The Definite Integral

Section 5.2-5.3

Outline

- The Definite Integral
  - Definition
  - Properties
- The Fundamental Theorem of Calculus

The Definite Integral

- The limit $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ appears in solving the area problem.
- It turns out that this same type of limit occurs in a wide variety of situations even when f is not necessarily a positive function. Hence, we give this type of limit a special name and notation.

The Definite Integral

$\boxed{2}$ **Definition of a Definite Integral** If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 \, (= a), x_1, x_2, \ldots, x_n \, (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

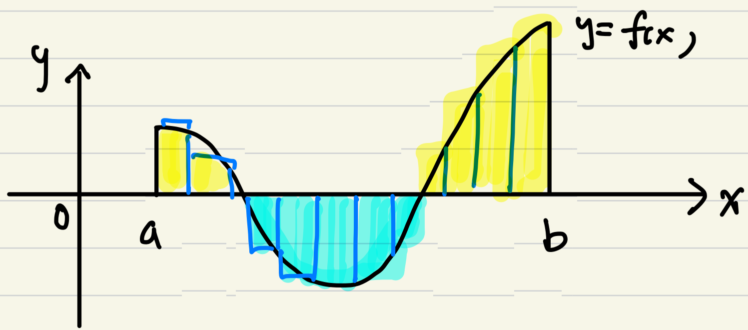
provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Ex: Recognize the limit of Riemann Sums

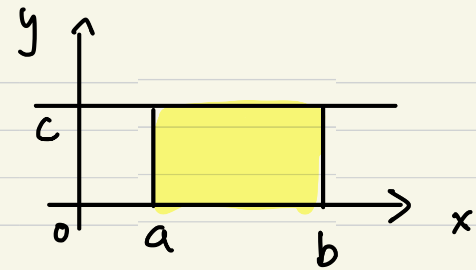
$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\ln(n^2+k^2) - 2\ln n}{n+k}$$ as a definite integral $\int_a^b f(x)\,dx$.

The geometrical meaning of $\int_a^b f(x)\,dx$.

If $f(x)$ takes on both positive and negative values, then $\int_a^b f(x)\,dx = A_1 - A_2$ where A_1 is the area of the region above the x-axis and below the graph of $f(x)$, and A_2 is the area of the region below the x-axis and above the graph of $f(x)$.



Ex: Show that $\int_a^b c \, dx = c(b-a)$.



The Definite Integral

- The precise meaning of the limit that defines the integral is as follows:

  For every $\epsilon > 0$ there is an integer N such that
  $$\left| \int_a^b f(x)dx - \sum_{i=1}^{n} f(x_i^*)\Delta x \right| < \epsilon$$
  for every integer $n > N$ and for every choice x_i^* in $[x_{i-1}, x_i]$.

The Definite Integral

- **Note 1**: In the definite integral notation, $f(x)$ is called the integrand. a is the lower limit and b is the upper limit. The dx simply indicates that the independent variable is x .
- **Note 2**: The definite integral is a number; it does not depend on x . In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr$$

The Definite Integral

- **Note 3**: The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ that occurs in Definition 2 is called a **Riemann sum**.
- **Note 4**: We don't need to divide the interval $[a, b]$ into equal width subintervals. If the subinterval widths are $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, we only have to ensure that all these widths approach 0 in the limiting process. In this case the definition of a definite integral becomes

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

The Definite Integral

- **Note 5**: We have defined the definite integral for an integrable function, but not all functions are integrable.

3 Theorem If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

- In the previous theorem, we can replace the assumption "f has only a finite number of *jump* discontinuities" by "f is bounded and has only a finite number of discontinuities."

Ex: Let $f(x) = \begin{cases} 0 & , \text{ if } x \in Q \\ 1 & , \text{ if } x \in Q^c \end{cases}$. Show that $f(x)$ is not integrable on $[0, 1]$.

The Definite Integral

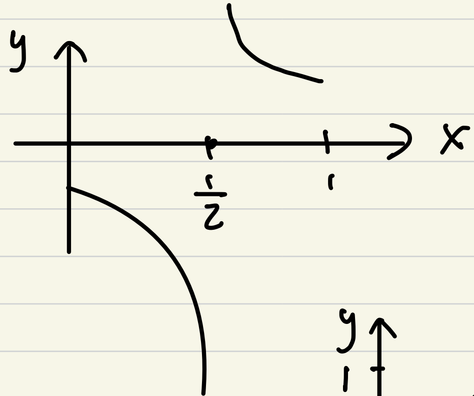
- Property:
- If $f(x)$ is not bounded on $[a, b]$, then $f(x)$ is *not* integrable on $[a, b]$.

- Exercise: Show that if $\lim_{x \to a^+} f(x) = \infty$, then the Riemann Sum of $f(x)$ on $[a, b]$ does not tend to a limit.

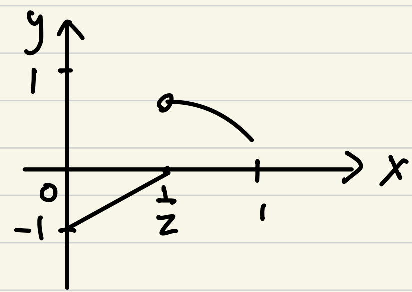
Ex: If $\lim_{x \to a^+} f(x) = \infty$, show that $f(x)$ is not integrable on $[a, b]$ for any $b > a$.

Ex: Choose integrable functions on the inteval $[0, 1]$

1. $f(x) = \begin{cases} \dfrac{1}{2x-1}, & \text{for } x \neq \frac{1}{2} \\ 0, & \text{for } x = \frac{1}{2}. \end{cases}$



2. $f(x) = \begin{cases} 2x-1, & x \in [0, \frac{1}{2}] \\ \cos x, & x \in (\frac{1}{2}, 1]. \end{cases}$



3. $f(x) = \begin{cases} \sin(\frac{1}{x}), & \text{for } x \in (0, 1] \\ 0, & \text{for } x = 0 \end{cases}$

The Definite Integral

- Compute / estimate a definite integral

4 Theorem If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\Delta x$

Properties of the Definite Integral

- $$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Properties of the Integral

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant

2. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Properties of the Definite Integral

- $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

Comparison Properties of the Integral

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.

7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a)$$

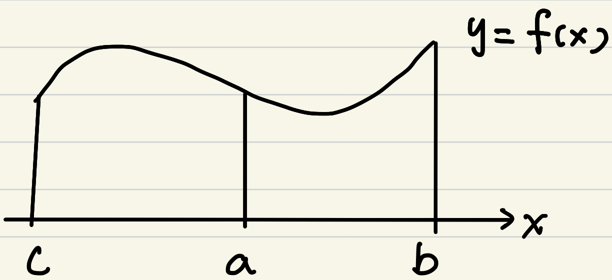
Ex: Show that $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

pf:

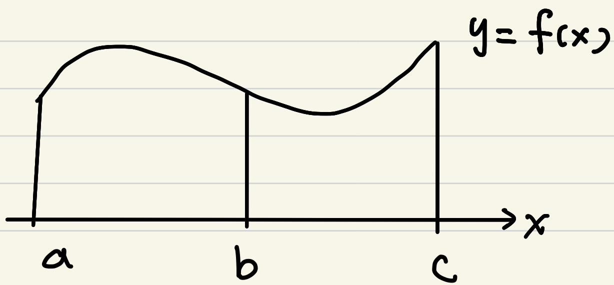
Suppose that $f(x) \geq 0$.

$\boxed{\text{Case 1: } a < c < b}$

$\boxed{\text{Case 2: } c < a < b}$



$\boxed{\text{Case 3: } a < b < c}$



Ex: Suppose that $\int_1^7 f(x)\,dx = 10$, $\int_5^{10} f(x)\,dx = 6$,

and $\int_1^{10} 2f(x)\,dx = 30$. Find $\int_1^5 f(x)\,dx$ and $\int_7^{10} f(x)\,dx$.

Ex: Compute $\displaystyle\lim_{x\to\infty}\int_x^{2x}\frac{1}{1+y^2}\,dy$.