Calculating Limits Using the Limit Laws

Section 2.3

Outline

- Limit Laws
  - Laws of algebraic operations
  - The squeeze theorem
  - Using One-Sided Limits

Limit Laws/ Laws of Algebraic Operations

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$

Limit Laws / Laws of Algebraic Operations

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

7. $\lim_{x \to a} c = c$

8. $\lim_{x \to a} x = a$

9. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Ex: Show that $\lim_{x \to a} P(x) = P(a)$ and $\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$

for all polynomials $P(x), Q(x)$ if $Q(a) \neq 0$.

Ex: Compute $\lim_{x \to 2} \frac{x^2+x-6}{x^2-9}$, $\lim_{x \to -3} \frac{x^2+x-6}{x^2-9}$.

Ex: Suppose that $\lim_{x \to 2} \frac{f(x) - x^2}{x - 2} = 5$. Show that $\lim_{x \to 2} f(x)$ exists and find $\lim_{x \to 2} f(x)$.

Ex: Show that if $\lim_{x \to a} \frac{f(x)}{g(x)} = L$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} f(x) = 0$.

Q: Show that if $\lim_{x \to a} f(x) = L \neq 0$ and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ doesn't exist.

Limit Laws / Laws of Algebraic Operations

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x \to a} f(x) > 0$.]

Ex: Find $\lim\limits_{h \to 0} \dfrac{\sqrt{4+h}-2}{h}$

Ex: Find $\lim\limits_{t \to 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$

Ex: Find $\lim\limits_{x \to -2} \dfrac{\sqrt{x^3+x^2+8}+x}{x+2}$

Ex: Find $\lim\limits_{x \to 1} \dfrac{x^2-1}{\sqrt[3]{x}-1}$

Limit Laws / The Squeeze Theorem

$\boxed{2}$ **Theorem** If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

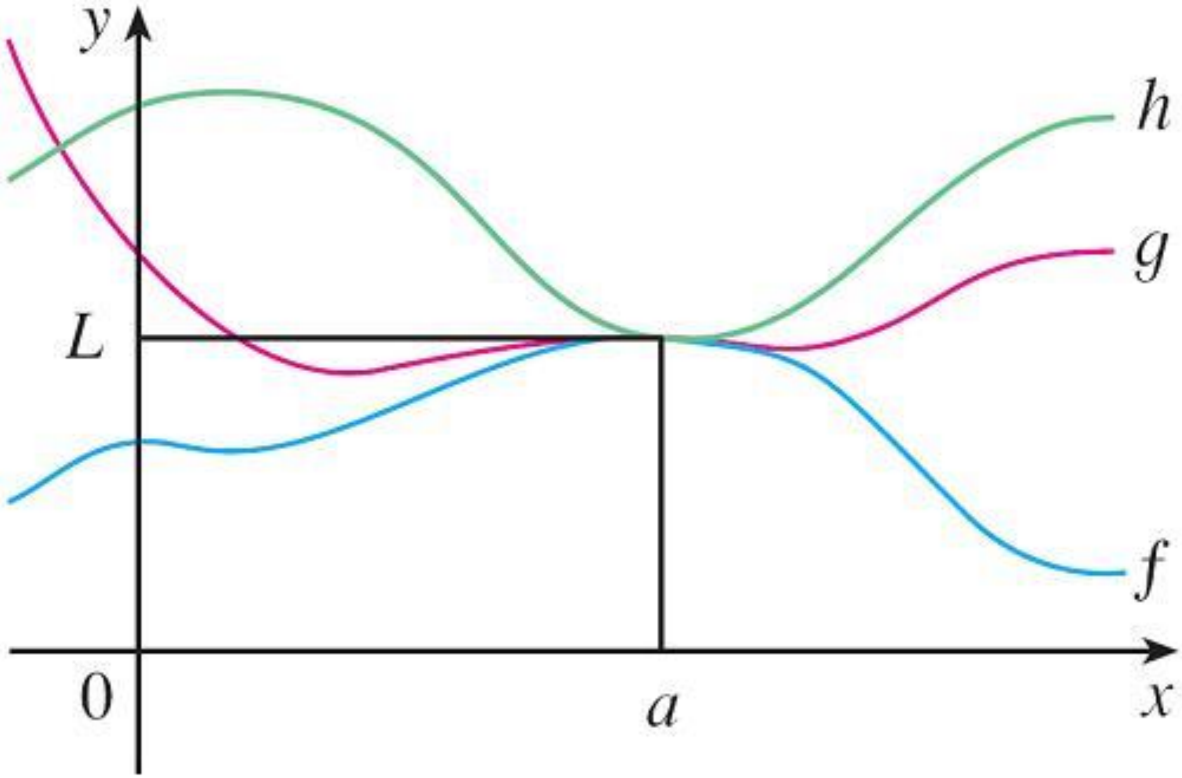
$\boxed{3}$ **The Squeeze Theorem** If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

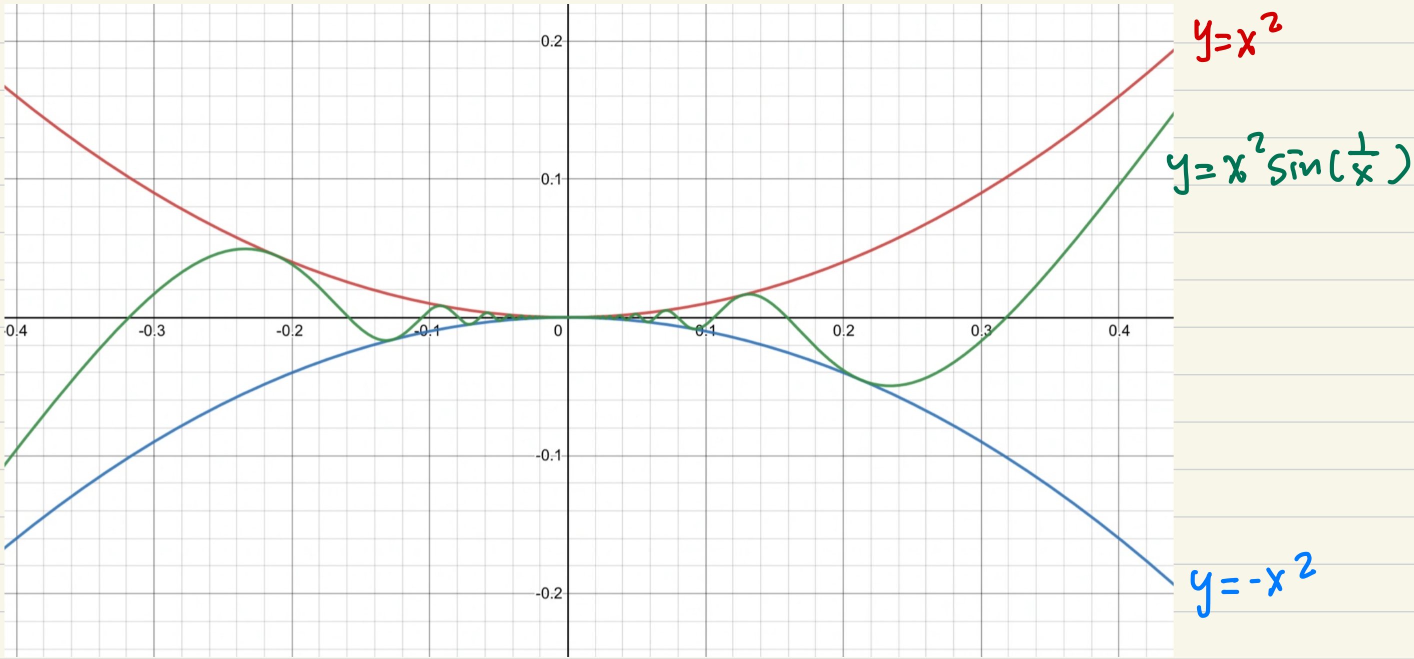
$$\lim_{x \to a} g(x) = L$$

Limit Laws / The Squeeze Theorem



Ex: Find $\lim_{x \to 0} x^2 \cdot \sin\left(\frac{1}{x}\right)$

Q: Find $\lim_{x \to 0} x \cos\left(\frac{1}{x}\right)$.



0.2
0.1
-0.1
-0.2
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
$y=x^2$
$y=x^2\sin(\frac{1}{x})$
$y=-x^2$

Ex: Find $\lim_{x \to 0} x \cdot \left[\frac{1}{x} \right]$.

Limit Laws / The Squeeze Theorem

- Property:

$$\lim_{x \to a} f(x) = 0 \Leftrightarrow \lim_{x \to a} |f(x)| = 0.$$

- Corollary:
- If $|g(x)| \leq M$ for some $M > 0$ and $\lim_{x \to a} f(x) = 0$, then $\lim_{x \to a} f(x)g(x) = 0$.

Ex: show that $\lim_{x \to a} f(x) = 0$ iff $\lim_{x \to a} |f(x)| = 0$.

Q: $f(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q} \\ -x^2, & \text{if } x \in \mathbb{Q}^c \end{cases}$. Find $\lim_{x \to 0} f(x)$.

Ex: Show that if $|g(x)| \le M$ for some $M > 0$ and $\lim_{x \to a} f(x) = 0$, then $\lim_{x \to a} f(x) \cdot g(x) = 0$.

Ex: Verify that $\lim_{x \to 0} x^2 2^{\cos\frac{1}{x}} = 0$, $\lim_{x \to 0} x\, 2^{\sin(\frac{1}{x})} = 0$

Using One-Sided Limits

- If $f(x)$ is defined differently on two sides of the point a, then the limit of $f(x)$ as x approaches a is computed by the following theorem.

3 $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$

Ex: $\lim\limits_{x \to 0} \left(\frac{1}{x} - \frac{1}{|x|} \right)$ sol:

Ex: $\lim\limits_{x \to 0} \frac{|x^2-2x|}{3x^2+|x|}$ sol:

Review

- State the Algebraic Limit Laws.
- State the Squeeze Theorem.
- When should we compute the limit by checking one-sided limits?