Agents mathematical modeling

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General problem statement

Here we focus on the problem of deriving the mathematical model of N identical LTI agents $S_i, \; i = 1, 2, \ldots, N$ described as

- Continuous-time LTI systems:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \; y_i(t) = Cx_i(t), \; t \in \mathbb{R}^+ \text{ (state space description)} \quad (1)$$

$$H(s) = C(sI - A)^{-1}B + D \text{ (transfer function description)} \quad (2)$$

- Discrete-time LTI systems:

$$x_i(k+1) = Ax_i(k) + Bu_i(k), \; y_i(k) = Cx_i(k), \; k \in \mathbb{N} \text{ (state space description)} \quad (3)$$

$$H(z) = C(zI - A)^{-1}B + D \text{ (transfer function description)} \quad (4)$$

General problem statement

Which kind of model?

Different approaches can be used to derive the model of a system from available physical insight and/or collected experimental data:

1. First-principle modeling
    - derived by applying fundamental principles of Physics
    - leads to *white-box models*
    - detailed structure of the equations derived from Physics
    - physical parameters values a-priori known and/or derived by suitable (dedicated) experimental procedures

General problem statement

Which kind of model?

Different approaches can be used to derive the model of a system from available physical insight and/or collected experimental data:

2 System identification

- derived from a set of input-output data experimentally collected, by applying suitable mathematical techniques/algorithms
- leads to *black-box models*
- structure of the equations based on some (mild) a-priori information/assumptions
- parameters values provided as output of the identification/estimation procedure
- parameters do not have (in general) a physical meaning

Which kind of model?

Different approaches can be used to derive the model of a system from available physical insight and/or collected experimental data:

3 Mixed approach

- leads to *gray-box models*
- structure of the equations (partially) from Physics
- physical parameters values provided as output of the identification/estimation procedure

System identification

generalities

- Objective: to obtain a mathematical model from input/output experimentally collected data (measurements)
- Collected data (being the output of a measurement procedure) are (typically) affected by uncertainty (measurements noise/errors)
- Model class/structure selected on the basis of a-priori physical insight (e.g., linearity/nonlinearity, (upper bound) on the system order, etc.)
- Since system identification (SysID) is typically performed using input-output data, the most natural description is by means of transfer function
- State-space models can be obtained from transfer functions by applying basic results on *realization theory*

System identification

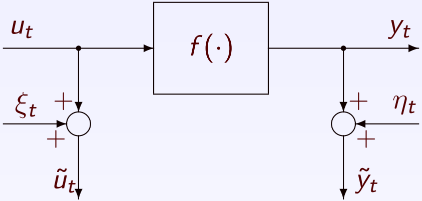
generalities

- Since experimental measurement procedure typically provides *samples* of input-output sequences, we start by considering identification of *discrete-time models*
- procedures for identifying continuous-time models from sampled input-output sequences will also be discussed
- we assume (without loss of generality) that the (nonlinear) system can be modeled by means of the following *regression form*

$$y(k) = f(y(k-1), y(k-2), \ldots, y(k-n), u(k), u(k-1), \ldots, u(k-m)) \quad (5)$$

- n is the *system order*
- $m \leq n$ (always true for physical systems)
- $u(k)$, $y(k)$, $k = 1, 2, \ldots, H$ are the (noise-free) samples of the input and output sequences

General black-box EIV set-up



- *Errors-in-variables* (EIV) problems refer to the most general case where both the input and the output collected samples are affected by noise
- *a-priori assumption on the model*: $f \in \mathcal{F}$ where $\mathcal{F}$ is a given class of functions
- *a-priori assumption on the noise* are available (e.g. statistical distribution, boundedness, etc.)

A naive noiseless example

- Let us consider the following second order LTI discrete-time system (agent):

$$\begin{aligned} y(k) &= f(y(k-1), y(k-2)), u(k), u(k-1), u(k-2)) \\ &= -\theta_1 y(k-1) - \theta_2 y(k-2) + \theta_3 u(k) + \theta_4 u(k-1) + \theta_5 u(k-2) \end{aligned} \tag{6}$$

  where $\theta = [\theta_1, \theta_2, \ldots, \theta_5]$, is the array of parameters to be estimated.
- By introducing the *backward shift operator* $q^{-r}(q^{-r}s(t) = s(k-r)$ we can rewrite the equation as

$$y(k) = -\theta_1 q^{-1} y(k) - \theta_2 q^{-2} y(k) + \theta_3 u(k) + \theta_4 q^{-1} u(k) + \theta_5 q^{-2} u(k) \tag{7}$$

- Solving the equation in the $y(k)$ we obtain:

$$y(k) = \frac{\theta_3 + \theta_4 q^{-1} + \theta_5 q^{-2}}{1 + \theta_1 q^{-1} + \theta_2 q^{-2}} u(k) \tag{8}$$

- By applying properties of the *Z-transform* it is possible to show that the system transfer function $G(z)$ can be obtained by simply replacing q^{-1} with z^{-1}:

$$G(z) \doteq \frac{Y(z)}{U(z)} = \frac{\theta_3 z^2 + \theta_4 z + \theta_5}{z^2 + \theta_1 z + \theta_2} \tag{9}$$

A naive noiseless example: parameter estimation

- Given H experimentally collected input-output samples equation (7) leads to the following system of linear equations:

$$y = A\theta \tag{10}$$

where $y = [y(3)\ y(4)\ \dots\ y(H)]'$, $\theta = [\theta_1\ \theta_2\ \dots\ \theta_5]$, and

$$A = \begin{bmatrix} -y(2) & -y(1) & u(3) & u(2) & u(1) \\ -y(3) & -y(2) & u(4) & u(3) & u(2) \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ -y(H-1) & -y(H-2) & u(H) & u(H-1) & u(H-2) \end{bmatrix} \tag{11}$$

System identification

A naive noiseless example: parameter estimation

- From equations (10) and (11) it appears clear that by collecting $H = 7$ samples we obtain a set of 5 linear equations in 5 unknown

- Assuming that the input sequence $u(k),\ k = 1, 2, \ldots, 7$ is such that A is invertible we can easily compute the parameter θ as:

$$\theta = A^{-1} y \tag{12}$$

- How to guarantee that A is invertible?

  - One possible solution: apply to the system a **random input** sequence u

System identification

A naive noiseless example: the role of the a-priori information on the system

- In the proposed solution we have started from the a-priori assumption that our agent can be modeled as a LTI discrete-time system of order 2
- This information is crucial since the model cannot be built by only using the collected input-output data
- In fact, the same 7 input-output sample can be perfectly fitted by an infinite number of different models depending of 5 unknown parameters
- For example, the model given by:

$$y(k) = \theta_1 u(k) + \theta_2 u(k)^2 + \theta_3 u(k)^3 + \theta_4 u(k)^4 + \theta_5 u(k)^5 \tag{13}$$

provided that the corresponding matrix A is still invertible.

Noise effect

- Let us know assume a more realistic situation where the collected data are corrupted by noise

- How can we attenuate the effect of the noise on the computed (estimated) parameter values?
  - Idea: let's collected more data ($H >> 2n + 1$, $n =$ system order)

- Matrix A in equation (11) is no more square (no more invertible)

- Naive idea: rely on the pseudo-inverse A^* of A

$$\hat{\theta} = (A^T A)^{-1} A^T y \tag{14}$$

where $A^* = (A^T A)^{-1} A^T$

Least-square approach

- The parameter estimate $\hat{\theta}$ computed in equation (14) is, in fact, the so-called *Least-square solution* θ_{LS} of the system of equation in (10) and (11), i.e.

$$\theta_{LS} = \arg\min_{\theta} = \|y - A\theta\|_2 \tag{15}$$

- In the context of system identification (but not only) θ_{LS} as what we refer to as the *Least square estimate* of the system parameter vector.
- Computation burden of the Least square algorithms is quite low also for large H
- The Least square estimate can be (also) computed recursively (i.e. online).

Least square estimate: Consistency property

- The LS estimate enjoy an interesting property if the following assumptions are satisfied:
  1. The effect of the uncertainties corrupting the collected data can be taken into account by introducing an additive term e (called *equation error*) as follows:

$$y(k) = -\theta_1 y(k-1) - \theta_2 y(k-2) + \theta_3 u(k) + \theta_4 u(k-1) + \theta_5 u(k-2) + e(k) \quad (16)$$

  2. $e(k), k = 1, 2, \ldots, H$ are independent and identically distributed (iid) random variables (typically it is assumed that e can be modeled as a white, zero-mean Gaussian noise)

Consistency property

Under assumptions 1 and 2:

$$\lim_{H \to \infty} E[\theta_{LS}] = \theta$$

where $E[\cdot]$ is the *expected value*

Least square estimate: limits of application

- What are the performance of the LS estimator if assumptions 1 and 2 are not satisfied? ($\Rightarrow$ Lab activity 1, next week)

- What can be done if assumptions 1 and 2 are not satisfied? ($\Rightarrow$ **Set-membership identification/estimation**)