Functions of Several variables

Section 14.1-14.3

Outline

- Functions of Several Variables
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  - Level Curves
- Limits and Continuity
- Partial Derivatives
  - Definition
  - Geometric Interpretation
  - Higher Derivatives and Clairaut's Theorem

Limits and Continuity

1 Definition Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

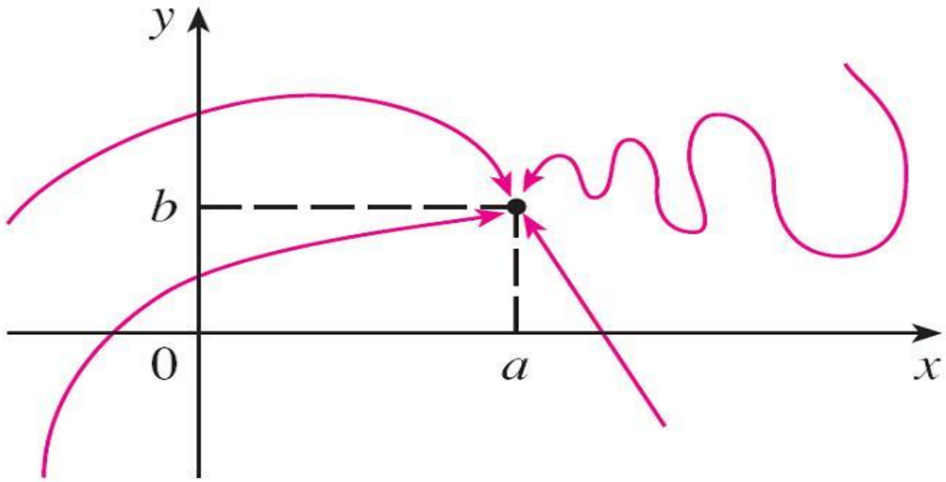
if $\quad (x, y) \in D \quad$ and $\quad 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \quad$ then $\quad |f(x, y) - L| < \varepsilon$

Other notations for the limit are $\displaystyle\lim_{x \to a\ y \to b} f(x, y) = L$

and $f(x, y) \to L$ as $(x, y) \to (a, b)$.

Limits and Continuity

- The definition refers only to the *distance* between (x, y) and (a, b). It does not refer to the direction of approach. Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b).



Limits and Continuity

- Hence we can show that $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist by the following argument.

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x, y)\to(a, b)} f(x, y)$ does not exist.

- To show that the limit exists we use the definition of limit or polar coordinates expression.

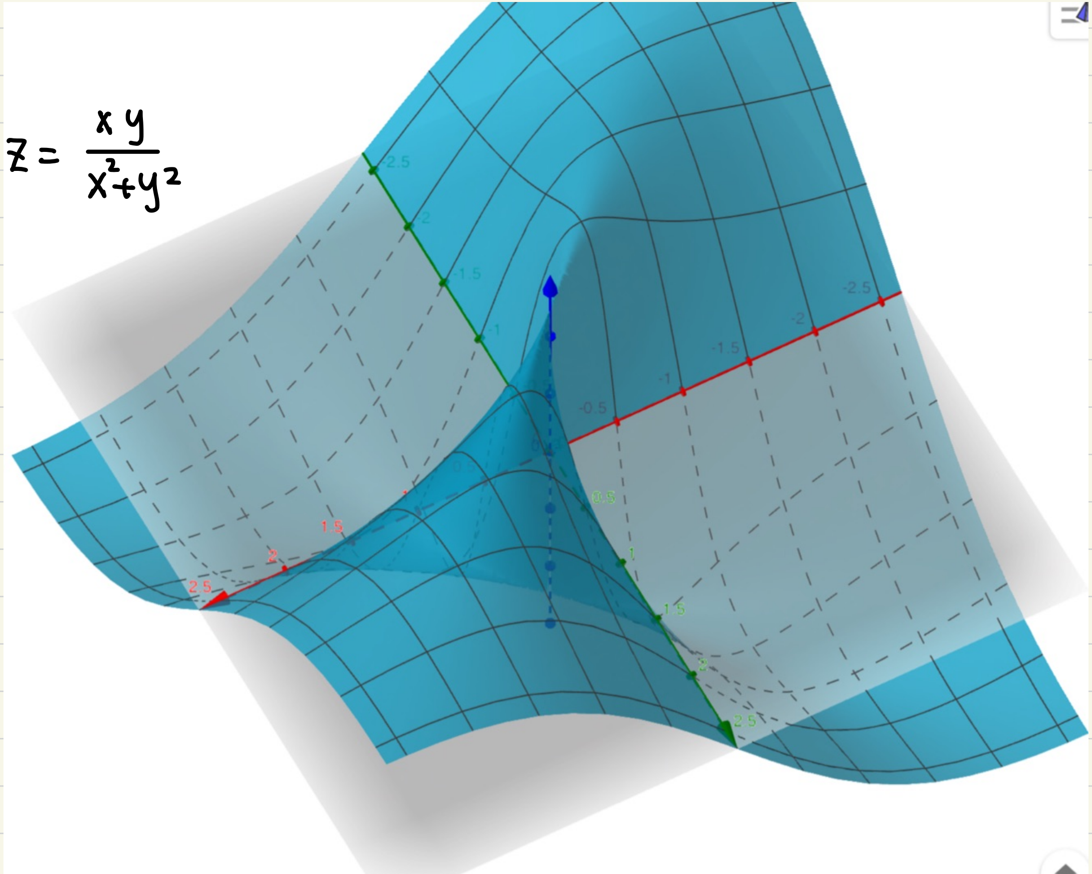
How to prove that the limit doesn't exist

Ex: $f(x,y) = \dfrac{x^2+y^3}{x^2+y^2}$. Check whether $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists.

Ex: $f(x,y) = \dfrac{xy}{x^2+y^2}$. Check whether $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists.

Sol:

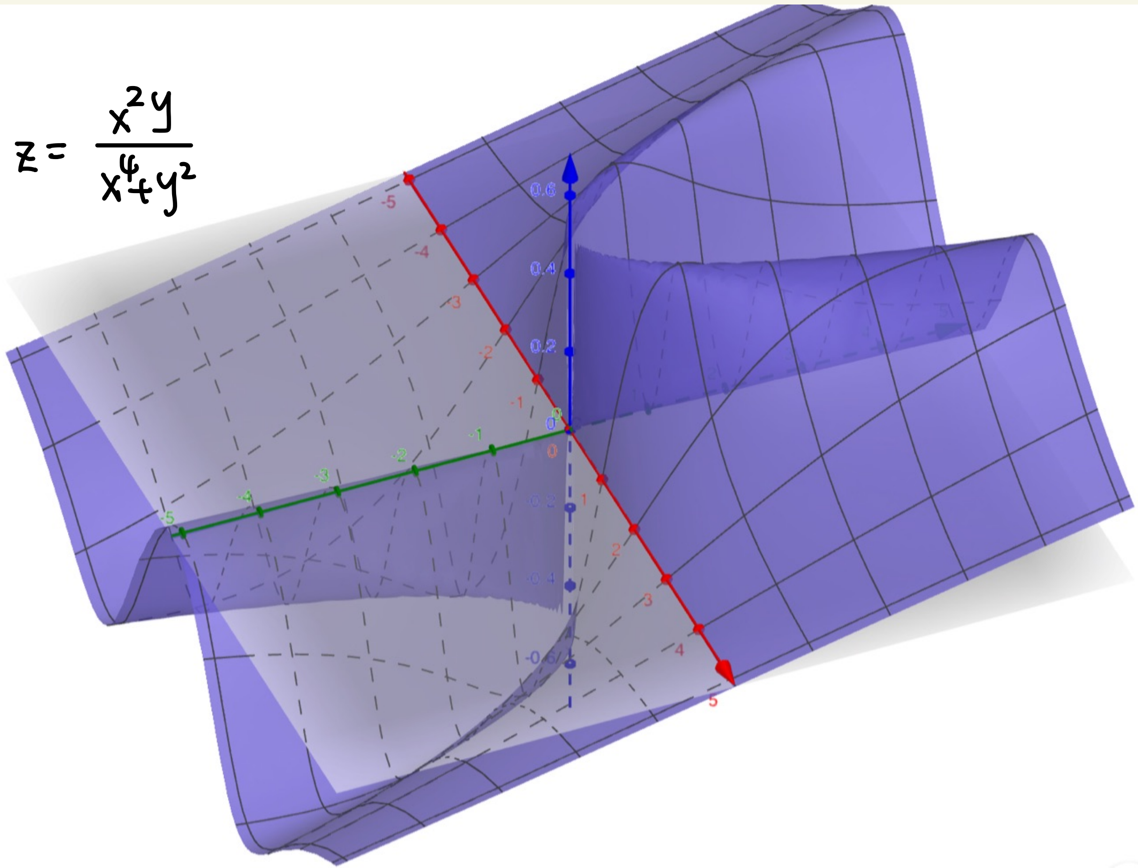
$$z = \frac{xy}{x^2+y^2}$$



Ex: $f(x,y) = \dfrac{x^2 y}{x^4 + y^2}$. Check whether $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists.

Sol:

$$z = \frac{x^2 y}{x^4 + y^2}$$



Ex: Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^6}$.

Sol:

Theorem:

$$\lim_{(x,y)\to(0,0)} \frac{x^a y^b}{x^c+y^d} = 0 \text{ if}$$

c, d are even integers and $\frac{a}{c}+\frac{b}{d}>1$.

Ex: Does $\lim\limits_{(x,y)\to(0,0)} x^y$ exist?

Sol:

How to prove that the limit exists

1. By Definition

Ex: Show that $\lim_{(x,y)\to(a,b)} x = a$.

Ex: Find the limit $\lim_{(x,y)\to(0,1)} \dfrac{x^2(y-1)}{x^2+(y-1)^2}$.

2. Use Polar Coordinates

Ex: Find the limit $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$.

Sol:

Ex: Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{(2x^2+2y^2)}-1}{\sin(x^2+y^2)}$.

Sol:

Ex: Use polar coordinates to find $\lim_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^4+y^2}$.

Limits and Continuity

- The Limit Laws can be extended to functions of two variables. The Squeeze Theorem also holds.
- In particular,

$$\lim_{(x,y)\to(a,b)} x = a \qquad \lim_{(x,y)\to(a,b)} y = b \qquad \lim_{(x,y)\to(a,b)} c = c$$

3. By Squeeze Theorem, Limit Laws

Ex: Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2\sin(x)}{x^2+xy+y^2}$.

Limits and Continuity

4 Definition A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D.

1. All polynomials are continuous on R^2 .
2. Any rational function is continuous on its domain because it is a quotient of continuous functions.

Limits and Continuity

- If f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also continuous.

Ex: let $g(x,y) = \begin{cases} \dfrac{\sin(x^2-y^2)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$.

Determine the set of points at which $g(x,y)$ is continuous.

Ex: let $f(x,y) = \begin{cases} x \cdot \arctan\left(\frac{y}{x^2}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$.

Determine the set of points on which $f(x,y)$ is continuous.

Ex: Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{e^{x+y}} \cdot \sin\left(\frac{1-y^2}{x^2+y^2}\right)$.

Limits and Continuity

- For functions of n variables, we define the limit of the function as follows.

$\boxed{5}$ If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad \mathbf{x} \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta \quad \text{then} \quad |f(\mathbf{x}) - L| < \varepsilon$$

Ex: Let $f(x, y, z) = \dfrac{x^2 z}{x^2+y^2+z^2}$. Check whether $\lim_{(x,y,z)\to(0,0,0)} f(x,y,z)$ exists.

Sol: