1. Summary

Def: a clothoid is a plane curve with curvature a linear function of arc length.

Problem: We have a clothoid segment of fixed arc length $s = 1$ and angles relative to the chord θ_0 and θ_1 and we wish to understand the curvature of this segment.

$$\kappa(s) = \kappa_0 + \kappa_1 s \text{ and } -\frac{1}{2} \leq s \leq \frac{1}{2}$$

integrating we find

$$\theta(s) = \kappa_0 s + \frac{\kappa_1}{2} s^2$$

$$z(s) = x + iy = \int_0^s e^{i\theta(t)} dt$$

Let's look at the drawing:

$$\psi = \arctan(\frac{y(\frac{1}{2}) - y(-\frac{1}{2})}{x(\frac{1}{2}) - x(-\frac{1}{2})})$$

$$\theta_0 = \psi - \theta(-\frac{1}{2}), \theta_1 = \theta(\frac{1}{2}) - \psi$$

Substituting our expression for θ yields

$$\theta_0 = \frac{\kappa_0}{2} - \frac{\kappa_1}{8} + \psi$$

$$\theta_1 = \frac{\kappa_0}{2} + \frac{\kappa_1}{8} - \psi$$

An expression for κ_0 follows

$$\kappa_0 = \theta_0 + \theta_1$$

We now only need to solve for a single variable κ_1

$$0 = \kappa_1 - 8\psi + 4(\theta_0 - \theta_1)$$

This is our first task i.e. finding a numerical recipe for κ_1

1.1. implementation for fix point iteration.

First consider this as a fixed point iteration for κ_1. Starting with

$$\kappa_1 = 8\psi - 4(\theta_0 - \theta_1). \tag{1.1}$$

Let $F(\kappa_1)$ be defined as follows

$$F(\kappa_1) = 8 \arctan \left(\frac{\mathfrak{Im} \displaystyle\int_{-1/2}^{1/2} e^{\mathrm{i}(\kappa_0 s + \frac{\kappa_1}{2} s^2)} ds}{\mathfrak{Re} \displaystyle\int_{-1/2}^{1/2} e^{\mathrm{i}(\kappa_0 s + \frac{\kappa_1}{2} s^2)} ds} \right) - 4(\theta_0 - \theta_1) \tag{1.2}$$

We can write this fixed point iteration as:

$$\kappa_1^{n+1} = F(\kappa_1^n) \tag{1.3}$$

Using a first order Taylor approximation we have the equation

$$\frac{\kappa_1^3}{5184} + \frac{2\kappa_1}{3} + 4\theta_0 - 4\theta_1 = 0 \tag{1.4}$$

We can use the real root of this as the initial guess. That is saying

$$\kappa_1^0 - F(\kappa_1^0) \approx 0 \tag{1.5}$$

Using the Python code, it seems that the above setup converges very fast. The trapozoid rule gives good convergence results.

1.2. **implementation for netwons method.** Using the above equation we can consider $G(\kappa_1) = \kappa_1 - F(\kappa_1) = 0$ and apply Newton's method. The derivative of $G(\kappa_1)$ is computed as

$$G'(\kappa_1) = 1 - 8\frac{g^2}{f^2+g^2}\left(\frac{f'g - g'f}{g^2}\right) \tag{1.6}$$

Where $f = f(\kappa_1)$ and $g = g(\kappa_1)$ are the following

$$f(\kappa_1) = \mathfrak{Re}\int_{-1/2}^{1/2} e^{\mathrm{i}(\kappa_0 s + \frac{\kappa_1}{2}s^2)}ds \tag{1.7}$$

$$g(\kappa_1) = \mathfrak{Im}\int_{-1/2}^{1/2} e^{\mathrm{i}(\kappa_0 s + \frac{\kappa_1}{2}s^2)}ds \tag{1.8}$$

With derivatives computed to be

$$f'(\kappa_1) = \mathfrak{Re}\int_{-1/2}^{1/2} \mathrm{i}\frac{s^2}{2}e^{\mathrm{i}(\kappa_0 s + \frac{\kappa_1}{2}s^2)}ds \tag{1.9}$$

$$g'(\kappa_1) = \mathfrak{Im}\int_{-1/2}^{1/2} \mathrm{i}\frac{s^2}{2}e^{\mathrm{i}(\kappa_0 s + \frac{\kappa_1}{2}s^2)}ds \tag{1.10}$$

Newtons iteration can then be written as

$$\kappa_1^{n+1} = \kappa_1^n - \frac{G(\kappa_1^n)}{G'(\kappa_1^n)} \tag{1.11}$$

Using the built in functions, this converges faster than the above fixed point iteration.

1.3. **Clothoid Interpolation.** The clothoid primitive is

$$\begin{aligned} z_p(\alpha, s) &= s\ {}_1F_1(1/2, 3/2, \mathrm{i}\alpha s^2/2) \\ T_p(\alpha, s) &= \exp(\mathrm{i}\alpha s^2/2) \\ \kappa(s) &= \alpha s \end{aligned} \tag{1.12}$$

A clothoid segment is a piece of a clothoid primitive with a rotation and a translation:

$$z(s) = T_0 z_p(\alpha, s) + z_0, \qquad s \in [s_i, s_f] \tag{1.13}$$

To find the parameters first we look at the unit length.

$$\kappa_0 = \arg(T_f/T_i) \tag{1.14}$$

κ_1 is obtained with a fixed point iteration, making sure that the total variation of θ is less than 2π. Now we use this information to find our true segment.

$$\lambda = \frac{\|z_f - z_i\|}{\|z_p(\kappa_1, \kappa_0/\kappa_1 + 1/2) - z_p(\kappa_1, \kappa_0/\kappa_1 - 1/2)} \tag{1.15}$$

$$\kappa_i = (\kappa_0 - \kappa_1/2)/\lambda \tag{1.16}$$

$$\kappa_f = (\kappa_0 + \kappa_1/2)/\lambda \tag{1.17}$$

$$\alpha = \kappa_1/\lambda^2 \tag{1.18}$$

$$s_i = \lambda(\kappa_0/\kappa_1 - 1/2) \tag{1.19}$$

$$s_f = \lambda(\kappa_0/\kappa_1 + 1/2) \tag{1.20}$$

$$T_0 = T_i/T_p(\alpha, s_i) \tag{1.21}$$

$$z_0 = z_i - T_0 z_p(\alpha, s_i) \tag{1.22}$$

1.4. G_2 **continuity.** Given a set of points, G_2 continuity requires the curvature be continuous. We will look at minimizing the bending energy this is

$$\int_\gamma \kappa(s)^2 \, ds \tag{1.23}$$

with γ being the interpolating curve. Given a set of points $\{p_i\}, i = 0$ to n, and clothoid segments $C_i, i = 0$ to $n-1$, where C_i is the segment from p_i to p_{i+1}. we can realize this discretely as

$$\sum_{i=0}^{n-1} |C_{i+1}(\kappa_i) - C_i(\kappa_f)|^2 \tag{1.24}$$

where$C_{i+1}(\kappa_i)$ is the initial curvature value for C_{i+1} at p_{i+1} and $C_i(\kappa_f)$ is curvature at the endpoint for C_i. With the above we can start by using CG. This will form a next set of iterations. The outer iterations will consider the tangent the inner will consider the curvature. The inner set is already done with the python code.