Techniques of Integration

Section 7.1-7.3

Outline

- Integration by Parts
- Trigonometric Integrals
- Trigonometric Substitution

Integration by Parts

- The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

- Hence, we have

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

which is called the **formula for integration by parts**.

Integration by Parts

- Let $u = f(x)$ and $v = g(x)$. Then the differentials are $dv = g'(x)dx$ and $du = f'(x)dx$, and the formula for integration by parts becomes

$$\int u\,dv = uv - \int v\,du$$

- For definite integral, we have the formula

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$$

Ex : Compute $\int x \cdot \sin x \, dx$

Ex: Compute $\int_{-1}^{1} x^2 e^{-x}\, dx$

Ex: Compute $\int e^x \sin x \, dx$

Ex: Compute $\int x \cdot \ln x \, dx$.

Conclusion: Integration by parts is powerful for computing the following integrations.

$$\int x^n \cdot \sin x \, dx$$

$$\int x^n \cdot \cos x \, dx$$

$$\int x^n e^x \, dx$$

$$\int e^x \sin x \, dx, \quad \int e^x \cos x \, dx$$

$$\int x^n \ln x \, dx$$

$$\boxed{\int \ln x \, dx \; , \quad \int \tan^{-1} x \, dx \; , \quad \int \sin^{-1} x \, dx}$$

Ex: Compute $\int \ln x \, dx$.

Ex: Compute $\int \tan^{-1} x \, dx$.

Ex: Compute $\int \sin^{-1} x \, dx$.

Derive Reduction Formula

Ex: Prove that $\int \sin^n x \, dx = \frac{-1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$ for $n \geq 2$.

Ex: Show that $\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$ and

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \cdot \frac{\pi}{2}, \quad \text{for } n \in \mathbb{N}.$$

Ex: Prove that $\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$ for $n \geq 2$.

Ex: Prove that $\int \sec^n x \, dx = \dfrac{\tan x \cdot \sec^{n-2} x}{n-1} + \dfrac{n-2}{n-1} \int \sec^{n-2} x \, dx$.

Ex: Prove the induction formula.

$$\int_0^1 (1-x^2)^n \, dx = \frac{2^{2n}(n!)^2}{(2n+1)!} \quad \text{for } n \in \mathbb{N}.$$

Combine the Substitution Rule and Integration by Parts

Ex: Compute $\int \sin\theta \ln(\cos\theta)\, d\theta$.

Ex: Compute $\int \cos\sqrt{x}\, dx$.

Ex: Compute $\int \sin(\ln x)\,dx$.

Ex: Compute $\int (\sin^{-1} x)^2 \, dx$.

Ex: Taylor Series.

From $f(x) = f(a) + \int_a^x f'(t)\,dt$, show that

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$$

for $n \in \mathbb{N}$.