L'Hospital's Rule

Section 4.4

Outline

- 1. L'Hospital's Rule
- 2. Indeterminate Forms
  - Type 0/0, Type ∞/∞
  - Indeterminate Product
  - Indeterminate Differences
  - Indeterminate Powers

L'Hospital's Rule

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

L'Hospital's Rule

- **Note 1:**
  L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of f and g before using l'Hospital's Rule.

L'Hospital's Rule

- **Note 2:**
  L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, " $x \to a$ " can be replaced by any of the symbols $x \to a^+$, $x \to a^-$
  $x \to \infty$, or $x \to -\infty$.

L'Hospital's Rule

- Note 3:
- If $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ does not exist, it doesn't imply that $\lim_{x \to a} \frac{f(x)}{g(x)}$ doesn't exist !

Ex: Compute $\lim_{x\to\infty} \frac{\ln x + \cos x}{\ln x + \sin x}$

Ex: $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$, $g(x) = \sin x$.

Compute $\lim_{x\to a} \frac{f(x)}{g(x)}$ and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$.

L'Hospital's Rule

- Proof the l'Hospital's Rule
- Cauchy's Mean Value Theorem:
- If $f(x)$, $g(x)$ are continuous on $[a,b]$ and differentiable on (a,b) with $g'(x) \neq 0$ for all $x \in (a,b)$, then there is some $c \in (a,b)$ such that $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$.

pf: Consider $h(x) = f(x) - f(a) - \dfrac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a))$.

Pf of L'Hospital's Rule :

Suppose that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

Let $F(x) = \begin{cases} f(x) & , \text{ for } x \in I \setminus \{a\} \\ 0 & , \text{ for } x = a \end{cases}$ $G(x) = \begin{cases} g(x) & , \text{ for } x \in I \setminus \{a\} \\ 0 & , \text{ for } x = a \end{cases}$

Indeterminate Form

- 1. Type $0/0$, Type ∞/∞
- 2. Indeterminate Product: Type $0 \cdot \infty$
- 3. Indeterminate Difference: Type $\infty - \infty$
- 4. Indeterminate Powers: Type 0^0 ∞^0 1^∞

$\boxed{\text{Type } \frac{0}{0}, \frac{\infty}{\infty}}$

Ex: Find $\lim_{x \to 0} \frac{\tan x - x}{x^3}$

Ex: Find $\lim\limits_{x \to 0^-} \dfrac{e^{\frac{1}{x}}}{x}$.

Ex: Find $\lim_{x \to \infty} \frac{e^x}{x^n}$ for some $n \in \mathbb{N}$.

Ex: Find $\lim_{x \to \infty} \frac{\ln x}{x^a}$ for some $a > 0$.

Ex: Find $\lim_{h \to 0} \dfrac{f(1+h) + f(1-h) - 2f(1)}{h^2}$ where $f(x)$ is differentiable near $x=1$ and $f''(1)$ exists.

Ex: Suppose that f is differentiable and $f''(a)$ exists.

Find $\displaystyle\lim_{x \to a} \frac{f(x) - [f(a) + f'(a)(x-a)]}{(x-a)^2}$

Type $0 \cdot \infty$

Ex: Find $\lim\limits_{x \to 0^+} x \ln x$.

Ex: Find $\lim_{x \to 1^+} \ln x \cdot \tan\left(\frac{\pi}{2} x\right)$.

Ex: $\lim_{x \to -\infty} x\left(e^{\frac{1}{x}} - 1\right)$

Type $\infty - \infty$

Ex: Find $\lim\limits_{x \to (\frac{\pi}{2})^-} \sec x - \tan x$.

Ex: Find $\lim_{x\to\infty} x - \ln x$.

Ex: Find $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$.

Ex: $\lim_{x \to \infty} \left(x - x^2 \sin\left(\frac{1}{x}\right) \right)$

Indeterminate Powers 0^0, 1^∞, ∞^0

$\lim\limits_{x\to a} f(x)^{g(x)}$ is an $\boxed{\text{indeterminate power}}$ if and only if

$\lim\limits_{x\to a} g(x)\ln(f(x))$ is an $\boxed{\text{indeterminate product}}$.

$f(x)^{g(x)}$	$g(x)\cdot\ln(f(x))$	
0^0	$0\cdot\infty$	$\because \lim\limits_{x\to a} g(x)=0,\ \lim\limits_{x\to a}\ln f(x)=-\infty$.
1^∞	$\infty\cdot 0$	$\because \lim\limits_{x\to a} g(x)=\infty,\ \lim\limits_{x\to a}\ln f(x)=0$
∞^0	$0\cdot\infty$	$\because \lim\limits_{x\to a} g(x)=0,\ \lim\limits_{x\to a}\ln f(x)=\infty$

Ex: Find $\lim_{x \to 0^+} x^x$.

Ex: $\lim_{x \to 0^+} x^{x^x}$

Ex: Find $\lim_{x\to 0}\left(\frac{2^x+3^x}{2}\right)^{\frac{1}{x}}$, $\lim_{x\to\infty}\left(\frac{2^x+3^x}{2}\right)^{\frac{1}{x}}$.

Ex: Find $\lim_{x \to 0^+} (1 + \sin 2x)^{\cot 3x}$

Ex: Find $\lim_{x \to 0} (1+x)^{\frac{1}{x}}$ and $\lim_{x \to \infty} (1+x)^{\frac{1}{x}}$

Ex: $\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{\frac{1}{x^2}}$

Ex: Suppose that $f(x) = \begin{cases} |x|^x, & \text{if } x \neq 0 \\ A, & \text{if } x = 0 \end{cases}$ is continuous.

Find the constant A. Is $f(x)$ differentiable at $x=0$?

Review

- State l'Hospital's Rule and review its assumptions.
- Recall all indeterminate forms, and compute their limits by l'Hospital's Rule.