Change of Variables in Multiple Integrals

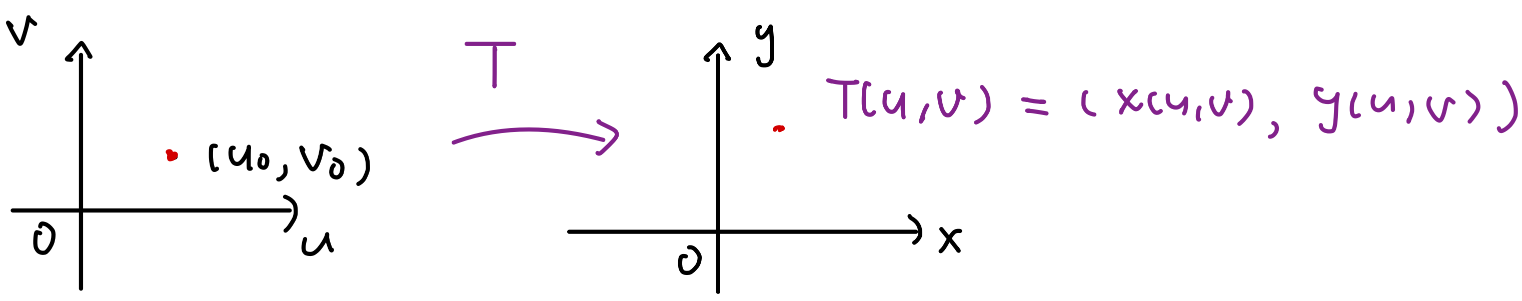
Section 15.7-15.9

Outline

- Triple Integrals in Cylindrical Coordinates
- Triple Integrals in Spherical Coordinates
- Change of Variables in Multiple Integrals

Change of Variables in Multiple Integrals

- We consider a change of variables that is given by a **transformation** T from the uv-plane to the xy-plane: $T(u,v)=(x,y)$ where x and y are related to u and v by the equations $x=g(u,v)$, $y=h(u,v)$, or as we sometimes write $x=x(u,v)$, $y=y(u,v)$.

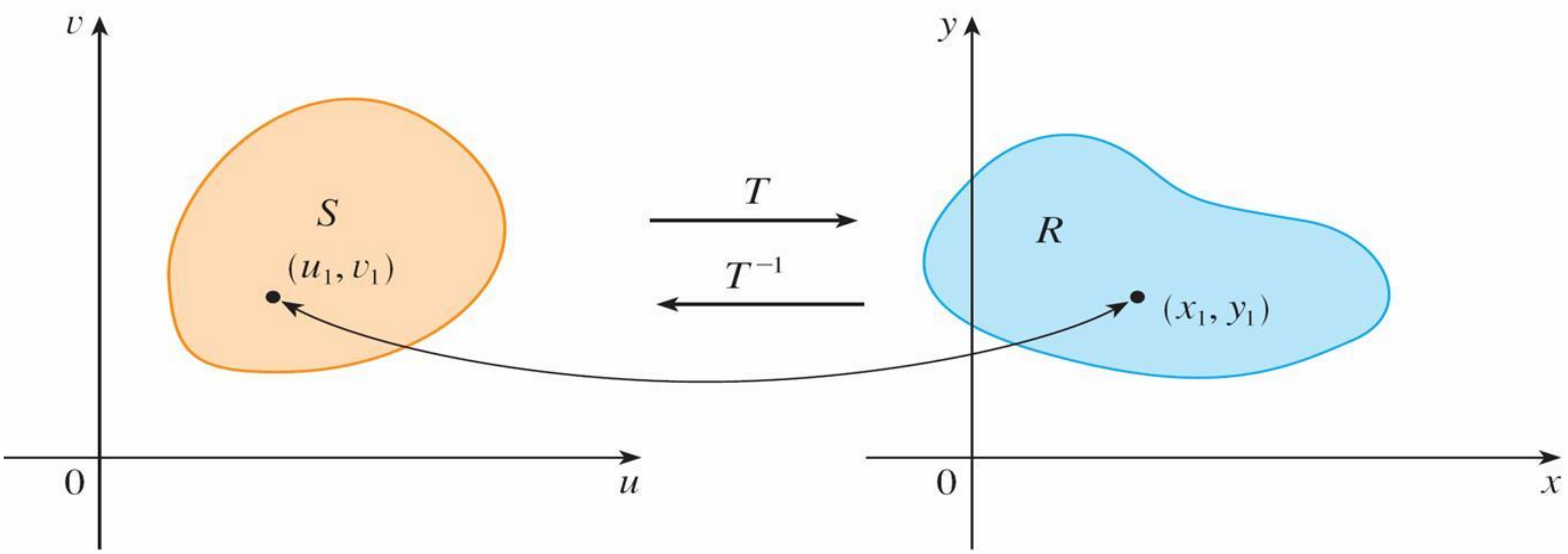


Change of Variables in Multiple Integrals

- We usually assume that T is a C^1 **transformation**, which means that g and h have continuous first-order partial derivatives.
- If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the **image** of the point (u_1, v_1) .
- If no two points have the same image, T is called **one-to-one**.

Change of Variables in Multiple Integrals

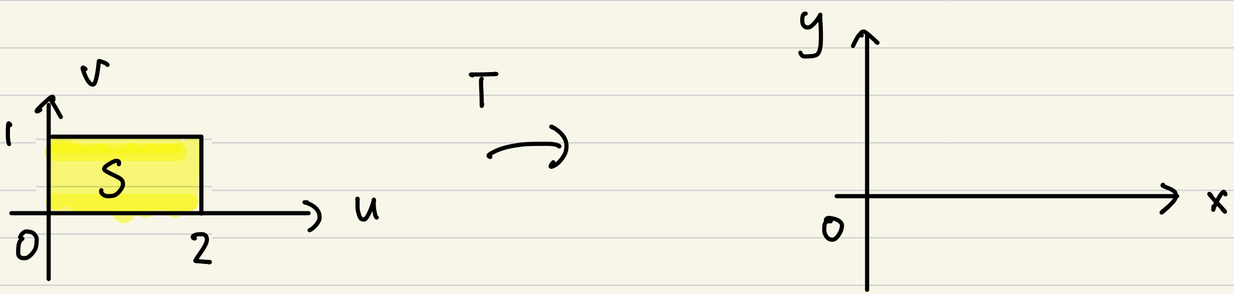
- T transforms a region S in the uv-plane into a region R in the xy-plane called the **image of *S***, consisting of the images of all points in S.

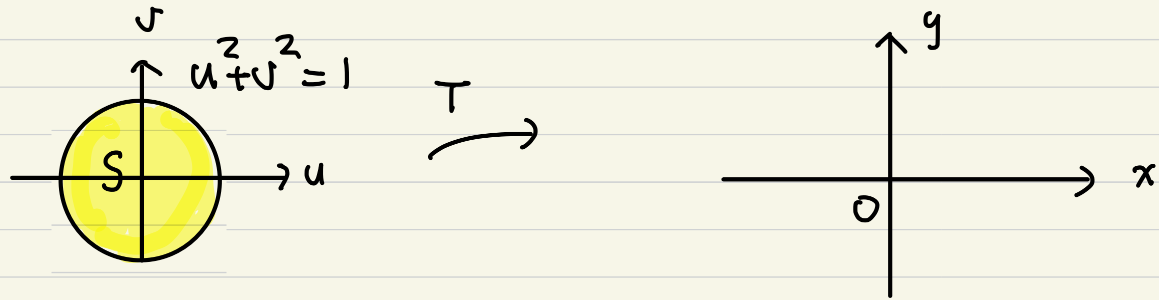


Change of Variables in Multiple Integrals

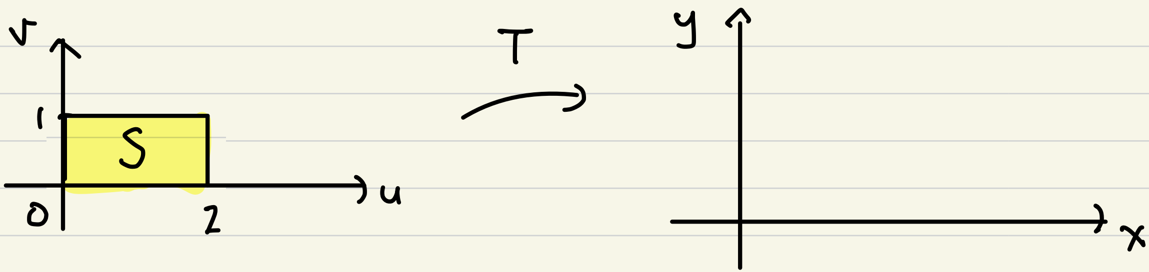
- If T is a one-to-one transformation, then it has an **inverse transformation** T^{-1} from the xy -plane to the uv -plane and it may be possible to solve for u and v in terms of x and y: $u = G(x, y)$, $v = H(x, y)$.

Ex: $T(u,v) = (au, bv)$ where $ab \neq 0$.



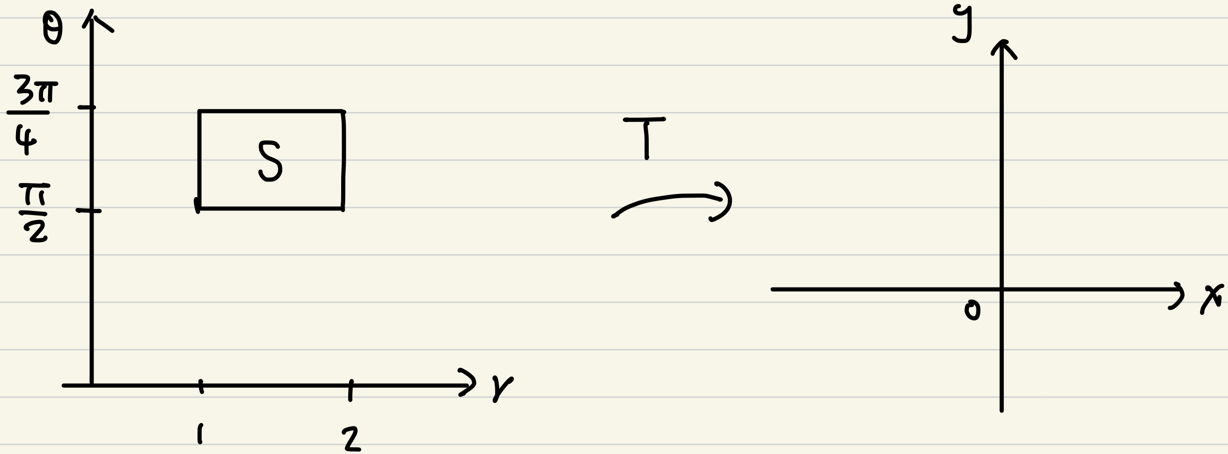


Ex: $T(u,v) = (au+bv,\ cu+dv)$, $ad-bc \neq 0$.

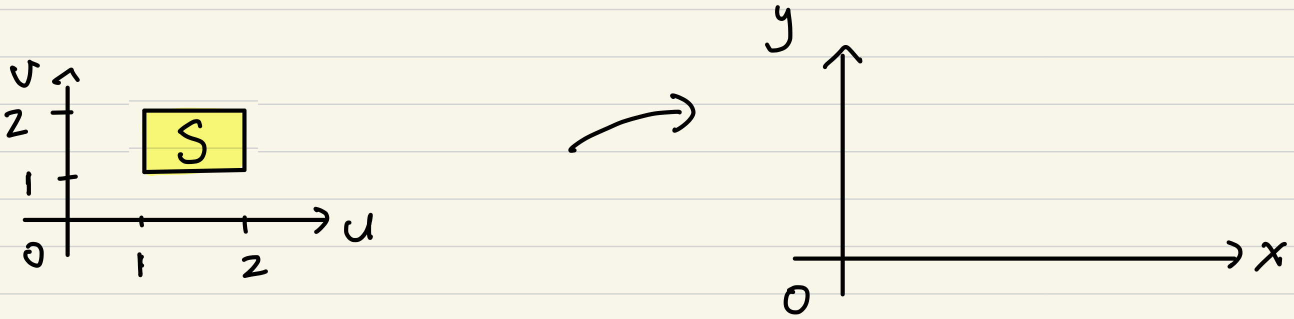


$$T: \begin{cases} x = au+bv \\ y = cu+dv \end{cases}$$

Ex: $T(r, \theta) = (r\cos\theta,\ r\sin\theta)$

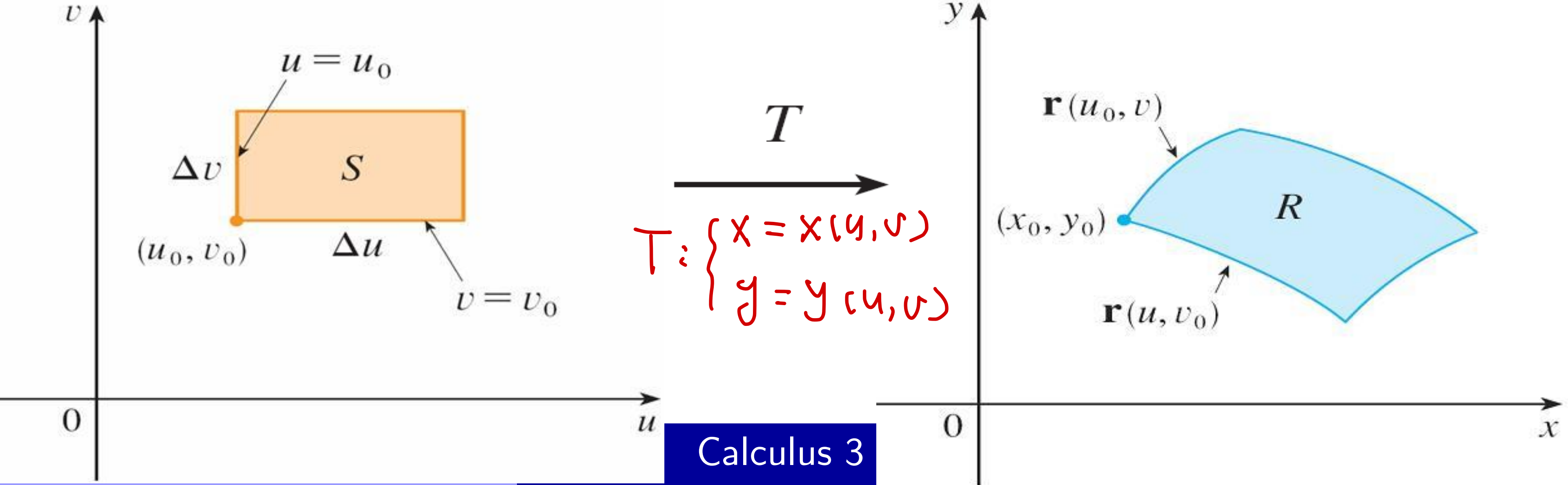


Ex: $T(u, v) = \left(uv, \frac{u}{v}\right)$



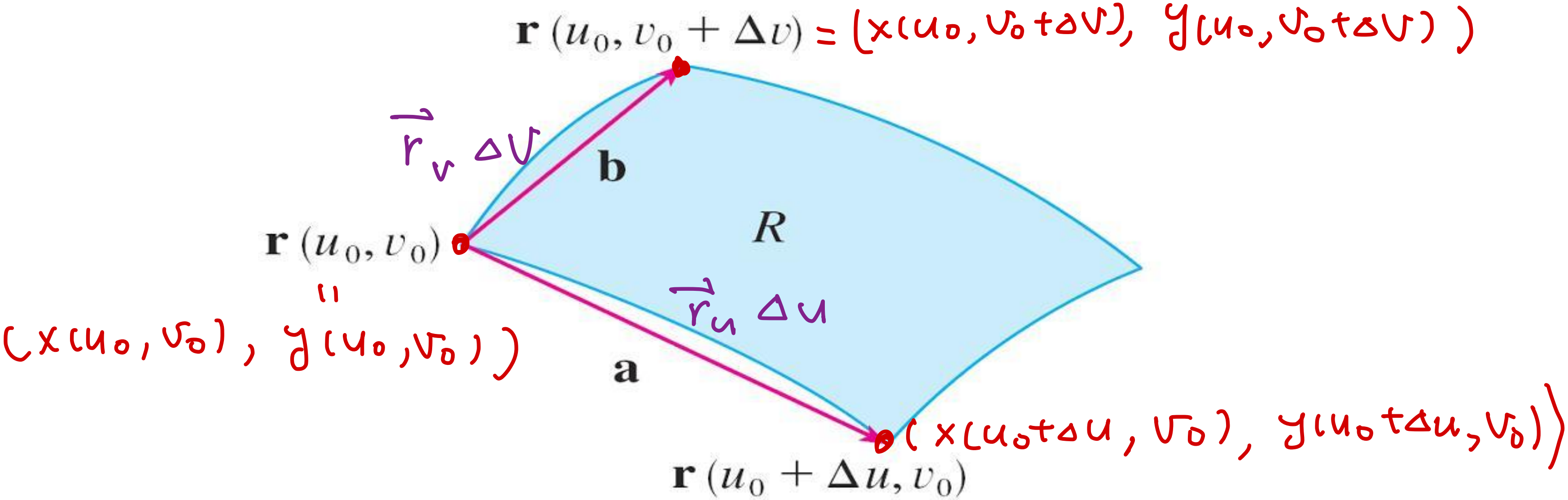
Change of Variables in Multiple Integrals

- Now let's see how a change of variables affects a double integral. We start with a small rectangle S in the uv-plane whose lower left corner is the point (u_0, v_0) and whose dimensions are Δu and Δv.



Change of Variables in Multiple Integrals

- We can approximate the image region $R = T(S)$ by a parallelogram determined by the secant vectors : $\vec{a} = \vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0)$,
$$\vec{b} = \vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0)$$



$$A(R) \approx |\vec{a} \times \vec{b}|$$

Change of Variables in Multiple Integrals

- Therefore we can approximate the area of R by
$$|(\Delta u \vec{r}_u) \times (\Delta v \vec{r}_v)| = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$
- Computing the cross product, we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

Change of Variables in Multiple Integrals

- The determinant that arises in this calculation is called the *Jacobian* of the transformation and is given a special notation.

7 Definition The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\boxed{\frac{\partial(x, y)}{\partial(u, v)}} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Ex: $T(u,v) = (au+bv,\ cu+dv)$. Find $\dfrac{\partial(x,y)}{\partial(u,v)}$.

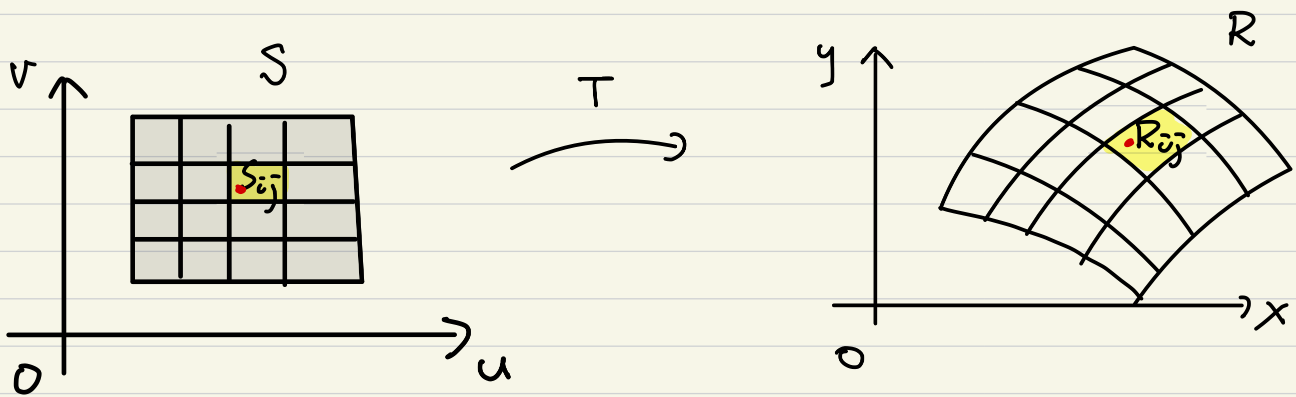
Sol:

Ex: $T(r,\theta) = (r\cos\theta,\ r\sin\theta)$, Find $\dfrac{\partial(x,y)}{\partial(r,\theta)}$.

Sol:

Change of variables

$T(u,v) = (x(u,v),\ y(u,v))$. $\quad T(S)=R$.



$$\iint_R f(x,y)\,dA = \lim \sum_i \sum_j f(x(u_i^*, v_j^*), y(u_i^*, v_j^*))\, A(R_{ij})$$

$$= \lim \sum_i \sum_j f(x(u_i^*, v_j^*), y(u_i^*, v_j^*)) \left\| \begin{matrix} x_u & x_v \\ y_u & y_v \end{matrix} \right\| \Delta u \Delta v$$

Change of Variables in Multiple Integrals

- And after taking the limit of the double Riemann sum, we obtain the following formula.

9 Change of Variables in a Double Integral Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x, y)\, dA = \iint_S f\big(x(u, v), y(u, v)\big) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

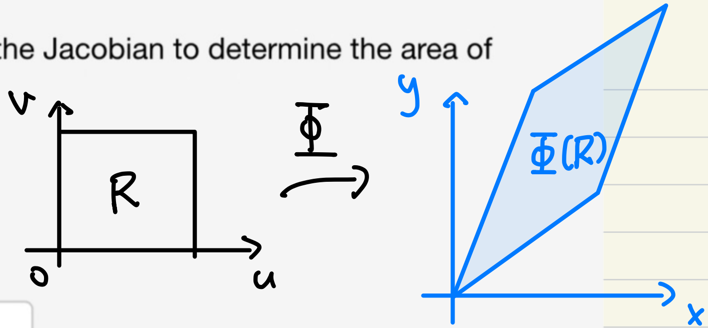
① $f(x,y) \longrightarrow f(x(u,v),\, y(u,v))$

② $dx\,dy \longrightarrow \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$

③ $R \longrightarrow S$

Ex:

Let
$\Phi(u, v) = (2u + 2v, 7u + 5v)$. Use the Jacobian to determine the area of $\Phi(\mathcal{R})$ for:



$(\mathbf{a})\mathcal{R} = [0, 5] \times [0, 6]$

$(\mathbf{b})\mathcal{R} = [9, 18] \times [4, 13]$

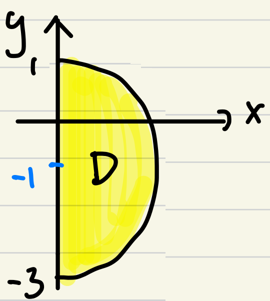
$(\mathbf{a})\text{Area}\,(\Phi(\mathcal{R})) =$ ______

$(\mathbf{b})\text{Area}\,(\Phi(\mathcal{R})) =$ ______

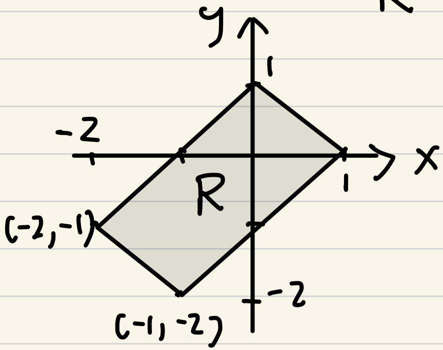
Sol:

Ex: Compute $\iint_D 4x^2+(y+1)^2 \, dA$, where D is the right half ellipse $\{(x,y) \mid 4x^2+(y+1)^2 \le 4 \text{ and } x \ge 0\}$.

Sol:

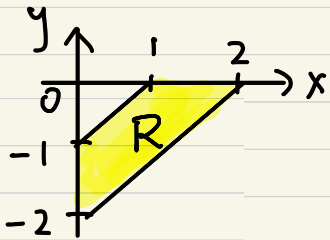


Ex: Compute $\iint_R e^x \, dA$, where R is a parallelogram with vertices $(1,0)$, $(0,1)$, $(-2,-1)$ and $(-1,-2)$



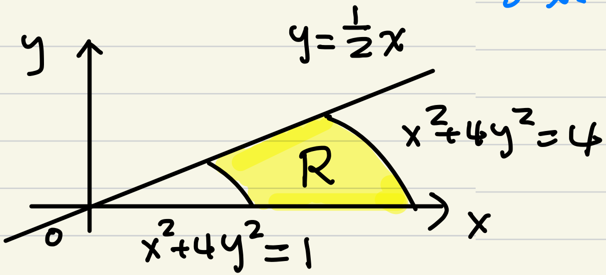
Sol:

Ex: Compute $\displaystyle\iint_R e^{\frac{x+y}{x-y}}\, dA$, where R is a trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-1)$, $(0,-2)$.



Sol:

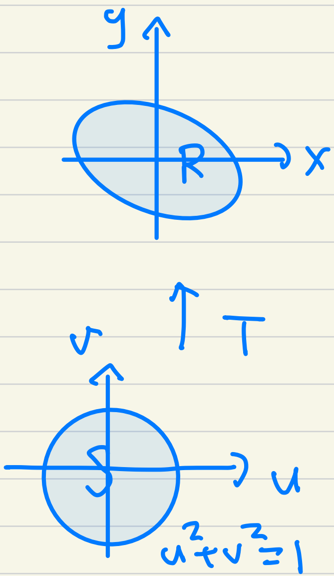
Ex: Compute $\iint_R \frac{y}{x}\,dA$, where R is the region shown in the figure.



sol:

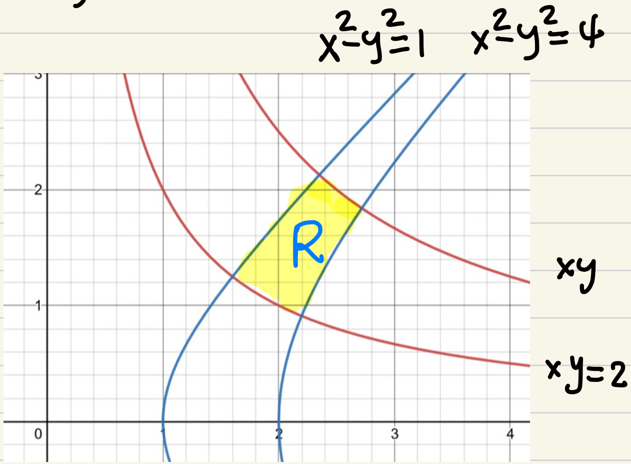
Ex: $\iint_R (x+y)^2 \, dA$, where R is bounded by $x^2+2xy+3y^2=1$.

sol:



Ex: $\int_0^1 \int_x^{3x} (3x-y)e^{(y-x)}\,dy\,dx + \int_1^2 \int_{3x-2}^{x+2} (3x-y)\,e^{(y-x)}\,dy\,dx$

Ex: Compute $\iint\limits_R (x^4-y^4)\,e^{xy}\,dA$, where R is the region in the first quadrant bounded by $xy=2$, $xy=5$, $x^2-y^2=1$, $x^2-y^2=4$.



Change of Variables in Multiple Integrals

- Sometimes, we have the " inverse"

T^{-1}: transformation $u = G(x, y)$ and $v = H(x, y)$ instead. By the chain rule, we derive that

$$\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Hence, $\dfrac{\partial(x, y)}{\partial(u, v)} \dfrac{\partial(u, v)}{\partial(x, y)} = 1$. Therefore

$$\frac{\partial(x, y)}{\partial(u, v)} = \left(\frac{\partial(u, v)}{\partial(x, y)} \right)^{-1}$$

Change of Variables in Multiple Integrals

- There is a similar change of variables formula for triple integrals. Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space by means of the equations $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$.

Change of Variables in Multiple Integrals

- The **Jacobian** of T is the following 3×3 determinant:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

- Under hypotheses similar to those in double integrals, we have the formula:

$$\boxed{13} \quad \iiint_R f(x, y, z)\, dV = \iiint_S f\big(x(u, v, w), y(u, v, w), z(u, v, w)\big) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

Ex: Find the Jacobian of $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$.

sol:

Ex: Compute the Jacobian of spherical coordinates

$$x=\rho\sin\varphi\cos\theta,\quad y=\rho\sin\varphi\sin\theta,\quad z=\rho\cos\varphi.$$

Sol:

Ex: Compute $\iiint_E x \, dV$, where E is the tetrahedron bounded by $x+y+z=1$, $x+y-z=0$, $x-y+z=0$, $-x+y+z=0$.

Sol:

Ex: Compute the integral $\iiint_E y^2 \, dV$, where E is the solid bounded by $\frac{x^2}{4} + (y-1)^2 + 9z^2 = 1$.

Sol:

Review

- How do we represent a point by cylindrical coordinates? How do we do integrations in cylindrical coordinates?
- How do we represent a point by spherical coordinates? How do we do integrations in spherical coordinates?
- What is the Jacobian of a transformation from uv-plane to xy-plane (from uvw-space to xyz-space)? How do we change of variables in multiple integrals?