Surface Integrals

Section 16.6, 16.7

Outline

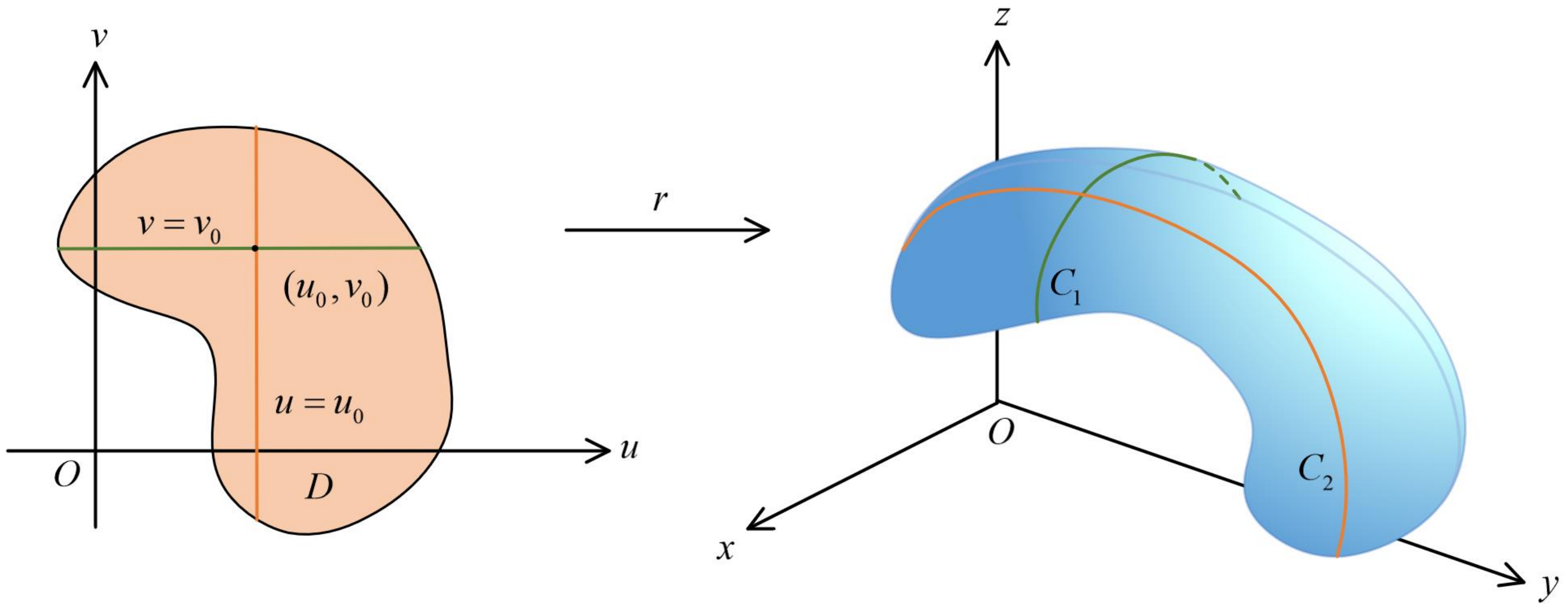
- Parametric Surfaces
  - Tangent Planes
  - Surface Area
- Surface Integrals
  - Surface Integral of a Scalar Function
  - Oriented Surfaces
  - Surface Integrals of Vector Fields

Parametric Surfaces

- We can describe a surface by a vector function $\vec{r}(u,v)$ of two parameters u and v .
- We suppose that
  $$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$
  is a vector-valued function defined on a region D in the uv-plane.
- The set of the image of $\vec{r}(u,v)$ is called a **parametric surface** S and equations $\vec{r}(u,v)$ are called **parametric equations** of S.

Parametric Surfaces

- There are two useful families of curves that lie on S, one family with u constant and the other with v constant. We call these curves **grid curves**.

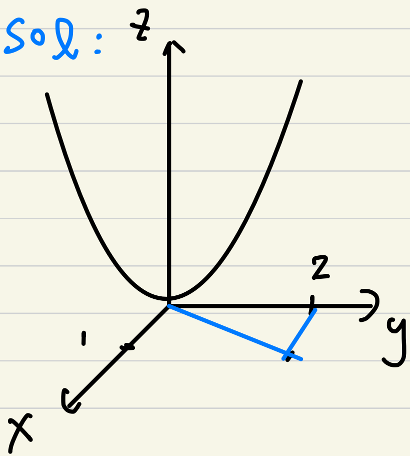


Ex: $\boxed{\text{Graphs of functions}}$. $z = f(x,y)$, $(x,y) \in D$.

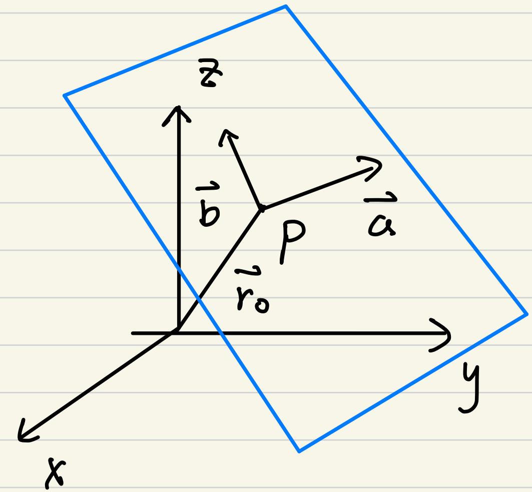
sol:

Ex: Parametrize the surface $z = x^2 + 2y^2$ above the triangle T on the xy-plane with vertices $(0,0)$, $(1,2)$, $(0,2)$.

sol:



Ex: Parametrize the plane passing through $P = \vec{r}_0$ containing nonparallel vectors $\vec{a}$, $\vec{b}$.



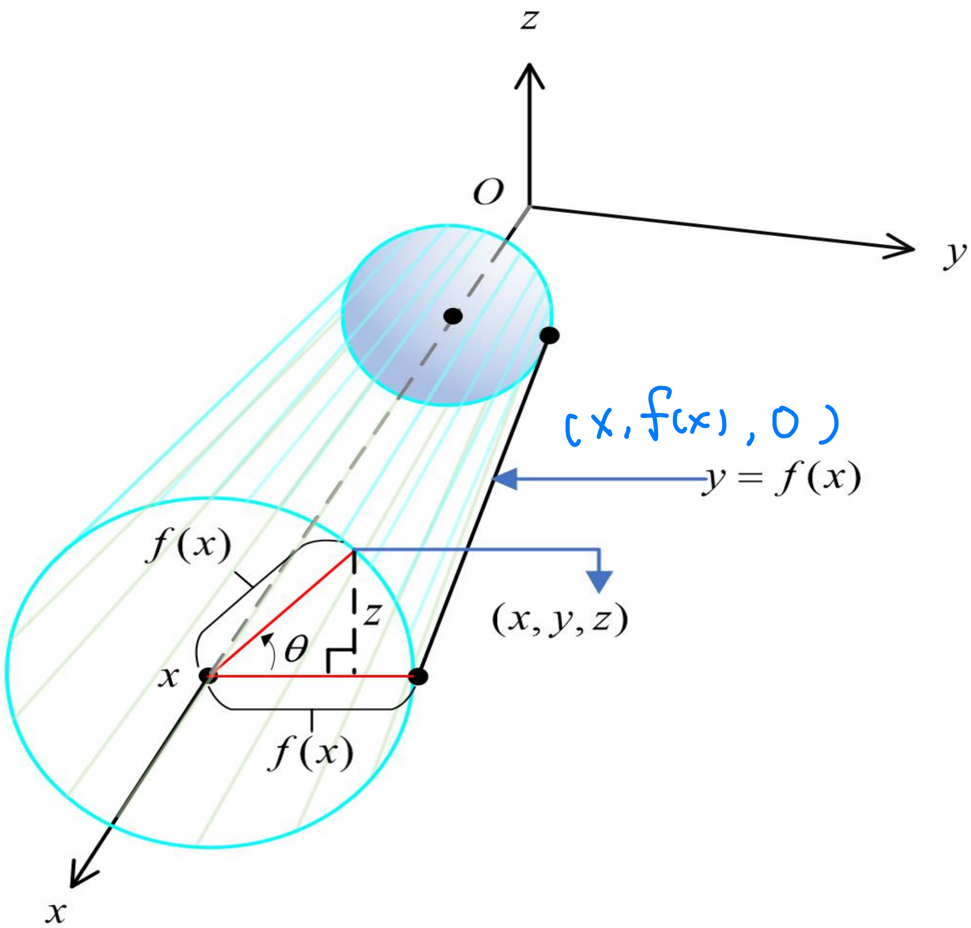
Sol:

Parametric Surfaces

- Example:
- Surfaces of revolution can be represented parametrically. For instance, let's consider the surface S obtained by rotating the curve $y = f(x), a \leq x \leq b$ about the x -axis, $f(x) \geq 0$ Let θ be the angle of rotation.
- If (x, y, z) is a point on S, then

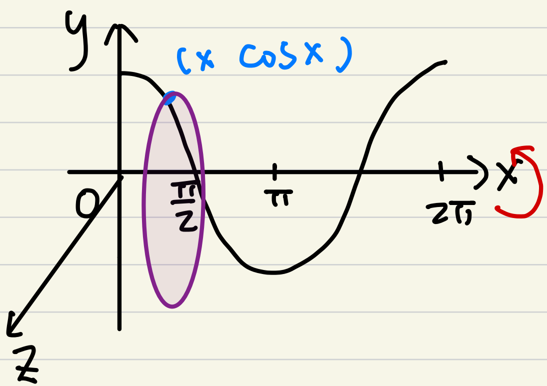
$$x = x \qquad y = f(x)\cos\theta \qquad z = f(x)\sin\theta$$

- Surfaces of Revolution

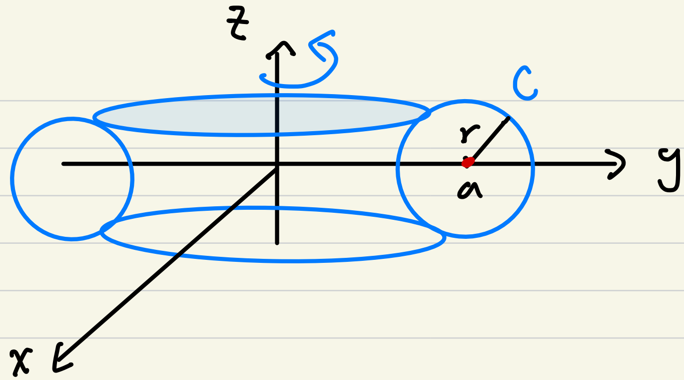


Ex: S is obtained by rotating the curve $y = \cos x$, $0 \le x \le 2\pi$, about the x-axis. Parametrize S.

Sol:



Ex: Parametrize torus



, $0 < r < a$.

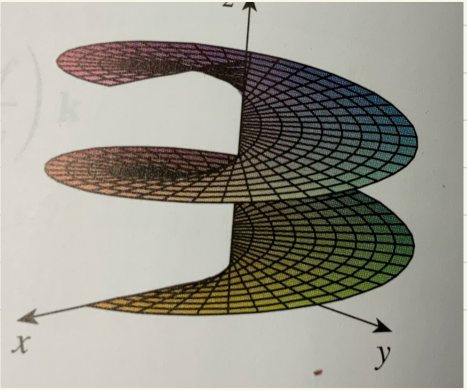
Sol:

Ex: $\vec{r}(u,v) = 2\cos u\,\vec{i} + v\,\vec{j} + 2\sin u\,\vec{k}$, $0 \le u \le \frac{\pi}{2}$, $0 \le v \le 2$.

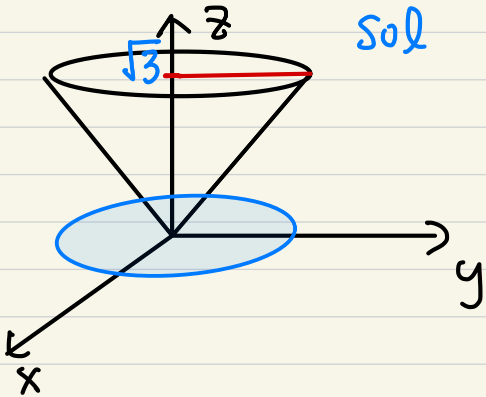
sol :

Ex: $\vec{r}(u,v) = u\cos v\,\vec{i} + u\sin v\,\vec{j} + v\,\vec{k}$, $0 \le u \le 2$, $0 \le v \le 2\pi$

sol:



Ex: Parametrize the cone $z = \sqrt{3}\sqrt{x^2+y^2}$, $0 \le z \le \sqrt{3}$.

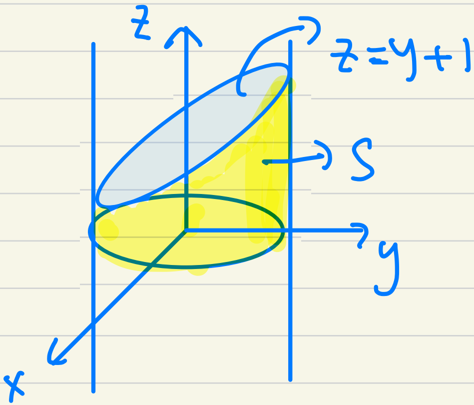


Sol

Ex: Parametrize the surface $z=x^2$ in the first octant and under the paraboloid $z=1-8x^2-y^2$.

Ex: Parametrize the cylinder $x^2+y^2=1$ between $z=0$ and $z=y+1$.

Parametrize the plane $z=y+1$ inside the cylinder $x^2+y^2=1$.

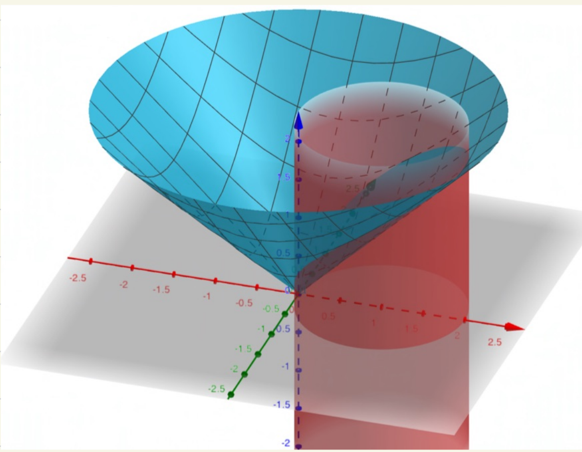


Sol:

b) The part of the cone $z = \sqrt{2(x^2 + y^2)}$ that lies below the plane $z = 1 + y$. (Hint. First find the projection of the surface onto xy-plane)

Ex: Parametrize the part of the cylinder $(x-1)^2 + y^2 = 1$ that is above the xy-plane and under $z = \sqrt{x^2+y^2}$.

sol:



Tangent Planes

- We now find the tangent plane to a parametric surface S traced out by a vector function
$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$
at a point P_0 with position vector $\vec{r}(u_0,v_0)$.
- These are the tangent vectors of the grid curves at P_0 :

$$\vec{r}_u = \frac{\partial x}{\partial u}(u_0,v_0)\vec{i} + \frac{\partial y}{\partial u}(u_0,v_0)\vec{j} + \frac{\partial z}{\partial u}(u_0,v_0)\vec{k}$$
$$\vec{r}_v = \frac{\partial x}{\partial v}(u_0,v_0)\vec{i} + \frac{\partial y}{\partial v}(u_0,v_0)\vec{j} + \frac{\partial z}{\partial v}(u_0,v_0)\vec{k}$$

Tangent Planes

- If $\vec{r}_u \times \vec{r}_v$ is not $\vec{0}$, then the surface S is called **smooth** (it has no "corners").
- For a smooth surface, the **tangent plane** is the plane that contains the tangent vectors $\vec{r}_u$ and $\vec{r}_v$, and the vector $\vec{r}_u \times \vec{r}_v$ is a normal vector to the tangent plane.

Ex: Find the tangent plane equation of the surface

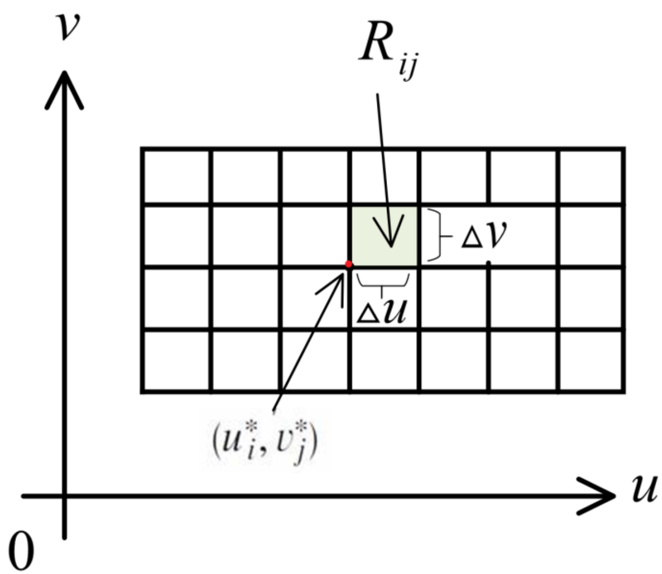
$\vec{r}(r,\theta) = (r\cos\theta,\ r\sin\theta,\ r)$ at $(1,0,1)$.

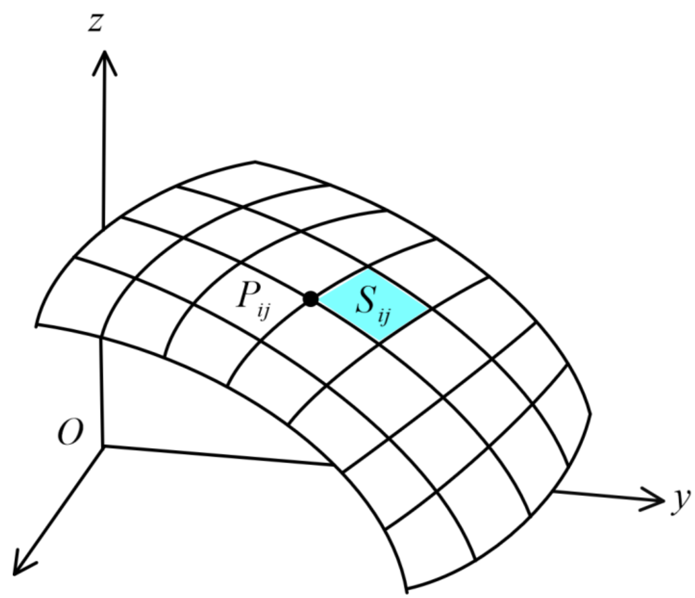
sol:

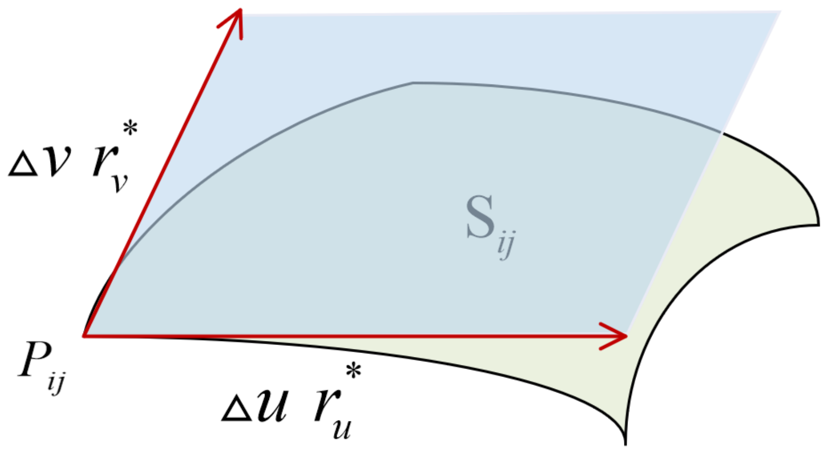
Surface Area

- To find the area of a parametric surface, we start by considering a surface whose parameter domain R is a rectangle, and we divide it into subrectangles R_{ij} . Let's choose (u_i^*, v_j^*) to be the lower left corner of R_{ij} .
- The part of S_{ij} the surface that corresponds to R_{ij} is called a *patch* and has the point P_{ij} with position vector $\vec{r}(u_i^*, v_j^*)$ as one of its corners.

Surface Area







Surface Area

- Let $\vec{r^*}_u = \vec{r}_u(u_i^*, v_j^*)$ and $\vec{r^*}_v = \vec{r}_v(u_i^*, v_j^*)$ be the tangent vectors at P_{ij} .
- The two edges of the patch that meet at P_{ij} can be approximated by vectors $\Delta u\, \vec{r^*}_u$ and $\Delta v\, \vec{r^*}_v$.
- So we approximate S_{ij} by the parallelogram determined by the vectors $\Delta u\, \vec{r^*}_u$ and $\Delta v\, \vec{r^*}_v$.

Surface Area

- And the area of S_{ij} can be approximated by $|(\Delta u\, \vec{r^*}_u) \times (\Delta v \vec{r^*}_v)| = |\vec{r^*}_u \times \vec{r^*}_v| \Delta u \Delta v$
- So an approximation to the area of S is $$\sum_{i=1}^{m} \sum_{j=1}^{n} |\vec{r^*}_u \times \vec{r^*}_v| \Delta u \Delta v$$
- After taking the limit as the partition becomes finer and finer, we have the area formula.

Surface Area

- Definition: If a smooth parametric surface S is given by the equation

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}, \quad (u,v) \in D$$

and S is covered just once as (u,v) ranges throughout D, then the **surface area** of S is

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$$

Ex: Find the area of the surface $S: z = f(x, y)$, $(x, y) \in D$.

Sol:

Surface Area

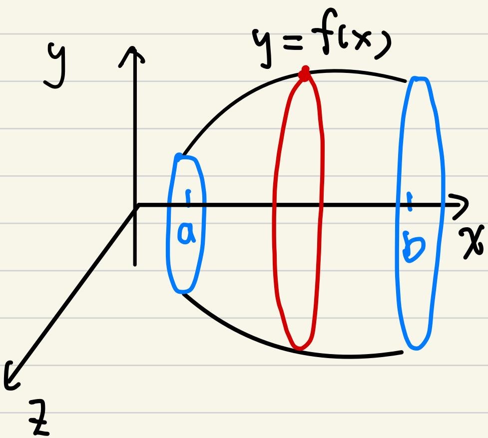
- For the special case of a surface S with equation $z = f(x, y)$, where (x, y) lies in D and f has continuous partial derivatives, we take x and y as parameters. The parametric equations are $x = x$, $y = y$, and $z = f(x, y)$. Then $|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}$

$$A(S) = \iint_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} dA$$

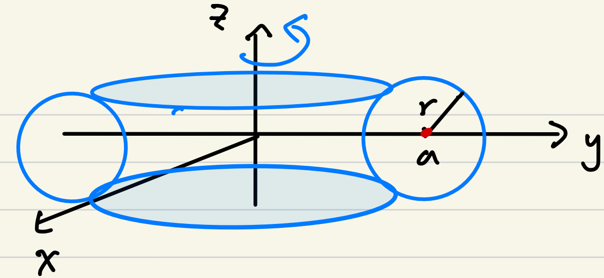
Ex: Find the area of the surface of revolution :

The curve $y=f(x)$, $a \le x \le b$, $f(x) \ge 0$ is rotated about the x-axis.

Sol: $\vec{r}(x,\theta) = (x, f(x)\cos\theta, f(x)\sin\theta)$



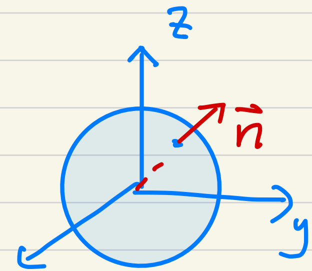
Ex: Find the area of the torus where $0 < r < a$.



sol:

Ex: Find the area of a sphere with radius $a>0$.

Sol:



Ex: Find the area of the cylinder $(x-1)^2+y^2=1$ that is above the xy-plane and under $z=\sqrt{x^2+y^2}$.

sol: