Derivatives of Logarithmic and Inverse Trigonometric Functions

Section 3.6

Outline

- Inverse Trigonometric Functions
- Derivatives of Inverse Trigonometric Functions
- Logarithmic Functions
- Derivatives of Logarithmic Functions
- Logarithmic Differentiation
- The Number e as a Limit

Inverse Trigonometric Functions

- Define $\arcsin x$, $\sin^{-1} x$:

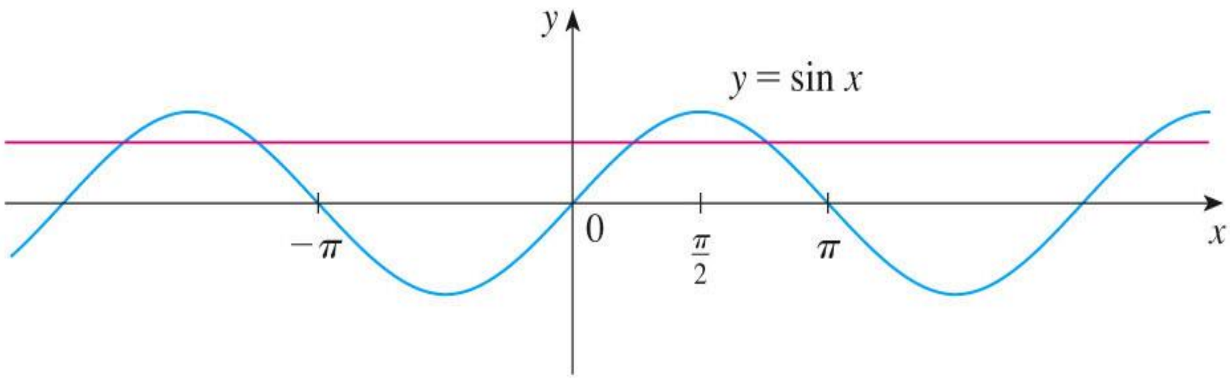


FIGURE 17

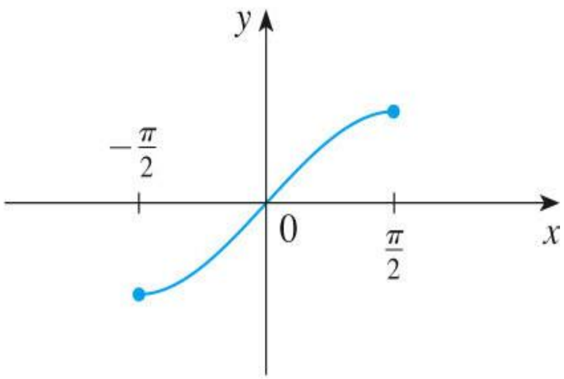


FIGURE 18 $y = \sin x, -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

Inverse Trigonometric Functions

- $y = \sin^{-1} x \Leftrightarrow \sin y = x$ and $y \in [-\pi/2, \pi/2]$
- Ex: $\sin^{-1}(-\frac{1}{2}) =$

  $\sin^{-1}(\sin(\pi)) =$

  $\tan(\sin^{-1}(\frac{1}{4})) =$

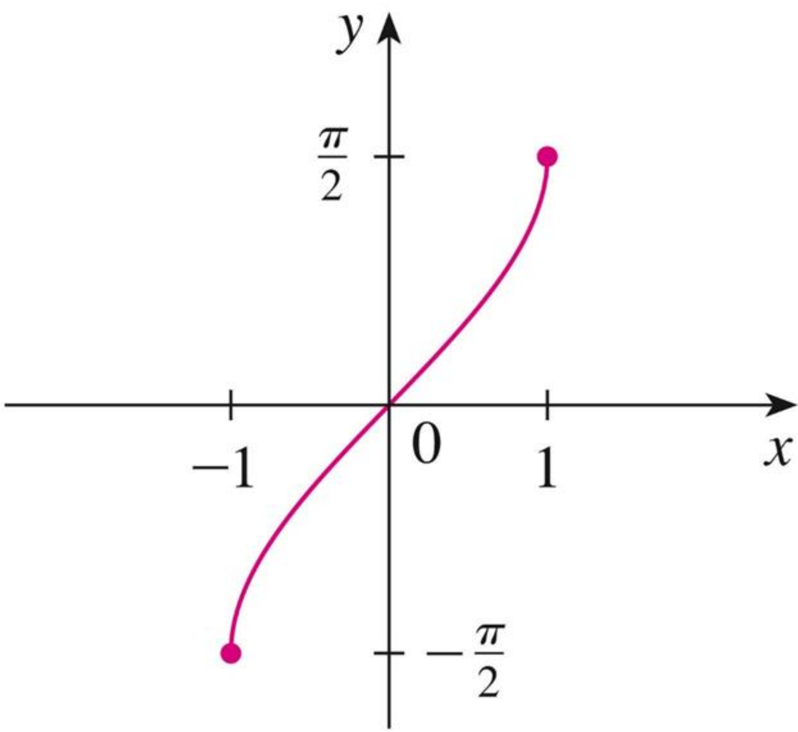


FIGURE 20
$y = \sin^{-1} x = \arcsin x$

Inverse Trigonometric Functions

- Define $\arccos x$, $\cos^{-1} x$:
- $y = \cos^{-1} x \Leftrightarrow \cos y = x$ and $y \in [0, \pi]$

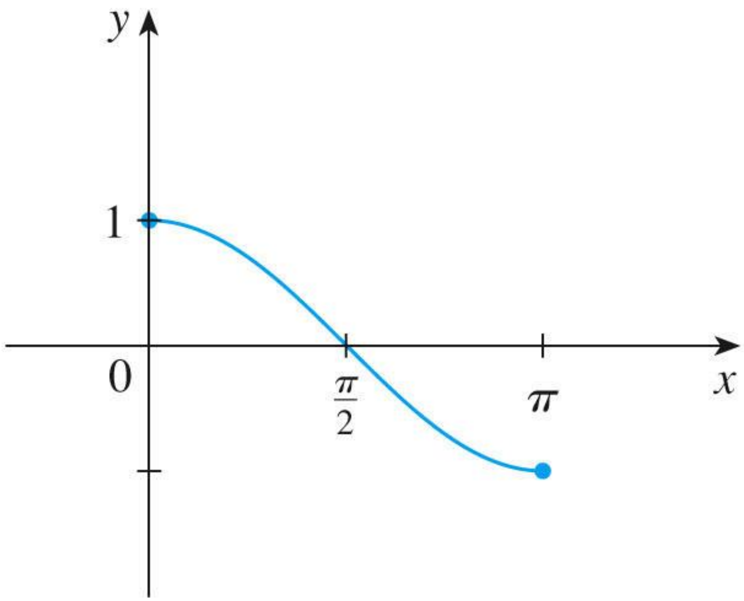


FIGURE 21

$y = \cos x,\ 0 \leqslant x \leqslant \pi$

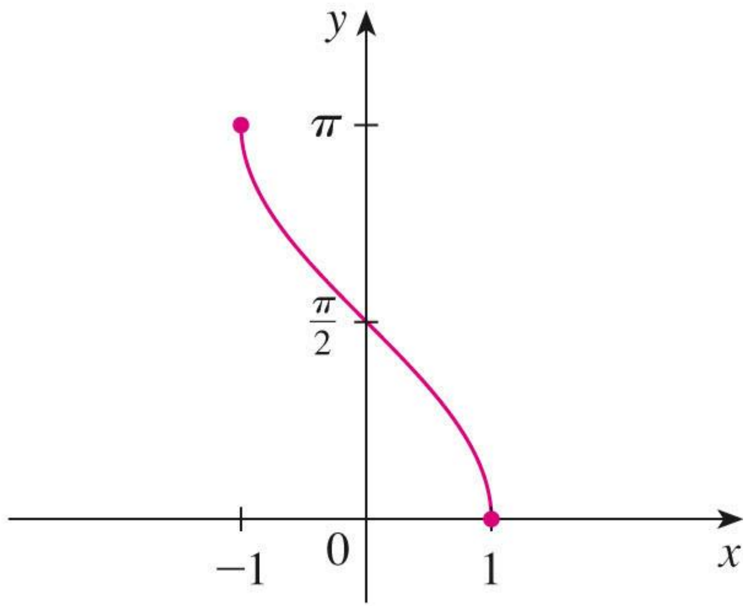


FIGURE 22

$y = \cos^{-1} x = \arccos x$

Inverse Trigonometric Functions

- Define $\arctan x,\ \tan^{-1} x$:
  $y = \tan^{-1} x \Leftrightarrow \tan y = x$
  and $y \in (-\pi/2, \pi/2)$

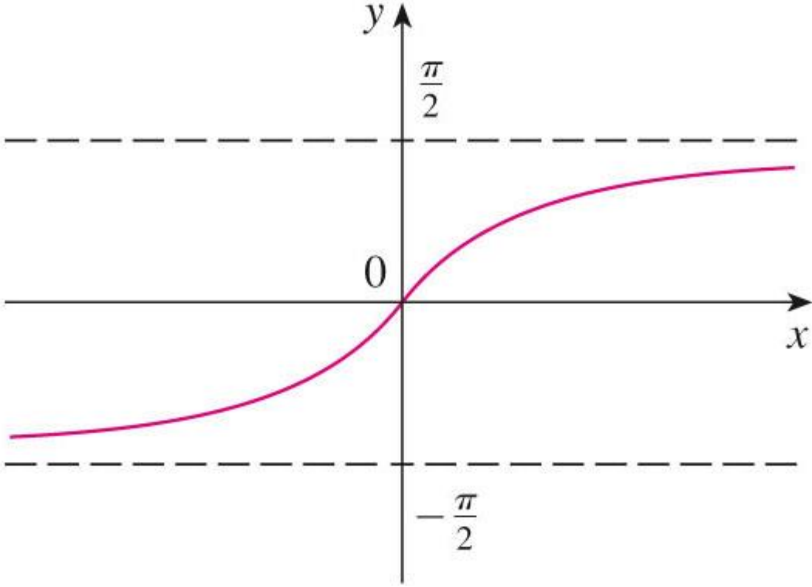


FIGURE 25
$y = \tan^{-1} x = \arctan x$

Inverse Trigonometric Functions

- Define $\sec^{-1} x$: (The definition is not universally agreed upon.)

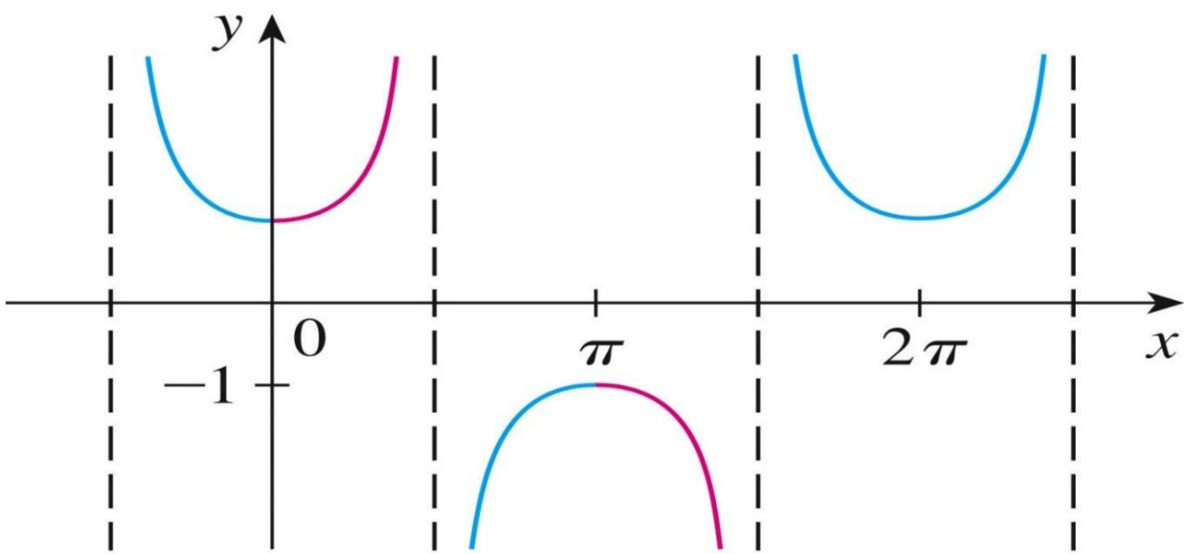
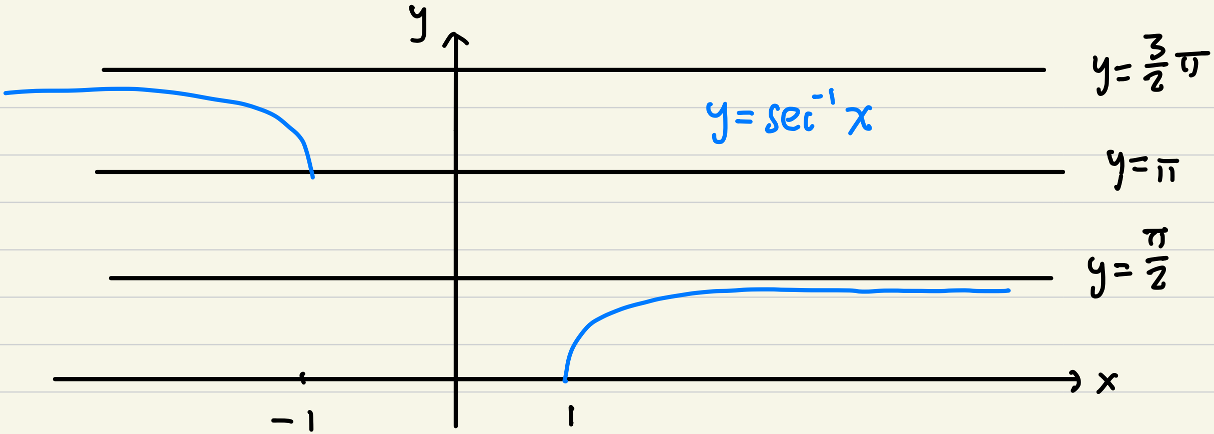


FIGURE 26

$y = \sec x$

$y = \sec^{-1} x \iff \sec y = x$ and

$y \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3}{2}\pi)$

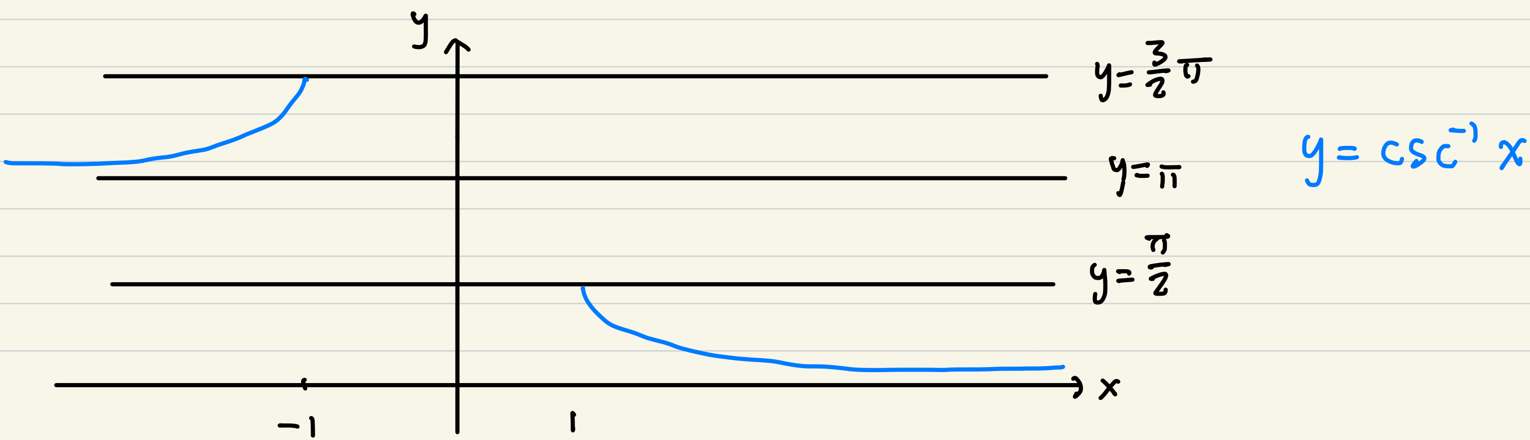


$\lim_{x\to\infty} \sec^{-1}x =$

$\lim_{x\to-\infty} \sec^{-1}x =$

Define $\csc^{-1}x$

$y=\csc^{-1}x \iff \csc y = x$ and $y\in(0,\frac{\pi}{2}]\cup(\pi,\frac{3}{2}\pi]$



Ex: Derive $\frac{d}{dx} \sin^{-1} x$.

Ex: Derive $\frac{d}{dx}\cos^{-1}x$.

Ex: Derive $\frac{d}{dx} \tan^{-1} x$.

Ex: Derive $\frac{d}{dx} \sec^{-1} x$.

Derivatives of Inverse Trigonometric Functions

- Applications: Derivatives of inverse trigonometric functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

Ex: Compute $\frac{d}{dx}\left(\sin^{-1}(\sqrt{x})\right)^2$.

Ex: Compute $\frac{d}{dx} \tan^{-1}(ax)$ where $a \in \mathbb{R}$, $a \neq 0$.

Ex: $\frac{d}{dx} \tan^{-1}\left(x - \sqrt{1+x^2} \right)$

Logarithmic Functions

- If $a > 0$ and $a \neq 1$, the exponential function $f(x) = a^x$ is either increasing or decreasing and so it is *one-to-one* by the Horizontal Line Test. It therefore has an inverse function which is called the **logarithmic function with base** a and is denoted by $\log_a x$.

Properties of Logarithmic Functions

$$\log_a x = y \iff a^y = x$$

$$\log_a(a^x) = x \quad \text{for every } x \in \mathbb{R}$$

$$a^{\log_a x} = x \quad \text{for every } x > 0$$

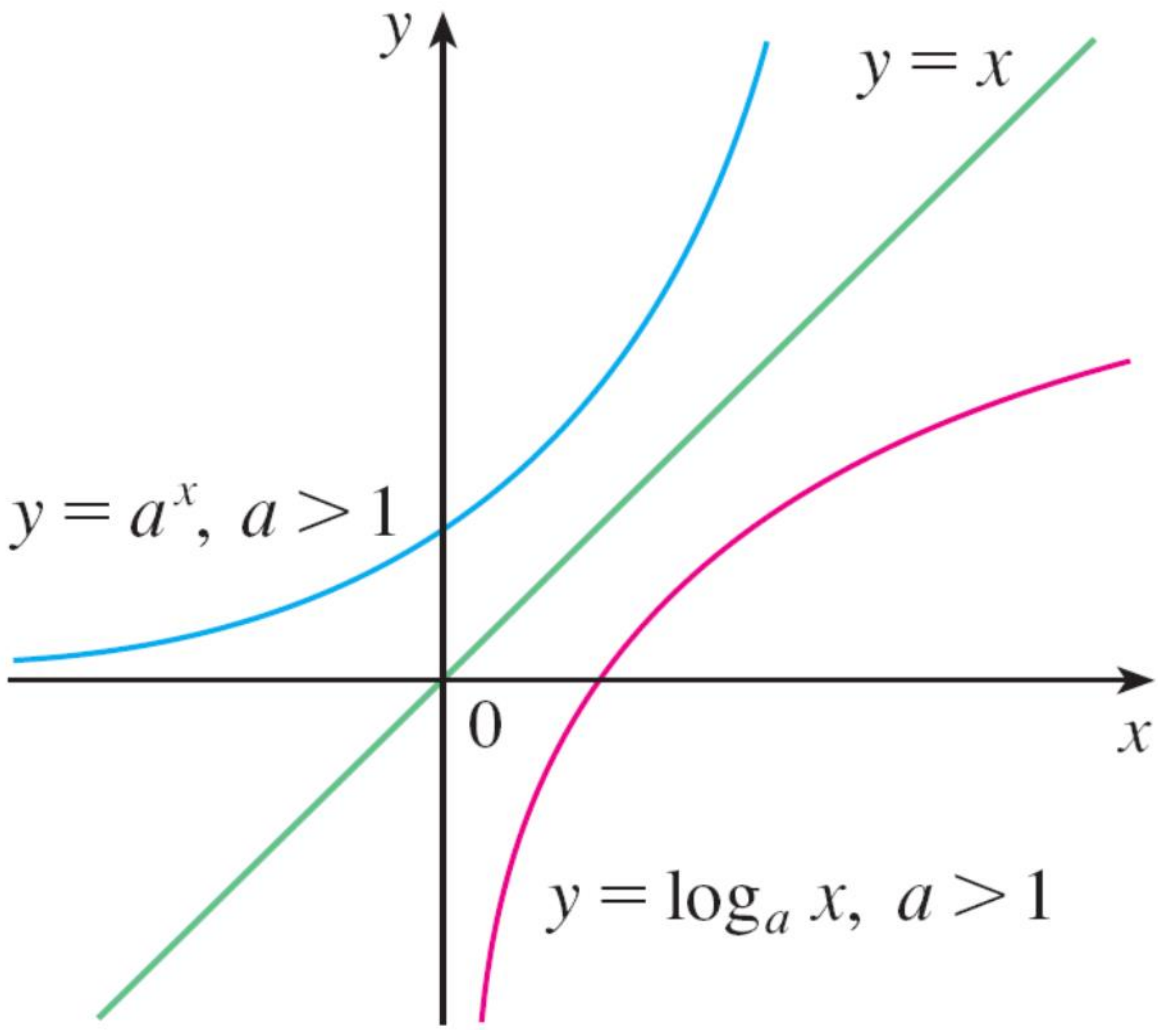
Laws of Logarithms If x and y are positive numbers, then

1. $\log_a(xy) = \log_a x + \log_a y$

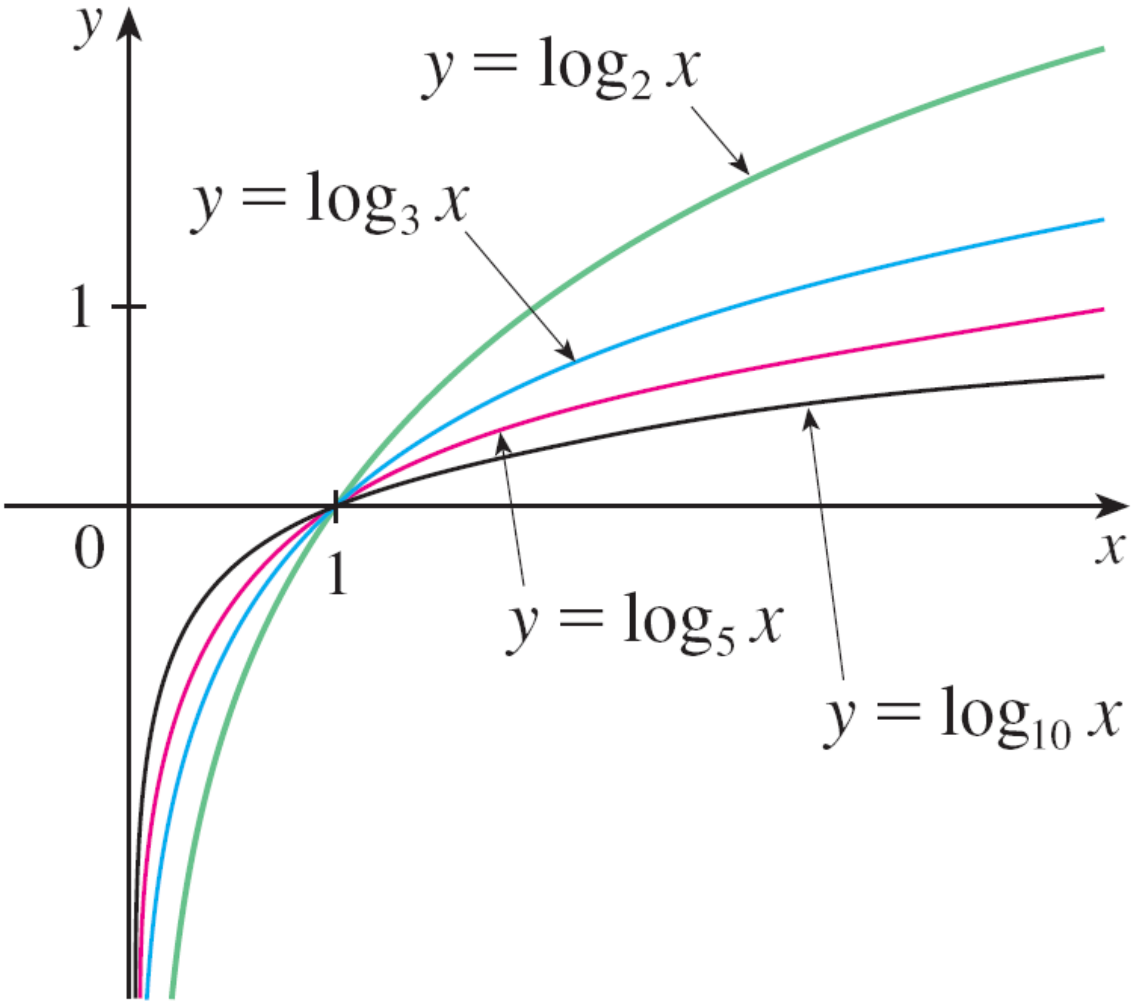
2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

3. $\log_a(x^r) = r\log_a x$ (where r is any real number)

Graphs of Logarithmic Functions



Graphs of Logarithmic Functions



Natural Logarithm

- The logarithm with base e is called the **natural logarithm** and has a special notation:

$$\boxed{\log_e x = \ln x}$$

10 **Change of Base Formula** For any positive number a $(a \neq 1)$, we have

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

Ex: Compute $\frac{d}{dx} \ln x$.

sol:

Ex: Compute $\frac{d}{dx} \log_a x$.

sol:

Ex : Compute $\frac{d}{dx} \ln|x|$

sol:

Ex : Compute $\frac{d}{dx} \ln|f(x)|$.

sol:

Derivatives of Logarithmic Functions

- Application:

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

- Combine the above formulas with the chain rule, we get

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \qquad \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

$$\frac{d}{dx}(\ln |x|) = \frac{1}{x}$$

Logarithmic Differentiation

- The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. This method is called ***logarithmic differentiation.***

Logarithmic Differentiation

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the properties of logarithms to simplify. $\ln|y| = \ln|f(x)|$
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

Logarithmic Differentiation

The Power Rule If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

$$\frac{d}{dx} f(x)^{g(x)} = ?$$

$$\frac{d}{dx} \log_{f(x)} g(x) = ?$$

Ex: Prove the power rule $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$ for an $n \in \mathbb{R}$.

Sol:

Ex: $f(x) = x^x$, for $x > 0$. Find $f'(x)$.

Ex: Derive $\frac{d}{dx}\left(f(x)^{g(x)}\right)$.

Ex: $f(x) = x^{2^x} + x^{x^2} + 2^{x^x}$, for $x>0$. Find $f'(x)$.

Ex: Compute $\frac{d}{dx}\left[(1+x^3)^x \, (\ln x)^{\cos x}\right]$, for $x > 1$.

Ex: Find $\frac{d}{dx}\left(\log_{f(x)} g(x)\right)$

The number e

Ex: Try to compute $\frac{d}{dx} \ln x \Big|_{x=1}$.

The number e as a limit

$$e = \lim_{x \to 0} (1 + x)^{1/x}$$

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^{n}$$

$$e = \lim_{x \to \infty} (1 + \frac{1}{x})^{x}$$

$$e = \lim_{x \to -\infty} (1 + \frac{1}{x})^{x}$$

Ex: Compute $\lim_{n\to\infty}\left(1-\frac{5}{n}\right)^{3n}$.

Sol:

Ex: Find $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}$

Review

- Define inverse trigonometric functions.
- Differentiate inverse trigonometric functions.
- Define logarithmic functions.
- Differentiate logarithmic functions.
- Describe the process of logarithmic differentiation.
- Write e as a limit.