The Area Problems

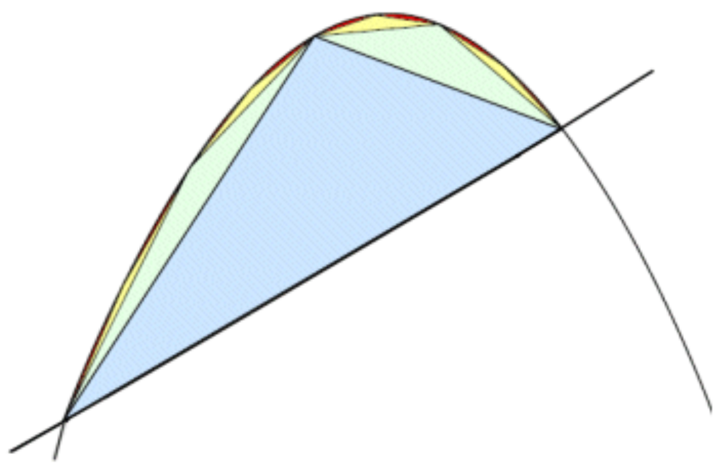
Section 5.1

Outline

- The Area Problem
  - Summation formulas

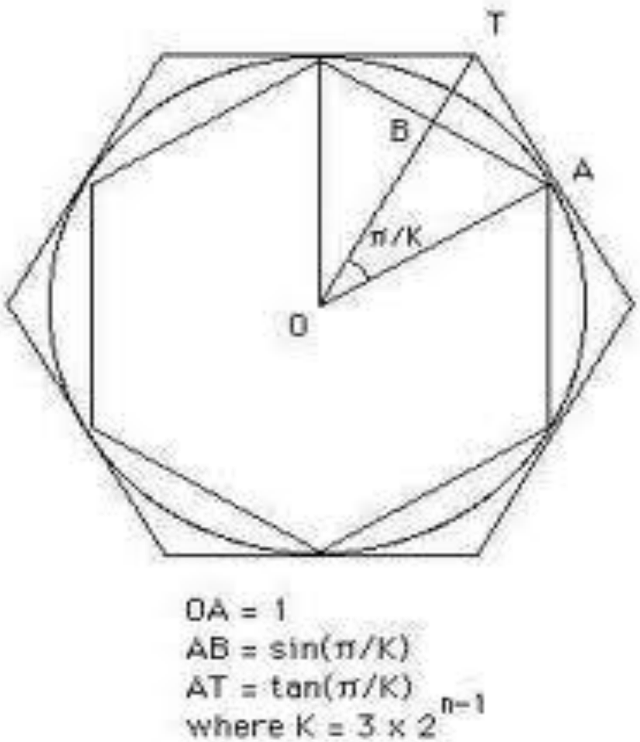
The Area Problem

- In the year about 214 B.C. Archimedes discovered several methods to solve the area bounded by a parabola.
- Ex: Archimedes' Quadrature of the Parabola



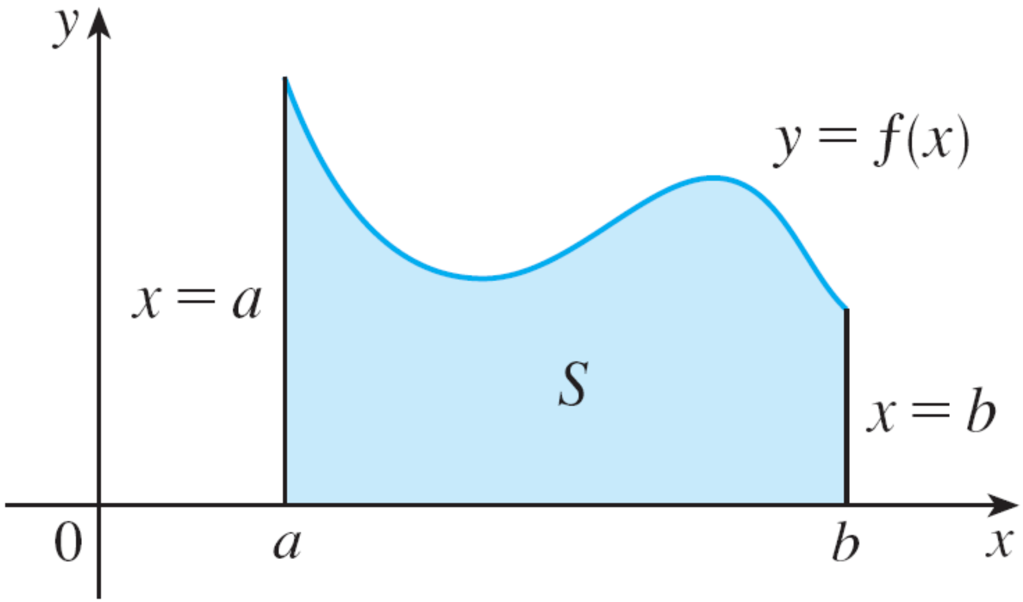
The Area Problem

- Archimedes also found ways to approximate the area inside a circle.



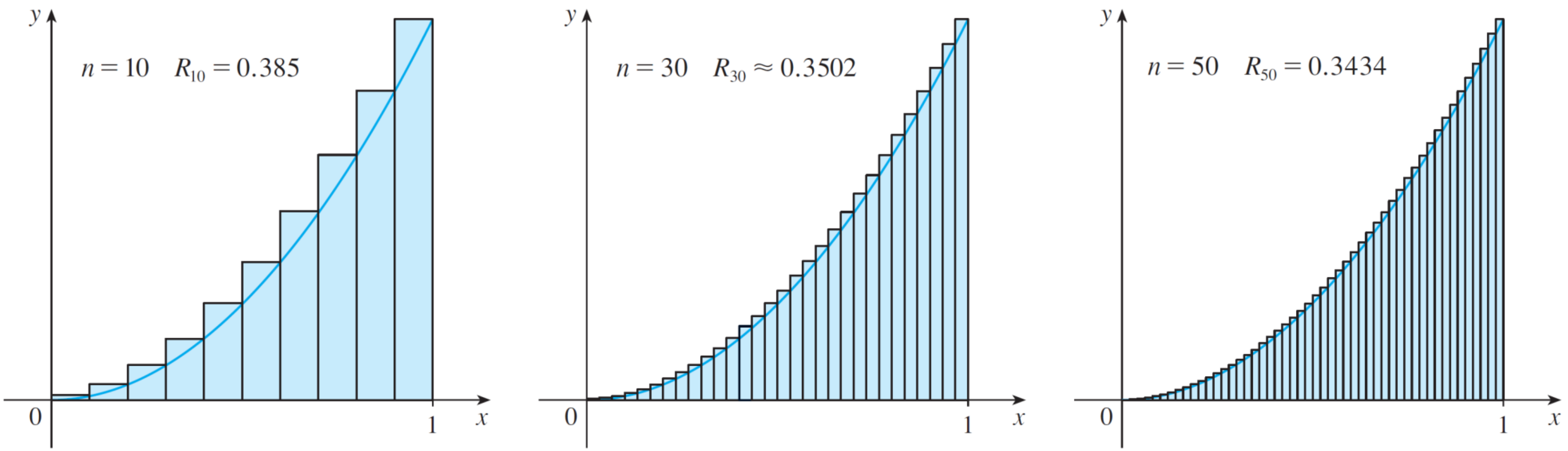
The Area Problem

- Is there an *universal simple method* to compute the area of a region bounded by *general curves*?



The Area Problem

- We all have an intuitive idea of what the area of a region is. Part of the area problem is to make this intuitive idea precise by ***giving an exact definition of area***.



The Area Problem

- We start by subdividing interval $[a, b]$ into n subintervals, $[x_0, x_1]$, $[x_1, x_2]$..., $[x_{n-1}, x_n]$ with equal width$\Delta x = (b - a)/n$, where $x_0 = a$ and $x_n = b$.
- The right endpoints of the subintervals are $x_1 = a + \Delta x,\ x_2 = a + 2\Delta x,\ x_i = a + i\Delta x$.
- What we think of intuitively as the area of S is approximated by the sum of the areas of the "rectangles", which is

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$$

The Area Problem

$\boxed{2}$ **Definition** The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

- Note 1: It can be proved that the limit in Definition 2 always exists if we assume that f is continuous.
- Note 2: It can also be shown that we get the same value if we use left endpoints

$$A = \lim_{n \to \infty} [f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x]$$

The Area Problem

- Note 3: In fact, instead of using left endpoints or right endpoints, we could take the height of the i th rectangle to be the value of f at *any number* x_i^* in the i th subinterval $[x_{i-1}, x_i]$. We call the numbers $x_1^*, x_2^*, \ldots, x_n^*$ the **sample points**. So a more general expression for the area of S is

$$A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x]$$

The Area Problem

- Note 4: It can be shown that an equivalent definition of area is the following: *A is the unique number that is smaller than all the upper sums and bigger than all the lower sums.* In general, we form **lower** (or **upper**) **sums** by choosing the sample points x_i^* so that $f(x_i^*)$ is the minimum (or maximum) value of f on the ith subinterval.

The Area Problem

- We often use **sigma notation** to write sums with many terms more compactly.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

The Area Problem

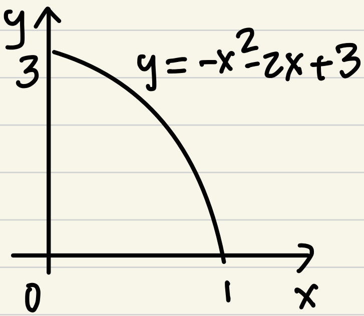
- To compute to sum, we may need the following formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Ex: Compute area of the region under $y = f(x) = -x^2 - 2x + 3$ above the interval $[0, 1]$



Ex: compute area of the region under the curve $y = f(x) = e^x$ above the interval $[1, 3]$.

Ex: Determine a region whose area is equal to

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n^2}{(n+2k)^3}.$$

Review

- How do we define the area under a graph?
- Review some summation formulas.