Second-Order Linear Differential Equations

Section 17.1-17.2

Outline

- Definitions and Basic Properties
- Solve Homogeneous Equations
  - Two Distinct Real Roots
  - One Real Root
  - Two Complex Roots
- Initial-Value and Boundary-Value Problems
- Solve Nonhomogeneous Equations (Find a Particular Solution)
  - The Method of Undetermined Coefficients
  - The Method of Variation of Parameters

Some Definitions

- A **second-order linear differential equation** has the form
$$P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y=G(x)$$
where P, Q, R, and G are continuous functions.
- When $G(x)=0$, such equations are called **homogeneous** linear equations.
- If $G(x)\neq 0$ for some x, it is a **nonhomogeneous** differential equation.

Basic Properties

3 Theorem If $y_1(x)$ and $y_2(x)$ are both solutions of the linear homogeneous equation $\boxed{2}$ and c_1 and c_2 are any constants, then the function

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is also a solution of Equation 2.

4 Theorem If y_1 and y_2 are linearly independent solutions of Equation 2 on an interval, and $P(x)$ is never 0, then the general solution is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1 and c_2 are arbitrary constants.

Solve Homogeneous Equations

- Now we focus on second-order linear differential equations with constant coefficient

$$\boxed{ay'' + by' + cy = 0}$$

where a, b, and c are constants and $a \neq 0$.

- We know that the exponential function $y = e^{rx}$ (where r is a constant) has the property that its derivative is a constant multiple of itself. We see that $y = e^{rx}$ is a solution if

$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$ i.e. $ar^2 + br + c = 0$

Solve Homogeneous Equations

- Hence $$\boxed{ar^2 + br + c = 0}$$
- This equation is called the **auxiliary equation** (or **characteristic equation**) of the differential equation $ay'' + by' + cy = 0$.
- **Case I:** $b^2 - 4ac > 0$

$\boxed{8}$ If the roots r_1 and r_2 of the auxiliary equation $ar^2 + br + c = 0$ are real and unequal, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Ex: Solve $y'' - 5y' + 4y = 0$.

Ex: Solve $y'' + 2y' - 5 = 0$

Solve Homogeneous Equations

- **Case II:** $b^2 - 4ac = 0$

10 If the auxiliary equation $ar^2 + br + c = 0$ has only one real root r, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

- **Case III:** $b^2 - 4ac < 0$

11 If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

Ex: Consider $ay'' + by' + cy = 0$ with $b^2 = 4ac$. Let $r = \frac{-b}{2a}$.
Show that $y(x) = x \cdot e^{rx}$ is a solution.

Ex: Solve $4y'' + 4y' + 1y = 0$.

Sol:

$\boxed{\text{Case 3 : } b^2 - 4ac < 0}$

The roots of $ar^2 + br + c = 0$ are $r = \alpha \pm \beta i$, where $\alpha = \frac{-b}{2a}$, $\beta = \frac{\sqrt{4ac - b^2}}{2a}$. Solutions of $ay'' + by' + cy = 0$ are $e^{(\alpha \pm i\beta)x} = e^{\alpha x} \cdot e^{\pm i\beta x} = e^{\alpha x}(\cos\beta x \pm i\sin\beta x)$

Hence $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are solutions.

Ex: Solve $y'' + 4y = 0$

Ex: Solve $y'' - 6y' + 13y = 0$

Ex: Motion of a spring.

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} \quad \text{where } m, k > 0, \; c \geq 0.$$

① Discuss the solution according to the value of c.

② Find $\lim_{t\to\infty} x(t)$.

Initial-Value Problem

- An **initial-value problem** for the second-order Equation 1 or 2 consists of finding a solution y of the differential equation that also satisfies initial conditions of the form

$$y(x_0) = y_0 \,, \quad y'(x_0) = y_1$$

where y_0 and y_1 are given constants.

Boundary-Value Problem

- A **boundary-value problem** for Equation 1 or 2 consists of finding a solution y of the differential equation that also satisfies boundary conditions of the form

  $y(x_0) = y_0$, $y(x_1) = y_1$.

- In contrast with the situation for initial-value problems, a boundary-value problem does not always have a solution.