Modeling with Differential Equations

Section 9.1

Outline

- Models of Population Growth
- A Model for the Motion of a Spring
- General Differential Equations

Modeling with Differential Equations

- Scientists usually formulate a mathematical model of a real world problem based on physical laws or intuitive reasoning about the phenomenon.
- The mathematical model often takes the form of a *differential equation*, that is, an equation that contains an unknown function and some of its derivatives.

Models of Population Growth

- One model for the growth of a population is based on the assumption that the population grows at a rate proportional to the size of the population. This is reasonable under ideal conditions (unlimited environment, adequate nutrition, absence of predators, immunity from disease).
- $\dfrac{dP}{dt} = kP$ where k is the proportionality constant.

Models of Population Growth

- A more realistic model must reflect the fact that a given environment has limited resources.
- We observe that many populations start by increasing in an exponential manner but the population levels off when it approaches its *carrying capacity M* (or decreases toward M if it ever exceeds M).

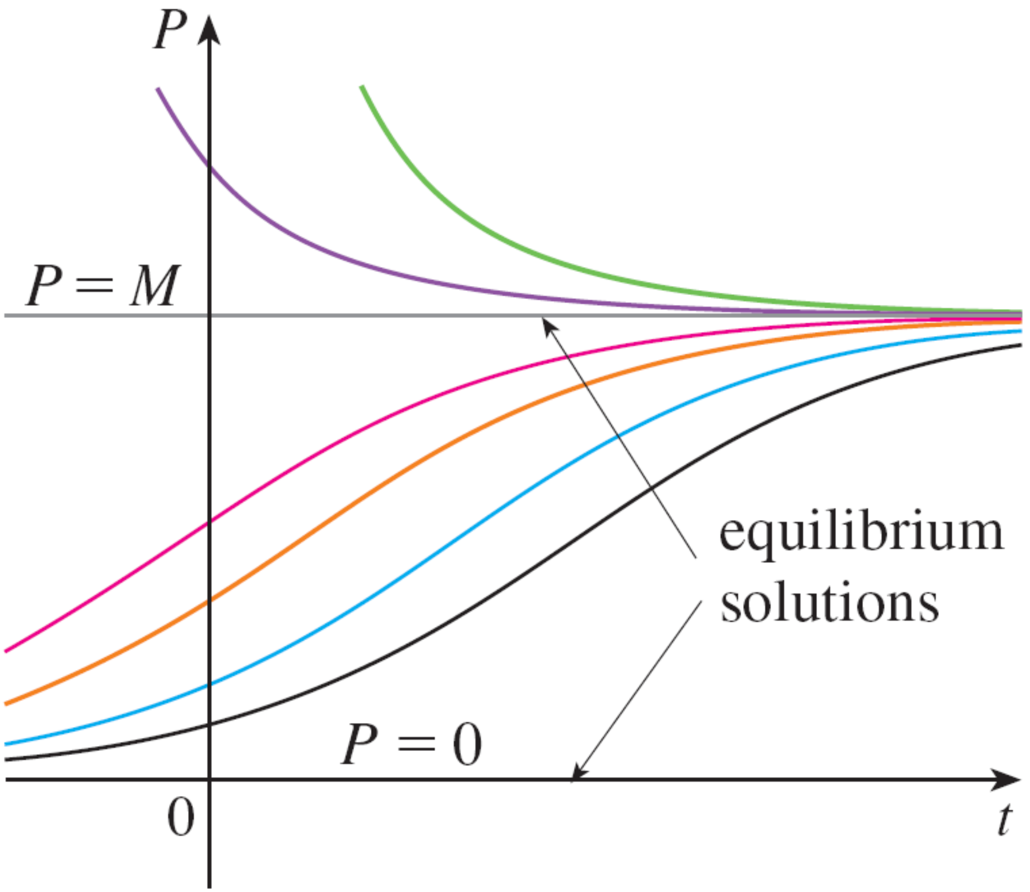
Models of Population Growth

- We make two assumptions:
- $\frac{dP}{dt} \approx kP$ if P is small (Initially, the growth rate is proportional to P.)
- $\frac{dP}{dt} < 0$ if $P > M$ (P decreases if it ever exceeds M.)
- The *logistic differential equation:*

$$\frac{dP}{dt} = kP(1 - \frac{P}{M})$$

Models of Population Growth

- Solutions for the *logistic differential equation:*



$\boxed{\text{Prop}}$ Suppose that $y'(t) = f(y)$ where f is continuous.
$y(t) = y_0$ is a constant solution of the D.E. iff $f(y_0) = 0$.

sol:

Ex:

The function $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = y^4 - y^3 - 56y^2$$

Find all constant solutions.

A Model for the Motion of a Spring

- Hooke's Law says that if the spring is stretched (or compressed) x units from its natural length, then it exerts a force that is proportional to x : restoring force $= -kx$.
- By Newton's Second Law (force equals mass times acceleration), we have

$$m\frac{d^2x}{dt^2} = -kx$$

General Differential Equations

- In general, a **differential equation** is an equation that contains an unknown function and one or more of its derivatives.
- The **order** of a differential equation is the order of the highest derivative that occurs in the equation.

General Differential Equations

- A function f is called a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.
- When we are asked to *solve* a differential equation we are expected to find all possible solutions of the equation.

General Differential Equations

- In many physical problems we need to find the particular solution that satisfies a condition of the form $y(t_0) = y_0$.
- This is called an **initial condition**, and the problem of finding a solution of the differential equation that satisfies the initial condition is called an **initial-value problem**.

Ex: Suppose that there is a chemical reaction

$A + B \longrightarrow C$. Let $x(t)$ be $[C]$ at time t.

Assume that $x'(t)$ is proportional to the product

of $[A]$ and $[B]$ and $[A](0) = a$, $[B](0) = b$.

Derive a differential equation for $x(t)$.

Review

- What is a differential equation?
- What is the order of a differential equation? What is an initial value problem?