Double Integrals

Section 15.1-15.3

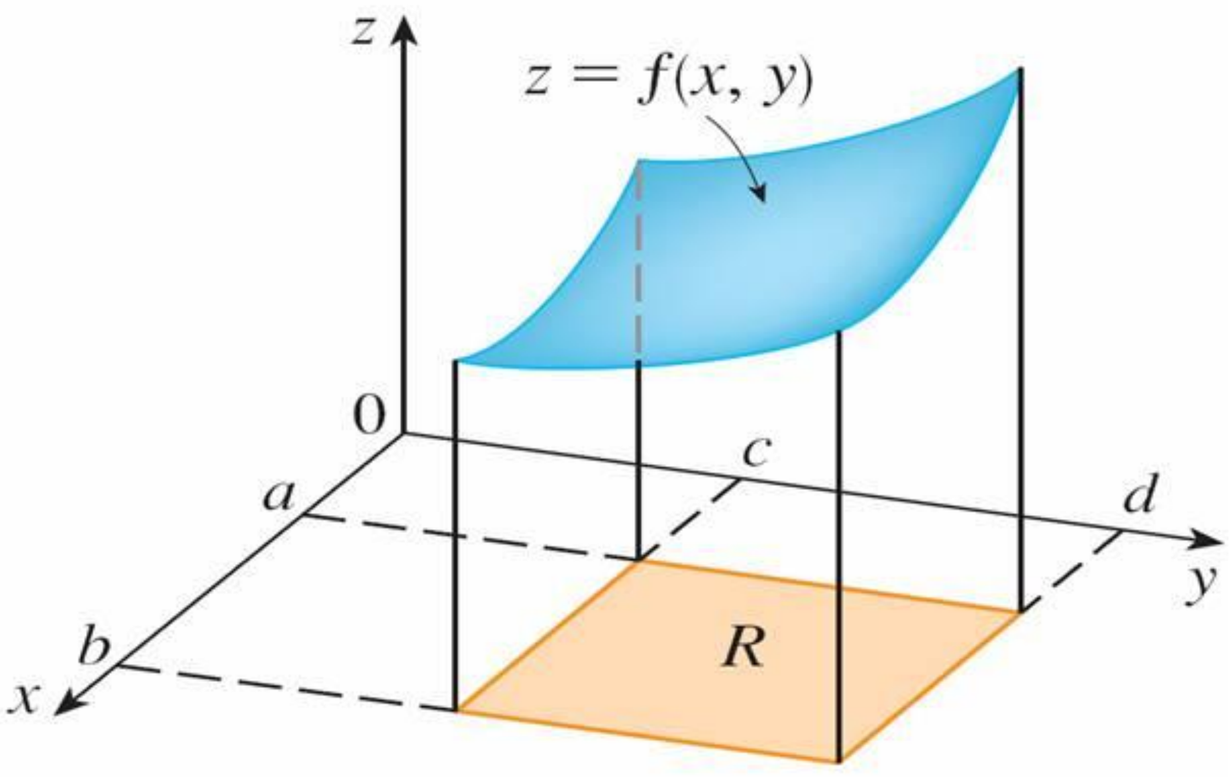
Outline

- Double Integrals over Rectangles
- Iterated Integrals
  - Fubini's Theorem
- Double Integrals over General Regions
  - Type I Regions
  - Type II Regions
- Double Integrals in Polar Coordinates

Double Integrals over Rectangles

- We consider a function f of two variables defined on a closed rectangle
  $$R = \{(x, y) \in R^2 | a \leq x \leq b, c \leq y \leq d\}$$
  and suppose that $f(x, y) \geq 0$.
- Let S be the solid that lies above R and under the graph of f , that is,
  $$S = \{(x, y, z) \in R^3 | 0 \leq z \leq f(x, y), (x, y) \in R\}$$
- We want to find the volume of S .

Double Integrals over Rectangles



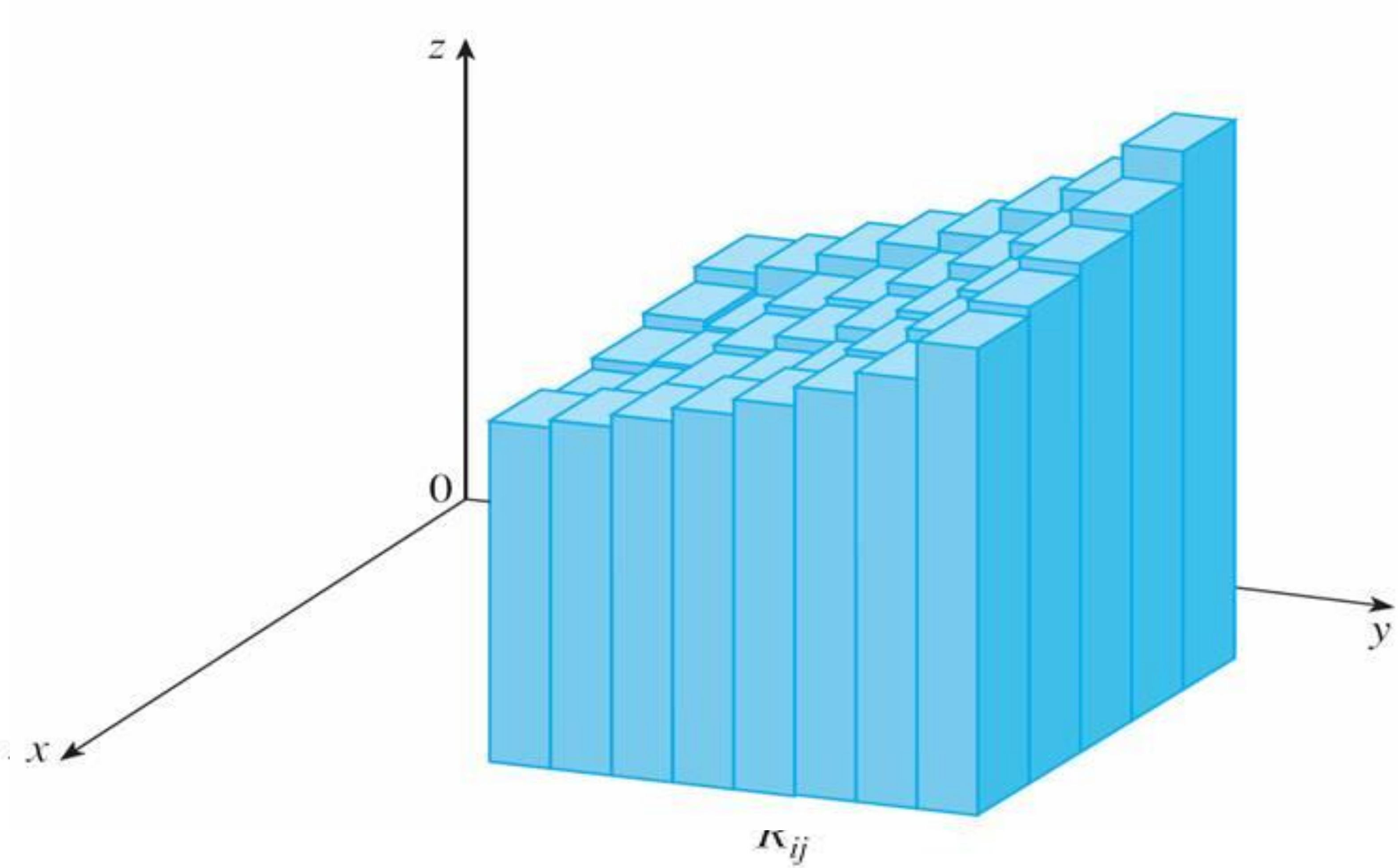
Double Integrals over Rectangles

- The first step is to divide the rectangle R into subrectangles (by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{m}$ and dividing $[c, d]$ into n subintervals $[y_{i-1}, y_i]$ of equal width $\Delta y = \frac{d-c}{n}$).
- Hence, there are subrectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ each with area $\Delta A = \Delta x \Delta y$.

Double Integrals over Rectangles

- If we choose a **sample point** (x_{ij}^*, y_{ij}^*) in each R_{ij} , then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box (or "column") with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$ which has volume $f(x_{ij}^*, y_{ij}^*)\Delta A$.
- Thus we can approximate the volume of S by $V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$.

Double Integrals over Rectangles



Double Integrals over Rectangles

- Our intuition is that the approximation becomes better as m and n become larger and so we would expect that
$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$
- We use the above expression to define the **volume** of the solid S that lies under the graph of f and above the rectangle R.

Double Integrals over Rectangles

- Limits of this type occur frequently, not just in finding volumes but in a variety of other situations even when f is not a positive function. So we make the following definition.

5 Definition The **double integral** of f over the rectangle R is

$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

if this limit exists.

Double Integrals over Rectangles

- The precise meaning of the previous limit is that for every number $\epsilon > 0$ there is an integer N such that

$$\left| \iint_R f(x,y)\,dA - \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A \right| < \epsilon$$

  for all integers m and n greater than N and for any choice of sample points (x_{ij}^*, y_{ij}^*) in R_{ij} .
- The sum in the definition is called a **double Riemann sum**.

Double Integrals over Rectangles

- A function f is called **integrable** if the limit exists.
- Theorem: If f is continuous on R , then f is integrable over R .
- Theorem: If f is bounded on R and f is continuous there except on a finite number of smooth curves, then f is integrable over R.

Iterated Integrals

- Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$
- We use the notation $\int_c^d f(x, y)dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called *partial integration with respect to y.*
- Now $\int_c^d f(x, y)dy$ is a number that depends on the value of x, so it defines a function of x:

$$A(x) = \int_c^d f(x, y)dy$$

Iterated Integrals

- If we now integrate the function $A(x)$ with respect to x from $x = a$ to $x = b$, we get
$$\int_a^b A(x)\ dx = \int_a^b [\int_c^d f(x,y)dy]dx$$
- This integral is called an **iterated integral**.
- Similarly, we can define another iterated integral
$$\int_c^d \int_a^b f(x,y)dxdy = \int_c^d [\int_a^b f(x,y)dx]dy$$

Ex: Compute $\int_0^2 \int_0^1 x^2y + y \, dx \, dy$, $\int_0^1 \int_0^2 x^2y + y \, dy \, dx$

Iterated Integrals

- The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

$\boxed{4}$ **Fubini's Theorem** If f is continuous on the rectangle $R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

- Geometric Meanings of Fubini's Theorem

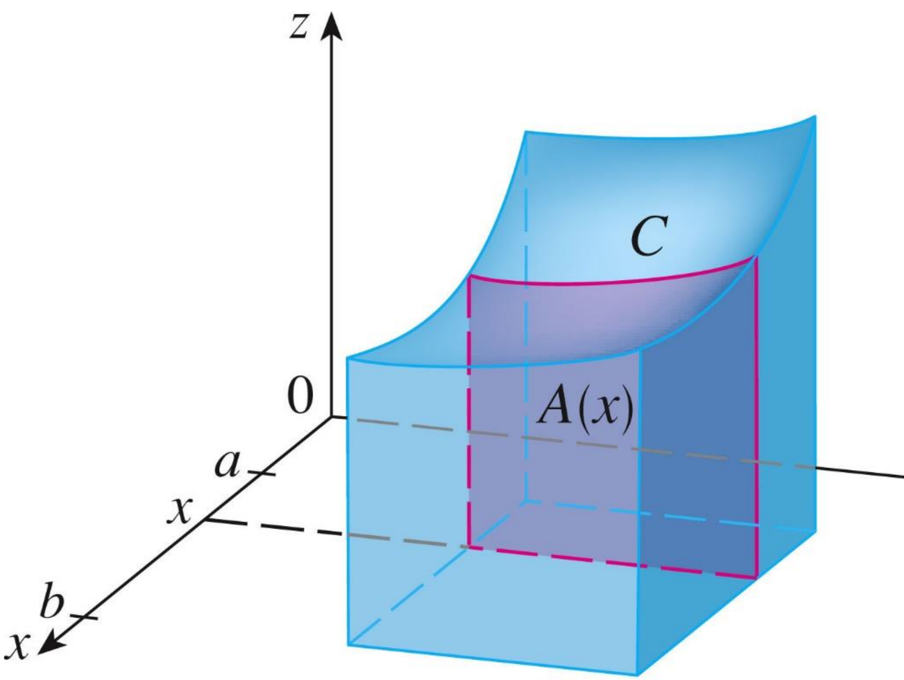


FIGURE 1

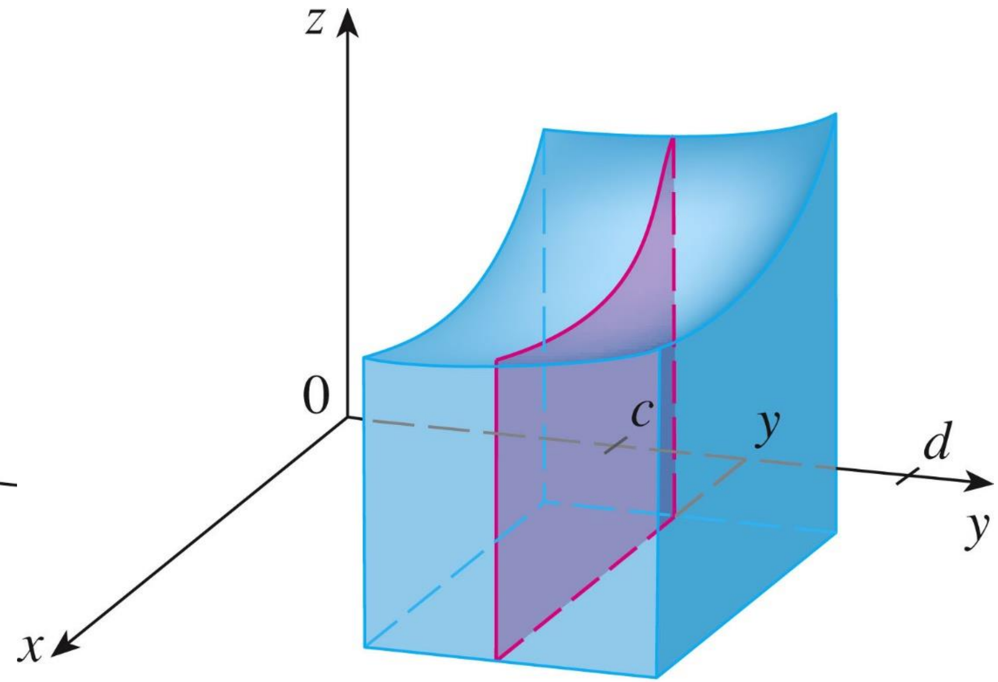


FIGURE 2

Ex: Compute $\int_0^1 \int_0^2 x e^{xy} \, dx \, dy$.

Ex: Compute $\int_0^1 \int_1^2 \frac{x}{(x^2+y^2)^2}\, dy\, dx$

Ex: Find the volume of the solid S bounded by the surface $x^2+2y^2+z=4$, the planes $x=1$, $x=-1$, $y=1$, the xz plane and the xy plane.

Ex: Compute $I = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx$, where $a, b > 0$.

(Note that $\int_a^b e^{-xy}\,dy = \frac{e^{-ax} - e^{-bx}}{x}$.)

Ex: Compute $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\, dx\, dy$ and $\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\, dy\, dx$.