Async Multi-Sensor Kalman Filter

0) Model and initial state

- State: $x = [x \quad y \quad V_x \quad V_y]^T$

- State transition for Δt:

$$F(\Delta t) = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Sensors (position only, linear)

↳ $H_{cam} = H_{rad} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

↳ $R_{cam} = \text{diag}(0.25, 0.25)$

$R_{rad} = \text{diag}(1.00, 1.00)$

Initial estimate at $t_0 = 1.000s$:

$$\hat{x}_0 = \begin{bmatrix} 10.0 \\ 5.0 \\ 1.0 \\ 0.0 \end{bmatrix}, \quad P_0 = \text{diag}(0.50, 0.50, 0.20, 0.20)$$

(ie. assume accel noise std $\sigma_a = 1 \, m/s^2$)

Process noise: use $q = 1$ and $\Delta t = 0.1s$

$$Q(\Delta t = 0.1) = \begin{bmatrix} \frac{0.1^4}{4} & 0 & \frac{0.1^3}{2} & 0 \\ 0 & \frac{0.1^4}{4} & 0 & \frac{0.1^3}{2} \\ \frac{0.1^3}{2} & 0 & 0.1^2 & 0 \\ 0 & \frac{0.1^3}{2} & 0 & 0.1^2 \end{bmatrix} = \begin{bmatrix} 0.000025 & 0 & 0.0005 & 0 \\ 0 & 0.000025 & 0 & 0.0005 \\ 0.0005 & 0 & 0.01 & 0 \\ 0 & 0.0005 & 0 & 0.01 \end{bmatrix}$$

Timeline: Radar at $t_1 = 1.100s$ with $z_{rad} = \begin{bmatrix} 10.6 \\ 4.7 \end{bmatrix}$

Camera at $t_2 = 1.200s$ with $z_{cam} = \begin{bmatrix} 10.2 \\ 5.1 \end{bmatrix}$

1) Process Radar @ $t_1 = 1.100s$

1.1 Predict to t_1 ($\Delta t = 0.0100s$)

$$\hat{x}^- = F_{0.1} \cdot \hat{x}_0 = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 5.0 \\ 1.0 \\ 0.0 \end{bmatrix} = \begin{bmatrix} 10.1 \\ 5.0 \\ 1.0 \\ 0.0 \end{bmatrix}$$

Covariance

For each axis [p, v] with

$$P_{0,axis} \begin{bmatrix} 0.50 & 0 \\ 0 & 0.20 \end{bmatrix}, \quad F_{axis} \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}$$

First FPF^T

$$P' = \begin{bmatrix} P_{pp} + 0.1^2 P_{vv} & 0.1 P_{vv} \\ 0.1 P_{vv} & P_{vv} \end{bmatrix} = \begin{bmatrix} 0.50 + 0.01 \cdot 0.20 & 0.1 \cdot 0.20 \\ 0.1 \cdot 0.20 & 0.20 \end{bmatrix}$$

$$= \begin{bmatrix} 0.502 & 0.020 \\ 0.020 & 0.200 \end{bmatrix}$$

Now add $Q_{axis} = \begin{bmatrix} 0.000025 & 0.0005 \\ 0.0005 & 0.01 \end{bmatrix}$

$$P^-_{1,axis} = \begin{bmatrix} 0.502025 & 0.0205 \\ 0.0205 & 0.210 \end{bmatrix}$$

Do for x & y axis → assemble the 4×4 P^-_1 with these blocks on the diagonal

1.2 Gate (NIS) and update

Innovation (positions only):

$$H\hat{x}^-_1 = \begin{bmatrix} 10.1 \\ 5.0 \end{bmatrix}, \quad V_{rad} = z_{rad} - H\hat{x}^-_1 = \begin{bmatrix} 10.6 - 10.1 \\ 4.7 - 5.0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.3 \end{bmatrix}$$

Innovation covariance

$$S_{rad} = HP^-_1 H^T + R_{rad}$$

$$= \text{diag}(0.5025 + 1.00, \; 0.502025 + 1.00)$$

$$= \text{diag}(1.502025, \; 1.502025)$$

$$S^{-1}_{rad} \approx \text{diag}(0.6658, 0.6658) \quad (\text{ie. } \frac{1}{1.502025})$$

NIS:

$NIS = v^T S^{-1} v$

$= (0.5^2 + 0.3^2) \cdot 0.6658$

$= 0.34 \times 0.6658 = 0.226$

$0.226 < 9.21 \Rightarrow$ ACCEPT

(ie: NIS = $(\text{Innovation})^T (\text{Innovation covariance})^{-1} (\text{Innovation})$

$$= \begin{bmatrix} 0.5 & -0.3 \end{bmatrix} \begin{bmatrix} 0.6658 & 0 \\ 0 & 0.6658 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.3 \end{bmatrix} = 0.226)$$

Kalman Gain (per axis, same numbers for x & y)

$$K_{axis} = \begin{bmatrix} k_p \\ k_v \end{bmatrix} = \begin{bmatrix} \frac{P_{pp}}{S} \\ \frac{P_{vp}}{S} \end{bmatrix} = \begin{bmatrix} \frac{0.502025}{1.502025} \\ \frac{0.0205}{1.502025} \end{bmatrix} \approx \begin{bmatrix} 0.334 \\ 0.01364 \end{bmatrix}$$

State update (per axis, apply independently on x & y)

x-axis: $p_x = 10.1 + 0.334(0.5) = 10.267$

$v_x = 1.0 + 0.01364(0.5) = 1.00682$

y-axis: $p_y = 5.0 + 0.334(-0.3) = 4.8998$

$v_y = 0.0 + 0.01364(-0.3) = -0.004092$

$$\Rightarrow \hat{x}_1 = \begin{bmatrix} 10.267 \\ 1.00682 \\ 4.8998 \\ -0.004092 \end{bmatrix}$$

Covariance update

$P_{pp}^{new} = (1 - k_p) P_{pp}^- \approx (1 - 0.334)(0.502025) = 0.334$

$P_{pv}^{new} = (1 - k_p) P_{pv}^- = 0.01365$

$P_{vv}^{new} = P_{vv}^- - k_v (P_{pv}^-) = 0.210 - 0.01364(0.0205)$

$= 0.20972$

Set $t_{state} = 1.100s$