Differentiability of Functions of Several Variables

Section 14.4–14.5

Outline

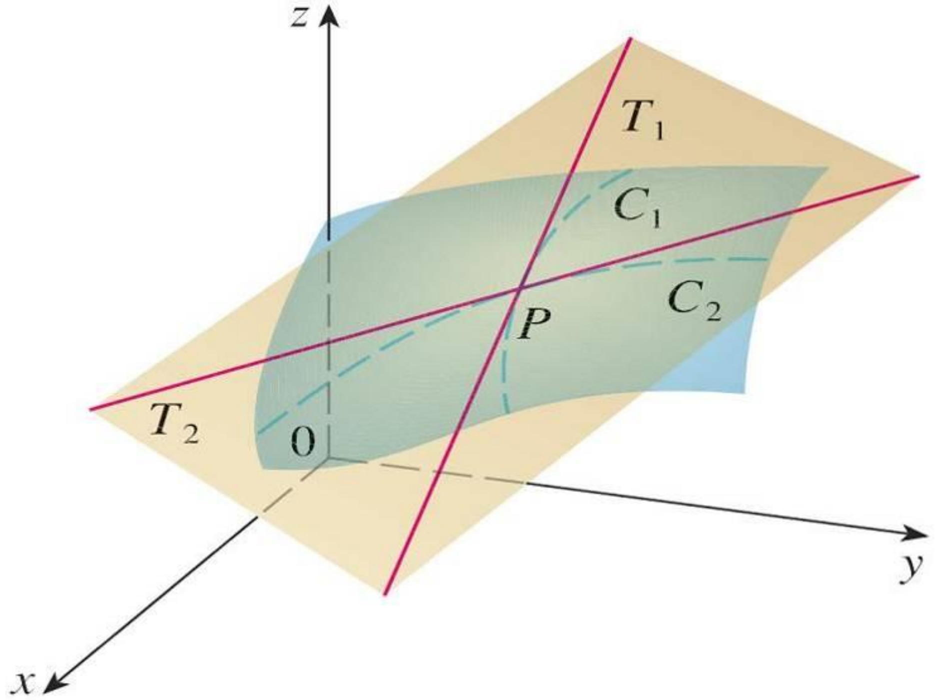
- Definition of Differentiability
  - Tangent Planes
  - Linear Approximations
  - Differentials
- The Chain Rules
  - Implicit Differentiation

Definition of Differentiability

- Tangent Planes:
- Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S.
- Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S.
- Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.

Definition of Differentiability

- Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2 .



The tangent lines of C_1 and C_2 at $(x_0, y_0, f(x_0, y_0))$ are parallel to $\vec{T}_1 //$, $\vec{T}_2 //$

The tangent plane of S at $(x_0, y_0, f(x_0, y_0))$ is the plane that contains $\vec{T}_1$ and $\vec{T}_2$ with normal vector

$\vec{n} =$

Definition of Differentiability

- We can show that the tangent plane equation must be as follows.

$\boxed{2}$ Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$\because z_0 = f(x_0, y_0)$

$\therefore$ The tangent plane equation is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Definition of Differentiability

- Linear Approximation:
- The linear function whose graph is the tangent plane, namely

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of f at (a, b).

- The approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b).

$f(x)$	$f(x,y)$	$f(x_1,\dots,x_n)$
The tangent line of $y=f(x)$ at $x=x_0$ is	The tangent plane of $z=f(x,y)$ at $(x_0,y_0,f(x_0,y_0))$ is	The hyper tangent plane of $x_{n+1}=f(x_1,\dots,x_n)$ at $(a_1,\dots,a_n,f(a_1,\dots,a_n))$ is
the linearization of $f(x)$ at $x=x_0$ is	The linearization of $f(x,y)$ at $(x,y)=(x_0,y_0)$ is	The linearization of $f(x_1,\dots,x_n)$ at $(a_1,\dots,a_n)$ is

Ex: $f(x,y) = x^2 + 4y^2$. Find the tangent plane to $S: z = f(x,y)$ at $(0,0,0)$

Use the linearization of $f(x,y)$ at $(0,0)$ to estimate $f(0.1, -0.2)$.

Sol:

Ex: Find the tangent plane to $z = f(x,y) = \sqrt{x^2+4y^2}$ at $(0,0,0)$.

Ex: Find the tangent plane to $S: z = f(x,y) = (2x)^y$ at $(1, 2, 4)$.

Estimate $f(0.95, 2.1)$ by the linearization of f at $(1, 2)$.

Ex: Find the "linearization" of $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

at $(0,0)$.

Differentiability

For a single variable function $f(x)$, $f(x)$ is differentiable at $x = a$ $\iff$

Definition of Differentiability

- A *differentiable* function is one for which the linear approximation should be a good approximation when (x, y) is near (a, b).

$\boxed{7}$ **Definition** If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form $(\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b))$

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x,\ \Delta y) \to (0, 0)$.

Definition of Differentiability

- Definition : We say that a function $f(x, y)$ is *differentiable* at the point (a, b) if

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - L(x,y)|}{\sqrt{(x-a)^2 + (y-b)^2}} = 0$$

where $L(x, y)$ is the linearization of $f(x, y)$ at (a, b) , i.e.

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b)|}{\sqrt{(x-a)^2 + (y-b)^2}} = 0$$

$$\lim_{(h,k)\to(0,0)} \frac{|f(a+h,b+k) - f(a,b) - f_x(a,b)h - f_y(a,b)k|}{\sqrt{h^2 + k^2}} = 0$$

$f(x)$	$f(x,y)$	$f(x_1,\dots,x_n)$
$f(x)$ is differentiable at $x=a$ if $\boxed{\lim\limits_{x\to a}\frac{\lvert f(x)-L(x)\rvert}{\lvert x-a\rvert}=0}$ where $L(x)=$	$f(x,y)$ is differentiable at $(x,y)=(a,b)$ if $\lim\limits_{(x,y)\to(a,b)}\frac{\lvert f(x,y)-L(x,y)\rvert}{\sqrt{(x-a)^2+(y-b)^2}}=0$ where $L(x,y)=$	f is differentiable at $(x_1,\dots,x_n)=(a_1,\dots,a_n)$ if

Ex: Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & \text{for } (x,y) \neq (0,0) \\ 0, & \text{for } (x,y) = (0,0). \end{cases}$ Is f differentiable at $(0,0)$?

Sol:

Ex: $f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0). \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$ Is $f(x,y)$ differentiable at $(0,0)$?

Theorem: If $f(x,y)$ is differentiable at (a,b) then f is continuous at (a,b)

Definition of Differentiability

- It's sometimes hard to use the definition directly to check the differentiability of a function, but the following theorem provides a convenient ***sufficient*** condition (not a necessary condition) for differentiability.

8 Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

$\boxed{\text{Theorem :}}$ If $f_x(x,y)$ and $f_y(x,y)$ exist near (a,b) and $f_x(x,y)$, $f_y(x,y)$ are continuous at (a,b), then $f(x,y)$ is differentiable at (a,b).

Ex: Find points at which $f(x,y) = \sqrt[3]{(x+1)^2 + y^2}$ is differentiable.

Sol:

Definition of Differentiability

- For a differentiable function of two variables $z = f(x, y)$, we define the differentials dx and dy to be independent variables. Then the **differential** dz, also called the **total differential**, is defined by

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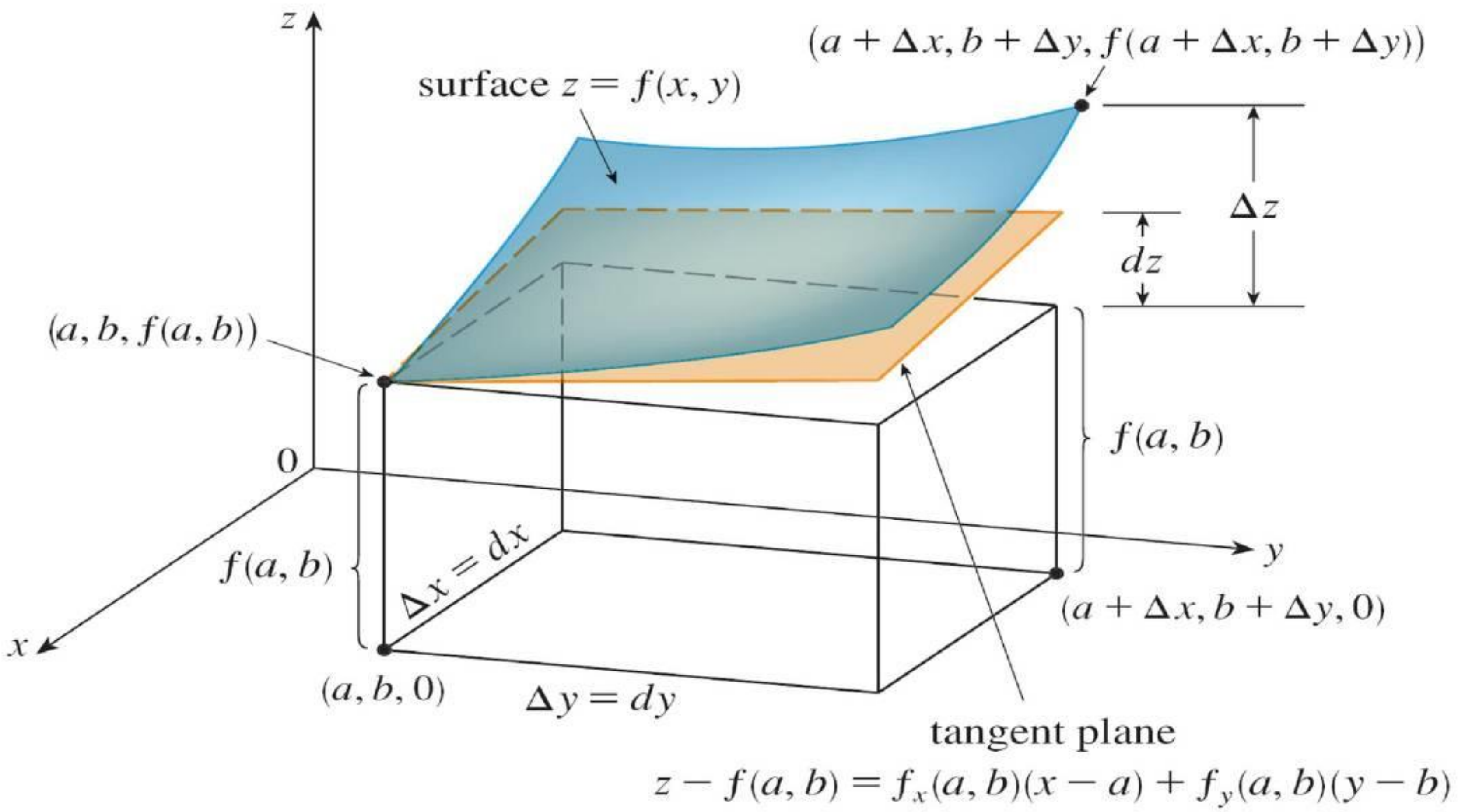
$$\boxed{dz = f_x(x, y)\, dx + f_y(x, y)\, dy = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy}$$

- Sometimes the notation df is used instead of dz.

Definition of Differentiability

- The following graph shows the geometric interpretation of the differential dz and the increment Δz : dz represents the change in height of the tangent plane, whereas Δz represents the change in height of the surface $z = f(x, y)$ when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

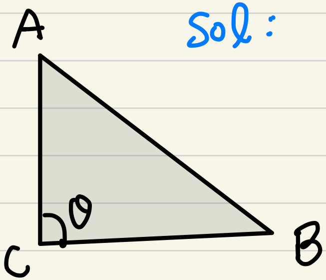
Definition of Differentiability



Differential

$f(x)$	$f(x,y)$	$f(x_1,\ldots,x_n)$
Consider $y=f(x)$. The differential $dy=$	Consider $z=f(x,y)$. The differential $dz=$	Consider $x_{n+1}=f(x_1,\ldots,x_n)$. The differential $dx_{n+1}=$

Ex: Use differential to estimate the change of $\overline{AB}^2$ as $\overline{AC}$ changes from 3 to 3.02, $\overline{BC}$ changes from 4 to 3.99 and θ changes from $\frac{\pi}{2}$ to $\frac{24}{50}\pi$.



Sol:

Definition of Differentiability

- Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables.
- Suppose that $f(\vec{x})$ is a function of n- variables, where $\vec{x} = (x_1, \ldots, x_n)$. Then the linear approximation of $f(\vec{x})$ at $\vec{a} = (a_1, \ldots, a_n)$ is
$$L(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} f_{x_i}(\vec{a})(x_i - a_i)$$

Definition of Differentiability

- And we say that $f(\vec{x})$ is *differentiable* at $\vec{a}$ if
$$\lim_{\vec{x}\to\vec{a}} \frac{|f(\vec{x}) - L(\vec{x})|}{|\vec{x} - \vec{a}|} = 0 .$$
- The differential df depends on $\vec{x}$ as well as differentials $dx_1, \ldots, dx_n$, and is defined as
$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x})dx_i .$$