Stokes' Theorem

Section 16.8

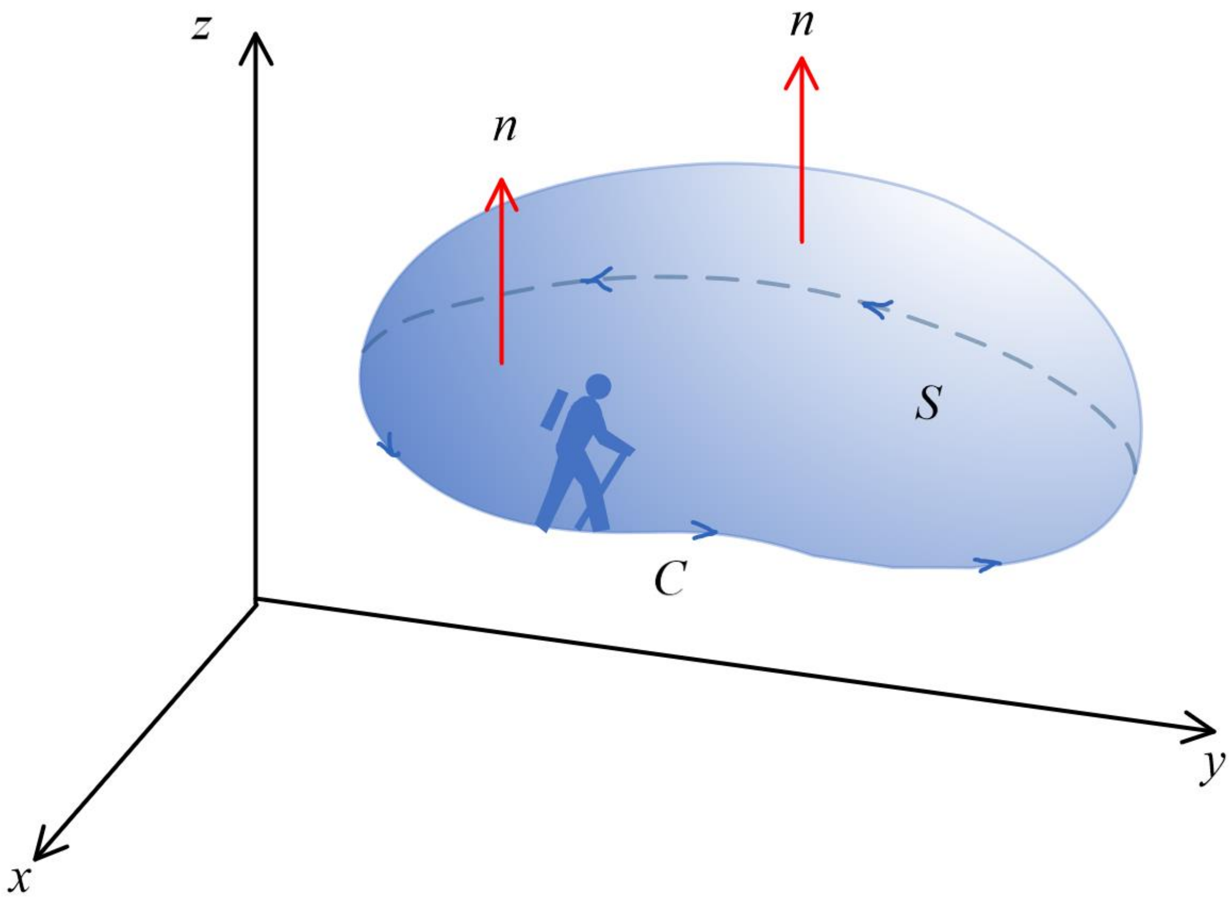
Stokes' Theorem

- Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem.
- Whereas Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve, Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (which is a space curve).

Stokes' Theorem

- The orientation of S induces the **positive orientation of the boundary curve** C shown in the following figure. This means that if you walk in the positive direction around C with your head pointing in the direction of $\vec{n}$, then the surface will always be on your left.

Stokes' Theorem



Stokes' Theorem

- Theorem:
- Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region that contains S. Then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$$

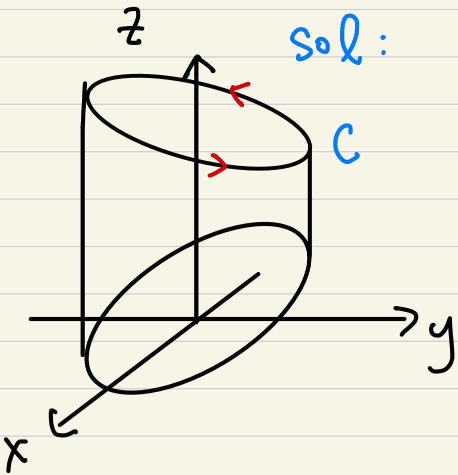
Stokes' Theorem

- The positively oriented boundary curve of the oriented surface S is often written as ∂S , so Stokes' Theorem can be expressed as

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

- As before, there is an integral involving derivatives on the left side of the equation (recall that $\operatorname{curl}\vec{F}$ is a sort of derivative of $\vec{F}$) and the right side involves the values of $\vec{F}$ only on the boundary of S .

Ex: Find $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y,z) = -y^2\vec{i} + 2x\vec{j} + z^2\vec{k}$ and C is the curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$. C is oriented counterclockwise when viewed from above.



Sol:

Stokes' Theorem

- Remark 1: Green's Theorem is a special case of Stokes' Theorem.
- Remark 2: If S_1 and S_2 are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' Theorem, then

$$\iint_{S_1} \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \iint_{S_2} \operatorname{curl} \vec{F} \cdot d\vec{S}$$

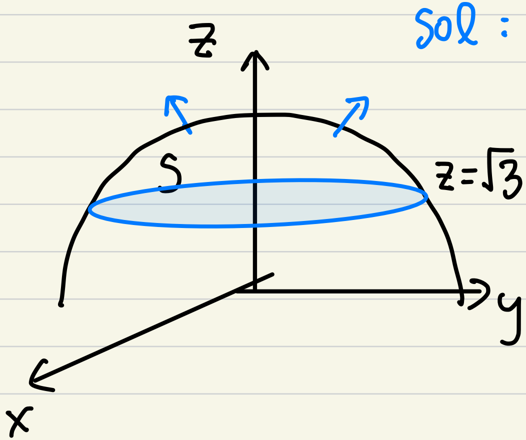
Ex: Show that Stokes' Theorem implies Green's Theorem.

Sol: Consider $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j} + 0\,\vec{k}$ and a xy-plane region D with positively oriented boundary C.

Then $\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = (Q_x - P_y)\,\vec{k}.$

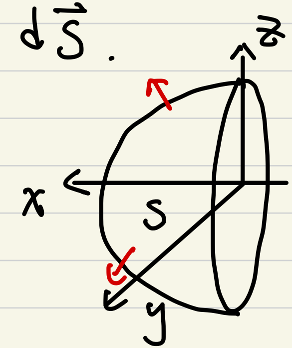
Ex: Compute $\iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$, where $\vec{F} = yz\,\vec{i} + z\,\vec{j} + y\,\vec{k}$ and

$S = \{(x,y,z) \mid x^2+y^2+z^2=4,\ z \geq 0 \text{ and } x^2+y^2 \leq 1\}$ with upward orientation.



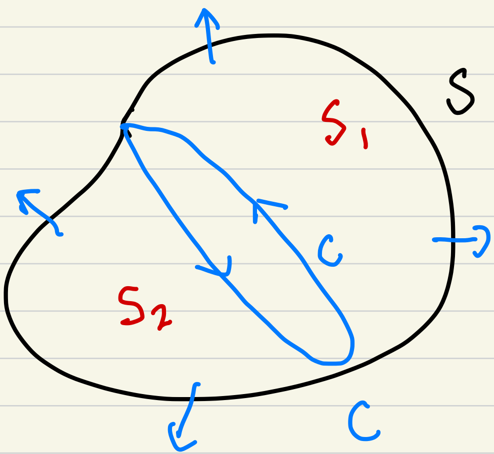
Sol :

Ex: $\vec{F}(x,y,z) = (e^{xy}\cos z,\ x^2+y^2,\ e^z)$. S: $x = \sqrt{1-y^2-z^2}$ where $\vec{n}$ has positive x-component. Compute $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$.



Sol

Remark: If S is a closed surface (the boundary surface of a solid) and $\text{curl}\,\vec{F}$ is continuous, then $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = 0$.



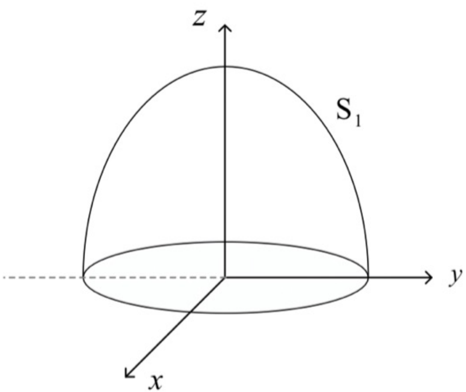
$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_{S_1} \text{curl}\,\vec{F}\cdot d\vec{S} + \iint_{S_2} \text{curl}\,\vec{F}\cdot d\vec{S}$$

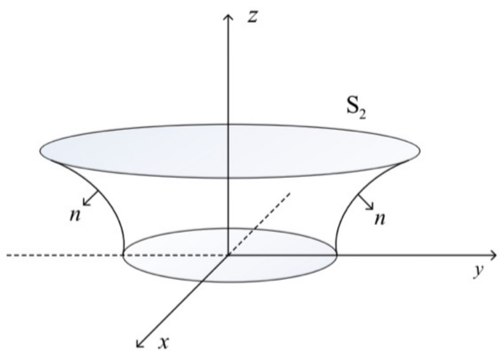
7. (14 points) Let $\boldsymbol{F} = (x-y)\boldsymbol{i} + (y-z)\boldsymbol{j} + (z-x)\boldsymbol{k}$ be a vector field on $\mathbb{R}^3$.

(a) (2 points) Compute $\operatorname{curl} \boldsymbol{F}$ on $\mathbb{R}^3$.

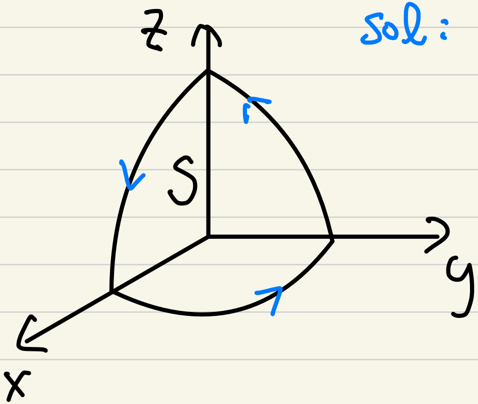
(b) (6 points) Let S_1 be a parametric surface given by $\boldsymbol{r}(r,\theta) = r\cos\theta\boldsymbol{i} + 2r\sin\theta\boldsymbol{j} + (9-r^2)\boldsymbol{k}$ for $r \in [0,3]$ and $\theta \in [0,2\pi]$, which comes with the standard orientation given by the normal vector $\boldsymbol{r}_r \times \boldsymbol{r}_\theta$. Find the flux of $\operatorname{curl} \boldsymbol{F}$ across S_1.

(c) (6 points) Let S_2 be a surface defined by the equation $\dfrac{x^2}{9} + \dfrac{y^2}{36} - z^2 = 1$ for $z \in [0,1]$ and endowed with the orientation given by the downward normal vector. Find the flux of $\operatorname{curl} \boldsymbol{F}$ across S_2.





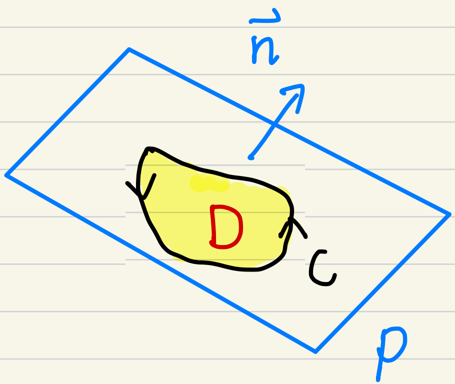
Ex: $\vec{F}(x,y,z) = (y, z, x)$, $S: z = 1-x^2-y^2$ in the first octant with upward orientation. Compute $\int_{\partial S} \vec{F} \cdot d\vec{r}$.



Sol:

Ex: There is a plane P in the space with $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$, $|\vec{n}| = 1$, and C is a smooth closed curve on P with orientation inherited from $\vec{n}$. Show that $\oint_C bz\,dx + cx\,dy + ay\,dz$ depends only on the area enclosed by C.

Sol:



Stokes' Theorem

- We now use Stokes' Theorem to throw some light on the meaning of the curl vector.
- Suppose that C is an oriented closed curve and $\vec{v}$ represents the velocity field in fluid flow. Consider the line integral
$$\int_C \vec{v} \cdot d\vec{r} = \int_C \vec{v} \cdot \vec{T}\, ds$$
- It is a measure of the tendency of the fluid to move around C and is called the **circulation of $\vec{v}$ around C**.

Stokes' Theorem

- Now let $P_0(x_0, y_0, z_0)$ be a point in the fluid and let S_a be a small disk with radius a and center P_0. Then $\text{curl } \vec{v}(P) \approx \text{curl } \vec{v}(P_0)$ for all points P on S_a because $\text{curl } \vec{v}$ is continuous.
- Thus, by Stokes' Theorem, we get the following approximation to the circulation around the boundary circle C_a :

Stokes' Theorem

- $$\int_{C_a} \vec{v} \cdot d\vec{r} = \iint_{S_a} \text{curl } \vec{v} \cdot d\vec{S}$$
  $$\approx \iint_{S_a} \text{curl } \vec{v}(P_0) \cdot \vec{n}(P_0) dS = \text{curl } \vec{v}(P_0) \cdot \vec{n}(P_0) \pi a^2$$
  $$\text{curl } \vec{v}(P_0) \cdot \vec{n}(P_0) = \lim_{a \to 0} \frac{1}{\pi a^2} \int_{C_a} \vec{v} \cdot d\vec{r}$$
  It shows that $\text{curl } \vec{v} \cdot \vec{n}$ is a measure of the rotating effect of the fluid about the axis $\vec{n}$.

(1 point) **Library/FortLewis/Calc3/20-3-Curl/HGM4-20-3-20-Curl.pg**
Three small circles
C_1,
C_2, and
C_3, each with radius
0.1 and centered at the origin are in the xy-, yz-, and xz-planes, respectively. The circles are oriented cour
viewed from the positive z-, x-, and y-axes, respectively. A vector field
$\vec{F}$ has circulation around
C_1 of
0.04π, around
C_2 of
0.4π, and around
C_3 of
4π. Estimate
$\operatorname{curl}(\vec{F})$ at the origin.

Stokes' Theorem

- We know that $\vec{F}$ is conservative if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C. Given C, suppose we can find an orientable surface S whose boundary is C. curl $\vec{F} = \vec{0}$
- If $\vec{F}$ is irrotational, then for every closed path C

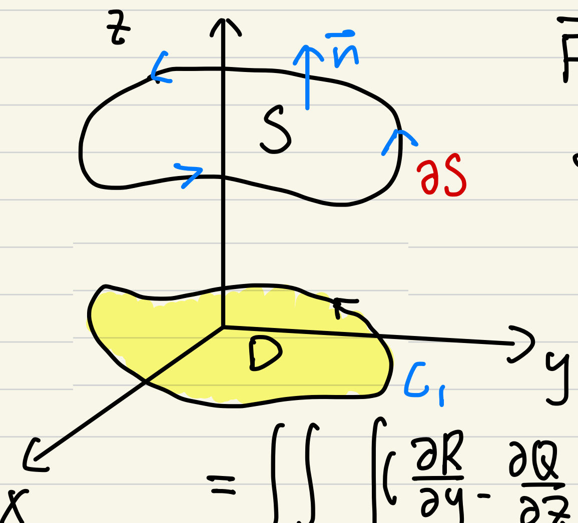
$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S} = \iint_S \vec{0} \cdot d\vec{S} = 0$$

- Hence, $\vec{F}$ is conservative.

Pf of Stokes' Theorem

Suppose that $S: z = z(x,y)$, $(x,y) \in D \subset xy$-plane.

$\partial D = C_1 : \vec{r}_1(t) = (x(t), y(t))$, $a \le t \le b$, oriented counterclockwise



$\vec{F} = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$.

$S: \vec{r}(x,y) = (x, y, z(x,y))$, $(x,y) \in D$

$\vec{r}_x \times \vec{r}_y = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right)$.

Then $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$

$$= \iint_D \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}\right] \cdot (\vec{r}_x \times \vec{r}_y)\, dx\, dy$$

$\partial S: \vec{r}(t) = (x(t), y(t), z(x(t), y(t)))$, $a \le t \le b$.

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \int_a^b (P, Q, R) \cdot \vec{r}'(t)\, dt = \int_a^b (P, Q, R) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \right) dt$$

Review

- State the Stokes' Theorem. What is the relation between the orientation of the surface and the orientation of the boundary curve in the Stokes' Theorem?