Curl and Divergence

Section 16.5

Outline

- Curl
- Interpretation of Curl
- Divergence
- Interpretation of Divergence
- Vector Forms of Green's Theorem

Curl

- If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on R^3 and the partial derivatives of P, Q, and R all exist, then the **curl** of $\vec{F}$ is the vector field on R^3 defined by

$$\mathrm{curl}\vec{F} = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\vec{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\vec{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\vec{k}$$

- Let's rewrite the equation using operator notation. We introduce the *vector differential operator* ∇ ("del") as $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

Curl

- It has meaning when it operates on a scalar function to produce the gradient of f :

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

- If we think of ∇ as a vector with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, we can also consider the formal cross product of ∇ with the vector field $\vec{F}$ as follows:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Curl

- So the easiest way to remember the curl is by means of the symbolic expression
$$\text{curl}\,\vec{F} = \nabla \times \vec{F}$$
- Theorem: If f is a function of three variables that has continuous second-order partial derivatives, then $\text{curl}(\nabla f) = \vec{0}$.
- Theorem: If $\vec{F}$ is conservative, then $\text{curl}\,\vec{F} = \vec{0}$
- This gives us a way of verifying that a vector field is not conservative.

Ex: Compute curl $\vec{F}$, where $\vec{F} = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$.

sol:

Ex: Compute curl $\vec{F}$, where $\vec{F}(x,y,z) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$.

sol:

Ex: If $f(x,y,z)$ has continuous 2nd order partial derivatives, then $\text{curl}(\vec{\nabla} f) = \vec{0}$.

Sol:

Ex: The field $\vec{F} = (axy+z)\vec{i} + x^2\vec{j} + (bx+2z)\vec{k}$ is conservative. Find a and b. Find a potential function of $\vec{F}$.

Sol:

Curl

- The converse of the above theorem is not true in general, but the following theorem says the converse is true if $\vec{F}$ is defined everywhere. (More generally it is true if the domain is *simply-connected*.)
- **Theorem:** If $\vec{F}$ is a vector field defined on all of R^3 whose component functions have continuous partial derivatives and $\operatorname{curl} \vec{F} = \vec{0}$, then $\vec{F}$ is a conservative vector field.

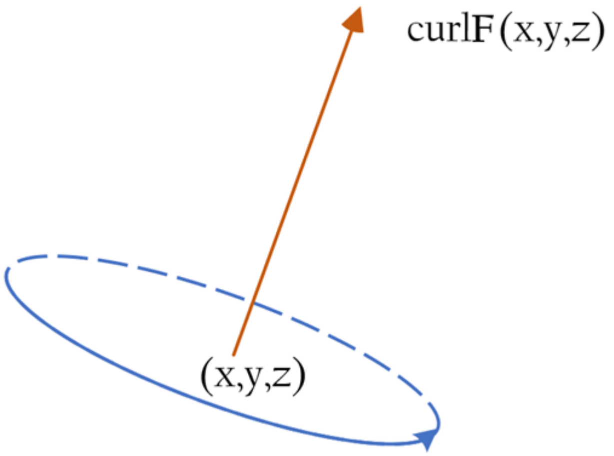
Interpretation of Curl

- The reason for the name *curl* is that the curl vector is associated with *rotations*.
- Theorem: If $\vec{F}$ is a smooth vector field and C_ϵ is a circle of radius ϵ centered at point P and bounding a disc S_ϵ with unit normal $\vec{N}$ (the orientation inherited from C_ϵ), then

$$\lim_{\epsilon \to 0} \frac{1}{\pi \epsilon^2} \oint_{C_\epsilon} \vec{F} \cdot d\vec{r} = \text{curl } \vec{F}(P) \cdot \vec{N}$$

Interpretation of Curl

- Hence, when $\vec{F}$ represents the velocity field in fluid flow, particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of $\text{curl } \vec{F}(x, y, z)$, and the length of this curl vector is a measure of how quickly the particles move around the axis.



Interpretation of Curl

- If $\operatorname{curl}\vec{F}=\vec{0}$ at a point P, then the fluid is free from rotations at P and $\vec{F}$ is called **irrotational** at P.
- If $\operatorname{curl}\vec{F}\neq\vec{0}$, the fluid rotates about the axis with direction $\operatorname{curl}\vec{F}$.
- Example: The velocity vector field of rotation is $\vec{F}=\vec{w}\times\vec{r}$, where $\vec{w}$ is in the direction of rotation axis, $|\vec{w}|$ is the angular speed, and $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$. Compute $\operatorname{curl}\vec{F}$.

Ex: The velocity field of rotation: $\vec{F}(x,y,z) = \vec{\omega} \times \vec{r}$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{\omega} = a\vec{i} + b\vec{j} + c\vec{k}$ is fixed. Compute curl $\vec{F}$.

Sol: $$\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\vec{i} + (cx - az)\vec{j} + (ay - bx)\vec{k}$$

Divergence

- If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on R^3 and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of** $\vec{F}$ is the function of three variables defined by

$$\mathrm{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \vec{F}$$

- Note: Observe that $\mathrm{curl}\ \vec{F}$ is a vector field but $\mathrm{div}\ \vec{F}$ is a scalar field.

Divergence

- Theorem: If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is defined on R^3 and P, Q, R have continuous second-order partial derivatives, then $\text{div curl } \vec{F} = 0$
- $$\text{div } (\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

  and this expression occurs so often that we abbreviate it as $\nabla^2 f$. The operator $\nabla^2 = \nabla \cdot \nabla$ is called the **Laplace operator**. Another notation for ∇^2 is Δ.

$\boxed{\text{Thm}}$: If $\vec{F}(x,y,z) = P\vec{i} + Q\vec{j} + R\vec{k}$ and P, Q, R have continuous 2nd order partial derivatives, then $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$.

Pf:

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k}$$

Divergence

- We can also apply the Laplace operator ∇^2 to a vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ in terms of its components:

$$\nabla^2 \vec{F} = \nabla^2 P\vec{i} + \nabla^2 Q\vec{j} + \nabla^2 R\vec{k} \quad .$$

Interpretation of Divergence

- Theorem:
- If $\vec{N}$ is the unit outward normal on the sphere S_ϵ of radius ϵ centered at point P, and if $\vec{F}$ is a smooth three-dimensional vector field, then

$$\lim_{\epsilon \to 0} \frac{3}{4\pi\epsilon^3} \iint_{S_\epsilon} \vec{F} \cdot d\vec{S} = \operatorname{div} \vec{F}(P)$$

Interpretation of Divergence

- If $\vec{F}(x, y, z)$ is the velocity of a fluid (or gas), then $\operatorname{div} \vec{F}(x, y, z)$ represents the *net rate of change* (with respect to time) *of the mass of fluid* (or gas) flowing from the point (x, y, z) per unit volume.
- If $\operatorname{div} \vec{F} = 0$, then $\vec{F}$ is said to be **incompressible**.

Vector Differential Identities

- Let f be a scalar function and $\vec{F}$, $\vec{G}$ be vector fields, all assumed to be sufficiently smooth that the partial derivatives in the identities are continuous. Then the following identities hold.

$$\nabla \cdot (f\vec{F}) = f(\nabla \cdot \vec{F}) + (\nabla f) \cdot \vec{F}$$

$$\nabla \times (f\vec{F}) = f(\nabla \times \vec{F}) + (\nabla f) \times \vec{F}$$

Vector Differential Identities

$$\nabla \cdot (\vec{F} \times \vec{G}) = (\nabla \times \vec{F}) \cdot \vec{G} - \vec{F} \cdot (\nabla \times \vec{G})$$

$$\nabla \times (\vec{F} \times \vec{G}) = (\nabla \cdot \vec{G})\vec{F} + (\vec{G} \cdot \nabla)\vec{F} - (\nabla \cdot \vec{F})\vec{G} - (\vec{F} \cdot \nabla)\vec{G}$$

$$\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

(curl curl =grad div –Laplacian)

Ex: Given $\begin{cases} \operatorname{div} \vec{E} = 0 \\ \operatorname{curl} \vec{E} = -\frac{1}{c} \frac{\partial H}{\partial t} \end{cases}$ $\begin{cases} \operatorname{div} \vec{H} = 0 \\ \operatorname{curl} \vec{H} = \frac{1}{c} \frac{\partial E}{\partial t} \end{cases}$, show that

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}.$$

sol: $\vec{\nabla}^2 \vec{E} = \vec{\nabla} \cdot (\operatorname{div} \vec{E}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{E})$

Vector Forms of Green's Theorem

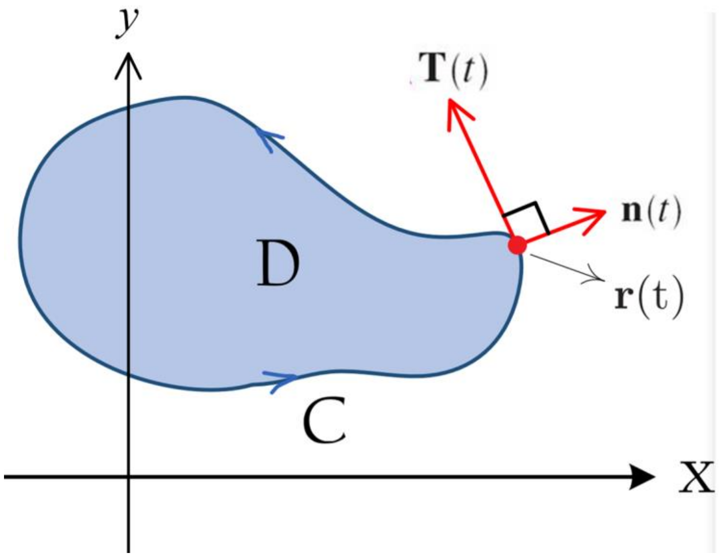
- We suppose that the plane region D , its boundary curve C , and the functions $P$and Q satisfy the hypotheses of Green's Theorem. Then we consider the vector field $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$.
- We can now rewrite the equation in Green's Theorem in the vector form

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\mathrm{curl}\vec{F}) \cdot \vec{k}\, dA$$

$\mathrm{curl}\,\vec{F} = (Q_x - P_y)\,\vec{k}$

Vector Forms of Green's Theorem

- If C has a parametrization $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ for $a \le t \le b$, then the unit tangent vector is
$$\vec{T}(t) = \frac{x'(t)}{|\vec{r'}(t)|}\vec{i} + \frac{y'(t)}{|\vec{r'}(t)|}\vec{j}$$
, and the outward unit normal vector is
$$\vec{n}(t) = \frac{y'(t)}{|\vec{r'}(t)|}\vec{i} - \frac{x'(t)}{|\vec{r'}(t)|}\vec{j}$$



Vector Forms of Green's Theorem

- Then $\int_C \vec{F} \cdot \vec{n}\, ds = \int_a^b (\vec{F} \cdot \vec{n})(t)|\vec{r'}|\, dt$

  $= \int_a^b P(x(t), y(t))y'(t) - Q(x(t), y(t))x'(t)\, dt$

  $= \int_C P\, dy - Q\, dx = \iint_D (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y})\, dA$
- Hence,

$$\oint_C \vec{F} \cdot d\vec{n} = \iint_D (\mathrm{div}\vec{F})(x, y)\, dA$$

Review

- Given a vector field $\vec{F}(x,y,z)$, what is $\text{curl } \vec{F}$?
- What is the interpretation of $\text{curl } \vec{F}$?
- Given a vector field $\vec{F}(x,y,z)$, what is $\text{div } \vec{F}$?
- What is the interpretation of $\text{div } \vec{F}$?