Distributed optimal SVFB cooperative control of multi-agents systems

Diego Regruto and Sophie M. Fosson

Politecnico di Torino

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Department of Control and Computer Engineering

DAUIN

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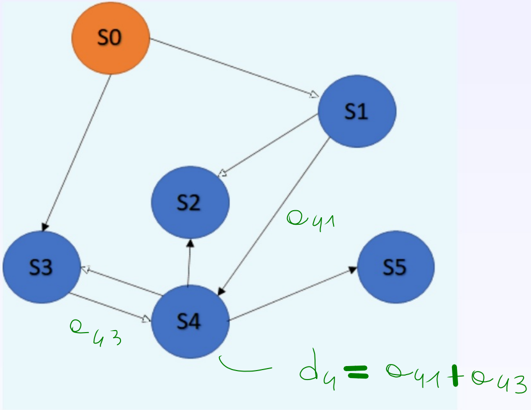
General formulation

- Here we will focus on CPS described as multi-agents systems modeled by means of directed graphs (digraphs)



- Each node S_i of the graph is a dynamical LTI system
- Edges are used to model communication between agents

General formulation

We will consider multi-agents control problems (generically called here *synchronization problems*) where the CPS is made up of

- 1 *leader node* S_0
- N *follower nodes* S_i $(i = 1, 2, \ldots, N)$
- The agents cooperate in order to perform a task possibly dictated (to some extent) by the leader node
- In order to perform the assigned task the follower nodes exploit information shared on the communication network represented by the digraph $\boxed{\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}}$
- Examples: autonomous vehicles platooning, unmanned air vehicles (UAVs) formation control, control of autonomous mobile robots team, ...

General formulation: cooperative tracking problems

- In this part we will mainly focus on *cooperative tracking problem* where the task to be performed is dictated by the leader node S_0 which generates the desired target trajectory (acting as a command generator exosystem)

- We will use the term *cooperative regulator problem* when the consensus is sought among agents in the absence of a leader node

- Assumption: the N follower agent are identical

Reference book

B5 *Cooperative Control of Multi-Agent Systems*, F. Lewis et al., Springer London , 2013

Agents mathematical models

- The dynamics of the **leader node** S_0 is described by

$$\boxed{\dot{x}_0 = Ax_0, \quad y_0 = Cx_0} \tag{1}$$

where $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^p$

- The dynamics of the N identical **follower nodes** S_i is described by

$$\boxed{\dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i} \tag{2}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_{\cancel{0}\,i} \in \mathbb{R}^p$ and $i \in \mathcal{N} = \{1, 2, \dots, N\}$

- The triple (A, B, C) is *stabilizable* and *detectable*

Communication network modeling

- The **follower nodes** S_i share information through a communication network represented as a digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with N nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ and a set of edges (arcs) $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$
- The **adjacency matrix** associated to $\mathcal{G}$ is $\mathcal{A} = [a_{ij}] \in \mathbb{R}^N$
- $a_{ij} > 0$ **is the weight for edge** (v_j, v_i) implying that node i can get information from node j (node j is a neighbor of node i)
- The neighbor set of node i is denoted as $\boxed{\mathcal{N}_i = \{j | a_{ij} > 0\}}$
- We assume there is no self-loop ($a_{ii} = 0, \forall i$)
- To describe connection between the leader node and the graph $\mathcal{G}$ of follower nodes we define the **augmented graph** $\overline{\mathcal{G}} = \{\overline{\mathcal{V}}, \overline{\mathcal{E}}\}$ where $\overline{\mathcal{V}} = \{v_0, v_1, \ldots, v_N\}$, $\overline{\mathcal{E}} \subset \overline{\mathcal{V}} \times \overline{\mathcal{V}}$

Cooperative state variables feedback (SBVF) control

Fundamental assumptions

- All the **state variables** of the agents are assumed to be measurable ($C = I$ in equation (2))
- The **leader node** S_0 can only be observed by a small subset of the follower nodes
- **If node i observes the leader**, node i is said to be **pinned to the leader**. Therefore an edge (v_0, v_i) **exists** in the augmented graph $\overline{\mathcal{G}}$ with **weight** g_i **(pinning gain)**
- **The control law of each agent can only use the local neighborhood information** of that agent, according to the graph topology (fully distributed control scheme)
- There exists at least **one directed path from the leader node to every follower node**.

Cooperative state variables feedback (SBVF) control

Local controller at each node i

Neighborhood tracking error of node i

i

$$\varepsilon_i = \sum_{j=1}^{N} a_{ij}(x_j - x_i) + \underbrace{g_i(x_0 - x_i)}_{} \quad (3)$$

SVFB control protocol for each node i

$$\boxed{u_i = cK\varepsilon_i} \quad (4)$$

$c \in \mathbb{R}$

- coupling gain: $c > 0$
- feedback gain matrix: $K \in \mathbb{R}^{m \times n}$

Cooperative state variables feedback (SBVF) control

Closed-loop control system

Closed-loop system of each node i

identity matrix $N \times N$

$$\dot{x}_i = Ax_i + cBK\left(\sum_{j=1}^{N} a_{ij}(x_j - x_i) + g_i(x_0 - x_i)\right) \quad (5)$$

Global closed-loop system dynamics

$$\dot{x} = (I_N \otimes A - c(L+G) \otimes BK)x + (c(L+G) \otimes BK)\underline{x}_0 \quad (6)$$

- $x = col(x_1, x_2, \ldots, x_N) \in \mathbb{R}^{nN}$, $\underline{x}_0 = col(x_0, x_0, \ldots, x_0) \in \mathbb{R}^{nN}$
- Laplacian matrix of $\mathcal{G}$, $L = [l_{ij}] = D - \mathcal{A}$, $D = diag(d_1, d_2, \ldots, d_N)$, d_i in-degree of node i

adjacency matrix

- Pinning matrix $G = diag(g_1, g_2, \ldots, g_N)$, $\otimes$ is the Kronecker product

- Kronecker product :

$A \otimes B$

$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$A \otimes B = \begin{bmatrix} a_{11} \cdot B & a_{12} \cdot B \\ a_{21} \cdot B & a_{22} \cdot B \end{bmatrix} =$$

$$= \begin{bmatrix} 4 & 8 \\ 1 & 2 \\ -4 & 12 \\ -1 & 3 \end{bmatrix}$$

Cooperative state variables feedback (SBVF) control

Disagreement error δ

Local disagreement error at each node i

i

$$\boxed{\delta_i(t) = x_i(t) - x_0(t)} \tag{7}$$

Global disagreement error and global disagreement error dynamics

$$\delta(t) = x(t) - \underline{x}_0(t) = col(\delta_1, \delta_2, \ldots, \delta_N) \tag{8}$$

$$\dot{\delta} = \dot{x} - \underline{\dot{x}}_0 = A_c \delta \tag{9}$$

$\dot{\delta} = A_c \cdot \delta$

$$\boxed{A_c = I_N \otimes A - c(L + G) \otimes BK} \tag{10}$$

Cooperative state variables feedback (SBVF) control

Objective and solution

Objective of the cooperative tracking problem

The cooperative tracking problem is solved if

$$\boxed{\lim_{t \to \infty} \delta(t) = 0} \tag{11}$$

Cooperative tracking problem solution

The global disagreement error converges to 0 if and only if matrix

$$A_c = I_N \otimes A - c(L + G) \otimes BK \tag{12}$$

is *Hurwitz* (i.e., it has all the eigenvalues with strictly negative real part) !

Cooperative state variables feedback (SBVF) control

Closed-loop CPS eigenvalues

- The following Lemma provides useful insight about the eigenvalues of the closed-loop multi-agents system

Lemma 1 (closed-loop eigenvalues)

$$eig(A_c) = \bigcup_{i=1}^{N} eig(A - c\lambda_i BK) \qquad (13)$$

Laplacian of G

where λ_i, $i = 1, \ldots, N$ are the eigenvalues of the matrix $L + G$

Pinning matrix

- Lemma 1 can be easily proved by analyzing the structure of matrix A_c in (12)

Controller design (I)

The following crucial considerations are directly obtained from Lemma 1:

- The closed-loop dynamics of the whole multi-agents system depends both on the local controller parameters K and c and on the eigenvalues of matrix $L + G$ accounting for the communication network effect

- Stability of the single agent dynamics does not imply stability of the multi-agents systems

$$(A - BK) \text{ Hurwitz} \nRightarrow A_c \text{ Hurwitz} \tag{14}$$

- The coupling parameter c is introduced to cope with the effect of λ_i on the global multi-agent system dynamics

Cooperative state variables feedback (SBVF) control

Controller design (II)

Theorem 1 (Cooperative controller design)

Consider the local distributed control protocols given in equation (4). Design the SVFB control gain K as

$$\boxed{K = R^{-1}B^{T}P} \tag{15}$$

R is a m × m (diagonal) matrix $R > \phi$

where P is the unique positive definite solution of the algebraic Riccati equation (ARE)

$P > \phi$

$$\boxed{A^{T}P + PA + Q - PBR^{-1}B^{T}P = 0} \tag{16}$$

where $Q > \phi$ (and diagonal)

can be solved with the MATLAB command 'care'

Then

$$\boxed{\lim_{t\to\infty} \delta(t) = 0} \tag{17}$$

Q, R user selected

if

$$\boxed{c \geq \frac{1}{2\min_{i\in\mathcal{N}} Re(\lambda_i)}} \tag{18}$$

(as discussed in the next notes)

$\boxed{\text{• Optimal linear quadratic control of LTI systems}}$ (LQR)

linear quadratic regulator

Let's focus on the problem of controlling a single LTI system described by the standard state-space model:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

under the assumption that all the state variables are measured!

$$\Downarrow$$

- Optimal quadratic control problem (LQ)

Compute the control input $u(t)$ which solves the following optimization problem:

$$(1) \qquad u(t) = \arg\min_{u(t)} \left[\frac{1}{2} \cdot \int_{0}^{+\infty} \left(x^T(t) \cdot Q\, x(t) + u^T(t) R\, u(t) \right) dt \right]$$

s.t.

$$\dot{x}(t) = A x(t) + B u(t)$$

The functional in equation (1) is given by the weighted sum of the 2-norm ($\Leftrightarrow$ energy) of the states $x(t)$ and the command inputs $u(t)$

- Why we want to minimize such a functional?

(i) by minimizing the energy of $x(t)$ $\Rightarrow$ $\begin{cases} \bullet \text{ shorter transient} \\ \bullet \text{ damped oscillations} \end{cases}$

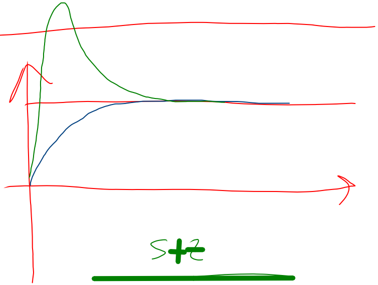
(ii) by minimizing the energy of $u(t)$ $\Rightarrow$ we minimize the energy required from our actuators

Matrices Q and R (which are symmetric matrices) are typically selected as diagonal matrices and $Q > 0$, $R > 0$. The elements on the diagonal of such matrices are selected by the designer!

$$Q = \begin{bmatrix} q_1 & & & & 0 \\ & q_2 & & & \\ & & q_3 & & \\ & & & \ddots & \\ 0 & & & & q_m \end{bmatrix}, \quad R = \begin{bmatrix} r_1 & & & 0 \\ & r_2 & & \\ & & \ddots & \\ 0 & & & r_m \end{bmatrix}$$

$q_i > 0 \quad \forall i$

$r_j > 0 \quad \forall j$



$$\frac{s+z}{(s+p_1)(s+p_2)}$$

$|z| < |p_1|, |p_2|$

Result 1 (solution of the optimal LQ problem)

The optimal solution to problem (1) is given by:

$$\boxed{u(t) = -K \cdot x(t)}$$

where:

$$K = R^{-1} \cdot B^T \cdot P$$

with $P > 0$, symmetric, solution of the algebraic Riccati equation (ARE):

$$A^T P + PA + Q - PBR^{-1} B^T P = 0 \quad (2)$$

Before proving Theorem 1 in the slide, we need to state (review) an important result due to Lyapunov:

$\boxed{\text{Result 2}}$ (Lyapunov stability of a single LTI system)

- Given an LTI system described by the following equation:

$$\dot{x}(t) = A x(t) + B u(t) \quad (3)$$

The system is asymptotically stable (i.e. A is Hurwitz $\equiv$ all eigenvalues of A have strictly negative real part) if and only if, the following Lyapunov equation:

$$A^* P + PA = -Q, \quad Q > 0 \quad (4)$$

has a solution P which is symmetric and $P > 0$.

("$*$" = conjugate transpose)