Change of Variables in Multiple Integrals

Section 15.7-15.9

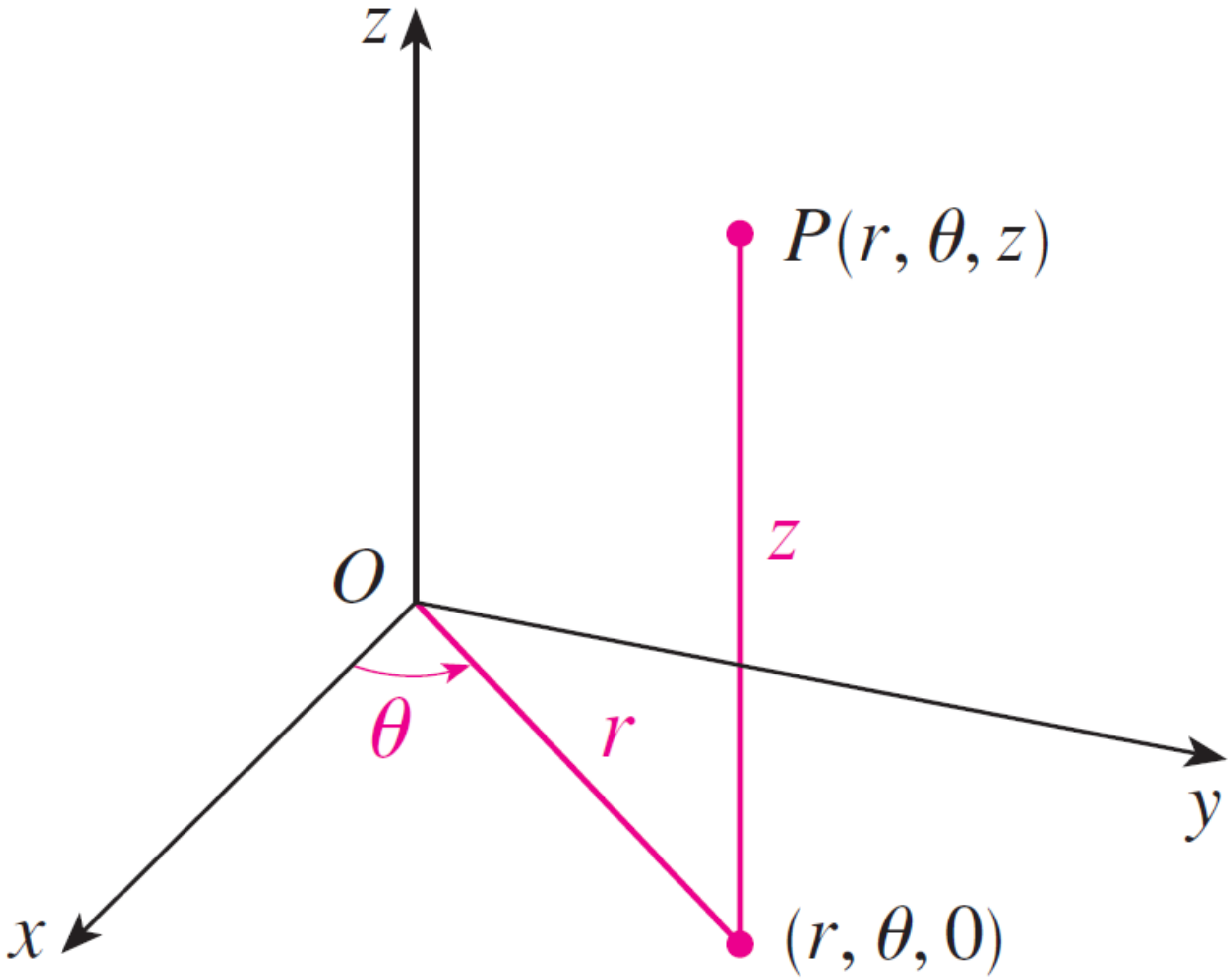
Outline

- Triple Integrals in Cylindrical Coordinates
- Triple Integrals in Spherical Coordinates
- Change of Variables in Multiple Integrals

Triple Integrals in Cylindrical Coordinates

- Definition:
- In the **cylindrical coordinate system**, a point P in three-dimensional space is represented by the ordered triple (r, θ, z) where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P.

Triple Integrals in Cylindrical Coordinates



Triple Integrals in Cylindrical Coordinates

- To convert from cylindrical to rectangular coordinates, we use the equations

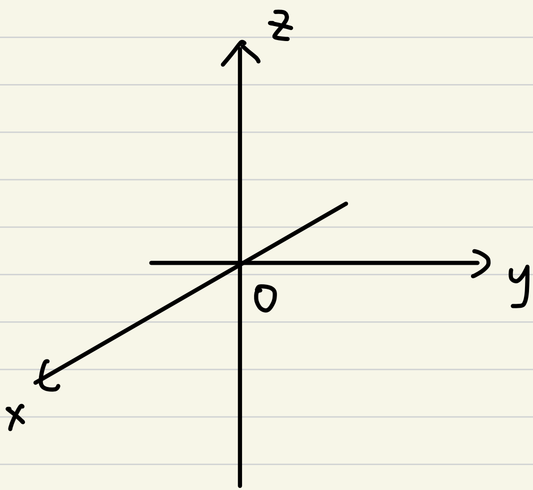
$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

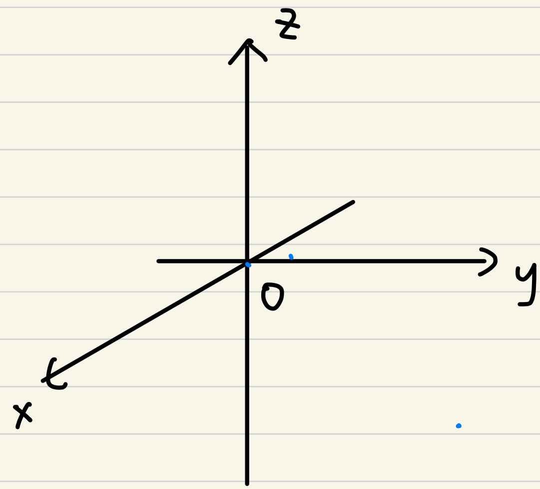
- while to convert from rectangular to cylindrical coordinates, we use

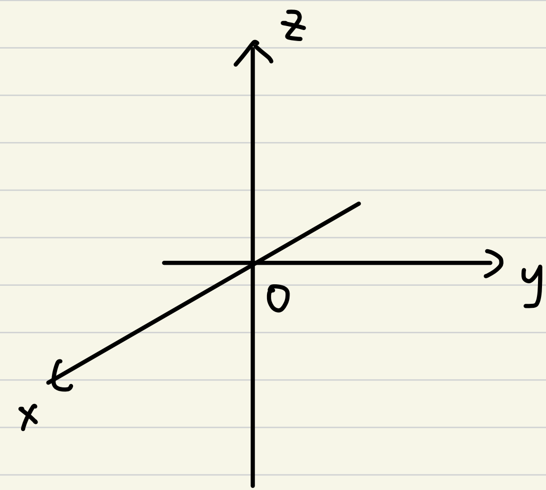
$$r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x} \qquad z = z$$

Ex: Draw $r = c$, $\theta = c$, $z = c$.

Sol:







Triple Integrals in Cylindrical Coordinates

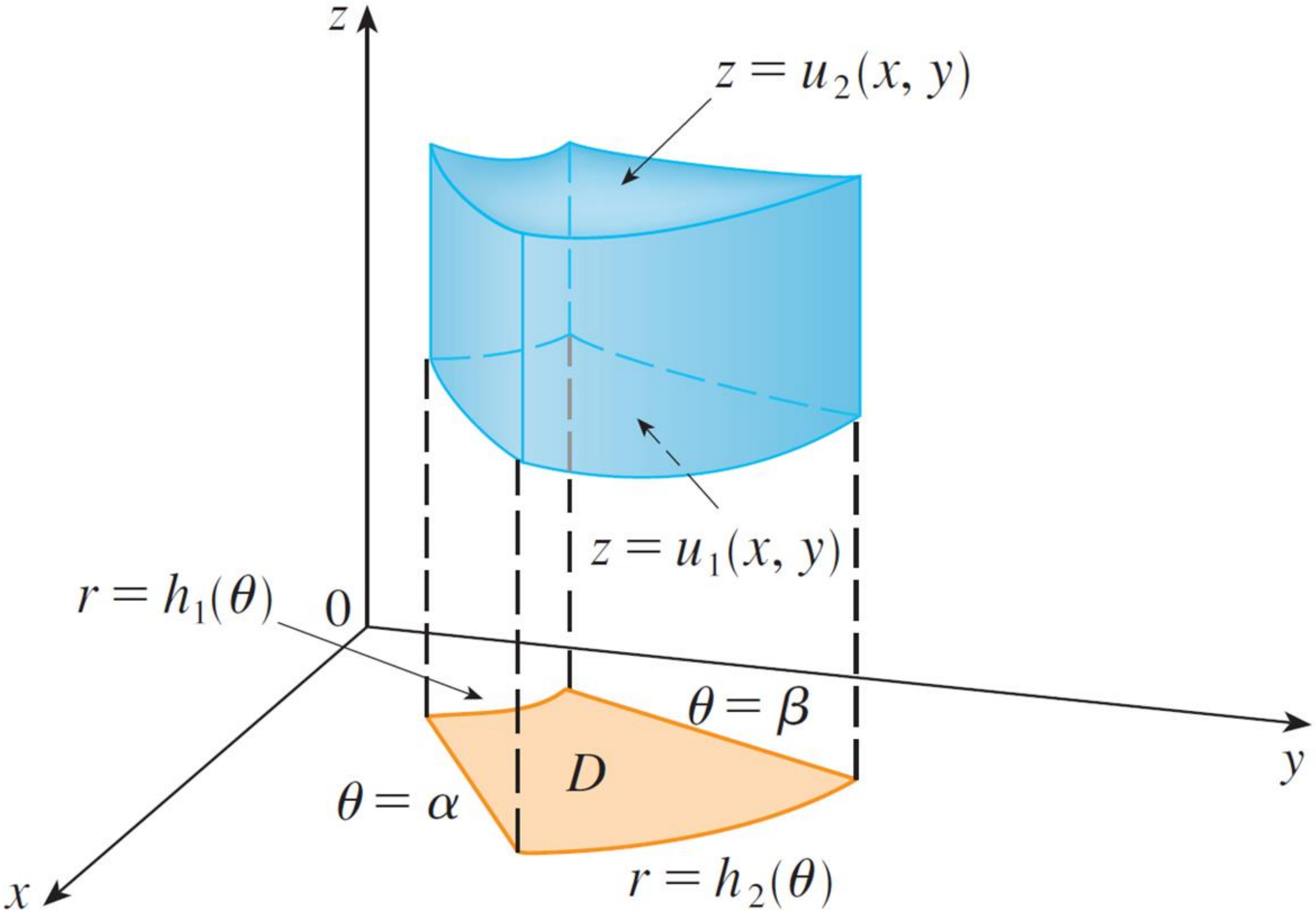
- Evaluating triple integrals with cylindrical coordinates:
- Suppose that E is a type I region whose projection D onto the xy-plane is conveniently described in polar coordinates, say

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$

where D is given in polar coordinates by

$$D = \{(r, \theta) | \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\} \cdot$$

Triple Integrals in Cylindrical Coordinates



Triple Integrals in Cylindrical Coordinates

- Suppose that f is a continuous function on E. Then we have

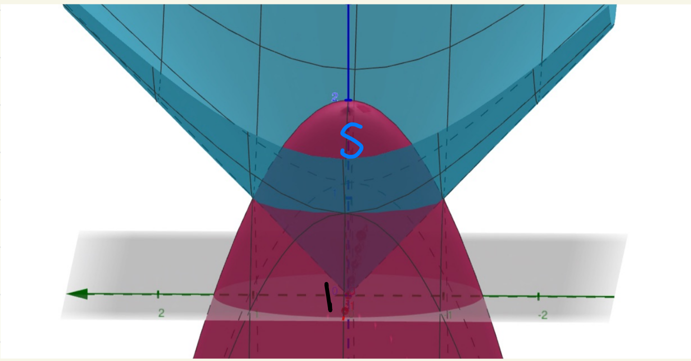
3 $$\iiint\limits_E f(x, y, z)\, dV = \iint\limits_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

- Next, we evaluate the "outer" double integrals in polar coordinates, and derive the formula

4 $$\iiint\limits_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Ex: Compute $\iiint_S \sqrt{x^2+y^2}\, dV$, where S is the solid that lies above the cone $z=\sqrt{x^2+y^2}$ and under the paraboloid $z=2-x^2-y^2$.

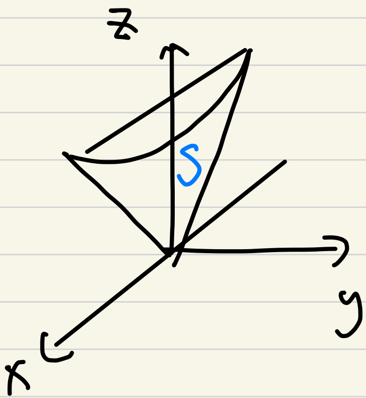
Sol:



Ex: Use Cylindrical Coordinates to compute the triple integral

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{1-x^2-y^2}^{\sqrt{1-x^2-y^2}} x^2+y^2 \; dz\, dy\, dx .$$

Ex: Compute $I = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{2} e^z \, dz \, dx \, dy$.



sol: