Double Integrals

Section 15.1-15.3

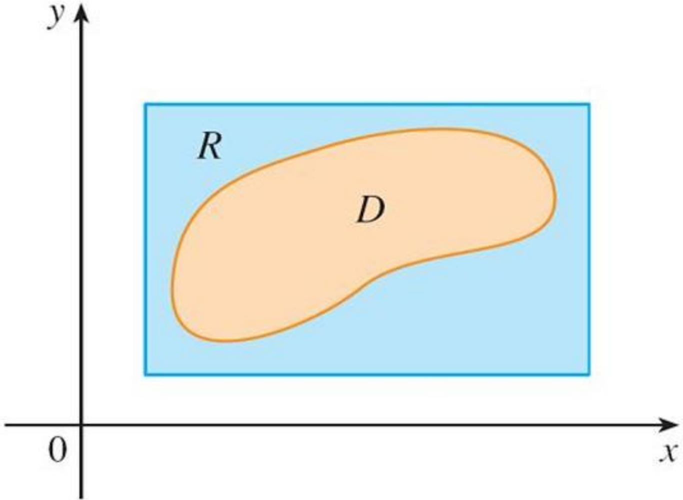
Outline

- Double Integrals over Rectangles
- Iterated Integrals
  - Fubini's Theorem
- Double Integrals over General Regions
  - Type I Regions
  - Type II Regions
- Double Integrals in Polar Coordinates

Double Integrals over General Regions

- For a general region D in R^2 , we suppose that D is bounded, which means that D can be enclosed in a rectangular region R .
- We define a new function F with domain R :

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in D} \\ 0 & \text{if } (x,y) \text{ is in R but not in D} \end{cases}$$



Double Integrals over General Regions

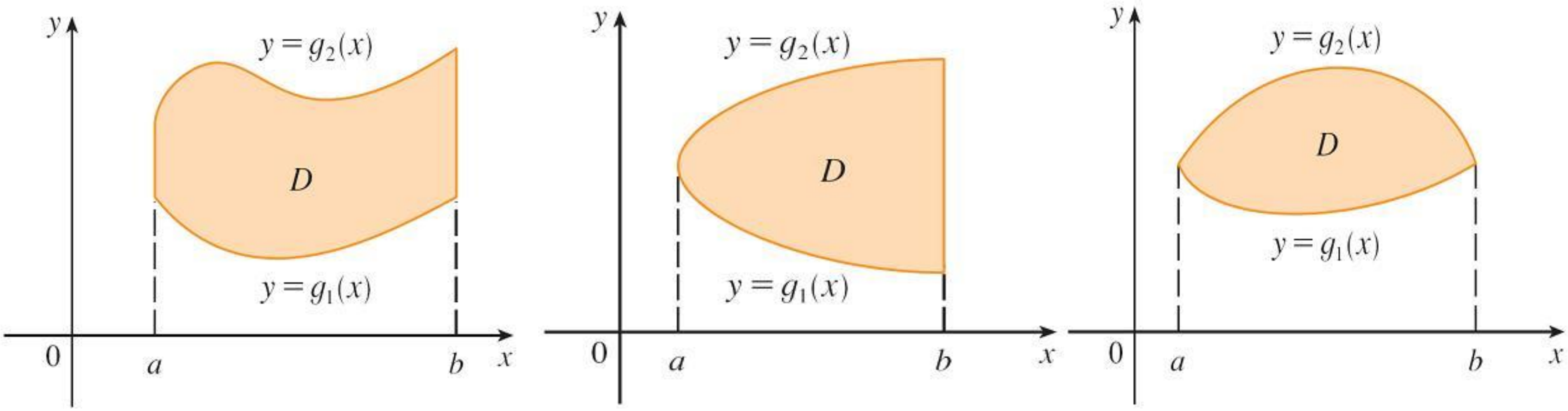
- If F is integrable over R, then we define the **double integral of f over D** by

$$\boxed{2} \qquad \iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA \qquad \text{where } F \text{ is given by Equation 1}$$

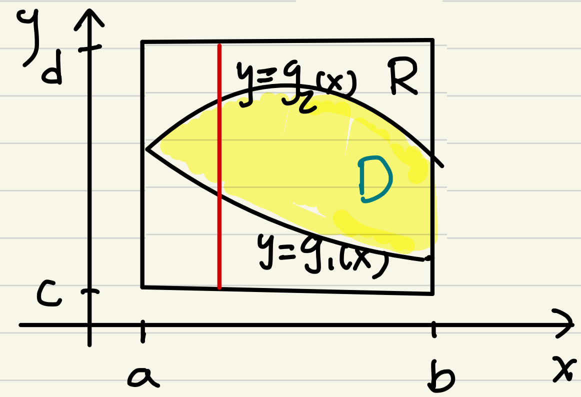
- If f is continuous on D and the boundary curve of D is "well behaved", then it can be shown that $\iint_R F(x, y) dA$ exists and therefore $\iint_D f(x, y) dA$ exists.
- In particular, this is the case for **type I** and **type II** regions.

Double Integrals over General Regions

- A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x , that is,
$$D = \{(x, y) | a \le x \le b, g_1(x) \le y \le g_2(x)\}$$
where g_1 and g_2 are continuous on $[a, b]$.



$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$. Choose a rectangle $R = [a,b] \times [c,d]$ that contains D.



Let $F(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D \\ 0, & \text{if } (x,y) \in R \setminus D. \end{cases}$

$$\boxed{\text{Define } \iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA.}$$

However, $\displaystyle\iint_R F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx$

Double Integrals over General Regions

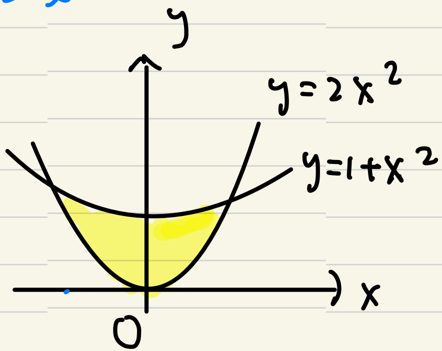
$\boxed{3}$ If f is continuous on a type I region D such that

$$D = \left\{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\right\}$$

then
$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

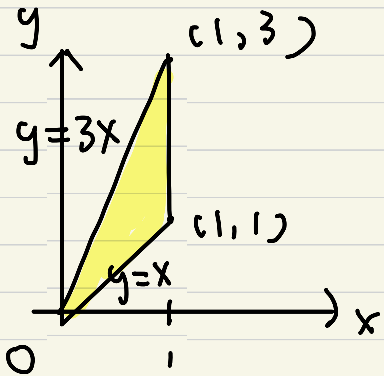
Ex: Compute $\iint_D x+2y \, dA$, where D is bounded by $y=2x^2$ and $y=1+x^2$.

Sol:



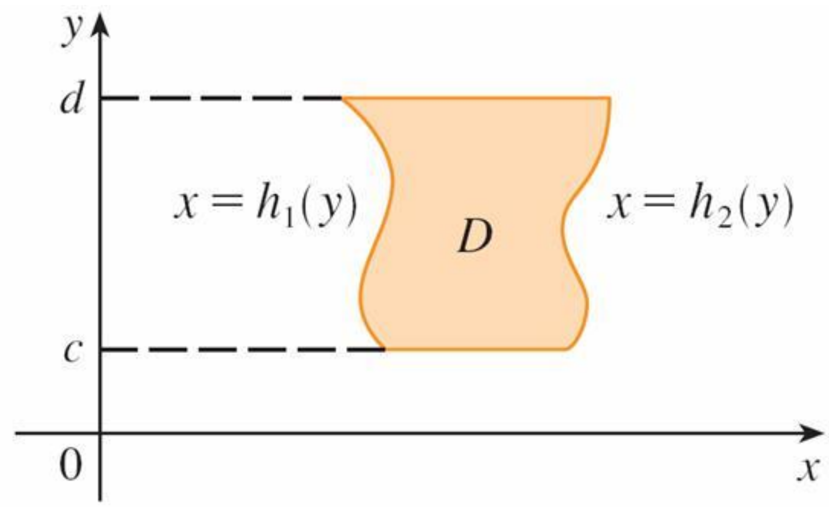
Ex: Compute $\iint_T \frac{xy}{1+x^4}\,dA$, where T is the triangle with vertices $(0,0)$, $(1,1)$ and $(1,3)$

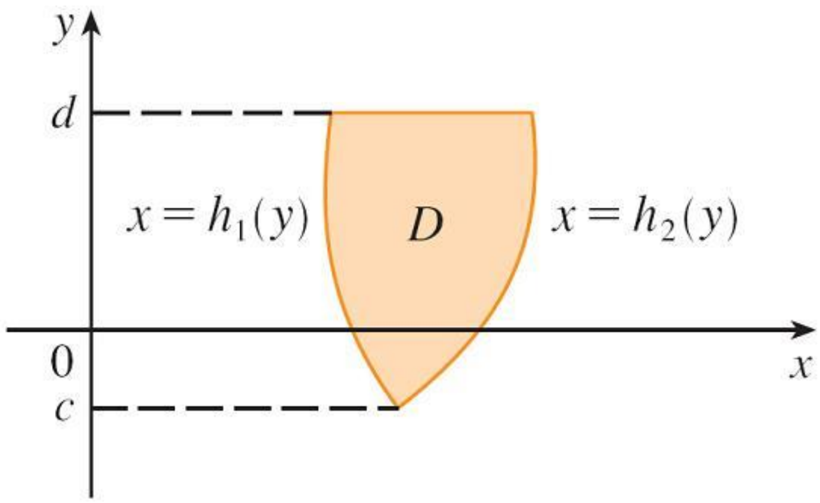
sol:



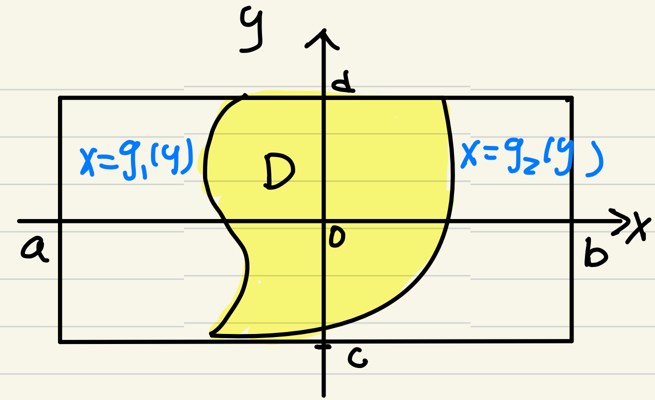
Double Integrals over General Regions

- We also consider plane regions of **type II**, which can be expressed as
$$D = \{(x, y) | c \le y \le d, h_1(y) \le x \le h_2(y)\}$$
where h_1 and h_2 are continuous on $[c, d]$.





$$D = \{(x,y) \mid c \le y \le d,\ g_1(y) \le x \le g_2(y)\}$$



$$D \subset [a,b]\times[c,d]$$

Then $\displaystyle\iint_D f(x,y)\,dA$

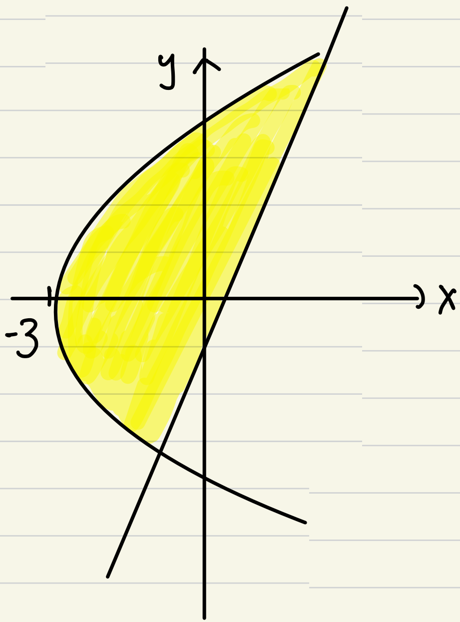
$$= \iint_{[a,b]\times[c,d]} F(x,y)\,dA = \int_c^d \int_a^b F(x,y)\,dx\,dy$$

Double Integrals over General Regions

5
$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

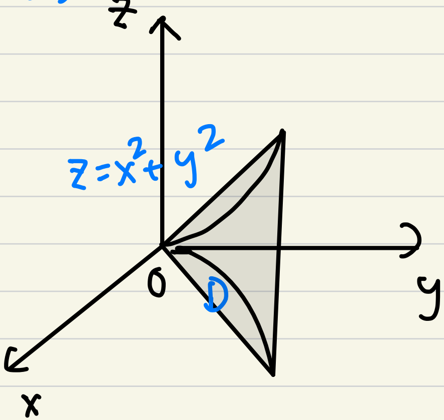
where D is a type II region given by Equation 4.

Ex: Evaluate $\iint_D xy^2\, dA$, where D is a region bounded by $y = x - 1$ and $y^2 = 2x + 6$



Ex: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by $y = \sqrt{x}$ and $y = x$.

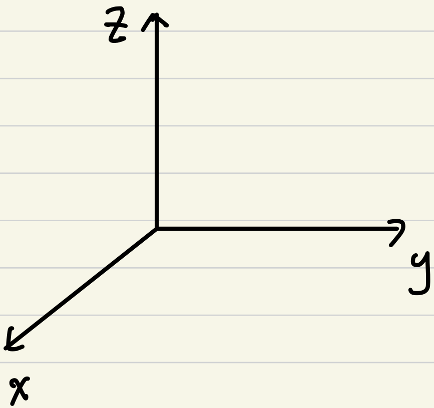
Sol:



Ex: Find the volume of the tetrahedron bounded by planes

$2x + y + z = 6$, $4x = y$, $x = 0$ and $z = 0$.

sol:



Change the Order of Integration

Change $I = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$ into the iterated integral of the other order.

Step 1: Write $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$ as $\iint_D f(x,y)\,dA$.

Draw $y = g_1(x)$, $y = g_2(x)$, $x = a$, $x = b$ and determine D.

Step 2: Write D as a type II region.

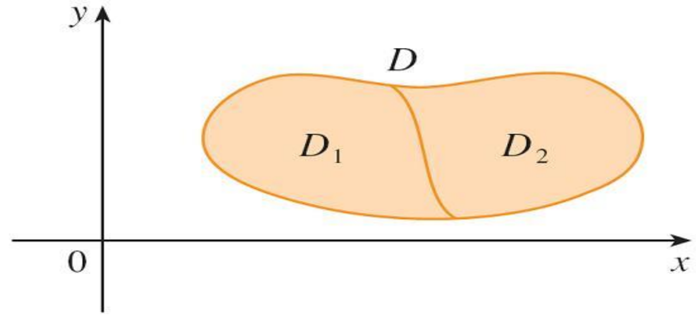
Step 3: Write $\iint_D f(x,y)\,dA$ as $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

Ex: Compute $I = \int_0^9 \int_{\sqrt{x}}^3 \frac{1}{1+y^3}\,dy\,dx$.

Ex: Compute $I = \int_0^1 \int_x^{x^{\frac{1}{3}}} \sqrt{1-y^4}\, dy\, dx$.

Ex: Compute $I = \int_0^1 \int_x^1 \frac{y^\lambda}{x^2+y^2}\,dy\,dx$, where $\lambda > 0$.

Double Integrals over General



- If $D = D_1 \cup D_2$, where $D_1$and D_2 don't overlap except perhaps on their boundaries, then

$$\boxed{\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA}$$

- We can use this property to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II.