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The Laplace distribution PDF, CDF, Median, Mode, Mean and Variance

The Laplace distribution is defined by the following PDF:

$$f_X(x) = \frac{1}{\sigma\sqrt{2}} \exp\left(-\frac{|x-\mu|}{\sigma}\sqrt{2}\right) \Leftrightarrow X \sim Laplace(\mu, \sigma)$$

With support on $\mathbb{R}$ for $\mu \in \mathbb{R}$ and $\sigma > 0$. Let us check if this is indeed a PDF (note that $f(x)$ is symmetric about the parameter μ):

$$\int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2}} \exp\left(-\frac{|x-\mu|}{\sigma}\sqrt{2}\right) dx = \frac{\sqrt{2}}{\sigma} \int_{\mu}^{+\infty} \exp\left(-\frac{x-\mu}{\sigma}\sqrt{2}\right) dx$$
$$= \begin{bmatrix} z = \frac{x-\mu}{\sigma}\sqrt{2} & dx = \frac{\sigma}{\sqrt{2}} dz \\ x = \frac{\sigma}{\sqrt{2}} z + \mu & \begin{matrix} x \to +\infty, z \to +\infty \\ x \to \mu, z \to 0 \end{matrix} \end{bmatrix} = \int_{0}^{+\infty} \exp(-z)\, dz$$
$$= -e^{-z}\Big|_0^{+\infty} = e^{-z}\Big|_{+\infty}^{0} = 1$$

So, this is indeed a valid PDF. Now let us compute the CDF:

$$
\begin{aligned}
F_X(x) = &\int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2}} \exp\left(-\frac{|t-\mu|}{\sigma}\sqrt{2}\right) dt \\
&= \begin{cases} \dfrac{1}{2} + \displaystyle\int_{\mu}^{x} \frac{1}{\sigma\sqrt{2}} \exp\left(-\frac{|t-\mu|}{\sigma}\sqrt{2}\right) dt, & \text{when } x \geq \mu \\ \dfrac{1}{2} - \displaystyle\int_{\mu}^{x} \frac{1}{\sigma\sqrt{2}} \exp\left(-\frac{|t-\mu|}{\sigma}\sqrt{2}\right) dt, & \text{when } x < \mu \end{cases} \\
&= \frac{1}{2} + \operatorname{sgn}(x-\mu) \frac{1}{\sigma\sqrt{2}} \int_{\mu}^{x} \exp\left(-\frac{|t-\mu|}{\sigma}\sqrt{2}\right) dt \\
&= \begin{bmatrix} z = \dfrac{x-\mu}{\sigma}\sqrt{2} & dx = \dfrac{\sigma}{\sqrt{2}} dz \\ x = \dfrac{\sigma}{\sqrt{2}} z + \mu & x \to \mu, z \to 0 \end{bmatrix} \\
&= \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(x-\mu) \int_{0}^{\frac{x-\mu}{\sigma}\sqrt{2}} \exp(-|z|)\, dt \\
&= \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(x-\mu)\left(1 - \exp\left(-\frac{|x-\mu|}{\sigma}\sqrt{2}\right)\right)
\end{aligned}
$$

The median $\operatorname{med}(X)$ of the distribution is obviously μ, since

$$F_X(\mu) = \frac{1}{2}$$

The μ parameter is also the mode, since the function $f_X(x)$ is maximized at $x = \mu$:

$$f_X(x) = \frac{1}{\sigma\sqrt{2}} \exp\left(-\frac{|x-\mu|}{\sigma}\sqrt{2}\right) \leq \frac{1}{\sigma\sqrt{2}} = f_X(\mu)$$

Let us find the MGF of the distribution:

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \\ -1, & \text{for } x < 0 \end{cases}$$

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2}} \exp(tx) \exp\left(-\frac{|x-\mu|}{\sigma}\sqrt{2}\right) dx$$
$$= \frac{1}{\sigma\sqrt{2}} \int_{-\infty}^{+\infty} \exp\left(tx - \frac{|x-\mu|}{\sigma}\sqrt{2}\right) dx$$
$$= \begin{bmatrix} z = \frac{x-\mu}{\sigma}\sqrt{2} & dx = \frac{\sigma}{\sqrt{2}} dz \\ x = \frac{\sigma}{\sqrt{2}} z + \mu & \begin{matrix} x \to +\infty, z \to +\infty \\ x \to -\infty, z \to -\infty \end{matrix} \end{bmatrix}$$
$$= \frac{1}{2} \int_{-\infty}^{+\infty} \exp\left(t\frac{\sigma}{\sqrt{2}} z + t\mu - |z|\right) dx$$
$$= \frac{\exp(t\mu)}{2} \int_{-\infty}^{+\infty} \exp\left(t\frac{\sigma}{\sqrt{2}} z - |z|\right) dx =$$
$$= \frac{\exp(t\mu)}{2}\left(\int_{-\infty}^{0} \exp\left(\left(t\frac{\sigma}{\sqrt{2}} + 1\right) z\right) dx + \int_{0}^{+\infty} \exp\left(\left(t\frac{\sigma}{\sqrt{2}} - 1\right) z\right) dx \right)$$

These two integrals only converge for the values of $|t| < \frac{\sqrt{2}}{\sigma}$:

$$M_X(t) = \frac{\exp(t\mu)}{2}\left(\frac{1}{1 + t\frac{\sigma}{\sqrt{2}}} + \frac{1}{1 - t\frac{\sigma}{\sqrt{2}}} \right) = \frac{\exp(t\mu)}{2} \cdot \frac{1 - t\frac{\sigma}{\sqrt{2}} + 1 + t\frac{\sigma}{\sqrt{2}}}{1 - t^2\frac{\sigma^2}{2}}$$
$$= \frac{\exp(t\mu)}{1 - t^2\frac{\sigma^2}{2}}$$

Thus,

$$\mathbb{E}(X) = \frac{d}{dt} M_X(t)\bigg|_{t=0} = \frac{d}{dt}\left(\frac{\exp(t\mu)}{1 - t^2\frac{\sigma^2}{2}} \right)\Bigg|_{t=0}$$
$$= \frac{\mu \exp(t\mu)\left(1 - t^2\frac{\sigma^2}{2}\right) + 2t\frac{\sigma^2}{2}\exp(t\mu)}{\left(1 - t^2\frac{\sigma^2}{2}\right)^2}\Bigg|_{t=0} = \mu$$

So, the location parameter μ is the expected value, mode and the median of the Laplace distribution.

$$\mathbb{E}(X^2) = \frac{d^2}{dt^2} M_X(t)\bigg|_{t=0} = \frac{d}{dt}\left(\frac{\mu \exp(t\mu)\left(1 - t^2 \frac{\sigma^2}{2}\right) + 2t\frac{\sigma^2}{2}\exp(t\mu)}{\left(1 - t^2\frac{\sigma^2}{2}\right)^2}\right)\Bigg|_{t=0} =$$

$$= \frac{d}{dt}\left(\mu \frac{\exp(t\mu)}{\left(1 - t^2\frac{\sigma^2}{2}\right)} + \sigma^2 \frac{t\exp(t\mu)}{\left(1 - t^2\frac{\sigma^2}{2}\right)^2}\right)\Bigg|_{t=0}$$

$$= \mu^2 + \sigma^2\left(\frac{(1+\mu t)\exp(t\mu)\left(1 - t^2\frac{\sigma^2}{2}\right)^2 + 2t\sigma^2\left(1 - t^2\frac{\sigma^2}{2}\right)t\exp(t\mu)}{\left(1 - t^2\frac{\sigma^2}{2}\right)^4}\right)\Bigg|_{t=0}$$

$$= \mu^2 + \sigma^2$$

Therefore,

$$\mathbb{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

As expected the scale parameter σ is also the standard deviation of the distribution.

The Laplace distribution parameter estimation

Let us find the maximum likelihood estimators for this distribution. The likelihood function for a sample $x = (x_1, x_2, \dots, x_n)^T$ of size n is:

$$L(x|\hat{\mu},\hat{\sigma}) = \prod_{i=1}^{n} f_X(x_i|\hat{\mu},\hat{\sigma})$$

$$l(x|\hat{\mu},\hat{\sigma}) = \ln\big(L(x|\hat{\mu},\hat{\sigma})\big) = \sum_{i=1}^{n} \ln\big(f_X(x_i|\hat{\mu},\hat{\sigma})\big)$$

$$= \sum_{i=1}^{n} \ln\left(\frac{1}{\hat{\sigma}\sqrt{2}}\exp\left(-\frac{|x_i - \hat{\mu}|}{\hat{\sigma}}\sqrt{2}\right)\right)$$

$$= \sum_{i=1}^{n}\left(-\ln\big(\hat{\sigma}\sqrt{2}\big) - \frac{|x_i - \hat{\mu}|}{\sigma}\sqrt{2}\right) = -\frac{\sqrt{2}}{\hat{\sigma}}\sum_{i=1}^{n}|x_i - \hat{\mu}| - n\ln\big(\hat{\sigma}\sqrt{2}\big)$$

$$\frac{d}{d\hat{\mu}} l(x|\hat{\mu},\hat{\sigma}) = -\frac{\sqrt{2}}{\hat{\sigma}}\sum_{i=1}^{n}\frac{x_i - \hat{\mu}}{|x_i - \hat{\mu}|} = -\frac{\sqrt{2}}{\hat{\sigma}}\sum_{i=1}^{n}\operatorname{sgn}(x_i - \hat{\mu}) = 0$$

$$\Rightarrow \sum_{i=1}^{n} \operatorname{sgn}(x_i - \hat{\mu}) = 0$$

Technically, the derivative of the absolute value is not exactly sgn since the absolute value function is not differentiable at $x_i = \hat{\mu}$, but $\operatorname{sgn}(\hat{\mu} - \hat{\mu}) = 0$ is well-defined. Let us make a continuation and state that $\frac{d}{dx}|x| \stackrel{\text{def}}{=} \operatorname{sgn}(x)$. So, we basically define $\frac{d}{dx}|x|\Big|_{x=0} \stackrel{\text{def}}{=} 0$

If the sample size n is odd, then choose $\hat{\mu} = x_{med}$, since there would be exactly $\frac{n-1}{2}$ cases where $\operatorname{sgn}(x_i - \hat{\mu}) = -1$ and $\frac{n-1}{2}$ cases where $\operatorname{sgn}(x_i - \hat{\mu}) = 1$ and one case where $\operatorname{sgn}(0) = 0$. So, this estimator satisfies the maximum likelihood constraint.

If the sample size n is even, then we can still choose $\hat{\mu} = x_{med}$, in which case exactly $\frac{n}{2}$ cases where $\operatorname{sgn}(x_i - \hat{\mu}) = -1$ and $\frac{n}{2}$ cases where $\operatorname{sgn}(x_i - \hat{\mu}) = 1$

So, $\hat{\mu} = x_{med}$ is the maximum likelihood estimator.

$$\frac{d}{d\hat{\sigma}} l(x|\hat{\mu}, \hat{\sigma}) = \frac{\sqrt{2}}{\hat{\sigma}^2} \sum_{i=1}^{n} |x_i - \hat{\mu}| - \frac{n}{\hat{\sigma}} = 0$$

$$\Rightarrow \sqrt{2} \sum_{i=1}^{n} |x_i - \hat{\mu}| - n\hat{\sigma} = 0 \Rightarrow \hat{\sigma} = \frac{\sqrt{2}}{n} \sum_{i=1}^{n} |x_i - \hat{\mu}|$$

And we have determined that $\hat{\mu} = x_{med}$, so in the end:

$$\begin{cases} \hat{\mu} = x_{med} \\ \hat{\sigma} = \dfrac{\sqrt{2}}{n} \displaystyle\sum_{i=1}^{n} |x_i - x_{med}| \end{cases}$$

If we wish to estimate the CDF of the Laplace distribution, then we would do that by

$$\hat{F}_X(x) = \frac{1}{2} + \frac{1}{2} \operatorname{sgn}(x - \hat{\mu}) \left(1 - \exp\left(-\frac{|x - \hat{\mu}|}{\hat{\sigma}} \sqrt{2}\right)\right) =$$

$$= \frac{1}{2} + \frac{1}{2} \operatorname{sgn}(x - x_{med}) \left(1 - \exp\left(-|x - x_{med}| \left(\frac{1}{n} \sum_{i=1}^{n} |x_i - x_{med}|\right)^{-1}\right)\right)$$

The related distributions

Note that $X \sim Laplace(\mu, \sigma) \Leftrightarrow X - \mu \sim Laplace(0, \sigma)$ and if we consider $|X - \mu|$, then

$$f_{|X-\mu|}(x) = 2 \cdot \frac{1}{\sigma\sqrt{2}} \exp\left(-\frac{\sqrt{2}}{\sigma} x\right) = \frac{\sqrt{2}}{\sigma} \exp\left(-\frac{\sqrt{2}}{\sigma} x\right)$$

With support on $\mathbb{R}_{\geq 0} = [0; +\infty)$. This is a PDF of an Exponential distribution with parameter $\lambda = \frac{\sqrt{2}}{\sigma}$, so

$$X \sim Laplace(\mu, \sigma) \Rightarrow |X - \mu| \sim Exp\left(\frac{\sqrt{2}}{\sigma}\right)$$

If $X_i \sim Exp(\lambda)$ iid, then

$$\sum_{i=1}^{n} |X_i| \sim Gamma(n, \lambda)$$

So,

$$X_i \sim Laplace(\mu, \sigma) \Rightarrow \sum_{i=1}^{n} |X_i - \mu| \sim Gamma\left(n, \frac{\sqrt{2}}{\sigma}\right)$$

Note that the following is true (for $C \in \mathbb{R} \setminus \{0\}$):

$$X \sim Gamma(\alpha, \beta) \Rightarrow C \cdot X \sim Gamma\left(\alpha, \frac{\beta}{C}\right)$$

Thus,

$$X_i \sim Laplace(\mu, \sigma) \Rightarrow \frac{2\sqrt{2}}{\sigma} \sum_{i=1}^{n} |X_i - \mu| \sim Gamma\left(n, \frac{1}{2}\right)$$

If we define $\frac{k}{2} = n \Rightarrow k = 2n$, then

$$X_i \sim Laplace(\mu, \sigma) \Rightarrow \frac{2\sqrt{2}}{\sigma} \sum_{i=1}^{n} |X_i - \mu| \sim \chi^2_{2n}$$

So, if we have an iid sample $x = (x_1, x_2, \ldots, x_n)^T$ that is drawn from the $Laplace(\mu, \sigma)$, then the statistic

$$\frac{2\sqrt{2}}{\sigma}\sum_{i=1}^{n}|x_i-\mu| \sim \chi^2_{2n}$$

Another important fact is that suppose $X, Y \sim Exp(\lambda)$ for $\lambda > 0$ are iid and $Z = X - Y$, then

$$f_Z(z) = f_{X-Y}(z) = \int_0^{+\infty} f_X(z+y) f_Y(y) dy = \lambda^2 \int_0^{+\infty} e^{-\lambda(z+y)} e^{-\lambda y} dy$$

$$= \lambda^2 e^{-\lambda z} \int_0^{+\infty} e^{-2\lambda y} dy = \lambda^2 e^{-\lambda z}\left(\frac{0-1}{-2\lambda}\right) = \frac{\lambda}{2} e^{-\lambda z}$$

However, this is only true in the case where $z \geq 0$. We need to also check when $z < 0$. Note that $Z = X - Y \Rightarrow -Z = Y - X$ and since X and Y are iid, the distribution of Z and $-Z$ must be the same. Thus, when $z < 0$, $f_Z(z)$ should still equal $\frac{\lambda}{2}e^{-\lambda z}$ for when $z \geq 0$, thus, for $z \in \mathbb{R}$:

$$f_{X-Y}(z) = \frac{\lambda}{2} e^{-\lambda|z|}$$

Recall that we defined for a Laplace distributed variable X

$$f_X(x) = \frac{1}{\sigma\sqrt{2}} \exp\left(-\frac{|x-\mu|}{\sigma}\sqrt{2}\right) \Leftrightarrow X \sim Laplace(\mu, \sigma)$$

In this case $\mu = 0$ and $\frac{\sqrt{2}}{\sigma} = \lambda \Rightarrow \sigma = \frac{\sqrt{2}}{\lambda}$

$X, Y \sim Exp(\lambda)$ for $\lambda > 0$ are iid $\Rightarrow X - Y \sim Laplace\left(0, \frac{\sqrt{2}}{\lambda}\right)$