Vector Fields and Line Integrals

Section 16.1-16.3

Outline

- Vector Field, Gradient Vector Fields
- Line Integrals
  - With Respect to Arc Length
  - With Respect to Variables
  - Integrate Vector Fields Along a Curve
- The Fundamental Theorem for Line Integrals
  - Independence of Path

Vector Fields

- A vector field is a function whose domain is a set of points in R^2 (or R^3) and whose range is a set of vectors in R^2 (or R^3).
- Definition: A vector field on $D \subset R^2$ is a function $\vec{F}$ that assigns to each point $(x, y) \in D$ a two-dimensional vector $\vec{F}(x, y)$.
- Definition: A vector field on $E \subset R^3$ is a function $\vec{F}$ that assigns to each point $(x, y, z) \in E$ a three-dimensional vector $\vec{F}(x, y, z)$.

Vector Fields

- Since $\vec{F}(x,y)$ (or $\vec{F}(x,y,z)$) is a two (three)-dimensional vector, we can write it in terms of its **component functions** P and Q (and R) as follows :

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j} = (P(x,y), Q(x,y))$$

$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

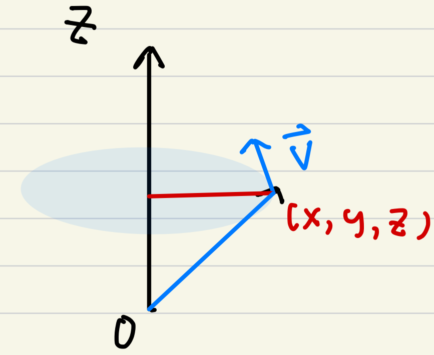
- P, Q , R are sometimes called **scalar fields** to distinguish them from vector fields.

Ex: A particle of mass M is placed at $\vec{x_0} = (x_0, y_0, z_0)$. Find the gravitational force field due to this particle.

sol:

Ex: The velocity field of a solid rotating about the z-axis with angular velocity $\omega\vec{k}$ is

$$\vec{V}(x,y,z) = \omega\vec{k} \times (x\vec{i} + y\vec{j} + z\vec{k})$$

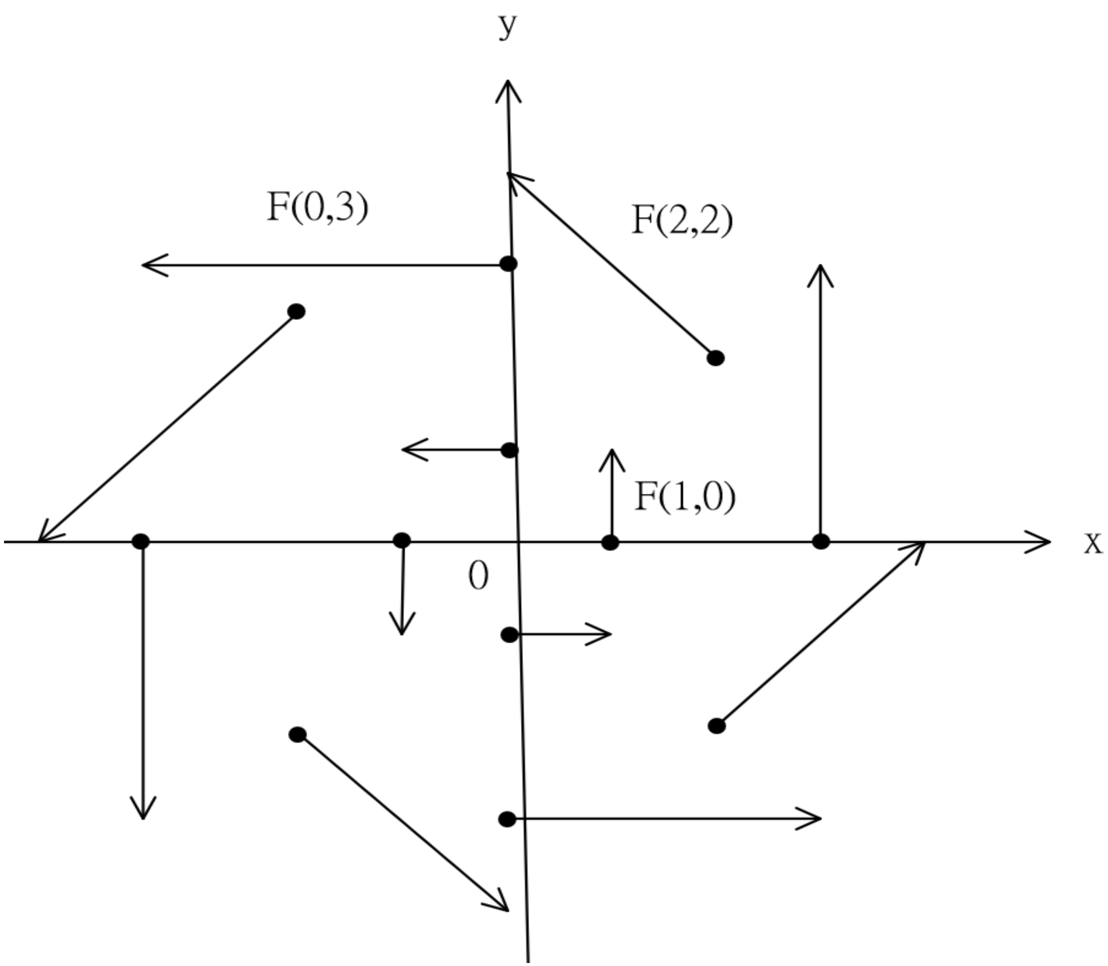


Vector Fields

- As with the vector functions, we can define continuity of vector fields and show that $\vec{F}$ is continuous if and only if its component functions P , Q, and R are continuous.
- Example: $\vec{F}(x, y) = -y\vec{i} + x\vec{j}$.
- Example: The Gravitational Force:

$$\vec{F}(\vec{x}) = -\frac{mMG}{|\vec{x}|^3}\vec{x}$$

Vector Fields



Gradient Vector Fields

- If f is a scalar function of two variables, ∇f is a vector field on R^2 and is called a **gradient vector field**.
- Likewise, if f is a scalar function of three variables, its gradient is a vector field on R^3 .

Gradient Vector Fields

- Definition: A vector field $\vec{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\vec{F} = \nabla f$.
- In this situation f is called a **potential function** for $\vec{F}$.

Ex: Compute $\vec{\nabla} f(x,y,z)$, where $f(x,y,z) = \dfrac{GMm}{\sqrt{x^2+y^2+z^2}}$.

sol:

Ex: Is $\vec{V}(x,y,z) = \omega\vec{k} \times (x\vec{i} + y\vec{j} + z\vec{k})$ a conservative vector field?

Sol: