Past Roots of Polynomials Questions (2012-2023)

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Q1

The equation $x^4 - 2x^3 - 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^3, \beta^3, \gamma^3, \delta^3$ and state the value of $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$. [4]

(b) Find the value of $\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3}$. [3]

(c) Find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

Q2

The cubic equation $2x^3 - 4x^2 + 3 = 0$ has roots α, β, γ. Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(a) State the value of S_1 and find the value of S_2. [3]

(b) (i) Express S_{n+3} in terms of S_{n+2} and S_n. [1]

(ii) Hence, or otherwise, find the value of S_4. [2]

(c) Use the substitution $y = S_1 - x$, where S_1 is the numerical value found in part (a), to find and simplify an equation whose roots are $\alpha + \beta, \beta + \gamma, \gamma + \alpha$. [3]

(d) Find the value of $\frac{1}{\alpha+\beta} + \frac{1}{\beta+\gamma} + \frac{1}{\gamma+\alpha}$. [2]

Q3

It is given that

$$\alpha + \beta + \gamma = 3, \quad \alpha^2 + \beta^2 + \gamma^2 = 5, \quad \alpha^3 + \beta^3 + \gamma^3 = 6.$$

The cubic equation $x^3 + bx^2 + cx + d = 0$ has roots α, β, γ.
Find the values of b, c and d. [6]

Q4

The cubic equation $x^3 + 2x^2 + 3x + 3 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

(b) Show that $\alpha^3 + \beta^3 + \gamma^3 = 1$. [2]

(c) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n} \left((\alpha + r)^3 + (\beta + r)^3 + (\gamma + r)^3\right) = n + \frac{1}{4}n(n+1)\left(an^2 + bn + c\right),$$

where a, b and c are constants to be determined. [6]

Q5

The cubic equation $x^3 + cx + 1 = 0$, where c is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^3, β^3, γ^3. [3]

(b) Show that $\alpha^6 + \beta^6 + \gamma^6 = 3 - 2c^3$. [3]

(c) Find the real value of c for which the matrix

$$\begin{pmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{pmatrix}$$

is singular. [5]

Q6

The cubic equation $x^3 + bx^2 + cx + d = 0$, where b, c and d are constants, has roots α, β, γ. It is given that $\alpha\beta\gamma = -1$.

(a) State the value of d. [1]

(b) Find a cubic equation, with coefficients in terms of b and c, whose roots are $\alpha + 1, \beta + 1, \gamma + 1$. [3]

(c) Given also that $\gamma + 1 = -\alpha - 1$, deduce that $(c - 2b + 3)(b - 3) = b - c$. [4]

Q7

The cubic equation $\alpha^3 + px^2 - 3x - 5 = 0$, where p is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^2, \beta^2, \gamma^2$. [3]

(b) It is given that $\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha + \beta + \gamma)$.

(i) Find the value of p. [3]

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

Q8

The cubic equation $7x^3 + 3x^2 + 5x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$. [3]

(b) Find the value of $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$. [2]

(c) Find the value of $\alpha^{-3} + \beta^{-3} + \gamma^{-3}$. [2]

Q9

The equation $x^3 + 2x^2 + x + 7 = 0$ has roots α, β, γ.

(i) Use the relation $x^2 = -7y$ to show that the equation

$$49y^3 + 14y^2 - 27y + 7 = 0$$

has roots $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\alpha\beta}$. [4]

(ii) Show that

$$\frac{\alpha^2}{\beta^2\gamma^2} + \frac{\beta^2}{\gamma^2\alpha^2} + \frac{\gamma^2}{\alpha^2\beta^2} = \frac{58}{49}.$$

[3]

(iii) Find the exact value of

$$\frac{\alpha^3}{\beta^3\gamma^3} + \frac{\beta^3}{\gamma^3\alpha^3} + \frac{\gamma^3}{\alpha^3\beta^3}.$$

[2]

Q10

The equation

$$x^3 - x + 1 = 0$$

has roots α, β, γ.

(i) Use the relation $x = y^{\frac{1}{3}}$ to show that the equation

$$y^3 + 3y^2 + 2y + 1 = 0$$

has roots $\alpha^3, \beta^3, \gamma^3$. Hence write down the value of $\alpha^3 + \beta^3 + \gamma^3$. [3]

Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(ii) Find the value of S_{-3}. [2]

(iii) Show that $S_6 = 5$ and find the value of S_9. [4]

Q11

A cubic equation $x^3 + bx^2 + cx + d = 0$ has real roots α, β and γ such that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -\frac{5}{12},$$
$$\alpha\beta\gamma = -12,$$
$$\alpha^3 + \beta^3 + \gamma^3 = 90.$$

(i) Find the values of c and d. [3]

(ii) Express $\alpha^2 + \beta^2 + \gamma^2$ in terms of b. [2]

(iii) Show that $b^3 - 15b + 126 = 0$. [4]

(iv) Given that $3 + \mathrm{i}\sqrt{(12)}$ is a root of $y^3 - 15y + 126 = 0$, deduce the value of b. [2]

Q12

It is given that the equation

$$x^3 - 21x^2 + kx - 216 = 0,$$

where k is a constant, has real roots α, ar and ar^{-1}.

(i) Find the numerical values of the roots. [6]

(ii) Deduce the value of k. [2]

Q13

The equation

$$9x^3 - 9x^2 + x - 2 = 0$$

has roots α, β, γ.

(i) Use the substitution $y = 3x - 1$ to show that $3\alpha - 1$, $3\beta - 1$, $3\gamma - 1$ are the roots of the equation

$$y^3 - 2y - 7 = 0.$$

[2]

The sum $(3\alpha - 1)^n + (3\beta - 1)^n + (3\gamma - 1)^n$ is denoted by S_n.

(ii) Find the value of S_3. [2]

(iii) Find the value of S_{-2}. [4]

Q14

The roots of the equation $x^3 + px^2 + qx + r = 0$ are $\alpha, 2\alpha, 4\alpha$, where p, q, r and α are non-zero real constants.

(i) Show that $2p\alpha + q = 0$. [4]

(ii) Show that $p^3 r - q^3 = 0$. [2]

Q15

The roots of the cubic equation

$$x^3 - 5x^2 + 13x - 4 = 0$$

are α, β, γ.

(i) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [3]

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

Q16

By finding a cubic equation whose roots are α, β and γ, solve the set of simultaneous equations

$$\begin{aligned} \alpha + \beta + \gamma &= -1, \\ \alpha^2 + \beta^2 + \gamma^2 &= 29, \\ \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= -1. \end{aligned}$$

[8]

Q17

The roots of the cubic equation $x^3 + 2x^2 - 3 = 0$ are α, β and γ.

(i) By using the substitution $y = \frac{1}{x^2}$, find the cubic equation with roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$. [3]

(ii) Hence find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$. [1]

(iii) Find also the value of $\frac{1}{\alpha^2\beta^2} + \frac{1}{\beta^2\gamma^2} + \frac{1}{\gamma^2\alpha^2}$. [1]

Q18

The cubic equation $2x^3 - 3x^2 + 4x - 10 = 0$ has roots α, β and γ.

(i) Find the value of $(\alpha+1)(\beta+1)(\gamma+1)$. [4]

(ii) Find the value of $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$. [4]

Q19

The roots of the cubic equation $2x^3 + x^2 - 7 = 0$ are α, β and γ. Using the substitution $y = 1 + \frac{1}{x}$, or otherwise, find the cubic equation whose roots are $1 + \frac{1}{\alpha}, 1 + \frac{1}{\beta}$ and $1 + \frac{1}{\gamma}$, giving your answer in the form $ay^3 + by^2 + cy + d = 0$, where a, b, c and d are constants to be found. [4]

Q20

The cubic equation

$$z^3 - z^2 - z - 5 = 0$$

has roots α, β and γ. Show that the value of $\alpha^3 + \beta^3 + \gamma^3$ is 19. [4]
Find the value of $\alpha^4 + \beta^4 + \gamma^4$. [2]
Show that the cubic equation with roots $\frac{\alpha-1}{\alpha}$, $\frac{\beta-1}{\beta}$ and $\frac{\gamma-1}{\gamma}$ may be found using the substitution $z = \frac{1}{1-x}$, and find this equation, giving your answer in the form $px^3 + qx^2 + rx + s = 0$, where p, q, r and s are constants to be determined. [4]

Q21

Find the cubic equation with roots α, β and γ such that

$$\alpha + \beta + \gamma = 3,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 1,$$
$$\alpha^3 + \beta^3 + \gamma^3 = -30,$$

giving your answer in the form $x^3 + px^2 + qx + r = 0$, where p, q and r are integers to be found. [6]

Q22

The roots of the cubic equation $x^3 - 7x^2 + 2x - 3 = 0$ are α, β and γ. Find the values of

(i)

$$\frac{1}{(\alpha\beta)(\beta\gamma)(\gamma\alpha)}$$

(ii)

$$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}$$

(iii)

$$\frac{1}{\alpha^2\beta\gamma} + \frac{1}{\alpha\beta^2\gamma} + \frac{1}{\alpha\beta\gamma^2}$$

[6]

Deduce a cubic equation, with integer coefficients, having roots $\frac{1}{\alpha\beta}, \frac{1}{\beta\gamma}$, and $\frac{1}{\gamma\alpha}$. [2]

Q23

The quartic equation $x^4 - px^2 + qx - r = 0$, where p, q and r are real constants, has two pairs of equal roots. Show that $p^2 + 4r = 0$ and state the value of q. [6]

Q24

The cubic equation $x^3 + px^2 + qx + r = 0$, where p, q and r are integers, has roots α, β and γ, such that

$$\alpha + \beta + \gamma = 15,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 83.$$

Write down the value of p and find the value of q. [3]
Given that α, β and γ are all real and that $\alpha\beta + \alpha\gamma = 36$, find α and hence find the value of r. [5]

Q25

The equation $x^3 + px + q = 0$, where p and q are constants, with $q \neq 0$, has one root which is the reciprocal of another root. Prove that $p + q^2 = 1$. [5]

Q26

The roots of the quartic equation $x^4 + 4x^3 + 2x^2 - 4x + 1 = 0$ are α, β, γ and δ. Find the values of

(i) $\alpha + \beta + \gamma + \delta$, [1]

(ii) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$, [2]

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$, [2]

(iv) $\frac{\alpha}{\beta\gamma\delta} + \frac{\beta}{\alpha\gamma\delta} + \frac{\gamma}{\alpha\beta\delta} + \frac{\delta}{\alpha\beta\gamma}$. [2]

Using the substitution $y = x + 1$, find a quartic equation in y. Solve this quartic equation and hence find the roots of the equation $x^4 + 4x^3 + 2x^2 - 4x + 1 = 0$. [7]

Q27

The cubic equation $x^3 - 2x^2 - 3x + 4 = 0$ has roots α, β, γ.
Given that $c = \alpha + \beta + \gamma$, state the value of c. [1]
Use the substitution $y = c - x$ to find a cubic equation whose roots are $\alpha + \beta$, $\beta + \gamma$, $\gamma + \alpha$. [3]
Find a cubic equation whose roots are $\dfrac{1}{\alpha+\beta}, \dfrac{1}{\beta+\gamma}, \dfrac{1}{\gamma+\alpha}$. [2]
Hence evaluate $\dfrac{1}{(\alpha+\beta)^2} + \dfrac{1}{(\beta+\gamma)^2} + \dfrac{1}{(\gamma+\alpha)^2}$. [2]

Q28

The roots of the equation $x^4 - 4x^2 + 3x - 2 = 0$ are α, β, γ and δ; the sum $\alpha'' + \beta'' + \gamma'' + \delta''$ is denoted by S_n. By using the relation $y = x^2$, or otherwise, show that $\alpha^2, \beta^2, \gamma^2$ and δ^2 are the roots of the equation

$$y^4 - 8y^3 + 12y^2 + 7y + 4 = 0.$$

[3]

State the value of S_2 and hence show that

$$S_8 = 8S_6 - 12S_4 - 72.$$

[3]

Q29

The cubic equation $x^3 - px - q = 0$, where p and q are constants, has roots α, β, γ. Show that

(i) $\alpha^2 + \beta^2 + \gamma^2 = 2p$, [1]

(ii) $\alpha^3 + \beta^3 + \gamma^3 = 3q$, [2]

(iii) $6(\alpha^5 + \beta^5 + \gamma^5) = 5(\alpha^3 + \beta^3 + \gamma^3)(\alpha^2 + \beta^2 + \gamma^2)$. [3]

Q30

The equation

$$8x^3 + 36x^2 + kx - 21 = 0,$$

where k is a constant, has roots $a - d, a, a + d$. Find the numerical values of the roots and determine the value of k. [8]

Q31

The roots of the cubic equation $x^3 - 7x^2 + 2x - 3 = 0$ are α, β, γ. Find the values of

(i) $\alpha^2 + \beta^2 + \gamma^2$, [2]

(ii) $\alpha^3 + \beta^3 + \gamma^3$. [3]

Q32

The roots of the equation $x^4 - 3x^2 + 5x - 2 = 0$ are $\alpha, \beta, \gamma, \delta$, and $\alpha^n + \beta^n + \gamma^n + \delta^n$ is denoted by S_n. Show that

$$S_{n+4} - 3S_{n+2} + 5S_{n+1} - 2S_n = 0.$$

[2]

Find the values of

(i) S_2 and S_4. [3]

(ii) S_3 and S_5. [6]

Hence find the value of

$$\alpha^2(\beta^3 + \gamma^3 + \delta^3) + \beta^2(\gamma^3 + \delta^3 + \alpha^3) + \gamma^2(\delta^3 + \alpha^3 + \beta^3) + \delta^2(\alpha^3 + \beta^3 + \gamma^3).$$

[3]

Q33

A cubic equation has roots α, β and γ such that

$$\begin{aligned} \alpha+\beta+\gamma &= 4, \\ \alpha^2+\beta^2+\gamma^2 &= 14, \\ \alpha^3+\beta^3+\gamma^3 &= 34. \end{aligned}$$

Find the value of $\alpha\beta+\beta\gamma+\gamma\alpha$. [2]
Show that the cubic equation is

$$x^3-4x^2+x+6=0,$$

and solve this equation. [6]

Q34

The cubic equation $x^3-x^2-3x-10=0$ has roots α, β, γ.

(i) Let $u=-\alpha+\beta+\gamma$. Show that $u+2\alpha=1$, and hence find a cubic equation having roots $-\alpha+\beta+\gamma$, $\alpha-\beta+\gamma, \alpha+\beta-\gamma$. [5]

(ii) State the value of $\alpha\beta\gamma$ and hence find a cubic equation having roots $\frac{1}{\beta\gamma}, \frac{1}{\gamma\alpha}, \frac{1}{\alpha\beta}$. [5]

Q35

The cubic equation $2x^3+5x^2-6=0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\frac{1}{\alpha^3}, \frac{1}{\beta^3}, \frac{1}{\gamma^3}$. [3]

(b) Find the value of $\frac{1}{\alpha^6}+\frac{1}{\beta^6}+\frac{1}{\gamma^6}$. [3]

(c) Find also the value of $\frac{1}{\alpha^9}+\frac{1}{\beta^9}+\frac{1}{\gamma^9}$. [2]

Q36

The equation $x^4+3x^2+2x+6=0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}, \frac{1}{\delta^2}$ and state the value of $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}$. [4]

(b) Find the value of $\beta^2\gamma^2\delta^2+\alpha^2\gamma^2\delta^2+\alpha^2\beta^2\delta^2+\alpha^2\beta^2\gamma^2$. [2]

(c) Find the value of $\frac{1}{\alpha^4}+\frac{1}{\beta^4}+\frac{1}{\gamma^4}+\frac{1}{\delta^4}$. [2]

Q37

The cubic equation $x^3+bx^2+d=0$ has roots α, β, γ, where $\alpha=\beta$ and $d \neq 0$.

(a) Show that $4b^3+27d=0$. [5]

(b) Given that $2\alpha^2+\gamma^2=3b$, find the values of b and d. [3]

Q38

The cubic equation $x^3 + 4x^2 + 6x + 1 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

(b) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n} \left((\alpha + r)^2 + (\beta + r)^2 + (\gamma + r)^2\right) = n(n^2 + an + b),$$

where a and b are constants to be determined. [6]

Q39

The cubic equation $27x^3 + 18x^2 + 6x - 1 = 0$ has roots α, β, γ.

(a) Show that a cubic equation with roots $3\alpha + 1, 3\beta + 1, 3\gamma + 1$ is

$$y^3 - y^2 + y - 2 = 0.$$

[3]

The sum $(3\alpha + 1)^n + (3\beta + 1)^n + (3\gamma + 1)^n$ is denoted by S_n.

(b) Find the values of S_2 and S_3. [4]

(c) Find the values of S_{-1} and S_{-2}. [3]

Q40

The quartic equation $x^4 + bx^3 + cx^2 + dx - 2 = 0$ has roots $\alpha, \beta, \gamma, \delta$. It is given that

$$\alpha + \beta + \gamma + \delta = 3, \quad \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 5, \quad \alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1} = 6.$$

(a) Find the values of b, c and d. [6]

(b) Given also that $\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = -27$, find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]