The Indefinite Integral and the Substitution Rule

Section 5.4-5.5

Outline

- 1. Indefinite Integrals
  - Notation
  - Application
- 2. The Substitution Rule
  - For Indefinite Integrals
  - For Definite Integrals
  - Application

The Substitution Rule

- For indefinite integrals:

4 **The Substitution Rule** If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

pf of the substitution Rule :

The Substitution Rule

- Note 1: The Substitution Rule for integration is proved using the Chain Rule for differentiation.
- Note 2: If $u = g(x)$, then $\boxed{du = g'(x)dx}$, so a way to remember the Substitution Rule is to think of dx and du in the formula as *differentials*.

Ex: Compute $\int 2x\sqrt{3+x^2}\,dx$.

Ex: Compute $\int \dfrac{x}{\sqrt{1+4x^2}}\,dx$

Ex: $\int \cos x \sin(\sin x)\, dx$

Ex: $\int \tan x \sec^2 x \, dx$

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx}$$

Ex: $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$

Ex: $\displaystyle\int \tan x\,dx$

Ex: $\int \frac{2^x}{2^x+1}\,dx$

Ex: $\int \frac{1}{x\ln x}\,dx$

Ex: $\int \sec x\,dx$

$$\boxed{\int f(ax)\,dx}$$

Ex: $\int f(ax)\,dx$

Ex: $\int \frac{1}{a^2+x^2}\,dx$

Ex: $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx$, where $a>0$.

Ex: $\displaystyle\int \frac{dx}{x^2+2x+10}$

Complicated Integrals

Ex: $\int (1+e^x)^e \; e^{2x} \, dx$

Ex: $$\int \frac{x^{\frac{1}{2}}}{1+x^{\frac{3}{4}}}\,dx$$

Ex: Find $\int \frac{x^2+x^5}{1+x^6}\,dx$

The Substitution Rule

- For definite integrals:

$\boxed{6}$ **The Substitution Rule for Definite Integrals** If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

- Note: The formula is true even if $g(b) < g(a)$.

Ex: Prove that $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ if

$f(x)$ and $g'(x)$ are continuous.

pf:

Ex: Compute $\int_{e}^{e^4} \frac{1}{x\sqrt{\ln x}}\, dx$.

Ex: By the substitution $u = \frac{1}{x}$, show that for any $a > 1$,

$$\int_{\frac{1}{a}}^{a} \frac{\ln x}{1+x+x^2}\,dx = 0$$

Ex: $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^6 x}{\sin^6 x + \cos^6 x}\, dx$

Ex: Compute $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + 2\cos x}\,dx$ and $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + 2\cos x}\,dx$

Ex: If $a, b > 0$ are constants, show that

$$\int_0^1 x^a (1-x)^b \, dx = \int_0^1 x^b (1-x)^a \, dx .$$

Ex: $\frac{d}{dx} \int_{x}^{x^2} f(x+t)\, dt$

- Application:

7 Integrals of Symmetric Functions Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

Pf:

Ex: Compute $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} x^4 \sin x \, dx$.

Ex: Given $\int_0^2 e^{-x^2} dx = K$, find $\int_0^4 x \cdot e^{-(x-2)^2} dx$.

Review

- What is an indefinite integral? What is the notation for indefinite integrals?
- Write down the formula for the substitution rule (for both indefinite integral and definite integral).