Change of Variables in Multiple Integrals

Section 15.7-15.9

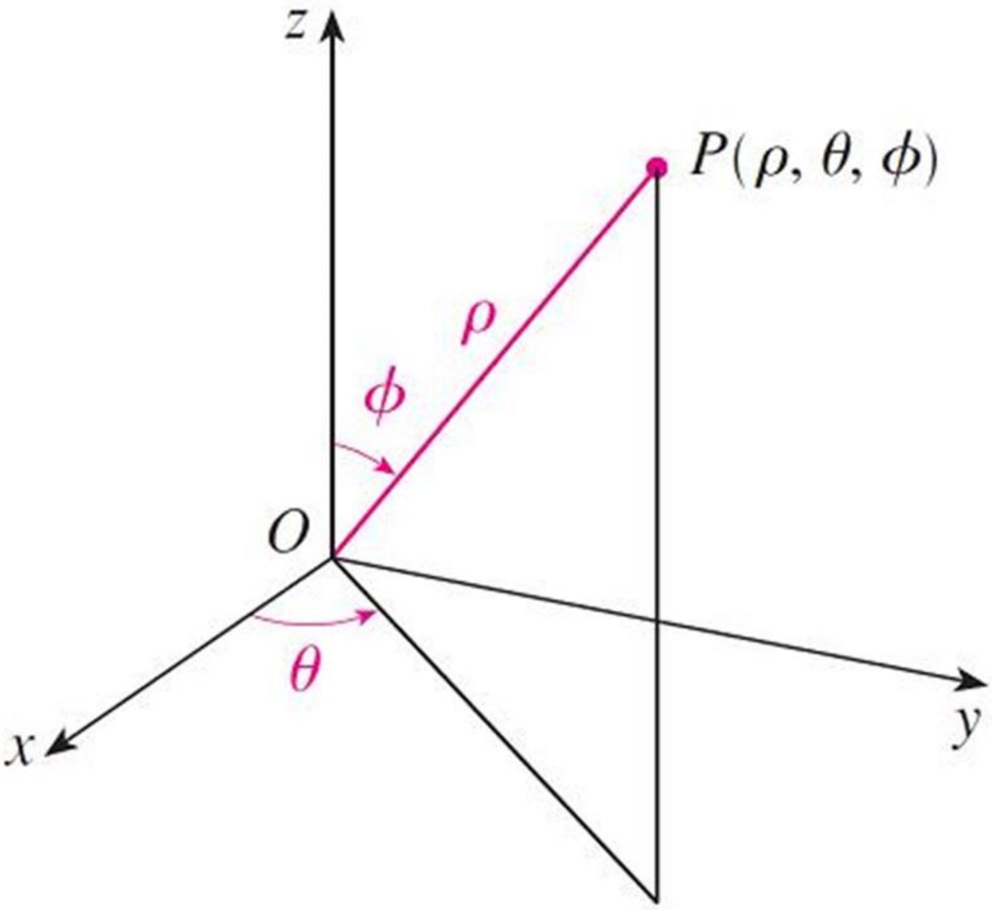
Outline

- Triple Integrals in Cylindrical Coordinates
- Triple Integrals in Spherical Coordinates
- Change of Variables in Multiple Integrals

Triple Integrals in Spherical Coordinates

- Definition:
- The **spherical coordinates** (ρ, θ, ϕ) of a point P in space are as follows.
- $\rho = |OP|$ is the distance from the origin to P.
- θ is the same angle as in cylindrical coordinates.
- ϕ is the angle between the positive z-axis and the line segment OP.

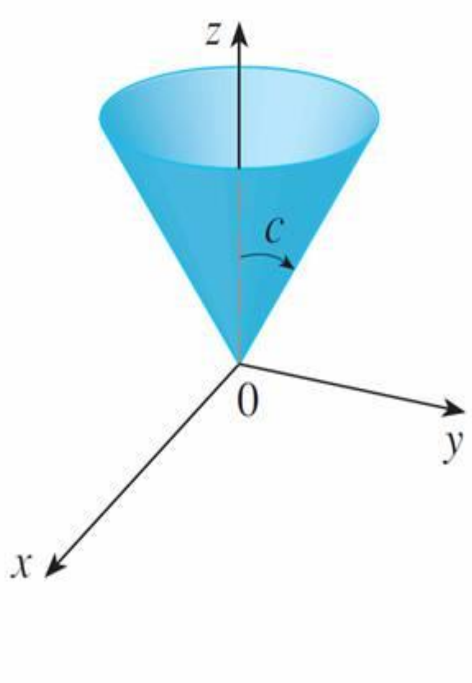
Triple Integrals in Spherical Coordinates



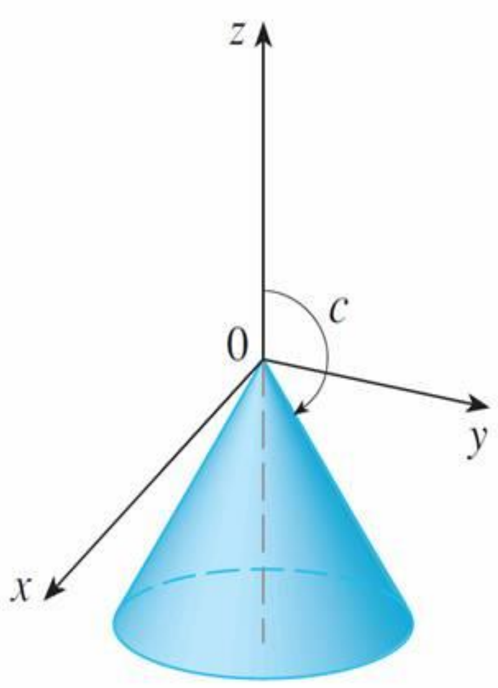
Triple Integrals in Spherical Coordinates

- The sphere centered at the origin with radius c satisfies the equation $\rho = c$.
- The graph of the equation $\theta = c$ is a vertical half-plane.
- The equation $\phi = c$ represents a half-cone with the z-axis as its axis.

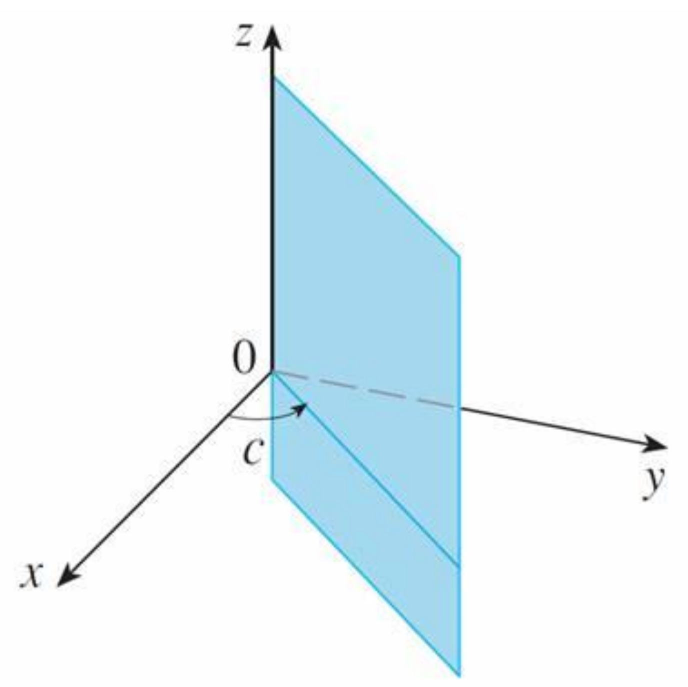
Triple Integrals in Spherical Coordinates



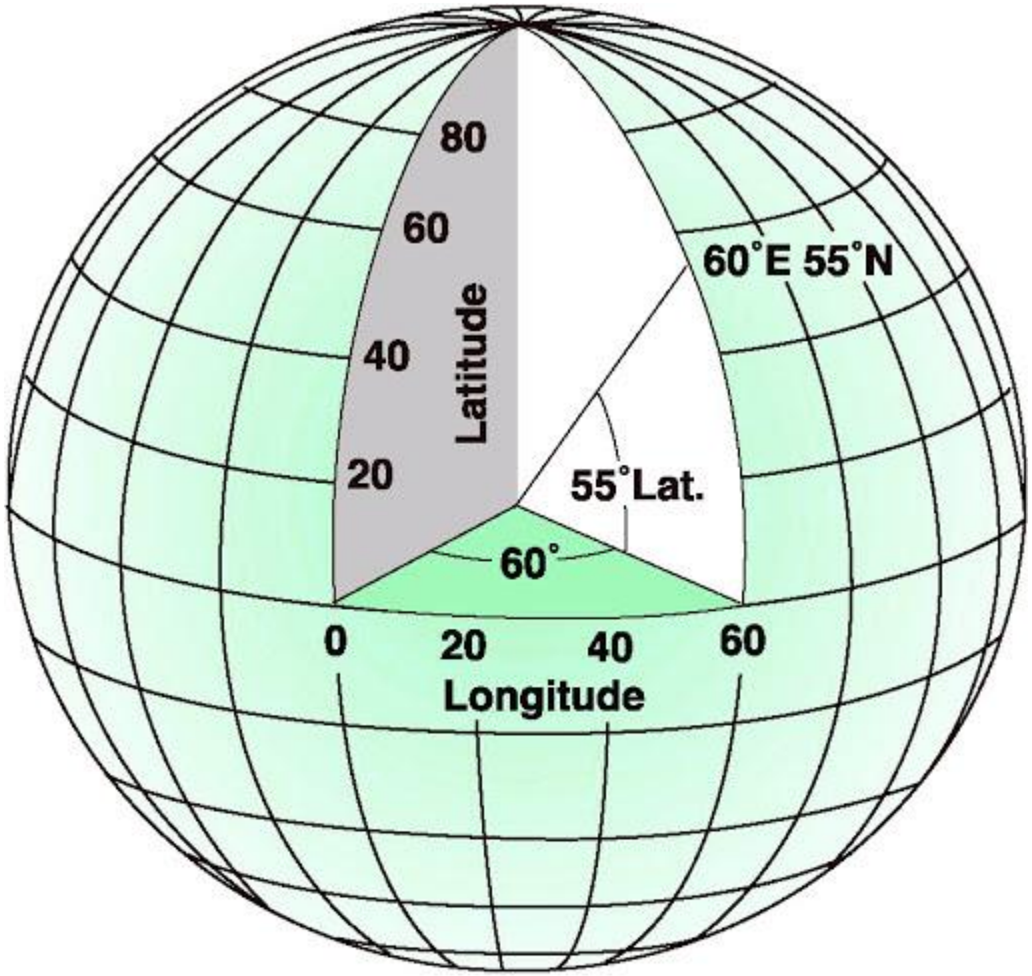
$0 < c < \pi/2$



$\pi/2 < c < \pi$



Triple Integrals in Spherical Coordinates



Triple Integrals in Spherical Coordinates

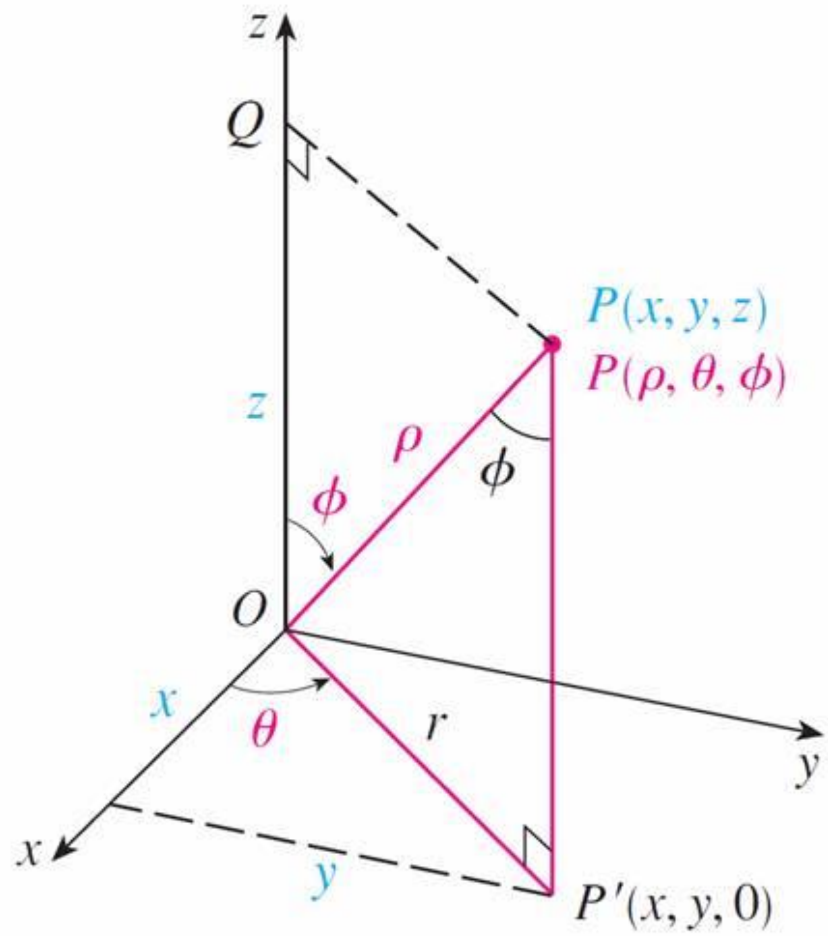
- The relationships between rectangular and spherical coordinates

$$\boxed{x = \rho \sin \phi \cos \theta \qquad y = \rho \sin \phi \sin \theta \qquad z = \rho \cos \phi}$$

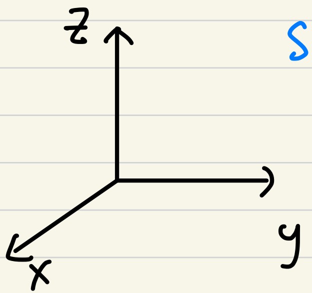
$0 \le \rho$ $\qquad 0 \le \theta \le 2\pi$ $\qquad 0 \le \phi \le \pi$

$$\rho^2 = x^2 + y^2 + z^2 \quad \tan\theta = \frac{y}{x} \quad (\tan\phi)^2 = \frac{x^2+y^2}{z^2}$$

Triple Integrals in Spherical Coordinates



Ex: $(\rho, \theta, \varphi) = (1, \frac{\pi}{3}, \frac{\pi}{4})$, find its rectangular coordinates.



Sol:

Ex: $(x, y, z) = (-\sqrt{3}, 0, -1)$, find its spherical coordinates.

Sol:

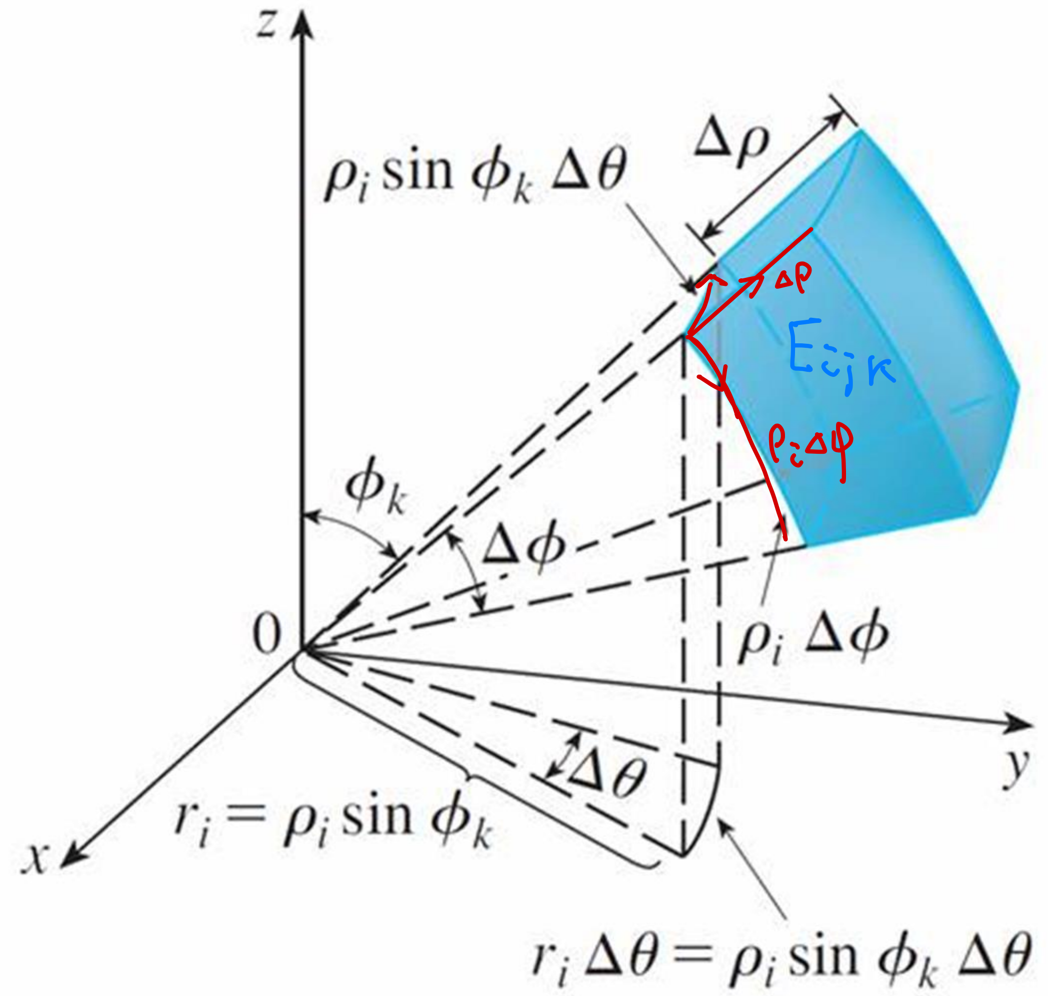
Triple Integrals in Spherical Coordinates

Triple Integrals in Spherical Coordinates

- To do triple integrals with spherical coordinates, we consider a region called a **spherical wedge** $E = \{(\rho, \theta, \phi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$
- Given a spherical wedge E, we divide E into smaller spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$ and half-cones $\phi = \phi_k$.

Triple Integrals in Spherical Coordinates

$V(E_{ijk}) \approx$



Triple Integrals in Spherical Coordinates

- E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho$, $\rho_i\Delta\phi$ (arc of a circle with radius ρ_i, angle $\Delta\phi$), and $\rho_i \sin\phi_k \Delta\theta$ (arc of a circle with radius $\rho_i \sin\phi_k$, angle $\Delta\theta$).
- So an approximation to the volume of E_{ijk} is

$$\Delta V_{ijk} \approx (\Delta\rho)(\rho_i\Delta\phi)(\rho_i \sin\phi_k \Delta\theta) = \rho_i^2 \sin\phi_k \Delta\rho\Delta\theta\Delta\phi$$

Triple Integrals in Spherical Coordinates

- Consequently, we have arrived at the following formula for triple integration in spherical coordinates.

3 $$\iiint_E f(x, y, z)\, dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \boxed{\rho^2 \sin\phi}\, d\rho\, d\theta\, d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$$

Triple Integrals in Spherical Coordinates

- This formula can be extended to include more general spherical regions such as

$$E = \{(\rho, \theta, \phi) | \alpha \leq \theta \leq \beta, c \leq \phi \leq d, g_1(\theta, \phi) \leq \rho \leq g_2(\theta, \phi)\}$$

- Usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

Ex: Compute the volume of a ball B with radius a.

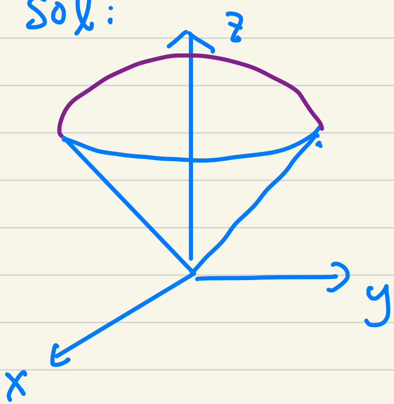
Sol:

Ex: Compute $\iiint_E \cos\left((x^2+y^2+z^2)^{\frac{3}{2}}\right) dV$, where E is the part of the unit ball $x^2+y^2+z^2 \leq 1$ in the first octant.

Sol:

Ex: Compute $\iiint_E \sqrt{x^2+y^2}\, dV$, where E is above the cone $z=\frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$ and under the sphere $x^2+y^2+z^2=4$.

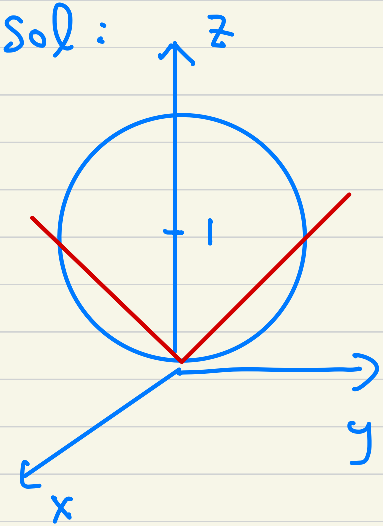
sol:



Ex: E is above the cone $z=\sqrt{x^2+y^2}$, under the sphere $x^2+y^2+z^2=2z$.

Compute the volume of E.

Sol:



Ex: Compute $I = \iiint_B \frac{1}{x^2+y^2+(z-2)^2} \, dV$, where B is the unit ball $x^2+y^2+z^2 \le 1$.

Sol:

Ex:
$$\int_{0}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_{\sqrt{\frac{x^2+y^2}{3}}}^{1}\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$