Power Series

Section 11.8, 11.9

Outline

- Power Series:
  - Definition
  - The Radius of Convergence and the Interval of Convergence
- Representations of Functions as Power Series
  - Geometric Power Series
  - Differentiation and Integration of Power Series

Power Series

- A **power series** is a series of the form
$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$
$\boxed{x^0 = 1}$

  where x is a variable and the c_ns are constants called the **coefficients** of the series.
- The sum of the series is a function
$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$
  whose domain is the set of all x for which the series converges.

Power Series

- More generally, a series of the form
$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$
is called a **power series in** $(x-a)$ or a **power series centered at *a*** or a **power series about *a***.
- Notice that when $x=a$ all of the terms are 0 for $n \geq 1$ and so the power series always converges when $x=a$.

Ex: Find x s.t. $\sum_{n=0}^{\infty} x^n$ converges.

sol:

Ex: Find x s.t. $\sum_{n=0}^{\infty} n!\,x^n$ converges. (Note: $0! = 1$)

sol:

Ex: Bessel function: $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$. Find x s.t. the power series converges.

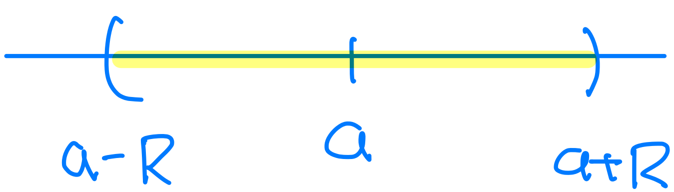
Sol:

Ex: Find x s.t. $\sum_{n=1}^{\infty} \frac{(2n+1)^{2n}}{3^n}$ converges.

sol:

The Radius of Convergence

- $\boxed{\text{Theorem:}}$ For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ only one of the following is true.
- 1. It converges at only one point $x = a$.
- 2. It converges for all x .
- 3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.



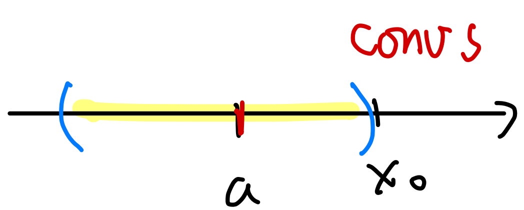
The Radius of Convergence

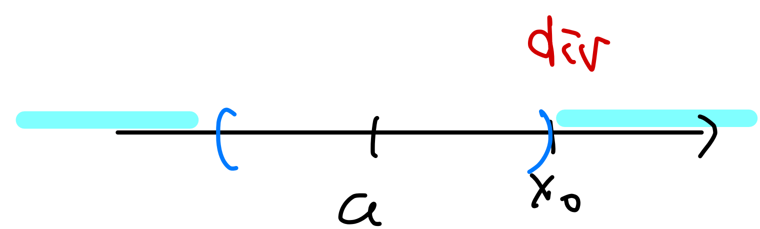
- The number R in case 3 is called the **radius of convergence** of the power series. By convention, $R = 0$ in case 1 and $R = \infty$ in case 2 .

The Radius of Convergence

- Theorem: If $\sum_{n=0}^{\infty} c_n(x_0 - a)^n$ converges, then for any x such that $|x - a| < |x_0 - a|$, $\sum_{n=0}^{\infty} c_n(x - a)^n$ converges absolutely.

- Corollary: If $\sum_{n=0}^{\infty} c_n(x_0 - a)^n$ diverges, then for any x such that $|x - a| > |x_0 - a|$, $\sum_{n=0}^{\infty} c_n(x - a)^n$ diverges.





pf of the theorem:

Suppose that $\sum_{n=0}^{\infty} c_n (x_0 - a)^n$ converges and x is a real number s.t. $|x-a| < |x_0 - a|$.

The Interval of Convergence

- The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges.
- It may be the single point $\{a\}$, or the whole real line or a finite interval centered at a.
- When x is an *endpoint* of the interval, that is, $x = a \pm R$, anything can happen.

Examples of Power Series

Series	Radius of convergence	Interval of convergence
$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n}$	$R = 1$	$[1, 3)$
$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$R = \infty$	$(-\infty, \infty)$

Ex: Find the interval of convergence of $\sum_{n=0}^{\infty} \left(1+\frac{1}{2}+\cdots+\frac{1}{2n}\right)(3x-1)^n$.