Double Integrals

Section 15.1-15.3

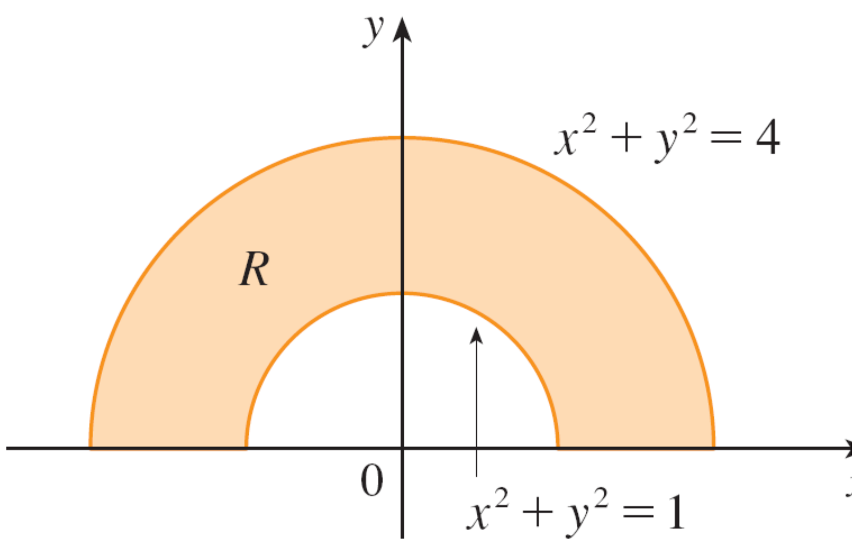
Outline

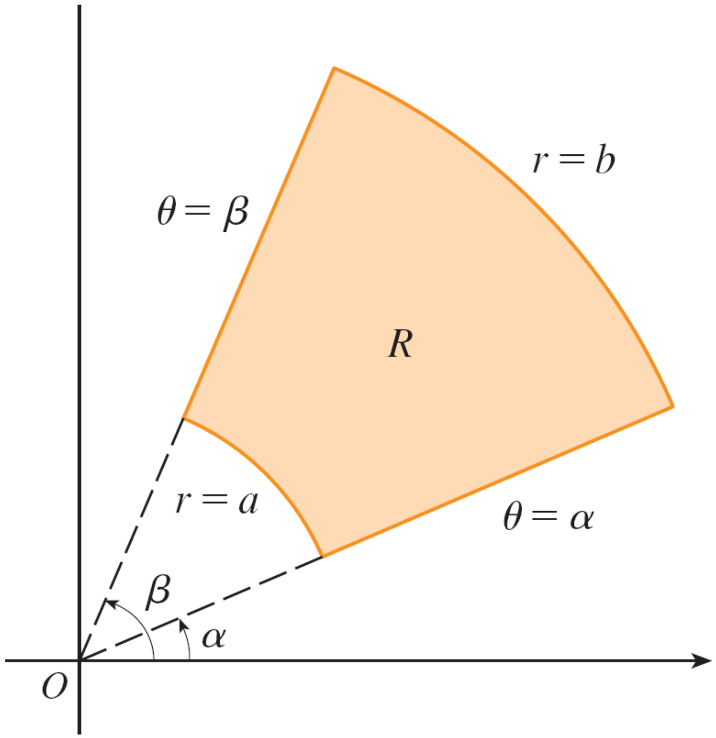
- Double Integrals over Rectangles
- Iterated Integrals
  - Fubini's Theorem
- Double Integrals over General Regions
  - Type I Regions
  - Type II Regions
- Double Integrals in Polar Coordinates

Double Integrals in Polar Coordinates

- Consider regions that are **polar rectangles**

$$R = \{(r, \theta) | a \leq r \leq b, \alpha \leq \theta \leq \beta\} \ .$$

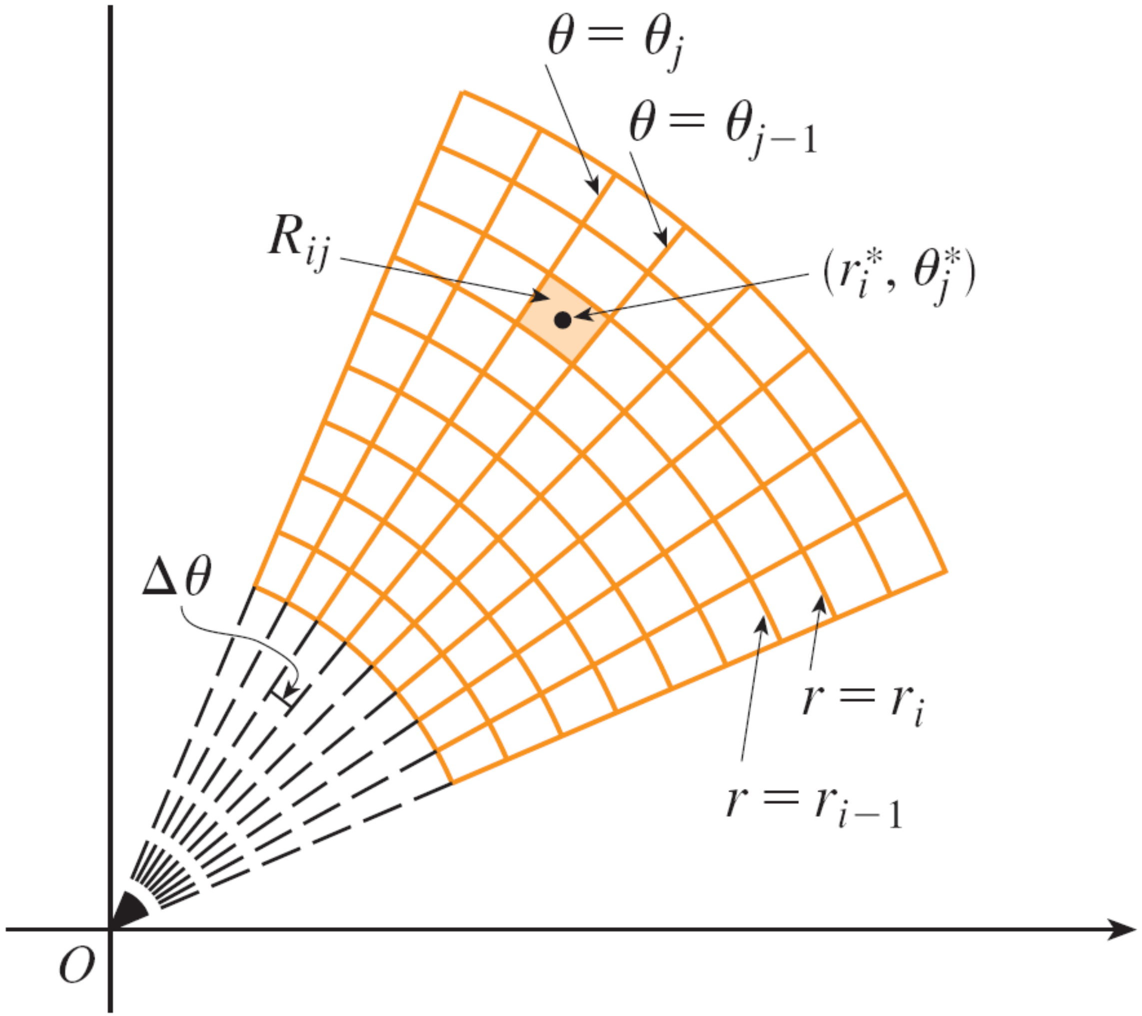




Double Integrals in Polar Coordinates

- In order to compute the double integral $\iint_R f(x,y)dA$, where R is a polar rectangle, we divide the interval $[a,b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = (b-a)/m$ and divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta = (\beta - \alpha)/n$.

Double Integrals in Polar Coordinates



Double Integrals in Polar Coordinates

- The area of polar subrectangle
  $R_{ij} = \{(r,\theta) | r_{i-1} \le r \le r_i, \theta_{j-1} \le \theta \le \theta_j\}$
  is $\Delta A_{ij} = \frac{1}{2} r_i^2 \Delta\theta - \frac{1}{2} r_{i-1}^2 \Delta\theta = r_i^* \Delta r \Delta\theta$
  where $r_i^* = (r_{i-1} + r_i)/2$.
- Let $r_i^* = (r_{i-1} + r_i)/2$, $\theta_j^* = (\theta_{j-1} + \theta_j)/2$
  be the sample points we choose from R_{ij}.

Double Integrals in Polar Coordinates

- So the Riemann sum is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\Delta A_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*) r_i^* \Delta r \Delta\theta$$

which is a Riemann sum for the double integral $\int_{\alpha}^{\beta}\int_{a}^{b} g(r,\theta)\,dr\,d\theta$, where

$$g(r,\theta) = r f(r\cos\theta, r\sin\theta)$$

Double Integrals in Polar Coordinates

$\boxed{2}$ **Change to Polar Coordinates in a Double Integral** If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

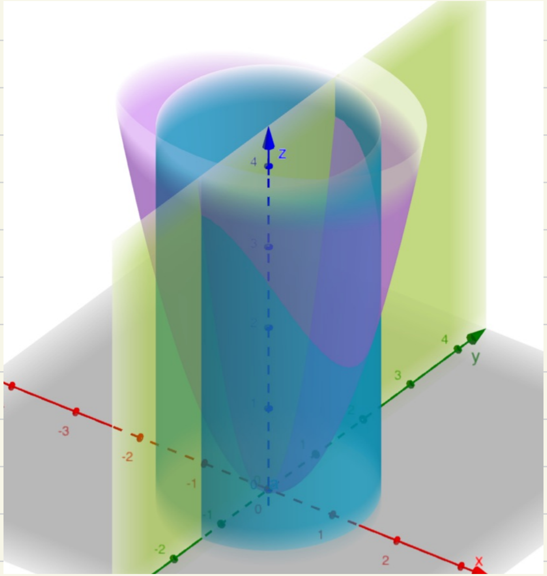
$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Ex: Compute $\iint_R x+2y \, dA$, where R is bounded by

$x^2+y^2=1$, $x^2+y^2=4$, with $y \geq 0$, $x \leq y$.

Ex: Find the volume of the solid which is under the surface $z = x^2 + 2y^2$, above the xy-plane, within the cylinder $x^2 + y^2 = 2$, and to the right of the xz-plane.

Sol:

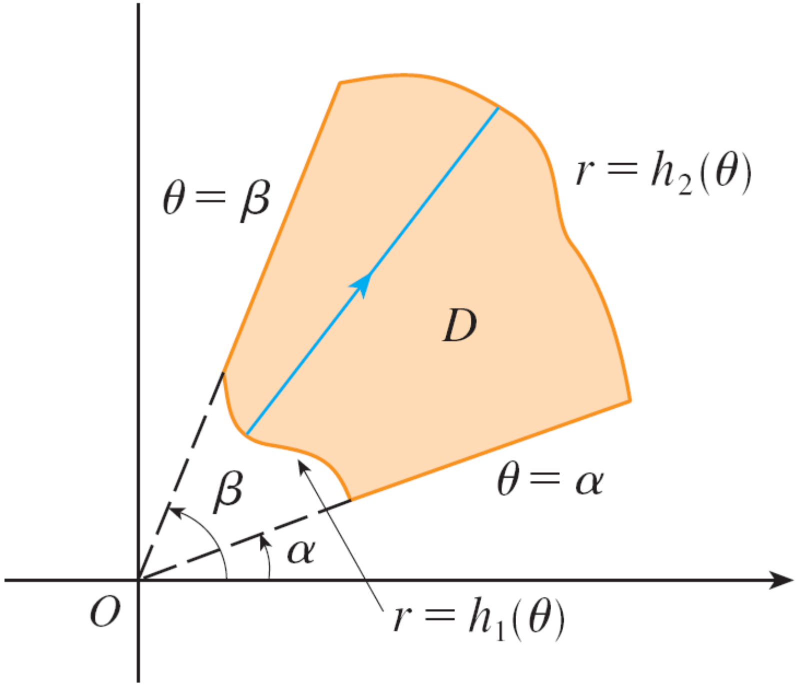


Ex: Compute $\displaystyle\iint_{\mathbb{R}^2} e^{-x^2-y^2}\, dA$.

Ex: Compute $\int_0^{\frac{a}{\sqrt{2}}} \int_{-\sqrt{a^2-y^2}}^{y} \cos(x^2+y^2)\,dx\,dy + \int_{-\frac{a}{\sqrt{2}}}^{\frac{a}{\sqrt{2}}} \int_{\frac{a}{\sqrt{2}}}^{\sqrt{a^2-x^2}} \cos(x^2+y^2)\,dy\,dx$

Double Integrals in Polar Coordinates

- What we have done so far can be extended to the more complicated type of region $D = \{(r, \theta) | \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$.



Double Integrals in Polar Coordinates

3 If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

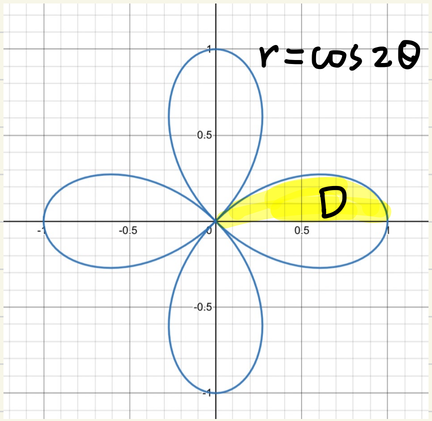
then
$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Ex: $D = \{(r,\theta) \mid \alpha \le \theta \le \beta, \; h_1(\theta) \le r \le h_2(\theta)\}$.

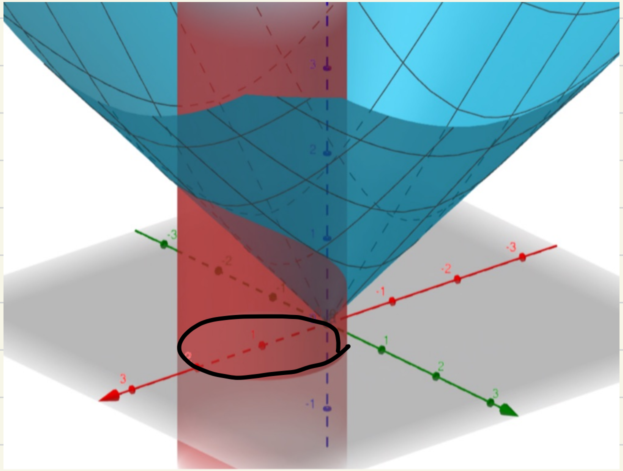
Find $A(D) = \iint_D 1 \, dA$

Ex: Compute $\iint_D xy\, dA$, where D is the shaded region.

sol:



Ex: Find the volume of the solid that lies under the cone $z=\sqrt{x^2+y^2}$ above the xy-plane and inside the cylinder $x^2+y^2=2x$.



Review

- How do we define a double integral over a finite rectangle?
- What is an iterated integral?
- State the Fubini's Theorem.
- How do we integrate a function over a type I or type II region?
- Write down the formula for double integrals in polar coordinates.