Finding Extreme Values

Section 14.7-14.8

Outline

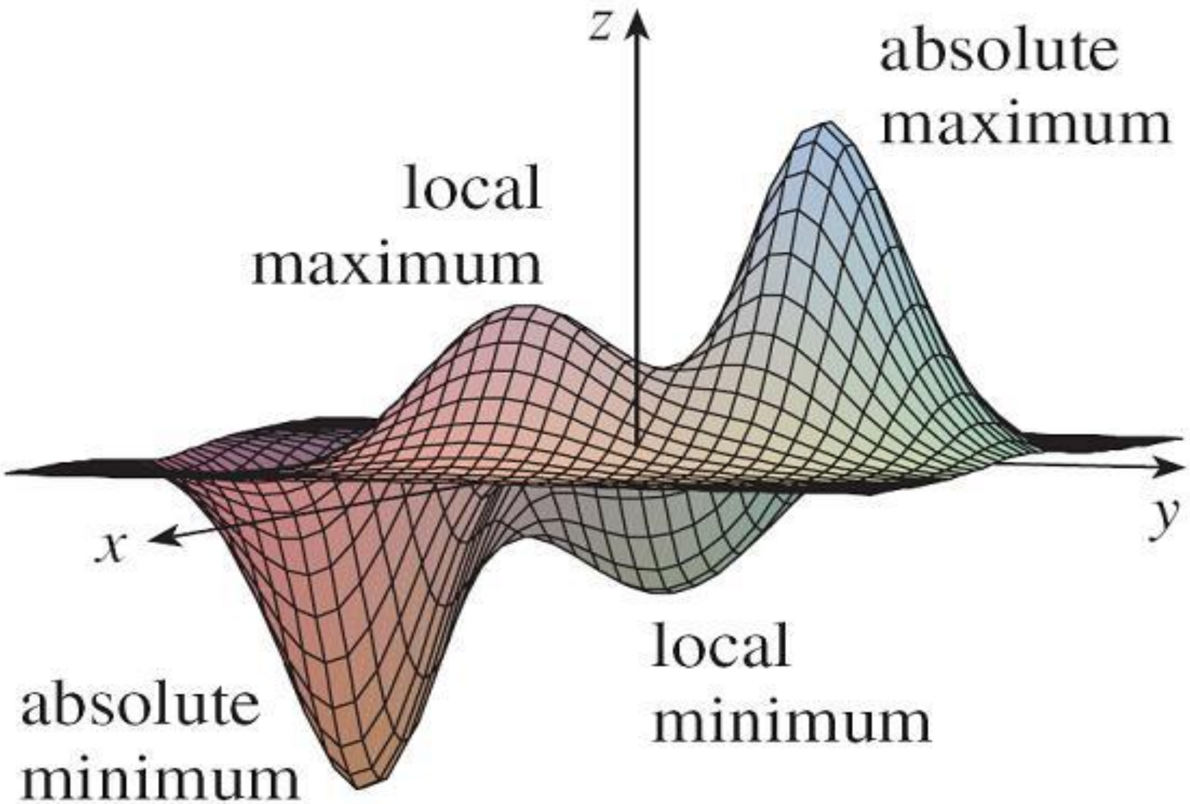
- Definitions of Extreme Values
- Tests for Finding Extreme Values
  - Fermat's Theorem
  - The Second Derivatives Test
- Strategies for Finding Extreme Values
- Finding Extreme Values under Constraints (Lagrange Multipliers)

Definitions of Extreme Values

1 Definition A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

- Definition:
- If the inequalities in Definition 1 hold for all points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).

Definitions of Extreme Values



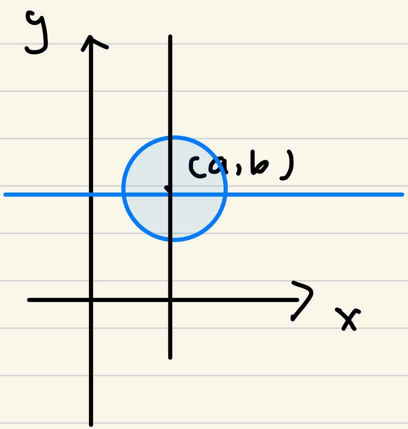
Tests for Finding Extreme Values

2 **Fermat's Theorem for Functions of Two Variables** If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

- Definition: A point (a, b) is called a **critical point** (or stationary point) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.
- Thus, if f has a local maximum or minimum at (a, b), then (a, b) is a critical point of f.

Theorem: If f has a local maximum or minimum at (a,b) and its first partial derivatives exist at (a,b), then $\vec{\nabla} f(a,b) = \vec{0}$.

pf:

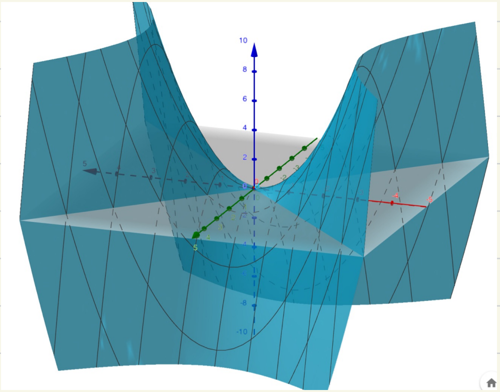


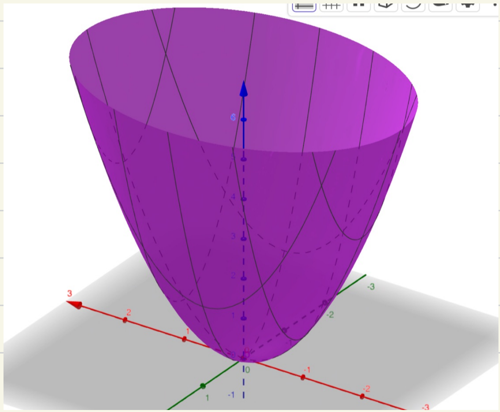
Ex: Find critical points of $f(x,y) = \cos x + \sin(x+2y)$

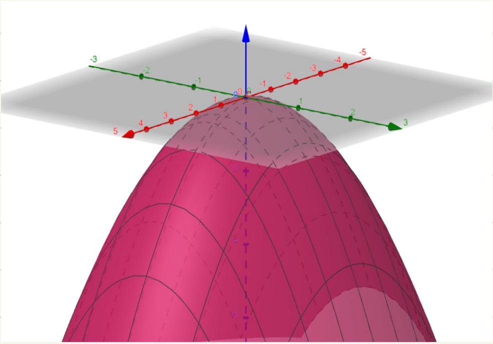
Sol:

Ex: Find critical points of $f(x,y) = x^2 - y^2$, $g(x,y) = x^2 + 2y^2$
$h(x,y) = -2x^2 - y^2$.

sol:







Tests for Finding Extreme Values

$\boxed{3}$ **Second Derivatives Test** Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

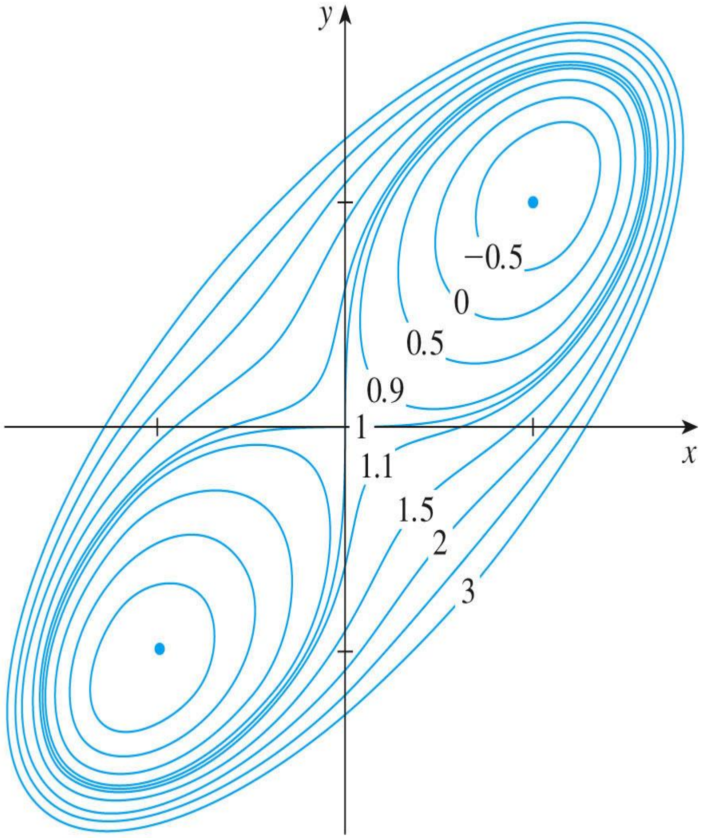
(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

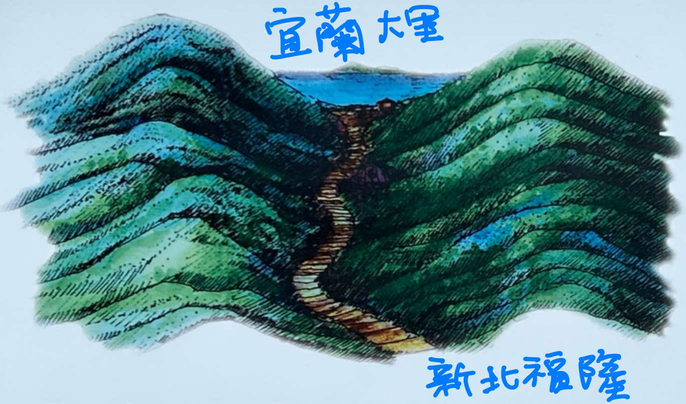
In case (c) the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b).

Tests for Finding Extreme Values



埡口

所謂「埡口」，是位於兩山交會點，自然形成的凹地，也就是地理學上所稱的「鞍部」。鞍部地形時有狂風出現與地勢有絕大關係，面迎台灣北部強大的東北季風，再加上雪山山脈高山阻擋，季風沿山谷湧昇，使得風勢增強，終年不息。



因爲風勢強大，使得高大的喬木難以生長，只有較低矮的芒草及地被植物能夠生存下來。所以在此你看到盡是一片芒草的景觀。入秋之後，當芒花綻放時，又是另外一種動人的景致。

Pf of the 2nd derivatives Test :

Ex: Find and classify critical points of $f(x,y) = x^4 + y^4 - 4xy + 1$

sol:

Strategies for Finding Absolute Extreme Values

- We have following theorem to guarantee the existence of absolute extreme values.

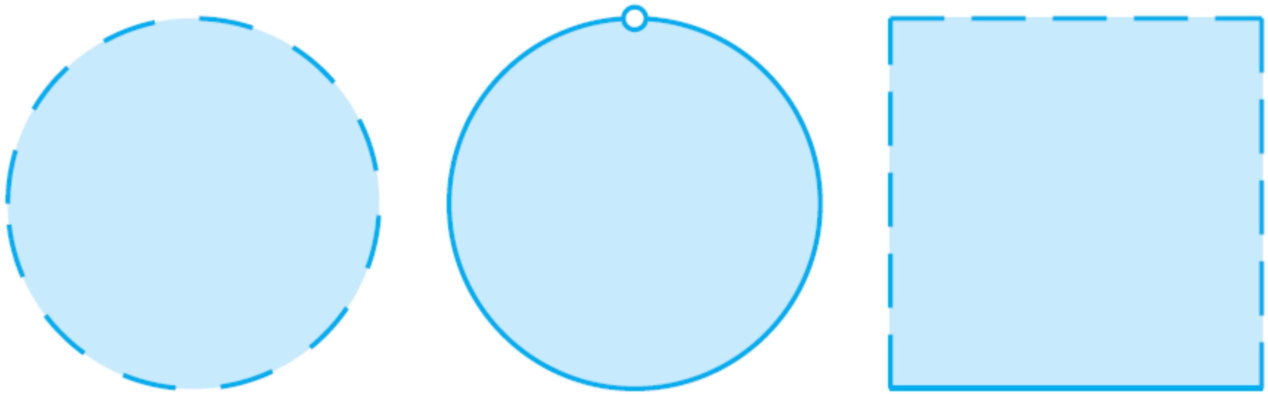
> **8 Extreme Value Theorem for Functions of Two Variables** If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Strategies for Finding Absolute Extreme Values

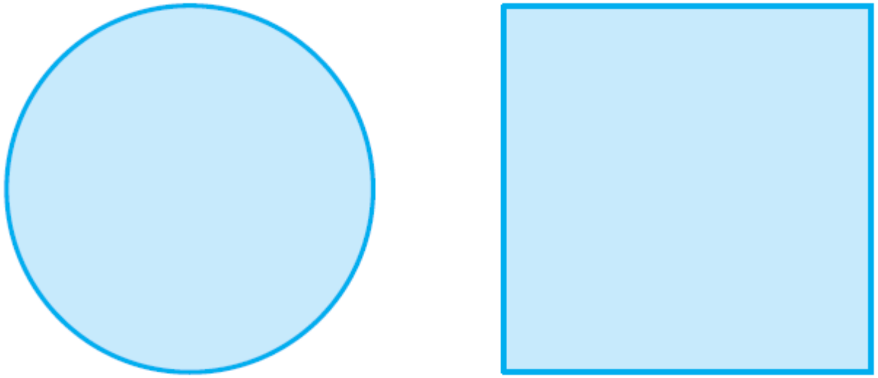
- We say that a set in R^2 is closed if it contains all its *boundary points*. [A *boundary point* of D is a point (a, b) such that every disc with center (a, b) contains points in D and also points not in D .]
- We say that a set S in R^2 is bounded if there is a finite disc D such that S is contained in D .

Strategies for Finding Absolute Extreme Values

Bounded but not closed sets:



Bounded and closed sets:



Strategies for Finding Absolute Extreme Values

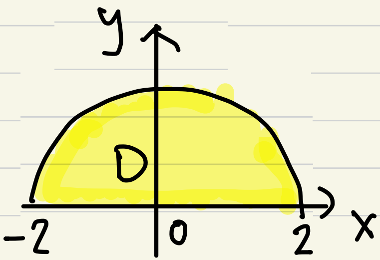
$\boxed{9}$ To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Ex: Find absolute extreme values of $f(x,y) = xy^2 - x$ on

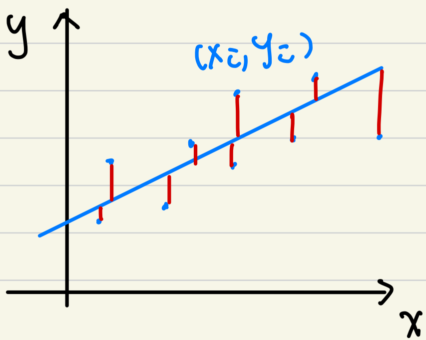
$D = \{(x,y) \mid 0 \le y,\ x^2 + y^2 \le 4\}$

Sol:



Application : Method of Least Squares

We believe that two quantities x and y are approximately related linearly i.e. $y = mx + b$. Now we collect data $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ and want to find the line $y = mx + b$ that fits the data the best. Let $E(m, b) = \sum_{i=1}^{n} (mx_i + b - y_i)^2$. Find m, b that minimizes E.



Review

- What are local extreme values and absolute extreme values of a function of several variables?
- State Fermat's theorem and the second derivatives test for functions of two variables.
- How do we find extreme values of a function on a closed and bounded set?
- State the method of Lagrange multiplier(s) for one constraint as well as two constraints.