Statistics of Financial Markets

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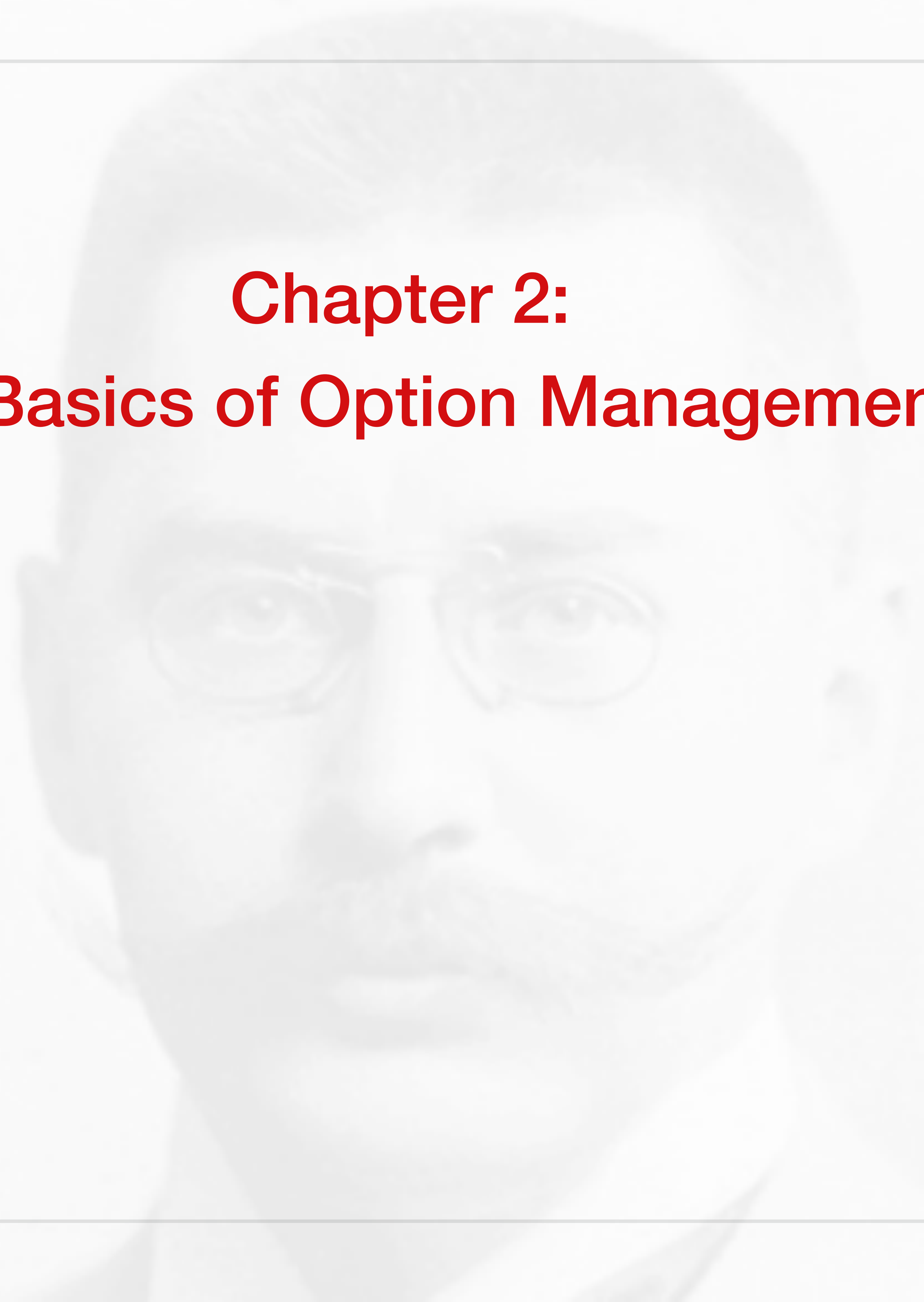
Chapter 2:
The Basics of Option Management



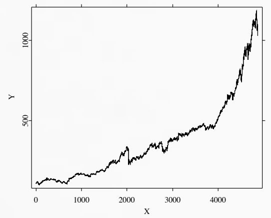
Arbitrage Theory

Arbitrage

Arbitrage exists if there is a trading strategy which produces a riskless profit with non zero probability.

Example

- buy and sell an asset on different stock markets for different prices
- two portfolios with the same current value, same maturity, but different values at maturity



Arbitrage Strategy

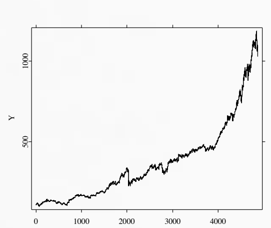
Consider two portfolios A and B with current value $W_A(t)$ and $W_B(t)$, respectively. Suppose that for some T the terminal value $W_A(T) = W_B(T)$, and that for some $t < T$:

$$W_A(t) \leq W_B(t)$$

By short selling B and buying A, and investing the difference

$$\Delta(t) = W_B(t) - W_A(t) \geq 0$$

into a savings account with interest r, the investor does not invest proprietary capital.



However, in T there will be risk-less profit of:

$$W_A(T) - W_B(T) + \Delta(T) = \Delta(t)e^{r(T-t)} > 0$$

He even could be infinitely rich!!

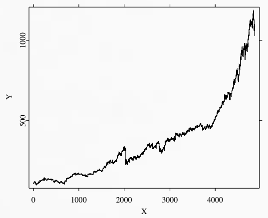
This violates the principle of no arbitrage, and thus it must hold:

$$W_A(t) = W_B(t), \text{ for all t}$$

Basic assumption for the remainder:

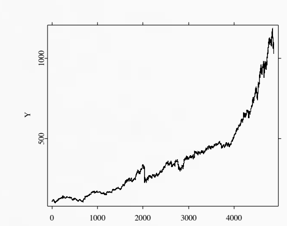
No Arbitrage

i.e., prices of derivatives must not permit arbitrage.



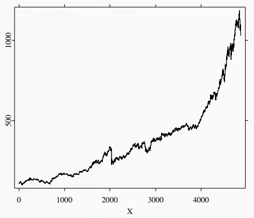
Perfect Financial Market

- debit interest rate = credit interest rate
- no transaction costs
- no taxes
- short-selling is allowed
- stocks are unlimitedly divisible
- no arbitrage



Price of a forward contract

Delivery price:	K
Maturity date:	T
Underlying price:	S_t
Price at time t:	$V_{K,T}(S_t, \tau)$



Proposition 2.1 *Under the assumption of a perfect market and the additional assumption of a constant interest rate r during the time interval $\tau = T - t$ the following holds:*

1. *With no dividend payments or costs of carry, we have*

$$V(S_t, \tau) = V_{K,T}(S_t, \tau) = S_t - Ke^{-r\tau}$$

and $F_t = S_t e^{r\tau}$.

2. *If D_t is the value of all earnings and costs related to the underlying during the time period τ calculated at time t, we have:*

$$V(S_t, \tau) = V_{K,T}(S_t, \tau) = S_t - D_t - Ke^{-r\tau}$$

and $F_t = (S_t - D_t)e^{r\tau}$.

3. *If we have continuous costs of carry, b, then the value of the forward contract at time t is:*

$$V(S_t, \tau) = V_{K,T}(S_t, \tau) = S_t e^{(b-r)\tau} - Ke^{-r\tau}$$

and $F_t = S_t e^{b\tau}$.

Proof:

[case 1] There are no dividend payments.

Portfolio A a long position in a forward contract with strike K, maturity date T and a long position in a zero bond with nominal value K and expiration T.

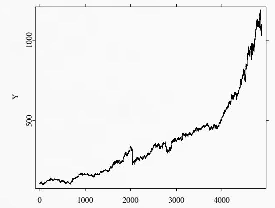
Portfolio B a long position in the asset.

Both portfolios have the same value S_T at time T, and hence by **no arbitrage** must have the same value at time t,

$$V_{K,T}(S_t, \tau) + Ke^{-r\tau} = S_t \tag{2.1}$$

and

Portfolio A = Portfolio B



We have to show that $V_{F_t,T}(\,\cdot\,) = 0$:

Inserting $F_T = S_t e^{t\tau}$ in (2.1) for K yields:

$$V_{F_t,T}(S_t, \tau) + F_t e^{-r\tau} = S_t$$

Thus, $V_{F_t,T}(S_t, \tau) + S_t e^{-r\tau} e^{r\tau} = S_t$

and hence $V_{F_t,\tau}(S_t, \tau) = 0$.

[cases 2 and 3] Assume the asset pays discrete dividends D_t.

Portfolio A a long position in a forward contract with strike K, maturity date T and a long position in a zerobond with nominal value K and expiration T.

Portfolio B a long position in the asset and a short position in a zerobond with current value D_t at time t and expiration date T.

Both portfolios have the same value S_T at time T, and hence by no arbitrage must have the same value at time t,

$$V_{K,T}(S_t, \tau) + Ke^{-r\tau} = S_t - D_t \tag{2.2}$$

and

Portfolio A = Portfolio B

We have to show that $V_{F_t,T}(\,\cdot\,) = 0$:

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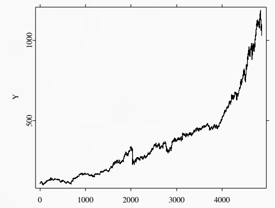
Inserting $F_t = S_t e^{t\tau}$ in (2.1) for K yields:

$$V_{F_t,T}(S_t,\tau) + F_t e^{-r\tau} = S_t$$

Thus, $V_{F_t,T}(S_t,\tau) + S_t e^{-r\tau} e^{r\tau} = S_t$

and hence $V_{F_t,T}(S_t,\tau) = 0$.

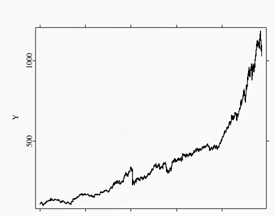


If we have continuous dividend payment d we change portfolio B to **Portfolio B**, buy $e^{(b-r)\tau} = e^{-d\tau}$ shares of the asset and reinvest all dividend payments in the asset.

The result follows by similar arguments as in the discrete dividend case.

$$V_{K,T}(S_t, \tau) + Ke^{-r\tau} = e^{-d\tau} S_t$$

Again for $F_t = S_t e^{b\tau}$ we have $V_{K,T}(S_t, \tau) = 0$.



Example: Forward contract with discrete dividends

Consider a forward contract on a five-year bond with:

- Bond price at time t: $S_t = 900$
- Delivery price: $K = 910$
- Time to maturity: $\tau = T - t = 1$ year
- Coupon payments of 60 in 6 and 12 months from now
- Continuous interest rate for 6 (12) months: $r_6 = 9\,\%$, $r_{12} = 10\,\%$

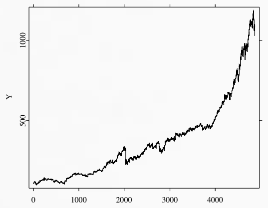
We have for D_t:

$$D_t = 60e^{-0.09\cdot\frac{1}{2}} + 60e^{-0.10} = 111.65 \qquad (2.3)$$

and the value of the forward contract is:

$$V_{K,T}(S_t, \tau) = 900 - 111.65 - 910e^{-0.10} = -35.05 \qquad (2.4)$$

The forward price F_t is $F_t = (S_t - D_t)e^{r\tau} = 871.26$.



Example: Forward contract with continuous dividends

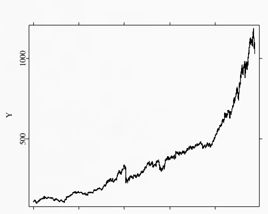
d	Dividend yield
$r(r_A)$	Domestic (American) interest rate
Assume	$r_A = d$
Cost of carry	$b = r - d$
S_t	USD exchange rate

Then the forward price is $F_t = S_t e^{b\tau} = S_t e^{(r-d)\tau}$.

For $r > d$ we have a premium $S_t < F_t$ if $r < d$ a discount $S_t > F_t$.

If $r > d$ and $K = S_t$ then $V_{S_t,T} = S_t(e^{-d\tau} - e^{-r\tau}) > 0$.

The forward contract at delivery price S_t is more expensive than the spot price, as the investor profits from the higher interest rates in the domestic currency till T.



Proposition 2.2 *If the interest rate is constant during the maturity of the contract, the future and the forward price are equal.*

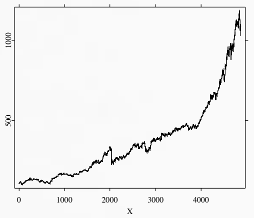
Proof:

Let $\tau = N$ days, F be the forward price at the end of day 0, F_t be the future price at the end of day $t = 0, \ldots, N$ and let ρ be the daily interest rate, at which daily profit loss calculations occur.

We consider two portfolios:

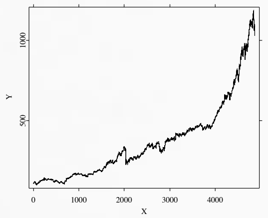
Portfolio A buy $e^{N\rho}$ forward contracts and buy a zerobond with nominal value $Fe^{N\rho}$ and with N days to expiration.

Portfolio B buy future contracts such that there are $e^{(t+1)\rho}$ contracts in the portfolio at the end of day t and buy a zerobond with nominal value $e^{N\rho}F_0$ and N days to expiration.



Portfolio A

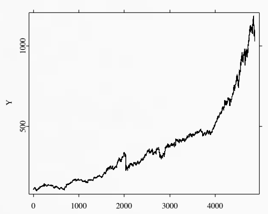
After day N, i.e. at the expiration date, portfolio A contains a total of $Fe^{N\rho}$ zerobonds which are used to fulfil the forward contract, i.e. to buy $e^{N\rho}$ underlying at the delivery price F. Thus the total value of portfolio A at the end of day N is $S_N e^{N\rho}$.



Portfolio B

Now consider portfolio B with future contracts. The profit-loss calculation of the $e^{t\rho}$ futures at the beginning of day t yields $(F_t - F_{t-1})e^{t\rho}$. Throughout the day more futures are bought into the portfolio so that at the end of the day a total of $e^{(t+1)\rho}$ futures are in the portfolio (remember that no cash-flow is triggered by this purchase). The change of the portfolio value has a future value on day N of:

$$(F_t - F_{t-1})e^{t\rho} \cdot e^{(N-t)\rho} = (F_t - F_{t-1})e^{N\rho} \qquad (2.5)$$



Since changes in the value of the future are settled every day, the cumulative return on day N is:

$$\sum_{t=1}^{N} (F_t - F_{t-1})e^{N\rho} = (F_N - F_0)e^{N\rho} \qquad (2.6)$$

Together with the zerobond it follows that the value of portfolio B at $t = N$ is $F_N - F_0 e^{N\rho} + F_0 e^{N\rho}$, which is exactly $F_N e^{N\rho} = S_N e^{N\rho}$. The same holds for portfolio A. Thus assuming no arbitrage the two portfolios must have the same value at day $t = 0$.

As the future and the forward contract have a value of zero at day $t = 0$, only the prices of zerobonds have to be considered.
It follows that:

$$0 + Fe^{N\rho} = 0 + F_0 e^{N\rho}$$

and

$$F = F_0$$

Option Pricing

Payoff functions

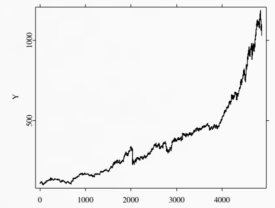
The payoff function of a call is $\max(S_T - K,\ 0)$

The payoff function of a put is $\max(K - S_T,\ 0)$.

Intrinsic value

Intrinsic value of a call is $\max(S_t - K,\ 0)$

Intrinsic value of a put $\max(K - S_t,\ 0)$.



In the money (ITM)

An option is called ITM if its intrinsic value is nonzero.

At the money (ATM)

An option is called ATM if $S_t \approx K$.

Out of the money (OTM)

An option is called OTM if its intrinsic value is zero.

Elementary Properties of Option Prices

- option prices are non-negative
- the price of a European option at maturity T is equal to the price of an American option at maturity T
- the price of an American option is larger or equal than its intrinsic value (this is not true for European options)

- for two American options C_1^{am} and C_2^{am} differing only by time to maturity, $T_1 \leq T_2$, the following holds:

$$C_{K,T_1}^{am}(S_t, T_1 - t) \leq C_{K,T_2}^{am}(S_t, T_2 - t)$$
$$P_{K,T_1}^{am}(S_t, T_1 - t) \leq P_{K,T_2}^{am}(S_t, T_2 - t)$$

Since:

$$C_{K,T_2}^{am}(S_t, T_2 - T_1) \geq \text{intrinsic value } = \max(S_t - K,\ 0)$$
$$= C_{K,T_1}^{am}(S_t,\ 0)\ ,\ \forall t$$

(not true in general for European options)

- the price of an American option is always bigger than or equal to the price of the corresponding European option
- call prices are monotone decreasing functions of strike prices
- put prices are monotone increasing functions of strike prices