The DivergenceTheorem

Section 16.9

The Divergence Theorem

- We state the Divergence Theorem for regions E that are simultaneously of types 1, 2, and 3 and we call such regions **simple solid regions**.
- The boundary of E is a closed surface, and we use the convention, that the positive orientation is outward; that is, the unit normal vector $\vec{n}$ is directed outward from E.

The Divergence Theorem

- The Divergence Theorem:
- Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$$

The Divergence Theorem

- Thus the Divergence Theorem states that, under the given conditions, *the flux of* $\vec{F}$ *across the boundary surface of* E *is equal to the triple integral of the divergence of* $\vec{F}$ *over* E *.*
- Notice its similarity to Green's Theorem and Stokes' Theorem in that it relates the integral of a derivative of a function ($\mathrm{div}\vec{F}$ in this case) over a region to the integral of the original function $\vec{F}$ over the boundary of the region.

Proof of the Divergence Theorem

pf: Suppose that $\vec{F}(x,y,z) = P\vec{i} + Q\vec{j} + R\vec{k}$.

Then $\iint_{\partial E} \vec{F} \cdot d\vec{S} = \iint_{\partial E} \vec{F} \cdot \vec{n}\, dS = \iint_{\partial E} P\vec{i}\cdot\vec{n} + Q\vec{j}\cdot\vec{n} + R\vec{k}\cdot\vec{n}\, dS$

And $\iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E P_x + Q_y + R_z\, dV$.

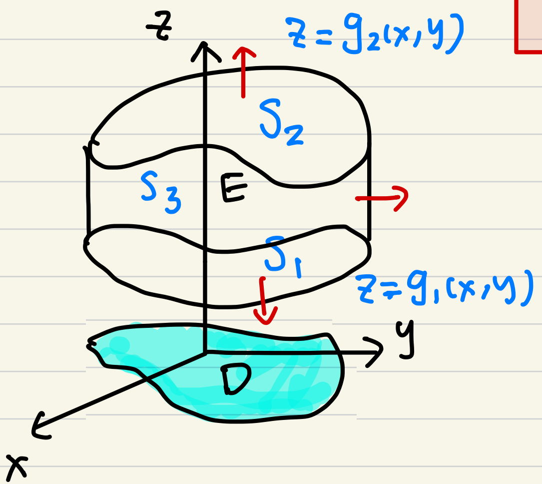
We want to show that $\iint_{\partial E} P\vec{i}\cdot\vec{n}\, dS = \iiint_E P_x\, dV$,

$\iint_{\partial E} Q\vec{j}\cdot\vec{n}\, dS = \iiint_E Q_y\, dV$, $\iint_{\partial E} R\vec{k}\cdot\vec{n}\, dS = \iiint_E R_z\, dV$

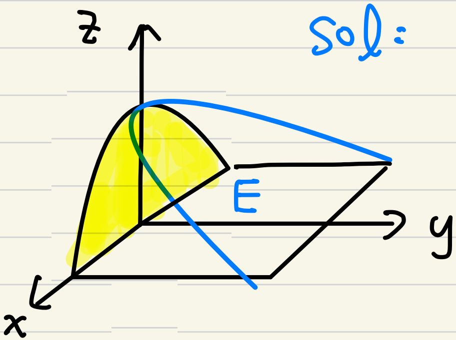
for all C^1 functions $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$.

Write E as type I region, $E = \{(x,y,z) \mid (x,y) \in D,\ g_1(x,y) \le z \le g_2(x)\}$.

We want to show $\boxed{\iint_{\partial E} R\,\vec{k}\cdot\vec{n}\,dS = \iiint_E \frac{\partial R}{\partial z}\,dV.}$



Ex: Evaluate $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x,y,z) = (xy + \cos z,\ y + e^{z^3},\ \tan^{-1} xy)$ and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and planes $z=0$, $y=0$ and $y+z=1$.



Sol:

Ex:

Evaluate the flux of

$$\mathbf{V}(x,y,z) = (z^2x + y^2z)\mathbf{i} + \left(\frac{1}{3}y^3 + z\tan x\right)\mathbf{j} + (x^2z + 2y^2 + 1)\mathbf{k}$$

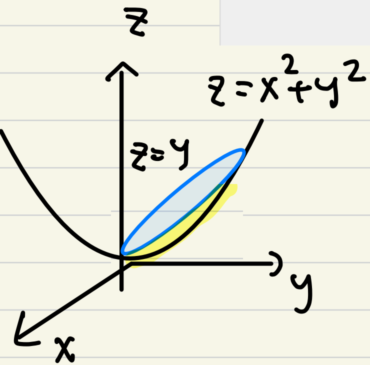
across S: the upper half sphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ with normal pointing away from the origin.

Ex:

Let S_1 be the surface $\{(x, y, z) | \ z = x^2 + y^2, \ z \leq y\}$, S_2 be the surface $\{(x, y, z) | \ z = y, \ x^2 + y^2 \leq z\}$, and $\mathbf{V}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$.

a. Compute directly the downward flux of $\mathbf{V}$ across S_1.

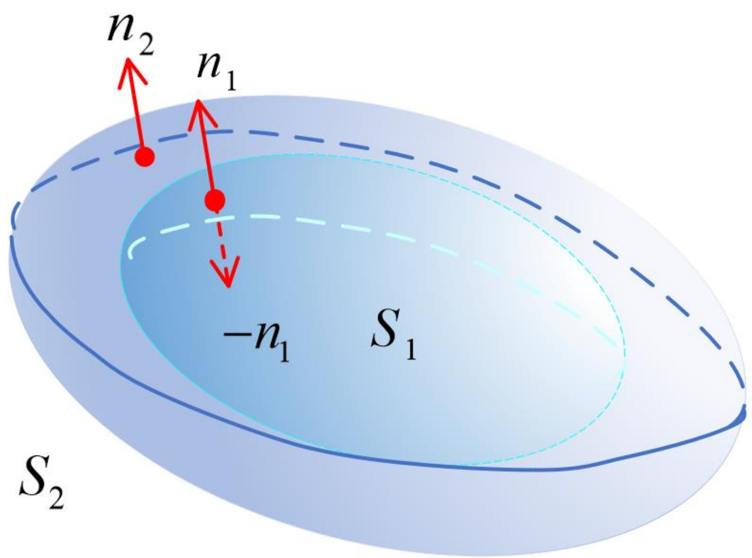
b. Use the divergence theorem to compute the upward flux of $\mathbf{V}$ across S_2.



Ex: Show that if curl $\vec{F}$ is still a C^1 vector field, then $\iint_S \text{curl}\, \vec{F} \cdot d\vec{S} = 0$ for all closed surface S with outward orientation.

The Divergence Theorem

- Let's consider the region E that lies between the closed surfaces S_1 and S_2 , where S_1 lies inside S_2. Let $\vec{n}_1$ and $\vec{n}_2$ be outward normal vectors of S_1 and S_2 .



The Divergence Theorem

- The boundary surface of E is $S = S_1 \cup S_2$ and its normal $\vec{n}$ is given by $\vec{n} = -\vec{n}_1$ on S_1 and $\vec{n} = \vec{n}_2$ on S_2 .
- By dividing E into finite unions of simple solid regions, we can show that the divergence theorem still holds for E .

$$\iiint_E \operatorname{div}\vec{F}dV = \iint_{S_1} \vec{F}\cdot(-\vec{n}_1)dS + \iint_{S_2} \vec{F}\cdot\vec{n}_2 dS$$

$$= \iint_S \vec{F}\cdot dS$$

The Divergence Theorem

- Example: Gauss's Law
- Consider the electric field: $\vec{E}(\vec{x}) = \dfrac{Q}{4\pi\epsilon_0|\vec{x}|^3}\vec{x}$ where the electric charge Q is located at the origin and $\vec{x} = (x, y, z)$ is the position vector.
- With the Divergence Theorem, we can show that the electric flux of $\vec{E}$ through any closed surface S that encloses the origin is

$$\iint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0}$$

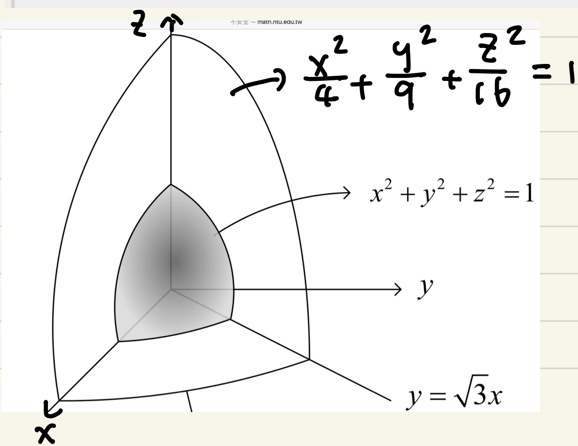
Ex: $\vec{E}(\vec{x}) = \frac{Q}{4\pi\varepsilon_0} \frac{\vec{x}}{|\vec{x}|^3}$. Show that for any closed surface S, $\vec{0} \notin S$,

$$\iint_S \vec{E} \cdot d\vec{S} = \begin{cases} \frac{Q}{\varepsilon_0}, & \text{if } S \text{ encloses } \vec{0}. \\ 0, & \text{if } S \text{ doesn't enclose } \vec{0}. \end{cases}$$

Let $\mathbf{F}(x, y, z) = \frac{1}{(x^2+y^2+z^2)^{3/2}}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$.

a. Let S_1 be the part of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant bounded by the planes $y = 0$, $y = \sqrt{3}x$ and $z = 0$, oriented upward. Find the flux of $\mathbf{F}$ across S_1.

b. Let S_2 be the part of the surface $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$ in the first octant bounded by the planes $y = 0$, $y = \sqrt{3}x$ and $z = 0$, oriented upward. Find the flux of $\mathbf{F}$ across S_2.



The Divergence Theorem

- Another application of the Divergence Theorem occurs in fluid flow. Let $\vec{v}(x, y, z)$ be the velocity field of a fluid with constant density ρ. Then $\vec{F} = \rho\vec{v}$ is the rate of flow per unit area.
- If $P_0(x_0, y_0, z_0)$ is a point in the fluid and B_a is a ball with center P_0 and very small radius a, then $\mathrm{div}\vec{F}(P) \approx \mathrm{div}\vec{F}(P_0)$ for all points in B_a since $\mathrm{div}\vec{F}$ is continuous.

The Divergence Theorem

- We approximate the flux over the boundary sphere S_a as follows:
  $\iint_{S_a} \vec{F} \cdot d\vec{S} = \iiint_{B_a} \text{div } \vec{F}\, dV$
  $\approx \iiint_{B_a} \text{div } \vec{F}(P_0)\, dV = \text{div } \vec{F}(P_0) V(B_a)$
- This approximation becomes better as $a \to 0$ and suggests that

$$\boxed{\text{div}\vec{F}(P_0) = \lim_{a \to 0} \frac{1}{V(B_a)} \iint_{S_a} \vec{F} \cdot d\vec{S}}$$

The Divergence Theorem

- Hence $\mathrm{div}\vec{F}(P_0)$ is the net rate of outward flux per unit volume at P_0. (This is the reason for the name *divergence*.)
- If $\mathrm{div}\vec{F}(P) > 0$, the net flow is outward near P and P is called a **source**.
- If $\mathrm{div}\vec{F}(P) < 0$, the net flow is inward near P and P is called a **sink**.

Review

- State the Divergence Theorem.
- Prove Gauss's Law by the divergence theorem.

Review of Fundamental Theorems

0-dim	1-dim	2-dim	3-dim