Finding Extreme Values

Section 14.7-14.8

Outline

- Definitions of Extreme Values
- Tests for Finding Extreme Values
  - Fermat's Theorem
  - The Second Derivatives Test
- Strategies for Finding Extreme Values
- Finding Extreme Values under Constraints (Lagrange Multipliers)

Finding Extreme Values under Constraints

- We want to find the extreme values of $f(x, y)$ subject to a constraint of the form $g(x, y) = k$. In other words, we seek the extreme values of $f(x, y)$ when (x, y) is restricted on the level curve $g(x, y) = k$.
- We can show that when the extreme value occurs at (x_0, y_0), level curves $g(x, y) = k$ and $f(x, y) = f(x_0, y_0)$ have a common tangent line at (x_0, y_0) if $\vec{\nabla} g(x_0, y_0) \neq \vec{0}$.

Finding Extreme Values under Constraints

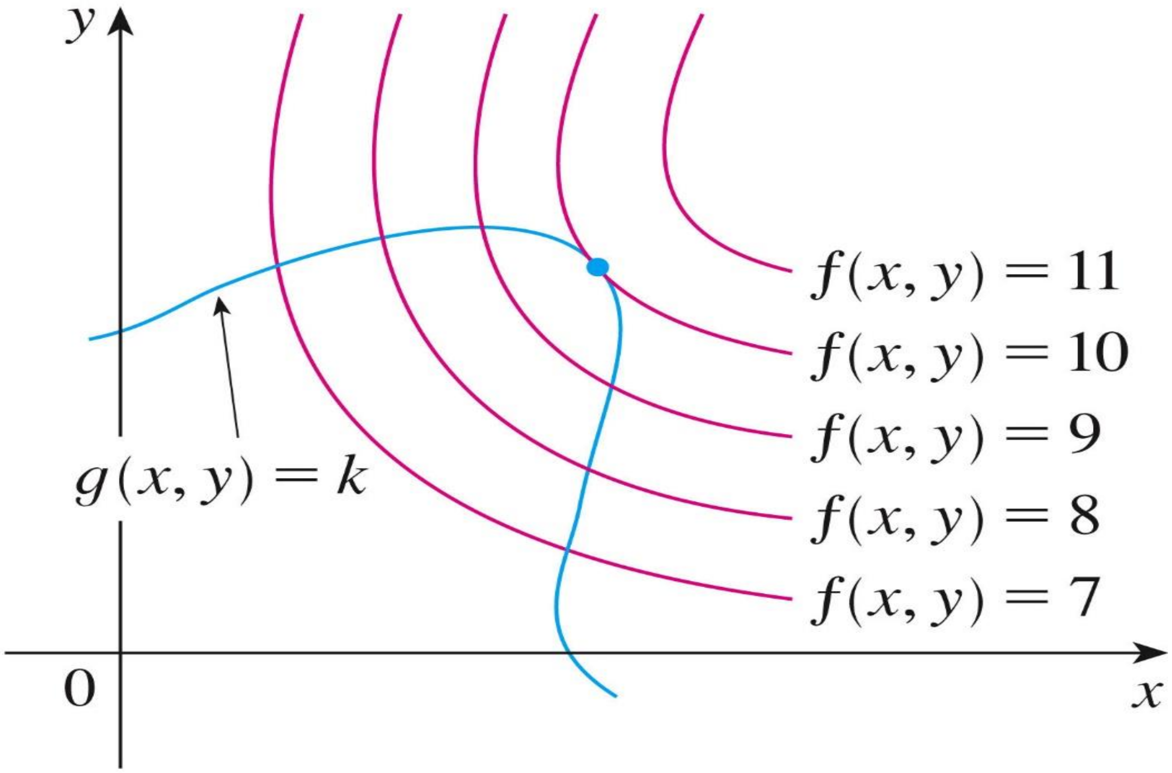
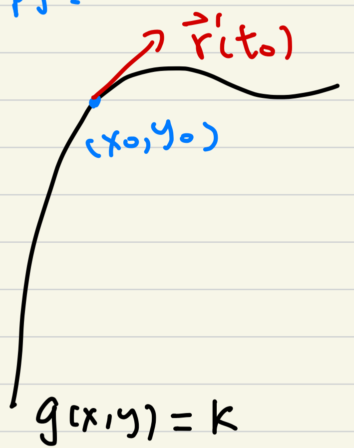


FIGURE 1

Prop: Suppose that C is the level curve $g(x,y)=k$ where $\vec{\nabla} g \neq \vec{0}$ and another differentiable function $f(x,y)$ has local extreme values at (x_0, y_0) when restricted to C. Then at (x_0, y_0), $\vec{\nabla} f(x_0,y_0) = \lambda \vec{\nabla} g(x_0,y_0)$ for some constant λ.

pf:



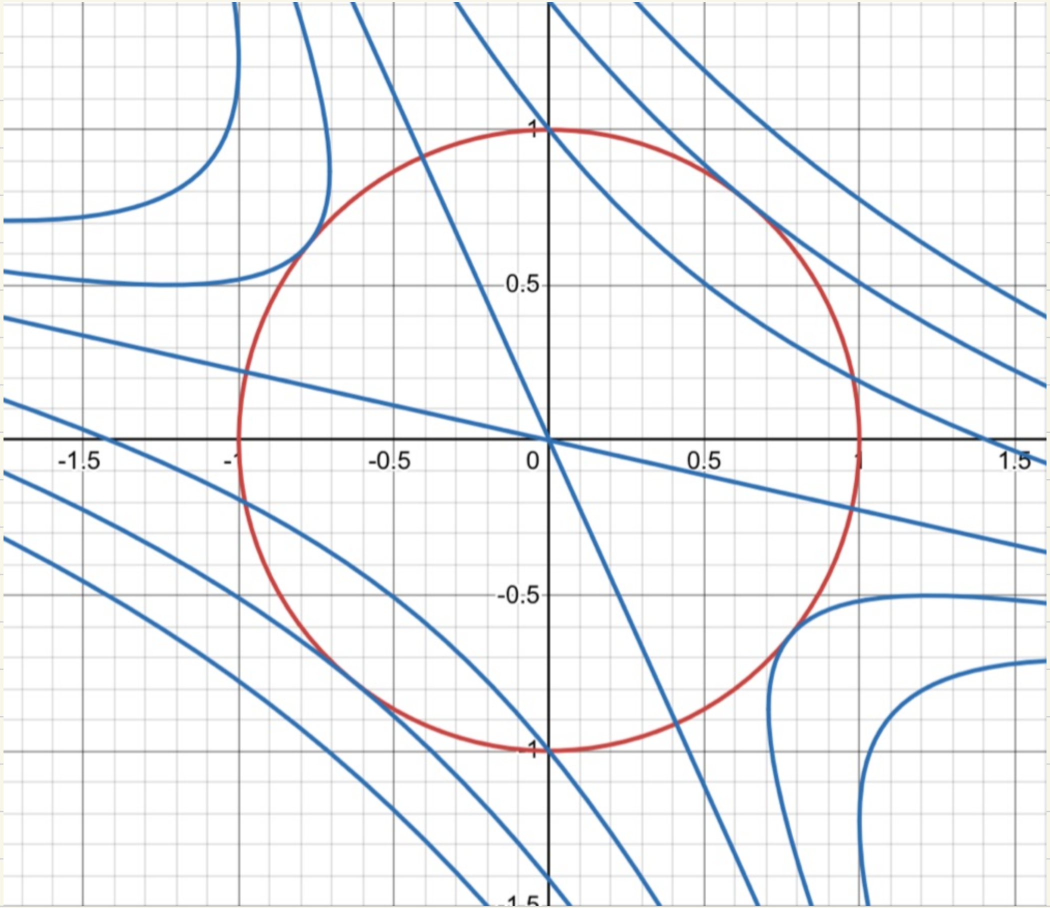
Finding Extreme Values under Constraints

- Hence, we should solve the following system of equations. $$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = k \end{cases}$$

- The extreme values of $f(x,y)$ occur among the solution set.

Ex: Find extreme values of $f(x,y) = x^2 + 2\sqrt{6}\,xy + 2y^2$ on the unit circle $x^2+y^2=1$.

$x^2 + 2\sqrt{6}\,xy + 2y^2 = k$



$k = 6$

$k = 4$

$k = 2$

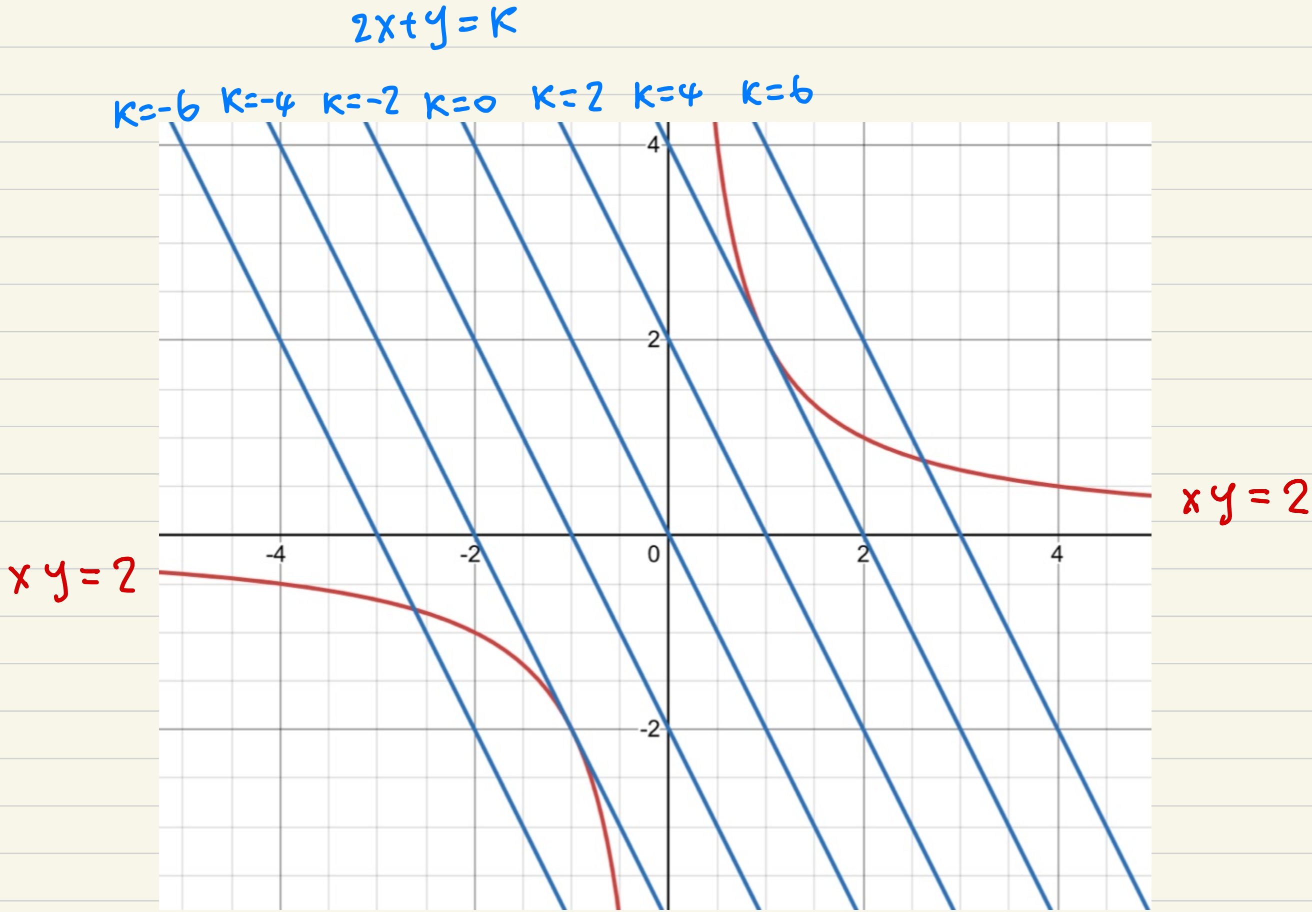
$k = 0$

$k = -1$

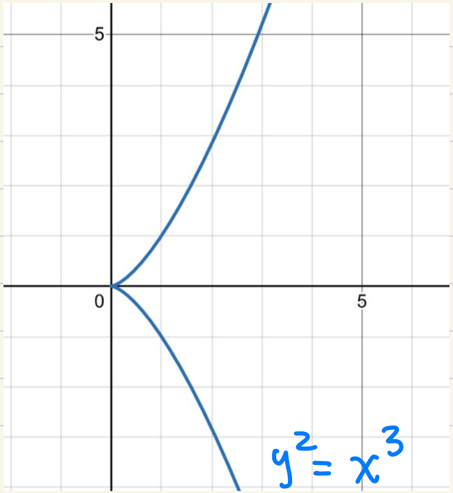
$k = -2$

Ex: Find extreme values of $f(x,y) = 2x + y$ on the curve

$g(x,y) = xy = 2$.

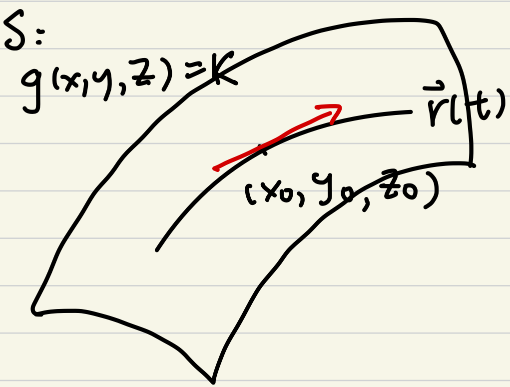

2x+y=k
k=-6
k=-4
k=-2
k=0
k=2
k=4
k=6
xy=2
xy=2
4
2
-2
-4
0

Ex: Minimize $f(x,y) = x$ on the curve $y^2 = x^3$.



$\boxed{\text{Prop}}$: Suppose that S is the level surface $g(x,y,z)=k$ where $\vec{\nabla} g \neq \vec{0}$ and another differentiable function $f(x,y,z)$ has local extreme values at (x_0, y_0, z_0) when restricted to S. Then at (x_0,y_0,z_0), $\vec{\nabla} f(x_0,y_0,z_0) = \lambda \vec{\nabla} g(x_0,y_0,z_0)$ for some constant λ.

Pf: Consider a diff curve $\vec{r}(t) \subset S$ such that $\vec{r}(t_0) = (x_0,y_0,z_0)$.



Finding Extreme Values under Constraints

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

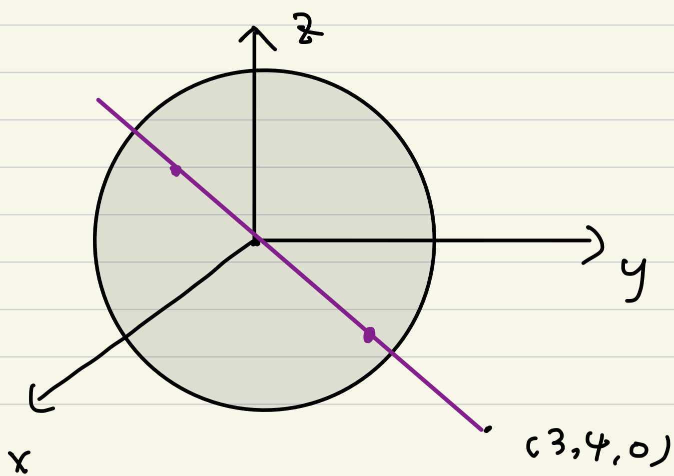
and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

The number λ here is called a **Lagrange multiplier**.

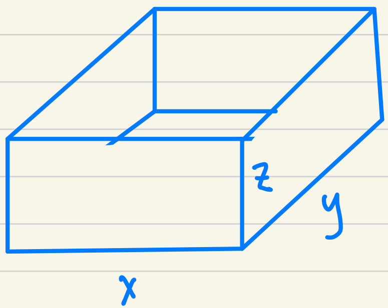
Ex: Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest from the point $(3,4,0)$.



z
y
x
(3,4,0)

Ex: A rectangular box without a lid is to be made from $12\ m^2$ of cardboard. Find the maximum volume of such a box.

Sol:



Finding Extreme Values under Constraints

- Two Constraints:
- Suppose that we want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints (side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$ given that $\vec{\nabla} g$ and $\vec{\nabla} h$ are linearly independent.

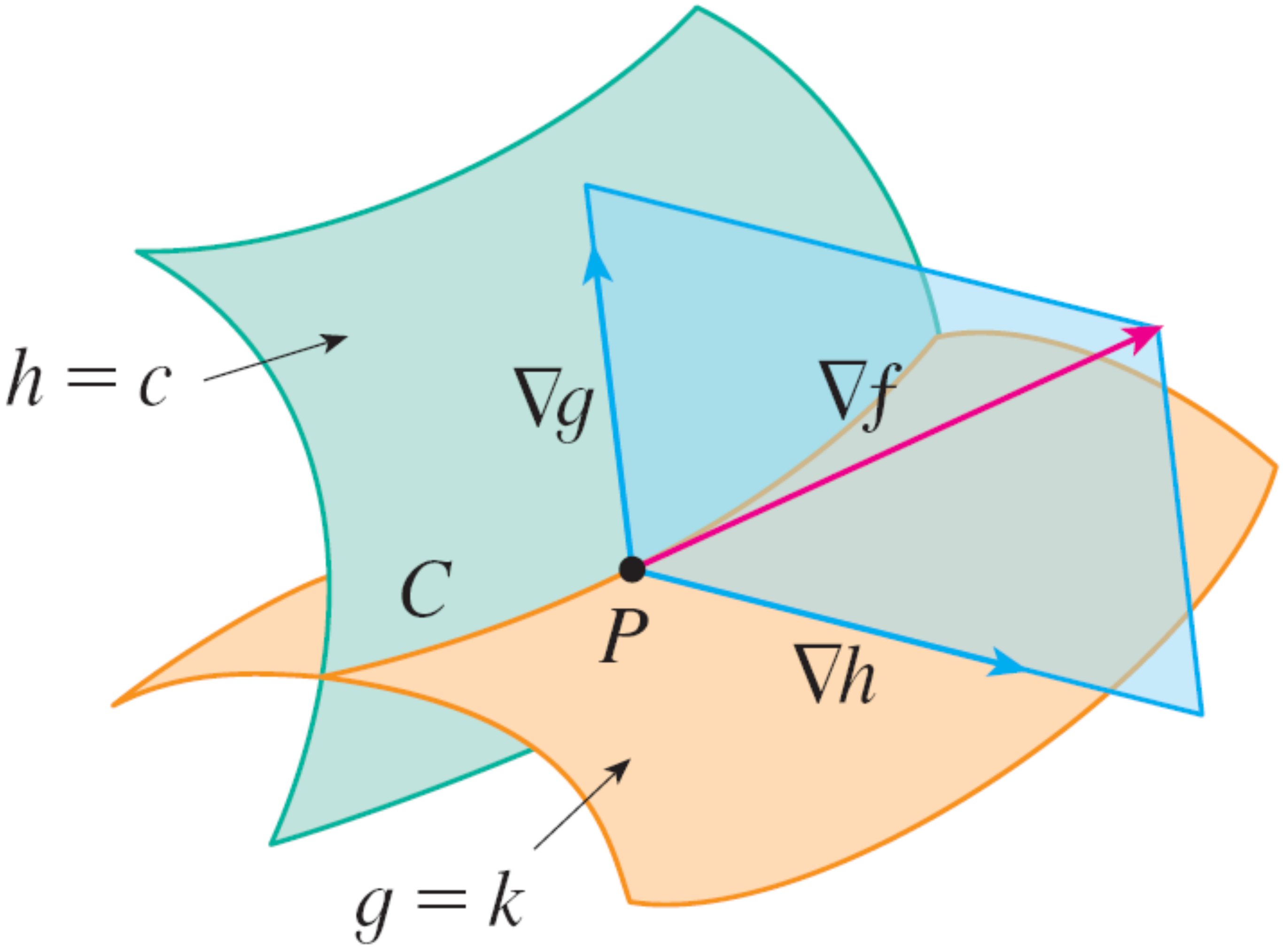
Two Constraints

Suppose that $f(x,y,z)$, $g(x,y,z)$, $h(x,y,z)$ are differentiable and $f(x,y,z)$ obtains local extreme value at (x_0,y_0,z_0) when restricted to the curve of intersection, C, of level surfaces $g(x,y,z)=k$ and $h(x,y,z)=l$. If $\vec{\nabla} g(x_0,y_0,z_0)$ and $\vec{\nabla} h(x_0,y_0,z_0)$ are linearly independent, then there are constants λ and μ such that $\vec{\nabla} f(x_0,y_0,z_0) = \lambda \vec{\nabla} g(x_0,y_0,z_0) + \mu \vec{\nabla} h(x_0,y_0,z_0)$.

pf: let $S_1 : g=k$. $S_2 : h=l$. $C : S_1 \cap S_2$.

$C : \vec{r}(t) = (x(t), y(t), z(t))$. $\vec{r}(t_0) = (x_0, y_0, z_0)$.

Finding Extreme Values under Constraints

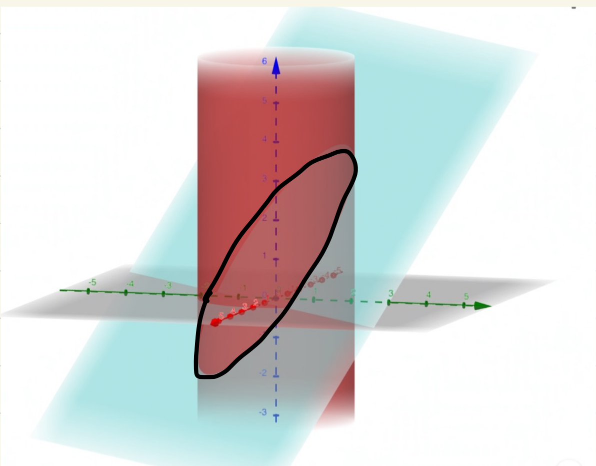


Finding Extreme Values under Constraints

- In this case Lagrange's method is to look for extreme values by solving five equations which are $g(x, y, z) = k$, $h(x, y, z) = c$, and

16 $$\boxed{\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)}$$

Ex: Find the maximum value of $f(x,y,z) = x+2y+3z$ on the curve of intersection of the plane $x-y+z=1$ and the cylinder $x^2+y^2=1$.



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The maximum value of $f(x, y)$ subject to the constraint $g(x, y) = 290$ is 6000. The method of Lagrange multipliers gives $\lambda = 30$. Find an approximate value for the maximum of $f(x, y)$ subject to the constraint $g(x, y) = 292$.}

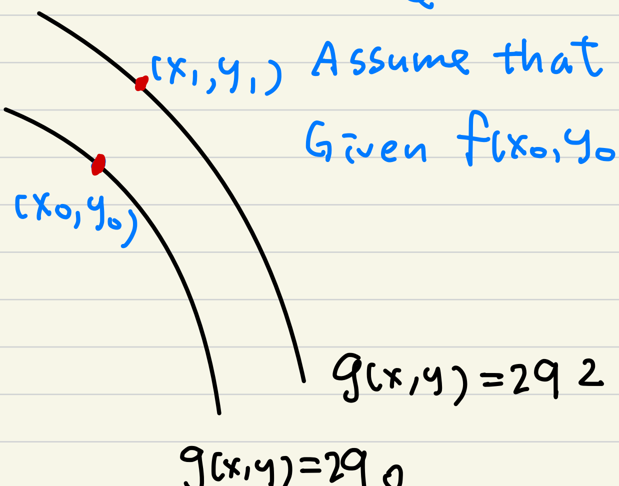
$f_{max} \approx$

$$f(x_1,y_1) - f(x_0,y_0) \approx f_x(x_0,y_0)(x_1-x_0)$$
$$+ f_y(x_0,y_0)(y_1-y_0)$$
$$= \vec{\nabla} f(x_0,y_0) \cdot (x_1-x_0,\ y_1-y_0)$$

sol: Assume that on $g=290$, f obtains maxi at (x_0,y_0).

Assume that on $g=292$, f obtains maxi at (x_1,y_1).

Given $f(x_0,y_0)=6000$, estimate $f(x_1,y_1)$.



Finding Extreme Values under Constraints

- The method of Lagrange multipliers can be applied to find extreme values of a function of n variables, say $f(\vec{x})$ where $\vec{x}$ is a vector of n variables, $\vec{x} = (x_1, \ldots, x_n)$ subject to $m \leq n-1$ constraints ,

$$g_1(\vec{x}) = k_1, \ldots, g_m(\vec{x}) = k_m \,.$$

Finding Extreme Values under Constraints

- Assume that f and all of the function g_j have continuous first derivatives in a neighborhood of the point P where the extreme value occurs, and the intersection of the constraint surface is **smooth** near P. Then P is the critical point of the $(n+m)$-variable *Lagrangian function :*

$$L(\vec{x}, \lambda_1, \lambda_1, \ldots, \lambda_m) = f(\vec{x}) - \sum_{j=1}^{m} \lambda_j (g_j(\vec{x}) - k_j)$$

Finding Extreme Values under

- At P, $\frac{\partial}{\partial \lambda_j} L = -(g_j(\vec{x}) - k_j) = 0$, for all j. Hence, at P , the constraints $g_j(\vec{x}) = k_j$ are satisfied.

- At P, $\frac{\partial}{\partial x_i} L = \frac{\partial}{\partial x_i} f - \sum_{j=1}^{m} \lambda_j \frac{\partial}{\partial x_i} g_j = 0$, for all i. Hence, at P, $\vec{\nabla} f = \sum_{j=1}^{m} \lambda_j \vec{\nabla} g_j$
i.e. $\vec{\nabla} f$ is a linear combination of $\{\vec{\nabla} g_j\}_{j=1}^{m}$.

Review

- What are local extreme values and absolute extreme values of a function of several variables?
- State Fermat's theorem and the second derivatives test for functions of two variables.
- How do we find extreme values of a function on a closed and bounded set?
- State the method of Lagrange multiplier(s) for one constraint as well as two constraints.