The Definite Integral

Section 5.2-5.3

Outline

- The Definite Integral
  - Definition
  - Properties
- The Fundamental Theorem of Calculus

Properties of the Definite Integral

- The Mean Value Theorem for Integrals:
- If $f(x)$ is continuous on $[a,b]$, then there is a number $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

that is, $\int_a^b f(x)dx = f(c)(b-a)$.

Proof of the Mean Value Theorem for Integrals

The Fundamental Theorem of Calculus

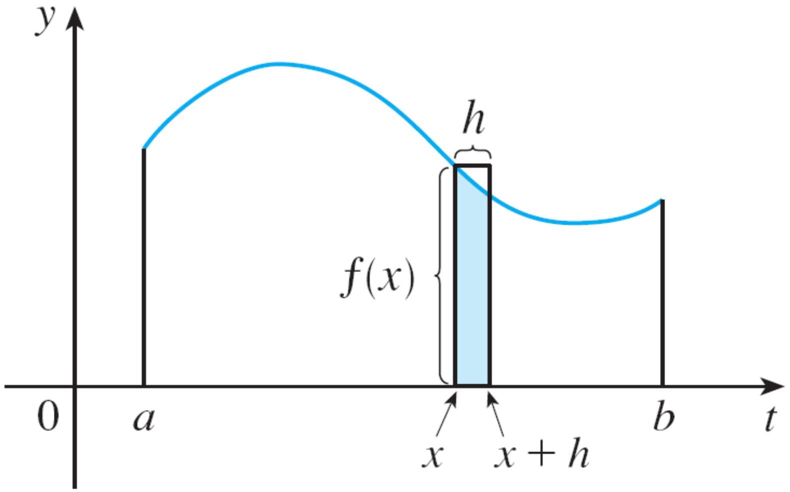
- The Fundamental Theorem of Calculus gives the precise inverse relationship between the derivative and the integral.
- If f is a continuous function, consider the function $g(x) = \int_a^x f(t)dt$.

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.



$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Proof of the Fundamental Theorem of Calculus, Part 1.

Ex: $S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt$ (Fresnel function). Find $S'(x)$.

Ex: $f(x) = \int_x^2 \sqrt{t}\, \sin t \, dt$. Find $f'(x)$.

$f_1(x) = \int_x^{10} \sqrt{t}\, \sin t \, dt$. Find $f_1'(x)$.

Ex: $f(x) = \int_0^{\tan x} \sqrt{1+t^2}\, dt$. Find $f'(x)$.

Ex: $F(x) = \int_{g(x)}^{h(x)} f(t)\,dt$. Find $F'(x)$

Ex: $f(x) = \int_{x^2}^{0} (\sqrt{x} + \sqrt{t})\, g(t)\, dt$, where $g(t)$ is continuous.

Find $f'(x)$, for $x > 0$.

Ex: Find $\displaystyle \lim_{x \to 0} \frac{\int_{\sin x}^{x} \sqrt{1-t^2}\, dt}{x^3}$

Ex: $f(x) = \int_x^{2x} e^{-t^2}\,dt$.

a) Compute $\lim_{x \to \pm\infty} f(x)$.

b) Compute $f'(x)$. Find intervals of increase and intervals of decreases.

c) Compute $f''(x)$. Discuss concavity of $y = f(x)$.

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Proof of the Fundamental Theorem of Calculus, Part 2.

Ex: Find $\int_0^{\frac{\pi}{2}} \sin x \, dx$

Ex: Find $\int_1^3 e^x \, dx$

Ex: Find $\int_{-1}^2 \frac{1}{x^2} \, dx$

Ex: $g(x) = \int_0^x f(t)\,dt$, where $f(t) = \begin{cases} 1, & \text{for } t < 0 \\ t+1, & \text{for } 0 \le t < 2 \\ \frac{1}{t}, & \text{for } t \ge 2 \end{cases}$.

Write $g(x)$ as a piecewised defined function.

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

Review

- What is the Riemann sum of a function $f(x)$ from a to b? What is the definite integral of a function $f(x)$ from a to b? State the precise definition of the limit of the Riemann sum.
- Review the properties of definite integrals.
- State the Fundamental Theorem of Calculus.