Set-membership identification of LTI discrete-time models

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Main ingredients

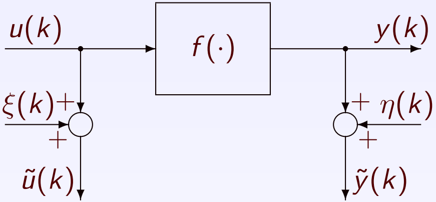
- We consider a discrete-time system described in following parameterized *regression form*

$$y(k) = f(y(k-1), y(k-2), \ldots, y(k-n), u(k), u(k-1), \ldots, u(k-m), \theta) \quad (1)$$

  where $m \leq n$
- A-priori information on the system
  - n and m are known
  - $f \in \mathcal{F}$ where $\mathcal{F}$ is the class of model selected on the basis of our *physical insights*
- A-priori information on the noise
  - the noise structure (i.e. the way the uncertainty affects the input-output data) is known
  - the noise is assumed to belong to a known bounded set $\mathcal{B}$

Noise structures

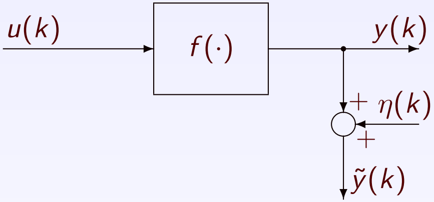
EIV set-up



- *Errors-in-variables* (EIV) problems refer to the most general case where both the input and the output collected samples are affected by noise
- $\xi = [\xi(1), \ldots, \xi(H)] \in \mathcal{B}_\xi$
- $\eta = [\eta(1), \ldots, \eta(H)] \in \mathcal{B}_\eta$
- $\mathcal{B}_\xi$ and $\mathcal{B}_\eta$ are bounded set described by *polynomial constraints*
- Most common case: $\mathcal{B}_\eta = \{\eta : |\eta(k)| \leq \Delta_\eta\}$, $\mathcal{B}_\xi = \{\xi : |\xi(k)| \leq \Delta_\xi\}$

Noise structures

Output-error (OE) set-up



- *Output error* (OE) problems refer to the case where only the output collected samples are affected by noise, while the input is assumed to be exactly known
- $\eta = [\eta(1), \ldots, \eta(H)] \in \mathcal{B}_\eta$
- $\mathcal{B}_\eta$ is a bounded set described by *polynomial constraints*
- Most common case: $\mathcal{B}_\eta = \{\eta : |\eta(k)| \leq \Delta_\eta\}$

Feasible parameter set (FPS)

- In the framework of set-membership (SM) identification, all the parameter values consistent with the a-priori information on the model, the a-priori information on the noise, and the collected input-output data, are considered as feasible solution of the system identification problem
- The set of all such values is called the *feasible parameter set* (FPS) $\mathcal{D}_\theta$
- The FPS for the general EIV problem is implicitly defined as follows

$$
\begin{aligned}
\mathcal{D}_\theta = \Big\{ \theta \in \mathbb{R}^p : \; & y(k) = f(y(k-1), \ldots, y(k-n)), u(k), \ldots, u(k-m)), \\
& k = n+1, \ldots, H, \\
& y(k) = \tilde{y}(k) - \eta(k), \; u(k) = \tilde{u}(k) - \xi(k), \; k = 1, \ldots, H \\
& \mid \xi(k) \mid \leq \Delta\xi(k), \;\; \mid \eta(k) \mid \leq \Delta\eta(k), \; k = 1, \ldots, H \Big\},
\end{aligned}
\qquad (2)
$$

Feasible parameter set (FPS)

LTI EIV discrete-time systems

- The FPS for the case of LTI discrete-time systems with EIV noise structure is given by

$$
\begin{aligned}
\mathcal{D}_\theta = & \Big\{ \theta \in \mathbb{R}^p : (y(k) - \eta(k)) + \sum_{i=1}^{n} a_i \left(y(k-i) - \eta(k-i) \right) = \\
& = \sum_{j=0}^{m} b_j \left(u(k-j) - \xi(k-j) \right), \quad t = n+1, \dots, H, \\
& \mid \xi(k) \mid \leq \Delta\xi(k), \quad \mid \eta(k) \mid \leq \Delta\eta(k), \qquad k = 1, \dots, H \Big\},
\end{aligned}
\tag{3}
$$

- The FPS enjoys the following interesting features:
  - The *true* value of the parameter vector θ is guaranteed to belong to $\mathcal{D}_\theta$
  - $\mathcal{D}_\theta$ implicitly quantify the uncertainty affecting our mathematical model

Parameter Uncertainty Intervals

- The FPS $\mathcal{D}_\theta$ allows us to compute the minimum and the maximum possible values of each single component θ_k of the parameter vector θ
- Such bounds implicitly provide the so called *parameter uncertainty intervals* (PUI) for θ_k formally defined as
$$PUI_k = \left[\underline{\theta}_k;\ \overline{\theta}_k\right] \tag{4}$$
where
$$\underline{\theta}_k = \min_{\theta \in \mathcal{D}_\theta} \theta_k, \qquad \overline{\theta}_k = \max_{\theta \in \mathcal{D}_\theta} \theta_k. \tag{5}$$
- Computation of PUI_k requires to compute the **global optimal solution** of optimization problems in equation (5).

Extended feasible parameter set (EFPS)

LTI EIV discrete-time systems

- The FPS defined in equation (3) depends on some additional unknowns (all the samples of the noise sequences)
- The constraints defining $\mathcal{D}_\theta$ cannot be rearranged in such a way to eliminate dependencies on such additional variables
- To solve problems (5) we need to extend the space of decision variables in by defining the *Extended feasible parameter set* (EFPS)

$$
\begin{aligned}
\mathcal{D}_{\theta,\eta,\xi} =& \Big\{\theta \in \mathbb{R}^p, \eta \in \mathbb{R}^H, \xi \in \mathbb{R}^H : (y(k) - \eta(k)) + \sum_{i=1}^{n} a_i \left(y(k-i) - \eta(k-i)\right) = \\
& = \sum_{j=0}^{m} b_j \left(u(k-j) - \xi(k-j)\right), \quad t = n+1, \dots, H, \\
& \mid \xi(k) \mid \leq \Delta\xi(k), \quad \mid \eta(k) \mid \leq \Delta\eta(k), \quad k = 1, \dots, H\Big\},
\end{aligned} \tag{6}
$$

Extended feasible parameter set (EFPS)

LTI EIV discrete-time systems

- $\mathcal{D}_{\theta,\eta,\xi}$ is the set of all the parameter values and noise samples values that are consistent with the a-priori on assumption on the system, the noise structure and the noise bounds
- $\mathcal{D}_{\theta}$ is the projection on the parameter space $\mathbb{R}^p$ of $\mathcal{D}_{\theta,\eta,\xi}$
- Computation of *PUIs* can be performed on the extended space of parameter and noise samples as follows

$$\underline{\theta}_k = \min_{\theta,\eta,\xi \in \mathcal{D}_{\theta,\eta,\xi}} \theta_k, \qquad \overline{\theta}_k = \max_{\theta,\eta,\xi \in \mathcal{D}_{\theta,\eta,\xi}} \theta_k. \tag{7}$$

- $\mathcal{D}_{\theta,\eta,\xi}$ is a non-convex set defined by polynomial (bilinear) constraints

Convex relaxations for PUIs computation

LTI EIV discrete-time systems

- Computation of parameter bounds (PUIs) require the solution to the global optimum of non-convex polynomial optimization problems
- By applying *Moments theory* (Lasserre) or *Sum-of-squares (SOS) theory* (Chesi, Parrillo) it is possible to build a sequence of convex semidefinite (SDP) optimization whose size/complexity depends on a parameter called *order of relaxation* δ
- For any given δ the following inequalities hold

$$\underline{\theta}_k^{\delta} \leq \underline{\theta}_k, \; \overline{\theta}_k^{\delta} \geq \overline{\theta}_k \tag{8}$$

- Furthermore

$$\lim_{\delta \to \infty} \underline{\theta}_k^{\delta} = \underline{\theta}_k, \; \lim_{\delta \to \infty} \overline{\theta}_k^{\delta} = \overline{\theta}_k \tag{9}$$