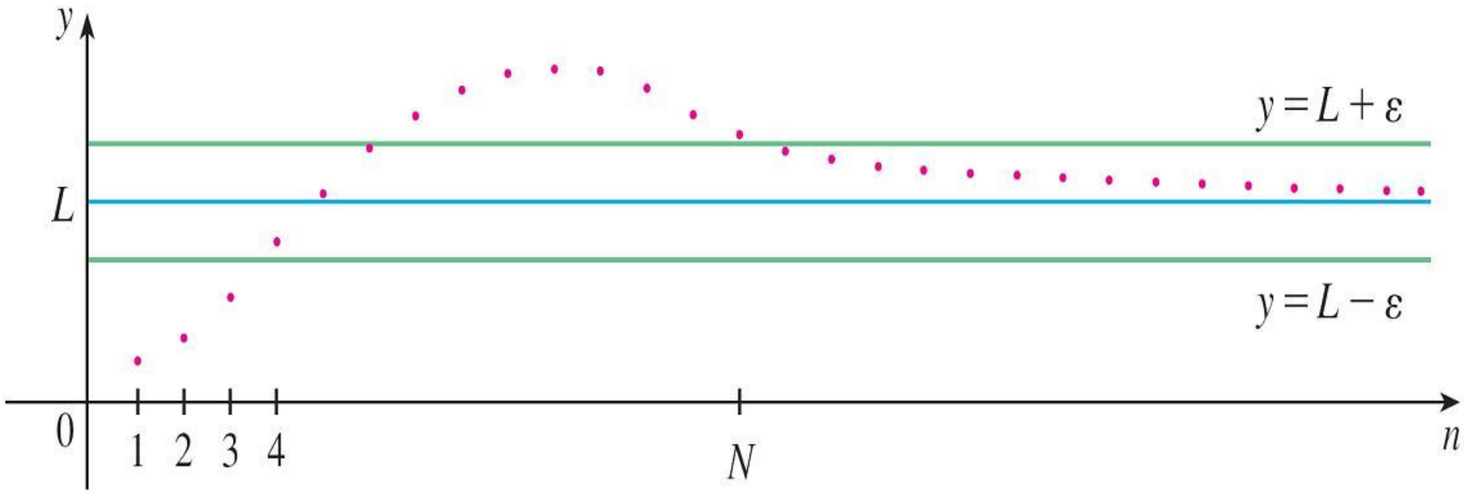
The Limit of a Sequence

2 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

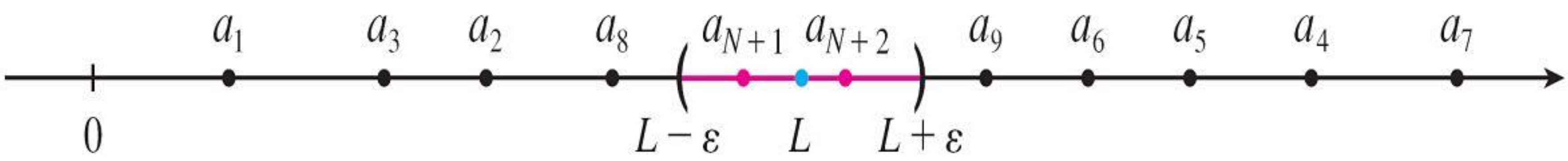
if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \varepsilon$$



The Limit of a Sequence

- If all the terms $a_1, a_2, a_3, \cdots$ are plotted on a number line, we have the following picture.



- No matter how small an interval $(L-\epsilon, L+\epsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.

The Limit of a Sequence

5 Definition $\lim_{n \to \infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if} \quad n > N \quad \text{then} \quad a_n > M$$

- Theorem: If $\lim_{x \to \infty} f(x) = L$ and $f(n) = a_n$ for $n \in N$, then $\lim_{n \to \infty} a_n = L$.

Ex: Find $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$. sol:

Ex: Find $\lim_{n\to\infty}\frac{n^2}{2^n}$. sol:

Ex: Find $\lim_{n\to\infty}\sqrt[n]{n}$ sol:

Properties of Limits

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

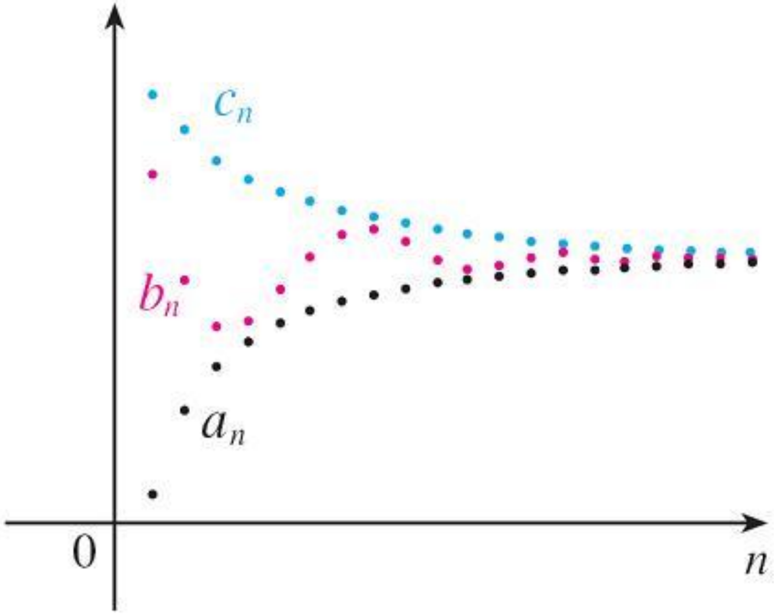
$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Properties of Limits

- Squeeze Theorem for Sequences:

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.



Ex: $a_n = \frac{n!}{n^n}$. Find $\lim_{n\to\infty} a_n$.

Sol.

Ex: Show that if $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

Sol:

Properties of Limits

6 **Theorem** If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

7 **Theorem** If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

9 The sequence $\{r^n\}$ is convergent if $-1 < r \leqslant 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Ex: Find $\lim_{n\to\infty} e^{n\sin(\frac{2}{n})}$

Ex: $\lim_{n\to\infty} \dfrac{3^n - 2^{2n}}{e^n + (-5)^n}$

Properties of Limits

- Theorem:
- If a sequence $\{a_n\}$ converges to the limit L , then every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ converges to L.
- Theorem:
- Given a sequence $\{a_n\}$, if both $\{a_{2n}\}$ and $\{a_{2n+1}\}$ converge to L, then $\lim_{n\to\infty} a_n = L$.

Monotonic Sequence Theorem

12 **Monotonic Sequence Theorem** Every bounded, monotonic sequence is convergent.

10 **Definition** A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.

11 **Definition** A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

It is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

Series

- Definition: When we try to add the terms of an infinite sequence, we get an expression of the form $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ which is called an **infinite series** (or just a **series**) and is denoted, for short, by the symbol $\sum a_n$ or $\sum_{n=0}^{\infty} a_n$.

Series

- What is the **sum** of a series?

2 Definition Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s = \lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k\right)$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Series: Examples

- The **geometric series:** For $a \neq 0$, consider

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

$\boxed{4}$ The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

Ex: Compute $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$, where $a \neq 0$.

Sol:

Ex: Find $\sum_{n=0}^{\infty} \frac{(-3)^{n-2}}{2^{2n+1}}$

Telescoping

write a_n as $b_n - b_{n-1}$

Ex: Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Ex: Find $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

Series: Examples

- The **harmonic series**:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

- Actually, we can show that $S_{2^n} > 1 + \dfrac{n}{2}$, and $S_n > \ln(n+1)$ for $n > 1$.

Ex: Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

pf:

① $S_2 = 1 + \frac{1}{2}$

$S_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2 \cdot \frac{1}{2}$

$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) >$

$1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + 3 \cdot \frac{1}{2}$

$S_{2^n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}}\right) + \cdots$

$\left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right)$

② $\because e^x > 1+x$ for $x>0$ $\therefore e^{\frac{1}{n}} > 1+\frac{1}{n} = \frac{n+1}{n}$ for $n \in \mathbb{N}$.

Therefore

$$e^{S_n} = e^{1+\frac{1}{2}+\cdots+\frac{1}{n}} = e^1 \cdot e^{\frac{1}{2}} \cdots e^{\frac{1}{n}} > \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n+1}{n} = n+1$$

Properties of Series

6 **Theorem** If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0$.

7 **The Test for Divergence** If $\lim_{n \to \infty} a_n$ does not exist or if $\lim_{n \to \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

8 **Theorem** If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum (a_n + b_n)$, and $\sum (a_n - b_n)$, and

(i) $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Remark: The converse is not true!

Ex: check whether the series converges?

(a) $\sum_{n=1}^{\infty} (2)^{\frac{1}{n}}$ (b) $\sum_{n=1}^{\infty} \sin(n^2)$ (c) $\sum_{n=1}^{\infty} \left(\frac{1}{2n}\right) - (\cos 112)^n$

Summary

- What is an infinite sequence? What is the limit of a sequence?
- Review properties of the limits of sequences.
- State the Monotonic Sequence Theorem.
- What is a series? What is its sum?
- Review the geometric series and the harmonic series.