The Limit of a Function

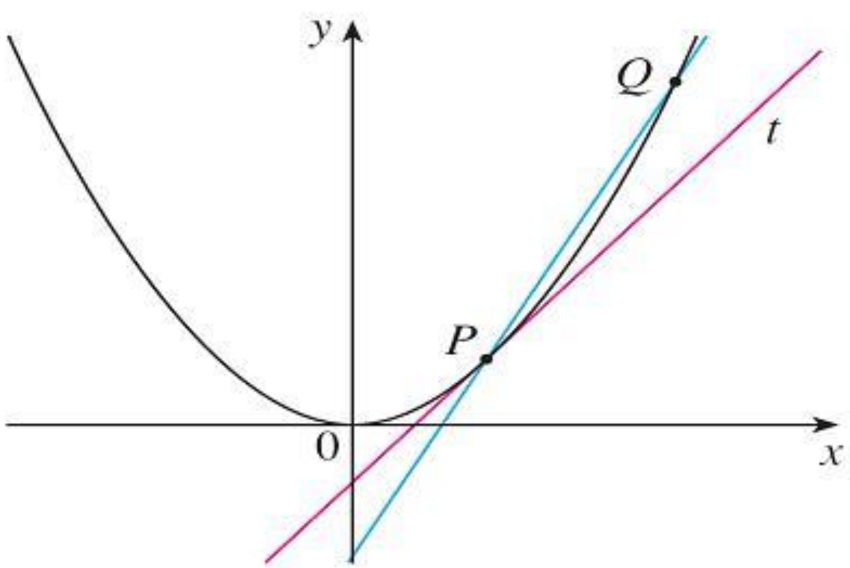
Section 2.1-2.2

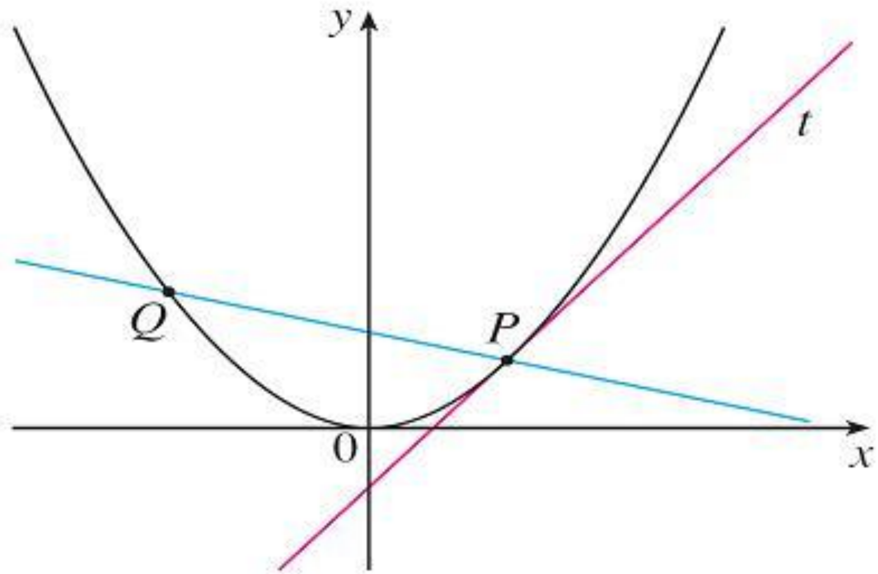
Outline

- The Tangent and Velocity Problems
- The Limit of a Function
  - Definition
  - Examples
  - One-Sided Limits
  - Infinite Limits

The Tangent and Velocity Problems

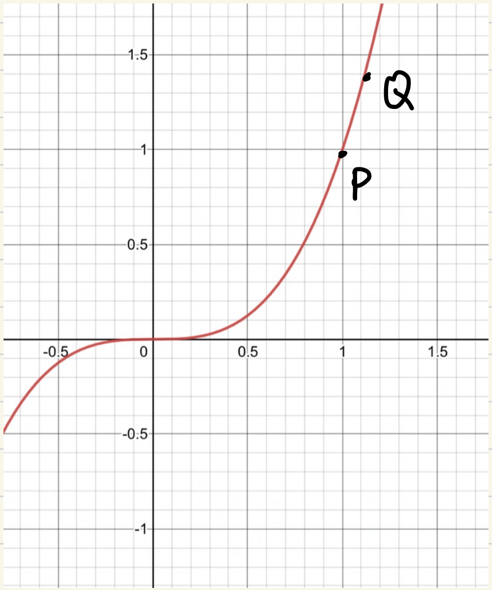
- Finding " tangent lines " of a curve





Ex: Find the tangent line to the curve $y=x^3$ at the point $(1,1)$.

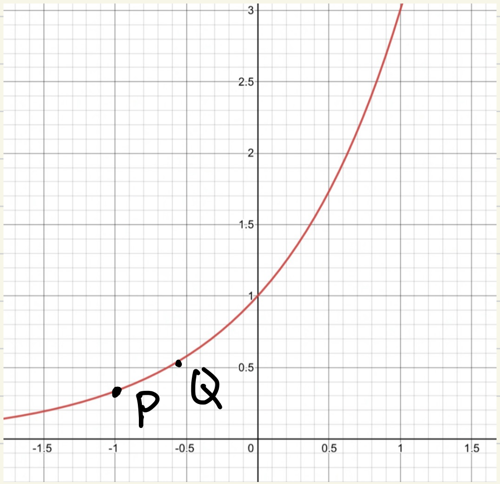
Sol:



Q: Find the tangent line of $y=\sqrt{x}$ at $(1,1)$.

Ex: Find the slope of the tangent line to the curve $y = a^x$ at (x_0, a^{x_0}), where $a > 0$.

Sol:



The Tangent and Velocity Problems

- Find the **instantaneous velocity** by taking the limit value of average velocities over shorter and shorter time periods
- The **method of taking limits** is used in solving both the tangent and velocity problems.

The Limit of a Function

$\boxed{1}$ **Definition** We write

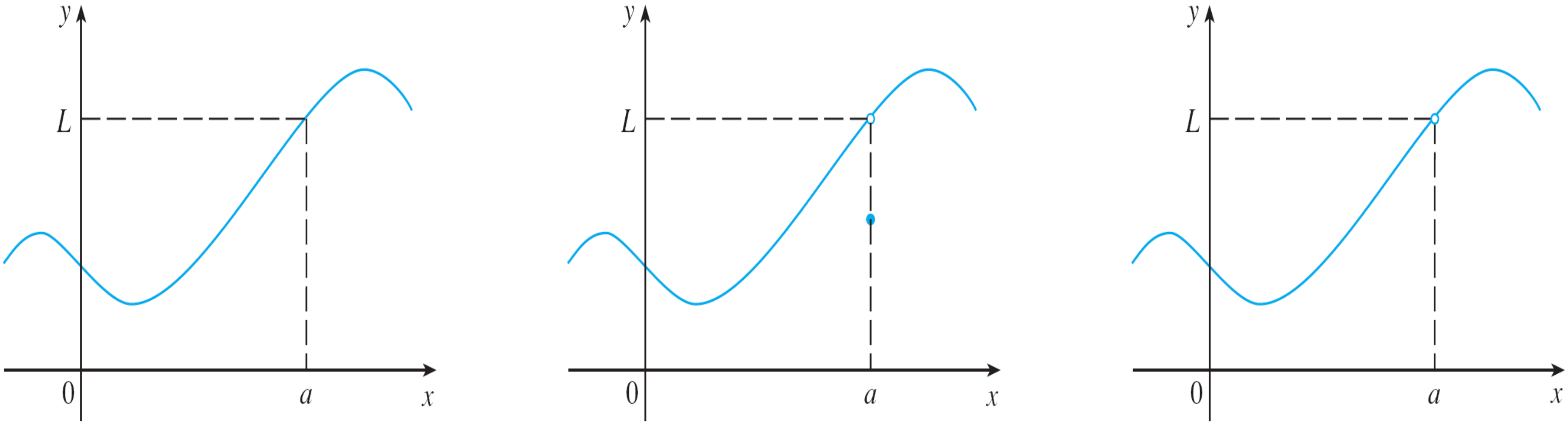
$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

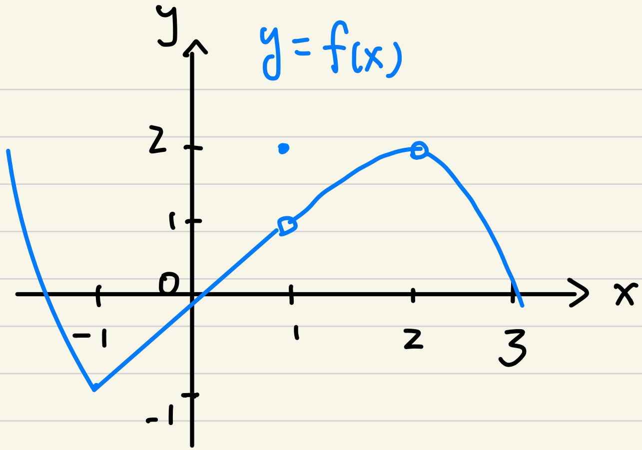
An alternative notation is " $f(x) \to L$ as $x \to a$ " which is usually read " $f(x)$ approaches L as x approaches a."

- **Remark:** In the definition of limit, we have the phrase " $\boxed{x \neq a}$ ".



(a) (b) (c)

Ex:



$\lim_{x \to -1} f(x) =$ $\qquad$ $f(-1) =$

$\lim_{x \to 1} f(x) =$ $\qquad$ $f(1) =$

$\lim_{x \to 2} f(x) =$ $\qquad$ $f(2) =$

The Limit of a Function

- Corollary:
- If $f(x) = g(x)$ for all $x \neq a$ and $\lim_{x\to a} f(x) = L$, then the limit of $g(x)$ as x approaches a exists and $\lim_{x\to a} g(x) = L$.

Ex : Find $\lim\limits_{x \to -2} \dfrac{x^2 - 4}{x + 2}$.

Sol :

The Limit of a Function

- Examples:
- Guess the value of $\lim_{x \to 1} \dfrac{x-1}{x^2-1}$.

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

The Limit of a Function

- Guess the value of $\displaystyle\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}$

- Guess the value of $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$

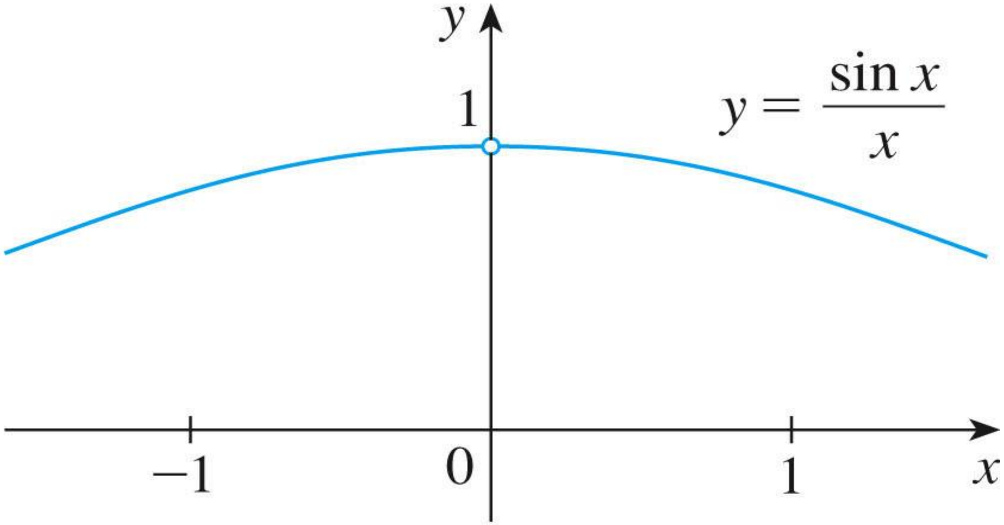
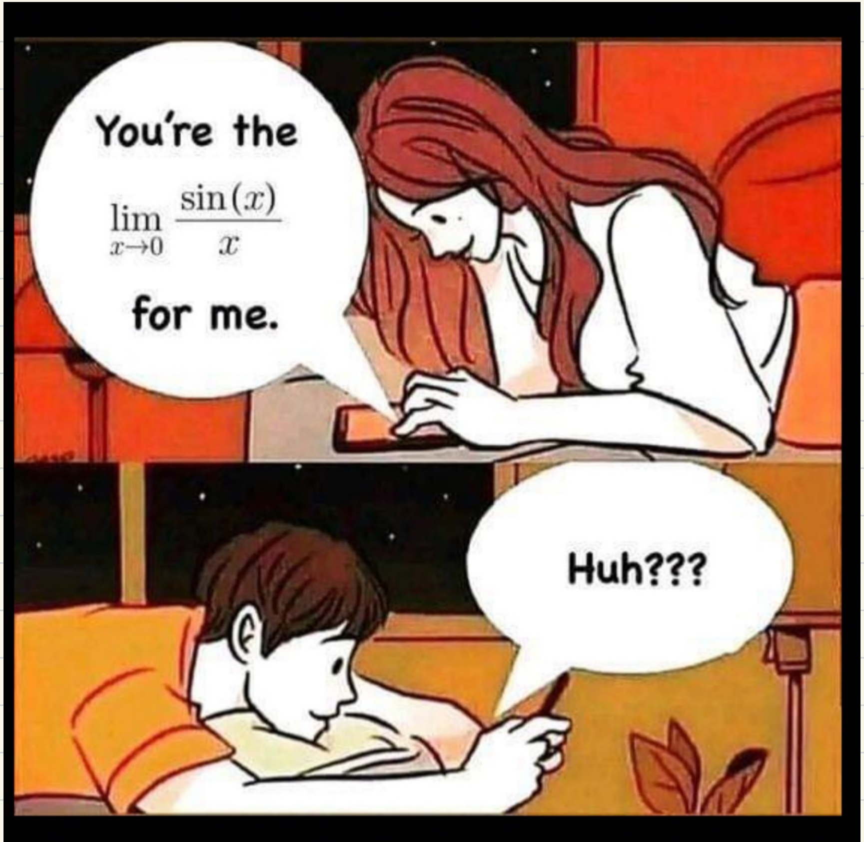


FIGURE 6



I feel like

$\boxed{\sin\left(\frac{\pi}{x}\right) \text{ near } x=0}$

in the first week of the semester!

The Limit of a Function

- Investigate $\lim_{x \to 0} \sin \frac{\pi}{x}$

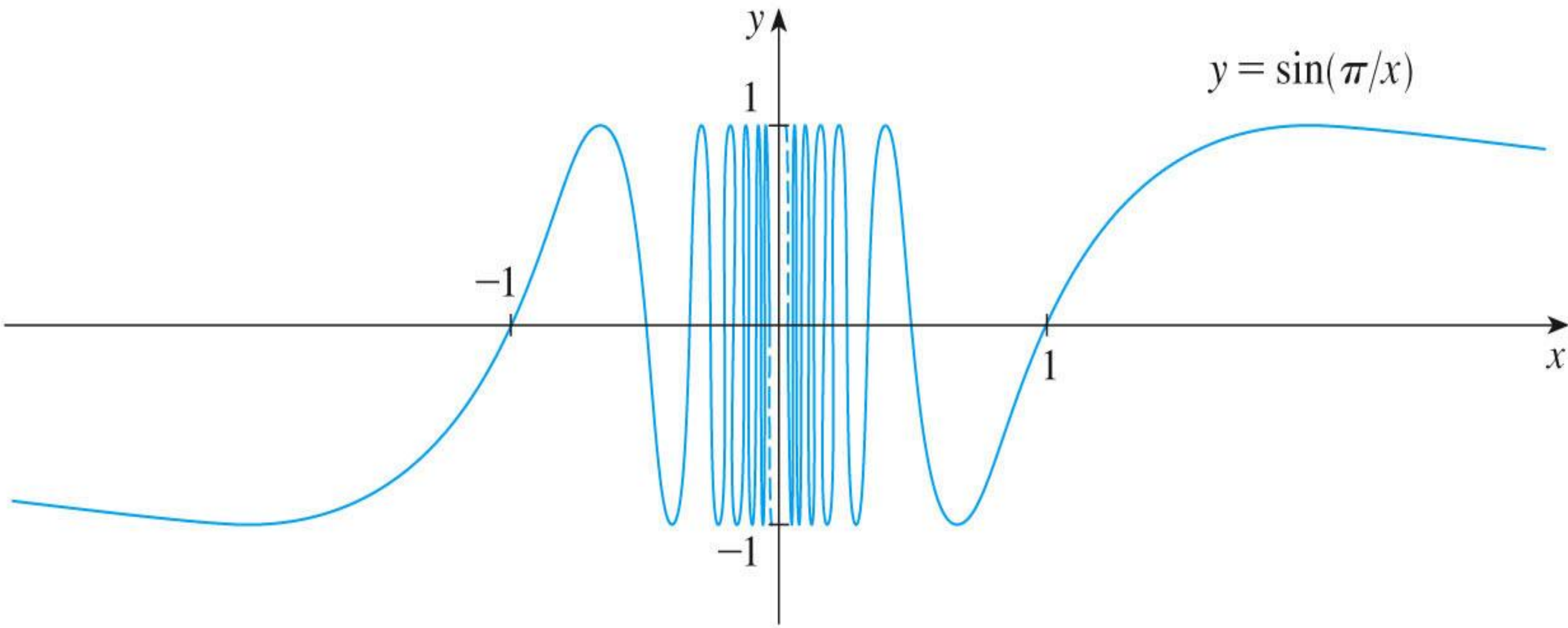


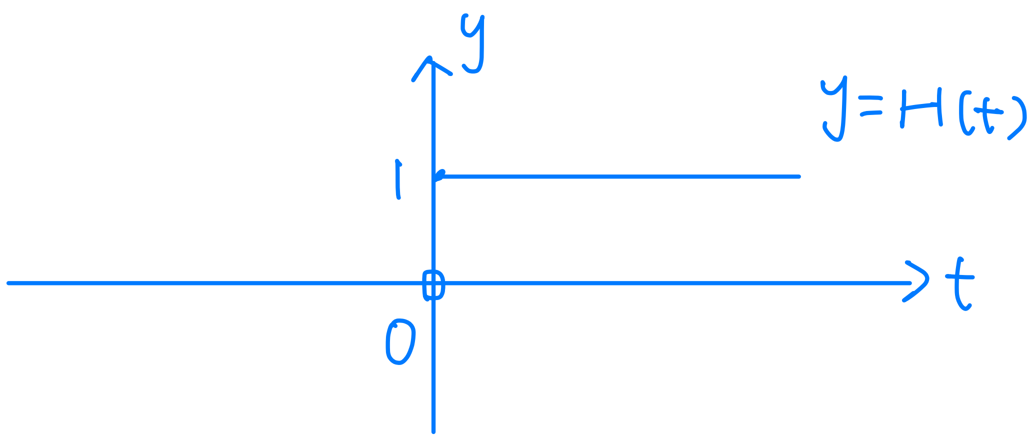
FIGURE 7

The Limit of a Function

- The Heaviside function, $H(t)$, is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

- Investigate $\lim_{t \to 0} H(t)$.



One-Sided Limits

2 Definition We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

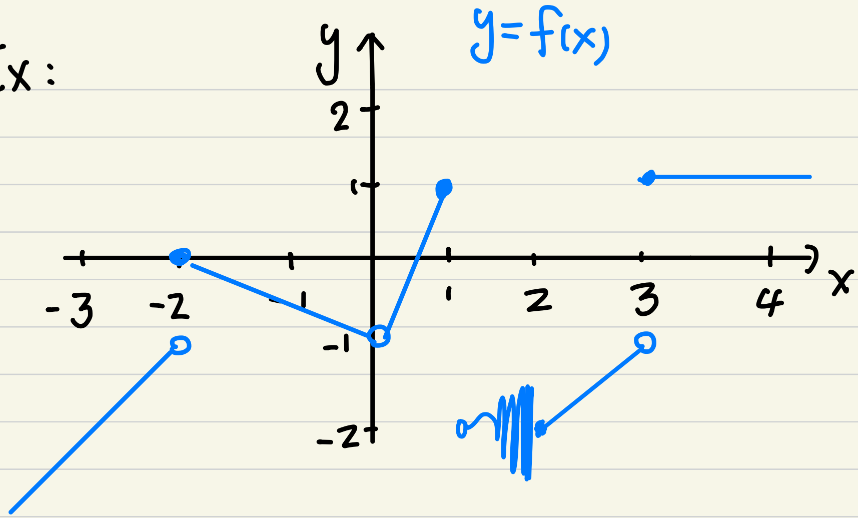
One-Sided Limits

- Similarly, if we require that x be greater than a, we get " the right-hand limit of $f(x)$ as x approaches a is equal to L " and we write $\lim_{x \to a^+} f(x) = L$.

- Property:

3 $\quad \lim_{x \to a} f(x) = L \quad$ if and only if $\quad \lim_{x \to a^-} f(x) = L \quad$ and $\quad \lim_{x \to a^+} f(x) = L$

Ex:



Find $\lim_{x \to -2^-} f(x)$

$\lim_{x \to -2^+} f(x)$

$\lim_{x \to 0} f(x)$

$\lim_{x \to 1^-} f(x)$

$\lim_{x \to 1^+} f(x)$

$\lim_{x \to 2^-} f(x)$

$\lim_{x \to 2^+} f(x)$

$\lim_{x \to 3^-} f(x)$

$\lim_{x \to 3^+} f(x)$

Find integers n s.t.

$\lim_{x \to n^+} f(x) = f(n)$.

Ex: The floor function. Gauss function.

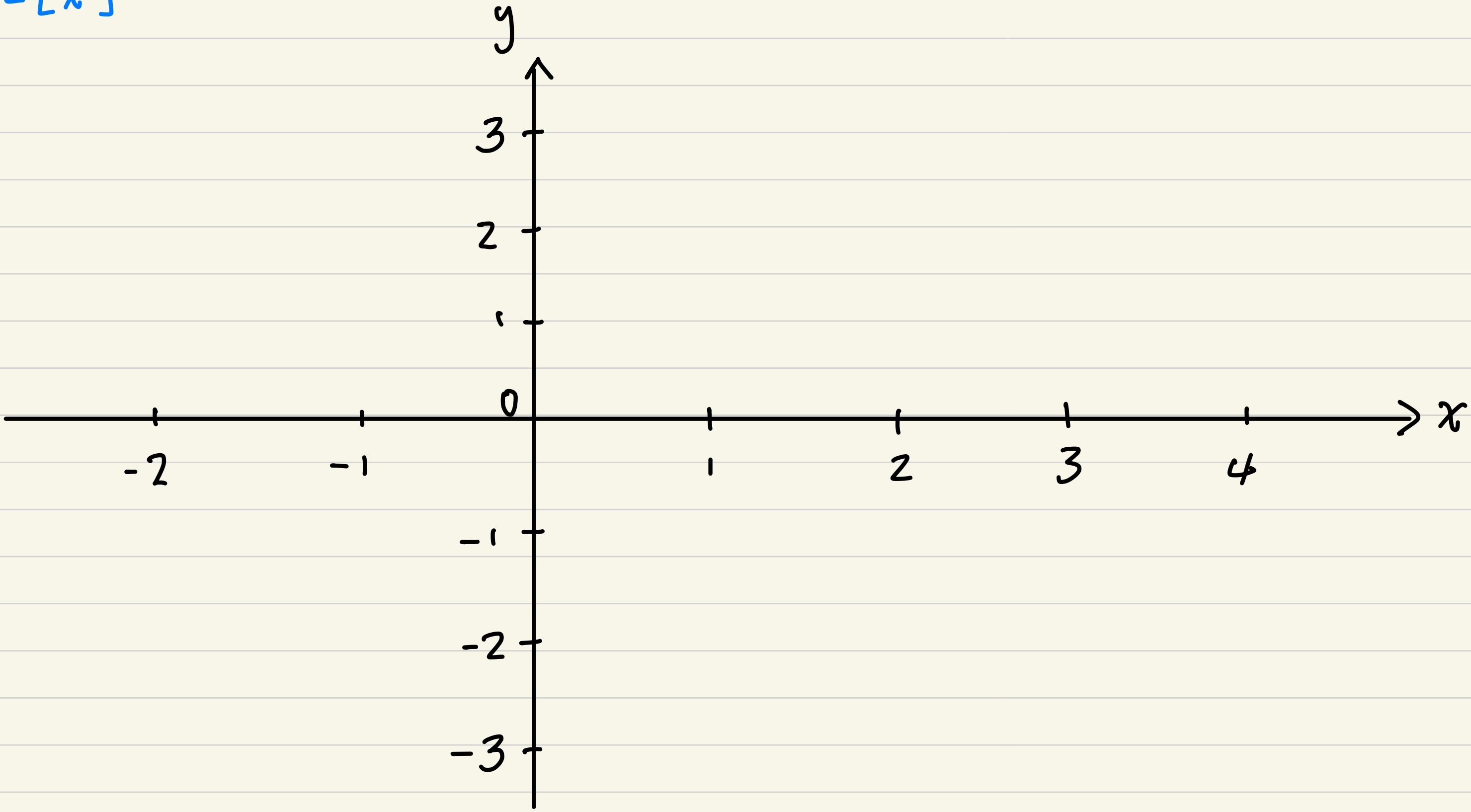
$f(x) = [x]$ = the largest integer that is less than or equal to x.

Ex: For all $n \in \mathbb{Z}$, $[n] =$ $\quad$ $[e] =$ $\quad$ $[-\pi] =$

Ex: Find $\lim\limits_{x \to -\frac{1}{2}} [x]$

Ex: For an $n \in \mathbb{Z}$, find $\lim\limits_{x \to n^+} [x]$, $\lim\limits_{x \to n^-} [x]$

$y = [x]$



Ex: Let $f(x) = [x] + [-x]$.

Find $\lim_{x \to 2^+} f(x)$, $\lim_{x \to 2^-} f(x)$, $\lim_{x \to 2} f(x)$ and $f(2)$.

Sol:

Q: Find $\lim_{x \to a} f(x)$, where $n_0 < a < n_0 + 1$ for some $n_0 \in \mathbb{Z}$.

Ex: Suppose that $f(x) = \begin{cases} \dfrac{x^2+x-2}{x+2}, & \text{if } x > -2 \\ a[x] + b[-2x] & \text{if } x \leq -2 \end{cases}$

Find constant a, b such that $\lim_{x \to -2} f(x) = f(-2)$.

Infinite Limits

4 Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Another notation for $\lim_{x \to a} f(x) = \infty$ is

"$f(x) \to \infty$ as $x \to a$"

which is read as " the limit of $f(x)$, as x approaches a, is infinity "

Infinite Limits

5 Definition Let f be defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

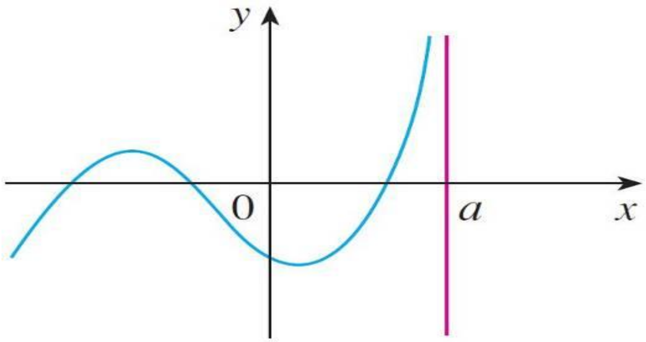
means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

Similar definitions can be given for the one-sided infinite limits $\lim_{x \to a^-} f(x) = \infty$

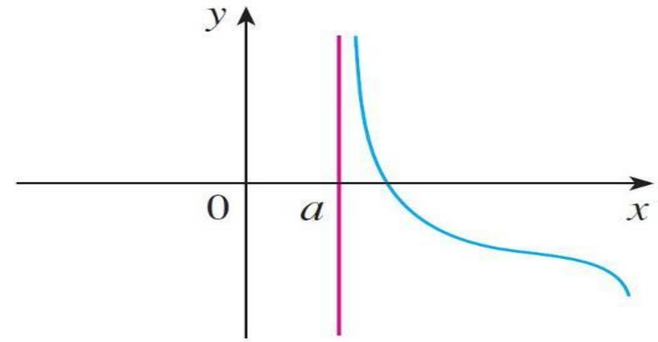
$\lim_{x \to a^+} f(x) = \infty$ $\lim_{x \to a^-} f(x) = -\infty$

$\lim_{x \to a^+} f(x) = -\infty$

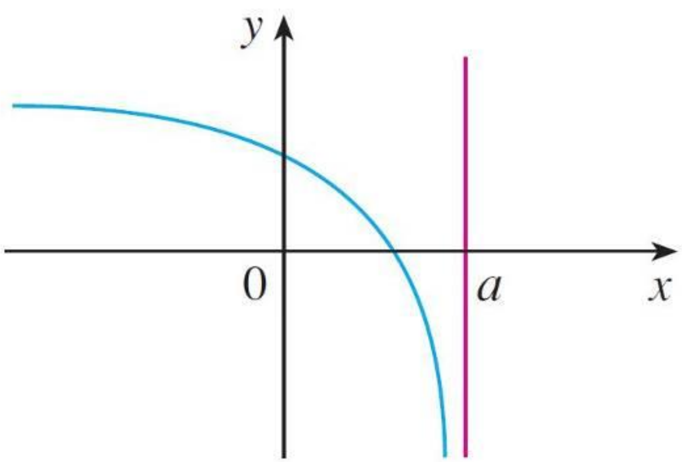
Infinite Limits



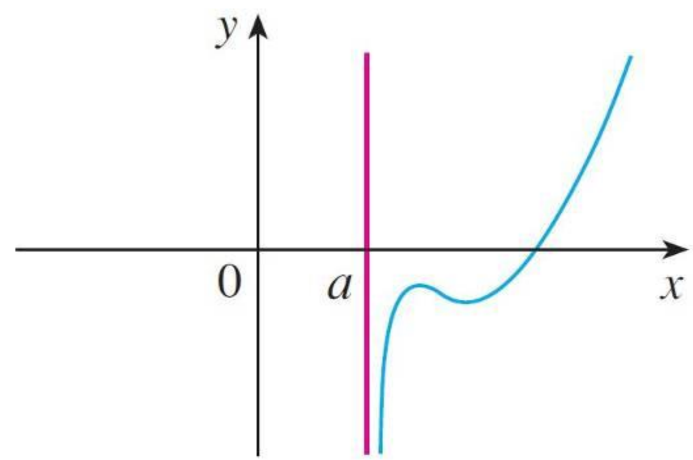
(a) $\lim_{x \to a^-} f(x) = \infty$



(b) $\lim_{x \to a^+} f(x) = \infty$



(c) $\lim_{x \to a^-} f(x) = -\infty$



(d) $\lim_{x \to a^+} f(x) = -\infty$

Infinite Limits

Ex: Find $\lim_{x \to -1} \frac{1}{(x+1)^2}$.

Sol:

One-Sided Infinite Limits

Ex: Find $\lim_{x \to 1^+} \frac{1}{1-x}$.

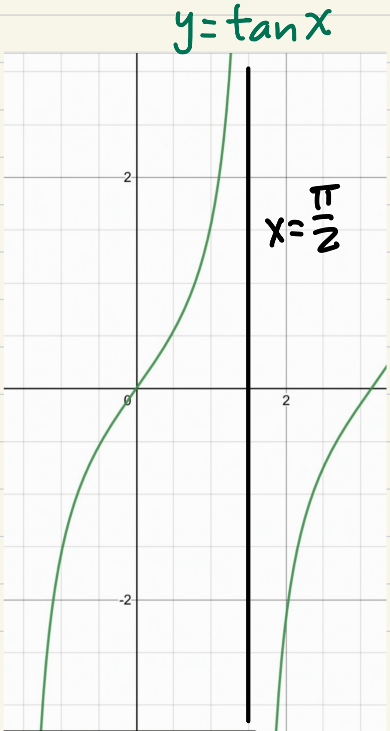
Sol:

Ex: Find $\lim\limits_{x \to 2^-} \dfrac{x^2-4x+3}{(x-2)^3}$.

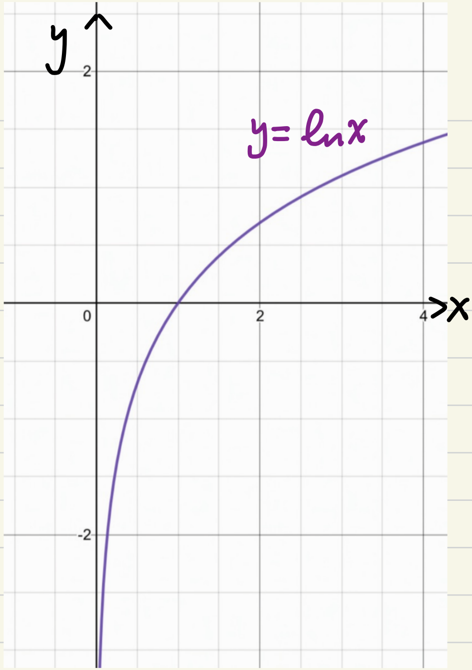
Sol:

Ex: Find $\lim\limits_{x \to \frac{\pi}{2}^-} \tan x$, $\lim\limits_{x \to \frac{\pi}{2}^+} \tan x$

Sol:



Ex: Find $\lim_{x \to 0^+} \log x$



Ex: Find $\lim_{x \to 0^+} 2^{\frac{1}{x}}$, $\lim_{x \to 0^-} 2^{\frac{1}{x}}$

Infinite Limits

$\boxed{6}$ **Definition** The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Ex: Find vertical asymptote(s) of $y = f(x) = \dfrac{x^2-4x+3}{x^3-x}$.

Sol:

Review

- Explain the following equations.

$$\lim_{x \to a} f(x) = L$$

$$\lim_{x \to a^{\pm}} f(x) = L$$

$$\lim_{x \to a^{\pm}} f(x) = \pm\infty$$

- Recall the definition of vertical asymptotes of the graph of a function $f(x)$.