Triple Integrals

Section 15.6

Outline

- Definition: The Limit of Riemann Sums
- Iterated Integrals
- Over a General Bounded Region
  - Type I
  - Type II
  - Type III
- Applications

Definition

- Suppose that f is defined over a rectangular box $B = \{(x, y, z) | a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$
- To find the integral of f on B, the first step is to divide B into sub-boxes. We do this by dividing the interval $[a, b]$ into l subintervals of equal width Δx, dividing $[c, d]$ into m subintervals of width Δy, and dividing $[r, s]$ into n subintervals of width Δz. Thus we divide the box B into lmn sub-boxes each of which has volume $\Delta V = \Delta x \Delta y \Delta z$.

Definition

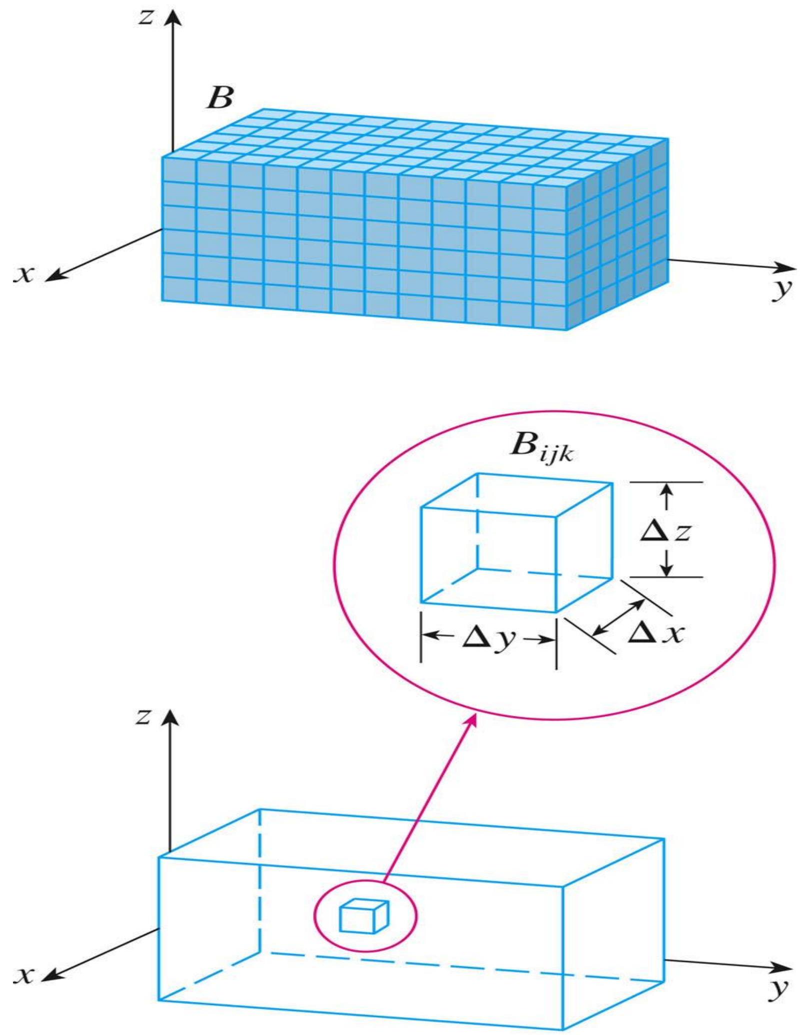


FIGURE 1

Definition

- Then we form the **triple Riemann sum**

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V$$

where the sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ is in

$$B_{ijk} = \{(x, y, z)|x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j, z_{k-1} \le z \le z_k\}$$

- We define the triple integral as the limit of the triple Riemann sums.

3 Definition The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\,dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\Delta V$$

if this limit exists.

Iterated Integrals

- Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals.

$\boxed{4}$ **Fubini's Theorem for Triple Integrals** If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

- There are ***five*** other possible orders in which we can integrate, all of which give the same value.

Over a General Bounded Region

- Now we define the triple integral over a *general bounded region* E in three-dimensional space. We enclose E in a box B. Then we define F so that it agrees with f on E but is 0 for points in B that are outside E. Then

$$\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV$$

- The triple integral of the right hand side exists if f is continuous and the boundary of E is "reasonably smooth."

Over a General Bounded Region: Type I

- A solid region E is said to be of **type I** if it lies between the graphs of two continuous functions of x and y, that is,

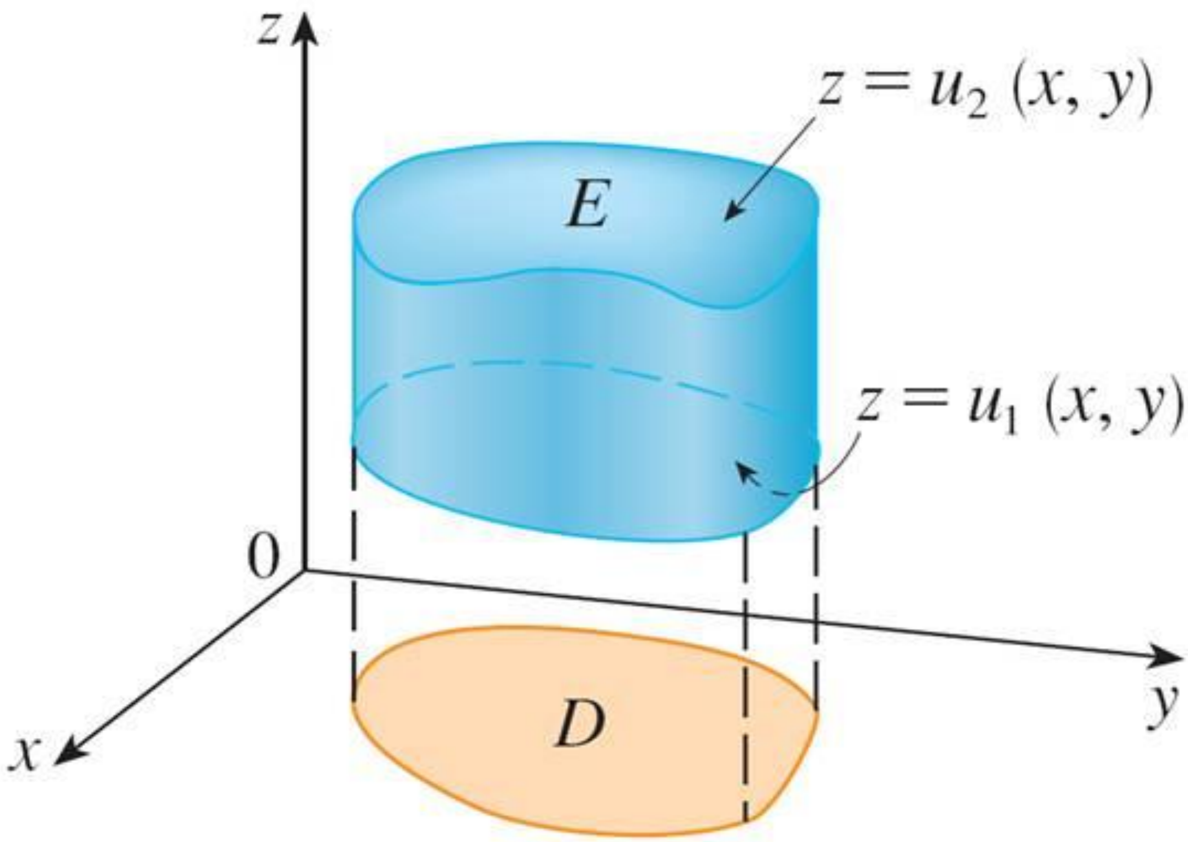
$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is the projection of E onto the xy-plane.

- Then the triple integral over E can be evaluated as

$$\iiint_E f(x, y, z)\, dV = \iint_D \Big[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\Big] dA$$

Over a General Bounded Region: Type I



Over a General Bounded Region: Type I

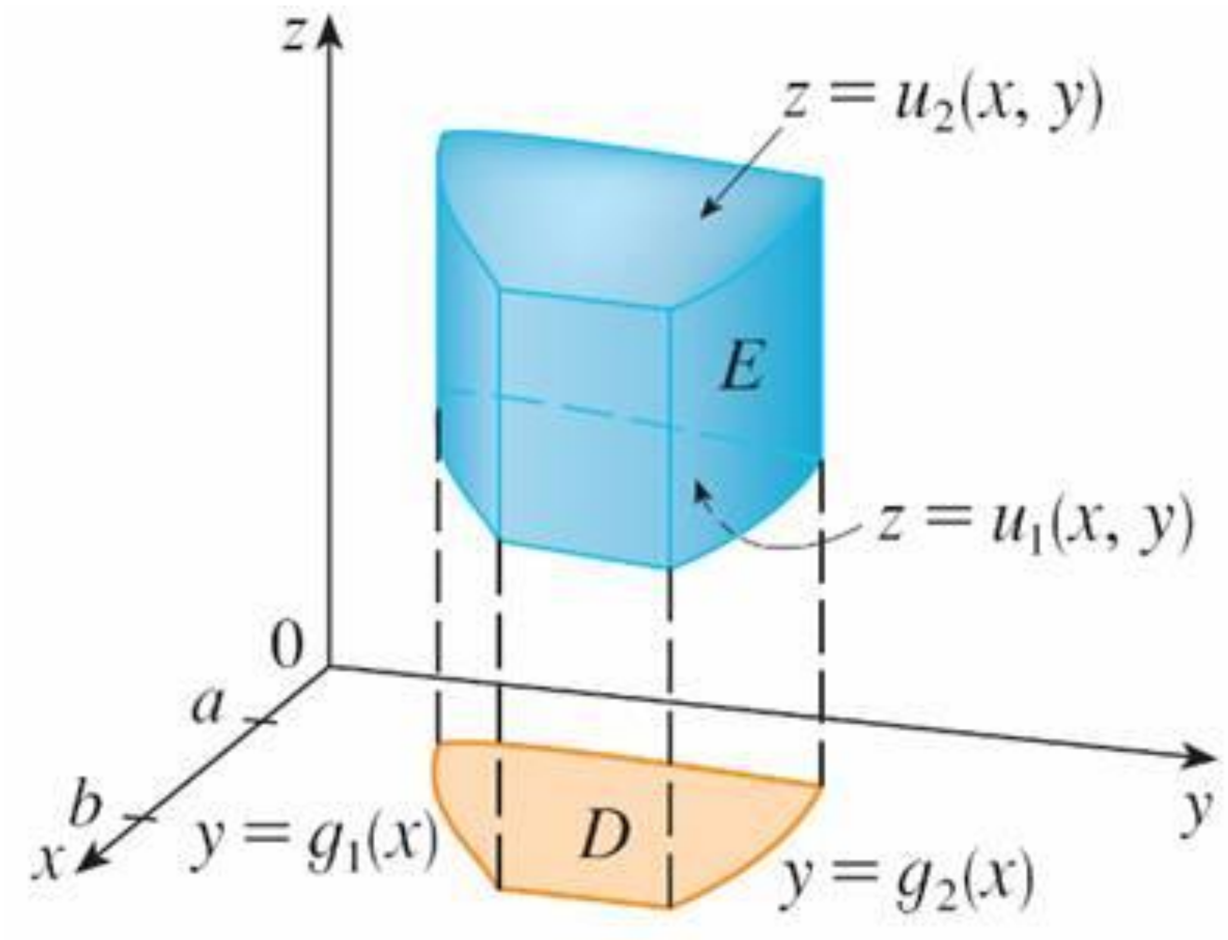
- In particular, if the projection of E onto the xy-plane, D, is a type I plane region, then

$$E = \{(x, y, z) | a \le x \le b, g_1(x) \le y \le g_2(x), \\ u_1(x, y) \le z \le u_2(x, y)\}$$

and the triple integral becomes

$$\iiint_E f(x, y, z) dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) dz dy dx$$

Over a General Bounded Region: Type I



Over a General Bounded Region: Type I

- On the other hand, if D is a type II plane region, then

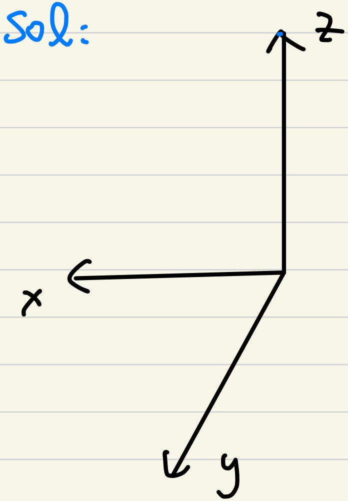
$$E = \{(x, y, z) | c \le y \le d, h_1(y) \le x \le h_2(y), \\ u_1(x, y) \le z \le u_2(x, y)\}$$

- And the triple integral is

$$\iiint_E f(x, y, z) dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) dz dx dy$$

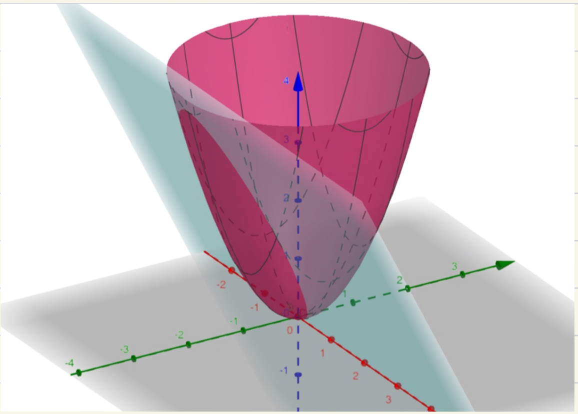
Ex: E is a tetrahedron bounded by $x+2y+z=2$, $x=2y$, $y=0$, and $z=0$. Compute $\iiint_E x\, dV$.

Sol:



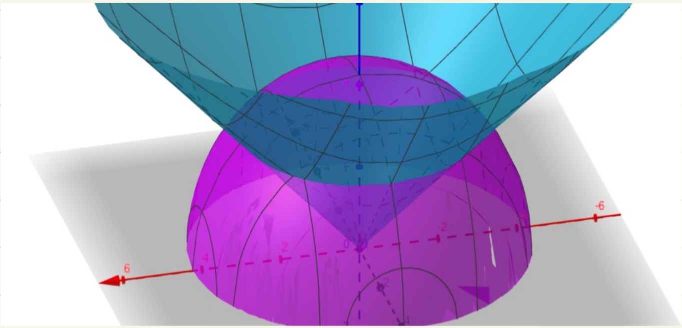
Ex: Find the volume of the solid S lying below the plane $z=-2y$ and above the paraboloid $z=x^2+y^2$.

Sol:



Ex :

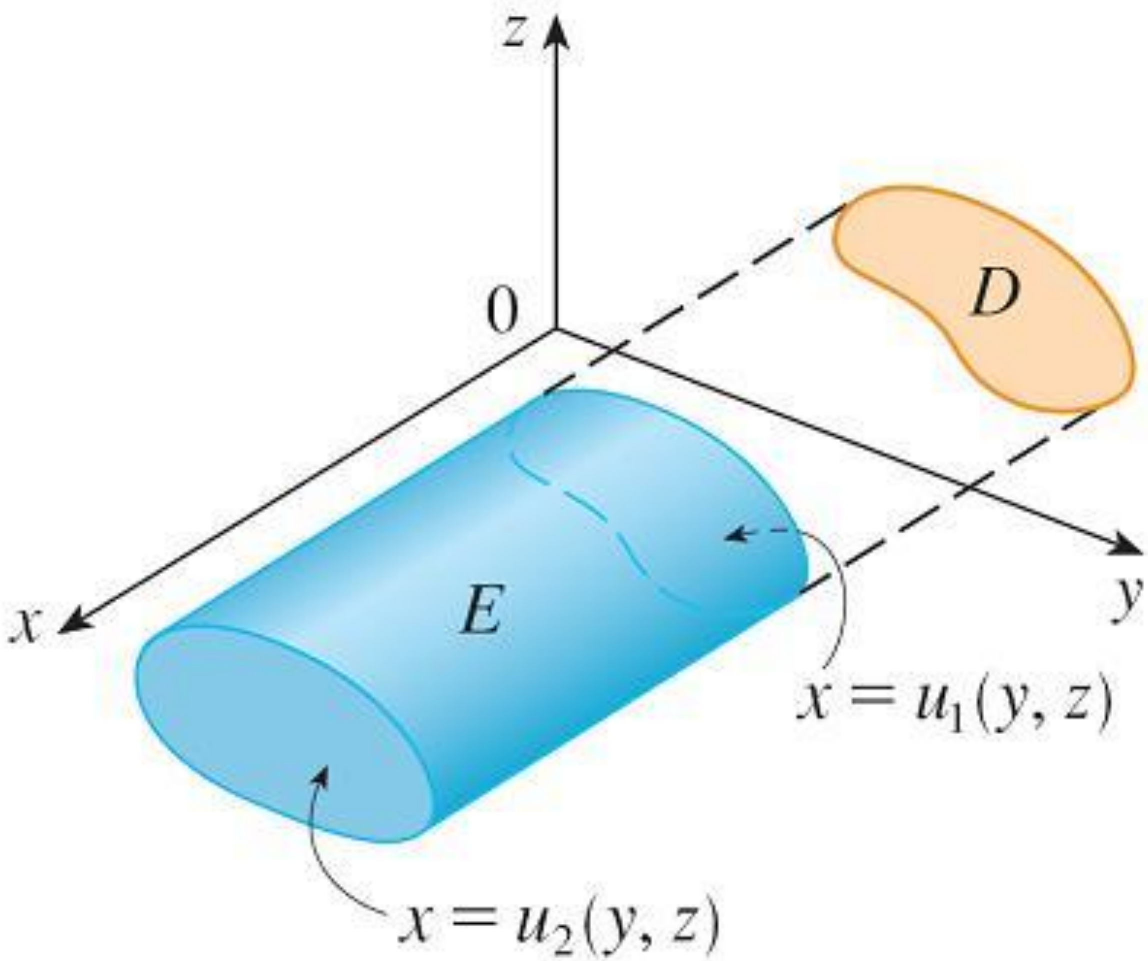
Consider the solid shaped like an ice cream cone that is bounded by the functions $z = \sqrt{x^2 + y^2}$ and $z = \sqrt{18 - x^2 - y^2}$. Set up an integral in polar coordinates to find the volume of this ice cream cone.



Over a General Bounded Region: Type II

- A solid region E is of **type II** if it is of the form
$$E = \{(x, y, z) | (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$
where D is the projection of E onto the yz -plane.
- Then we have
$$\iiint_E f(x, y, z)\, dV = \iint_D [\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx] dA$$

Over a General Bounded Region: Type II



Over a General Bounded Region: Type III

- Finally, a **type III** region is of the form

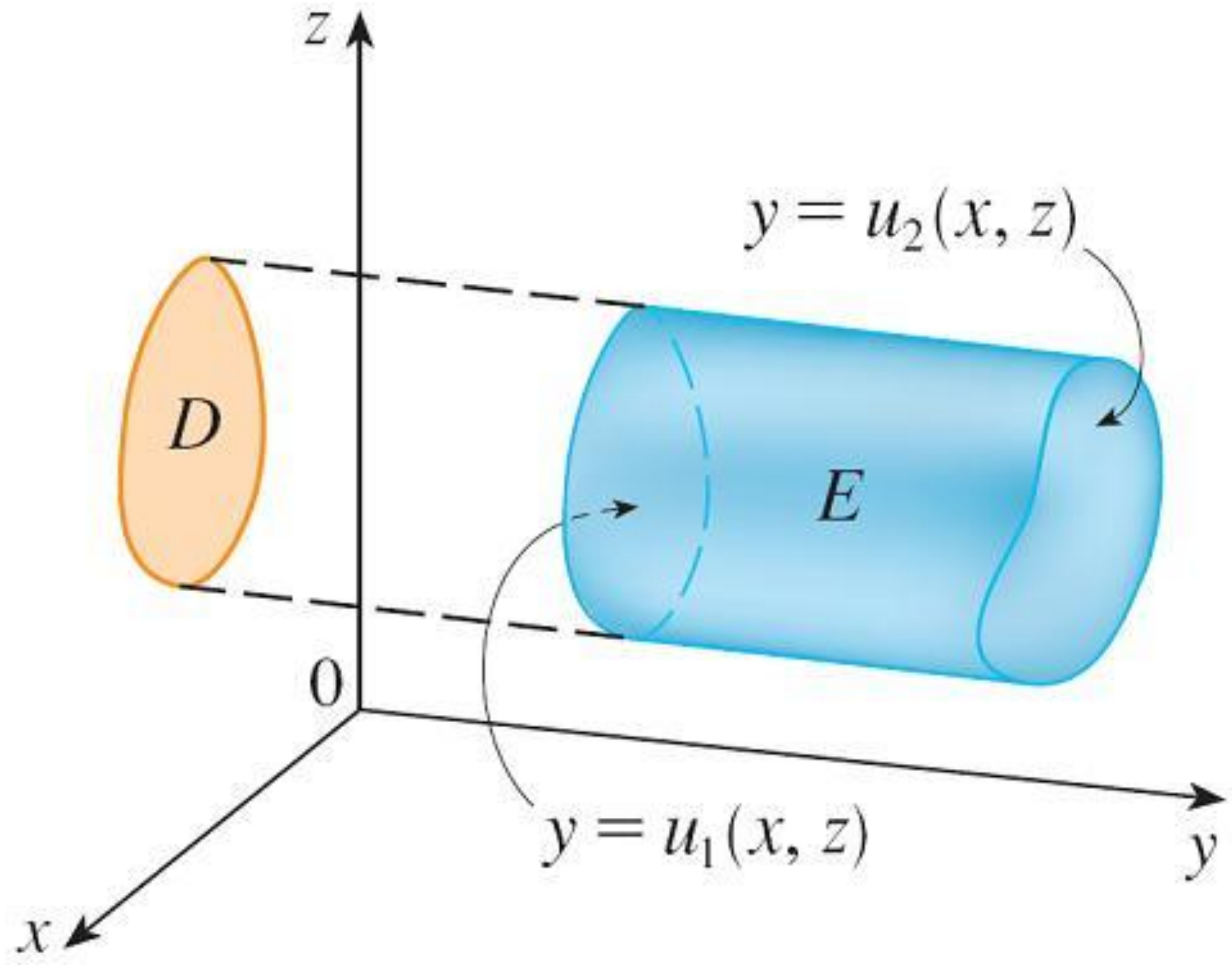
$$E = \{(x, y, z) | (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$

Where D is the projection of E onto the xz-plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface.

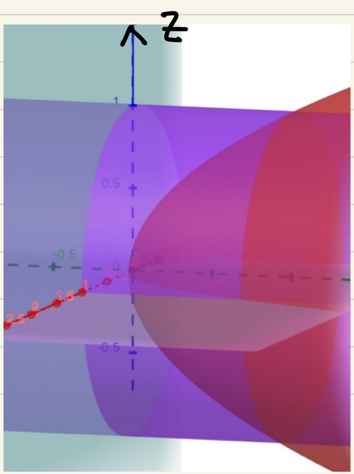
- And the triple integral is

$$\iiint_E f(x, y, z)\, dV = \iint_D [\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy] dA$$

Over a General Bounded Region: Type III

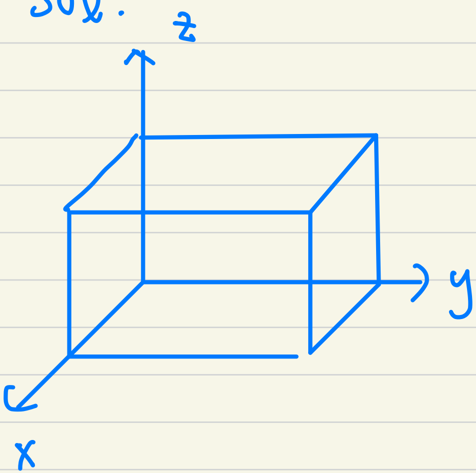


Ex: Compute $\iiint_E y \, dV$, where E is bounded by $y = x^2 + z^2$, $y = 0$ and $x^2 + z^2 = 1$.



Ex: Compute $\iiint_R y \, dV$, where R is the solid that is inside the cube $0 \le x, y, z \le 1$, lying above $y+z=1$ and below $x+y+z=2$.

Sol:



Change the Order of Integration

Ex: Change the order of integration of $\int_0^1 \int_y^1 \int_0^y f(x,y,z)\, dz\, dx\, dy$

Strategy for Changing the Order of Integration

1. Draw the solid on which the triple integral is computed.
2. Describe the solid as different types.
3. Write the triple integral as iterated integrals.

Trick / Shortcut

Change adjacent double integrals at a time.

Ex: Write $\int_0^1 \int_z^1 \int_0^{x-z} f(x,y,z)\, dy\, dx\, dz$ in other orders.

Applications

- Let's begin with the special case where $f = 1$ for all points in E. Then the triple integral does represent the volume of E :

$$V(E) = \iiint_E dV$$

Applications

- If the density function of a solid object that occupies the region E is $\rho(x,y,z)$, in units of mass per unit volume, at any given point then its **mass** is
$$m = \iiint_E \rho(x,y,z)\,dV$$

- And its **moments** about the three coordinate planes are
$$M_{yz} = \iiint_E x\rho(x,y,z)\,dV$$
$$M_{xz} = \iiint_E y\rho(x,y,z)\,dV \quad M_{xy} = \iiint_E z\rho(x,y,z)\,dV$$

Applications

- The **center of mass** is located at the point $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = M_{yz}/m$, $\overline{y} = M_{xz}/m$, and $\overline{z} = M_{xy}/m$.
- If the density is constant, the center of mass of the solid is called the **centroid** of E.

Review

- State the definition of triple integrals (as the limit of triple Riemann sums).
- State Fubini's Theorem for triple integrals.
- How do we compute triple integrals over general regions (type I, type II, or type III)?
- What are the center of mass, and moments about the coordinate planes?