More Techniques of Integration

Section 7.4-7.5

Outline

- Integration of Rational Functions by Partial Fractions
  - Partial Fractions
  - Rationalizing Substitution
- Strategy for Integration

Partial Fractions

- In section 7.4 we show how to integrate any rational function by expressing it as a sum of simpler fractions, called *partial fractions*, that we already know how to integrate.

Partial Fractions

- Given any rational function $f(x) = \frac{P(x)}{Q(x)}$
- Step 1: Express f as the sum of a *proper* rational function and a polynomial

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where the degree of R is smaller than the degree of Q.

Partial Fractions

- Step 2: Factorized Q as a product of linear factors (of the form $ax+b$) and irreducible quadratic factors (of the form ax^2+bx+c , where $b^2-4ac<0$) by the Fundamental Theorem of Algebra.

Partial Fractions

- Step 3: Express the proper rational function $R(x)/Q(x)$ as a sum of **partial fractions** of the form $\dfrac{A}{(ax+b)^i}$ or $\dfrac{Ax+B}{(ax^2+bx+c)^j}$

Ex: $\displaystyle\int \frac{1}{(ax+b)^k}\,dx$

Ex: $\displaystyle\int \frac{x}{(x^2+a^2)^n}\,dx$

Ex: $\displaystyle \int \frac{1}{(x^2+a^2)^n}\,dx$

Partial Fractions

- **Case I: The denominator $Q(x)$ is a product of distinct linear factors.**
- $Q(x) = (a_1 x + b_1)(a_2 x + b_2) \ldots (a_k x + b_k)$
  where no factor is repeated.
- Then the partial fraction theorem states that there exist constants $A_1, A_2, \ldots A_k$ such that
- $$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_k}{a_k x + b_k}$$

Ex: Compute $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$.

Ex: Compute $\int \frac{1}{x^2+k}\,dx$, where

(a) $k>0$ (b) $k=0$ (c) $k<0$.

Partial Fractions

- **Case II: $Q(x)$ is a product of linear factors, some of which are repeated.**
- Suppose the linear factor (a_1x+b_1) is repeated r times; that is, $(a_1x+b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x+b_1)$ in the partial fractions, we would use

$$\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_1x+b_1)^2}+\cdots+\frac{A_r}{(a_1x+b_1)^r}$$

Ex: Compute $\displaystyle\int \frac{x^3-2x^2+2x+1}{x^3-2x^2+x}\,dx$.

Ex: Compute $\int \frac{1}{(x^2-1)^2}\,dx$

Partial Fractions

- **Case III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.**
- If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions due to the linear factors, the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

Ex: Compute $\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx$

Ex: Compute $\int \frac{3}{x^3+1}\,dx$.

Ex: Decompose $\dfrac{1}{x^4-1}$ as a sum of partial fractions.

Ex: Decompose $\dfrac{1}{x^4+4}$ as a sum of partial fractions.

Partial Fractions

- **Case IV: $Q(x)$ contains a repeated irreducible quadratic factor.**
- If $Q(x)$ has the factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then instead of the single partial fraction, we have the sum

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

in the partial fraction of $R(x)/Q(x)$.

Ex: Compute $\displaystyle\int \frac{x^3+x^2+3x-1}{(x-1)(x^2+1)^2}\,dx$

Rationalizing Substitutions

- Some non-rational functions can be changed into rational functions by means of appropriate substitutions.
- In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$ then the substitution $u = \sqrt[n]{g(x)}$ may be effective.

Ex: Compute $\int \frac{\sqrt{x+4}}{x}\,dx$

Ex: Compute $\displaystyle\int \frac{dx}{2\sqrt{x+3}+x}$

Ex: Compute $\displaystyle\int \frac{dx}{x^{\frac{1}{2}}(1+x^{\frac{1}{3}})}$.

Ex: Compute $\int \frac{1}{(e^{2x}+1)\,e^{x}}\,dx$

Rationalizing Substitutions

- For rational functions of $\sin x$ and $\cos x$, we could try the substitution $t = \tan(x/2)$ where $-\pi < x < \pi$. Then

$$\cos(\frac{x}{2}) = \frac{1}{\sqrt{1+t^2}} \quad , \quad \sin(\frac{x}{2}) = \frac{t}{\sqrt{1+t^2}}$$

$$\cos(x) = \frac{1-t^2}{1+t^2} \quad , \quad \sin(x) = \frac{2t}{1+t^2}$$, and

$$dx = \frac{2}{1+t^2}dt$$

Ex: $\displaystyle\int \frac{1}{\sin x-\cos x}\,dx$

Ex: $\displaystyle\int \frac{1}{1+\sin x}\,dx$