# Surface Integrals

Section 16.6, 16.7

## Outline

- Parametric Surfaces
  - Tangent Planes
  - Surface Area
- Surface Integrals
  - Surface Integral of a Scalar Function
  - Oriented Surfaces
  - Surface Integrals of Vector Fields

## Surface Integrals

- Suppose $f$ is a function of three variables whose domain includes a surface $S$.
- We will define the surface integral of $f$ over $S$ in such a way that, in the case where $f = 1$, the value of the surface integral is equal to the surface area of $S$.
- Suppose that a surface $S$ has a vector equation

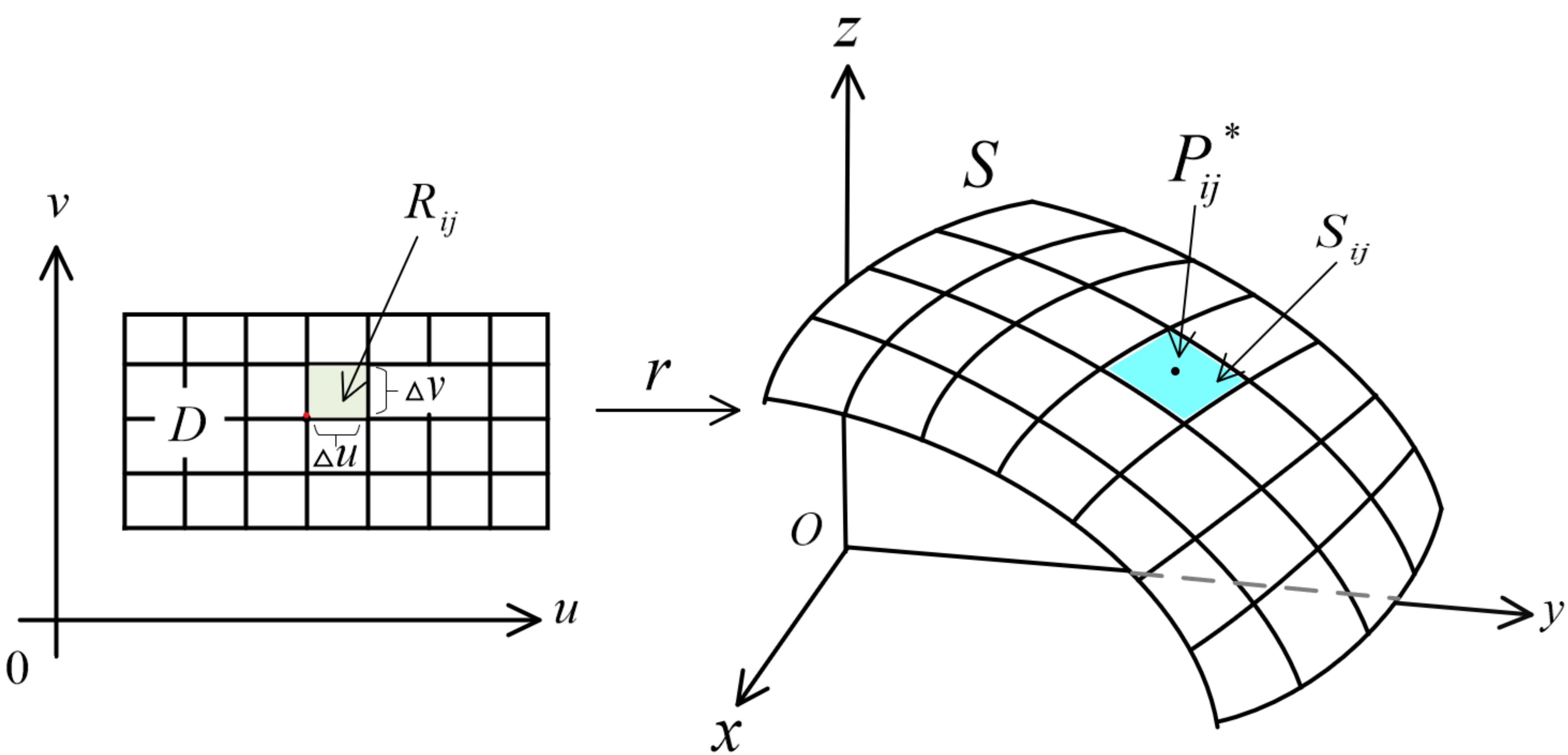
$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

$$(u,v) \in D$$

## Surface Integrals

- We first assume that the parameter domain $D$ is a rectangle and we divide it into subrectangles $D_{ij}$ with dimensions $\Delta u$ and $\Delta v$.
- Then the surface $S$ is divided into corresponding patches $S_{ij}$.
- We evaluate $f$ at a point $P_{ij}^*$ in each patch, multiply by the area $\Delta S_{ij}$ of the patch, and form the Riemann sum
$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*) \Delta S_{ij}$$

# Surface Integral



## Surface Integrals

- Then we take the limit as the number of patches increases and define the **surface integral of $f$ over the surface** $S$ as

$$\iint_S f(x,y,z)\,dS = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}$$

- We know that $\Delta S_{ij} \approx |\vec{r}_u \times \vec{r}_v|\Delta u \Delta v$ . Hence,

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|\,dA$$

## Surface Integrals

(x, y) ∈ D

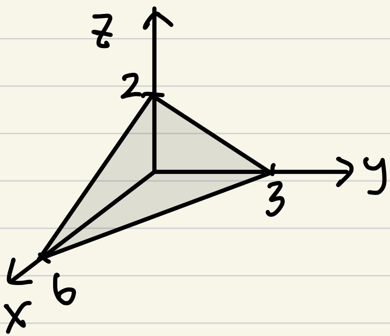
- Any surface $S$ with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations: $x = x$ , $y = y$, $z = g(x, y)$.
- Because $|\vec{r}_x \times \vec{r}_y| = \sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1}$ , in this case,

$$\iint_S f(x, y, z)dS =$$

$$\iint_D f(x, y, g(x, y))\sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1}\, dA$$

Ex: Compute $\iint_S y+1 \, dS$ where $S$ is the part of the plane $x+2y+3z=6$ in the first octant.

Sol:

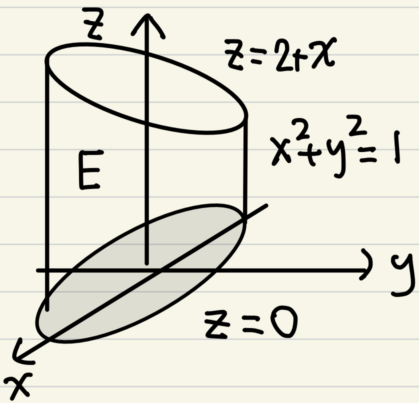


Ex: $\iint_S z\,dS$, where $S$ is the upper sphere, $x^2+y^2+z^2=a^2$, $z\geq 0$.

sol:

Ex: $\iint_S z \, dS$, where $S$ is the boundary surface of the solid $E$.

Sol:



Ex:

Compute the surface integral

$$\iint_S xz \, \mathrm{d}S,$$

where $S$ is the part of the cone $z = \sqrt{x^2 + y^2}$ inside the circular cylinder $x^2 + y^2 = 2x$.

Ex:

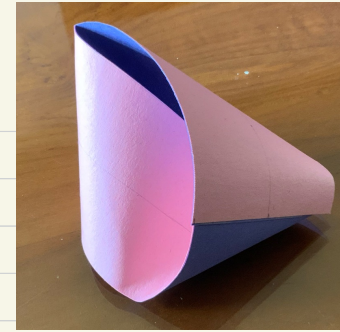
Evaluate the surface integral $\iint_S \sqrt{x^2+y^2}\,\mathrm{d}S$, where $S$ is the part of the surface $z=\tan^{-1}\left(\frac{y}{x}\right)$ inside the circular cylinder $x^2+y^2=1$ and in the first octant.

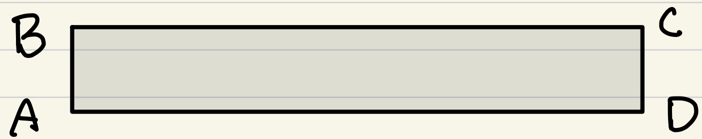
## Oriented Surfaces

- To define surface integrals of vector fields, we need to rule out nonorientable surfaces such as the Möbius strip. A Möbius strip really has only one side.

- From now on we consider only orientable (two-sided) surfaces.

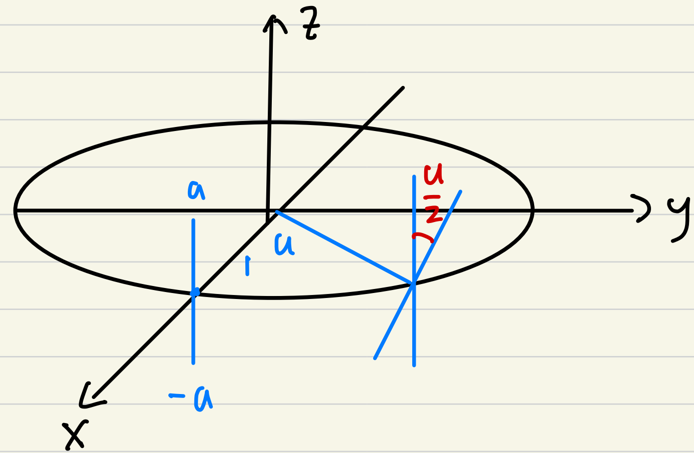
# Möbius Strip (Nonorientable parametric surface)






$$\vec{r}(u,v) = \left( \left(1 + v\sin\frac{u}{2}\right)\cos u,\ \left(1+v\sin\frac{u}{2}\right)\sin u,\ v\cos\frac{u}{2} \right)$$

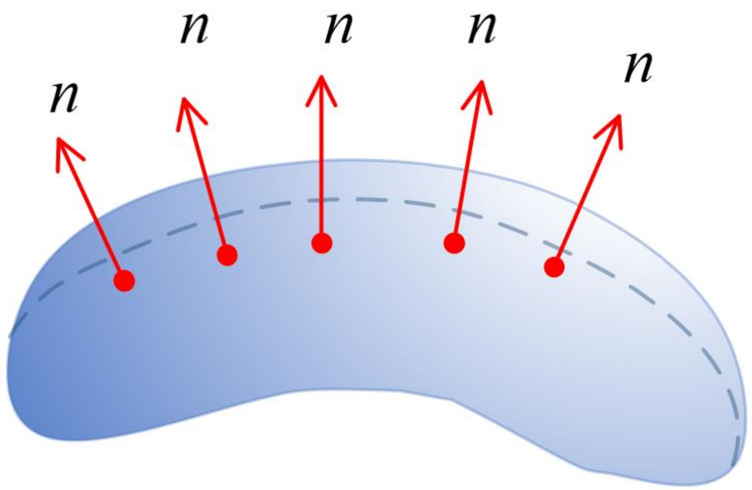
$0 \le u \le 2\pi, \quad -a \le v \le a.$

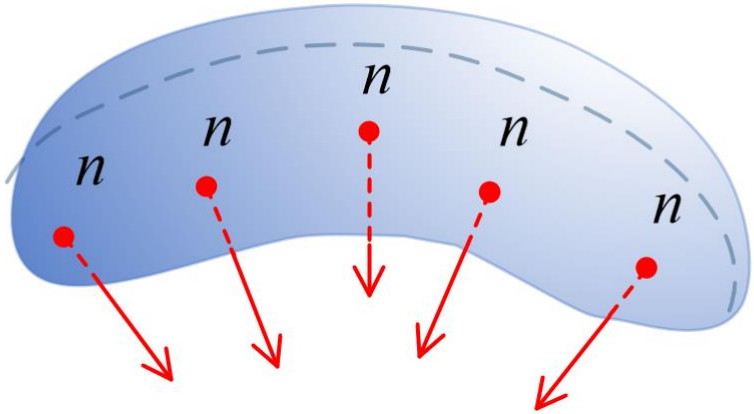


## Oriented Surfaces

- We start with a surface $S$ that has a tangent plane at every point $(x, y, z)$ on $S$ (except at any boundary point). There are two unit normal vectors $\vec{n}_1$ and $\vec{n}_2 = -\vec{n}_1$ at $(x, y, z)$ .
- Definition: If it is possible to choose a unit normal vector $\vec{n}$ at every such point $(x, y, z)$ so that $\vec{n}$ *varies continuously over* $S$ , then $S$ is called an **oriented surface** and the given choice of $\vec{n}$ provides $S$ with an **orientation**.

# Oriented Surfaces



## Oriented Surfaces

- For a surface $z = g(x, y)$, a natural orientation given by the unit normal vector

$$\vec{n} = \frac{-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}}{\sqrt{1 + (\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2}}$$

- Since the $\vec{k}$-component is positive, this gives the *upward* orientation of the surface.

## Oriented Surfaces

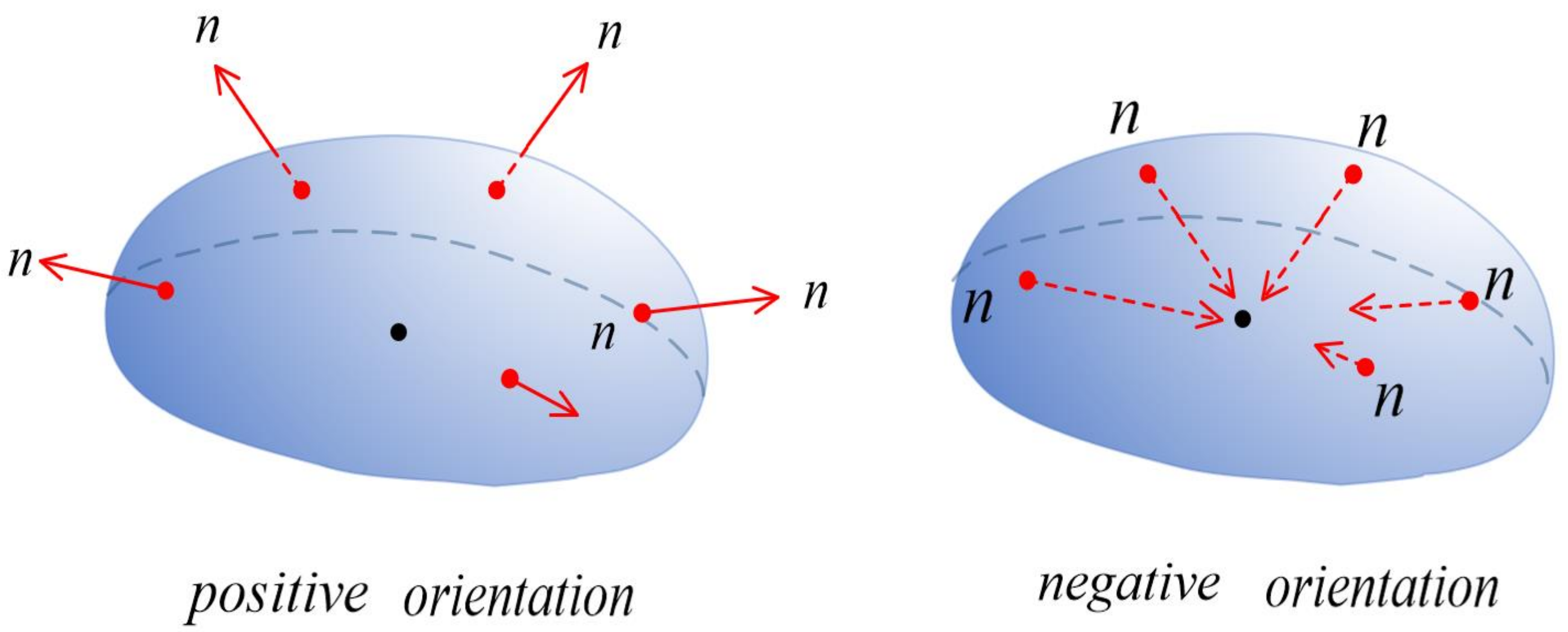
- If $S$ is a smooth orientable surface given in parametric form by a vector function $\vec{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector
$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$
and the opposite orientation is given by $-\vec{n}$.

## Oriented Surfaces

- For a **closed surface**, that is, a surface that is the boundary of a solid region $E$ , the convention is that the **positive orientation** is the one for which the normal vectors point *outward* from $E$ , and inward-pointing normal vectors give the negative orientation.

# Oriented Surfaces

- Closed surface:

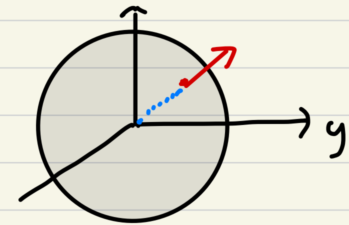


How do we choose the "orientation" of $S$.

| Surface | Orientation |
| --- | --- |
| $z = g(x,y)$ | |
| $S: \vec{r}(u,v)$ | |
| $S$: a closed surface, the boundary of a solid region. | |

Ex: $S$ : the sphere with radius $a>0$, center $(0,0,0)$.

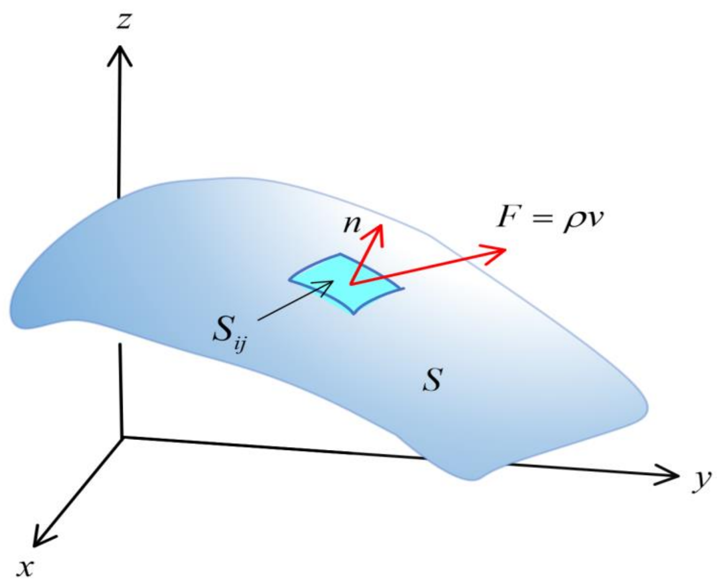
Find the outward $\vec{n}(x,y,z)$.




Sol:

# Surface Integrals of Vector Fields

- Suppose that $S$ is an oriented surface with unit normal vector $\vec{n}$, and imagine a fluid with density $\rho(x, y, z)$ and velocity field $\vec{v}(x, y, z)$ flowing through $S$. Then the rate of flow (mass per unit time) per unit area is $\rho\vec{v}$.



## Surface Integrals of Vector Fields

- If we divide $S$ into small patches $S_{ij}$, Then $S_{ij}$ is nearly planar and so we can approximate the mass of fluid per unit time crossing $S_{ij}$ in the direction of the normal $\vec{n}$ by the quantity $(\rho\vec{v}\cdot\vec{n})A(S_{ij})$ .

## Surface Integrals of Vector Fields

- By summing these quantities and taking the limit we get,

$$\iint_S \rho\vec{v}\cdot\vec{n}\,dS = \iint_S \rho(x,y,z)\vec{v}(x,y,z)\cdot\vec{n}(x,y,z)\,dS$$

  and this is interpreted physically as the rate of flow through $S$ .
- A surface integral of this form occurs frequently in physics, and is called the *surface integral* (or *flux integral*) of a vector field over $S$.

# Surface Integrals of Vector Fields

- Definition: If $\vec{F}$ is a continuous vector field defined on an oriented surface $S$ with unit normal vector $\vec{n}$, then the **surface integral of $\vec{F}$ over** $S$ is defined as $\iint_S \vec{F} \cdot \vec{n}\, dS$ and denoted by $\iint_S \vec{F} \cdot d\vec{S}$ .
- The integral is also called the **flux** of $\vec{F}$ across $S$ .

## Surface Integrals of Vector Fields

- If $S$ is given by a vector function $\vec{r}(u,v)$, we have

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\, dS$$

$$= \iint_D \vec{F}(\vec{r}(u,v))\cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\, |\vec{r}_u \times \vec{r}_v|\, dA$$

$$= \iint_D \vec{F}\cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

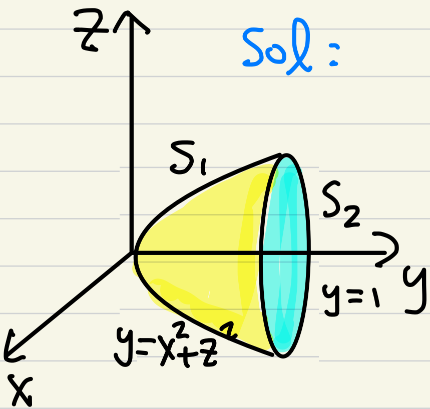
Ex: If $S: z = g(x,y)$, $(x,y) \in D$, $\vec{F}(x,y,z) = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$,

then $\iint\limits_S \vec{F} \cdot d\vec{S} =$

## Surface Integrals of Vector Fields

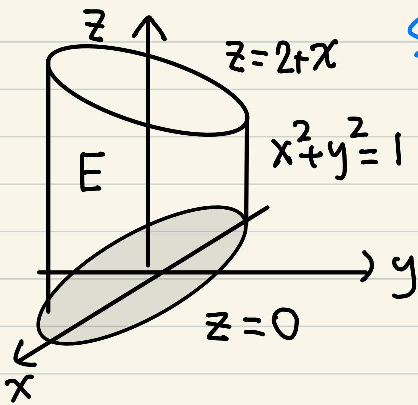
- In the case of a surface $S$ given by a graph $z = g(x, y)$, for $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$,
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$
- This formula assumes the upward orientation of $S$; for a downward orientation we multiply by $-1$.

Ex: $\vec{F}(x,y,z) = y\vec{j} - z\vec{k}$. $S$ consists of paraboloid $y = x^2 + z^2$, $0 \le y \le 1$, and the disc $x^2 + z^2 \le 1$, $y = 1$, with outward orientation. Compute $\iint_S \vec{F} \cdot d\vec{S}$.




Sol:

Ex: Compute $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = (y, x, z)$ and $S$ is the boundary of the solid $E$ with outward orientation.




Sol:

Ex: Find the flux of the electric field $\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \frac{Q\,\vec{x}}{|\vec{x}|^3}$ across the sphere $S: x^2+y^2+z^2=a^2$ (with outward orientation).

Sol:

Suppose

$\mathbf{F}$ is a radial force field, $\Leftrightarrow$ $\vec{F}(\vec{x}) = f(|\vec{x}|)\,\vec{x}$

$S_1$ is a sphere of radius

$5$ centered at the origin, and the flux integral

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = 9.$$

Let

$S_2$ be a sphere of radius

$40$ centered at the origin, and consider the flux integral

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

(B) If the magnitude of

$\mathbf{F}$ is inversely proportional to the cube of the distance from the origin, what is the value of

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}?$$

## Review

- How do we parametrize a surface?
- Given a smooth parametrization of a surface, how do we compute the tangent planes and the area of the surface?
- How do we compute the surface integral of a scalar function?
- What is an oriented surface?
- How do we compute surface integrals of vector fields over an oriented surface?