Vector Fields and Line Integrals

Section 16.1-16.3

Outline

- Vector Field, Gradient Vector Fields
- Line Integrals
  - With Respect to Arc Length
  - With Respect to Variables
  - Integrate Vector Fields Along a Curve
- The Fundamental Theorem for Line Integrals
  - Independence of Path

The Fundamental Theorem for Line Integrals

- Theorem:
- Let C be a smooth plane curve (or space curve) with parametrization $\vec{r}(t),\ a \le t \le b$. Let f be a differentiable function of two (or three) variables whose gradient ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) \; .$$

v.s. $\int_a^b g'(t)\,dt = g(b) - g(a)$

Proof of the theorem.

Suppose that $f(x,y)$ is a function of 2 variables and C has a parametrization $\vec{r}(t) = (x(t), y(t))$, $a \le t \le b$.

The Fundamental Theorem for Line Integrals

- Suppose C_1 and C_2 are two piecewise-smooth curves (which are called **paths**) that have the same initial point A and terminal point B. We know that, in general, $\int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r}$. But $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$.

- In other words, the line integral of a *conservative* vector field depends only on *the initial point and the terminal point of* the curve.

The Fundamental Theorem for Line Integrals

- **Definition**: If $\vec{F}$ is a continuous vector field with domain D, we say that the line integral $\int_C \vec{F} \cdot d\vec{r}$ is **independent of path** if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two paths C_1 and C_2 in D that have the same initial and terminal points.

- *Line integrals of conservative vector fields are independent of path.*

The Fundamental Theorem for Line Integrals

- A curve is called **closed** if its terminal point coincides with its initial point, that is, $\vec{r}(a) = \vec{r}(b)$.
- If line integral of $\vec{F}$ is independent of path in D and C is any closed path in D , then $\int_C \vec{F} \cdot d\vec{r} = 0$.
- Conversely, if $\int_C \vec{F} \cdot d\vec{r} = 0$ whenever C is a closed path in D, then the line integral of $\vec{F}$ is independent of path in D.

The Fundamental Theorem for Line Integrals

- **Theorem**: $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C in D.

- **Theorem**: Suppose $\vec{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D, then there is a scalar function f such that $\nabla f = \vec{F}$, which means that $\vec{F}$ is conservative.

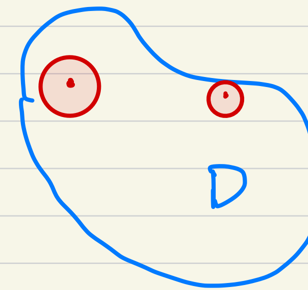
$\boxed{\text{Theorem}}$: $\int_C \vec{F} \cdot d\vec{r}$ is indep of path in D if and only if $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every closed path $C \subset D$.

Pf: "$\Leftarrow$" Suppose $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every closed path $C \subset D$.

"$\Longrightarrow$" Suppose that $\int_C \vec{F} \cdot d\vec{r}$ is indep of path in D.

Def: A plane region $D \subset \mathbb{R}^2$ is **open** if for any $\vec{x}_0 \in D$, there is some $\varepsilon > 0$ s.t.

$$D_\varepsilon(\vec{x}_0) = \{\vec{x} \mid |\vec{x} - \vec{x}_0| \le \varepsilon\} \subset D.$$

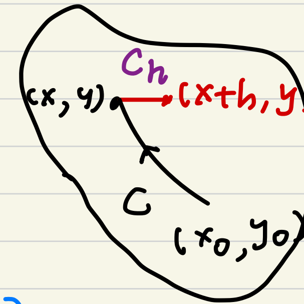


Def: A plane region $D \subset \mathbb{R}^2$ is **connected** if for any $\vec{x}_0, \vec{y}_0 \in D$, there is some piecewise smooth curve $C \subset D$ connecting $\vec{x}_0$ and $\vec{y}_0$.

not connected

Connected region

$\boxed{\text{Theorem}}$: Suppose $\vec{F}$ is a vector field that is continuous on an $\boxed{\text{open}}$ $\boxed{\text{connected}}$ region D. If $\int_C \vec{F} \cdot d\vec{r}$ is indep of path in D, then $\vec{F}$ is a conservative vector field on D.



pf: Fix a point $(x_0, y_0) \in D$.

For any $(x, y) \in D$, define $f(x, y) = \int_C \vec{F} \cdot d\vec{r}$

where C is a curve in D from (x_0, y_0) to (x, y).

Claim: $\vec{\nabla} f = \vec{F}$ i.e. if $\vec{F} = P\vec{i} + Q\vec{j}$, then $f_x = P$, $f_y = Q$.

The Fundamental Theorem for Line Integrals

- **Theorem**: If $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is a conservative vector field, where P and Q have continuous first order partial derivatives on a domain D , then throughout D , we have $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

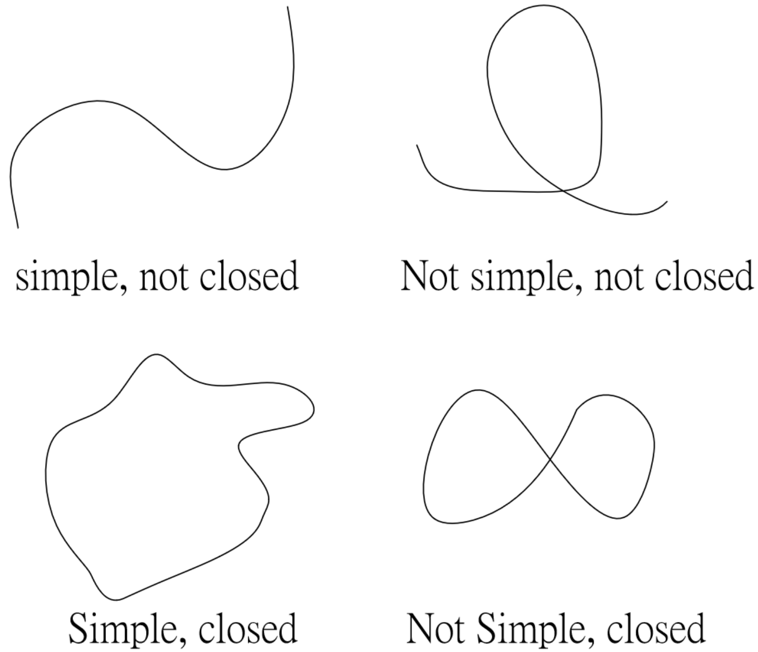
- The converse of the theorem is true only for a certain type of region!

$\boxed{\text{Theorem}}$: $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$ is a conservative vector field on a domain D where P & Q have continuous 1st order partial derivatives. Then $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ on D.

Pf: Suppose that $\vec{F} = P\,\vec{i} + Q\,\vec{j} = \vec{\nabla} f$ i.e. $P = f_x$, $Q = f_y$.

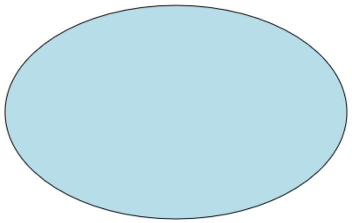
The Fundamental Theorem for Line Integrals

- To explain this, we first need the concept of a **simple curve**, which is a curve that doesn't intersect itself anywhere between its endpoints.

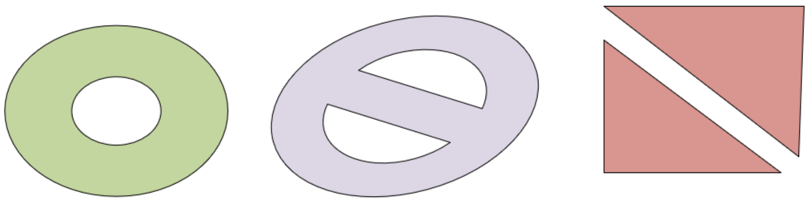


The Fundamental Theorem for Line Integrals

- A **simply-connected region** in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D.



Simply-connected region



Regions that are not simply-connected

Ex: Are the following regions simply-connected?

$D_1 = \{(x,y) \mid x>0,\ y>0\}$

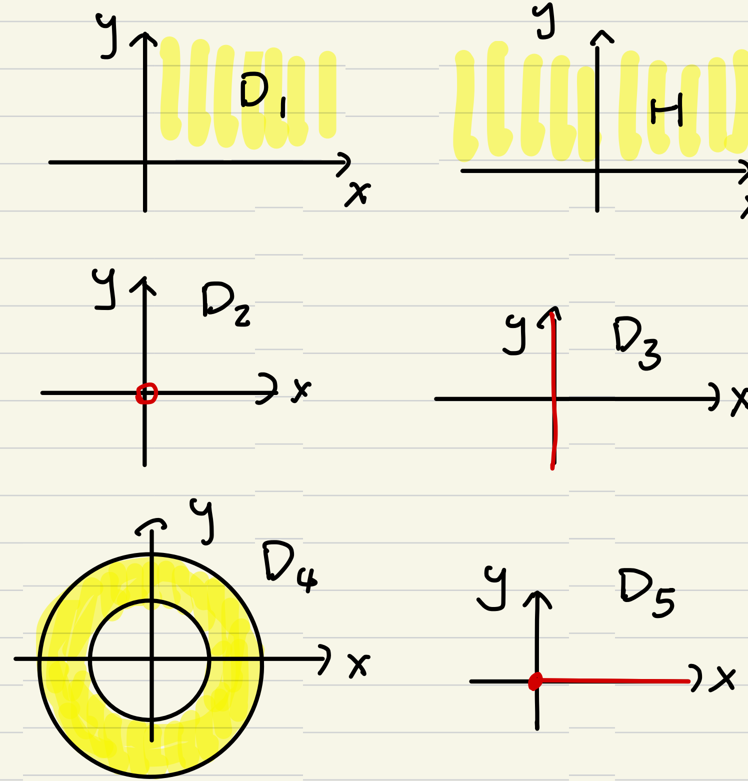
$H = \{(x,y) \mid y>0\}$

$D_2 = \{(x,y) \mid (x,y) \neq (0,0)\}$

$D_3 = \{(x,y) \mid x \neq 0\}$

$D_4 = \{(x,y) \mid 1 < x^2+y^2 < 4\}$.

$D_5 = \mathbb{R}^2 \setminus \{(x,0) \mid x \geq 0\}$



The Fundamental Theorem for Line Integrals

- In terms of simply-connected regions, we have a convenient method for verifying that a vector field on R^2 is conservative.
- **Theorem**: Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a vector field

1 on an open simply-connected region D.

2 Suppose that P and Q have continuous first-order partial derivatives and $\boxed{\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}}$ on D. Then $\vec{F}$ is conservative.

Ex: Determine whether or not the vector field is conservative.
If it is, find its potential function.

$$\vec{F}(x,y) = (\ln y + 2xy^3)\,\vec{i} + \left(3x^2y^2 + \frac{x}{y} + \cos y\right)\vec{j}$$

Sol:

Ex: Find $\int_C \vec{F} \cdot d\vec{r}$, where C is the part of the parabolic $y = x^2 + 1$ from $(-1, 2)$ to $(2, 5)$ and

a) $\vec{F}(x,y) = \left(\ln y + 2xy^3,\ 3x^2y^2 + \frac{x}{y} + \cos y \right)$

b) $\vec{F}(x,y) = \left(\ln y + 2xy^3 - 2,\ 3x^2y^2 + \frac{x}{y} + \cos y \right)$

Sol:

Ex: If $\vec{F}(x,y,z) = y^2\vec{i} + (2xy + e^{3z})\vec{j} + 3ye^{3z}\vec{k}$, find a scalar function f s.t. $\vec{\nabla}f = \vec{F}$.

sol:

Ex: Determine whether the vector field is conservative.

a) $\vec{F}(x,y) = \dfrac{x}{x^2+y^2}\vec{i} + \dfrac{y}{x^2+y^2}\vec{j}$

b) $\vec{F}(x,y) = \dfrac{-y}{x^2+y^2}\vec{i} + \dfrac{x}{x^2+y^2}\vec{j}$

Sol:

Ex: $\vec{F} = \left(\ln y + \dfrac{ax+by}{x^2+y^2} \right) \vec{i} + \left(\dfrac{x}{y} + \dfrac{x-4y}{x^2+y^2} \right) \vec{j}$ is conservative on its domain. Find a, b.

$\boxed{\text{Conservative of Energy}}$ A particle with mass m moves along a path C: $\vec{r}(t)$, $a \le t \le b$, under the influence of the force field $\vec{F}$ i.e. $\vec{F}(\vec{r}(t)) = m \cdot \vec{a}(t) = m\,\vec{r}''(t)$. Compute the work done by the force field. If $\vec{F}$ is conservative, show that the energy is conservative.

Sol:

Review

- What is a vector field? What is a gradient vector field or a conservative vector field?
- How do we do line integrals with respect to arc length or with respect to variables? How do we integrate a vector field along a curve?
- Given a vector field $\vec{F} = P\vec{i} + Q\vec{j}$, what are the relations between the following statements ?
  - The line integral of $\vec{F}$ is independent of path.
  - The line integral of $\vec{F}$ along any closed curve is 0.
  - $\vec{F}$ is conservative.
  - $\partial P/\partial y = \partial Q/\partial x$