Tests for Convergence of Series

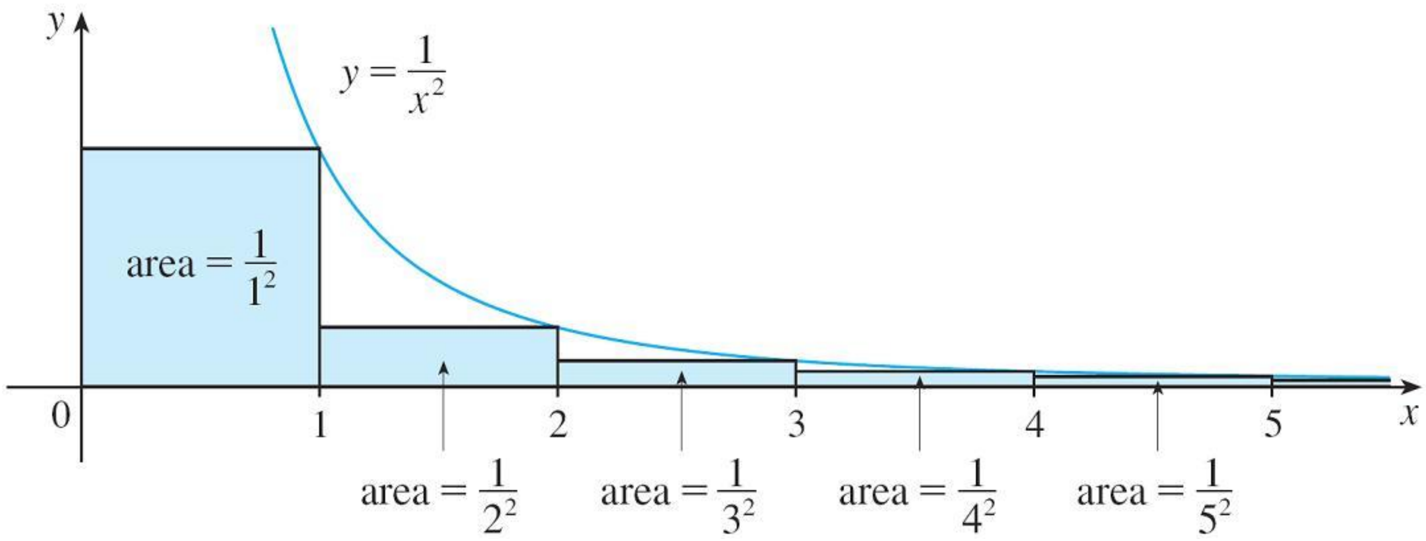
Section 11.3 - 11.6

Outline

The Integral Test

- Example:

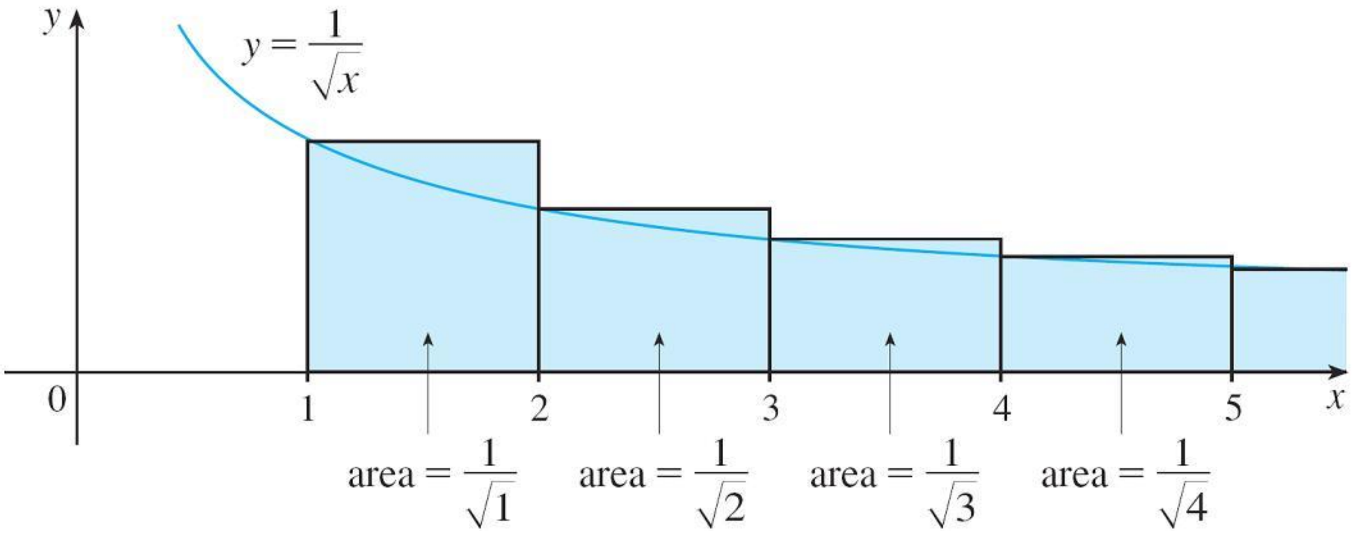
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$



The Integral Test

- Example:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$



The Integral Test

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(a) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

The Integral Test

- Example:

1 The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ are convergent.

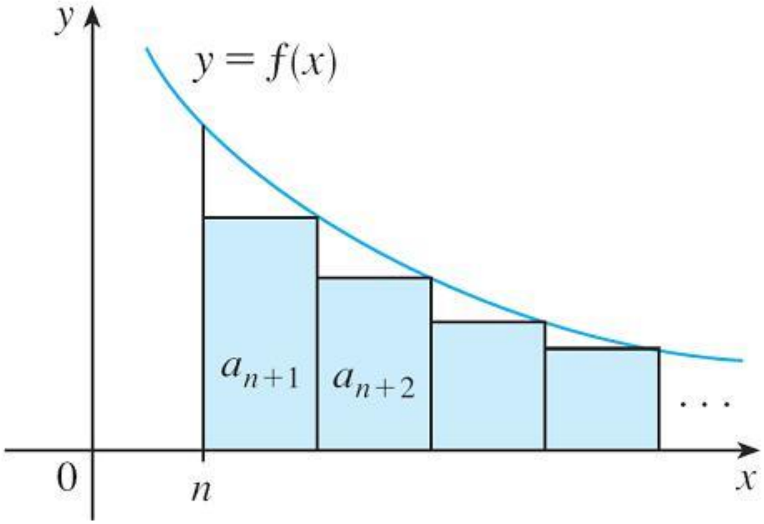
Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$ are divergent.

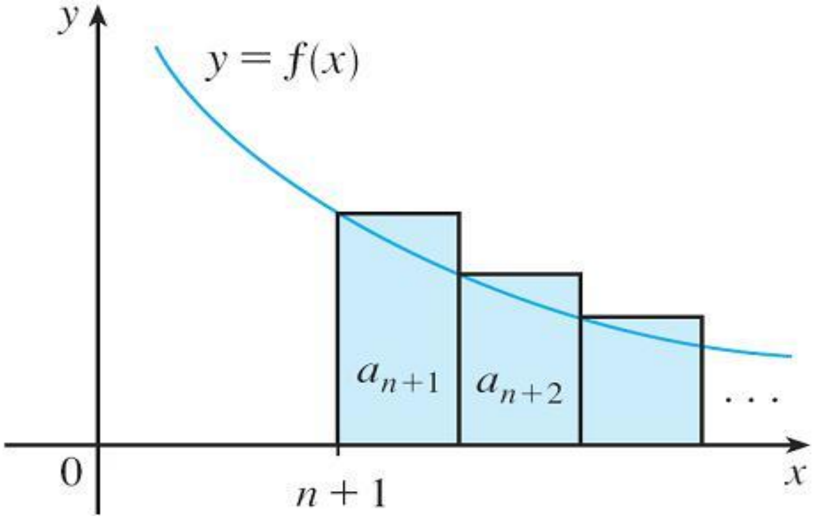
Estimate the Sum of a Series

- A partial sum s_n is an approximation to s. But how good is such an approximation? To find out, we need to estimate the size of the **remainder** $R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$
- The remainder R_n is the error made when s_n, the sum of the first n terms, is used as an approximation to the total sum.

Estimate the Sum of a Series

- $$R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^\infty f(x)dx$$





$$R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^\infty f(x)dx$$

Estimate the Sum of a Series

$\boxed{2}$ **Remainder Estimate for the Integral Test** Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geqslant n$ and $\Sigma\, a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\,dx \leqslant R_n \leqslant \int_{n}^{\infty} f(x)\,dx$$

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leqslant s \leqslant s_n + \int_{n}^{\infty} f(x)\,dx$$

Estimate Partial Sums of a Divergent Series

- Example 1:

$$\ln(n+1) < \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \ln n$$

- Example 2: Stirling's Formula

$$\sqrt{2\pi n}\,(\frac{n}{e})^n < n! < \sqrt{2\pi n}\,(\frac{n}{e})^n(1 + \frac{1}{4n})$$

The Comparison Test

The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\sum a_n$ is also divergent.

Remark: Although the condition $a_n \leq b_n$ or $a_n \geq b_n$ in the Comparison Test is given for all n, we only need to verify that it holds for $n \geq N$, where N is some fixed integer.

Ex: Prove the comparison test.

pf: ① Suppose that $0 < a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ converges.

$\because$ $a_n > 0$ $\therefore$ $\{ S_n = \sum_{k=1}^{n} a_k \}$ is an increasing sequence.

Moreover, $0 < S_n = \sum_{k=1}^{n} a_k \le \sum_{k=1}^{n} b_k < \sum_{k=1}^{\infty} b_k$.

This means that $\{S_n\}$ is a bounded sequence.

Hence by Monotonic Sequence Theorem, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ exists.

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② Suppose that $0 < b_n \le a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges.

Then $S_n = \sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} b_k$

The Comparison Test

- In using the Comparison Test, we often compare the unknown series with the following series:
- A p-series ($\sum 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$.)
- A geometric series ($\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$.)

Ex: Does $\sum_{n=1}^{\infty} \frac{1}{n \cdot \sqrt{n^2+2}}$ converge?

Ex: Does $\sum_{n=3}^{\infty} \frac{\ln n}{\sqrt{n^2-1}}$ converge?

The Limit Comparison Test

The Limit Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Estimate Sums

- If we have used the Comparison Test to show that a series $\sum a_n$ converges by comparing it with a series $\sum b_n$ then we may be able to estimate the sum $\sum a_n$ by comparing remainders.
- Let $R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$

$$T_n = t - t_n = b_{n+1} + b_{n+2} + \cdots$$

- Then R_n is smaller than T_n .

Alternating Series

- Definition:
- An **alternating series** is a series whose terms are alternately positive and negative.
- Example:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

Alternating Series Test

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

$$\text{(i)} \quad b_{n+1} \leqslant b_n \qquad \text{for all } n$$

$$\text{(ii)} \quad \lim_{n \to \infty} b_n = 0$$

then the series is convergent.

Alternating Series Test

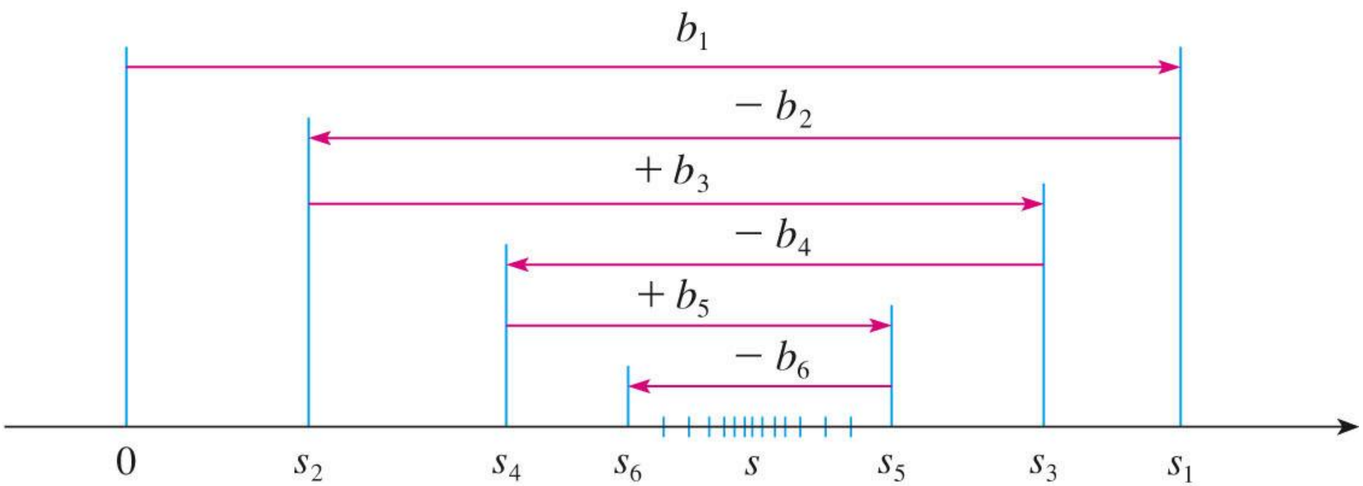

FIGURE 1

Estimating Sums

Alternating Series Estimation Theorem If $s = \Sigma\,(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies

$$\text{(i)}\ \ b_{n+1} \leqslant b_n \qquad \text{and} \qquad \text{(ii)}\ \ \lim_{n\to\infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \leqslant b_{n+1}$$

Absolute/Conditional Convergence

1 Definition A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

2 Definition A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

3 Theorem If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Ex: Prove that if $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

The Ratio Test

The Ratio Test

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

e.g. $\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Proof of the Ratio Test:

(i) Find an $\varepsilon > 0$ s.t. $L + \varepsilon = C < 1$.

There is some N_0 s.t. if $n > N_0$ then $\left|\frac{a_{n+1}}{a_n}\right| < L + \varepsilon = C < 1$

$\Rightarrow |a_{N_0+1}| < C \cdot |a_{N_0}|, \cdots, |a_{N_0+k}| < C^k |a_{N_0}|$ for $k \in \mathbb{N}$.

Hence $\sum_{k=1}^{\infty} |a_{N_0+k}|$ convs and $\sum_{n=1}^{\infty} |a_n|$ convs $\Rightarrow \sum_{n=1}^{\infty} a_n$ convs.

(ii) Find an $\varepsilon > 0$, s.t. $L - \varepsilon > 1$.

There is some N_0 s.t. if $n > N_0$, then $\left|\frac{a_{n+1}}{a_n}\right| > L - \varepsilon$
> 1.

$\Rightarrow |a_{n+1}| > |a_n|$ for $n > N_0$.

$\Rightarrow |a_n| > |a_{N_0}|$ for $n > N_0$

$\Rightarrow \lim_{n\to\infty} a_n \neq 0$ and $\sum_{n=1}^{\infty} a_n$ diverges.

Ex: For what values of r and p will $\sum_{n=1}^{\infty} r^n n^p$ converge absolutely?

Ex: Determine whether $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges or diverges.

Ex: Find values of r s.t. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} r^n$ converges.

The Root Test

The Root Test

(i) If $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n \to \infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n \to \infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Ex: Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{n+2}{3n+1}\right)^n \cdot n^3$

Ex: Test the convergence of the series $\sum_{n=2}^{\infty} \left(\frac{n-1}{n}\right)^n$, $\sum_{n=2}^{\infty} \left(\frac{n-1}{n}\right)^{n^2}$.

The Root Test

- Note:
- If $L = 1$ in the Ratio Test, don't try the Root Test because the limit will again be 1.
- If $L = 1$ in the Root Test, don't try the Ratio Test because the limit either won't exist or the limit will be 1.

Review

- State the Integral Test. How do we estimate the remainder term by the Integral Test?
- When will a p-series converge?
- State the Comparison Test.
- State the Alternating Series Test.
- How do we estimate the remainder term of an alternating series?
- State the Ratio Test and the Root Test.