Limits at Infinity

Section 2.6

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Outline

- 1. Limits at infinity
  - Definition
  - Horizontal Asymptotes
  - Examples
- 2. Infinite Limits at Infinity

Limits at Infinity

$\boxed{1}$ **Definition** Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

$\boxed{2}$ **Definition** Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

Limits at Infinity

7 Definition Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x > N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

8 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

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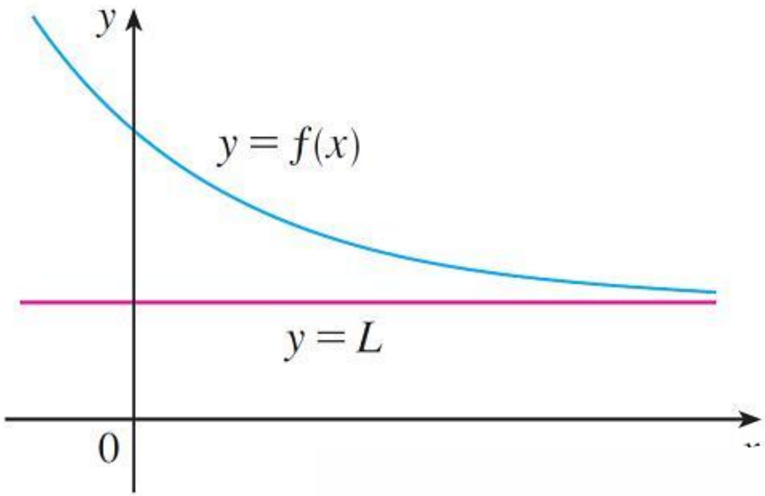
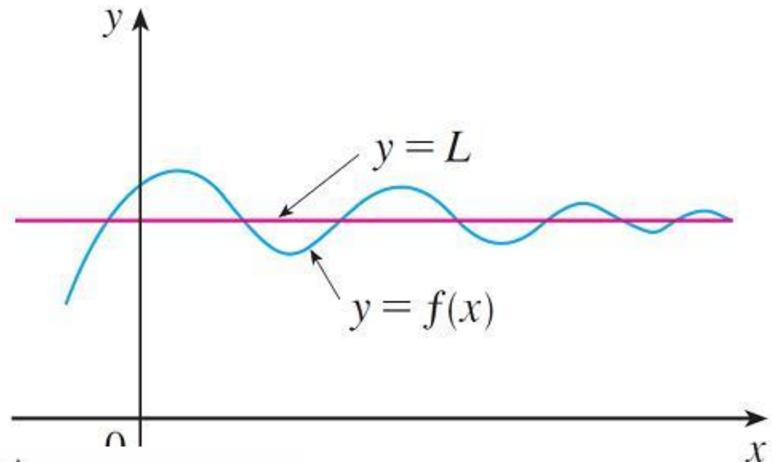
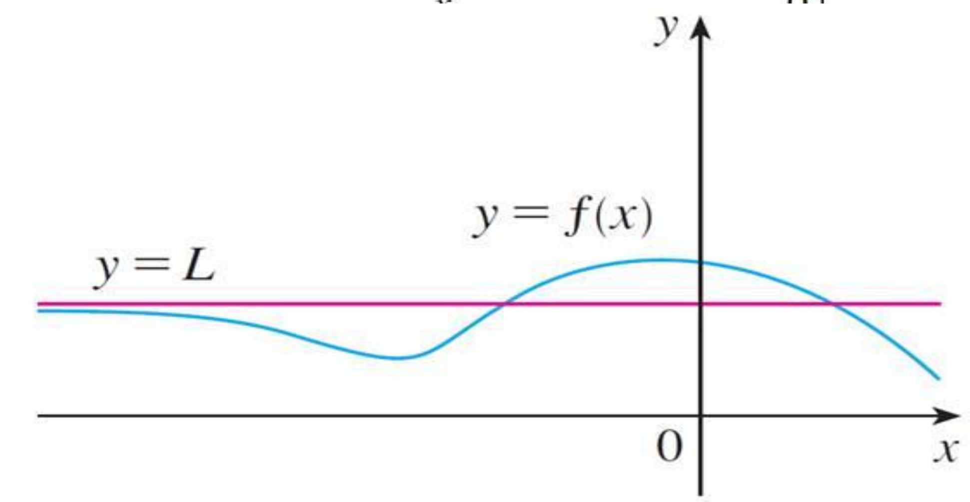
$$\text{if} \quad x < N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

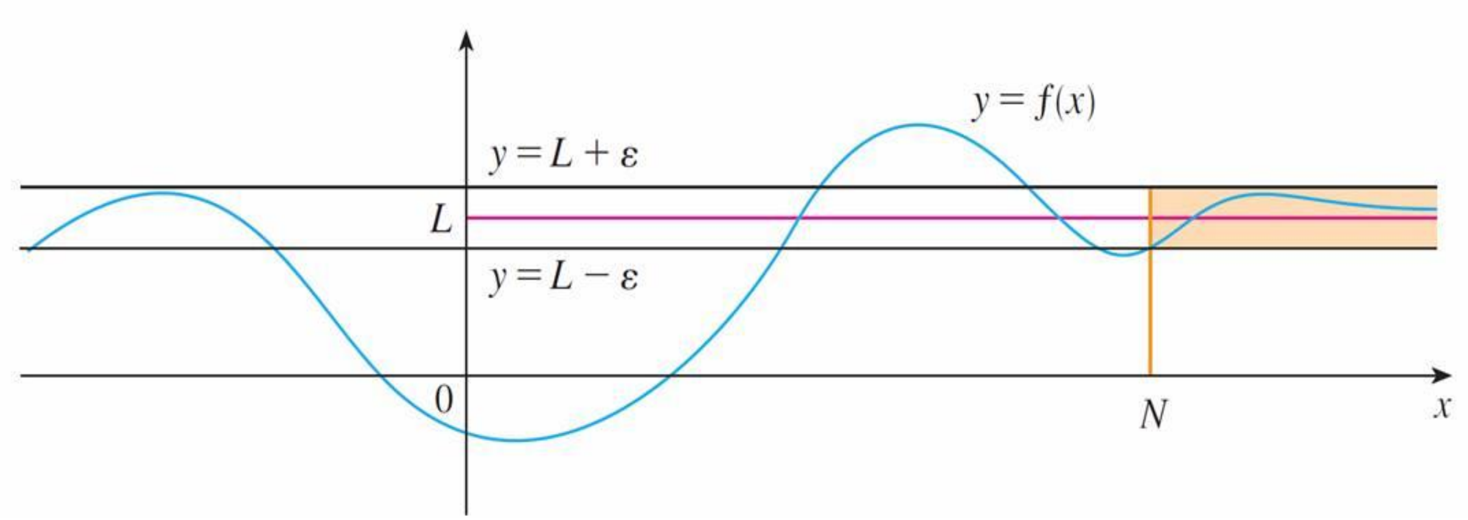
Limits at Infinity







Limits at Infinity



Limits at Infinity

3 **Definition** The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

Examples

$\boxed{5}$ **Theorem** If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Ex: Prove that $\lim_{x \to \infty} \frac{1}{x^r} = 0$ for all $r > 0$ and $r \in \mathbb{Q}$.

Pf: For any $\varepsilon > 0$, we need to find N s.t. if $x > N$ then $\left| \frac{1}{x^r} - 0 \right| < \varepsilon$.

Examples

- Limit Laws are valid for limits at infinity.
- We can use the Squeeze Theorem to find limits at infinity.
- Find limits at infinity of the composition with a continuous function.

Use Limit Laws

Ex: Find $\lim_{x \to \infty} \dfrac{2x^2 + 3x + 1}{5x^2 + 4x - 2}$

sol:

Ex: Find $\lim_{x \to -\infty} \frac{\sqrt{x^2+4x-3}}{2x+1}$

Sol:

Ex: Find $\lim_{x \to \infty} \sqrt{x^2+ax+b} - x$ and $\lim_{x \to -\infty} \sqrt{x^2+ax+b} - x$, where a, b are constants.

Sol:

Ex: Find all horizontal and vertical asymptotes of

$$y = f(x) = \frac{\sqrt{x^4+1} - \sqrt{x^6+1}}{x(x+1)^2}$$

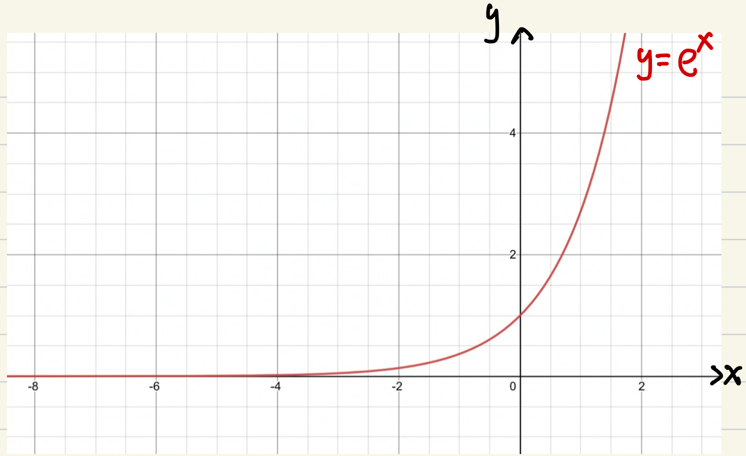
sol:

Ex: Find all horizontal and vertical asymptotes of

$$y = f(x) = \frac{1}{\sqrt{x^2+2x+2} - x}.$$

Use the Squeeze Theorem

Ex: Find $\lim_{x \to -\infty} e^x \sin(x^2)$



Composition of a continuous function

Ex: Find $\lim_{x\to\infty}[\ln(1+ax)-\ln(2+bx)]$, where $a, b>0$ are constants.

sol:

Ex: Find $\lim_{x \to \infty} \cos\left(\frac{1}{x}\right)$, $\lim_{x \to -\infty} \frac{[x^2]}{3x^2 - 2x}$

sol:

Ex: Find $\lim_{x \to \infty} \cos(x^2) \sin\left(\frac{1}{x}\right)$

sol:

Infinite Limits at Infinity

- The notation $\lim_{x\to\infty} f(x) = \infty$ is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x\to-\infty} f(x) = \infty \qquad \lim_{x\to\infty} f(x) = -\infty$$

$$\lim_{x\to-\infty} f(x) = -\infty$$

Infinite Limits at Infinity

$\boxed{9}$ **Definition** Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = \infty$$

means that for every positive number M there is a corresponding positive number N such that

$$\text{if} \quad x > N \qquad \text{then} \qquad f(x) > M$$

Infinite Limits at Infinity

Ex: $\lim_{x \to \infty} x^n = \infty$ for all $n \in \mathbb{N}$.

$\lim_{x \to -\infty} x^n =$

Ex: $\lim_{x \to \infty} x^2 - 3x$

Ex: Find $\lim\limits_{x \to -\infty} \dfrac{-2x^3 + 4x + 1}{x^2 - 3x + 5}$

Sol:

Q: Find $\lim\limits_{x \to 0^+} e^{\frac{1}{x(x-1)}}$, $\lim\limits_{x \to 0^-} e^{\frac{1}{x(x-1)}}$, $\lim\limits_{x \to \pm\infty} e^{\frac{1}{x(x-1)}}$

- Example:
- Discuss $\lim_{x\to\infty} \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials with
- (i) $\deg P(x) < \deg Q(x)$
- (ii) $\deg P(x) > \deg Q(x)$
- (iii) $\deg P(x) = \deg Q(x)$

Review

- Write down the precise definition of $\lim_{x \to \pm\infty} f(x) = L$
- Write down the precise definition of $\lim_{x \to \pm\infty} f(x) = \pm\infty$
- What is the horizontal asymptote of the curve $y = f(x)$?