First-Order Linear Differential Equations

Section 9.5

Outline

- First-Order Linear Differential Equations
- Integrating Factors

First-Order Linear Differential Equations

- A first-order **linear** differential equation is one that can be put into the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions on a given interval.

Motivation

Ex: Solve $x y'(x) + y(x) = 2x$.

First-Order Linear Differential Equations

- We try to find $I(x)$ so that the left side of the equation , when multiplied by $I(x)$, becomes the derivative of the product $I(x)y$:

$$I(x)(y' + P(x)y) = (I(x)y)'$$

- If we can find such a function $I(x)$, then the equation becomes

$$(I(x)y)' = I(x)Q(x)$$

First-Order Linear Differential Equations

- Such $I(x)$ is called an integrating factor.
- To find such an $I(x)$, we observe that

$$I(x)y' + I(x)P(x)y = (I(x)y)' = I'(x)y + I(x)y'$$

which means that $I(x)P(x) = I'(x)$.

- Hence, we can choose $I(x)$ as

$$I(x) = e^{\int P(x)dx}$$

Ex: Solve for $I(x)$ such that

$$I(x)\,(y'(x) + P(x)\,y(x)) = (I(x)\,y(x))'$$

sol: $I\,y' + I\,P\,y = I\,y' + I'\,y.$

$\Rightarrow \quad I(x)\,P(x) = I'(x).$

First-Order Linear Differential Equations

- Conclusion:

To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the **integrating factor** $I(x) = e^{\int P(x)\,dx}$ and integrate both sides.

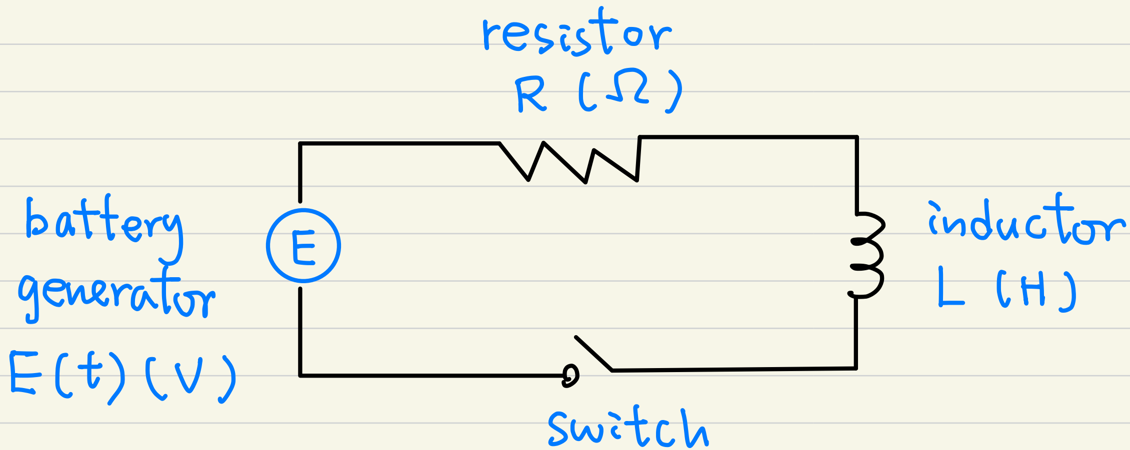
Ex: Solve $\frac{dy}{dx} + 2xy = 2x^3$.

Ex: Solve $x^2 y' + xy = \ln x$ for $x > 0$ and $y(1) = 2$.

Ex: Solve $\sec x \cdot y'(x) + y(x) = 3$, $y(0) = 1$.

Application to Electric Circuits.

In a simple electric circuit:



Let $I(t)$ be the current at time t (A).

$$\boxed{L\frac{dI}{dt} + RI = E(t).}$$

Ex: Suppose that $R = 12\,\Omega$, $L = 4\,H$ and $E(t) = 60 \sin(10t)\ V$.
If $I(0) = 0$ and the switch is closed at $t = 0$, find $I(t)$.

Integral Equations

Ex: $y(x) = 3 + \int_1^x t - \frac{1}{t} y(t)\, dt$

Ex: $y(x) = 1 + \int_0^{x^2} t\, y(\sqrt{t})\, dt$, for $x \geq 0$.

Review

- What is a First-order linear differential equation?
- What is an integrating factor for a First-order linear differential equation?
- How does an integrating factor help us to solve the equation?