The Mean Value Theorem and its Applications

Section 4.2-4.3

Outline

- 1. The Mean Value Theorem
  - Lemma: Rolle's Theorem
  - Theorem: The Mean Value Theorem
  - Corollary
- 2. How Derivatives Affect the Shape of a Graph
  - First Derivatives: Increasing / Decreasing Test
  - Second Derivatives: Concavity Test

How Derivatives Affect the Shape of a Graph

- The First Derivatives:

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

Ex: Prove the Increasing / Decreasing Test

Pf: Suppose that $f'(x) > 0$ on (a, b).

Ex: $f(x) = 2x^3 + 3x^2 - 12x + 7$. Find intervals of increase / decrease and local extreme values of $f(x)$.

Ex: $f(x) = x + 2\cos x$, $0 \le x \le 2\pi$. Find intervals of increase / decrease and local extreme values of $f(x)$.

Ex: $f(x) = x^{\frac{1}{3}}(x-6)^{2/3}$. Find intervals of increase / decrease and local extreme values of $f(x)$.

Ex: $f(x) = x(\ln|x|)^2$. Find intervals of increase / decrease and local extreme values of $f(x)$.

How Derivatives Affect the Shape of a Graph

- The Second Derivatives:

Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

- Theorem: $f(x)$ is concave upward on an open interval I if and only if $f'(x_1) \leq f'(x_2)$ for all $x_1 < x_2$, $x_1, x_2 \in I$.

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

How Derivatives Affect the Shape of a Graph

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

- Remark: From the definition, we have assumed that $f(x)$ is continuous at an inflection point and is differentiable on its both sides. But $f(x)$ may not be differentiable at an inflection point.

How Derivatives Affect the Shape of a Graph

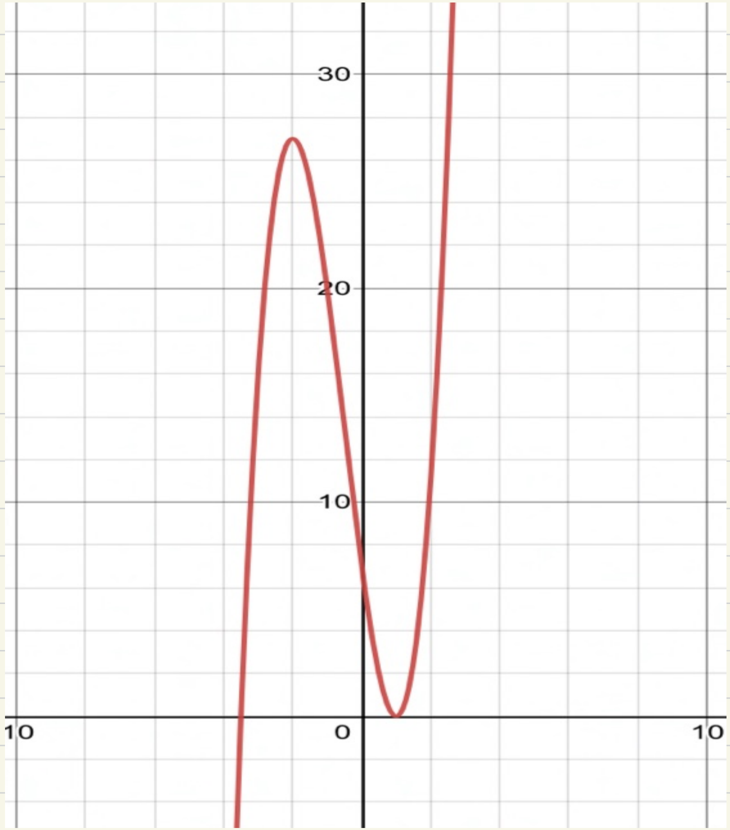
- Property: If $P = (a, f(a))$ is an inflection point of $y = f(x)$ and $f(x)$ is differentiable at $x = a$, then
- a) $f'(x)$ has local extreme value at $x = a$.
- b) The tangent line of $y = f(x)$ at P crosses the graph $y = f(x)$ there.
- c) If $f''(a)$ exists, then $f''(a) = 0$.

Sketch the Graph of a Function

f'' \ f'	$f' > 0$ Increasing	$f' < 0$ Decreasing
$f'' > 0$ concave upward		
$f'' < 0$ concave downward		

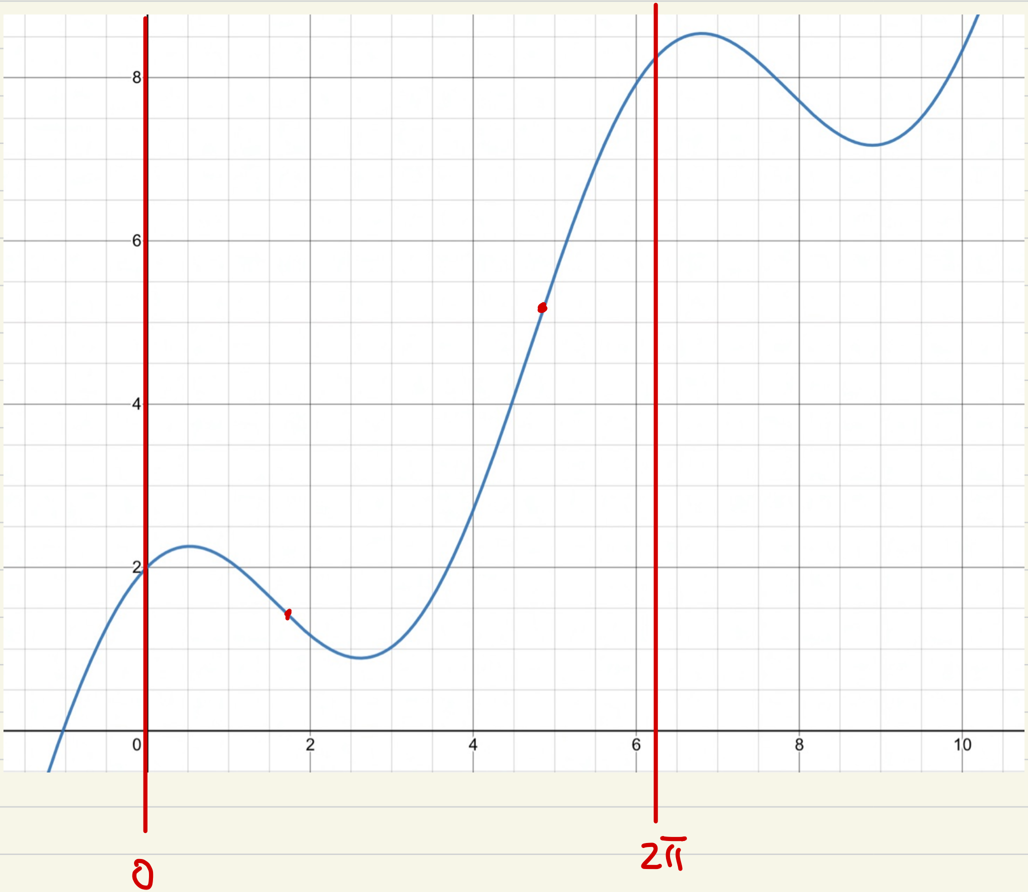
Ex: Determine Concavity of $f(x) = 2x^3 + 3x^2 - 12x + 7$ and sketch the graph of $f(x)$.

$y = 2x^3 + 3x^2 - 12x + 7$



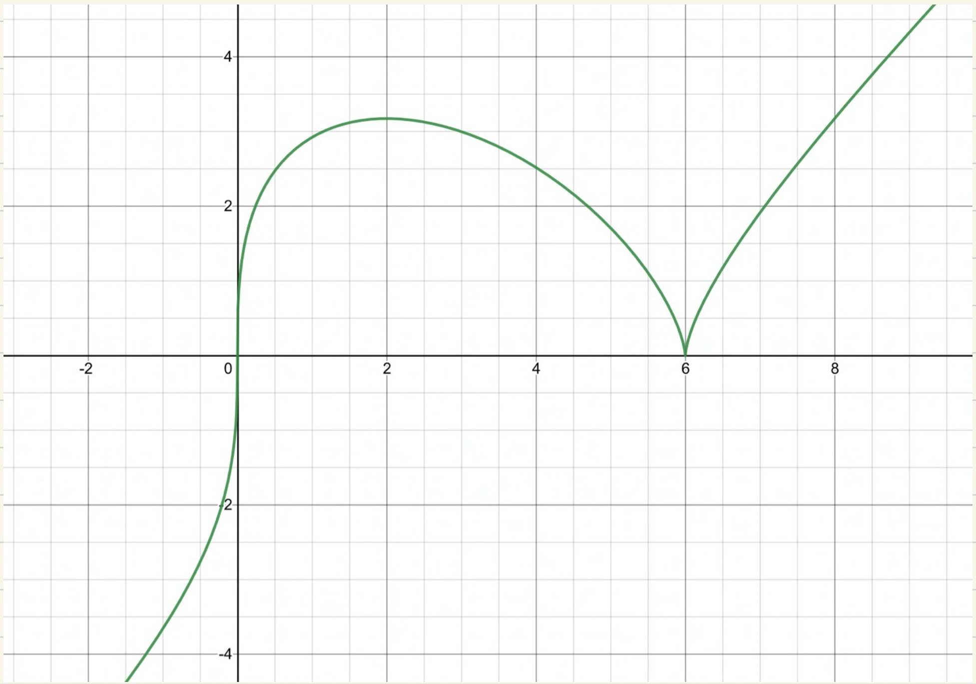
Ex: Determine Concavity of $f(x) = x + 2\cos x$, $0 \le x \le 2\pi$, and sketch graph of $f(x)$.

$y = x + 2\cos x$

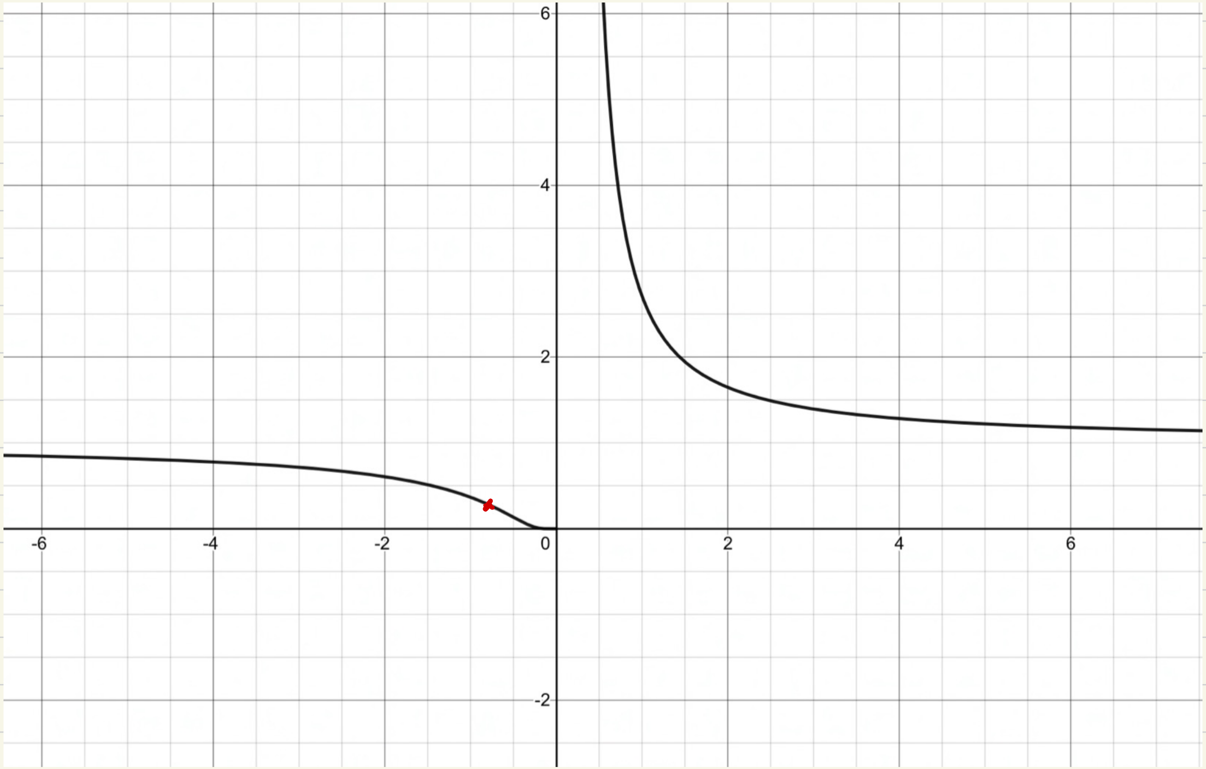


Ex: Determine Concavity of $f(x) = x^{\frac{1}{3}}(x-6)^{2/3}$ and sketch the graph of $f(x)$.

$$y = x^{\frac{1}{3}}(x-6)^{\frac{2}{3}}$$



Ex: Sketch the curve $y = e^{\frac{1}{x}}$



$y = e^{\frac{1}{x}}$

How Derivatives Affect the Shape of a Graph

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

- Question: Compare the first derivative test and the second derivative test.

Review

- State Rolle's Theorem and the Mean Value Theorem.
- What can the first derivative of a function tell us about the graph of it?
- What is the concavity of a function?
- What can the second derivative of a function tell us about the graph of it?
- What are the first derivative test and the second derivative test for the critical points?