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# Turtles

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Month ??, 202?

```
g++ -std=c++20 -Wshadow -Wall -o "%e" "%f" -O2 -Wno-unused-  
result -DONPC  
g++ -std=c++20 -Wshadow -Wall -o "%e" "%f" -g -fsanitize=  
address -fsanitize=undefined -D_GLIBCXX_DEBUG -DONPC
```

- **#pragma GCC target ("avx2")** can double performance of vectorized code, but causes crashes on old machines.
- **#pragma GCC optimize("unroll-loops")** enables aggressive loop unrolling, which reduces the number of branches and optimizes parallel computation.

[illegible]

# Data Structures (2)

|   |                  |
|---|------------------|
| dsu.hpp   | 0e8ecd, 33 lines |
| <pre>struct DSU {     int n;     VI par, siz;      DSU (int _n = 0)     {         n = _n;         par.resize(n);         iota(all(par), 0);         siz.assign(n, 1);     }     int find(int v)     {         if (v == par[v])             return v;         return par[v] = find(par[v]);     }     bool unite(int a, int b)     {         a = find(a);         b = find(b);         if (a != b)         {             if (siz[a] &lt; siz[b])                 swap(a, b);             par[b] = a;             siz[a] += siz[b];             return true;         }         return false;     } };</pre> |                  |
| fenwick.hpp   | 6c5f66, 44 lines |
| <pre>// methods work in 0-indexing struct Fenwick {     int n;     VL ar;      Fenwick (int _n = 0): n(_n + 1), ar(n) {}     Fenwick (const VL&amp; _ar)     {         n = sz(_ar) + 1;         ar.assign(n, 0);         FOR (i, 1, n)         {             ar[i] += _ar[i - 1];             int x = i + (i &amp; -i);             if (x &lt; n)                 ar[x] += ar[i];         }     }     void upd(int x, ll val)     {         x++;         while (x &lt; n)         {             ar[x] += val;             x += x &amp; -x;         }     }     ll getSum(int x)</pre>                     |                  |

|   |  |
|---|--|
| <pre>{     ll res = 0;     while (x)     {         res += ar[x];         x -= x &amp; -x;     }     return res; } // [l, r] ll query(int l, int r) {     return getSum(r + 1) - getSum(l); } };</pre> |  |
|---|--|

|   |                  |
|---|------------------|
| fenwick-lower-bound.hpp   | 1c8acb, 15 lines |
| <pre>// returns first index p such that sum on [0, p] &gt;= val or n if // not found int lower_bound(ll val) {     ll sm = 0;     int pos = 0;     for (int i = 1 &lt;&lt; (31 - __builtin_clz(n)); i; i &gt;&gt;= 1)     {         if (pos + i &lt; n &amp;&amp; sm + ar[pos + i] &lt; val)         {             sm += ar[pos + i];             pos += i;         }     }     return pos; }</pre> |                  |

## Minimum on a Segment

Maintain two Fenwick trees with  $n = 2^k$  — one for the original array and the other for the reversed array. Use: `n = __bit_ceil(n)`.

When querying for the minimum on the segment, only consider segments  $[(i \& (i + 1)), i]$  that are completely inside  $[l, r]$ .

## Add on a Segment

Maintain two Fenwick trees: `tMult` and `tAdd`.

To add  $x$  on the segment  $[l, r]$ , perform:

```
tMult.upd(l, x),
tMult.upd(r + 1, -x),
tAdd.upd(l, -x · l),
tAdd.upd(r + 1, x · (r + 1)).
```

Then, the sum on  $[l, r]$  is:

$$\text{sum}(l, r) = (r + 1) \text{tMult.getSum}(r + 1) + \text{tAdd.getSum}(r + 1) - \left( l \cdot \text{tMult.getSum}(l) + \text{tAdd.getSum}(l) \right).$$

|   |                  |
|---|------------------|
| segtree.hpp                                       | 461f46, 46 lines |
| <pre>struct SegTree {     int n;     VI ar;</pre> |                  |

|   |  |
|---|--|
| SegTree(int _n)   |  |
| <pre>{     n = __bit_ceil(_n);     ar.assign(2 * n, INF); } SegTree(const VI&amp; _ar) {     n = __bit_ceil(sz(_ar));     ar.assign(2 * n, INF);     FOR (i, 0, sz(_ar))         ar[i + n] = _ar[i];      RFOR (i, n, 1)         ar[i] = min(ar[i &lt;&lt; 1], ar[(i &lt;&lt; 1)   1]); } void upd(int p, int val) {     p += n;     ar[p] = val;     while (p &gt;&gt;= 1)     {         ar[p] = min(ar[p &lt;&lt; 1], ar[(p &lt;&lt; 1)   1]);     } } // [l, r] int query(int l, int r) {     l += n;     r += n;      int resL = INF, resR = INF;     while (l &lt; r)     {         if (l &amp; 1) resL = min(resL, ar[l++]);         if (r &amp; 1) resR = min(ar[--r], resR);         l &gt;&gt;= 1;         r &gt;&gt;= 1;     }     return min(resL, resR); } };</pre> |  |

|   |                  |
|---|------------------|
| segtree-minleft-maxright.hpp  | 16c856, 62 lines |
| <pre>// 7385d9 for min_left  // max right: find maximum r that. f(op over [l, r]) == true template &lt;class F&gt; int max_right(int l, F f) {     if (l == n)         return n;     l += n;     int sm = INF;     do     {         while ((l &amp; 1) == 0)             l &gt;&gt;= 1;         if (!f(min(sm, ar[l])))         {             while (l &lt; n)             {                 l = (l &lt;&lt; 1);                 if (f(min(sm, ar[l])))                 {                     sm = min(sm, ar[l]);                     l++;                 }             }         }     }     while (l &lt; n);     return l;</pre> |                  |

```
    }
    return l - n;
}
sm = min(sm, ar[l]);
l++;
} while ((l & -l) != l);
return n;
}

// min_left: find minimum l that f(op over [l, r)) == true
//template <class F>
//int min_left(int r, F f)
//{
//    if (r == 0)
//        return 0;
//    r += n;
//    int sm = INF;
//    do
//    {
//        r--;
//        while (r > 1 && (r & 1)) r >>= 1;
//        if (!f(min(ar[r], sm)))
//        {
//            while (r < n)
//            {
//                r = (r << 1) | 1;
//                if (f(min(ar[r], sm)))
//                {
//                    sm = min(ar[r], sm);
//                    r--;
//                }
//            }
//            return r + 1 - n;
//        }
//        sm = min(ar[r], sm);
//    } while ((r & -r) != r);
//    return 0;
//}
```

min= and sum with Segment Tree

Store in each node: max, cntMax, max2, sum.

In update check *l, r* conditions and:

- if (val ≥ max) return;
- else if (val > max2) update this node;
- else go to left and right

You can do max= and += on segment at the same time. Time: *O*(log *n*). Each extra descent decreases number of diferent elements in segment.

```
lazysegtree.hpp
Description: Supporte everything related to seg trees
2375f8, 106 lines

template<class S, S (*op)(S, S), S (*e)(),
        class F, S (*mapping)(F, S),
        F (*composition)(F, F), F (*id)()>
struct LazySegTree
{
    int n, size, log;
    vector<S> d;
    vector<F> lz;

    LazySegTree(int _n = 0) : LazySegTree(vector<S>(_n, e())) {
    }
    LazySegTree(const vector<S>& v)
    {
```

```
        n = sz(v);
        size = 1;
        while (size < n)
            size <=< 1;
        log = __builtin_ctz(size);
        d.assign(2 * size, e());
        lz.assign(size, id());
        FOR(i, 0, n) d[size + i] = v[i];
        RFOR(i, size, 1) update(i);
    }

    void update(int k) { d[k] = op(d[k << 1], d[k << 1 | 1]); }

    void all_apply(int k, F f)
    {
        d[k] = mapping(f, d[k]);
        if (k < size) lz[k] = composition(f, lz[k]);
    }

    void push(int k)
    {
        all_apply(k << 1, lz[k]);
        all_apply(k << 1 | 1, lz[k]);
        lz[k] = id();
    }

    void set(int p, S x)
    {
        p += size;
        RFOR(i, log + 1, 1) push(p >> i);
        d[p] = x;
        FOR (i, 1, log + 1) update(p >> i);
    }

    S get(int p)
    {
        p += size;
        RFOR(i, log + 1, 1) push(p >> i);
        return d[p];
    }

    // [l, r)
    S prod(int l, int r)
    {
        if (l == r) return e();
        l += size; r += size;
        RFOR(i, log + 1, 1)
        {
            if (((l >> i) << i) != l) push(l >> i);
            if (((r >> i) << i) != r) push((r - 1) >> i);
        }
        S sml = e(), smr = e();
        while (l < r)
        {
            if (l & 1) sml = op(sml, d[l++]);
            if (r & 1) smr = op(d[--r], smr);
            l >>= 1; r >>= 1;
        }
        return op(sml, smr);
    }

    S all_prod() { return d[1]; }

    void apply(int p, F f)
    {
        p += size;
        RFOR(i, log + 1, 1) push(p >> i);
        d[p] = mapping(f, d[p]);
        FOR(i, 1, log + 1) update(p >> i);
    }
```

```
    }

    void apply(int l, int r, F f)
    {
        if (l == r) return;
        l += size; r += size;
        RFOR(i, log + 1, 1)
        {
            if (((l >> i) << i) != l) push(l >> i);
            if (((r >> i) << i) != r) push((r - 1) >> i);
        }
        int l2 = l, r2 = r;
        while (l < r)
        {
            if (l & 1) all_apply(l++, f);
            if (r & 1) all_apply(--r, f);
            l >>= 1; r >>= 1;
        }
        FOR (i, 1, log + 1)
        {
            if (((l2 >> i) << i) != l2) update(l2 >> i);
            if (((r2 >> i) << i) != r2) update((r2 - 1) >> i);
        }
    }
};

lazy-minleft-maxright.hpp
f6217d, 67 lines

// 7d34d3 for min_left

// If f is monotone, this is the maximum r that satisfies
// f(op(a[l], a[l + 1], ..., a[r - 1])) = true
template<class G>
int max_right(int l, G g)
{
    if (l == n) return n;
    assert(g(e()));
    l += size;
    RFOR(i, log + 1, 1) push(l >> i);
    S sm = e();
    do
    {
        while ((l & 1) == 0) l >>= 1;
        if (!g(op(sm, d[l])))
        {
            while (l < size)
            {
                push(l);
                l = (l << 1);
                if (g(op(sm, d[l])))
                {
                    sm = op(sm, d[l]);
                    l++;
                }
            }
            return l - size;
        }
        sm = op(sm, d[l]);
        l++;
    } while ((l & -l) != l);
    return n;
}

// If f is monotone, this is the minimum l that satisfies
// f(op(a[l], a[l + 1], ..., a[r - 1])) = true
//template<class G>
//int min_left(int r, G g)
//{
//    if (r == 0) return 0;
```

```
//assert(g(e()));
//r += size;
//RFOR(i, log + 1, 1) push((r - 1) >> i);
//S sm = e();
//do
//{
//r--;
//while (r > 1 && (r & 1)) r >>= 1;
//if (!g(op(d[r], sm)))
//{
//while (r < size)
//{
//push(r);
//r = (r << 1) | 1;
//if (g(op(d[r], sm)))
//{
//sm = op(d[r], sm);
//r--;
//}
//}
//return r + 1 - size;
//}
//sm = op(d[r], sm);
//} while ((r & -r) != r);
//return 0;
//}
```

segtree-usage.hpp

add5af, 32 lines

```
// Example of (Sum + Range Add) with Lazy Segment Tree
struct S
{
    long long sum;
    int len;
};

using F = long long;

S op(S a, S b)
{
    return {a.sum + b.sum, a.len + b.len};
}

S e(){
    return {0, 0};
}

S mapping(F f, S x) {
    return {x.sum + f * x.len, x.len};
}

F composition(F f, F g) {
    return f + g;
}

F id() {
    return 0;
}
```

```
vector<S> v(n, {0, 1}); // each segment length = 1 initially
LazySegTree<S, op, e, F, mapping, composition, id> seg(v);
```

sparse-table.hpp

**Description:** Sparse table for minimum on the range  $[l, r), l < r$ . You can push back an element in  $O(\text{LOG})$  and query anytime.

e666cf, 19 lines

**struct** SparseTable

```
{
    VI t[LOG];
};
```

```
void push_back(int v)
{
    int i = sz(t[0]);
    t[0].pb(v);
    FOR(j, 0, LOG - 1)
        t[j + 1].pb(min(t[j][i], t[j][max(0, i - (1 << j))]]));
}
// [l, r)
int query(int l, int r)
{
    assert(l < r && r <= sz(t[0]));
    int i = 31 - __builtin_clz(r - l);
    return min(t[i][r - 1], t[i][l + (1 << i) - 1]);
}
};
```

LCA.hpp

ce66b1, 43 lines

**struct** LCA

```
{
    int n;
    VI I; // v -> po(v)
    VI RI;
    VI M; // to index mapping
    VI D;
    SparseTable st;

    LCA(const vector<vector<int>>& adj, int root)
    {
        n = sz(adj);
        I = vector<int>(n);
        RI = vector<int>(n);
        D = vector<int>(n, -1);
        M = vector<int>(2*n, -1);
        int ctr = 0;
        vector<int> a;
        function<void(int, int, int)> preorder = [&](int v, int pr, int d)
        {
            I[v] = ctr++;
            RI[I[v]] = v;
            a.pb(I[v]);
            D[v]=d;
            for(auto to: adj[v])
            {
                if(to != pr)
                {
                    preorder(to, v,d+1);
                    a.pb(I[v]);
                }
            }
        };
        preorder(root, -1,0);
        FOR(i,0,sz(a))st.pb(a[i]);
        FOR(i,0,sz(a)) M[a[i]] = i;
    }

    int lca(int u, int v)
    {
        return RI[st.query(min(M[I[u]], M[I[v]]), max(M[I[u]], M[I[v]])+1)];
    }
};
```

treap.hpp

**Description:** uncomment in split for explicit key or in merge for implicit priority. Minimum and reverse queries.

215374, 144 lines

```
mt19937 rng;
```

**struct** Node

```
{
    int l, r;
    int x, y;
    int cnt, par;
    int rev, mn;

    Node(int value)
    {
        l = r = -1;
        x = value;
        y = rng();
        cnt = 1;
        par = -1;
        rev = 0;
        mn = value;
    }
};
```

**struct** Treap

```
{
    vector<Node> t;

    int getCnt(int v)
    {
        if (v == -1)
            return 0;
        return t[v].cnt;
    }
    int getMn(int v)
    {
        if (v == -1)
            return INF;
        return t[v].mn;
    }
    int newNode(int val)
    {
        t.pb({val});
        return sz(t) - 1;
    }
    void upd(int v)
    {
        if (v == -1)
            return;
        // important!
        t[v].cnt = getCnt(t[v].l) +
            getCnt(t[v].r) + 1;

        t[v].mn = min(t[v].x, min(getMn(t[v].l), getMn(t[v].r)));
    }
    void reverse(int v)
    {
        if (v == -1)
            return;
        t[v].rev ^= 1;
    }
    void push(int v)
    {
        if (v == -1 || t[v].rev == 0)
            return;
        reverse(t[v].l);
        reverse(t[v].r);
        swap(t[v].l, t[v].r);
        t[v].rev = 0;
    }
    pii split(int v, int cnt)
    {
        if (v == -1)
            return {-1, -1};
    }
```

```

push(v);
int left = getCnt(t[v].l);
pii res;
// elements t[v].x == val will be in right part
// if (val <= t[v].x)
if (cnt <= left)
{
    if (t[v].l != -1)
        t[t[v].l].par = -1;
    // res = split(t[v].l, val);
    res = split(t[v].l, cnt);
    t[v].l = res.y;
    if (res.y != -1)
        t[res.y].par = v;
    res.y = v;
}
else
{
    if (t[v].r != -1)
        t[t[v].r].par = -1;
    // res = split(t[v].r, val);
    res = split(t[v].r, cnt - left - 1);
    t[v].r = res.x;
    if (res.x != -1)
        t[res.x].par = v;
    res.x = v;
}
upd(v);
return res;
}
int merge(int v, int u)
{
    if (v == -1) return u;
    if (u == -1) return v;
    int res;
    // if ((int)(rng() % (getCnt(v) + getCnt(u))) < getCnt(v))
    if (t[v].y > t[u].y)
    {
        push(v);
        if (t[v].r != -1)
            t[t[v].r].par = -1;
        res = merge(t[v].r, u);
        t[v].r = res;
        if (res != -1)
            t[res].par = v;
        res = v;
    }
    else
    {
        push(u);
        if(t[u].l != -1)
            t[t[u].l].par = -1;
        res = merge(v, t[u].l);
        t[u].l = res;
        if (res != -1)
            t[res].par = u;
        res = u;
    }
    upd(res);
    return res;
}
// returns index of element [0, n)
int getIdx(int v, int from = -1)
{
    if (v == -1)
        return 0;
    int x = getIdx(t[v].par, v);
    push(v);
    if (from == -1 || t[v].r == from)

```

```

        x += getCnt(t[v].l) + (from != -1);
        return x;
    }
};

lct.hpp
Description: Link-Cut Tree. Calculate any path queries. Change upd to maintain
what you need. Don't use upd in push(). Calculate non commutative functions in
both ways and swap them in push. cnt - number of nodes in current splay tree.
Don't touch rev, sub, vsub. v->access() brings v to the top and pushes it; its
left subtree will be the path from v to the root and its right subtree will be empty.
Only then sub will be the number of nodes in the connected component of v and
vsub will be the number of nodes under v. Change upd to calc sum in subtree of
other functions. Use makeRoot for arbitrary path queries.
Usage: FOR (i, 0, n) LCT[i] = new snode(i); link(LCT[u], LCT[v]);
Time: O(log n)
788027, 159 lines

typedef struct Snode* sn;
struct Snode
{
    sn p, c[2]; // parent, children
    bool rev = false; // subtree reversed or not (internal usage)
    int val, cnt; // value in node, # nodes in splay subtree
    int sub, vsub = 0; // vsub stores sum of virtual children

    Snode(int _val): val(_val)
    {
        p = c[0] = c[1] = 0;
        upd();
    }
    friend int getCnt(sn v)
    {
        return v ? v->cnt : 0;
    }
    friend int getSub(sn v)
    {
        return v ? v->sub : 0;
    }
    void push()
    {
        if (!rev)
            return;
        swap(c[0], c[1]);
        rev = false;
        FOR (i, 0, 2)
            if (c[i])
                c[i]->rev ^= 1;
    }
    void upd()
    {
        FOR (i, 0, 2)
            if (c[i])
                c[i]->push();
        cnt = 1 + getCnt(c[0]) + getCnt(c[1]);
        sub = 1 + getSub(c[0]) + getSub(c[1]) + vsub;
    }
    int dir()
    {
        if (!p) return -2;
        FOR (i, 0, 2)
            if (p->c[i] == this)
                return i;
        // p is path-parent pointer
        // -> not in current splay tree
        return -1;
    }

    // checks if root of current splay tree
    bool isRoot()
    {
        return dir() < 0;
    }

```

```

}
friend void setLink(sn p, sn v, int d)
{
    if (v)
        v->p = p;
    if (d >= 0)
        p->c[d] = v;
}
void rot()
{
    assert(!isRoot());
    int d = dir();
    sn pa = p;
    setLink(pa->p, this, pa->dir());
    setLink(pa, c[d ^ 1], d);
    setLink(this, pa, d ^ 1);
    pa->upd();
}
void splay()
{
    while (!isRoot() && !p->isRoot())
    {
        p->p->push();
        p->push();
        push();
        dir() == p->dir() ? p->rot() : rot();
        rot();
    }
    if (!isRoot())
        p->push(), push(), rot();
    push();
    upd();
}
// bring this to top of tree, propagate
void access()
{
    for (sn v = this, pre = 0; v; v = v->p)
    {
        v->splay();
        if (pre)
            v->vsub -= pre->sub;
        if (v->c[1])
            v->vsub += v->c[1]->sub;
        v->c[1] = pre;
        v->upd();
        pre = v;
    }
    splay();
    assert(!c[1]);
}
void makeRoot()
{
    access();
    rev ^= 1;
    access();
    assert(!c[0] && !c[1]);
}
friend sn lca(sn u, sn v)
{
    if (u == v)
        return u;
    u->access();
    v->access();
    if (!u->p)
        return 0;
    u->splay();
    return u->p ? u->p : u;
}
friend bool connected(sn u, sn v)

```

```
{
    return lca(u, v);
}
void set(int v)
{
    access();
    val = v;
    upd();
}
friend void link(sn u, sn v)
{
    assert(!connected(u, v));
    v->makeRoot();
    u->access();
    setLink(v, u, 0);
    v->upd();
}
// cut v from it's parent in LCT
// make sure about root or better use next function
friend void cut(sn v)
{
    v->access();
    assert(v->c[0]); // assert if not a root
    v->c[0]->p = 0;
    v->c[0] = 0;
    v->upd();
}
// u, v should be adjacent in tree
friend void cut(sn u, sn v)
{
    u->makeRoot();
    v->access();
    assert(v->c[0] == u && !u->c[0] && !u->c[1]);
    cut(v);
}
};
```

ordered-set.hpp16 lines

```
#include <ext/pb_ds/assoc_container.hpp>
using namespace __gnu_pbds;
using namespace std;
typedef tree<int, null_type, less<int>, rb_tree_tag,
    tree_order_statistics_node_update> ordered_set;
```

```
ordered_set s;
s.insert(47);
// Returns the number of elements less then k
s.order_of_key(k);
// Returns iterator to the k-th element or s.end()
s.find_by_order(k);
// Does not exist
s.count();
// Doesn't trigger RE. Returns 0 if compiled using F8
*s.end();
```

convex-hull-trick.hppDescription: add(a, b) adds a straight line  $y = ax + b$ . getMaxY(p) finds the maximum  $y$  at  $x = p$ .94e3d7, 72 lines

```
struct Line
{
    ll a, b, xLast;
    Line() {}
    Line(ll _a, ll _b): a(_a), b(_b) {}
    bool operator<(const Line& l) const
    {
        return MP(a, b) < MP(l.a, l.b);
    }
    bool operator<(int x) const
```

```
{
    return xLast < x;
}
__int128 getY(__int128 x) const
{
    return a * x + b;
}
ll intersect(const Line& l) const
{
    assert(a < l.a);
    ll dA = l.a - a, dB = b - l.b, x = dB / dA;
    if (dB < 0 && dB % dA != 0)
        x--;
    return x;
}
};
```

```
struct ConvexHull: set<Line, less<>>
{
    bool needErase(iterator it, const Line& l)
    {
        ll x = it->xLast;
        if (it->getY(x) > l.getY(x))
            return false;
        if (it == begin())
            return it->a >= l.a;
        x = prev(it)->xLast + 1;
        return it->getY(x) < l.getY(x);
    }
    void add(ll a, ll b)
    {
        Line l(a, b);
        auto it = lower_bound(l);
        if (it != end())
        {
            ll x = it == begin() ? -LINF :
                prev(it)->xLast;
            if ((it == begin()
                || prev(it)->getY(x) >= l.getY(x))
                && it->getY(x + 1) >= l.getY(x + 1))
                return;
        }
        while (it != end() && needErase(it, l))
            it = erase(it);
        while (it != begin() && needErase(prev(it), l))
            erase(prev(it));
        if (it != begin())
        {
            auto itP = prev(it);
            Line itL = *itP;
            itL.xLast = itP->intersect(l);
            erase(itP);
            insert(itL);
        }
        l.xLast = it == end() ? LINF : l.intersect(*it);
        insert(l);
    }
    ll getMaxY(ll p)
    {
        return lower_bound(p)->getY(p);
    }
};
```

Graphs (3)

Shortest paths

bellman-ford-moore.hppDescription: Computes shortest paths from a single source vertex to all of the other vertices in a weighted directed graph. Time:  $O(nm)$ eb281b, 35 lines

```
VL spfa(const vector<vector<pair<int, ll>>& g, int n, int s)
{
    VL dist(n, LINF);
    dist[s] = 0;
    queue<int> q;
    q.push(s);
    VI inQueue(n);
    inQueue[s] = true;
    VI cnt(n);
    bool negCycle = false;
    while (!q.empty())
    {
        int v = q.front();
        q.pop();
        cnt[v]++;
        negCycle |= cnt[v] > n;
        inQueue[v] = false;
        for (auto [to, w] : g[v])
        {
            ll newDist = dist[v] + w;
            if (newDist < dist[to])
            {
                dist[to] = newDist;
                if (!inQueue[to])
                {
                    q.push(to);
                    inQueue[to] = true;
                }
            }
        }
        if (negCycle)
            break;
    }
    return dist;
}
```

monge-shortest-path.hppDescription: Finds shortest paths from the vertex 0 to all vertices in a DAG with  $n$  vertices, where the edges weights  $c(i, j)$  satisfy the Monge property:  $\forall i, j, k, l, \quad 0 \leq i < j < k < l < n \implies c(i, l) + c(j, k) \geq c(i, k) + c(j, l)$ . Time:  $O(n \log n)$ 540e92, 34 lines

```
template<typename F>
VL mongeShortestPath(int n, const F& cost)
{
    VL dist(n, LINF);
    VI amin(n);
    dist[0] = 0;

    auto update = [&](int i, int k)
    {
        ll nd = dist[k] + cost(k, i);
        if (nd < dist[i])
        {
            dist[i] = nd;
            amin[i] = k;
        }
    };

    function<void(int, int)> solve = [&](int l, int r)
    {
        if (r - l == 1)
            return;
        int m = (l + r) / 2;
```

```
FOR(k, amin[l], min(m, amin[r] + 1))
    update(m, k);
solve(l, m);
FOR(k, l + 1, m + 1)
    update(r, k);
solve(m, r);
};

update(n - 1, 0);
solve(0, n - 1);
return dist;
}
```

Decompositions

centroid.hpp8c3d24, 51 lines

```
VI g[N];
int siz[N];
bool usedC[N];

int dfsSZ(int v, int par)
{
    siz[v] = 1;
    for (auto to : g[v])
    {
        if (to != par && !usedC[to])
            siz[v] += dfsSZ(to, v);
    }
    return siz[v];
}

void build(int u)
{
    dfsSZ(u, -1);
    int szAll = siz[u];
    int pr = u;
    while (true)
    {
        int v = -1;
        for (auto to : g[u])
        {
            if (to == pr || usedC[to])
                continue;
            if (siz[to] * 2 > szAll)
            {
                v = to;
                break;
            }
        }
        if (v == -1)
            break;
        pr = u;
        u = v;
    }
    int cent = u;
    usedC[cent] = true;

    // here calculate f(cent)

    for (auto to : g[cent])
    {
        if (!usedC[to])
        {
            build(to);
        }
    }
}
```

```
hld.hpp
Description: Run dfsSZ(root, -1, 0) and dfsHLD(root, -1, root) to build the
HLD. Each vertex v has an index tin[v]. To update on the path, use the process as
defined in get(). The values are stored in the vertices.
dc0437, 67 lines

VI g[N];
int siz[N];
int h[N];
int p[N];
int top[N];
int tin[N];
int tout[N];
int t = 0;

void dfsSZ(int v, int par, int hei)
{
    siz[v] = 1;
    h[v] = hei;
    p[v] = par;
    for (auto& to : g[v])
    {
        if (to == par)
            continue;
        dfsSZ(to, v, hei + 1);
        siz[v] += siz[to];
        if (g[v][0] == par || siz[g[v][0]] < siz[to])
            swap(g[v][0], to);
    }
}

void dfsHLD(int v, int par, int tp)
{
    tin[v] = t++;
    top[v] = tp;
    FOR (i, 0, sz(g[v]))
    {
        int to = g[v][i];
        if (to == par)
            continue;
        if (i == 0)
            dfsHLD(to, v, tp);
        else
            dfsHLD(to, v, to);
    }
    tout[v] = t - 1;
}

ll get(int u, int v)
{
    ll res = 0;
    while(true)
    {
        int tu = top[u];
        int tv = top[v];
        if (tu == tv)
        {
            int t1 = tin[u];
            int t2 = tin[v];
            if (t1 > t2)
                swap(t1, t2);
            // query [t1, t2] both inclusive
            //res += query(t1, t2);
            break;
        }
        if (h[tu] < h[tv])
        {
            swap(tu, tv);
            swap(u, v);
        }
        //res += query(tin[tu], tin[u]);
        u = p[tu];
    }
}
```

```
return res;
}

biconnected-components.hpp
Description: Colors the edges so that the vertices, connected with the same color
are still connected if you delete any vertex.
Time: O(m)
7d48ce, 117 lines

struct Graph
{
    int n, m;
    vector<pii> edges;
    vector<VI> g;

    VI used, par;
    VI tin, low, inComp;
    int t = 0, c = 0;
    VI st;

    // components of vertices
    // a vertex can be in several components
    vector<VI> verticesCol;
    // components of edges
    vector<VI> components;
    // col[i] - component of the i-th edge
    VI col;

    Graph(int _n = 0, int _m = 0): n(_n), m(_m), edges(m), g(n),
    used(n), par(n, -1), tin(n), low(n), inComp(n), col(m, -1) {}

    void addEdge(int a, int b, int i)
    {
        assert(0 <= a && a < n);
        assert(0 <= b && b < n);
        assert(0 <= i && i < m);

        edges[i] = MP(a, b);
        g[a].pb(i);
        g[b].pb(i);
    }

    void addComp()
    {
        unordered_set<int> s;
        s.reserve(7 * sz(components[c]));
        for (auto e : components[c])
        {
            s.insert(edges[e].x);
            s.insert(edges[e].y);
            inComp[edges[e].x] = true;
            inComp[edges[e].y] = true;
        }
        verticesCol.pb(VI(all(s)));
    }

    void dfs(int v, int p = -1)
    {
        used[v] = 1;
        par[v] = p;
        low[v] = tin[v] = t++;
        int cnt = 0;
        for (auto e : g[v])
        {
            int to = edges[e].x;
            if (to == v)
                to = edges[e].y;

            if (p == to) continue;
            if (!used[to])
            {

```

```

cnt++;
st.pb(e);
dfs(to, v);

low[v] = min(low[v], low[to]);

if ((par[v] == -1 && cnt > 1) ||
    (par[v] != -1 && low[to] >= tin[v]))
{
    components.pb({});
    while (st.back() != e)
    {
        components[c].pb(st.back());
        col[st.back()] = c;

        st.pop_back();
    }
    components[c].pb(st.back());
    addComp();
    col[st.back()] = c++;

    st.pop_back();
}
}
else
{
    low[v] = min(low[v], tin[to]);
    if (tin[to] < tin[v])
        st.pb(e);
}
}
}
}
void build()
{
    FOR (i, 0, n)
    {
        if (used[i]) continue;
        dfs(i, -1);
        if (st.empty()) continue;
        components.pb({});
        while (!st.empty())
        {
            int e = st.back();
            col[e] = c;
            components[c].pb(e);
            st.pop_back();
        }
        addComp();
        c++;
    }
    FOR (i, 0, n)
        if (!inComp[i])
            verticesCol.pb(VI(1, i));
}
};

```

scc.hpp

e8b50c, 61 lines

vector&lt;bool&gt; vis;

```

void dfs(int v, vector<VI> const& adj, vector<int> &output)
{
    vis[v] = true;
    for (auto u : adj[v])
        if (!vis[u])
            dfs(u, adj, output);
    output.pb(v);
}

```

```

// input: adj — adjacency list of G
// output: comps — the strongly connected components in G
// output: adj_cond — adjacency list of G^SCC (by root
//            vertices)
void scc(vector<vector<int>> const& adj,
         vector<vector<int>> &comps,
         vector<vector<int>> &adj_cond) {

    int n = sz(adj);
    comps.clear(), adj_cond.clear();

    vector<int> ord; // will be a sorted list of G's vertices
                     // by exit time

    vis.assign(n, false);

    // first series of depth first searches
    FOR (i, 0, n)
        if (!vis[i])
            dfs(i, adj, ord);

    // create adjacency list of G^T
    vector<vector<int>> adj_rev(n);
    FOR (v, 0, n)
        for (int u : adj[v])
            adj_rev[u].pb(v);

    vis.assign(n, false);
    reverse(all(ord));

    vector<int> roots(n, 0); // gives the root vertex of a
                             // vertex's SCC

    // second series of depth first searches
    for (auto v : ord)
    {
        if (!vis[v])
        {
            VI comp;
            dfs(v, adj_rev, comp);
            comps.pb(comp);
            int root = *min_element(all(comp));
            for (auto u : comp)
                roots[u] = root;
        }
    }

    // add edges to condensation graph
    adj_cond.assign(n, {});
    FOR (v, 0, n)
        for (auto u : adj[v])
            if (roots[v] != roots[u])
                adj_cond[roots[v]].pb(roots[u]);
}

```

## Hierholzer's algorithm

hierholzer.hpp

**Description:** Finds an Eulerian path in a directed or undirected graph.  $g$  is a graph with  $n$  vertices.  $g[u]$  is a vector of pairs  $(v, \text{edge\_id})$ .  $m$  is the number of edges in the graph. The vertices are numbered from 0 to  $n-1$ , and the edges - from 0 to  $m-1$ . If there is no Eulerian path, returns  $\{-1\}, \{-1\}$ . Otherwise, returns the path in the form (vertices, edges) with vertices containing  $m+1$  elements and edges containing  $m$  elements. If you need an Eulerian cycle, check vertices[0] = vertices.back().

fa6dc3, 101 lines

```

// f14a40 for undirected
tuple<bool, int, int> checkDirected(vector<vector<pii>& g)
{
    int n = sz(g), v1 = -1, v2 = -1;
    bool bad = false;

```

```

VI degIn(n);
FOR(u, 0, n)
    for (auto [v, e] : g[u])
        degIn[v]++;
FOR(u, 0, n)
{
    bad |= abs(degIn[u] - sz(g[u])) > 1;
    if (degIn[u] < sz(g[u]))
    {
        bad |= v2 != -1;
        v2 = u;
    }
    else if (degIn[u] > sz(g[u]))
    {
        bad |= v1 != -1;
        v1 = u;
    }
}
return {bad, v1, v2};
}

/*tuple<bool, int, int> checkUndirected(vector<vector<pii>& g)
{
    int n = sz(g), v1 = -1, v2 = -1;
    bool bad = false;
    FOR(u, 0, n)
    {
        if (sz(g[u]) & 1)
        {
            bad |= v2 != -1;
            if (v1 == -1)
                v1 = u;
            else
                v2 = u;
        }
    }
    return {bad, v1, v2};
}*/

pair<VI, VI> hierholzer(vector<vector<pii>> g, int m)
{
    // checkUndirected if undirected
    auto [bad, v1, v2] = checkDirected(g);
    if (bad)
        return {{-1}, {-1}};
    if (v1 != -1)
    {
        g[v1].pb({v2, m});
        // uncomment if undirected
        //g[v2].PB({v1, m});
        m++;
    }
    deque<pii> d;
    VI used(m);
    int v = 0, k = 0;
    while (m > 0 && g[v].empty())
        v++;
    while (sz(d) < m)
    {
        while (k < m)
        {
            while (!g[v].empty() && used[g[v].back().y])
                g[v].pop_back();
            if (!g[v].empty())
                break;
            d.push_front(d.back());
            d.pop_back();
            v = d.back().x;
            k++;

```



```

assert(0 <= s && s < n);
assert(0 <= t && t < n);
assert(s != t);
//initPotentials(s);
int flow = 0;
ll cost = 0;
for (int it = 0; ; it++)
{
    fill(all(d), LINF);
    fill(all(pred), -1);
    d[s] = 0;
    priority_queue<pair<ll, int>> q;
    q.push({0, s});
    while (!q.empty())
    {
        auto [dv, v] = q.top();
        q.pop();
        if (it > 0 && v == t)
            break;
        if (-dv != d[v])
            continue;

```

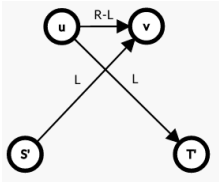
```
for (int i : g[v])
{
    if (edges[i].flow == edges[i].cap)
        continue;
    int to = edges[i].to;
    ll nd = d[v] + edges[i].cost + pi[v] - pi[to];
    if (nd < d[to])
    {
        d[to] = nd;
        pred[to] = i;
        q.push({-nd, to});
    }
}
}
if (d[t] == LINF)
    break;
int curFlow = INF;
for (int v = t; v != s; )
{
    int i = pred[v];
    curFlow = min(curFlow, edges[i].cap - edges[i].flow);
    v = edges[i].from;
}
for (int v = t; v != s; )
{
    int i = pred[v];
    edges[i].flow += curFlow;
    edges[i ^ 1].flow -= curFlow;
    v = edges[i].from;
}
flow += curFlow;
cost += (d[t] + pi[t] - pi[s]) * curFlow;
FOR(u, 0, n)
    if (it == 0 || d[u] <= d[t])
        pi[u] += d[u] - d[t];
}
return {flow, cost};
}
```

Maximum flow with minimum capacities

On the resulting graph, accumulate maximum flow in the following order:

- from  $S'$  to  $T'$
- from  $S'$  to  $T$
- from  $S$  to  $T'$
- from  $S$  to  $T$ .

An  $S - T$  flow that satisfies the minimum capacities exists if and only if, for all outgoing edges from  $S'$  and incoming edges to  $T'$ , the flow and capacity are equal.



Quadratic supermodular pseudoboolean optimization

$$\sum_i a_i x_i + \sum_i b_i \overline{x_i} + \sum_{i,j} c_{ij} x_i \overline{x_j} \rightarrow \min$$
$$c_{ij} x_i x_j = c_{ij} x_i - c_{ij} x_i \overline{x_j}$$

If  $a_i \leq b_i$ , add an edge from  $S$  to  $i$  of capacity  $b_i - a_i$  and add  $a_i$  to the answer.

Otherwise, add an edge from  $i$  to  $T$  of capacity  $a_i - b_i$  and add  $b_i$  to the answer.

Add an edge from  $i$  to  $j$  of capacity  $c_{ij}$ .

Add the  $S - T$  minimum cut to the answer.

Matching tricks

Minimum cut

To find the min-cut, search from vertex  $S$  on unsaturated edges. Original edges from used vertices to unused ones are in the min-cut.

Minimum vertex cover

The vertex cover problem is not NP-complete in bipartite graphs. The minimum number of vertices required to cover all **edges** is equal to the size of the maximum matching. To reconstruct the minimum vertex cover, create a directed graph:

- matched edges from the right part to the left part
- unmatched edges from the left part to the right part.

Start traversal from unmatched vertices in the left part. The cover includes vertices from the matching:

- unvisited vertices in the left part
- visited vertices in the right part.

Maximum independent set

The independent set problem is not NP-complete in bipartite graphs. It is the complement of the minimum vertex cover.

Minimum edge cover

A minimum edge cover can be found in **any** graph. The minimum number of edges required to cover all vertices can only be determined in graphs without isolated vertices. By utilizing one edge in the matching, we cover two vertices, while any other vertices are covered using one edge for each.

DAG paths

In a DAG, you can find the minimum number of non-intersecting paths that cover all vertices. Duplicate vertices and create a bipartite graph with edges  $u_L \rightarrow v_R$ . Edges in the matching correspond to edges in the paths.

Dominating set

A dominating set for a graph is a subset  $D$  of  $V$  such that any vertex is in  $D$ , or has a neighbor in  $D$ . The dominating set problem is NP-complete **even on bipartite graphs**. It can be found greedily on a tree.

Sqrt problems

3-cycles.hpp

**Description:** Finds all triangles in a graph. Each triangle  $(v, u, w)$  increments the cnt.  
**Time:**  $\mathcal{O}(m \cdot \sqrt{m})$

e5e996, 22 lines

```
int triangles(int n)
{
    vector<VI> ng(n);
    FOR (v, 0, n)
        for (auto u : adj[v])
            if (MP(sz(adj[v]), v) < MP(sz(adj[u]), u))
                ng[v].pb(u);
    int cnt = 0;
    VI used(n, 0);
    FOR (v, 0, n)
    {
        for (auto u : ng[v])
            used[u] = 1;
        for (auto u : ng[v])
            for(auto w : ng[u])
                if (used[w])
                    cnt++;
        for (auto u : ng[v])
            used[u] = 0;
    }
    return cnt;
}
```

4-cycles.hpp

**Description:** Sort  $d$  and add breaks to speed up. With breaks works 0.5s for  $m = 5 \cdot 10^5$ .  
**Time:**  $\mathcal{O}\left(\sum_{u,v \in E} \min(\deg(u), \deg(v))\right) = \mathcal{O}(m \cdot \sqrt{m})$

73a48f, 20 lines

```
ll rect(int n)
{
    ll cnt4 = 0;
    vector<pii> d(n);
    FOR (v, 0, n) d[v] = MP(sz(adj[v]), v);
    VI L(n);
    FOR (v, 0, n)
    {
        for (auto u : adj[v])
            if (d[u] < d[v])
                for (auto y : adj[u])
                    if (d[y] < d[v])
                        cnt4 += L[y], L[y]++;
        for (auto u : adj[v])
            if (d[u] < d[v])
                for (auto y : adj[u])
                    L[y] = 0;
    }
    return cnt4;
}
```

Strings (4)

aho-corasick.hpp

e59836, 64 lines

const int AL = 26;

```
struct Node
{
    int p;
    int c;
    int g[AL];
    int nxt[AL];
    int link;
```

```
Node(int _c, int _p)
{
    c = _c;
    p = _p;
    fill(g, g + AL, -1);
    fill(nxt, nxt + AL, -1);
    link = -1;
}

};

struct AC
{
    vector<Node> a;
    AC(): a(1, {-1, -1}) {}

    int addStr(const string& s)
    {
        int v = 0;
        FOR (i, 0, sz(s))
        {
            // change to [0 AL)
            int c = s[i] - 'a';
            if (a[v].nxt[c] == -1)
            {
                a[v].nxt[c] = sz(a);
                a.pb(Node(c, v));
            }
            v = a[v].nxt[c];
        }
        return v;
    }

    int go(int v, int c)
    {
        if (a[v].g[c] != -1)
            return a[v].g[c];

        if (a[v].nxt[c] != -1)
            a[v].g[c] = a[v].nxt[c];
        else if (v != 0)
            a[v].g[c] = go(getLink(v), c);
        else
            a[v].g[c] = 0;

        return a[v].g[c];
    }

    int getLink(int v)
    {
        if (a[v].link != -1)
            return a[v].link;
        if (v == 0 || a[v].p == 0)
            return 0;
        return a[v].link = go(getLink(a[v].p), a[v].c);
    }
};
```

```
suffix-automaton.hpp
183478, 57 lines

const int AL = 26;

struct Node
{
    int g[AL];
    int link;
    int len;
    int cnt;
    Node(): link(-1), len(0), cnt(1)
    {
        fill(g, g + AL, -1);
    }
};
```

```
};

struct Automaton
{
    vector<Node> a;
    int head;
    Automaton(): a(1), head(0) {}
    void add(char c)
    {
        // change to [0 AL)
        int ch = c - 'a';
        int nhead = sz(a);
        a.pb(Node());
        a[nhead].len = a[head].len + 1;
        int cur = head;
        head = nhead;
        while (cur != -1 && a[cur].g[ch] == -1)
        {
            a[cur].g[ch] = head;
            cur = a[cur].link;
        }
        if (cur == -1)
        {
            a[head].link = 0;
            return;
        }
        int p = a[cur].g[ch];
        if (a[p].len == a[cur].len + 1)
        {
            a[head].link = p;
            return;
        }
        int q = sz(a);
        a.pb(Node());
        a[q] = a[p];
        a[q].cnt = 0;
        a[q].len = a[cur].len + 1;
        a[p].link = a[head].link = q;
        while (cur != -1 && a[cur].g[ch] == p)
        {
            a[cur].g[ch] = q;
            cur = a[cur].link;
        }
    }
};

suffix-array.hpp
Description: Cast your string to vector. Don't forget about delimiters. No need
to add anything at the end. sa represents permutations of positions if you sort all
suffixes.
Time: O(n log n)
aa241e, 59 lines

void countSort(VI& p, const VI& c)
{
    int n = sz(p);
    VI cnt(n);
    FOR (i, 0, n)
        cnt[c[i]]++;
    VI pos(n);
    FOR (i, 1, n)
        pos[i] = pos[i - 1] + cnt[i - 1];
    VI p2(n);
    for (auto x : p)
    {
        int i = c[x];
        p2[pos[i]++] = x;
    }
    p = p2;
}
```

```
suffix-array.hpp
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    VI cnt(n);
    FOR (i, 0, n)
        cnt[c[i]]++;
    VI pos(n);
    FOR (i, 1, n)
        pos[i] = pos[i - 1] + cnt[i - 1];
    VI p2(n);
    for (auto x : p)
    {
        int i = c[x];
        p2[pos[i]++] = x;
    }
    p = p2;
}
```

```
VI suffixArray(VI s)
{
    // strictly smaller than any other element
    s.pb(-1);
    int n = sz(s);
    VI p(n), c(n);
    iota(all(p), 0);
    sort(all(p), [&](int i, int j)
    {
        return s[i] < s[j];
    });
    int x = 0;
    c[p[0]] = 0;
    FOR (i, 1, n)
    {
        if (s[p[i]] != s[p[i - 1]])
            x++;
        c[p[i]] = x;
    }
    int k = 0;
    while ((1 << k) < n)
    {
        FOR (i, 0, n)
            p[i] = (p[i] - (1 << k) + n) % n;
        countSort(p, c);
        VI c2(n);
        pii pr = {c[p[0]], c[(p[0] + (1 << k)) % n]};
        FOR (i, 1, n)
        {
            pii nx = {c[p[i]], c[(p[i] + (1 << k)) % n]};
            c2[p[i]] = c2[p[i - 1]];
            if (pr != nx)
                c2[p[i]]++;
            pr = nx;
        }
        c = c2;
        k++;
    }
    p.erase(p.begin());
    return p;
}
```

```
lcp.hpp
Description: queryLcp returns the longest common prefix of substrings starting
at i and j.
911c8c, 49 lines

struct LCP
{
    int n;
    VI s, sa, rnk, lcp;
    SparseTable st;

    LCP(VI _s): n(sz(_s)), s(_s)
    {
        sa = suffixArray(s);
        rnk.resize(n);
        FOR (i, 0, n)
            rnk[sa[i]] = i;
        lcpArray();
        FOR (i, 0, n - 1)
            st.pb(lcp[i]);
    }

    void lcpArray()
    {
        lcp.resize(n - 1);
        int h = 0;
        FOR (i, 0, n)
        {
            if (h > 0)
                h--;
```

```

    h--;
    if (rnk[i] == 0)
        continue;
    int j = sa[rnk[i] - 1];
    for (; j + h < n && i + h < n; h++)
    {
        if (s[j + h] != s[i + h])
            break;
    }
    lcp[rnk[i] - 1] = h;
}
}
int queryLcp(int i, int j)
{
    if (i == n || j == n)
        return 0;
    assert(i != j); // return n - i ???
    i = rnk[i];
    j = rnk[j];
    if (i > j)
        swap(i, j);
    // query [i, j)
    return st.query(i, j);
}
};

```

### run-enumerate.hpp

**Description:** Enumerate all tuples  $(t, l, r)$  with  $t$  being the minimum period of  $s[l, r)$  and  $r - l \geq 2 \cdot t$ .  $l$  and  $r$  are maximal. In other words  $(t, l - 1, r)$  and  $(t, l, r + 1)$  do not satisfy the previous condition.

The number of runs is  $\leq |s|$ . Other properties are stated at the end of the function.  
**Time:**  $\mathcal{O}(n \log n)$ , where  $n = |s|$ .

f9baf1, 62 lines

```

struct Run
{
    int t, l, r;
    bool operator<(const Run& p) const
    {
        return make_tuple(t, l, r) < make_tuple(p.t, p.l, p.r);
    }
    bool operator==(const Run& p) const
    {
        return !(*this < p) && !(p < *this);
    }
};
vector<Run> runEnumerate(VI s)
{
    int n = sz(s);
    LCP lcp(s); reverse(all(s));
    LCP rev(s); reverse(all(s));

    vector<Run> runs;
    FOR(inv, 0, 2)
    {
        VI st = {n};
        auto pop = [&](int i)
        {
            int j = st.back();
            int dist = j - i;
            int distPrev = st[sz(st) - 2] - j;
            int distMn = min(dist, distPrev);

            int len = lcp.queryLcp(i, j);
            if((len >= distMn && dist < distPrev) ||
                (len < distMn && ((s[i + len] < s[j + len]) ^ inv)))
                return true;
            return false;
        };

        RFOR(i, n, 0)

```

```

    {
        while(sz(st) > 1 && pop(i))
            st.pop_back();
        int j = st.back();
        int dist = j - i;
        st.pb(i);

        int x = rev.queryLcp(n - i, n - j);
        int y = lcp.queryLcp(i, j);
        if(x < dist && x + y >= dist)
            runs.pb({dist, i - x, j + y});
    }
}
sort(all(runs));
runs.resize(unique(all(runs)) - runs.begin());

//ll sumLen = 0, sumCnt = 0, sum = 0;
//for(auto [len, l, r] : runs)
//    sumLen += len, sumCnt += (r - l) / len, sum += r - l;
//assert(sz(runs) <= sz(s));
//assert(sumLen <= LOG * sz(s));
//assert(sumCnt <= 2 * sz(s));
//assert(sum <= 2 * LOG * sz(s));
return runs;
}

```

### suffix-tree.hpp

**Description:** Ukkonen's algorithm for building a suffix tree. Cast your string to vector. Don't forget about delimiters.  $a[v].g[c]$  is a transition in format  $(u, l, r)$ , that goes from  $v$  to  $u$  and the string spelled out by this transition is the substring  $s_l..r$ . For transitions that go to leaves,  $r = \text{INF}$ . For the root node which has number 0,  $\text{link} == -1$ . For leaves,  $\text{link} == -2$ . For all other nodes,  $\text{link}$  is maintained explicitly.  
**Time:**  $\mathcal{O}(n \log |\Sigma|)$ , where  $\Sigma$  is an alphabet

4aa61c, 85 lines

```

struct SuffixTree
{
    struct Transition
    {
        int u, l, r;
    };
    struct Node
    {
        map<int, Transition> g;
        int link;
        Node(): link(-2) {}
    };
    VI s;
    vector<Node> a;
    pair<bool, int> testAndSplit(int v, int l, int r, int c)
    {
        if (v == -1)
            return {true, -1};
        if (l <= r)
        {
            auto [nv, nl, nr] = a[v].g[s[l]];
            if (c == s[nl + r - l + 1])
                return {true, v};
            int newNode = sz(a);
            a.pb(Node());
            a[v].g[s[l]] = {newNode, nl, nl + r - l};
            a[newNode].g[s[nl + r - l + 1]] = {nv, nl + r - l + 1, nr};
        }
        return {false, newNode};
    }
    pii canonize(int v, int l, int r)
    {
        if (v == -1 && l <= r)

```

```

    {
        v = 0;
        l++;
    }
    if (r < l)
        return {v, l};
    Transition cur = a[v].g[s[l]];
    while (cur.r - cur.l <= r - l)
    {
        l += cur.r - cur.l + 1;
        v = cur.u;
        if (l <= r)
            cur = a[v].g[s[l]];
    }
    return {v, l};
}
pii update(int v, int l, int r)
{
    int oldu = 0;
    auto [endPoint, u] = testAndSplit(v, l, r - 1, s[r]);
    while (!endPoint)
    {
        int newNode = sz(a);
        a.pb(Node());
        a[u].g[s[r]] = {newNode, r, INF};
        if (oldu != 0)
            a[oldu].link = u;
        oldu = u;
        tie(v, l) = canonize(a[v].link, l, r - 1);
        tie(endPoint, u) = testAndSplit(v, l, r - 1, s[r]);
    }
    if (oldu != 0)
        a[oldu].link = v;
    return {v, l};
}
SuffixTree(const VI& _s)
{
    s = _s;
    // Add the symbol that was not present in 's'
    s.pb(-1);
    a.reserve(2 * sz(s));
    a = {Node()};
    a[0].link = -1;
    int v = 0, l = 0;
    FOR(i, 0, sz(s))
    {
        tie(v, l) = update(v, l, i);
        tie(v, l) = canonize(v, l, i);
    }
}
};

```

### z.hpp

9da7e8, 23 lines

```

VI zFunction(const string& s)
{
    int n = sz(s);
    VI z(n);

    int l = 0;
    int r = 0;
    FOR (i, 1, n)
    {
        z[i] = 0;
        if (i <= r)
            z[i] = min(r - i + 1, z[i - l]);

        while(i + z[i] < n && s[i + z[i]] == s[z[i]])
            z[i]++;
    }
}

```

```
    if(i + z[i] - 1 > r)
    {
        r = i + z[i] - 1;
        l = i;
    }
}
return z;
}
```

prefix.hpp5b81c4, 16 lines

```
VI prefixFunction(const string& s)
{
    int n = sz(s);
    VI p(n);
    p[0] = 0;
    FOR (i, 1, n)
    {
        int j = p[i - 1];
        while(j != 0 && s[i] != s[j])
            j = p[j - 1];

        if (s[i] == s[j]) j++;
        p[i] = j;
    }
    return p;
}
```

minimal-cyclic-shift.hpp  
Description:  $s_{shift}, s_{shift+1}, \dots$  is lexicographically smallest cyclic shift. If more than one answer it finds the minimum value of  $shift$ .  
Time:  $\mathcal{O}(n)$  time and memory complexity.

d4d30a, 29 lines

```
int minimalCyclicShift(VI s)
{
    int n = sz(s);
    s.resize(2 * n);
    FOR(i, 0, n)
        s[n + i] = s[i];

    int shift = 0;
    VI f(2 * n);
    FOR(i, 1, 2 * n)
    {
        int j = f[i - 1 - shift];
        while(j > 0 && s[shift + j] != s[i])
        {
            if(s[shift + j] > s[i])
                shift = i - j;
            j = f[j - 1];
        }
        if(j == 0 && s[shift] != s[i])
        {
            if(s[shift] > s[i])
                shift = i;
        }
        else
            j++;
        f[i - shift] = j;
    }
    return shift;
}
```

manacher.hpp  
Description:  $s[i - d0_i, i + d0_i - 1], s[i - d1_i + 1, i + d1_i - 1]$  are palindromes.

e64188, 20 lines

```
vector<VI> manacher(const string& s)
{
    int n = sz(s);
    vector<VI> d(2);
    FOR (t, 0, 2)
```

```
{
    d[t].resize(n);
    int l = -1;
    int r = -1;
    FOR (i, 0, n)
    {
        if (i <= r)
            d[t][i] = min(r - i + 1, d[t][l + (r - i) + 1 - t]);
        while (i + d[t][i] < n && i + t - d[t][i] - 1 >= 0
            && s[i + d[t][i]] == s[i + t - d[t][i] - 1])
            d[t][i]++;
        if (i + d[t][i] - t > r)
        {
            r = i + d[t][i] - 1;
            l = i - d[t][i] + t;
        }
    }
    return d;
}
```

palindromic-tree.hpp62993e, 54 lines

```
const int AL = 26;

struct Node
{
    int to[AL];
    int link;
    int len;
    Node(int _link, int _len)
    {
        fill(to, to + AL, -1);
        link = _link;
        len = _len;
    }
};

struct PalTree
{
    string s;
    vector<Node> a;
    int last;

    PalTree(string t = ""): s(t), a({{-1, -1}, {0, 0}}), last(1)
    {}

    void add(int idx)
    {
        // change to [0, AL)
        int ch = s[idx] - 'a';

        int cur = last;
        while (cur != -1)
        {
            int pos = idx - a[cur].len - 1;
            if (pos >= 0 && s[pos] == s[idx])
                break;
            cur = a[cur].link;
        }
        if (a[cur].to[ch] == -1)
        {
            a[cur].to[ch] = sz(a);
            int link = a[cur].link;
            while (link != -1)
            {
                int pos = idx - a[link].len - 1;
                if (pos >= 0 && s[pos] == s[idx])
                    break;
                link = a[link].link;
            }
        }
    }
}
```

```
    if (link == -1)
        link = 1;
    else
        link = a[link].to[ch];
    a.pb(Node(link, a[cur].len + 2));
}
last = a[cur].to[ch];
}
};
```

## Geometry (5)

point.hpp1a2063, 91 lines

```
struct Pt
{
    db x, y;
    Pt operator+(const Pt& p) const
    {
        return {x + p.x, y + p.y};
    }
    Pt operator-(const Pt& p) const
    {
        return {x - p.x, y - p.y};
    }
    Pt operator*(db d) const
    {
        return {x * d, y * d};
    }
    Pt operator/(db d) const
    {
        return {x / d, y / d};
    }
};

db sq(const Pt& p)
{
    return p.x * p.x + p.y * p.y;
}

db abs(const Pt& p)
{
    return sqrt(sq(p));
}

int sgn(db x)
{
    return (EPS < x) - (x < -EPS);
}

// Returns 'p' rotated counter-clockwise by 'a'
Pt rot(const Pt& p, db a)
{
    db co = cos(a), si = sin(a);
    return {p.x * co - p.y * si,
        p.x * si + p.y * co};
}

// Returns 'p' rotated counter-clockwise by 90 degrees
Pt perp(const Pt& p)
{
    return {-p.y, p.x};
}

db dot(const Pt& p, const Pt& q)
{
    return p.x * q.x + p.y * q.y;
}

// Returns the angle between 'p' and 'q' in [0, pi]
db angle(const Pt& p, const Pt& q)
{
    return acos(clamp(dot(p, q) / abs(p) /
        abs(q), (db)-1.0, (db)1.0));
}
```

```
db cross(const Pt& p, const Pt& q)
{
    return p.x * q.y - p.y * q.x;
}
// Positive if R is on the left side of PQ,
// negative on the right side,
// and zero if R is on the line containing PQ
db orient(const Pt& p, const Pt& q, const Pt& r)
{
    return cross(q - p, r - p) / abs(q - p);
}
// Checks if argument of 'p' is in [-pi, 0)
bool half(const Pt& p)
{
    assert(sgn(p.x) != 0 || sgn(p.y) != 0);
    return sgn(p.y) == -1 ||
        (sgn(p.y) == 0 && sgn(p.x) == -1);
}
void polarSortAround(const Pt& o, vector<Pt>& v)
{
    sort(all(v), [o](Pt p, Pt q)
    {
        p = p - o;
        q = q - o;
        bool hp = half(p), hq = half(q);
        if (hp != hq)
            return hp < hq;
        int s = sgn(cross(p, q));
        if (s != 0)
            return s == 1;
        return sq(p) < sq(q);
    });
}
ostream& operator<<(ostream& os, const Pt& p)
{
    return os << "(" << p.x << ", " << p.y << ")";
}
```

```
line.hpp
83c9af, 50 lines
struct Line
{
    // Equation of the line is dot(n, p) + c = 0
    Pt n;
    db c;
    Line(const Pt& _n, db _c): n(_n), c(_c) {}
    // n is the normal vector to the left of PQ
    Line(const Pt& p, const Pt& q):
        n(perp(q - p)), c(-dot(n, p)) {}
    // The "positive side": dot(n, p) + c > 0
    // The "negative side": dot(n, p) + c < 0
    db side(const Pt& p) const
    {
        return dot(n, p) + c;
    }
    db dist(const Pt& p) const
    {
        return abs(side(p)) / abs(n);
    }
    db sqDist(const Pt& p) const
    {
        return side(p) * side(p) / (db)sq(n);
    }
    Line perpThrough(const Pt& p) const
    {
        return {p, p + n};
    }
    bool cmpProj(const Pt& p, const Pt& q) const
    {

```

```
        return sgn(cross(p, n) - cross(q, n)) < 0;
    }
    Pt proj(const Pt& p) const
    {
        return p - n * side(p) / sq(n);
    }
    Pt reflect(const Pt& p) const
    {
        return p - n * 2 * side(p) / sq(n);
    }
};
bool parallel(const Line& l1, const Line& l2)
{
    return sgn(cross(l1.n, l2.n)) == 0;
}
Pt inter(const Line& l1, const Line& l2)
{
    db d = cross(l1.n, l2.n);
    assert(sgn(d) != 0);
    return perp(l2.n * l1.c - l1.n * l2.c) / d;
}
```

```
segment.hpp
687634, 39 lines
// Checks if 'p' is in the disk (the region in a plane
// bounded by a circle) of diameter [ab]
bool inDisk(const Pt& a, const Pt& b, const Pt& p)
{
    return sgn(dot(a - p, b - p)) <= 0;
}
// Checks if 'p' lies on segment [ab]
bool onSegment(const Pt& a, const Pt& b, const Pt& p)
{
    return sgn(orient(a, b, p)) == 0 && inDisk(a, b, p);
}
// Checks if the segments [ab] and [cd] intersect
// properly (their intersection is one point
// which is not an endpoint of either segment)
bool properInter(const Pt& a, const Pt& b, const Pt& c, const
Pt& d)
{
    db oa = orient(c, d, a);
    db ob = orient(c, d, b);
    db oc = orient(a, b, c);
    db od = orient(a, b, d);
    return sgn(oa) * sgn(ob) == -1 && sgn(oc) * sgn(od) == -1;
}
// Returns the distance between [ab] and 'p'
db segPt(const Pt& a, const Pt& b, const Pt& p)
{
    Line l(a, b);
    assert(sgn(sq(l.n)) != 0);
    if (l.cmpProj(a, p) && l.cmpProj(p, b))
        return l.dist(p);
    return min(abs(p - a), abs(p - b));
}
// Returns the distance between [ab] and [cd]
db segSeg(const Pt& a, const Pt& b, const Pt& c, const Pt& d)
{
    if (properInter(a, b, c, d))
        return 0;
    return min({segPt(a, b, c), segPt(a, b, d),
        segPt(c, d, a), segPt(c, d, b)});
}
```

```
polygon.hpp
d2cc47, 67 lines
bool isConvex(const vector<Pt>& v)
{
    bool hasPos = false, hasNeg = false;
```

```
    int n = sz(v);
    FOR(i, 0, n)
    {
        int s = sgn(orient(v[i], v[(i + 1) % n], v[(i + 2) % n]));
        hasPos |= s > 0;
        hasNeg |= s < 0;
    }
    return !(hasPos && hasNeg);
}
db areaTriangle(const Pt& a, const Pt& b, const Pt& c)
{
    return abs(cross(b - a, c - a)) / 2.0;
}
db areaPolygon(const vector<Pt>& v)
{
    db area = 0.0;
    int n = sz(v);
    FOR(i, 0, n)
        area += cross(v[i], v[(i + 1) % n]);
    return abs(area) / 2.0;
}
// Checks if point 'a' is inside the convex
// polygon 'v'. Returns true if on the boundary.
// 'v' must not contain duplicated vertices.
// Time: O(log n)
bool inConvexPolygon(const vector<Pt>& v, const Pt& a)
{
    assert(sz(v) >= 2);
    if (sz(v) == 2)
        return onSegment(v[0], v[1], a);
    if (sgn(orient(v.back(), v[0], a)) < 0
        || sgn(orient(v[0], v[1], a)) < 0)
        return false;
    int i = lower_bound(v.begin() + 2, v.end(), a,
        [&](const Pt& p, const Pt& q)
        {
            return sgn(orient(v[0], p, q)) > 0;
        }) - v.begin();
    return sgn(orient(v[i - 1], v[i], a)) >= 0;
}
bool above(const Pt& a, const Pt& p)
{
    return sgn(p.y - a.y) >= 0;
}
bool crossesRay(const Pt& a, const Pt& p,
    const Pt& q)
{
    return sgn((above(a, q) - above(a, p))
        * orient(a, p, q)) == 1;
}
// Checks if point 'a' is inside the polygon
// If 'strict', false when 'a' is on the boundary
bool inPolygon(const vector<Pt>& v, const Pt& a, bool strict =
true)
{
    int numCrossings = 0;
    int n = sz(v);
    FOR(i, 0, n)
    {
        if (onSegment(v[i], v[(i + 1) % n], a))
            return !strict;
        numCrossings += crossesRay(a, v[i], v[(i + 1) % n]);
    }
    return numCrossings & 1;
}
```

```
vector<Pt> convexHull(vector<Pt> v, bool include_collinear = false)
{
    if (sz(v) <= 1)
        return v;
    sort(all(v), [](const Pt& p, const Pt& q)
    {
        int dx = sgn(p.x - q.x);
        if (dx != 0)
            return dx < 0;
        return sgn(p.y - q.y) < 0;
    });
    vector<Pt> lower, upper;
    for (const Pt& p : v)
    {
        while (sz(lower) > 1 &&
            sgn(orient(lower[sz(lower) - 2], lower.back(), p)) < (
                include_collinear ? 0 : 1))
            lower.pop_back();
        while (sz(upper) > 1 &&
            sgn(orient(upper[sz(upper) - 2], upper.back(), p)) > (
                include_collinear ? 0 : -1))
            upper.pop_back();
        lower.pb(p);
        upper.pb(p);
    }
    reverse(all(upper));
    lower.insert(lower.end(), next(upper.begin()), prev(upper.end()
    ()));
    return lower;
}
```

#### tangents-to-convex-polygon.hpp

**Description:** Returns the indices of tangent points from  $p$ .  $p$  must be strictly outside the polygon.

32608c, 38 lines

```
pii tangentsToConvexPolygon(const vector<Pt>& v, const Pt& p)
{
    int n = sz(v), i = 0;
    if (n == 2)
        return {0, 1};
    while (sgn(orient(p, v[i], v[(i + 1) % n]))
        * sgn(orient(p, v[i], v[(i + n - 1) % n])) > 0)
        i++;
    int s1 = 1, s2 = -1;
    if (sgn(orient(p, v[i], v[(i + 1) % n])) == s1
        || sgn(orient(p, v[i], v[(i + n - 1) % n])) == s2)
        swap(s1, s2);
    pii res;
    int l = i, r = i + n - 1;
    while (r - l > 1)
    {
        int m = (l + r) / 2;
        if (sgn(orient(p, v[i], v[m % n])) != s1
            && sgn(orient(p, v[m % n], v[(m + 1) % n])) != s1)
            l = m;
        else
            r = m;
    }
    res.x = r % n;
    l = i;
    r = i + n - 1;
    while (r - l > 1)
    {
        int m = (l + r) / 2;
        if (sgn(orient(p, v[i], v[m % n])) == s2
            || sgn(orient(p, v[m % n], v[(m + 1) % n])) != s2)
            l = m;
        else
            r = m;
    }
}
```

```
}
res.y = r % n;
return res;
}
```

#### minkowski-sum.hpp

**Description:** Returns the Minkowski sum of two convex polygons.

dbcd43, 40 lines

```
vector<Pt> minkowskiSum(const vector<Pt>& v1, const vector<Pt>& v2)
{
    if (v1.empty() || v2.empty())
        return {};
    if (sz(v1) == 1 && sz(v2) == 1)
        return {v1[0] + v2[0]};
    auto comp = [](const Pt& p, const Pt& q)
    {
        return sgn(p.x - q.x) < 0
            || (sgn(p.x - q.x) == 0
                && sgn(p.y - q.y) < 0);
    };
    int i1 = min_element(all(v1), comp) - v1.begin();
    int i2 = min_element(all(v2), comp) - v2.begin();
    vector<Pt> res;
    int n1 = sz(v1), n2 = sz(v2),
        j1 = 0, j2 = 0;
    while (j1 < n1 || j2 < n2)
    {
        const Pt& p1 = v1[(i1 + j1) % n1];
        const Pt& q1 = v1[(i1 + j1 + 1) % n1];
        const Pt& p2 = v2[(i2 + j2) % n2];
        const Pt& q2 = v2[(i2 + j2 + 1) % n2];
        if (sz(res) >= 2 && onSegment(res[sz(res) - 2], p1 + p2,
            res.back()))
            res.pop_back();
        res.pb(p1 + p2);
        int s = sgn(cross(q1 - p1, q2 - p2));
        if (j1 < n1 && (j2 == n2 || s > 0
            || (s == 0 && (sz(res) < 2
                || sgn(dot(res.back()
                    - res[sz(res) - 2],
                        q1 + p2 - res.back())) > 0))))
            j1++;
        else
            j2++;
    }
    if (sz(res) > 2 && onSegment(res[sz(res) - 2], res[0], res.
        back()))
        res.pop_back();
    return res;
}
```

#### ear-clipping.hpp

**Description:** Finds an arbitrary triangulation of a simple polygon with no three collinear vertices.

0252d5, 55 lines

```
vector<tuple<int, int, int>> earClipping(const vector<Pt>& v)
{
    int n = sz(v);
    vector<tuple<int, int, int>> res;
    VI indices(n, ear(n), reflex(n);
    iota(all(indices), 0);
    auto updReflexStatus = [&](int i)
    {
        int sz = sz(indices),
            pos = find(all(indices), i) - indices.begin();
        int iPrev = indices[(pos + sz - 1) % sz],
            iNext = indices[(pos + 1) % sz];
        reflex[i] = orient(v[iPrev], v[i], v[iNext]) < 0;
    };
}
```

```
auto updEarStatus = [&](int i)
{
    if (reflex[i])
    {
        ear[i] = 0;
        return;
    }
    int sz = sz(indices),
        pos = find(all(indices), i) - indices.begin();
    int iPrev = indices[(pos + sz - 1) % sz],
        iNext = indices[(pos + 1) % sz];
    ear[i] = 1;
    for (int j : indices)
    {
        if (j != iPrev && j != i && j != iNext && reflex[j]
            && inConvexPolygon({v[iPrev], v[i], v[iNext]}, v[j]))
        {
            ear[i] = 0;
            break;
        }
    }
};
FOR(i, 0, n)
    updReflexStatus(i);
FOR(i, 0, n)
    updEarStatus(i);
RFOR(sz, n + 1, 3)
{
    int i = 0;
    while (!ear[indices[i]])
        i++;
    int iPrev = indices[(i + sz - 1) % sz], iNext = indices[(i
        + 1) % sz];
    res.pb({iPrev, indices[i], iNext});
    indices.erase(indices.begin() + i);
    updReflexStatus(iPrev);
    updReflexStatus(iNext);
    updEarStatus(iPrev);
    updEarStatus(iNext);
}
return res;
}
```

#### halfplane-intersection.hpp

**Description:** Returns the counter-clockwise ordered vertices of the half-plane intersection. Returns empty if the intersection is empty. Adds a bounding box to ensure a finite area.

cf6d03, 47 lines

```
vector<Pt> hplaneInter(vector<Line> lines)
{
    const db C = 1e9;
    lines.pb({{-C, C}, {-C, -C}});
    lines.pb({{-C, -C}, {C, -C}});
    lines.pb({{C, -C}, {C, C}});
    lines.pb({{C, C}, {-C, C}});
    sort(all(lines), [](const Line& l1, const Line& l2)
    {
        bool h1 = half(l1.n), h2 = half(l2.n);
        if (h1 != h2)
            return h1 < h2;
        int p = sgn(cross(l1.n, l2.n));
        if (p != 0)
            return p > 0;
        return sgn(l1.c / abs(l1.n) - l2.c / abs(l2.n)) < 0;
    });
    lines.erase(unique(all(lines), parallel), lines.end());
    deque<pair<Line, Pt>> d;
    for (const Line& l : lines)
    {
        while (sz(d) > 1 && sgn(l.side((d.end() - 1)->y)) < 0)
            d.pop_back();
        while (sz(d) > 1 && sgn(l.side((d[0]->y))) < 0)
            d.pop_front();
        d.pb({l, l.p});
    }
}
```

```
    d.pop_back();
    while (sz(d) > 1 && sgn(l.side((d.begin() + 1)->y)) < 0)
        d.pop_front();
    if (!d.empty() && sgn(cross(d.back().x.n, l.n)) <= 0)
        return {};
    if (sz(d) < 2 || sgn(d.front().x.side(inter(l, d.back().x))
        ) >= 0)
    {
        Pt p;
        if (!d.empty())
        {
            p = inter(l, d.back().x);
            if (!parallel(l, d.front().x))
                d.front().y = inter(l, d.front().x);
        }
        d.pb({l, p});
    }
}
vector<Pt> res;
for (auto [l, p] : d)
{
    if (res.empty() || sgn(sq(p - res.back())) > 0)
        res.pb(p);
}
return res;
}
```

circle.hpp

ab2e8c, 42 lines

```
// Returns the circumcenter of triangle abc.
// The circumcircle of a triangle is a circle that passes
// through all three vertices.
Pt circumCenter(const Pt& a, Pt b, Pt c)
{
    b = b - a;
    c = c - a;
    assert(sgn(cross(b, c)) != 0);
    return a + perp(b * sq(c) - c * sq(b)) / cross(b, c) / 2;
}
// Returns circle-line intersection points
vector<Pt> circleLine(const Pt& o, db r, const Line& l)
{
    db h2 = r * r - l.sqDist(o);
    if (sgn(h2) == -1)
        return {};
    Pt p = l.proj(o);
    if (sgn(h2) == 0)
        return {p};
    Pt h = perp(l.n) * sqrt(h2) / abs(l.n);
    return {p - h, p + h};
}
// Returns circle-circle intersection points
vector<Pt> circleCircle(const Pt& o1, db r1, const Pt& o2, db
r2)
{
    Pt d = o2 - o1;
    db d2 = sq(d);
    if (sgn(d2) == 0)
    {
        // assuming the circles don't coincide
        assert(sgn(r2 - r1) != 0);
        return {};
    }
    db pd = (d2 + r1 * r1 - r2 * r2) / 2;
    db h2 = r1 * r1 - pd * pd / d2;
    if (sgn(h2) == -1)
        return {};
    Pt p = o1 + d * pd / d2;
    if (sgn(h2) == 0)
```

```
    return {p};
    Pt h = perp(d) * sqrt(h2 / d2);
    return {p - h, p + h};
}

tangents.hpp
Description: Finds common tangents (outer or inner) to two circles. If there are
two tangents, returns the pairs of tangency points on each circle (p1, p2). If there
is one tangent, the circles are tangent to each other at some point p, res contains p
four times, and the tangent line can be found as line(o1, p).perpThrough(p). The
same code can be used to find the tangent to a circle through a point by setting r2
to 0 (in which case inner doesn't matter).
82f1dc, 20 lines
vector<pair<Pt, Pt>> tangents(const Pt& o1, db r1,
const Pt& o2, db r2, bool inner)
{
    if (inner)
        r2 = -r2;
    Pt d = o2 - o1;
    db dr = r1 - r2, d2 = sq(d), h2 = d2 - dr * dr;
    if (sgn(d2) == 0 || sgn(h2) < 0)
    {
        assert(sgn(h2) != 0);
        return {};
    }
    vector<pair<Pt, Pt>> res;
    for (db sign : {-1, 1})
    {
        Pt v = (d * dr + perp(d) * sqrt(h2) * sign) / d2;
        res.pb({o1 + v * r1, o2 + v * r2});
    }
    return res;
}

welzl.hpp
Description: Returns the smallest enclosing circle of points in v
Time: O(n) (expected)
f6000c, 36 lines
pair<Pt, db> welzl(vector<Pt> v)
{
    int n = sz(v), k = 0, idxes[2];
    mt19937 rng;
    shuffle(all(v), rng);
    Pt c = v[0];
    db r = 0;
    while (true)
    {
        FOR(i, k, n)
        {
            if (sgn(abs(v[i] - c) - r) > 0)
            {
                swap(v[i], v[k]);
                if (k == 0)
                    c = v[0];
                else if (k == 1)
                    c = (v[0] + v[1]) / 2;
                else
                    c = circumCenter(v[0], v[1], v[2]);
                r = abs(v[0] - c);
                if (k < i)
                {
                    if (k < 2)
                        idxes[k++] = i;
                    shuffle(v.begin() + k, v.begin() + i + 1, rng);
                    break;
                }
            }
        }
        while (k > 0 && idxes[k - 1] == i)
            k--;
        if (i == n - 1)
            return {c, r};
    }
}
```

```
    }
    }
}

closest-pair.hpp
Description: Returns the distance between the closest points
Time: O(n log n)
678ecf, 23 lines
db closestPair(vector<Pt> v)
{
    sort(all(v), [](const Pt& p, const Pt& q)
    {
        return sgn(p.x - q.x) < 0;
    });
    set<pdd> s;
    int n = sz(v), ptr = 0;
    db h = 1e18;
    FOR(i, 0, n)
    {
        for (auto it = s.lower_bound(MP(v[i].y - h, v[i].x));
            it != s.end() && sgn(it->x - (v[i].y + h)) <= 0; it++)
        {
            Pt q = {it->y, it->x};
            h = min(h, abs(v[i] - q));
        }
        for (; sgn(v[ptr].x - (v[i].x - h)) <= 0; ptr++)
            s.erase({v[ptr].y, v[ptr].x});
        s.insert({v[i].y, v[i].x});
    }
    return h;
}

planar-graph.hpp
Description: Finds faces in a planar graph. Use addVertex() and addEdge() for
initializing the graph and addQueryPoint() for initializing the queries. After ini-
tialization, call findFaces() before using other functions. getIncidentFaces(i)
returns the pair of faces (u, v) (possibly u = v) such that the i-th edge lies on the
boundary of these faces. getFaceOfQueryPoint(i) returns the face where the i-th
query point lies.
939539, 169 lines
namespace PlanarGraph
{
    struct IndexedPt
    {
        Pt p;
        int index;
        bool operator<(const IndexedPt& q) const
        {
            return p.x < q.p.x;
        }
    };
    struct Edge
    {
        // cross(vertices[j].p - vertices[i].p, l.n) > 0
        int i, j;
        Line l;
    };
    vector<IndexedPt> vertices, queryPoints;
    vector<Edge> edges;
    struct Comparator
    {
        using is_transparent = void;
        static IndexedPt vertex;
        db getY(const Line& l) const
        {
            return -(l.n.x * vertex.p.x + l.c) / l.n.y;
        }
        bool operator()(int i, int j) const
        {
            auto [u1, v1, l1] = edges[i];
            auto [u2, v2, l2] = edges[j];
```

```

    if (u1 == vertex.index && u2 == vertex.index)
        return sgn(cross(l1.n, l2.n)) > 0;
    if (v1 == vertex.index && v2 == vertex.index)
        return sgn(cross(l1.n, l2.n)) < 0;
    int dy = sgn(getY(l1) - getY(l2));
    assert(dy != 0);
    return dy < 0;
}
bool operator()(int i, const Pt& p) const
{
    int dy = sgn(getY(edges[i].l) - p.y);
    assert(dy != 0);
    return dy < 0;
}
} comparator;
IndexedPt Comparator::vertex;
DSU dsu;
VI upperFace, queryAns;

void addVertex(const Pt& p)
{
    vertices.pb({p, sz(vertices)});
}
void addEdge(int i, int j, const Line& l)
{
    assert(0 <= i && i < sz(vertices));
    assert(0 <= j && j < sz(vertices));
    assert(i != j);
    assert(vertices[i].index == i);
    assert(vertices[j].index == j);
    edges.pb({i, j, l});
}
void addEdge(int i, int j)
{
    addEdge(i, j, {vertices[i].p, vertices[j].p});
}
void addQueryPoint(const Pt& p)
{
    queryPoints.pb({p, sz(queryPoints)});
}
void findFaces()
{
    int n = sz(vertices), m = sz(edges);
    const db ROT_ANGLE = 4;
    for (auto& p : vertices)
        p.p = rot(p.p, ROT_ANGLE);
    for (auto& p : queryPoints)
        p.p = rot(p.p, ROT_ANGLE);
    vector<VI> edgesL(n), edgesR(n);
    FOR(k, 0, m)
    {
        auto& [i, j, l] = edges[k];
        l.n = rot(l.n, ROT_ANGLE);
        if (vertices[i].p.x > vertices[j].p.x)
        {
            swap(i, j);
            l.n = l.n * (-1);
            l.c *= -1;
        }
        edgesL[j].pb(k);
        edgesR[i].pb(k);
    }
    sort(all(vertices));
    sort(all(queryPoints));
    // when choosing INF, remember that we rotate the plane
    addVertex({-INF, INF});
    addVertex({INF, INF});
    addEdge(n, n + 1);
    dsu = DSU(m + 1);

```

```

    set<int, Comparator> s;
    s.insert(m);
    upperFace.resize(m);
    int ptr = 0;
    queryAns.resize(sz(queryPoints));
    for (const IndexedPt& vertex : vertices)
    {
        int i = vertex.index;
        while (ptr < sz(queryPoints)
            && (i >= n || queryPoints[ptr] < vertex))
        {
            const auto& [pt, j] = queryPoints[ptr++];
            Comparator::vertex = {pt, -1};
            queryAns[j] = *s.lower_bound(pt);
        }
        if (i >= n)
            break;
        Comparator::vertex = vertex;
        int upper = -1, lower = -1;
        if (!edgesL[i].empty())
        {
            sort(all(edgesL[i]), comparator);
            auto it = s.lower_bound(edgesL[i][0]);
            lower = edgesL[i][0];
            for (int e : edgesL[i])
            {
                assert(*it == e);
                assert(next(it) != s.end());
                upperFace[e] = *next(it);
                it = s.erase(it);
            }
            assert(it != s.end());
            upper = *it;
        }
        if (!edgesR[i].empty())
        {
            sort(all(edgesR[i]), comparator);
            if (upper == -1)
            {
                upper = *s.lower_bound(edgesR[i][0]);
            }
            int prv = -1;
            for (int e : edgesR[i])
            {
                s.insert(e);
                if (prv != -1)
                {
                    upperFace[prv] = e;
                }
                prv = e;
            }
            upperFace[edgesR[i].back()] = upper;
            dsu.unite(edgesL[i].empty() ? upper : lower, edgesR[i][0]);
        }
        else if (lower != -1 && upper != -1)
        {
            dsu.unite(upper, lower);
        }
    }
}
pii getIncidentFaces(int i)
{
    return {dsu.find(i), dsu.find(upperFace[i])};
}
int getFaceOfQueryPoint(int i)
{
    return dsu.find(queryAns[i]);
}

```

```

};

```

## Mathematics (6)

### Number-theoretic algorithms

modular-arithmetics.hpp

6271b9, 67 lines

```

const int MOD = 998244353;

int add(int a, int b)
{
    return a + b < MOD ? a + b : a + b - MOD;
}

void updAdd(int& a, int b)
{
    a += b;
    if (a >= MOD)
        a -= MOD;
}

int sub(int a, int b)
{
    return a - b >= 0 ? a - b : a - b + MOD;
}

void updSub(int& a, int b)
{
    a -= b;
    if (a < 0)
        a += MOD;
}

int mult(int a, int b)
{
    return (ll)a * b % MOD;
}

int binPow(int a, ll n)
{
    int res = 1;
    while (n)
    {
        if (n & 1)
            res = mult(res, a);
        a = mult(a, a);
        n >>= 1;
    }
    return res;
}

int inv[N], fact[N], ifact[N];

void init()
{
    inv[1] = 1;
    FOR(i, 2, N)
    {
        inv[i] = mult(MOD - MOD / i, inv[MOD % i]);
    }
    fact[0] = ifact[0] = 1;
    FOR(i, 1, N)
    {
        fact[i] = mult(fact[i - 1], i);
        ifact[i] = mult(ifact[i - 1], inv[i]);
    }
}

```

```
int C(int n, int k)
{
    if (k < 0 || k > n)
        return 0;
    return mult(fact[n], mult(ifact[n - k], ifact[k]));
}
```

gcd.hpp  
**Description:**  $ax + by = d$ ,  $\gcd(a, b) = |d| \rightarrow (d, x, y)$ .  
Minimizes  $|x| + |y|$ . And minimizes  $|x - y|$  for  $a > 0, b > 0$ .  
bcd80c, 16 lines

```
tuple<ll, ll, ll> gcdExt(ll a, ll b)
{
    ll x1 = 1, y1 = 0;
    ll x2 = 0, y2 = 1;
    while (b)
    {
        ll k = a / b;
        x1 -= k * x2;
        y1 -= k * y2;
        a %= b;
        swap(a, b);
        swap(x1, x2);
        swap(y1, y2);
    }
    return {a, x1, y1};
}
```

fast-chinese.hpp  
**Description:**  $x \% p_i = m_i, \text{lcm}(p_i) \leq 10^{18}, 0 \leq x < \text{lcm}(p_i) \rightarrow x$  or  $-1$ .  
**Time:**  $\mathcal{O}(n \log(\text{lcm}(p_i)))$   
046449, 24 lines

```
ll fastChinese(vector<ll> m, vector<ll> p)
{
    assert(sz(m) == sz(p));
    ll aa = p[0];
    ll bb = m[0];
    FOR(i, 1, sz(m))
    {
        ll b = (m[i] - bb % p[i] + p[i]) % p[i];
        ll a = aa % p[i];
        ll c = p[i];

        auto [d, x, y] = gcdExt(a, c);
        if(b % d != 0)
            return -1;
        a /= d;
        b /= d;
        c /= d;
        b = (b * (__int128)x % c + c) % c;

        bb = aa * b + bb;
        aa = aa * c;
    }
    return bb;
}
```

miller-rabin.hpp  
**Description:** To speed up change candidates to at least 4 random values `rng() % (n - 3) + 2`. Use `__int128` in mult.  
**Time:**  $\mathcal{O}(|\text{candidates}| \cdot \log n)$   
2f89bb, 33 lines

```
VI candidates = {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 47};
bool millerRabin(ll n)
{
    if (n == 1)
        return false;
    if (n == 2 || n == 3)
        return true;
    ll d = n - 1;
    int s = __builtin_ctzll(d);
```

```
    d >>= s;

    for (ll b : candidates)
    {
        if (b >= n)
            break;
        b = binpow(b, d, n);
        if (b == 1)
            continue;
        bool ok = false;
        FOR (i, 0, s)
        {
            if (b + 1 == n)
            {
                ok = true;
                break;
            }
            b = mult(b, b, n);
        }
        if (!ok)
            return false;
    }
    return true;
}
```

pollard.hpp  
**Description:** Uses the Miller-Rabin test. rho finds a divisor of  $n$ . Use `__int128` in mult.  
**Time:**  $\mathcal{O}(n^{1/4} \cdot \log n)$ .  
69a916, 62 lines

```
ll f(ll x, ll c, ll n)
{
    return add(mult(x, x, n), c, n);
}

ll rho(ll n)
{
    const int iter = 47 * pow(n, 0.25);
    while (true)
    {
        ll x0 = rng() % n;
        ll c = rng() % n;
        ll x = x0;
        ll y = x0;
        ll g = 1;
        FOR (i, 0, iter)
        {
            x = f(x, c, n);
            y = f(y, c, n);
            y = f(y, c, n);
            g = gcd(abs(x - y), n);
            if (g != 1)
                break;
        }
        if (g > 1 && g < n)
            return g;
    }
}

VI primes = {2, 3, 5, 7, 11, 13, 17, 19, 23};
```

```
VL factorize(ll n)
{
    VL ans;

    for (auto p : primes)
    {
        while (n % p == 0)
        {
            ans.pb(p);
            n /= p;
```

```
        }
    }
    queue<ll> q;
    q.push(n);

    while (!q.empty())
    {
        ll x = q.front();
        q.pop();
        if (x == 1)
            continue;
        if (millerRabin(x))
            ans.pb(x);
        else
        {
            ll y = rho(x);
            q.push(y);
            q.push(x / y);
        }
    }
    return ans;
}
```

floor-sum.hpp  
**Description:** Computes  $\sum_{i=0}^{n-1} \left\lfloor \frac{a \cdot i + b}{m} \right\rfloor$ .  
**Time:**  $\mathcal{O}(\log m)$ .  
9517db, 16 lines

```
ll floorSum(ll n, ll m, ll a, ll b)
{
    ll ans = 0;
    while (true)
    {
        ans += (a / m) * n * (n - 1) / 2 + (b / m) * n;
        a %= m;
        b %= m;
        if (a == 0)
            return ans;
        return ans;
        ll k = (a * (n - 1) + b) / m;
        b = a * n - m * k + b;
        n = k;
        swap(a, m);
    }
}
```

min-mod-linear.hpp  
**Description:** Finds  $\min\{(ax + b) \bmod m \mid 0 \leq x < n\}$ .  
**Time:**  $\mathcal{O}(\log m)$ .  
03b25c, 14 lines

```
int minModLinear(ll n, ll m, ll a, ll b)
{
    ll res = m;
    while (n > 0)
    {
        a %= m;
        b = (b % m + m) % m;
        res = min(res, b);
        n = (a * (n - 1) + b) / m;
        b -= m * n;
        swap(a, m);
    }
    return res;
}
```

mod-inequality.hpp  
**Description:** Finds the smallest  $x \geq 0$  such that  $(ax + b) \bmod m \geq c$ . Returns  $-1$ , if the solution does not exist.  
**Time:**  $\mathcal{O}(\log m)$ .  
4a4b4a, 15 lines

```
int modInequality(ll m, ll a, ll b, ll c)
{
```

```
a %= m;
b %= m;
if (b >= c)
    return 0;
if (a == 0)
    return -1;
if (c + a < m)
    return (c - b + a - 1) / a;
int k = modInequality(a, m, c - b - 1, c + a - m);
if (k == -1)
    return -1;
return (k * m + c - b + a - 1) / a;
}
```

disLog.hpp a986d8, 23 lines

```
// Returns minimum x for which (a ^ x) % MOD = b % MOD, a and
MOD are coprime.
int disLog(int a, int b)
{
    int n = sqrt(MOD) + 1;

    int an = binPow(a, n);
    unordered_map<int, int> vals;
    for (int q = 0, cur = b; q <= n; ++q)
    {
        vals[cur] = q;
        cur = mult(cur, a);
    }

    for (int p = 1, cur = 1; p <= n; ++p)
    {
        cur = mult(cur, an);
        if (vals.count(cur))
        {
            return n * p - vals[cur];
        }
    }
    return -1;
}
```

Matrices

gaussian.hpp  
**Description:** Solves the system  $Ax = b$ . Returns  $(v, w)$  such that every solution  $x$  can be represented as  $v + c_1 w_1 + c_2 w_2 + \dots + c_k w_k$ , where  $v$  is arbitrary solution,  $c_i$  are scalars and  $w$  is basis. If there is no solution, returns an empty pair. If the solution is unique, then  $w$  is empty.  
**Time:**  $\mathcal{O}(nm \min(n, m))$

3fa52c, 66 lines

```
pair<VI, vector<VI>> solveLinearSystem(vector<VI> a, VI b)
{
    int n = sz(a), m = sz(a[0]);
    assert(sz(b) == n);
    FOR(i, 0, n)
    {
        assert(sz(a[i]) == m);
        a[i].pb(b[i]);
    }
    int p = 0;
    VI pivots;
    FOR(j, 0, m)
    {
        // with doubles, abs(a[p][j]) -> max
        if (a[p][j] == 0)
        {
            int l = -1;
            FOR(i, p, n)
                if (a[i][j] != 0)
                    l = i;
            if (l == -1)

```

```
                continue;
                swap(a[p], a[l]);
            }
            int in = binPow(a[p][j], MOD - 2);
            FOR(i, p + 1, n)
            {
                int c = mult(a[i][j], in);
                FOR(k, j, m + 1)
                    updSub(a[i][k], mult(c, a[p][k]));
            }
            pivots.pb(j);
            p++;
            if (p == n)
                break;
        }
    }
    FOR(i, p, n)
        if (a[i].back() != 0)
            return {};
    VI v(m);
    RFOR(i, p, 0)
    {
        int j = pivots[i];
        v[j] = a[i].back();
        FOR(k, j + 1, m)
            updSub(v[j], mult(a[i][k], v[k]));
        v[j] = mult(v[j], binPow(a[i][j], MOD - 2));
    }
    vector<VI> w;
    FOR(q, 0, m)
    {
        if (find(all(pivots), q) != pivots.end())
            continue;
        VI d(m);
        d[q] = 1;
        RFOR(i, p, 0)
        {
            int j = pivots[i];
            FOR(k, j + 1, m)
                updSub(d[j], mult(a[i][k], d[k]));
            d[j] = mult(d[j], binPow(a[i][j], MOD - 2));
        }
        w.pb(d);
    }
    return {v, w};
}
```

hungarian.hpp  
**Description:** Finds a maximum matching that has the minimum weight in a weighted bipartite graph.  
**Time:**  $\mathcal{O}(n^2 m)$

792894, 63 lines

```
ll hungarian(const vector<VL>& a)
{
    int n = sz(a), m = sz(a[0]);
    assert(n <= m);
    VL u(n + 1), v(m + 1);
    VI p(m + 1, n), way(m + 1);
    FOR(i, 0, n)
    {
        p[m] = i;
        int j0 = m;
        VL minv(m + 1, LINF);
        VI used(m + 1);
        while (p[j0] != n)
        {
            used[j0] = true;
            int i0 = p[j0], j1 = -1;
            ll delta = LINF;
            FOR(j, 0, m)

```

```

        {
            if (!used[j])
            {
                ll cur = a[i0][j] - u[i0] - v[j];
                if (cur < minv[j])
                {
                    minv[j] = cur;
                    way[j] = j0;
                }
                if (minv[j] < delta)
                {
                    delta = minv[j];
                    j1 = j;
                }
            }
        }
    }
    assert(j1 != -1);
    FOR(j, 0, m + 1)
    {
        if (used[j])
        {
            u[p[j]] += delta;
            v[j] -= delta;
        }
        else
            minv[j] -= delta;
    }
    j0 = j1;
}
while (j0 != m)
{
    int j1 = way[j0];
    p[j0] = p[j1];
    j0 = j1;
}
}
VI ans(n + 1);
FOR(j, 0, m)
    ans[p[j]] = j;
ll res = 0;
FOR(i, 0, n)
    res += a[i][ans[i]];
assert(res == -v[m]);
return res;
}
```

simplex.hpp  
**Description:**  $c^T x \rightarrow \max, Ax \leq b, x \geq 0$ .

aa2614, 142 lines

```
typedef vector<db> VD;

struct Simplex
{
    void pivot(int l, int e)
    {
        assert(0 <= l && l < m);
        assert(0 <= e && e < n);
        assert(abs(a[l][e]) > EPS);
        b[l] /= a[l][e];
        FOR(j, 0, n)
            if (j != e)
                a[l][j] /= a[l][e];
        a[l][e] = 1 / a[l][e];
        FOR(i, 0, m)
        {
            if (i != l)
            {
                b[i] -= a[i][e] * b[l];
                FOR(j, 0, n)

```

```
        if (j != e)
            a[i][j] -= a[i][e] * a[l][j];
        a[i][e] *= -a[l][e];
    }
}
v += c[e] * b[l];
FOR(j, 0, n)
    if (j != e)
        c[j] -= c[e] * a[l][j];
c[e] *= -a[l][e];
swap(nonBasic[e], basic[l]);
}
void findOptimal()
{
    VD delta(m);
    while (true)
    {
        int e = -1;
        FOR(j, 0, n)
            if (c[j] > EPS && (e == -1 || nonBasic[j] < nonBasic[e]))
                e = j;
        if (e == -1)
            break;
        FOR(i, 0, m)
            delta[i] = a[i][e] > EPS ? b[i] / a[i][e] : LINF;
        int l = min_element(all(delta)) - delta.begin();
        if (delta[l] == LINF)
        {
            // unbounded
            assert(false);
        }
        pivot(l, e);
    }
}
void initializeSimplex(const vector<VD>& _a, const VD& _b, const VD& _c)
{
    m = sz(_b);
    n = sz(_c);
    nonBasic.resize(n);
    iota(all(nonBasic), 0);
    basic.resize(m);
    iota(all(basic), n);
    a = _a;
    b = _b;
    c = _c;
    v = 0;
    int k = min_element(all(b)) - b.begin();
    if (b[k] > -EPS)
        return;
    nonBasic.pb(n);
    iota(all(basic), n + 1);
    FOR(i, 0, m)
        a[i].pb(-1);
    c.assign(n, 0);
    c.pb(-1);
    n++;
    pivot(k, n - 1);
    findOptimal();
    if (v < -EPS)
    {
        // infeasible
        assert(false);
    }
    int l = find(all(basic), n - 1) - basic.begin();
    if (l != m)
    {
        int e = -1;
```

```
        while (abs(a[l][e]) < EPS)
            e++;
        pivot(l, e);
    }
}
n--;
int p = find(all(nonBasic), n) - nonBasic.begin();
assert(p < n + 1);
nonBasic.erase(nonBasic.begin() + p);
FOR(i, 0, m)
    a[i].erase(a[i].begin() + p);
c.assign(n, 0);
FOR(j, 0, n)
{
    if (nonBasic[j] < n)
        c[j] = _c[nonBasic[j]];
    else
        nonBasic[j]--;
}
FOR(i, 0, m)
{
    if (basic[i] < n)
    {
        v += _c[basic[i]] * b[i];
        FOR(j, 0, n)
            c[j] -= _c[basic[i]] * a[i][j];
    }
    else
        basic[i]--;
}
pair<VD, db> simplex(const vector<VD>& _a, const VD& _b, const VD& _c)
{
    initializeSimplex(_a, _b, _c);
    assert(sz(a) == m);
    FOR(i, 0, m)
        assert(sz(a[i]) == n);
    assert(sz(b) == m);
    assert(sz(c) == n);
    assert(sz(nonBasic) == n);
    assert(sz(basic) == m);
    findOptimal();
    VD x(n);
    FOR(i, 0, m)
        if (basic[i] < n)
            x[basic[i]] = b[i];
    return {x, v};
}
private:
    int m, n;
    VI nonBasic, basic;
    vector<VD> a;
    VD b;
    VD c;
    db v;
};
```

Convolutions

```
conv-xor.hpp
Description:  $c_k = \sum_{i \oplus j = k} a_i b_j$ .
075f59, 24 lines

void convXor(VI& a, int k)
{
    FOR(i, 0, k)
        FOR(j, 0, 1 << k)
            if ((j & (1 << i)) == 0)
            {
                int u = a[j];
                int v = a[j + (1 << i)];
```

```
                a[j] = add(u, v);
                a[j + (1 << i)] = sub(u, v);
            }
}
VI multXor(VI a, VI b, int k)
{
    convXor(a, k);
    convXor(b, k);
    FOR(i, 0, 1 << k)
        a[i] = mult(a[i], b[i]);
    convXor(a, k);
    int d = binPow(1 << k, MOD - 2);
    FOR(i, 0, 1 << k)
        a[i] = mult(a[i], d);
    return a;
}
```

```
conv-and.hpp
Description:  $c_i \wedge j + = a_i * b_j$ .
662d5e, 21 lines

void convAnd(VI& a, int k, bool inverse)
{
    FOR(i, 0, k)
        FOR(j, 0, 1 << k)
            if ((j & (1 << i)) == 0)
            {
                if (inverse)
                    updSub(a[j], a[j + (1 << i)]);
                else
                    updAdd(a[j], a[j + (1 << i)]);
            }
}
VI multAnd(VI a, VI b, int k)
{
    convAnd(a, k, false);
    convAnd(b, k, false);
    FOR(i, 0, 1 << k)
        a[i] = mult(a[i], b[i]);
    convAnd(a, k, true);
    return a;
}
```

```
conv-or.hpp
Description:  $c_k = \sum_i \text{OR }_{j=k} a_i b_j$ .
e4e659, 21 lines

void convOr(VI& a, int k, bool inverse)
{
    FOR(i, 0, k)
        FOR(j, 0, 1 << k)
            if ((j & (1 << i)) == 0)
            {
                if (inverse)
                    updSub(a[j + (1 << i)], a[j]);
                else
                    updAdd(a[j + (1 << i)], a[j]);
            }
}
VI multOr(VI a, VI b, int k)
{
    convOr(a, k, false);
    convOr(b, k, false);
    FOR(i, 0, 1 << k)
        a[i] = mult(a[i], b[i]);
    convOr(a, k, true);
    return a;
}
```

```
subset-convolution.hpp
Description:  $c[S] = \sum_{T \subseteq S} a[T] \cdot b[S \setminus T]$ .
```

Time:  $\mathcal{O}\left(n^2 \cdot 2^n\right)$ , 1.5s for  $n = 20$ .

5f8849, 27 lines

vector<VI> rankedMobius(VI a, int n)

{

vector<VI> res(n + 1, VI(1 << n));

FOR(mask, 0, 1 << n)

res[\_\_builtin\_popcount(mask)][mask] = a[mask];

FOR(sz, 0, n + 1)

conv0r(res[sz], n, false);

return res;

}

VI subsetConvolution(VI a, VI b, int n)

{

auto f = rankedMobius(a, n);

auto g = rankedMobius(b, n);

vector<VI> conv(n + 1, VI(1 << n));

FOR(sz, 0, n + 1)

{

FOR(i, 0, sz + 1)

FOR(mask, 0, 1 << n)

updAdd(conv[sz][mask], mult(f[i][mask], g[sz - i][mask]));

conv0r(conv[sz], n, true);

}

VI res(1 << n);

FOR(mask, 0, 1 << n)

res[mask] = conv[\_\_builtin\_popcount(mask)][mask];

return res;

}

Polynomials and FFT

fft.hpp  
Description: Number-theoretic transform. If you need complex-valued FFT, use the commented out code.  
Time:  $\mathcal{O}(n \log n)$

const int LEN = 1 << 23;

const int GEN = 31;

1a18a5, 73 lines

/\*typedef complex<db> com;

com pw[LEN];

void init()

{

db phi = (db)2 \* PI / LEN;

FOR(i, 0, LEN)

pw[i] = com(cos(phi \* i), sin(phi \* i));

}/

void fft(VI& a, bool inverse)

{

const int IGEN = binPow(GEN, MOD - 2);

int lg = \_\_builtin\_ctz(sz(a));

FOR(i, 0, sz(a))

{

int k = 0;

FOR(j, 0, lg)

k |= ((i >> j) & 1) << (lg - j - 1);

if(i < k)

swap(a[i], a[k]);

}

for(int len = 2; len <= sz(a); len \*= 2)

{

// int diff = inv ? LEN - LEN / len : LEN / len;

int ml = binPow(inverse ? IGEN : GEN, LEN / len);

for(int i = 0; i < sz(a); i += len)

{

// int pos = 0;

int pw = 1;

FOR(j, 0, len / 2)

{

int u = a[i + j];

int v = mult(a[i + j + len / 2], pw); // \* pw[pos]

a[i + j] = add(u, v);

a[i + j + len / 2] = sub(u, v);

// pos = (pos + diff) % LEN;

pw = mult(pw, ml);

}

}

if (inverse)

{

int m = binPow(sz(a), MOD - 2);

FOR(i, 0, sz(a))

// a[i] /= SZ(a);

a[i] = mult(a[i], m);

}

}

VI mult(VI a, VI b)

{

int n = sz(a), m = sz(b);

if (n == 0 || m == 0)

return {};

int sz = 1, szRes = n + m - 1;

while(sz < szRes)

sz \*= 2;

a.resize(sz);

b.resize(sz);

fft(a, false);

fft(b, false);

FOR(i, 0, sz)

a[i] = mult(a[i], b[i]);

fft(a, true);

a.resize(szRes);

return a;

}

mult-arbitrary-mod.hpp

Description: Multiplies polynomials modulo arbitrary mod (or without modulo). Add the modulo parameter to the modular arithmetics functions (int add(int a, int b, int m = mod)). LEN must be 2<sup>24</sup>. Change signature of the fft function into void fft(VI& a, bool inverse, int nttMod, int GEN). GEN will not be a constant anymore. You must add nttMod inside the fft function 10 times in 8 lines of code. Change signature of the original mult function into VI mult(VI a, VI b, int nttMod, int GEN). You must add nttMod inside the original mult function 4 times in 4 lines of code.

6ef40a, 32 lines

VI mult(const VI& a, const VI& b)

{

int n = sz(a), m = sz(b);

if (n == 0 || m == 0)

return {};

const int mods[3] = {754974721, 167772161, 469762049};

const int invs[3] = {190329765, 58587104, 187290749};

const int gens[3] = {362, 2, 40};

vector<VI> fa(3, VI(n)), fb(3, VI(m));

vector<VI> c(3);

FOR(i, 0, 3)

{

FOR(j, 0, n)

fa[i][j] = a[j] % mods[i];

FOR(j, 0, m)

fb[i][j] = b[j] % mods[i];

c[i] = mult(fa[i], fb[i], mods[i], gens[i]);

}

\_\_int128 modsProd = (\_\_int128)mods[0] \* mods[1] \* mods[2];

VI res(n + m - 1);

FOR(i, 0, n + m - 1)

{

\_\_int128 cur = 0;

FOR(j, 0, 3)

{

cur += (\_\_int128)mods[(j + 1) % 3] \* mods[(j + 2) % 3]

\* mult(invs[j], c[j][i], mods[j]);

}

res[i] = cur % modsProd % mod;

}

return res;

}

inverse.hpp  
Description:  $\frac{1}{A(x)}$  modulo  $x^n$ .  
Time:  $\mathcal{O}(n \log n)$

VI inverse(const VI& a, int n)

{

assert(sz(a) == n && a[0] != 0);

if(n == 1)

return {binPow(a[0], MOD - 2)};

VI ra = a;

FOR(i, 0, sz(ra))

if(i & 1)

ra[i] = sub(0, ra[i]);

int nn = (n + 1) / 2;

VI t = mult(a, ra);

t.resize(n);

FOR(i, 0, nn)

t[i] = t[2 \* i];

t.resize(nn);

t = inverse(t, nn);

t.resize(n);

RFOR(i, nn, 1)

{

t[2 \* i] = t[i];

t[i] = 0;

}

VI res = mult(ra, t);

res.resize(n);

return res;

}

log.hpp  
Description:  $\log(A(x))$  modulo  $x^n$ .  
Time:  $\mathcal{O}(n \log n)$

VI deriv(const VI& a)

{

int n = sz(a);

VI res(max(0, n - 1));

FOR(i, 0, n - 1)

res[i] = mult(a[i + 1], i + 1);

return res;

}

VI integr(const VI& a)

{

int n = sz(a);

VI res(n + 1);

RFOR(i, n, 1)

res[i] = mult(a[i - 1], inv[i]);

}

```
    res[0] = 0;
    return res;
}

VI log(const VI& a, int n)
{
    assert(sz(a) == n && a[0] == 1);
    VI res = integr(mult(deriv(a), inverse(a, n)));
    res.resize(n);
    return res;
}
```

```
exp.hpp
Description: exp(A(x)) modulo x^n.
Time: O(n log n)
865aca, 21 lines

VI exp(const VI& a, int n)
{
    assert(sz(a) == n && a[0] == 0);
    VI q = {1};
    for (int k = 2; k <= 2 * n; k *= 2)
    {
        q.resize(k);
        VI lnQ = log(q, k);
        FOR(i, 0, k)
        {
            if(i < n)
                lnQ[i] = sub(a[i], lnQ[i]);
            else
                lnQ[i] = sub(0, lnQ[i]);
        }
        lnQ[0] = add(lnQ[0], 1);
        q = mult(q, lnQ);
    }
    q.resize(n);
    return q;
}
```

```
divide.hpp
Description: Finds Q(x) and R(x) such that A(x) = Q(x)B(x) + R(x) and
deg R < deg B.
Time: O(n log n)
7ff56ad, 28 lines

void removeLeadingZeros(VI& a)
{
    while(sz(a) > 0 && a.back() == 0)
        a.pop_back();
}

pair<VI, VI> divide(VI a, VI b)
{
    removeLeadingZeros(a);
    removeLeadingZeros(b);
    int n = sz(a), m = sz(b);
    assert(m > 0);
    if(m > n)
        return {{}, a};
    reverse(all(a));
    reverse(all(b));
    VI q = b;
    q.resize(n - m + 1);
    q = mult(a, inverse(q, n - m + 1));
    q.resize(n - m + 1);
    reverse(all(a));
    reverse(all(b));
    reverse(all(q));
    VI r = mult(b, q);
    FOR(i, 0, n)
        r[i] = sub(a[i], r[i]);
    removeLeadingZeros(r);
    return {q, r};
}
```

```

}

multipoint-eval.hpp
Description: Evaluates the polynomial P(x) of degree m at points x_0, ..., x_{n-1}.
Time: O(n log^2 n + m log m)
df0c10, 44 lines

VI multipointEval(const VI& p, const VI& x)
{
    int n = sz(x);
    vector<VI> t;
    int _n = 1;
    while (_n < 2 * n)
        _n *= 2;
    t.resize(_n);

    function<void(int, int, int)> build = [&](int v, int tl, int tr)
    {
        if(tl + 1 == tr)
        {
            t[v] = {sub(0, x[tl]), 1};
            return;
        }
        int tm = (tl + tr) / 2;
        build(2 * v + 1, tl, tm);
        build(2 * v + 2, tm, tr);
        t[v] = mult(t[2 * v + 1], t[2 * v + 2]);
    };

    build(0, 0, n);
    VI ans(n);

    function<void(int, int, int, VI)> solve
    = [&](int v, int tl, int tr, VI q)
    {
        q = divide(q, t[v]).y;
        if (q.empty())
            return;
        if(tl + 1 == tr)
        {
            ans[tl] = q[0];
            return;
        }
        int tm = (tl + tr) / 2;
        solve(2 * v + 1, tl, tm, q);
        solve(2 * v + 2, tm, tr, q);
    };

    solve(0, 0, n, p);
    return ans;
}
```

```
shift-eval-values.hpp
Description: Let P(x) be the polynomial of degree at most n - 1. Given
P(0), P(1), ..., P(n - 1). Computes P(c), P(c + 1), ..., P(c + m - 1).
Time: O((n + m) log(n + m))
cc8c04, 35 lines

VI shiftEvalValues(VI a, int c, int m)
{
    int n = sz(a);
    VI q(n);
    FOR(i, 0, n)
    {
        q[i] = mult(a[i], mult(ifact[i], ifact[n - i - 1]));
        if ((n - i) % 2 == 0)
            q[i] = sub(0, q[i]);
    }
    VI s(n + m - 1);
    FOR(i, 0, sz(s))
        s[i] = binPow(sub(add(c, i), n - 1), MOD - 2);
}
```

```
VI res = mult(q, s);
res = {res.begin() + n - 1, res.begin() + n + m - 1};
int prod = 1;
FOR(i, 0, n)
{
    int cur = sub(c, i);
    if (cur != 0)
        prod = mult(prod, cur);
}
FOR(i, 0, m)
{
    int j = add(c, i);
    res[i] = j < n ? a[j] : mult(res[i], prod);
    int r = add(c, i + 1);
    if (r != 0)
        prod = mult(prod, r);
    int l = sub(add(c, i), n - 1);
    if (l != 0)
        prod = mult(prod, binPow(l, MOD - 2));
}
return res;
}
```

```
lagrange-eval.hpp
48f01b, 31 lines

// Evaluates P(k), where P is at most degree n-1 polynomial
// with values P[0..n-1]
int lagrange_eval(const vector<int>& P, int k)
{
    if (k < 0)
        return 0;
    int n = sz(P);
    if (k < n)
        return P[k];

    vector<int> pref(n + 1), suf(n + 1);
    pref[0] = suf[n] = 1;
    FOR (i, 0, n)
    {
        pref[i + 1] = mult(pref[i], sub(k, i));
    }
    RFOR (i, n, 0)
    {
        suf[i] = mult(suf[i + 1], sub(k, i));
    }

    int res = 0;
    FOR (i, 0, n)
    {
        int num = mult(P[i], mult(pref[i], suf[i + 1]));
        int den = mult(ifact[i], ifact[n - 1 - i]);
        if ((n - 1 - i) & 1)
            den = sub(0, den);
        res = add(res, mult(num, den));
    }
    return res;
}
```

## Newton’s method

Usable to find the solution of equation  $F(Q) = 0$ .

For example  $F(Q) = x \cdot Q^2 + A - Q = 0$ .

Newton’s method approximates the solution of the equation using the formula:

$$Q_{k+1} = Q_k - \frac{F(Q_k)}{F'(Q_k)}, \text{ where } F' = \frac{dF}{dQ}$$

Example of the derivative:  $F'(Q) = 2 \cdot x \cdot Q - 1$ .

Keep in mind that  $|Q_k| = 2^k$ .

## FFT tricks

## Two-dimensional FFT

The complexity is  $O(nm(\log n + \log m))$ . The main problem is to resize the matrix. You must add non-empty vectors.

## Divide-and-conquer FFT

Suppose we have the following DP relation:

$f(t) = g(t) - \sum_{0 \leq u < t} f(u)h(t - u)$ , where  $g(t)$  and  $h(t)$  are known and we want to compute  $f(t)$ . We can apply divide-and-conquer FFT.

Let  $m = \lfloor \frac{l+r}{2} \rfloor$ . We guarantee the following invariant conditions.

By the time we compute the values for the segment  $[l, r)$ , the following conditions are already met:

- The values for  $[0, l)$  on the DP is already determined.
- The sum of contributions from  $[0, l)$  through  $[l, r)$  is already applied to the DP in  $[l, r)$ .

When calculate the values for the segment  $[l, r)$  do:

- Calculate the values for the segment  $[l, m)$  recursively.
- Calculate the contributions from  $[l, m)$  to  $[m, r)$ .
- Calculate the values for the segment  $[m, r)$  recursively.

## Properties of the discrete Fourier transform

$$DFT(x)_k = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi \frac{k}{N} n}$$

Let  $x_n^R = x_{N-n \bmod N}$ .

$$DFT(x^R) = \overline{DFT(x)}.$$

For real  $x$ ,  $DFT(x)^R = \overline{DFT(x)}$ .

## Interpolation

When  $x_0, x_1, \dots, x_d$  and  $y_0, y_1, \dots, y_d$  are given (where  $x_i$  are pairwise distinct), a polynomial  $f(x)$  of degree no more than  $d$  such that  $f(x_i) = y_i (i = 0, \dots, d)$  is uniquely determined.

## Lagrange polynomial

Lagrange basis polynomial:  $L_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$ .

$$f(x) = y_0 L_0(x) + y_1 L_1(x) + \dots + y_d L_d(x).$$

## Newton polynomial

Divided differences:

$$[y_i] = y_i$$

$$[y_i, y_{i+1}] = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

$$[y_i, \dots, y_j] = \frac{[y_{i+1}, \dots, y_j] - [y_i, \dots, y_{j-1}]}{x_j - x_i}$$

Newton basis polynomial:  $N_i(x) = \prod_{j=0}^{i-1} (x - x_j)$ .

$$f(x) = [y_0]N_0(x) + \dots + [y_0, y_1, \dots, y_d]N_d(x).$$

## Linear recurrence

berlekamp-massey.hpp

**Description:** Finds a sequence of  $d$  integers  $c_1, \dots, c_d$  of the minimum length  $d$  such that  $a_i = \sum_{j=1}^d c_j a_{i-j}$ .

9979fe, 36 lines

```
VI berlekampMassey(const VI& a)
{
    VI c = {1}, bp = {1};
    int l = 0, b = 1, x = 1;
    FOR(j, 0, sz(a))
    {
        assert(sz(c) == l + 1);
        int d = a[j];
        FOR(i, 1, l + 1)
            updAdd(d, mult(c[i], a[j - i]));
        if (d == 0)
        {
            x++;
            continue;
        }
        VI t = c;
        int coef = mult(d, binPow(b, MOD - 2));
        if (sz(bp) + x > sz(c))
            c.resize(sz(bp) + x);
        FOR(i, 0, sz(bp))
            updSub(c[i + x], mult(coef, bp[i]));
        if (2 * l > j)
        {
            x++;
            continue;
        }
        l = j + 1 - l;
        bp = t;
        b = d;
        x = 1;
    }
    c.erase(c.begin());
    for (int& ci : c)
        ci = mult(ci, MOD - 1);
    return c;
}
```

bostan-mori.hpp

**Description:** Computes the  $n$ -th term of a given linearly recurrent sequence  $a_i = \sum_{j=1}^d c_j a_{i-j}$ . The first  $d$  terms  $a_0, a_1, \dots, a_{d-1}$  are given. The problem reduces to determining  $[x^n]P(x)/Q(x)$ .

$$\frac{P(x)}{Q(x)} = \frac{P(x)Q(-x)}{Q(x)Q(-x)} = \frac{U_e(x^2)}{V(x^2)} + x \cdot \frac{U_o(x^2)}{V(x^2)}.$$

**Time:**  $\mathcal{O}(d \log d \log n)$ .

e2a8cf, 25 lines

```
int bostanMori(const VI& c, VI a, ll n)
{
    int k = sz(c);
    assert(sz(a) == k);
    VI q(k + 1);
    q[0] = 1;
    FOR(i, 0, k)
        q[i + 1] = sub(0, c[i]);
    VI p = mult(a, q);
    p.resize(k);
    while (n)
    {
        VI qMinus = q;
        for (int i = 1; i <= k; i += 2)
            qMinus[i] = sub(0, qMinus[i]);
        VI newP = mult(p, qMinus);
        VI newQ = mult(q, qMinus);
```

```
        FOR(i, 0, k)
            p[i] = newP[2 * i + (n & 1)];
        FOR(i, 0, k + 1)
            q[i] = newQ[2 * i];
        n >>= 1;
    }
    return mult(p[0], binPow(q[0], MOD - 2));
}
```

## P-recursive sequences

find-coefs-of-p-recursive.hpp

**Description:** Finds the polynomials  $P_j$  such that  $\sum_{j=0}^d P_j(i) \cdot a_{i+d-j} = 0$ . Returns an empty vector if unable to find such polynomials. The first  $k$  terms  $a_0, a_1, \dots, a_{k-1}$  are given.

**Time:**  $\mathcal{O}(k^3)$

d2d417, 32 lines

```
const int LEN = 1 << 23;
const int GEN = 31;
vector<VI> findCoefsOfPRecursive(const VI& a, int d)
{
    int m = (sz(a) - d) / (d + 1) - 1;
    if (m < 0)
        return {};
    int n = (m + 1) * (d + 1);
    vector<VI> matr(sz(a) - d, VI(n));
    FOR(i, 0, sz(a) - d)
    {
        FOR(j, 0, d + 1)
        {
            int pw = 1;
            FOR(k, 0, m + 1)
            {
                matr[i][(m + 1) * j + k] = mult(pw, a[i + d - j]);
                pw = mult(pw, i);
            }
        }
    }
    auto [v, w] = solveLinearSystem(matr, VI(sz(a) - d));
    if (w.empty())
        return {};
    vector<VI> p(d + 1);
    FOR(j, 0, d + 1)
    {
        p[j] = {w[0].begin() + (m + 1) * j, w[0].begin() + (m + 1) * (j + 1)};
        removeLeadingZeros(p[j]);
    }
    return p;
}
```

find-nth-of-p-recursive.hpp

**Description:** Computes the  $n$ -th term of a given linearly recurrent sequence with polynomial coefficients  $\sum_{j=0}^d P_j(i) \cdot a_{i+d-j} = 0$ . The first  $d$  terms  $a_0, a_1, \dots, a_{d-1}$  are given. Let  $m$  be the maximum degree of  $P_j$ .

**Time:**  $\mathcal{O}(d^2 \sqrt{nm} \log nm + d^3 \sqrt{nm})$

a48f36, 134 lines

```
VI add(const VI& a, const VI& b)
{
    int n = sz(a), m = sz(b);
    VI c(max(n, m));
    FOR(i, 0, n)
        updAdd(c[i], a[i]);
    FOR(i, 0, m)
        updAdd(c[i], b[i]);
    return c;
}
```

```
int evalPoly(const VI& p, int x)
{
```

```
int res = 0;
RFOR(i, sz(p), 0)
    res = add(mult(res, x), p[i]);
return res;
}

VI mult(const vector<VI>& a, const VI& b)
{
    int n = sz(a);
    VI c(n);
    FOR(i, 0, n)
        FOR(j, 0, n)
            updAdd(c[i], mult(a[i][j], b[j]));
    return c;
}

vector<VI> mult(const vector<VI>& a, const vector<VI>& b)
{
    int n = sz(a);
    vector<VI> c(n, VI(n));
    FOR(i, 0, n)
        FOR(k, 0, n)
            FOR(j, 0, n)
                updAdd(c[i][j], mult(a[i][k], b[k][j]));
    return c;
}

typedef vector<vector<VI>> PolyMatr;

PolyMatr mult(const PolyMatr& a, const PolyMatr& b)
{
    int n = sz(a);
    PolyMatr c(n, vector<VI>(n));
    FOR(i, 0, n)
        FOR(k, 0, n)
            FOR(j, 0, n)
                c[i][j] = add(c[i][j], mult(a[i][k], b[k][j]));
    return c;
}

int findNthOfPRecursive(const vector<VI>& p, VI a, int n)
{
    int d = sz(p) - 1;
    assert(sz(a) == d);
    if (n < d)
        return a[n];
    auto polyMatrProd = [](const PolyMatr& polyMatr, int k, VI u)
    {
        int h = sz(polyMatr);

        auto shiftEvalMatrs =
            [&](const vector<vector<VI>>& matrices, int c, int m)
            {
                int cnt = sz(matrices);
                vector<vector<VI>> res(m, vector<VI>(h, VI(h)));
                FOR(i, 0, h)
                {
                    FOR(j, 0, h)
                    {
                        VI b(cnt);
                        FOR(l, 0, cnt)
                            b[l] = matrices[l][i][j];
                        b = shiftEvalValues(b, c, m);
                        FOR(l, 0, m)
                            res[l][i][j] = b[l];
                    }
                }
                return res;
            };
    };

    PolyMatr polyMatr(d, vector<VI>(d));
    FOR(i, 0, d - 1)
        polyMatr[i][i + 1] = p[0];
    FOR(i, 0, d)
    {
        polyMatr[d - 1][i] = p[d - i];
        for (int& coef : polyMatr[d - 1][i])
            coef = sub(0, coef);
    }
    PolyMatr denom = {{p[0]}};
    a = polyMatrProd(polyMatr, n - d + 1, a);
    const VI& x = polyMatrProd(denom, n - d + 1, {1});
    return mult(binPow(x[0], MOD - 2), a.back());
};
```

```
int m = 0;
FOR(i, 0, h)
    FOR(j, 0, h)
        m = max(m, sz(polyMatr[i][j]) - 1);
int s = 1;
while ((ll)m * s * s < k)
    s *= 2;
int invS = binPow(s, MOD - 2);
vector<vector<VI>> matrices(m + 1, vector<VI>(h, VI(h)));
FOR(l, 0, m + 1)
{
    FOR(i, 0, h)
        FOR(j, 0, h)
            matrices[l][i][j] = evalPoly(polyMatr[i][j], l * s);
}
for (int r = 1; r < s; r *= 2)
{
    auto sh = shiftEvalMatrs(matrices, r * m + 1, sz(matrices) - 1);
    matrices.insert(matrices.end(), all(sh));
    sh = shiftEvalMatrs(matrices, mult(r, invS), sz(matrices));
    FOR(l, 0, sz(matrices))
        matrices[l] = mult(sh[l], matrices[l]);
}
int l = 0;
for (; l + s <= k; l += s)
    u = mult(matrices[l / s], u);
vector<VI> matr(h, VI(h));
for (; l < k; l++)
{
    FOR(i, 0, h)
        FOR(j, 0, h)
            matr[i][j] = evalPoly(polyMatr[i][j], l);
    u = mult(matr, u);
}
return u;
};

PolyMatr polyMatr(d, vector<VI>(d));
FOR(i, 0, d - 1)
    polyMatr[i][i + 1] = p[0];
FOR(i, 0, d)
{
    polyMatr[d - 1][i] = p[d - i];
    for (int& coef : polyMatr[d - 1][i])
        coef = sub(0, coef);
}
PolyMatr denom = {{p[0]}};
a = polyMatrProd(polyMatr, n - d + 1, a);
const VI& x = polyMatrProd(denom, n - d + 1, {1});
return mult(binPow(x[0], MOD - 2), a.back());
};
```

Mathematical analysis and numerical methods  
Taylor series

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + o((x - x_0)^n)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$
$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n + 1)!}$$

Green’s theorem

$$\oint_C (Ldx + Mdy) = \iint_D \left( \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx dy$$

Runge-Kutta 4th Order

$$\frac{dy}{dx} = f(x, y), y(0) = y_0, x_{i+1} - x_i = h$$
$$y_{i+1} = y_i + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)h$$

$$k_1 = f(x_i, y_i) \qquad k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h)$$
$$k_3 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2h) \qquad k_4 = f(x_i + h, y_i + k_3h)$$

List of integrals

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctg \frac{x}{a} + C$$
$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{x + a}{x - a} \right| + C$$
$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{x^2 + a}} = \ln \left| x + \sqrt{x^2 + a} \right| + C$$
$$\int \frac{dx}{\cos^2 x} = \tg x + C$$
$$\int \frac{dx}{\sin^2 x} = -\ctg x + C$$

Simpson’s rule

$n - \text{even number}, h = \frac{b-a}{n}, x_i = a + ih$

$$\int_a^b f(x)dx \approx \frac{h}{3} \left[ f(x_0) + 4 \sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}) + \right. \\ \left. + 2 \sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}) + f(x_n) \right]$$

Vandermonde matrix

$$V = V(x_0, x_1, \dots, x_m) = \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^n \end{bmatrix}$$

$$V_{i,j} = x_i^j, \quad \det(V) = \prod_{0 \leq i < j \leq n} (x_j - x_i).$$

**Hadamard matrix**

$$H_1 = [1], \qquad H_{2^k} = \begin{bmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{bmatrix}$$

$\det(H_n) = \pm n^{\frac{n}{2}}$   
For a matrix  $M$  such that  $|M_{ij}| \leq 1$ , holds  $|\det(M)| \leq n^{n/2}$ .

**Number theory**

**Calculation of  $a^b \pmod m$**   
if  $b \geq \phi(m)$ , then value  $a^b \equiv a^{[b \pmod{\phi(m)}] + \phi(m)} \pmod m$ .

**Generators**

A generator exists only for  $n = 1, 2, 4, p^k, 2p^k$  for odd primes  $p$  and positive integers  $k$ .

$g$  is a generator modulo  $n$  if any number coprime with  $n$  can be represented as  $[g^i \pmod n], 0 \leq i < \phi(n)$ .

To find a generator:

- find  $\phi(n)$  and  $p_1, ..., p_m$  — the prime factors of  $\phi(n)$
- $g$  is generator only if  $g^{\frac{\phi(n)}{p_j}} \not\equiv 1 \pmod n$  for each  $j$
- check  $g = 2, 3, 4, ..., p - 1$

**Wilson’s theorem**

$p$  is prime if and only if  $(p - 1)! \equiv (p - 1) \pmod p$ .

**Quadratic residues**

$q$  is a quadratic residue modulo  $p$  if there exists an integer  $x$  such that  $x^2 \equiv q \pmod p$ . If  $p$  is odd prime then there exist  $\frac{p+1}{2}$  residues (including 0).

**Number theory functions**

$$n = p_1^{\alpha_1} \cdot \dots \cdot p_k^{\alpha_k}$$

$$\phi(n) = \prod p_i^{\alpha_i-1} (p_i - 1)$$
 – the number of coprimes

$$F(n) = \frac{n \cdot \phi(n)}{2}$$
 – the sum of coprimes for  $n > 1$

$$\mu(n) = (-1)^k$$
 if  $\max(\alpha_i) = 1$ , else 0

$$\sigma_k(n) = \sum_{d|n} d^k$$

$$\sigma_0(n) = \prod (\alpha_i + 1)$$

$$\sigma_{k>0}(n) = \prod \frac{p_i^{(\alpha_i+1) \cdot k} - 1}{p_i^k - 1}$$

**Möbius**

$$g(n) = \sum_{d|n} f(d) \iff f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right)$$

$$M(n) = \sum_{k=1}^n \mu(k), \quad \sum_{d=1}^n M\left(\left\lfloor \frac{n}{d} \right\rfloor\right) = 1$$

$$\sum_{d|n} \phi(d) = n, \quad \sum_{d|n} \mu(d) = [n = 1]$$

**Combinatorics**

**Binomials**

$$\sum_{k=0}^n C_n^k = 2^n$$

$$\sum_{k=0}^m C_{n+k}^k = C_{n+m+1}^m$$

$$\sum_{m=0}^n C_m^k = C_{n+1}^{k+1}$$

$$\sum_{k=0}^n (C_n^k)^2 = C_{2n}^n$$

$$\sum_{j=0}^k C_m^j C_{n-m}^{k-j} = C_n^k$$

$$\sum_{j=0}^m C_j^m C_{n-m}^{k-j} = C_{n+1}^{k+1}$$

$$\sum_{k=0}^n C_{n-k}^k = F_{n+1}$$

**Catalan numbers**

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k} = \frac{1}{n+1} C_{2n}^n = C_{2n}^n - C_{2n}^{n-1}$$

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786

**Fibonacci numbers**

$$F_1 = F_2 = 1$$

$$F_n = F_{n-1} + F_{n-2}$$

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$$

$$F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$$

$$\gcd(F_m, F_n) = F_{\gcd(n, m)}$$

$$F_{n+1} F_{n-1} - F_n^2 = (-1)^n$$

$$F_{47} \approx 2.9 \cdot 10^9$$

$$F_{88} \approx 1.1 \cdot 10^{18}$$

**Stirling numbers of the second kind**

$S(n, k)$  – the number of ways to divide  $n$  element into  $k$  non-empty groups.

$$S(n, n) = 1, n \geq 0$$

$$S(n, 0) = 0, n > 0$$

$$S(n, k) = S(n - 1, k - 1) + S(n - 1, k) \cdot k.$$

$$B_n = \sum_{k=0}^n S(n, k)$$
 from  $n = 0$ :

1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, 27644437, 190899322, 1382958545, 10480142147, 82864869804,...

**Generating functions**

$$[x^i](1+x)^n = C_n^i \quad [x^i](1-x)^{-n} = C_{n+i-1}^i$$

$$C_\alpha^n = \frac{\alpha(\alpha-1) \dots (\alpha-n+1)}{n!}$$

$$\prod_{n=1}^\infty (1-x^n) = \sum_{k=-\infty}^\infty (-1)^k x^{\frac{k(3k-1)}{2}}$$
 (pentagonal number theorem)

**Hook length formula**

A standard Young tableau is a filling of the  $n$  cells of the Young diagram with a permutation, such that each row and each column form increasing sequences. The **hook**  $h_\lambda(i, j)$  is number of cells  $(a, b)$  in diagram such that  $a = i$  and  $b \geq j$  or  $a \geq i$  and  $b = j$ .

The number of standard Young tableaux of shape  $\lambda$ :

$$f^\lambda = \frac{n!}{\prod h_\lambda(i, j)}$$

**Burnside’s lemma**

Let  $G$  be a finite group that acts on a set  $X$ .

The *orbit* of an element  $x$  in  $X$  is the set of elements in  $X$  to which  $x$  can be moved by the elements of  $G$ . The orbit of  $x$  is denoted by  $G \cdot x$ :

$$G \cdot x = \{g \cdot x \mid g \in G\}.$$

For each  $g$  in  $G$ , let  $X^g$  denote the set of elements in  $X$  that are fixed by  $g$  (also said to be left invariant by  $g$ ), that is,  $X^g = \{x \in X \mid g \cdot x = x\}$ . Burnside’s lemma asserts the following formula for the number of orbits, denoted  $|X/G|$ :

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

**Graphs**

**Prüfer sequence**

At step  $i$ , remove the leaf with the smallest label and set the  $i$ -th element of the Prüfer sequence to be the label of this leaf’s neighbour. The Prüfer sequence of a labeled tree is unique and has length  $n - 2$ .

The number of spanning trees of  $K_n$  is  $n^{n-2}$ .  
The number of spanning trees of  $K_{L,R}$  number is  $L^{R-1} \cdot R^{L-1}$ .

Let  $T_{n,k}$  be the number of labelled forests on  $n$  vertices with  $k$  connected components, such that vertices  $1, \dots, k$  all belong to different components.  $T_{n,k} = k \cdot n^{n-k-1}$ .

The number of spanning trees in a complete graph  $K_n$  with the fixed degrees  $d_i$  is equal to:  $\frac{(n-2)!}{\prod (d_i-1)}$

For a forest graph with connected components of sizes  $s_0, \dots, s_{k-1}$ , the number of ways to add edges to make a spanning tree is equal to:  $n^{k-2} \cdot \prod s_i$

|   |   |   |   |
|---|---|---|---|
| 7 | 4 | 3 | 1 |
| 5 | 2 | 1 |   |
| 2 |   |   |   |
| 1 |   |   |   |

A tableau listing the hook length of each cell in the Young diagram (4, 3, 1, 1)

### Chromatic polynomial

For a graph  $G$ ,  $\chi(G, \lambda) = \chi(\lambda)$  counts the number of its vertex  $\lambda$ -colorings. There is a unique polynomial  $\chi(\lambda)$ .  
Deletion-contraction:

- The graph  $G/uv$  is obtained by merging  $u$  and  $v$ .
- The graph  $G - uv$  is obtained by deleting the edge  $uv$ .
- $\chi(G, \lambda) = \chi(G - uv, \lambda) - \chi(G/uv, \lambda)$ .

|                    |   |
|--------------------|---|
| $G$ is tree        | $\chi(\lambda) = \lambda(\lambda - 1)^{n-1}$            |
| $G$ is cycle $C_n$ | $\chi(\lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$ |

**Proposition.**  $\chi(\lambda)$  is equal to the number of pairs  $(\sigma, O)$ , where  $\sigma$  is any map  $\sigma : V \rightarrow \{1, \dots, \lambda\}$  and  $O$  is an orientation of  $G$ , subject to the two conditions:

- The orientation  $O$  is acyclic.
- If  $u \rightarrow v$  in  $O$ , then  $\sigma(u) > \sigma(v)$ .

Define  $\bar{\chi}(\lambda)$  to be the number of pairs  $(\sigma, O)$ , where  $\sigma$  is any map  $\sigma : V \rightarrow \{1, \dots, \lambda\}$  and  $O$  is an orientation of  $G$ , subject to the two conditions:

- The orientation  $O$  is acyclic.
- If  $u \rightarrow v$  in  $O$ , then  $\sigma(u) \geq \sigma(v)$ .

**Theorem.** Suppose that  $|V| = n$ . Then for all non-negative integers  $\lambda$  holds:

$$\bar{\chi}(\lambda) = (-1)^n \chi(-\lambda)$$

**Corollary.**  $(-1)^n \chi(G, -1)$  is equal to the number of acyclic orientations of  $G$ .

### Kirchhoff’s theorem

Let  $G$  be a finite graph, allowing multiple edges but not loops.

The laplacian matrix  $L$  of  $G$  is the  $n \times n$  matrix whose  $(i, j)$ -entry  $L_{ij}$  is given by

$$L_{ij} = \begin{cases} -m_{ij}, & \text{if } i \neq j, m_{ij} \text{ edges between } v_i \text{ and } v_j, \\ \deg(v_i), & \text{if } i = j. \end{cases}$$

Let  $L_0$  denote  $L$  with the  $i$ -th row and column removed for any  $i$ . Then for a connected graph,  $\det(L_0)$  equals the number of spanning trees of  $G$ .

### Karp’s minimum mean-weight cycle algorithm

Let  $G = (V, E)$  be a directed graph with weight function  $w : E \rightarrow \mathbb{R}$ , and let  $n = |V|$ . We define the **mean weight** of a cycle  $c = \langle e_1, e_2, \dots, e_k \rangle$  of edges in  $E$  to be

$$\mu(c) = \frac{1}{k} \sum_{i=1}^k w(e_i).$$

Let  $\mu^* = \min_c \mu(c)$ , where  $c$  ranges over all directed cycles in  $G$ . We call a cycle  $c$  for which  $\mu(c) = \mu^*$  a **minimum mean-weight cycle**.

Assume without loss of generality that every vertex  $v \in V$  is reachable from a source vertex  $s \in V$ . Let  $\delta_k(s, v)$  be the weight of a shortest path from  $s$  to  $v$  consisting of *exactly*  $k$  edges. If there is no path from  $s$  to  $v$  with exactly  $k$  edges, then  $\delta_k(s, v) = \infty$ .

$$\mu^* = \min_{v \in V} \max_{0 \leq k \leq n-1} \frac{\delta_n(s, v) - \delta_k(s, v)}{n - k}.$$

This can be computed in time  $O(VE)$ .

### Erdős–Gallai theorem

A sequence of non-negative integers  $d_1 \geq \dots \geq d_n$  can be represented as the degree sequence of a finite simple graph on  $n$  vertices if and only if  $d_1 + \dots + d_n$  is even and  $\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(d_i, k)$  holds for every  $k$  in  $1 \leq k \leq n$ .

### Planar graph properties

For a simple, **connected**, planar graph with  $v$  vertices,  $e$  edges and  $f$  faces, the following simple conditions hold for  $v \geq 3$ :

- Theorem 1.  $e \leq 3 \cdot v - 6$ .
- Theorem 2. If there are no cycles of length 3, then  $e \leq 2 \cdot v - 4$ .
- Theorem 3.  $f \leq 2 \cdot v - 4$ .
- Euler’s formula.  $v - e + f = 2$ .
- Theorem 4.  $3 \cdot f \leq 2 \cdot e$ .
- Theorem 5. The dual graph is also planar.
- Theorem 6. There exists a vertex  $v$  with  $\deg(v) \leq 5$ .

### Dilworth’s theorem

A partially ordered set is a set  $S$  with a relation  $\leq$  on  $S$  satisfying:

1.  $a \leq a$  for all  $a \in S$  (reflexivity);
2. if  $a \leq b$  and  $b \leq a$ , then  $a = b$  (antisymmetry);
3. if  $a \leq b$  and  $b \leq c$ , then  $a \leq c$  (transitivity).

A chain is a subset of a set where each pair of distinct elements is comparable. An antichain is a subset of a set where every pair of elements is incomparable.

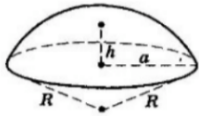
Dilworth’s theorem states that, in any finite partially ordered set, the **largest antichain** has the same size as the **smallest chain decomposition**. Here, the size of the antichain is its number of elements, and the size of the chain decomposition is its number of chains.

### Geometry

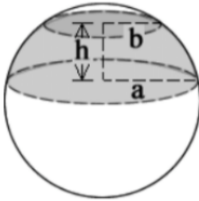
#### Trigonometry formulas

$$\begin{aligned} \sin(v + w) &= \sin v \cos w + \cos v \sin w \\ \sin(v - w) &= \sin v \cos w - \cos v \sin w \\ \tan(v + w) &= \frac{\tan v + \tan w}{1 - \tan v \tan w} \\ \sin v + \sin w &= 2 \sin \frac{v+w}{2} \cos \frac{v-w}{2} \\ \cos v + \cos w &= 2 \cos \frac{v+w}{2} \cos \frac{v-w}{2} \end{aligned}$$

### Ball formulas



$$\begin{aligned} a &= \sqrt{h \cdot (2R - h)} \\ V &= \pi \cdot h^2 \left(R - \frac{h}{3}\right) \end{aligned}$$



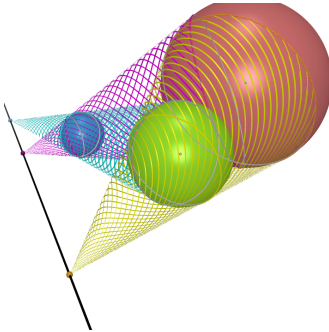
$$\begin{aligned} V &= \frac{1}{6} \pi h (3a^2 + 3b^2 + h^2) \\ R &= \sqrt{\frac{((a-b)^2 + h^2)((a+b)^2 + h^2)}{4h^2}} \end{aligned}$$

### Triangle formulas

$$\begin{aligned} S &= \sqrt{p(p-a)(p-b)(p-c)} = \frac{abc}{4R} \\ m_a^2 &= \frac{2b^2 + 2c^2 - a^2}{4} \text{ (median)} \\ u_a^2 &= \frac{bc((b+c)^2 - a^2)}{(b+c)^2} \text{ (bisector)} \\ \frac{a}{\sin A} &= \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \\ a^2 &= b^2 + c^2 - 2bc \cos A \end{aligned}$$

### Monge’s theorem

There are three circles(balls) of different radii, for each pair of circles find the point of intersection of the external tangents. All three obtained points **lie on a line**. The point from the pair of the largest and the smallest **lies between** the other two.



### Pick’s theorem

Suppose that a polygon has integer coordinates for all of its vertices. Let  $i$  be the number of integer points inside, and let  $b$  be the number of integer points on boundary. Then the area  $S = i + \frac{b}{2} - 1$ .

### Ptolemy’s theorem

For a general quadrilateral  $ABCD$  holds:  
 $AB \cdot CD + AD \cdot BC \geq AC \cdot BD$ .

Equality holds if and only if the quadrilateral is cyclic.

### Euler line

For a general triangle, the orthocenter  $H$ , the centroid  $G$ , and the circumcenter  $O$ , in this order, lie on the same line (Euler line) and  $\frac{|HG|}{|GO|} = \frac{2}{1}$ .

### Fermat point

In a given triangle  $\triangle ABC$  the Fermat point is the point  $X$ , which minimizes the sum of distances from  $A$ ,  $B$ , and  $C$ ,  $|AX| + |BX| + |CX|$ .

If all angles of the triangle are less than  $120^\circ$ , the the Fermat point is the interior point  $X$  from which each side subtends an angle of  $120^\circ$ , i.e.,  $\angle BXC = \angle CXA = \angle AXB = 120^\circ$ .

If any angle of the triangle formed by those points is  $120^\circ$  or more, then the Fermat point is the vertex of that angle.

## Various (7)

linear-basis.hpp 2ff3b8, 45 lines

```
const int MAX_BITS = 64;

struct LinearBasis
{
    bitset<MAX_BITS> basis[MAX_BITS];
    int size;

    LinearBasis()
    {
        size = 0;
    }

    void insert(bitset<MAX_BITS> x)
    {
        for (int i = MAX_BITS - 1; i >= 0; --i)
        {
            if (!x[i]) continue;
            if (basis[i].none())
            {
                basis[i] = x;
                ++size;
                return;
            }
            x ^= basis[i];
        }
    }

    bool canRepresent(bitset<MAX_BITS> x)
    {
        RFOR(i,MAX_BITS,0)
            if (x[i]) x ^= basis[i];
        return x.none();
    }
}
```

### linear-basis

```
bitset<MAX_BITS> getMaxXOR()
{
    bitset<MAX_BITS> res;
    RFOR(i,MAX_BITS,0)
    {
        if ((res ^ basis[i]).to_ullong() > res.to_ullong())
            res ^= basis[i];
    }
    return res;
};
```