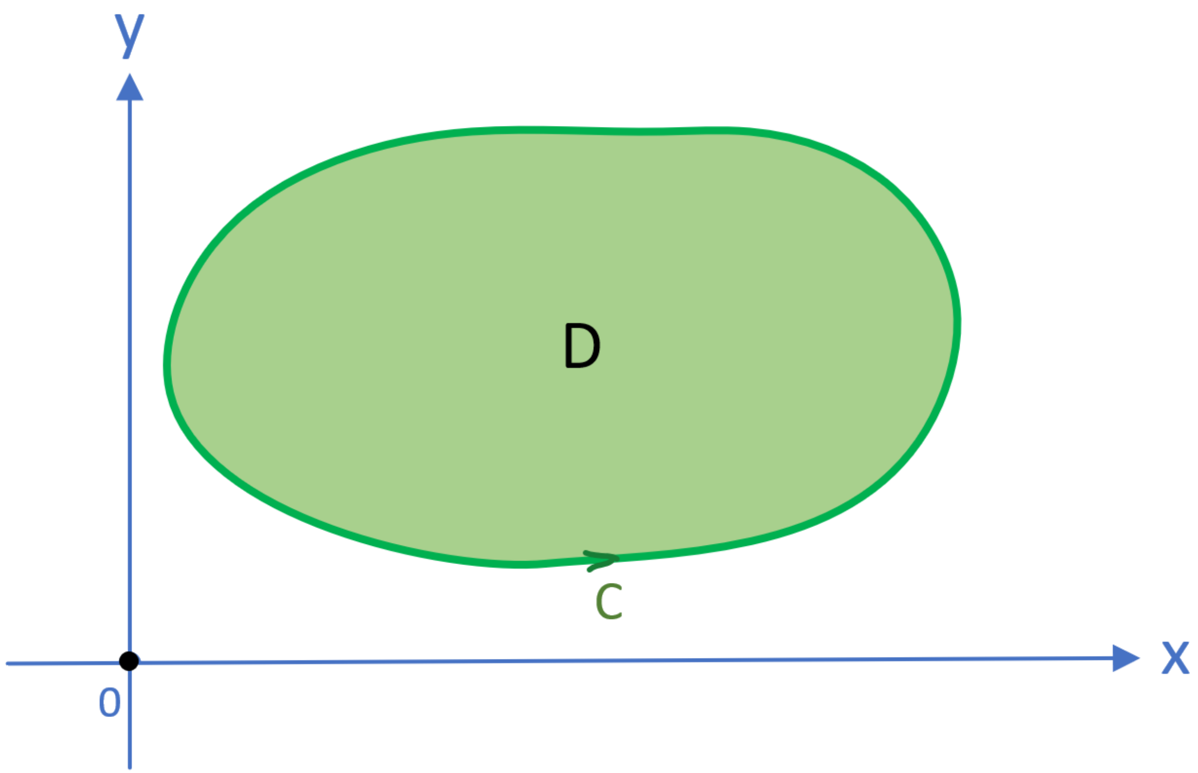
Green's Theorem

Section 16.4

Green's Theorem

- Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C.
- Hence it should be regarded as the **Fundamental Theorem of Calculus for double integrals**.

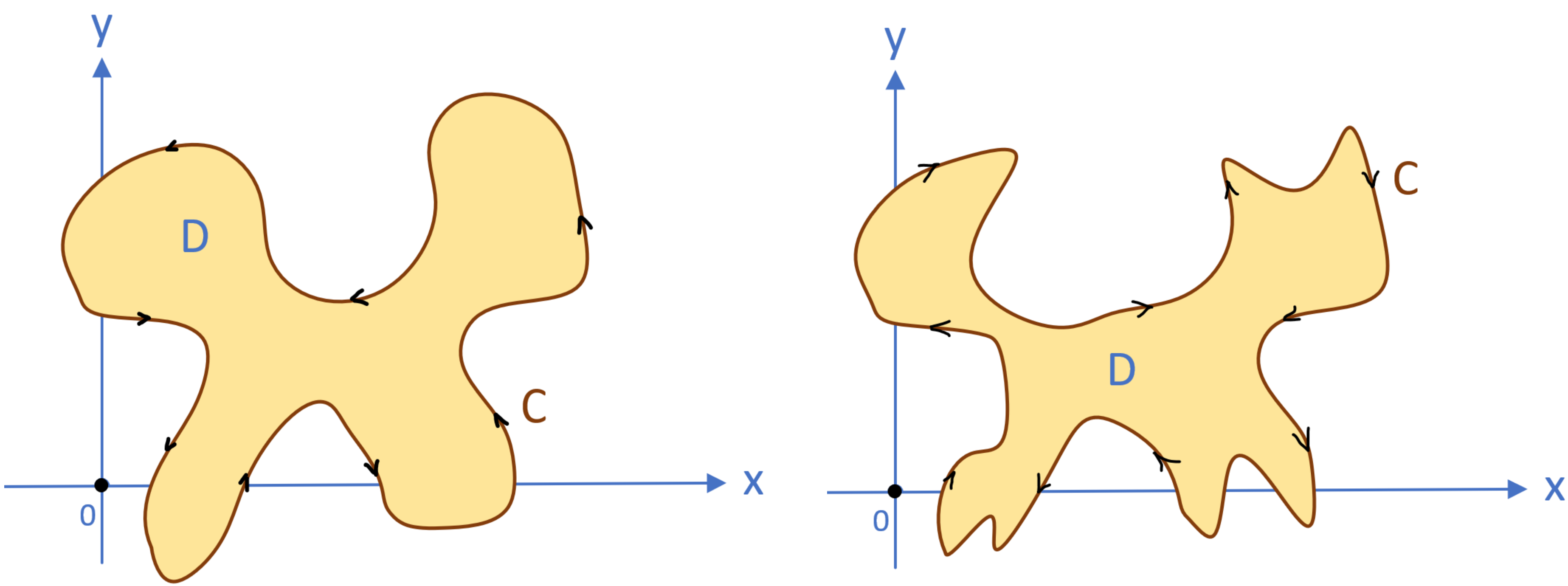
Green's Theorem



Green's Theorem

- Definition: The **positive orientation** of a simple closed curve C is a single counterclockwise traversal of C. Thus, if C is given by the vector function $\vec{r}(t), a \leq t \leq b$, then the region D is always on the *left* as the point $\vec{r}(t)$ traverses C .

Green's Theorem



(a) Positive orientation

(b) Negative orientation

Green's Theorem

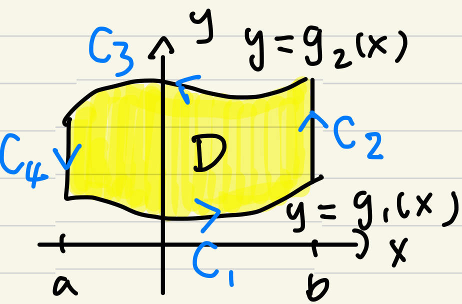
- Green's Theorem:
- Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D . Then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Proof of the theorem:

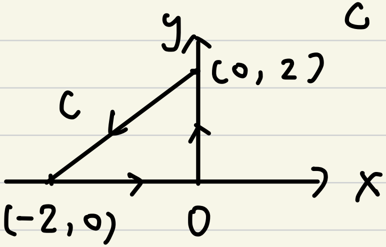
pf: We only prove Green's theorem when D is both type I and type II. Show that $\iint_D Q_x \, dA = \int_C Q \, dy$, and

$$\iint_D -P_y \, dA = \int_C P \, dx$$ for all smooth P, Q.



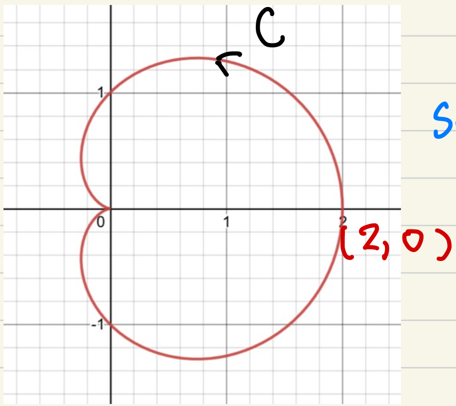
Write $D = \{(x,y) \mid a \le x \le b, \ g_1(x) \le y \le g_2(x)\}$.

Ex: Compute $\int_C xy\,dx + y^2\,dy$ where C is given in the figure.



Sol:

Ex: Compute $\int_C y\,dx + e^y \sin y\,dy$, where C is the curve $r = 1 + \cos\theta$, $0 \le \theta \le \pi$.



Sol:

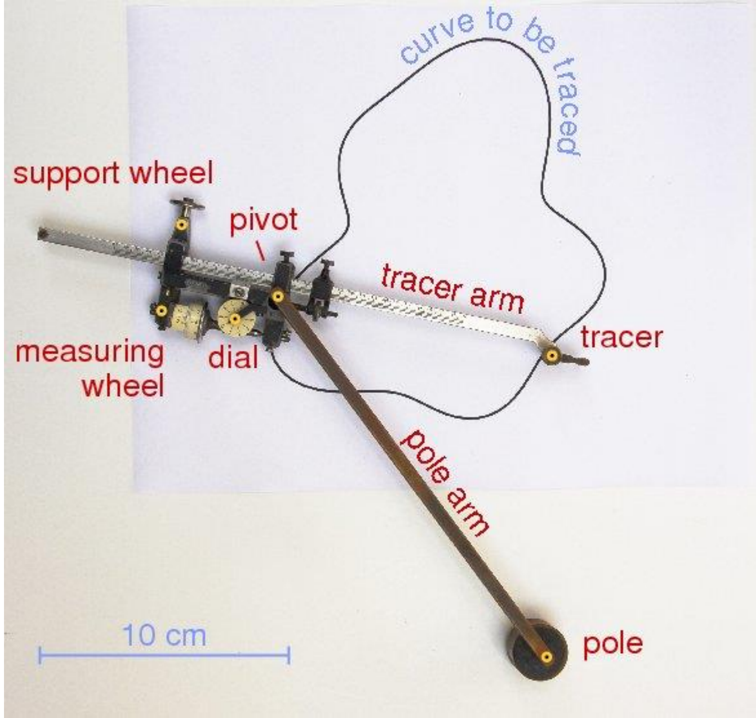
Applications of Green's Theorem

- An application of the reverse direction of Green's Theorem is in computing areas.

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

Applications of Green's Theorem

- A **planimeter** is a mechanical instrument used for measuring the area of a region by tracing its boundary curve.

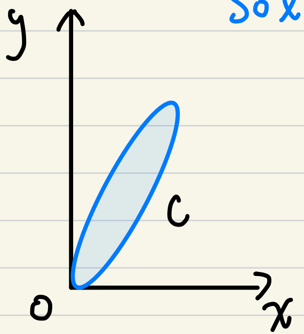


Ex: Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

sol:

Ex:

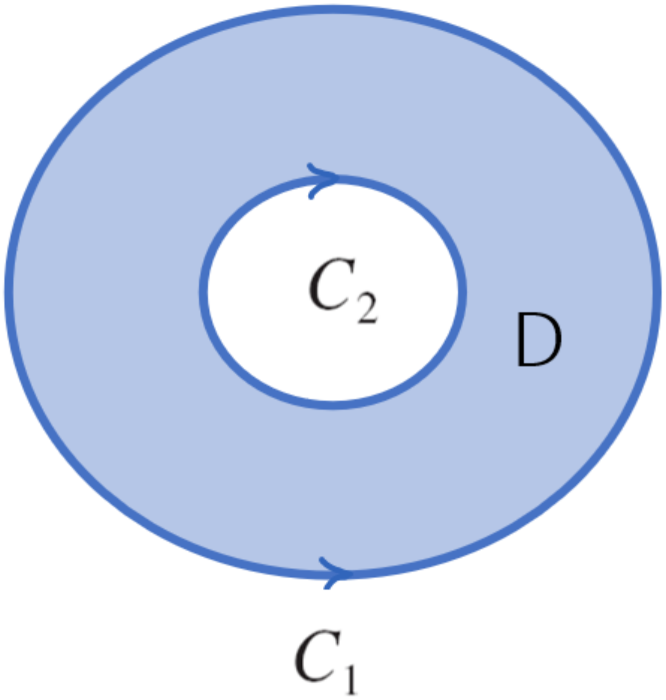
$C: \vec{r}(t) = (t-t^2, t-t^3)$, $0 \le t \le 1$. Find the area enclosed by C.



Sol:

Extended Versions of Green's Theorem

- Green's Theorem can be extended to apply to regions with holes, that is, regions that are not simply-connected.

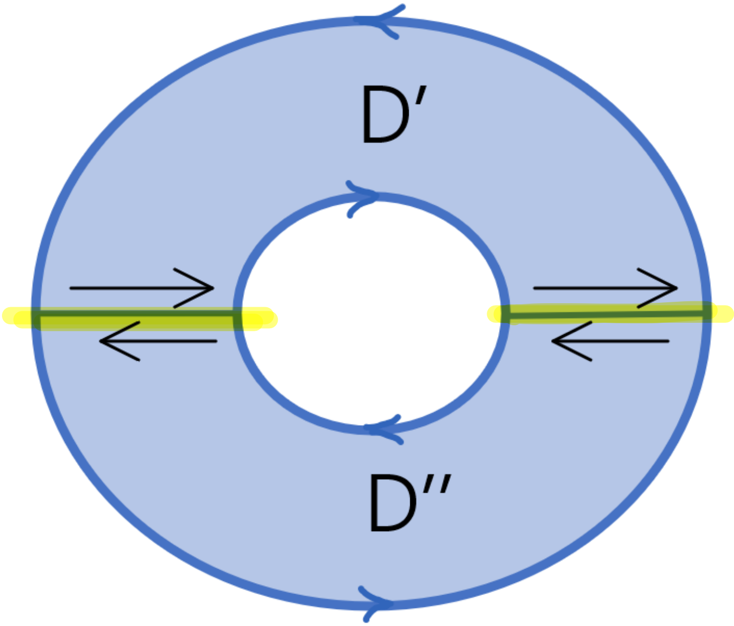


Extended Versions of Green's Theorem

- Observe that the boundary C of the region D in the above figure consists of two simple closed curves C_1 and C_2. We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed.
- Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2.

Extended Versions of Green's Theorem

- If we divide D into two regions D' and D'' as shown in the figure and then apply Green's Theorem to each of D' and D'', we get



$$\iint_D (Q_x - P_y)\,dA = \iint_{D'} (Q_x - P_y)\,dA + \iint_{D''} (Q_x - P_y)\,dA$$

Extended Versions of Green's Theorem

$$\iint_D (Q_x - P_y)dA =$$
$$\int_{\partial D'} Pdx + Qdy + \int_{\partial D''} Pdx + Qdy$$

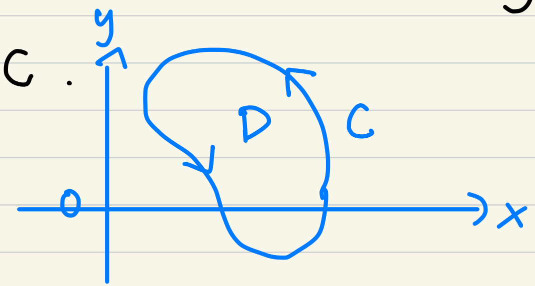
- Since the line integrals along the common boundary lines are in opposite directions, they cancel and we get

$$\iint_D (Q_x - P_y)dA = \int_{C_1} Pdx + Qdy + \int_{C_2} Pdx + Qdy$$
$$= \int_C Pdx + Qdy$$

- Hence, Green's Theorem is still true for the region D .

Ex: Show that $\oint_C \vec{F} \cdot d\vec{r} = \begin{cases} 0, & \text{if } C \text{ doesn't enclose } (0,0) \\ 2\pi, & \text{if } C \text{ encloses } (0,0) \end{cases}$,

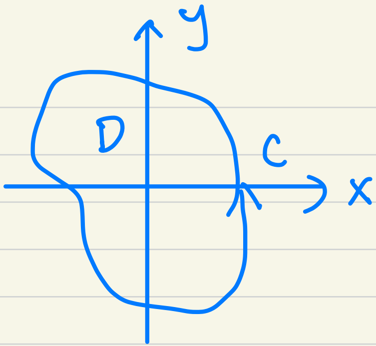
where $\vec{F}(x,y) = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j} = P\vec{i} + Q\vec{j}$ and C is a positively oriented simple closed curve and $(0,0) \notin C$.



Sol: Recall : $\boxed{P_y = Q_x}$

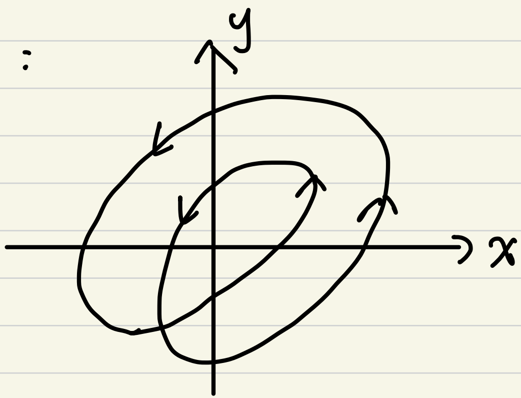
$\boxed{\text{Case 1:}}$ Suppose that C encloses D and $(0,0) \notin D$.

$\boxed{\text{Case 2}}$ C encloses D and $(0,0) \in D$.

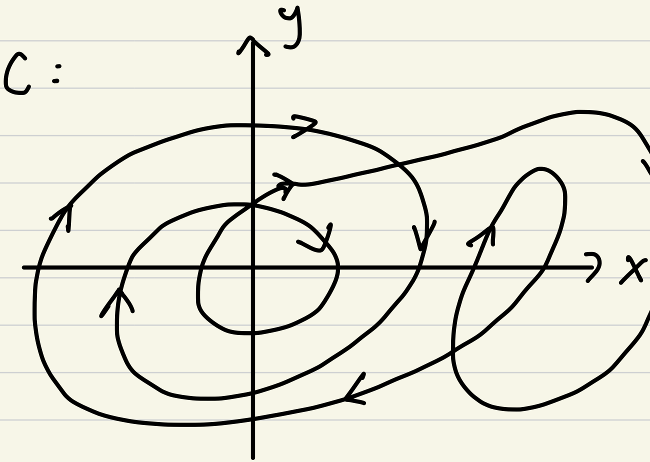

y
D
C
x

Ex: Compute $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = \dfrac{-y}{x^2+y^2}\,\vec{i} + \dfrac{x}{x^2+y^2}\,\vec{j}$.

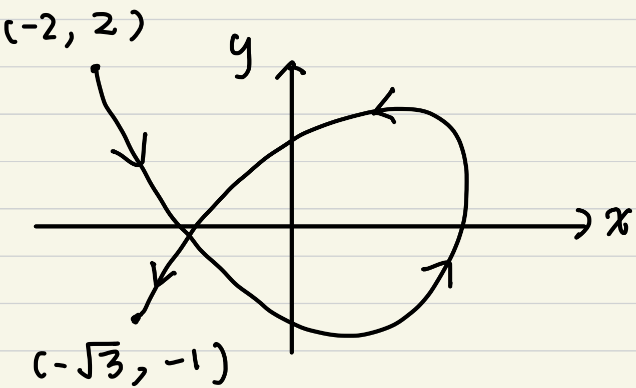
(a) C:



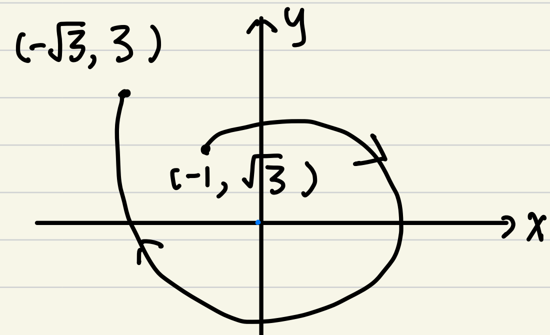
(b) C:



(c) C:



(d) C:



Conclusion : If $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ are defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ and P, Q have continuous partial derivatives on $\mathbb{R}^2 \setminus \{(0,0)\}$ with $\boxed{P_y = Q_x}$, then

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① $\vec{F}$ is conservative on any open simply connected region $D \subset \mathbb{R}^2 \setminus \{(0,0)\}$.

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② $\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy = 0$ for all simple closed curve C in $\mathbb{R}^2 \setminus \{(0,0)\}$ which doesn't enclose $(0,0)$.

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③ $\int_{C_r} \vec{F} \cdot d\vec{r}$ is constant for all C_r where C_r is the positively oriented circle $(r\cos t, r\sin t)$, $0 \le t \le 2\pi$.

④ If C is a positively oriented simple closed curve which encloses $(0,0)$, then $\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r}$.

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⑤ If $\boxed{\int_{C_1} \vec{F} \cdot d\vec{r} = 0}$, then $\int_C \vec{F} \cdot d\vec{r} = 0$ for all simple closed curve $C \subset \mathbb{R}^2 \setminus \{(0,0)\}$. Hence line integrals of $\vec{F}$ are independent of path on $\mathbb{R}^2 \setminus \{(0,0)\}$ and

$\boxed{\vec{F} \text{ is conservative on } \mathbb{R}^2 \setminus \{(0,0)\}.}$

(c) Let $\mathbf{G}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be a C^1-vector field on $\mathbb{R}^2 \setminus \{(0,0)\}$. It is known that $\mathbf{G}$ satisfies :

$$(1)\ Q_x = P_y, \qquad (2)\ \oint_E \mathbf{G} \cdot d\mathbf{r} = 4\pi.$$

(i) Find the value of k such that $\oint_E (\mathbf{G} + k\mathbf{F}) \cdot d\mathbf{r} = 0$.

(ii) For the value that we found in (c)(i), we are going to prove that $\mathbf{H} = \mathbf{G} + k\mathbf{F}$ is conservative on $\mathbb{R}^2 \setminus \{(0,0)\}$ by the following steps.

Applications of Green’s Theorem

- We can use Green’s Theorem to prove the necessary and sufficient condition for a vector field to be conservative provided that the domain of the vector field is simply connected.

Ex: Prove that if $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$ is defined on an $\boxed{\text{open simply-connected region}}$ D and $\boxed{\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}}$ on D, then $\vec{F}$ is conservative on D.

sol:

Review

- State Green's Theorem. How do we define the orientations of the boundary curves so that Green's Theorem is true?
- Review some applications of Green's Theorem.