Differentiability of Functions of Several Variables

Section 14.4-14.5

Outline

- Definition of Differentiability
  - Tangent Planes
  - Linear Approximations
  - Differentials
- The Chain Rules
  - Implicit Differentiation

The Chain Rules

- We want to compute the derivatives of functions of several variables. In particular, we need to know *the chain rules* for several variables functions.

$\boxed{2}$ **The Chain Rule (Case 1)** Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

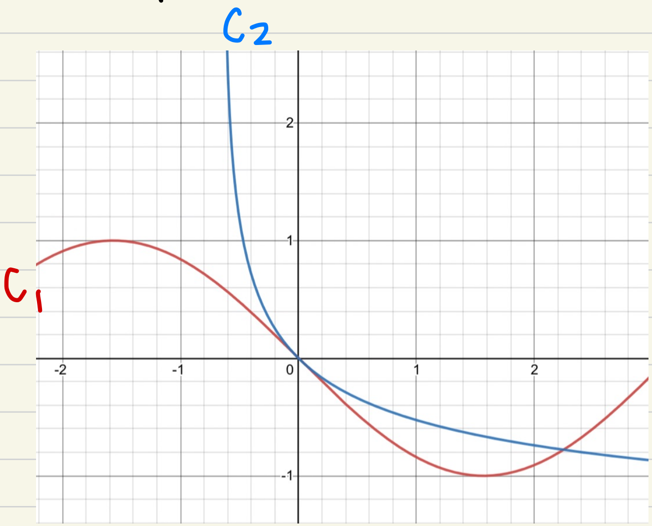
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Proof of the chain rule (case 1):

$f(x(t))$	$\frac{d}{dt} f(x(t)) =$
$f(x(t), y(t))$	$\frac{d}{dt} f(x(t), y(t)) =$
$f(x_1(t), \ldots, x_n(t))$	$\frac{d}{dt} f(x_1(t), \cdots, x_n(t)) =$

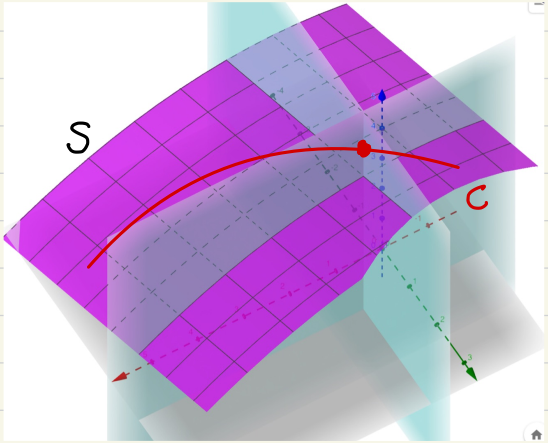
Ex: Let $z = x\cos 3y + e^{2x+y}$ and $C_1 : \begin{cases} x(t) = t, \\ y(t) = -\sin t, \end{cases}$

$C_2 : \begin{cases} x(t) = e^t - 1, \\ y(t) = -\ln(1+t). \end{cases}$ Find $\frac{d}{dt} z(x(t), y(t))$ on C_1 and C_2 at $t=0$.



The tangent plane

Suppose that $f(x,y)$ is $\boxed{\text{differentiable}}$. Let surface S be the graph of $f(x,y)$. Fix a point $P_0:(x_0,y_0,z_0)\in S$. For any differentiable curve $C:\begin{cases} x = x(t) \\ y = y(t) \\ z = f(x(t),y(t)) \end{cases}$ on S passing P_0, the tangent vector of C at P_0 lies on a same plane.



The Chain Rules

3 **The Chain Rule (Case 2)** Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

s and t are called **independent** variables while x and y are called **intermediate** variables and z is the **dependent** variable.

Proof of the chain rule (Case 2):

Ex: $z = f(x, y)$, $\begin{cases} x(r, \theta) = r\cos\theta \\ y(r, \theta) = r\sin\theta \end{cases}$. Find $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial \theta}$, $\frac{\partial^2 z}{\partial r^2}$, $\frac{\partial^2 z}{\partial r \partial \theta}$, $\frac{\partial^2 z}{\partial \theta^2}$.

Ex: $z = z(r, \theta)$, $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ in terms of $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial \theta}$. Find $\frac{\partial^2 z}{\partial x^2}$.

The Chain Rules

- We can write the Chain Rule in the matrix form.

$$\left(\frac{\partial z}{\partial s} \quad \frac{\partial z}{\partial t}\right) = \left(\frac{\partial z}{\partial x} \quad \frac{\partial z}{\partial y}\right) \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}.$$

The Chain Rules

- **Change of variables**: If $x = x(s,t)$, $y = y(s,t)$ and we also have the inverse expression $s = s(x,y)$ and $t = t(x,y)$, then we can apply the chain rule on functions $x(s(x,y),t(x,y)) = x$ and $y(s(x,y),t(x,y)) = y$.

  Then we obtain

$$\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} \begin{pmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Chain Rules

- Example:
- For $x = r\cos\theta$, $y = r\sin\theta$, the inverse functions are $r^2 = x^2 + y^2$, $\tan\theta = y/x$.

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}$$

2. $(5+10+10=25\%)$ Suppose that $z=f(x,y)$ is a differentiable function, where $x=u^2-v^2$ and $y=2uv$ and $(u,v)\neq(0,0)$.

(a) Find z_u and z_v in terms of z_x (i.e. f_x), z_y (i.e. f_y), u and v.

(b) Conversely, express z_x and z_y in terms of z_u, z_v, u, and v. (**Hint**. Use the result in (a).)

(c) For $z=f\left(x(u,v),y(u,v)\right)=u^2\cos(uv)$, find z_x (i.e. f_x) in terms of u and v.

The Chain Rules

- This is the most general version of the chain rules.

$\boxed{4}$ **The Chain Rule (General Version)** Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

Ex: If $u = e^{xz}\sin(\frac{z}{y}\pi)$, $x = rse^{t}$, $y = r + se^{-t}$, $z = r^2 + s^2 + \sin(2t)$.

Find $\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial s}$, $\frac{\partial u}{\partial t}$ when $(r, s, t) = (0, 2, 0)$.

The Chain Rules: Implicit Differentiation

- The **Implicit Function Theorem**:
- If F is defined on a disk containing (a, b), where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by

$$\boxed{\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}} \tag{6}$$

Suppose that $F_x(x, y)$, $F_y(x, y)$ are continuous on a disc containing (a, b) and $F(a, b) = 0$, $F_y(a, b) \neq 0$. Then $F(x, y) = 0$ defines y as an implicit function of x. Moreover, $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

The Chain Rules: Implicit Differentiation

- Now we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. If F and f are differentiable, then we can use the Chain Rule to compute the derivatives of f.

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$$\boxed{\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}}$$

Suppose that z is given implicitly as a function of x, y by an equation $F(x,y,z)=0$, where F is differentiable. Show that

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial x} = -\frac{F_y}{F_z} \quad \text{if } F_z \neq 0.$$

Ex: The equation $e^z = xy + \cos(x\sqrt{z}) + y$ defines z as an implicit function of x, y, $z(x,y)$, near $(0, 1, \ln 2)$. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ at $(0, 1, \ln 2)$. Estimate $z(0.02, 0.9)$.

Ex: Suppose that $F(x,y,z)=0$ implicitly defines each x, y, z as functions of the other two variables, $z=f(x,y)$, $y=g(x,z)$, $x=h(y,z)$. If $F(x,y,z)$ is differentiable and F_x, F_y, F_z are nonzero, show that $\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}=-1$.

3. (12 pts) Let $f(x,y) = xg(\frac{y}{x})$, where g is a differentiable function with $g(1) = -1$, $g'(1) = 2$.

(a) (4 pts) Use linear approximation to estimate the value of $f(2.01, 1.98)$.

(b) (4 pts) Suppose that at $(x,y) = (2,2)$, $g(\frac{y}{x})$ decreases most rapidly in the direction $\vec{u}$, where $|\vec{u}| = 1$. Find $D_{\vec{u}} f(2,2)$.

(c) (4 pts) If near the point $(2,2,-2)$, the surface $z = f(x,y)$ defines x implicitly as a function of y and z, $x = h(y,z)$. Find $\frac{\partial x}{\partial y}$ and $\frac{\partial x}{\partial z}$ when $(y,z) = (2,-2)$.

Review

- What is the linear approximation of a function of several variables at a point?
- What do we mean by saying that a function of several variables is differentiable at a point?
- Review the chain rules and the implicit differentiation for functions of several variables.