Techniques of Integration

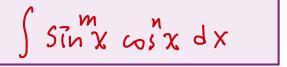
Section 7.1-7.3

Outline

- Integration by Parts
- Trigonometric Integrals
- Trigonometric Substitution

Trigonometric Integrals

- ▶ In section 7.2 we use trigonometric identities to integrate certain combinations of trigonometric functions.
- ightharpoonup Powers of $\sin x$ and $\cos x$:



Identity	
Differentials	

$$Ex: \int \omega_5^5 x dx$$

Ex:
$$\int \sin^3 x (\cos x)^{\frac{1}{3}} dx$$

$$Ex = \int \frac{\sin^3 x}{2 - \cos x} dx$$

$$Ex: \int x \omega_3^2 x dx$$

Ex: $\int \cos^2 x \, dx$, $\int \sin^2 x \, dx$

Ex: Sinx cos x dx

Recall that

$$\int \sin^{n} x \, dx = \frac{-1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx, \quad \text{for } n \ge 2.$$

$$\int \cos^{n} x \, dx = \frac{1}{n} \cos^{n} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx, \quad \text{for } n \ge 2.$$

$$Ex = \int_{0}^{\frac{11}{2}} \int 1 + \cos 2x \, dx$$

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd (n = 2k + 1), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$$
$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd (m = 2k + 1), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \, \sin x \, dx$$
$$= \int (1 - \cos^2 x)^k \cos^n x \, \sin x \, dx$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

Trigonometric Integrals

- ▶ Finally, we can make use of another set of trigonometric identities:
- To evaluate the integrals (a) $\int \sin mx \cos nx \, dx$, (b) $\int \sin mx \sin nx \, dx$, or (c) $\int \cos mx \cos nx \, dx$, use the corresponding identity:

(a)
$$\sin A \cos B = \frac{1}{2} [\sin(A - B) + \sin(A + B)]$$

(b)
$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

(c)
$$\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$$

Ex: Compute S sin4x cos x dx.

Ex: compute J cos2x cos4x dx.

Trigonometric Integrals

- \blacktriangleright Powers of $\tan x$ and $\sec x$:
- Since $(d/dx)\tan x=\sec^2 x$, we can separate a $\sec^2 x$ factor and convert the remaining even power of $\sec x$ to an expression involving $\tan x$ using the identity $\sec^2 x=1+\tan^2 x$.
- Or, since $(d/dx) \sec x = \sec x \tan x$, we can separate a $\sec x \tan x$ factor and convert the remaining even power of $\tan x$ to $\sec x$.

Ex:
$$\int \tan^3 x \sec^3 x \, dx$$

Ex:
$$\int \tan^5 x (\sec x) dx$$

Ex:
$$\int \frac{e^{-1}}{\tan x} \int \frac{-e^{-2}}{\sec x} dx$$

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(a) If the power of secant is even $(n = 2k, k \ge 2)$, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \, \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx$$
$$= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd (m = 2k + 1), save a factor of sec x tan x and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of sec x:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx$$
$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx$$

Then substitute $u = \sec x$.

Reduction Formula
$$\int tan^{n} x \, dx = \frac{1}{n-1} tan^{n-1} - \int tan^{n-2} \, dx$$

Reduction Formula
$$\int \sec^{n} x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

Other Trigonometric Integrals

Ex: S CSCX dx

Ex: \ cot x csc x dx

Ex=
$$\int \frac{1+\sin x}{1-\sin x} dx$$
,
$$\int \frac{\cos x}{1+\cos x} dx$$

$$Ex = \int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

Ex: \int x tan x dx