Power Series

Section 11.8, 11.9

Outline

- Power Series:
  - Definition
  - The Radius of Convergence and the Interval of Convergence
- Representations of Functions as Power Series
  - Geometric Power Series
  - Differentiation and Integration of Power Series

Representations of Functions as Power Series

- Geometric Series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

for $|x| < 1$.

- This equation expresses the function $f(x) = 1/(1-x)$ as a power series.

Ex: Find the sum $\sum_{n=0}^{\infty} x^n$

Sol:

Ex: Represent $\frac{1}{2+x}$ as a power series centered at $x=1$.

Sol:

Ex: Write $\dfrac{-2x+1}{x^2-3x+2}$ as power series centered at $x=0$.

Sol:

Ex: Write $\dfrac{1}{1+x^2}$ as power series centered at $x=0$.

Sol:

- Differentiation and Integration of Power Series:
- We would like to be able to differentiate and integrate the sum of power series, and the following theorem says that we can do so by differentiating or integrating each individual term in the series, just as for a polynomial.
- This is called **term-by-term differentiation** and **integration**.

Representations of Functions as Power Series

- Theorem: If the power series $\sum c_n(x-a)^n$ has radius of converges $R > 0$, then the function f defined by $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on the interval $(a-R, a+R)$ and $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

Moreover, the radius of convergence of the above power series are both R .

Theorem: If the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R>0$, then the power series $\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$ converges absolutely on $(a-R, a+R)$ and diverges for $|x-a|>R$.

Pf:

Ex: Write $\frac{1}{(1-x)^2}$ as a power series centered at $x=0$

sol:

Ex: Write $\ln(1+x)$ as a power series centered at $x=0$.

sol:

Ex: Write $\ln(1-x)$ as a power series centered at $x=0$.

sol:

Ex: Write $\ln(a-x)$ as a power series centered at $x=0$.

sol:

Ex: Write $\ln\left(\frac{1-x}{1+x}\right)$ as a power series centered at $x=0$.

Sol:

Ex: Write $\tan^{-1}x$ as a power series centered at $x=0$.

sol:

Representations of Functions as Power Series

- Examples: For $|x| < 1$,

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$$

$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Ex: Find the sum $\sum_{n=1}^{\infty} n \cdot x^n$.

sol:

Ex: Let $S_n = \sum_{k=1}^{n} \frac{1}{k}$, and $f(x) = \sum_{n=1}^{\infty} S_n x^n$.

(a) Find the interval of convergence of $f(x)$.

(b) Find the sum of the power series (i.e. express $f(x)$ as an elementary function.) (Hint: consider $f(x) - x f(x)$.)

Sol:

Review

- What is a power series?
- What are the radius of convergence and the interval of convergence of a power series?
- How do we differentiate or integrate a power series function?
- Review some representations of functions as power series.