Derivatives

Section 2.7-2.8

Outline:

- Derivatives:
  - As tangents and velocities
  - In general
- The Derivative as a Function:
  - Differentiable functions
  - Some cases of not differentiable functions
  - Higher derivatives

Derivatives

- The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit.
- This special type of limit is called a *derivative.*

Derivatives: Tangents

1 **Definition** The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

provided that this limit exists.

2

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Derivatives: In General

4 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

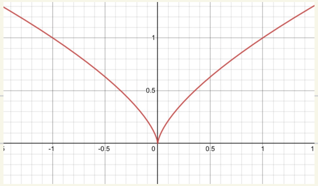
if this limit exists.

5

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

Ex: Find $f'(1)$ where $f(x)=\sqrt{x}$. Find the tangent line of $y=\sqrt{x}$ at $x=1$.

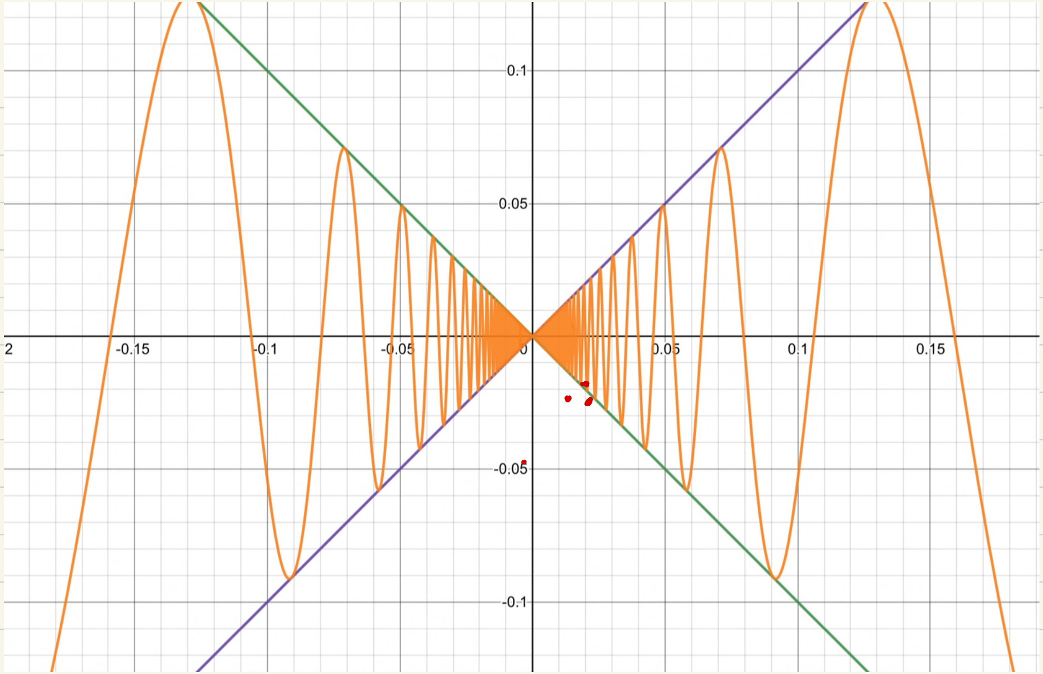
Ex: Find $f'(0)$, where $f(x) = x^{\frac{2}{3}}$.



Sol:

Ex: Find $f'(0)$, where $f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases}$

Sol:



$y = x \sin\frac{1}{x}$

Q: $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$. Find $f'(0)$?

Ex: Suppose that $\boxed{f \text{ is diff at } x=a.}$

Find $\lim\limits_{h \to 0} \dfrac{f(a+h)-f(a-h)}{2h}$.

$(a-h, f(a-h))$
和 $(a+h, f(a+h))$
間的割線斜率.

Q: Can we define $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h)-f(a-h)}{2h}$?

(Try $f(x)=|x|$ at $a=0$.)

Derivatives: In General

- Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. Then the derivative of f at $x = a$ means the **instantaneous rate of change** of y with respect to x at $x = a$.

$$\boxed{6} \qquad \text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The Derivative as a Function

- Let x be a variable. We can define

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

- Now we regard $f'(x)$ as a new function, called the **derivative** of f .

Ex: Find $f'(x)$, where $f(x) = \sqrt{x}$.

Ex: Find $f'(x)$ where $f(x) = |x|$.

Ex: Recognize $\lim\limits_{h \to 0} \dfrac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$, $\lim\limits_{h \to 0} \dfrac{a^h - 1}{h}$, $\lim\limits_{h \to 0} \dfrac{\sin h}{h}$

as derivatives.

The Derivative as a Function

- Notations:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

- The symbols D and $\dfrac{d}{dx}$ are called **differentiation operators** because they indicate the operation of **differentiation**.

The Derivative as a Function

- If we want to indicate the value of a derivative in Leibniz notation at a specific number a, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \quad \text{or} \quad \left.\frac{dy}{dx}\right]_{x=a}$$

which is a synonym for $f'(a)$.

Differentiable Functions

3 Definition A function f is **differentiable at *a*** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

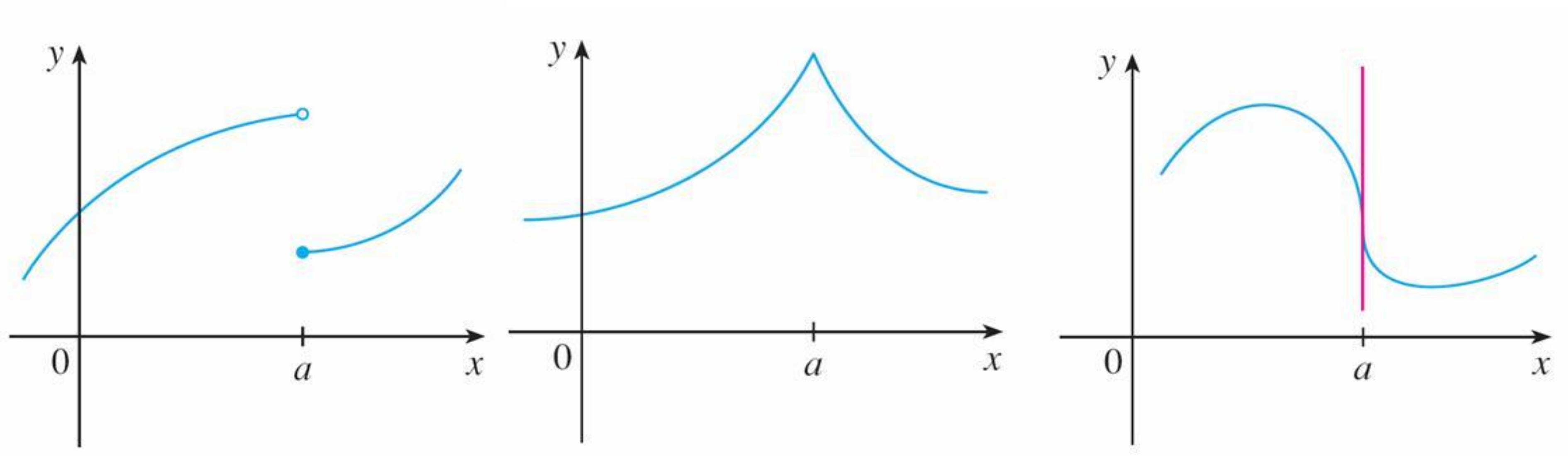
4 Theorem If f is differentiable at a, then f is continuous at a.

Note: The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable.

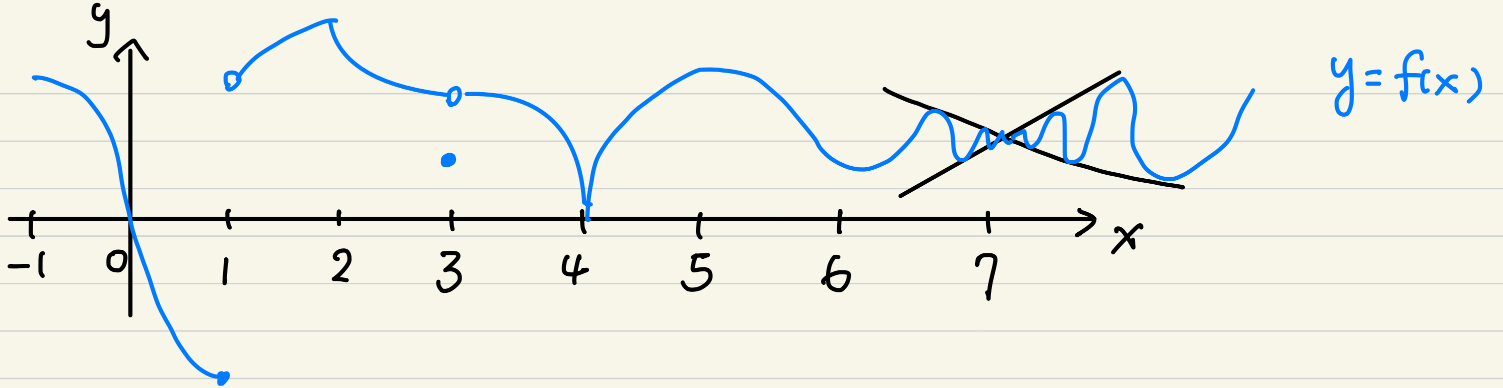
Ex: Show that if $f(x)$ is differentiable at $x=a$ then $f(x)$ is continuous at $x=a$.

Differentiable Functions

- Some cases of not differentiable functions:
- A discontinuous point, a corner, a vertical tangent, or an oscillating point.



Ex:



Find numbers a s.t. f(x) is not diff at a.

Ex: Suppose that $f(x)$ is continuous at $x = 2$ and

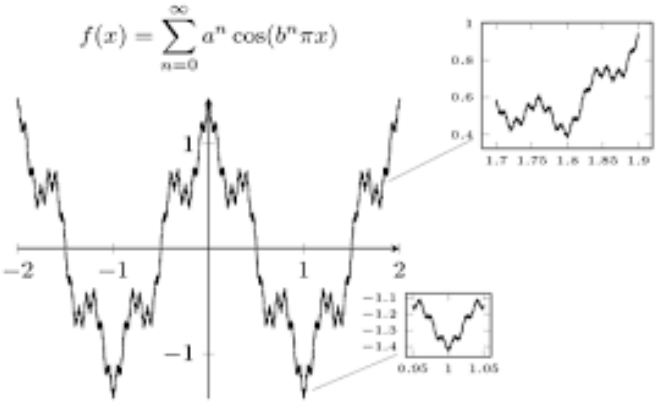
$$\lim_{x \to 2} \frac{f(x) - 3}{x^2 - 4} = -1.$$ Find $f(2)$ and $f'(2)$.

Continuous everywhere but differentiable nowhere

- **Weierstrass functions**

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

where $0 < a < 1$, b is a positive odd integer and $ab > 1 + \frac{3}{2}\pi$.



Higher Derivatives

- If $f(x)$ is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f .
- Leibniz notation: $\dfrac{d}{dx}(\dfrac{dy}{dx}) = \dfrac{d^2y}{dx^2}$

Higher Derivatives

- The higher derivatives of $f(x)$ can be defined in the similar way:

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}$$

Review

- What is the derivative of a function at a point $x = a$?
- What is the relation between the differentiability of a function and the continuity of the function ?
- What are the higher derivatives of a function ?