The Mean Value Theorem and its Applications

Section 4.2-4.3

Outline

- 1. The Mean Value Theorem
  - Lemma: Rolle's Theorem
  - Theorem: The Mean Value Theorem
  - Corollary
- 2. How Derivatives Affect the Shape of a Graph
  - First Derivatives: Increasing / Decreasing Test
  - Second Derivatives: Concavity Test

The Mean Value Theorem

- There is a lemma preparing for the Mean Value Theorem.

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Proof of Rolle's Theorem :

Ex: Prove that the equation $x^5 + 2x - 1 = 0$ has exactly one real root.

Ex: Prove that $f(x) = x + 2e^x$ is 1-1.

The Mean Value Theorem

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

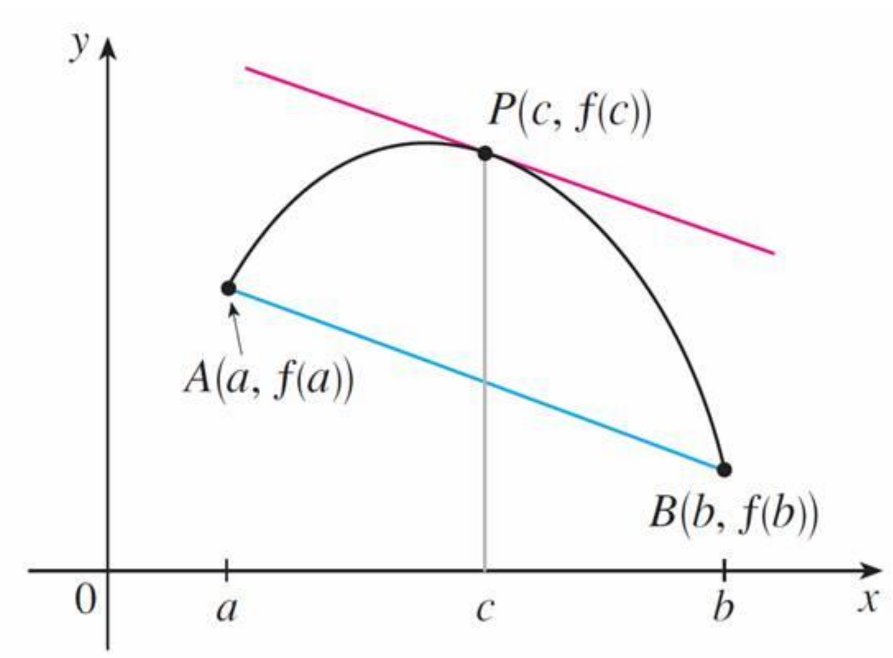
Then there is a number c in (a, b) such that

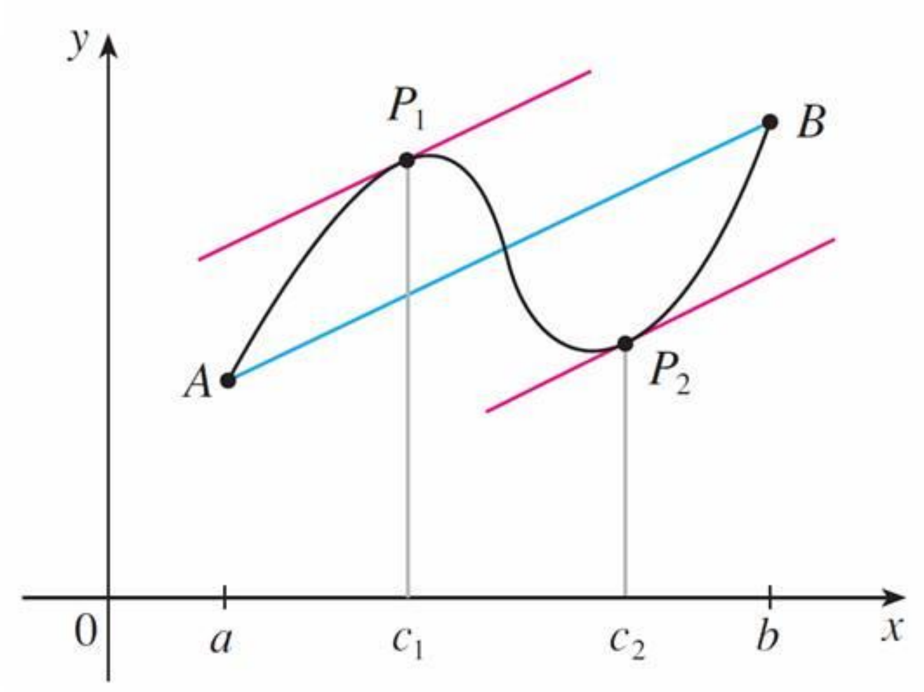
$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a) \tag{2}$$

The Mean Value Theorem





Proof of the Mean Value Theorem.

The Mean Value Theorem

- Applications:

5 Theorem If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

7 Corollary If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

Ex: Prove that if $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).

Ex: Simplify $\tan^{-1} x + \tan^{-1} \frac{1}{x}$.

Ex: Show that $|\sin a - \sin b| \leq |a-b|$.

The Mean Value Theorem

- Applications:
- The Racetrack Principle
- Suppose that $f(0) = g(0)$ and $f'(x) > g'(x)$ for $x > 0$, then $f(x) > g(x)$ for $x > 0$.

Proof of the Racetrack Principle.

Ex: Show that $(1+x)^r \geq 1+rx$ for $x>0$ if $r>1$.

Ex: a) Show that $e^x > 1+x$ for $x>0$

b) Show that $e^x > 1+x+\frac{x^2}{2}$ for $x>0$.

The Mean Value Theorem

- Applications
- Theorem: Suppose that $f(x)$ is continuous at $x = a$, and differentiable on an open interval containing a but possibly except a.
  If $\lim_{x \to a} f'(x) = L$, then $f(x)$ is differentiable at $x = a$ and $f'(a) = L$.
  If $\lim_{x \to a} f'(x) = \pm\infty$, then $f(x)$ has vertical tangent at $x = a$.

Ex: Suppose that $f(x)$ is differentiable on an open interval containing a except possiblely at $x=a$ and $f(x)$ is continuous at $x=a$. Prove that if $\lim_{x \to a} f'(x) = L$, then $f(x)$ is differentiable at $x=a$ and $f'(a) = L$.

Suppose that $f(x)$ is continuous at $x=a$, $f'(x)$ is discontinuous at $x=a$. Discuss some possible cases.

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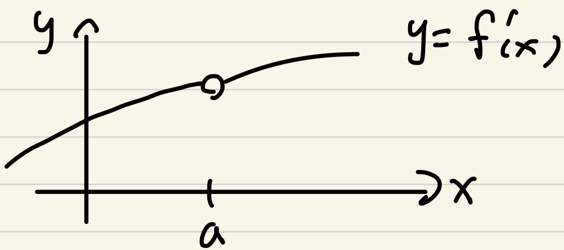

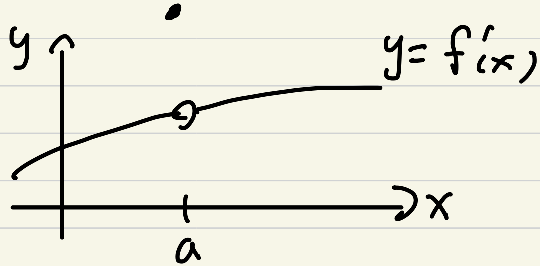

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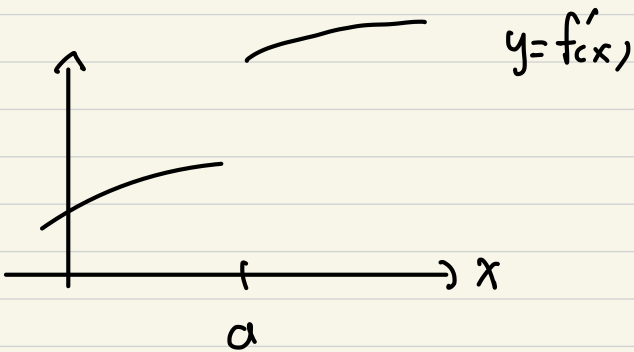

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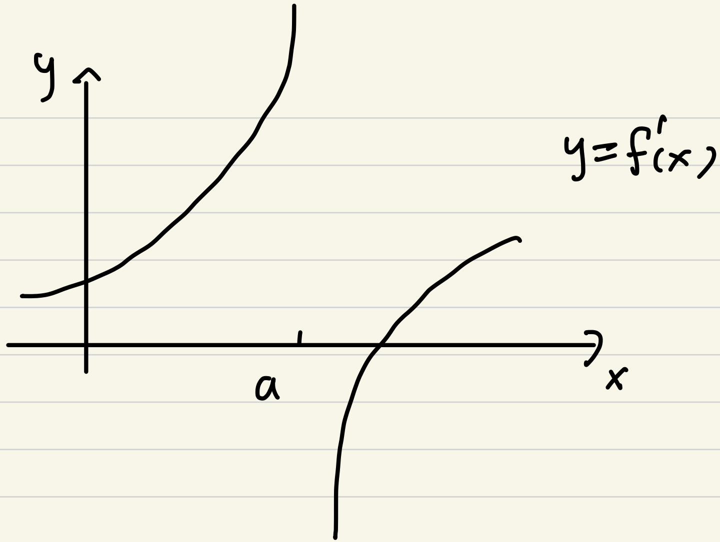

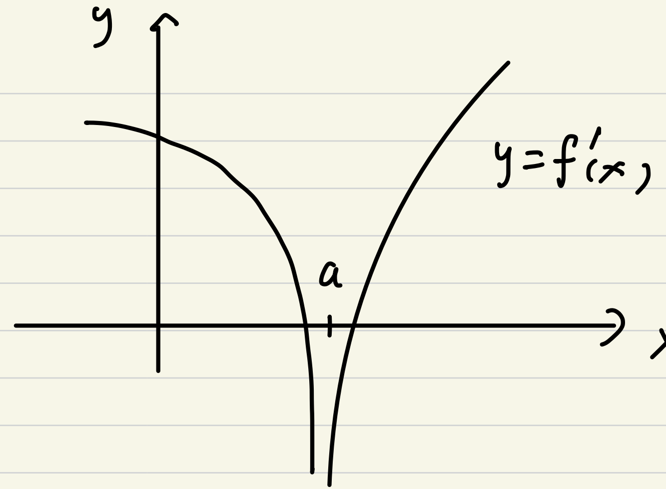

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