Taylor Series and Its Applications

Section 11.10, 11.11

Outline

- Taylor Series
  - Definitions of Taylor Series and Maclaurin Series
  - Taylor's Inequality
  - Examples
  - Multiplication and Division of Power Series
  - Applications

Taylor Series

- We have found power series representations for a certain class of functions. Now we want to investigate
  - Which functions have power series representations ?
  - How can we find such representations?

Taylor Series

- We start by assuming that f is any function that can be represented by a power series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

for $|x-a| < R$

- Then we can derive all the coefficients.

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Ex: If $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$ for $|x-a|<R$, then show

that $\boxed{c_n = \dfrac{f^{(n)}(a)}{n!}}$.

$\boxed{f^{(0)}(x) = f(x),\ (x-a)^0 = 1,\ 0! = 1}$

Sol:

Taylor Series

- If f has a power series expansion at $x = a$, then it must be of the following form.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

- The above series is called the **Taylor series** of the function f at a (or about a or centered at a).

Taylor Series

- For the special case $a = 0$ the Taylor series becomes
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$
$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$
- This case arises frequently enough that it is given the special name **Maclaurin series**.

Ex: Let $f(x) = \tan^{-1}(2x)$. Find $f^{(101)}(0)$.

Sol:

Taylor Series

- We have shown that $1/(1-x)$, $\arctan(x)$, and $\ln(1+x)$ have power series representations. Hence these functions are equal to the sums of their Taylor series.
- However, in general, a function's Taylor series may not converge to the function itself !
- Ex: $$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$ $f^{(n)}(0) = 0$

is not equal to its Maclaurin series.

Taylor's Inequality

- Under what circumstances is a function equal to the sum of its Taylor series?
- Let the nth partial sum of Taylor Series (the ***n*th-degree Taylor polynomial of f at a**) be
$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$
- Let the nth remainder term be
$$R_n(x) = f(x) - T_n(x) \; .$$

Taylor's Inequality

8 Theorem If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n \to \infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

9 Taylor's Inequality If $|f^{(n+1)}(x)| \leqslant M$ for $|x - a| \leqslant d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x - a|^{n+1} \qquad \text{for } |x - a| \leqslant d$$

Taylor's Inequality

- Taylor's inequality can help us to show that the remainder term $R_n(x)$ goes to 0 as n goes to infinity.
- Taylor's inequality can be proved by expressing $R_n(x)$ as the *integral form*. If $f^{(n+1)}(x)$ is continuous on an open interval containing x and a, then

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt$$

$$R_0(x) = f(x) - T_0(x) = f(x) - f(a) = \int_a^x f'(t)\,dt$$

$$= (t-x) f'(t) \Big|_{t=a}^{t=x} - \int_a^x (t-x) f''(t)\,dt$$

$$= -(a-x) f'(a) + \int_a^x (x-t) f''(t)\,dt$$

$$\Rightarrow \underbrace{f(x) - f(a) - f'(a)(x-a)}_{= f(x) - T_1(x)} = \int_a^x (x-t) f''(t)\,dt$$

Taylor's Inequality

- Another way to prove Taylor's Inequality is by the Lagrange's formula of the remainder term which says that there is some z between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

Ex: Find Taylor series of $f(x) = e^x$ at $x=0$.

Show that $f(x)$ equals to its Taylor series on $\mathbb{R}$.

Sol:

Examples of Taylor Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for all } \ x$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Ex: Find the Taylor Series of $f(x) = e^x$ at $x = a$.

Sol:

Ex: Find Taylor series of $f(x) = \sin x$ at $x=0$.

Show that $f(x)$ equals to its Taylor series on $\mathbb{R}$.

Sol:

Examples of Taylor Series

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \text{ for all } x.$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \text{ for all } x.$$

Ex: Find Taylor series of $f(x) = \cos x$ at $x = \frac{\pi}{3}$.

Sol:

Ex: Show that $e^{i\theta} = \cos\theta + i\sin\theta$ for $\theta \in \mathbb{R}$.

Sol:

Examples of Taylor Series

- The Binomial Series: For any k and $|x| < 1$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \qquad \binom{k}{0} = 1.$$

$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

- The binomial series converges at 1 if $-1 < k < 0$ and at both endpoints if $k \geq 0$.

Ex: Find Taylor series of $f(x) = (1+x)^k$ at $x=0$.

Sol:

Ex: Write down the Maclaurin Series for $(1+x)^{\frac{1}{2}}$.

Sol:

Ex: If $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n$, show that $(1+x)\,g'(x) = k \cdot g(x)$, for $|x|<1$.

Sol: $$g'(x) = \sum_{n=1}^{\infty} n \cdot \binom{k}{n} x^{n-1}$$

$$(1+x)\,g'(x) = \sum_{n=1}^{\infty} n \cdot \binom{k}{n} x^{n-1} + \sum_{n=1}^{\infty} n \binom{k}{n} x^n$$

$$= \sum_{n=0}^{\infty} (n+1) \binom{k}{n+1} x^n + \sum_{n=1}^{\infty} n \binom{k}{n} x^n .$$

$$= k + \sum_{n=1}^{\infty} \Big[\underbrace{(n+1)\binom{k}{n+1} + n \binom{k}{n}}_{k \binom{k}{n}} \Big] x^n = k \cdot g(x) .$$

Ex: Find Taylor series for $f(x) = \frac{1}{\sqrt{9-x}}$ at $x=0$

Sol:

Ex: Find Taylor series for $f(x) = (7+x)^{\frac{1}{3}}$ at $x=1$.

Sol:

Ex: Find the Maclaurin series for $f(x) = \sin^{-1} x$.

Sol:

Examples of Taylor Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots \qquad R = 1$$

Multiplication and Division of Power Series

- Power series can be multiplied and divided like polynomials.
- Examples:
- Find the first three terms of Maclaurin series for $e^x \sin x$ and $\tan x$.

Applications

- Approximating functions by polynomials:
- If f is the sum of its Taylor series, we know that $T_n(x)$ approaches $f(x)$ as n goes to infinity and so $T_n(x)$ can be used as an approximation to $f(x)$.
- Notice that the first-degree Taylor polynomial

$$T_1(x) = f(a) + f'(a)(x - a)$$

is just the linearization of f at a .

Applications

- In general, it can be shown that the derivatives of T_n at $x = a$ agree with those of f up to and including derivatives of order n.
- When using a Taylor polynomial T_n to approximate a function f, we have to ask the questions:
  How good an approximation is it?

Applications

- To answer these questions we need to look at the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|$$

- There are three possible methods for estimating the size of the error:

Applications

- **1.** If a graphing device is available, we can use it to graph $|R_n(x)|$ and thereby estimate the error.
- **2.** If the series happens to be an alternating series, we can use the Alternating Series Estimation Theorem.
- **3.** In all cases we can use Taylor's Inequality which says that if $|f^{(n+1)}(x)| \leq M$, then
$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

Ex: Find $\displaystyle\lim_{x\to 0} \frac{\ln(1-\frac{x^2}{2}) - \cos(x) + 1}{\tan^{-1}(\frac{1}{3}x^4)}$.

sol:

$$\ln(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \cdots$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

$$\tan^{-1} y = y - \frac{y^3}{3} + \frac{y^5}{5} - \cdots$$

$$\frac{\ln(1-\frac{x^2}{2}) - \cos x + 1}{\tan^{-1}(\frac{1}{3}x^4)} = \frac{\left(-\frac{x^2}{2} - \frac{1}{2}\left(\frac{x^2}{2}\right)^2 - \frac{1}{3}\left(\frac{x^2}{2}\right)^3 - \cdots\right) + \left(\frac{1}{2}x^2 - \frac{1}{4!}x^4 + \frac{1}{6!}x^6 \cdots\right)}{\frac{1}{3}x^4 - \frac{1}{3}\left(\frac{1}{3}x^4\right)^3 + \cdots}$$

$$= \frac{\left(-\frac{1}{4} - \frac{1}{4\times 6}\right)x^4 + O(x^6)}{\frac{1}{3}x^4 + O(x^{12})} \longrightarrow \frac{-\frac{7}{24}}{\frac{1}{3}} = -\frac{7}{8} \quad \text{as } x \to 0.$$

Applications to Physics

- Example: In Einstein's theory of special relativity the mass of an object moving with velocity v is $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$.
- where m_0 is the mass of the object when at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0c^2$$

Applications to Physics

- Show that when v is very small compared with c, this expression for K is very close to the classical Newtonian physics:
$$K = \frac{1}{2} m_0 v^2$$
- Use Taylor's Inequality to estimate the difference in these expressions for K when $|v| \leq 100 \text{ m/s}$.

Ex: Estimate the difference between $K = m_0 c^2 \left(\frac{1}{\sqrt{1-(\frac{v}{c})^2}} - 1 \right)$ and $\frac{1}{2} m_0 v^2$ when $|v| \leq 100\ m/s$.

Sol: Let $f(x) = (1-x)^{-\frac{1}{2}} - 1 = \left[1 + (-\frac{1}{2})(-x) + R_1(x)\right] - 1 = \frac{x}{2} + R_1(x)$.

We know that $R_1(x) = \frac{f''(z)}{2!} x^2$, for some z between 0 & x, and

$f''(x) = \frac{3}{4} (1-x)^{-\frac{5}{2}}$.

Hence $K = m_0 c^2 f((\frac{v}{c})^2) = m_0 c^2 \left(\frac{1}{2} (\frac{v}{c})^2 + R_1((\frac{v}{c})^2) \right)$

$= \frac{1}{2} m_0 v^2 + \boxed{m_0 c^2 \cdot R_1((\frac{v}{c})^2)}$.

$\varepsilon = m_0 c^2 R_1((\frac{v}{c})^2) = \frac{m_0 c^2}{2} \cdot \frac{3}{4} (1-z)^{-\frac{5}{2}} (\frac{v}{c})^4$, where $0 < z < (\frac{v}{c})^2 \leq \frac{10^{-12}}{9}$

$< \frac{3}{8} m_0 (1 - \frac{10^{-12}}{9})^{-\frac{5}{2}} \frac{v^4}{c^2} \leq \frac{6}{8} m_0 \frac{10^8}{9 \times 10^{16}} = \frac{m_0}{12} \times 10^{-8}$

Review

- What are the Taylor series (at $x = a$) and Maclaurin series of a function?
- State the Taylor's inequality for the remainder term.
- Write down the Maclaurin series for e^x, $\sin x$, $\cos x$, $\ln(1+x)$, $\arctan x$, and $(1+x)^k$.

Review

Function	Taylor Series	Radius of Convergence
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$R = \infty$
$\sin x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$R = \infty$
$\cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$R = \infty$
$(1+x)^k$	$\sum_{n=0}^{\infty} \left(\begin{array}{c} k \\ n \end{array} \right) x^n$	$R = 1$