Continuity

Section 2.5

Outline

- Continuity at a Point / on an Interval
- Discontinuity
- Some Operators That Preserve Continuity
- Examples
- Properties of Continuous Functions

Continuity at a Point

- The limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions with this property are called *continuous* at a.

$\boxed{1}$ **Definition** A function f is **continuous at a number *a*** if

$$\lim_{x \to a} f(x) = f(a)$$

Continuity at a Point

- Remark: If $f(x)$ is continuous at $x = a$, then
- **1.** $f(a)$ is defined (that is, a is in the domain of $f(x)$)
- **2.** $\lim_{x \to a} f(x)$ exists
- **3.** $\lim_{x \to a} f(x) = f(a)$

- Intuition: The graph of a continuous function has no hole or break.

Ex: $f(x) = \begin{cases} [\sqrt{x}] + cx, & \text{for } x \geq 3. \\ \dfrac{|-x^2+x+6|}{x-3}, & \text{for } x < 3. \end{cases}$ Find constant c

such that $f(x)$ is continuous at $x=3$.

Continuity on an Interval

2 Definition A function f is **continuous from the right at a number *a*** if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at *a*** if

$$\lim_{x \to a^-} f(x) = f(a)$$

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left*.)

Q: Find a function that is continuous only at one point.

Discontinuity

- If f is defined near a (i.e. f is defined on an open interval containing a, except perhaps at a), we say that f is **discontinuous at** a (or f has a **discontinuity** at a) if f is not continuous at a.
- Some types of discontinuity:
  - Removable discontinuity
  - Jump discontinuity
  - Infinite discontinuity
  - Essential discontinuity

		Ex:
Removable Discontinuity	$\lim_{x\to a} f(x)$ exists but $f(a)$ is not defined or $\lim_{x\to a} f(x) \neq f(a)$	
Jump Discontinuity	$\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$	
Infinit Discontinuity	$\lim_{x\to a} f(x) = \pm\infty$	
Essential Discontinuity	Near $x=a$ $f(x)$ varies rapidly.	

Some Operators That Preserve Continuity

$\boxed{4}$ **Theorem** If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$

$\boxed{8}$ **Theorem** If f is continuous at b and $\lim\limits_{x \to a} g(x) = b$, then $\lim\limits_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

$\boxed{9}$ **Theorem** If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Ex: Show that if $f(x)$ and $g(x)$ are continuous at $x=a$ and c is a constant then $f(x) \pm g(x)$, $cf(x)$, and $f(x) \cdot g(x)$ are continuous at $x=a$. If we further assume that $g(a) \neq 0$, then $\frac{f(x)}{g(x)}$ is continuous at $x=a$.

Remark: Theorem 8 can be proved by the precise definition of a limit. The continuity of f is necessary. In fact, there are examples such that $\lim_{x\to a} g(x) = b$ and $\lim_{x\to b} f(x) = L$ but $\lim_{x\to a} f(g(x)) \neq L$.

Remark: We can prove Theorem 9 by Theorem 8.

Ex: Show that $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$.

Examples of Continuous Functions

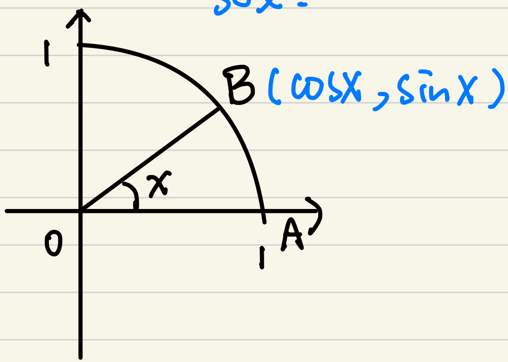
7 Theorem The following types of functions are continuous at every number in their domains:

polynomials	rational functions
root functions	trigonometric functions

- Exponential functions are also continuous on its domain.

Ex: Show that $\sin x$ and $\cos x$ are continuous at $x=0$.

Sol:



Ex: Show that $\sin x$ and $\cos x$ are continuous at any $x = x_0$.

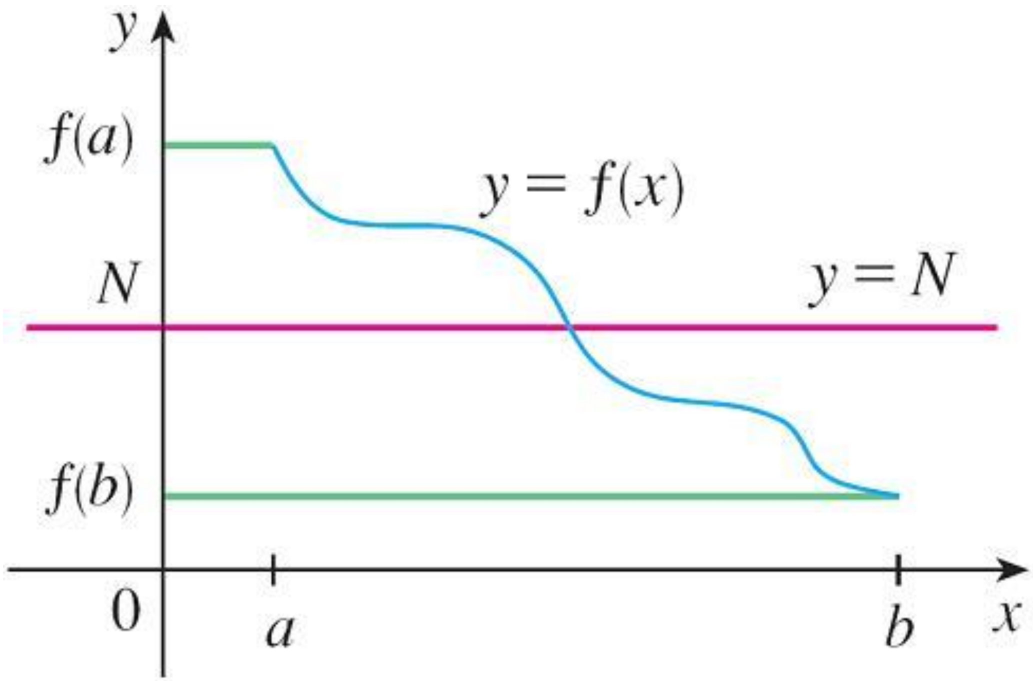
Sol:

Ex: Show that $\tan x$, $\cot x$, $\sec x$, $\csc x$ are continuous on their domains

Ex: compute $\lim_{x \to 1} \tan\left(\pi \frac{\sqrt[3]{x} - 1}{x - 1}\right)$.

Properties of Continuous Functions

10 **The Intermediate Value Theorem** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



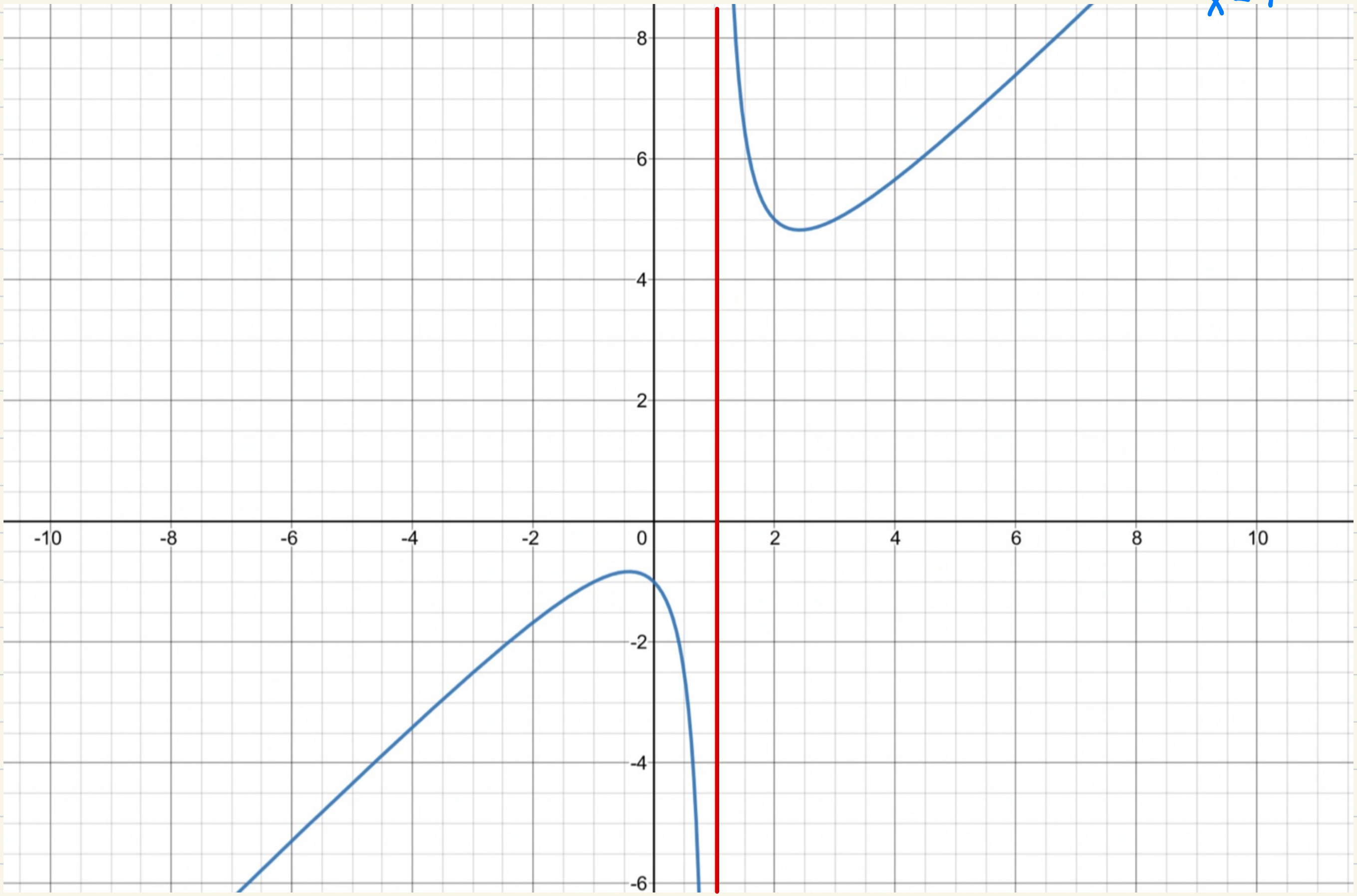
Properties of Continuous Functions

- Applications: The Intermediate Value Theorem can be used to locate roots of equations.
- Question:
- Show that a continuous 1-1 function is either increasing or decreasing. (Hence, we can show that the inverse function of any continuous function is also continuous.)

Ex: Show that $f(x) = x^2 - 3 + \frac{1}{x}$ has at least two real roots on the interval $[\frac{1}{3}, 2]$.

Ex: Does $f(x) = \dfrac{x^2+1}{x-1}$ have real roots on the interval $(0, 2)$?

$$y = \frac{x^2+1}{x-1}$$



Ex: Prove that if $f(x)$ is continuous with domain $[0,1]$ and range contained in $[0,1]$ then there is some $c \in [0,1]$ such that $f(c) = c$. (we call c the fixed point of f.)

Review

- Write the equation which expresses that f is continuous at the point a .
- Describe some types of discontinuity.
- List operations that preserve continuity.
- List some types of continuous functions.
- State the Intermediate Value Theorem.