Techniques of Integration

Section 7.1-7.3

Outline

- Integration by Parts
- Trigonometric Integrals
- Trigonometric Substitution

Trigonometric Substitution

- In general, we can make a substitution of the form $x = g(t)$ by using the Substitution Rule in reverse.
- To make our calculations simpler, we assume that $g(t)$ has an inverse function; that is, g is one-to-one.
- In this case, $\int f(x)dx = \int f(g(t))g'(t)dt$

Trigonometric Substitution

- In the following table we list trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities.

Trigonometric Substitution

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta, \quad -\dfrac{\pi}{2} \leqslant \theta \leqslant \dfrac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta, \quad -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta, \quad 0 \leqslant \theta < \dfrac{\pi}{2}$ or $\pi \leqslant \theta < \dfrac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Integrals of $\sqrt{a^2-x^2}$

Ex: Compute $\int \sqrt{a^2-x^2}\,dx$, where $a>0$ is a constant.

Ex: Compute $\int \frac{dx}{(a^2-x^2)^{3/2}}$, where $a>0$ is a constant.

Ex: Compute $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$.

Ex: Compute $\displaystyle\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{x\sqrt{1-x^2}}$.

Ex: Compute $\int \sqrt{-2x - x^2}\, dx$

Integrals of $\sqrt{a^2-x^2}$

Let $x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ where $a > 0$. Then

1. $\sqrt{a^2 - x^2} = \sqrt{a^2(1-\sin^2\theta)} = a\cos\theta$

2. $dx = a\cos\theta \, d\theta$

3. $\sin\theta = \frac{x}{a}$, $\cos\theta = \sqrt{1-\sin^2\theta} = \frac{1}{a}\sqrt{a^2-x^2}$.

Integrals of $\sqrt{a^2+x^2}$

Let $x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, where $a > 0$. Then

① $\sqrt{a^2+x^2} = \sqrt{a^2(1+\tan^2\theta)} = a\sec\theta$

② $dx = a\sec^2\theta \, d\theta$

③ $\tan\theta = \frac{x}{a}$, $\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{x}{a}\right)^2} = \frac{1}{a}\sqrt{a^2+x^2}$.

Ex: Compute $\int \frac{1}{\sqrt{4+x^2}}\, dx$.

Ex: Compute $\int \frac{dx}{x^2\sqrt{x^2+9}}$.

Ex: Compute $\int \sqrt{a^2+x^2}\,dx$, where $a>0$ is a constant.

Ex: $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+5}}$

Integrals of $\sqrt{x^2-a^2}$

Let $x = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$, $\pi \le \theta < \frac{3}{2}\pi$, where $a>0$. Then

① $\sqrt{x^2-a^2} = \sqrt{a^2(\sec^2\theta-1)} = a\tan\theta$

② $dx = a\sec\theta \cdot \tan\theta \, d\theta$

③ $\sec\theta = \frac{x}{a}$, $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{x}{a}\right)^2 - 1} = \frac{1}{a}\sqrt{x^2-a^2}$

Ex: Compute $\int \frac{dx}{\sqrt{x^2-a^2}}$, where $a>0$ is a constant.

Ex: Compute $\int \frac{1}{e^x\sqrt{e^{2x}-1}}\,dx$

Ex: Compute $\displaystyle\int \frac{x^3}{(x^2-1)^{3/2}}\,dx$

Ex: $\int x^2 \sqrt{x^6 - 1}\, dx$

Review

- Write down the formula for integration by parts.
- How do we integrate the powers of $\sin x$ and $\cos x$?
  How do we integrate the powers of $\tan x$ and $\sec x$?
- When should we use the trigonometric substitution?