Krylov Subspace Methods and Preconditioninggit

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1 Classical methods

1.1 Stationary methods

Given a non singular square matrix $A \in \mathbb{R}^{n \times n}$, a vector $b \in \mathbb{R}^n$ and a *splitting* $A = B - C$ of A the linear system

$$Ax = b$$

is equivalent to the fixed-point problem

$$x = B^1 Cx + B^{-1}b.$$

So, given an intial guess $x_0 \in \mathbb{R}^n$, the iterative method associated is given by the recursion

$$x_{k+1} = B^1 Cx + B^{-1}b = Tx + c$$

where $T = B^1 C$ and $c = B^1 b$. It holds the following

Proposizione: $\|e_k\|_2 \xrightarrow{k \to \infty} 0 \iff \rho(T) < 1$

1.2 Steepest descend (Gradient descend)

Given a linear system $Ax = b$, if A is s.p.d. the solution x^* satisfies

$$x^* = \arg\min_{x \in \mathbb{R}^n} J(x) \qquad \text{where} \qquad J(x) = \frac{1}{2} x^{\mathrm{T}} Ax - b^{\mathrm{T}} x.$$

The recursion given by

$$x_{k+1} = x_k + \alpha_k r_k \qquad \text{where} \qquad \alpha_k = \arg\min_{\alpha \in \mathbb{R}} J(x_k + \alpha r_k) = \cdots = \frac{r_k^{\mathrm{T}} r_k}{r_k^{\mathrm{T}} A r_k}$$

satisfies the following

Teorema: $\forall k \quad \|e_k\|_A \leq \left(\frac{\kappa_2(A) - 1}{\kappa_2(A) + 1} \right)^k \|e_0\|_A$

2 Krylov methods

2.1 Krylov space

Definizione: Given a square matrix $A \in \mathbb{K}^{n \times n}$ and a vector $v \in \mathbb{K}^n$ the m^{th} *(polynomial) Krylov subspace* of A and v is

$$\mathcal{K}_m(A, v) = \operatorname{span}\{v, Av, \ldots A^{m-1}v\}$$

Definizione: Given a square matrix $A \in \mathbb{K}^{n \times n}$ and a vector $v \in \mathbb{K}^n$ the *minimal polynomial of v w.r.t. A* is the monic polynomial p with the lowest degree s.t. $p(A) = 0$. The degree $\deg_A(v)$ of such p is called the *grade of v w.r.t. A*.

The following statements holds

Proposizione: Given a square matrix $A \in \mathbb{K}^{n \times n}$, a vector $v \in \mathbb{K}^n$ and $\mu = \deg_A(v)$ the grade of v w.r.t. A the μ^{th} Krylov subspace $\mathcal{K}_\mu(A, v)$ is A-invariant.

Proposizione: Given a square matrix $A \in \mathbb{K}^{n \times n}$ and a vector $v \in \mathbb{K}^n$, it holds

$$\forall m \quad \dim \mathcal{K}_m(A, v) = m \iff \deg_A(v) \geq m.$$

Corollario: $\forall m \quad \dim \mathcal{K}_m(A, v) = \min\{m, \deg_A(v)\}$.

2.2 Krylov iterations

Given a linear system $Ax = b$ and an initial guess $x_0 \in \mathbb{K}^n$ any Krylov method search the m^{th} approximate solution in the affine space

$$x_0 + \mathcal{K}_m(A, r_0) = \{x_0 + p_{m-1}(A)r_0 \mid p_{m-1} \in P^{m-1}\} = \{(I_n + Ap_{m-1}(A))\, x_0 + p_{m-1}(A)b \mid p_{m-1} \in P^{m-1}\}$$

which implies that the m^{th} residual r_m belongs to the space $r_0 + A\mathcal{K}_m(A, r_0)$.

In the special case of $x_0 = 0$ this approach consist in approximate the solution $x^* = A^{-1}b$ by the the action of a $m-1$ degre polinomial in A on the vector b, i.e. in approximate the inverse A^-1 with a polinomial in A. This is reasonable given the Cayley–Hamilton theorem and the following approximation theorems.

Teorema [Weistrass approximation]: Fixed a compact interval $I \subseteq \mathbb{R}$, $\forall f \in C^0(I) \quad \exists\{p_m \in P^m\} \ : \ \|f - p_m\|_\infty \xrightarrow{m\to\infty} 0$.

Teorema [Bernstein]: Fixed a compact interval $I \subseteq \mathbb{R}$, $\forall f$ analytic on an ellipse $\Omega \subseteq \mathbb{C}$ s.t. $\Omega \supset I \qquad \exists\{p_m \in P^m\}$ and $\alpha > 0 \ : \ \forall m \quad \|f - p_m\|_\infty < Ce^{-\alpha m}$.

Teorema: Given a s.p.d. matrix A, the function $f(x) = x^{-1}$ is analytic on an ellipse $\Omega \subseteq \mathbb{C}$ s.t. $I_A = [\lambda_{\min(A), \lambda_{\max(A)]\subset\Omega}}$ and $0 \notin \Omega$.
Corollario: $\exists\{p_m \in \mathbb{P}_m\} s.t. \|A^{-1} - p_m(A)\|_\infty \to 0$ exponentially.
Corollario: If $A = Q\Lambda Q^H$ then $\forall b \quad \|p_m(A)b - A^{-1}b\|_2 \leq C' e^{-\alpha m}\|b\|_2$.

Since the m^{th} is defined inside a m-dimensional subspace, at each step m independent conditions are needed to have an unique solution. This is achieved by the following

Teorema [Saad]: Given linear system $Ax = b$, an initial guess x_0 and an m s.t. $\dim \mathcal{K}_m(A, r_0) = m$, if
• (C) $\quad A = A^H$ and $\mathcal{C}_m = \mathcal{K}_m(A, r_0)$
or
• (M) $\quad A$ is non singular and $\mathcal{C}_m = A\mathcal{K}_m(A, r_0)$
then $\exists! x_m \in x_0 + \mathcal{K}_m(A, r_0) s.t. x_m \perp \mathcal{C}_{\Updownarrow}$.
Therefore this unique vector satisfies
$\|x_m - x^*\|_A = \min_{z \in x_0 + \mathcal{K}_m(A, r_0)} \|z - x^*\|_A = \min_{p\in\Pi_m} \|p(A)e_0\|_A$ in the (C) case, and
$\|b - Ax_m\|_2 = \min_{z \in x_0 + \mathcal{K}_m(A, r_0)} \|b - Az\|_2 = \min_{p\in\Pi_m} \|p(A)r_0\|_2$ in the (M) case.