Improper Integrals

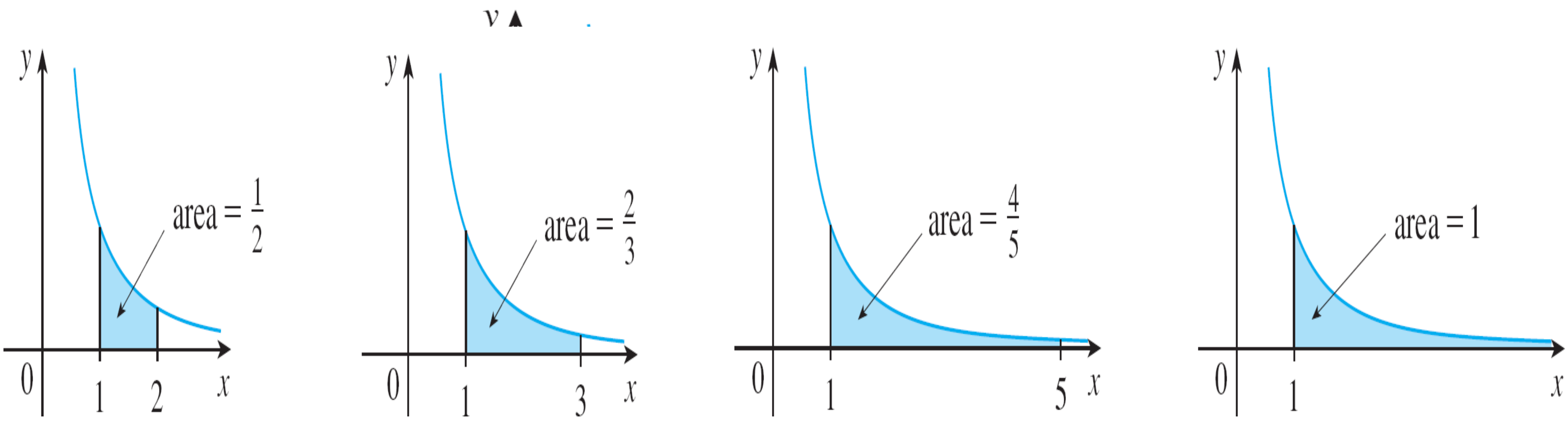
Section 7.8

Outline

- Improper Integrals
  - Type I: Infinite Intervals
  - Type II: Infinite Integrands
- Comparison Tests for Improper Integrals

Type I: Infinite Intervals

- Consider the infinite region S that lies under the curve $y = 1/x^2$, above the x-axis, and to the right of the line $x = 1$.



1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used.

Remark:

① Suppose that $f(x)$ is continuous on $\mathbb{R}$ and $\int_a^{\infty} f(x)\,dx$ is convergent. Then $\int_b^{\infty} f(x)\,dx = \int_b^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$ is also convergent for any $b \in \mathbb{R}$.

② If there is some $a \in \mathbb{R}$ s.t. both $\int_{-\infty}^{a} f(x)\,dx$ and $\int_a^{\infty} f(x)\,dx$ converge, then for any $b \in \mathbb{R}$, both $\int_{-\infty}^{b} f(x)\,dx$ and $\int_b^{\infty} f(x)\,dx$ converge and

$$\int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_b^{\infty} f(x)\,dx.$$

③ $\int_{-\infty}^{\infty} f(x)\,dx$ is NOT defined as $\lim_{t\to\infty}\int_{-t}^{t} f(x)\,dx$.

Ex: $\lim_{t\to\infty}\int_{-t}^{t} x\,dx = 0$ but $\int_{-\infty}^{\infty} x\,dx$ diverges.

Ex: Is $\int_1^{\infty} \frac{1}{x}\,dx$ convergent or divergent?

Sol:

Ex: For what values of p does $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

Ex: Is $\int_{-\infty}^{0} x e^{x}\, dx$ convergent or divergent?

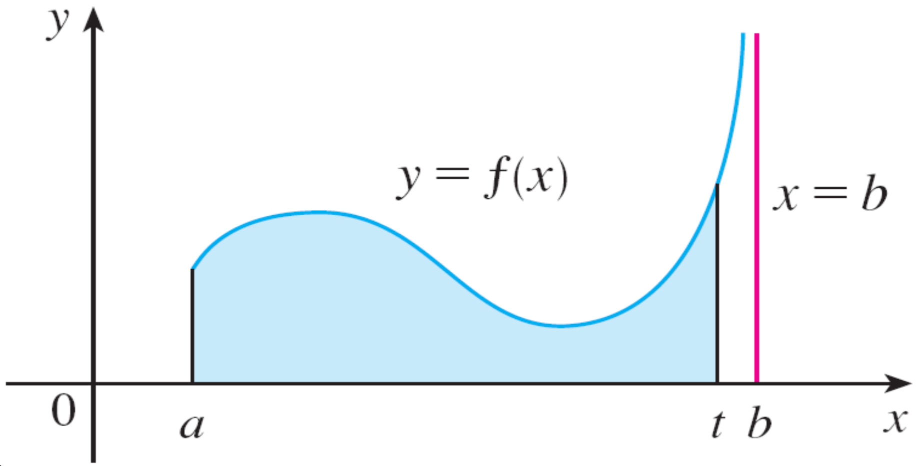
Ex: Is $\int_0^\infty \cos x \, dx$ convergent or divergent?

Ex: Is $\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2}$ convergent or divergent? $a>0$.

Ex: Compute $\displaystyle\int_0^\infty \frac{1}{1+x^3}\,dx$.

Type II: Infinite Integrands

- Suppose that f is a positive continuous function defined on a finite interval $[a, b)$ but has a vertical asymptote at b .



$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Ex: For what values of p does $\int_0^1 \frac{dx}{x^p}$ converge?

Ex: Compute $\int_1^2 \frac{dx}{\sqrt{x-1}}$ if it is convergent.

Ex: Compute $\int_0^{\frac{\pi}{2}} \tan x \, dx$ if it is convergent.

Ex: Find values of p such that $\int_0^1 x^p \ln x \, dx$ converges.

Improper Integrals of Both Types

Ex: Find values of p s.t. $\displaystyle\int_1^\infty \frac{1}{x(\ln x)^p}\,dx$ converges.

Comparison Tests for Improper Integrals

- Basic:

Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Ex: Determine whether $\int_0^\infty e^{-x^2}\,dx$ converges or diverges.

Ex: Show that $\displaystyle\int_{-1}^{0} \frac{1}{(x+1)(x^4+x^2+2)}\,dx$ is divergent.

Ex: Determine whether $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$ converges or diverges.

Comparison Tests for Improper Integrals

- The Limit Comparison Test:
- Suppose that $f(x), g(x) > 0$, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = c$ where c is a nonzero finite constant. Then $\int_a^\infty f(x)dx$ converges if and only if $\int_a^\infty g(x)dx$ converges.

Proof of the limit Comparison Test Theorem.

Pf: For $\varepsilon > 0$ such that $c - \varepsilon > 0$, there is some $N > 0$ such that

$$c - \varepsilon < \frac{f(x)}{g(x)} < c + \varepsilon \quad \text{for } x > N.$$

Hence $0 < (c-\varepsilon)\, g(x) < f(x) < (c+\varepsilon)\, g(x)$ for $x > N$.

Therefore by the basic comparison test,

$\int_N^\infty f(x)\,dx$ converges if and only if $\int_N^\infty g(x)\,dx$ converges.

Thus $\int_a^\infty f(x)\,dx$ converges if and only if $\int_a^\infty g(x)\,dx$ converges.

Ex: Is $\int_2^\infty \frac{1}{x^2 - e^{-x}}\,dx$ convergent?

Ex: For $b>0$, find values of a and b s.t. $\int_0^\infty \frac{x^a}{1+x^b}\,dx$ converges.

Laplace Transform

If $f(t)$ is continuous for $t \geq 0$, the Laplace Transform of $f(t)$ is the function $\mathcal{L}\{f(t)\}$ defined by

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

and the domain of $\mathcal{L}\{f(t)\}$ is the set consisting of all numbers s for which the integral converges.

Ex: Compute the Laplace transform of constant function $f(t) = 1$.

Ex: Compute the Laplace transform of $f(t) = e^{2t}$.

Ex: Compute $\mathcal{L}\{t\}$

Q: $\mathcal{L}\{t^n\} = ?$ $n \geq 1$.

Ex: Compute $\mathcal{L}\{\sin kt\}$

Ex: Compute $\mathcal{L}\{\cos kt\}$.

Review

- How do we define improper integrals on infinite intervals?
- How do we define improper integrals for infinite integrands?
- State the basic comparison test for improper integrals.
- State the limit comparison test for improper integrals.