Directional Derivatives

Section 14.6

Outline

- Directional Derivatives
- The Gradient Vector
  - Definition
  - Geometric Meanings of the Gradient Vector

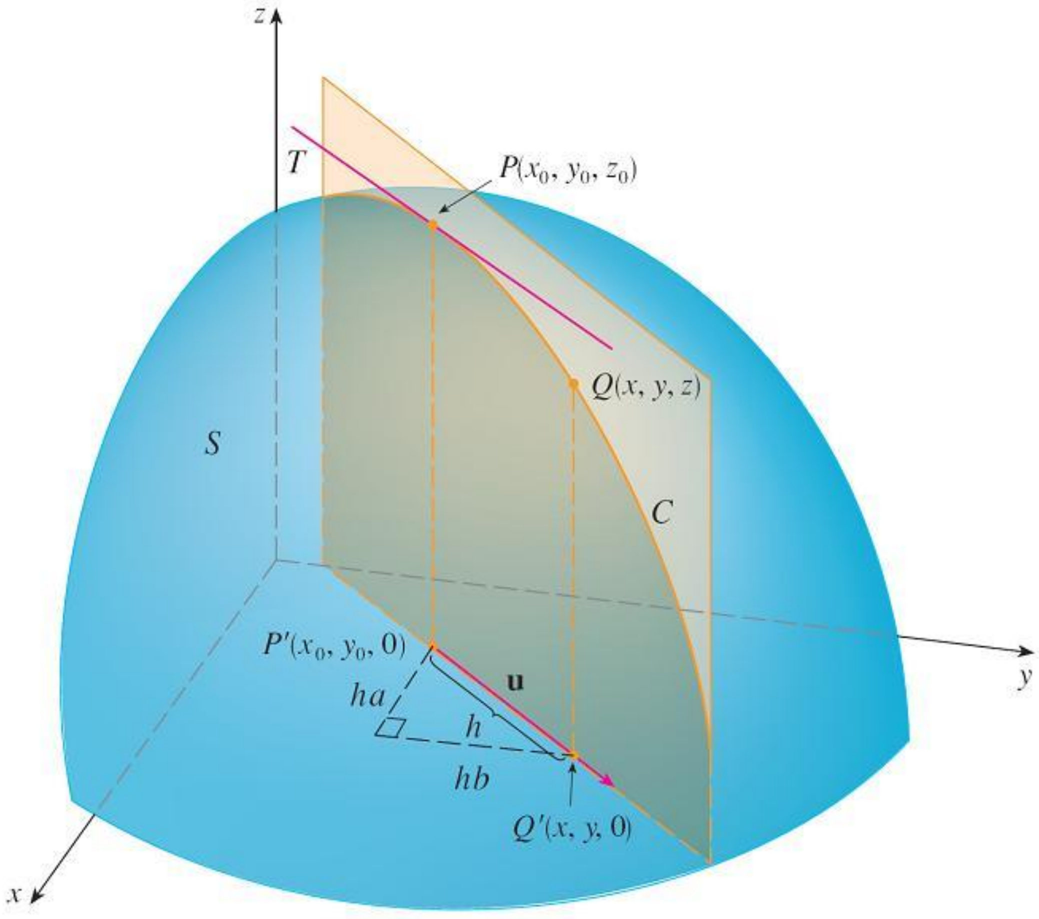
Directional Derivatives

- When you climb a mountain, you will experience different “slopes” along different paths. Which path will you choose?

Directional Derivatives

- Now we want to find the rate of change of $z = f(x, y)$ at (x_0, y_0) in the direction of an arbitrary unit vector $\vec{u} = (a, b)$.
- First draw a vertical plane that passes through the point $P(x_0, y_0, f(x_0, y_0))$ in the direction of $\vec{u}$ which intersects surface $S : z = f(x, y)$ in a curve C.

Directional Derivatives



Directional Derivatives

- The slope of the tangent line T to C at the point P is the rate of change of z in the direction of $\vec{u}$.

$\boxed{2}$ **Definition** The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Directional Derivatives

- For functions of n variables, we define the directional derivatives of f in the direction $\vec{u}$ by the vector notation:

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$$\boxed{D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}}$$

Ex: $f(x,y) = x^{\frac{1}{3}} y^{\frac{1}{3}}$. Find $D_{\vec{i}} f(0,0)$, $D_{\vec{j}} f(0,0)$ and $D_{\vec{u}} f(0,0)$

where $\vec{u} = (\cos\theta, \sin\theta)$.

Directional Derivatives

- If $\vec{u} = \vec{i} = (1, 0)$, then $D_{\vec{i}} f = f_x$.
- If $\vec{u} = \vec{j} = (0, 1)$, then $D_{\vec{j}} f = f_y$.
- In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivatives.

Directional Derivatives

$\boxed{3}$ **Theorem** If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

$$D_{\vec{u}} f(x) = (aD_{\vec{i}} + bD_{\vec{j}}) f(x) = (aD_x + bD_y) f(x)$$

i.e. $D_{\vec{u}} = aD_x + bD_y$.

If the unit vector $\vec{u}$ makes an angle θ with the positive x-axis, then we can write $\vec{u} = (\cos\theta, \sin\theta)$ and $D_{\vec{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$

Theorem If $f(x,y)$ is **differentiable**, then f has a directional derivative in any direction $\vec{u}=(a,b)$ (with $|\vec{u}|=1$) and

$$D_{\vec{u}}f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b.$$

Ex: $f(x,y) = xy^2 + x^{\frac{1}{y}}$. Find the direction derivative of f in the direction $\vec{u} = (-3, 4)$ at $(2, 1)$.

Ex: Let $f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$ and $\vec{u} = (\cos\theta, \sin\theta)$

for some $0 \leq \theta < 2\pi$. Compute $D_{\vec{u}} f(0,0)$.

The Gradient Vector

8 Definition If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

- Hence, if f is differentiable at (x_0, y_0) we can write the directional derivatives as

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$$\boxed{D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}}$$

The Gradient Vector

- If f is a function of n variables, then the gradient of f is defined as the vector

$$\nabla f = \frac{\partial f}{\partial x_1}\vec{e_1} + \frac{\partial f}{\partial x_2}\vec{e_2} + \cdots + \frac{\partial f}{\partial x_n}\vec{e_n}$$

- If f is differentiable at $\vec{x_0}$, then

$$D_{\vec{u}} f(\vec{x_0}) = \nabla f(\vec{x_0}) \cdot \vec{u}$$


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The Gradient Vector

- Meanings of the gradient vector:
- 1.Maximizing the directional derivatives:

15 Theorem Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

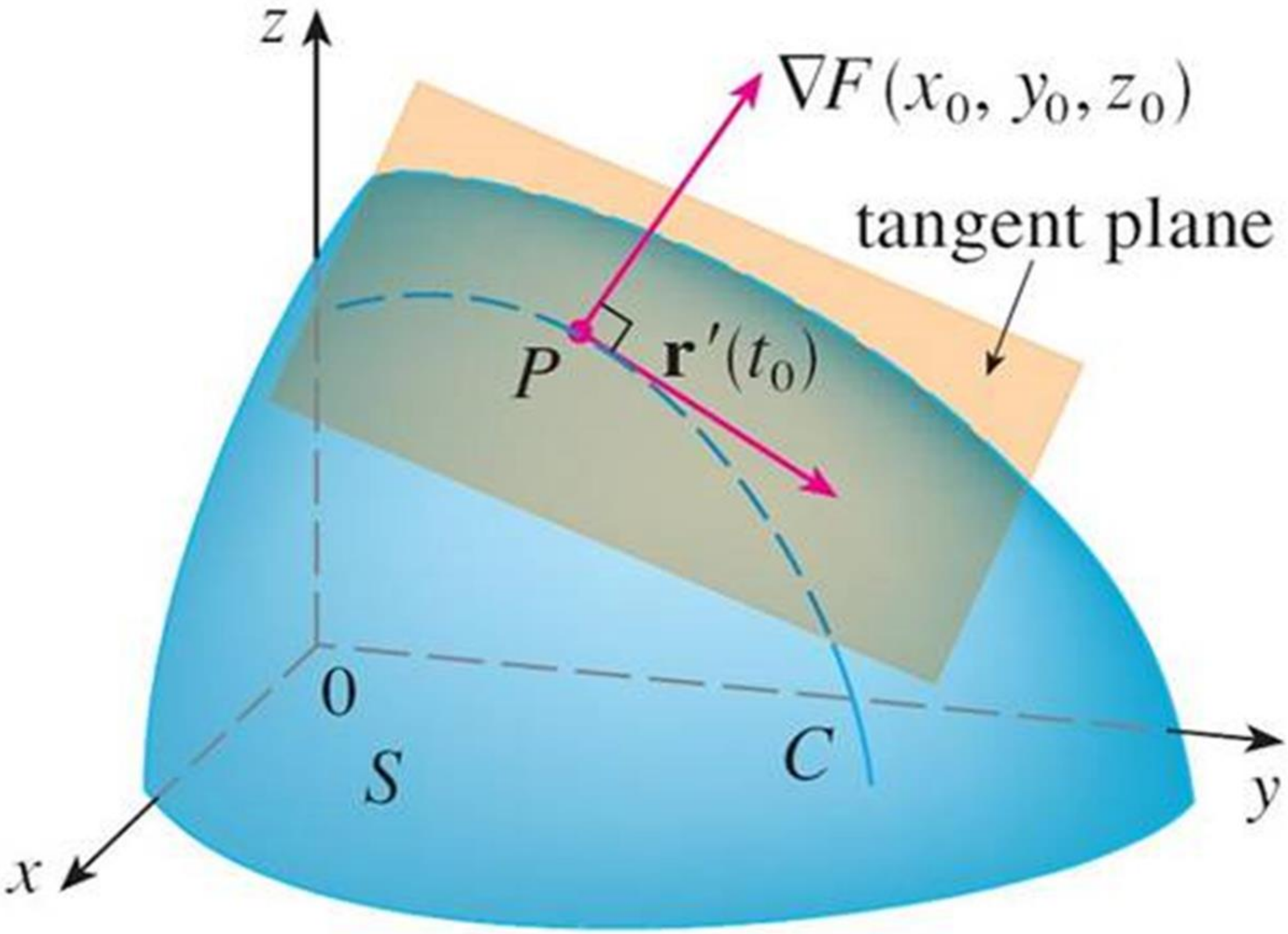
pf of the theorem :

Ex: Suppose that $D_{\vec{u}} f(1,0) = \frac{5}{2}\sqrt{2}$ for $\vec{u} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $D_{\vec{v}} f(1,0) = \frac{1}{5}$ for $\vec{v} = (\frac{3}{5}, -\frac{4}{5})$ and $f(x,y)$ is diff at $(1,0)$.

Find $\vec{\nabla} f(1,0)$ and maximum value of $D_{\vec{w}} f(1,0)$ for all $|\vec{w}| = 1$.

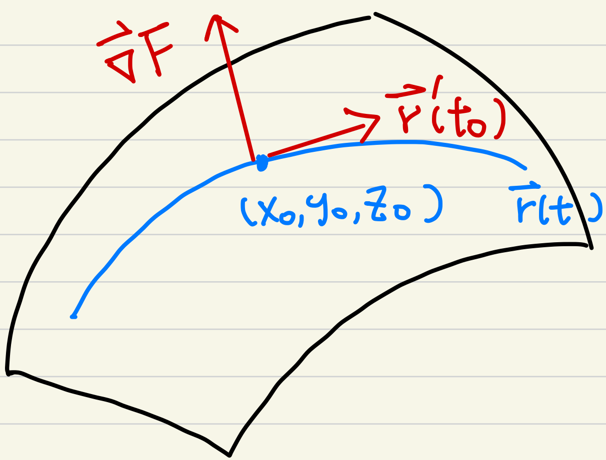
The Gradient Vector

- 2. As the normal vector to the level surface:
- Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F, and let $P(x_0, y_0, z_0)$ be a point on S. We can derive that the gradient vector at P, $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector to any curve C on S that passes through P .


z
$\nabla F(x_0, y_0, z_0)$
tangent plane
P
$\mathbf{r}'(t_0)$
0
S
C
y
x

Prop: Suppose that $F(x,y,z)$ is differentiable. Consider a level surface $S: F(x,y,z)=k$. For any differentiable curve $\vec{r}(t) = (x(t), y(t), z(t)) \subset S$ with $\vec{r}(t_0) = (x_0, y_0, z_0)$, we have $\vec{\nabla} F(\vec{x}_0) \cdot \vec{r}\,'(t_0) = 0$.

pf:



The Gradient Vector

- Therefore, if $\nabla F(x_0, y_0, z_0) \neq \vec{0}$, it is natural to define the **tangent plane to the level surface** $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. We can write the equation of this tangent plane as

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \tag{19}$$

Ex: Find an equation of the tangent plane to the surface

$S: \ x^2 + 4y^2 - z^2 = 1$ at (x_0, y_0, z_0).

Ex: $S_1: z = x^2 + 2y^2$, $S_2: x^2 + 2y^2 + z^2 = 12$

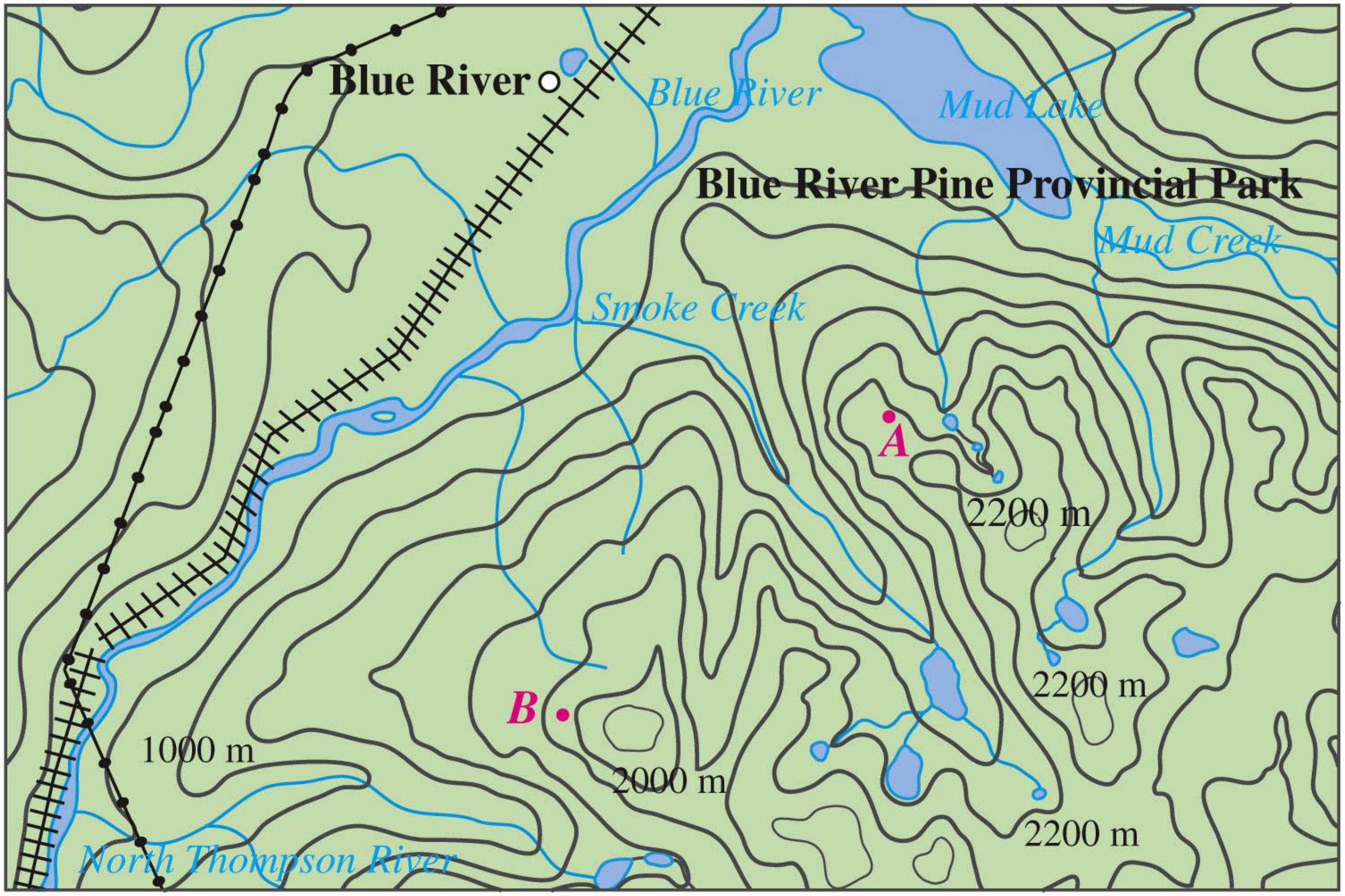
Let C be the curve of intersection $C = S_1 \cap S_2$.

Find the tangent line of C at $(1, -1, 3)$.

Prop: Suppose that $f(x,y)$ is differentiable. Let $C: f(x,y)=k$.

If $(x_0, y_0) \in C$, then $\vec{\nabla} f(x_0, y_0) \perp C$.

The Gradient Vector



Review

- How do we compute directional derivatives?
- What is the gradient vector of a function?
- What are the geometric meanings of the gradient vector ?
- Write down the equation of the tangent plane of a level surface $F(x, y, z) = k$ at a point $P(x_0, y_0, z_0)$.