Functions of Several variables

Section 14.1-14.3

Outline

- Functions of Several Variables
  - Graphs
  - Level Curves
- Limits and Continuity
- Partial Derivatives
  - Definition
  - Geometric Interpretation
  - Higher Derivatives and Clairaut's Theorem

Derivatives

$f(x)$	$f'(a) =$	$f'(x) =$
$f(x,y)$	$f_x(a,b) =$	
	$f_y(a,b) =$	
	$f_x(x,y) =$	
$f(x_1, \ldots x_n)$	$f_{x_i}(x_1, \ldots, x_n) =$	

Partial Derivatives

$\boxed{4}$ If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Partial Derivatives

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Partial Derivatives

- In general, if u is a function of n variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the *i*th variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, ., x_{i-1}, x_i + h, x_{i+1}, ., x_n) - f(x_1, ., x_i, ., x_n)}{h}$$

- We also denote it as

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

Ex: $f(x,y) = \sqrt{1+x^2y^4} + \tan^{-1}\left(\frac{y}{x}\right) + x\ln y + 2^{xy^2}$

Find $f_x(x,y)$ and $f_y(x,y)$.

Ex: $f(x, y, z) = (2x+y)^{yz}$. Find f_x, f_y, and f_z.

Ex: $f(x,y,z) = \int_{xy}^{\frac{y}{\sqrt{z}}} g(t)\,dt$, where $g(t)$ is continuous.

Find $f_x(x,y,z)$, $f_y(x,y,z)$ and $f_z(x,y,z)$.

Ex: $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$ Find $f_x(0,0)$ and $f_y(0,0)$.

Ex: $f(x,y) = \begin{cases} (x^2+y)\cos\dfrac{1}{x^2+y^2} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0). \end{cases}$

Find $f_x(0,0)$ and $f_y(0,0)$.

Ex: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation $e^z = xy + \sin(xz) + e^2$ at $(0, 1, 2)$.

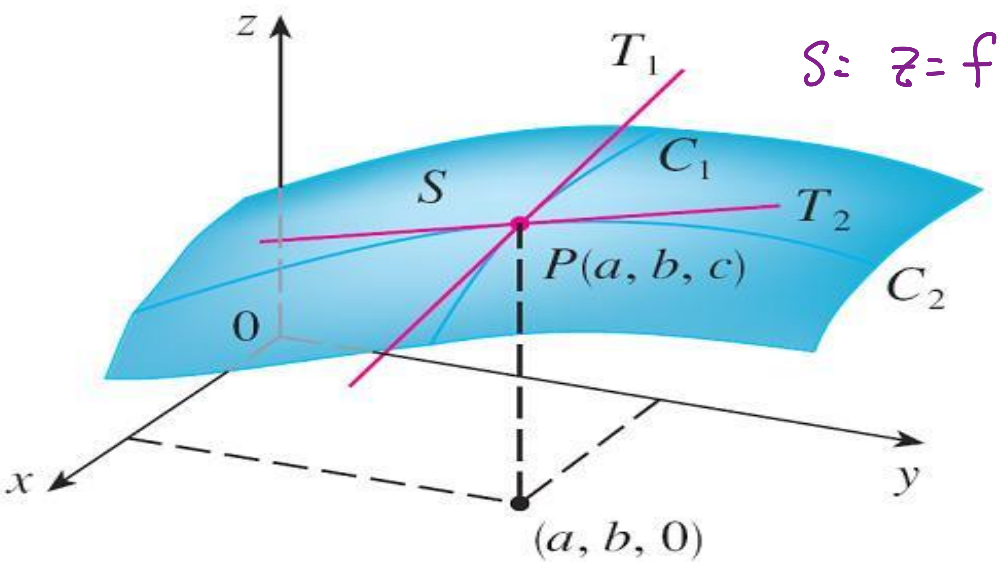
Partial Derivatives

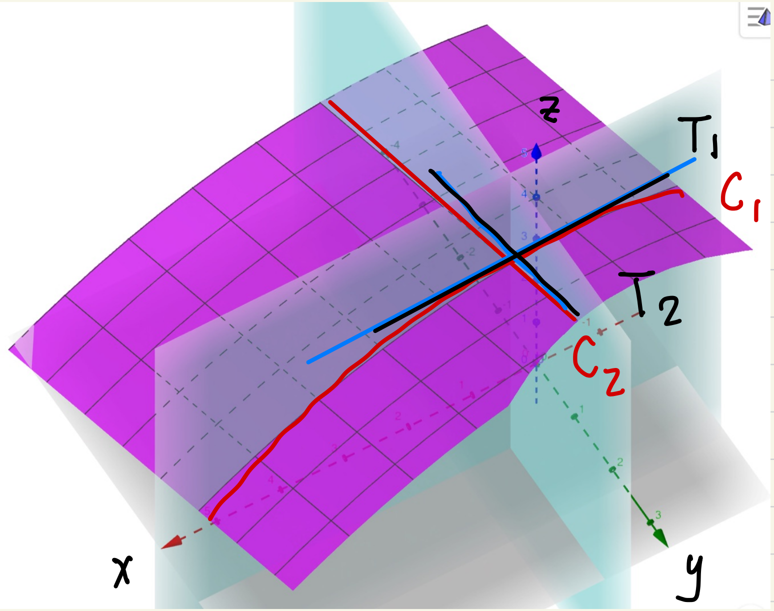
- The partial derivatives $f_x(a,b)$ and $f_y(a,b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a,b,c)$ to the traces C_1 and C_2 of the graph of f (surface S) in the planes $x = a$ and $y = b$.

$C_1: S \cap P_{y=b}$

$C_2: S \cap P_{x=a}$

$S: z = f(x,y)$





Parametric equations for C_1 and C_2

C_1 : $\vec{r}_1(x) = (x, b, f(x,b))$
$x \in \mathbb{R}$

C_2 : $\vec{r}_2(y) = (a, y, f(a,y))$
$y \in \mathbb{R}$

The tangent line of C_1 at $(a, b, f(a,b))$ is parallel to

The tangent line of C_2 at $(a, b, f(a,b))$ is parallel to

Ex: $S: z = 4 - \frac{x^2}{3} + \frac{y^2}{2}$. $C_1 = S \cap P_{y=2}$, $C_2 = S \cap P_{x=3}$.

Find tangent line equations of C_1 and C_2 at $(3, 2, 3)$.

Partial Derivatives

- We also define **second partial derivatives** of f.

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 z}{\partial y\,\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial x\,\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

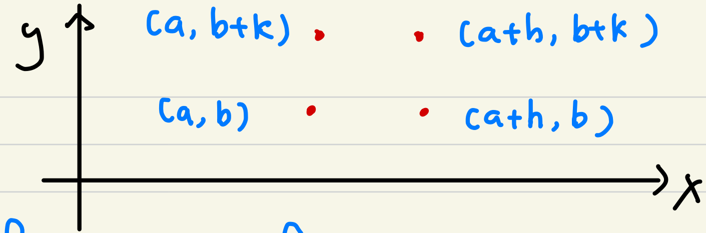
Ex: $f(x, y) = x^3y^2 + e^{xy}$. Find f_{xy} and f_{yx}.

Partial Derivatives

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Proof of Clairaut's Theorem:



For h, k close to 0, consider

$$Q(h,k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)$$

Review

- What are the graph and level curves of a function $f(x, y)$?
- How do we define and compute the limit of a function of several variables?
- What are the partial derivatives of a function of several variables? Describe their geometric meanings.
- State Clairaut's Theorem.