# Problem 1

(a) $A = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

matrix A has constant columns sum, so one eigenvalue is equal to its sum equal to 1.

(b) $A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$

$det(A) = 4 - 4 = 0$

Thus 0 is an eigenvalue and matrix is singular. By trace rule $tr(A) = 5$.

$\lambda_1 = 0 \quad \lambda_2 = 5$

(c) $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

Sum of each column is equal to one another, so one of eigenvalues is $\lambda_1 = 5$.

By trace rule $tr(A) = 4$

$\lambda_1 = 5 \quad \lambda_2 = -1$

(d) $A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$ Sum of each row $= 3$ thus $\lambda_1 = 3$

$tr(A) = \lambda_1 + \lambda_2 + \lambda_3 = 0$

$det(A) = \lambda_1 \lambda_2 \lambda_3 = 6$

$\lambda_2 + \lambda_3 = -3$

$\lambda_2 \lambda_3 = 2$

$\lambda_2 = -2 \quad \lambda_3 = -1$

(e) $A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 3 & 0 & 2 \end{pmatrix}$

tr(A) = 2 + 4 + 2 = 8

det(A) = -8

It is easy to see, that if we subtract 4I from A we will get singular matrix, thus we could claim that 4 is an eigenvalue.

$\lambda_1 = 4$

$\lambda_2 + \lambda_3 = 8 - \lambda_1 = 4$

$\lambda_2\lambda_3 = -2$

$\lambda^2 - 4\lambda - 2 = 0$

$D = 16 - 4(-2)1 = 32$

$\lambda_2 = 2 + 2\sqrt{2} \quad \lambda_3 = 2 - 2\sqrt{2}$

# Problem 2

(a) $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$

$\lambda_1 = 4 \quad \lambda_2 = -1$

$B_1 = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}$

$B_2 = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}$

$\left(\begin{array}{cc|c} -2 & 3 & 0 \\ 2 & -3 & 0 \end{array}\right) = -2x_1 + 3x_2 = 0$

$-2x_1 = -3x_2$

$x_1 = \frac{3}{2}x_2$

$x_1 = \frac{3}{2} \quad x_2 = 1$

$\vec{v_1} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$

$\left(\begin{array}{cc|c} 3 & 3 & 0 \\ 2 & 2 & 0 \end{array}\right) = 3x_1 + 3x_2 = 0$

$3x_1 = -3x_2$

$x_1 = -x_2$

$x_1 = -1 \quad x_2 = 1$

$\vec{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(b) $A = \begin{pmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{pmatrix}$

$tr(A) = 6 \quad det(A) = 6$

$\lambda_1 = 3$ because all row sums are equal to 3

$\lambda_2\lambda_3 = \frac{6}{3} = 2$

$\lambda_2 + \lambda_3 = 6 - \lambda_1 = 3$

$\lambda_2 = 2 \quad \lambda_3 = 1$

$$\begin{pmatrix}-2 & 2 & 3\\-2 & 3 & 2\\-4 & 2 & 5\end{pmatrix} - \begin{pmatrix}3 & 0 & 0\\0 & 3 & 0\\0 & 0 & 3\end{pmatrix} = \begin{pmatrix}-5 & 2 & 3\\-2 & 0 & 2\\-4 & 2 & 2\end{pmatrix} => \begin{pmatrix}1 & -\frac{2}{5} & -\frac{3}{5}\\0 & 1 & -1\\0 & 0 & 0\end{pmatrix}$$

$x_2 = x_3 = 1 \quad x_1 = \frac{2}{5} + \frac{3}{5}$

$\vec{v_1} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$$\begin{pmatrix}-2 & 2 & 3\\-2 & 3 & 2\\-4 & 2 & 5\end{pmatrix} - \begin{pmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{pmatrix} = \begin{pmatrix}-4 & 2 & 3\\-2 & 1 & 2\\-4 & 2 & 3\end{pmatrix} => \begin{pmatrix}1 & -\frac{1}{2} & 0\\0 & 0 & 1\\0 & 0 & 0\end{pmatrix}$$

$x_1 = x_2 = \frac{1}{2} \quad x_3 = 0$

$\vec{v_2} = \begin{pmatrix}\frac{1}{2}\\1\\0\end{pmatrix}$

$$\begin{pmatrix}-2 & 2 & 3\\-2 & 3 & 2\\-4 & 2 & 5\end{pmatrix} - \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} = \begin{pmatrix}-3 & 2 & 3\\-2 & 2 & 2\\-4 & 2 & 4\end{pmatrix} => \begin{pmatrix}1 & 0 & -1\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}$$

$x_1 = x_3 = 1 \quad x_2 = 0$

$\vec{v_3} = \begin{pmatrix}1\\0\\1\end{pmatrix}$

$A^2, A^{100}$:

(a) $\lambda_1 = 16 \quad \lambda_2 = 1 \qquad\qquad \lambda_1 = 4^{100} \quad \lambda_2 = 1$

(b) $\lambda_1 = 9 \quad \lambda_2 = 4 \quad \lambda_3 = 1 \qquad\qquad \lambda_1 = 3^{100} \quad \lambda_2 = 2^{100} \quad \lambda_3 = 1$

$Av = \lambda v$

$A^2 v = \lambda^2 v$

So, with increasing of power eigenvalue would change accordingly while eigenvectors would stay the same

$A^{-1}$: $A^{-1}Av = A^{-1}\lambda v$

$v = A^{-1}\lambda v$

$A^{-1}v = \frac{1}{\lambda}v$

So, as expected, eigenvalues will change, while eigenvectors would stay the same.

$e^{tA}$:

$Av = \lambda v$

$e^{tA}v = e^{t\lambda}v$

# Problem 3

If A $n$ x $n$ matrix has $n$ distinct eigenvalues, than $D$ is equal to matrix with eigenvalues of $A$ on its main diagonal.

(a) $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$

$\lambda_1 = 4 \quad \lambda_2 = -1$

$\vec{v_1} = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad \vec{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$

Knowing that $AP = PD$ lets check our results.

$P = \begin{pmatrix} \frac{3}{2} & -1 \\ 1 & 1 \end{pmatrix}$

$AP = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 4 & -1 \end{pmatrix}$

$PD = \begin{pmatrix} \frac{3}{2} & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 4 & -1 \end{pmatrix}$

$D$ is correctly found.

(b) $A = \begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix}$

$det(A) = \lambda_1\lambda_2 = -4 - 5 = -9$

$tr(A) = \lambda_1 + \lambda_2 = 0$

$\lambda_1 = 3 \quad \lambda_2 = -3$

$D = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} -\frac{1}{5} \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} 1 & -\frac{1}{5} \\ 1 & 1 \end{pmatrix}$

$AP = \begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{5} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & \frac{3}{5} \\ 3 & -3 \end{pmatrix}$

$PD = \begin{pmatrix} 1 & -\frac{1}{5} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 3 & \frac{3}{5} \\ 3 & -3 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$

$\lambda_1 = 4 \quad \lambda_2 = -1$

(d) $A = \begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix}$

Sums of rows $= 3$, thus $\lambda_1 = 3$

$tr(A) = 6$

$\lambda_2 = 3$

$P = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$P$ doesn't form a basis. $A$ is not diagonalizable.

# Problem 4

$A = \begin{pmatrix} 3 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix}$

$\lambda_{1,2,3} = 3$

Lets find eigenvectors

$A = \begin{pmatrix} 3 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

After row reduction we have:
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

This means, that our eigenvectors are in the form $v_{1,2} = \begin{pmatrix} a \\ 0 \\ b \end{pmatrix}$

But in order to form matrix P we must complete the basis, so we need to find generalized eigenvector $(A - \lambda I)v_k = v_{k-1}$
$\begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ b \end{pmatrix}$
So, as we could see, we need to find such a vector, that contains the $y$ value non zero (to increase rank). The simplest vector possible in this case is:
$v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
$v_2 = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$
$v_1 = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
With such a vector $v_1$ we aren't able to form a basis. So, lets change it (we are allowed to do this, as far as we are restricted only by form of a vector $\begin{pmatrix} a \\ 0 \\ b \end{pmatrix}$).

We will take $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ in order to eliminate $D_{13}$ entry.

Thus we have $P = \begin{pmatrix} 2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$P^{-1}AP = \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

# Problem 5

(a) $\vec{x_n} = A\vec{x_{n-1}}$

$\vec{x_n} = PD^nP^{-1}\vec{x_0} = A^n\vec{x_0}$

$$\vec{x_n} = \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} 2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3^n \\ 3^n \\ 3^n \end{pmatrix}$$

# Problem 7

(a) In order to have the same eigenvalues, $AB$ and $BA$ must have the same characteristic polynomial, means $AB$ and $BA$ must be similar.

If A is invertible then $A^{-1}(AB)A = BA$, so $AB$ and $BA$ are similar.

(b) $tr(AB) = tr(BA)$

$det(AB) = det(BA)$

So, that means they have the same characteristic polynomial, thus their eigenvalues are the same.

# Problem 8

(a) $A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP = PD^2P^{-1}$

$A^k = PD^kP^{-1}$

(b) Involutory matrix is the matrix that is equal to its own inverse. $n$ x $n$ matrix is diagonalizable $\iff$ it has n distinct eigenvalues. As far as entries of such a matrix are only 1 and -1 than the eigenvalues will be $\in$\{1, -1\}.

So $A^2 = P^{-1}D^2P = P^{-1}IP = P^{-1}P = I$

$A^2 = I$

# Problem 9

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$B^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$det(B) = -1 \quad tr(B) = 1$

$\lambda_1 = -1 \quad \lambda_2 = 1 \quad \lambda_3 = 1$

$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Lets convert it to RREF.

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so $\vec{v_1} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$A_2 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$, so $\vec{v_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

After all, we need generalized vector $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$P = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$P^{-1}BP = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$P^{-1}BP$ has diagonal form.

# Problem 11

$g_n = 3g_{n-1} - g_{n-2} - g_{n-3}$

Having $x_n = \begin{pmatrix} g_{n-2} \\ g_{n-1} \\ g_n \end{pmatrix}$

$$Ax_n = \begin{pmatrix} g_{n-2} \\ g_{n-1} \\ 3g_{n-1} - g_{n-2} - g_{n-3} \end{pmatrix} = x_{n+1}$$

Thus we could write $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 3 \end{pmatrix} = x_{n+1}$

As we have sums of each row equal, than 1 is an eigenvalue.

$det(A) = -1 \quad tr(A) = 3$, thus $\lambda_2\lambda_3 = -1 \quad \lambda_2 + \lambda_3 = 2$

$\lambda_2 = 1 + \sqrt{2}\ \lambda_3 = 1 - \sqrt{2}$

As far as $A$ has 3 distinct eigenvalues we could determine matrix $D$.

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\sqrt{2} & 0 \\ 0 & 0 & 1-\sqrt{2} \end{pmatrix}$$

Skipping eigenvectors calculation...

$$P = \begin{pmatrix} 1 & 1 & \sqrt{17} \\ 1 & \sqrt{2}-1 & -\sqrt{2}-1 \\ 1 & 1 & 1 \end{pmatrix}$$

# Problem 12

$A(x_1, x_2, x_3) = (x_3, x_1, x_2)$

Transformation a is a simple permutation matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$tr(A) = 0 \quad det(A) = 1$ Having same sums of rows and columns we could say that $\lambda_1 = 1$

**Eigenvalues**:

$\lambda_1 = 1 \quad \lambda_2 = \frac{1}{2}(-1 - i\sqrt{3}) \quad \lambda_3 = \frac{1}{2}(-1 + i\sqrt{3})$

**Eigenvectors**:

$$\vec{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{v_2} = \begin{pmatrix} \frac{1}{2}(-1 + i\sqrt{3}) \\ -1 + \frac{1}{2}(1 - i\sqrt{3}) \\ 1 \end{pmatrix} \quad \vec{v_3} = \begin{pmatrix} \frac{1}{2}(-1 - i\sqrt{3}) \\ -1 + \frac{1}{2}(1 + i\sqrt{3}) \\ 1 \end{pmatrix}$$

# Problem 14

(a) $det(A - \lambda I) = 0 \quad det(B - \lambda I) = 0$

$det(B - \lambda I) = det[P^{-1}(A - \lambda I)P] = det(P^{-1})det(A - \lambda I)det(P)$

Since $det(P^{-1})det(P) = det(P^{-1}P) = det(I) = 1$

$det(B - \lambda I) = det(A - \lambda I)$

(b) From assumption we know that $A$ and $D$ are similar, this means they share the same characteristic polynomial. Since triangular matrix has eigenvalues on its main diagonal, we could say than $D$, diagonal matrix, consists only of eigenvalues of $A$.

(c) **Determinant**:

If $A$ is $n$ x $n$ matrix and it is diagonalziable, than it has $n$ distinct eigenvalues.

$det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\dots(\lambda_n - \lambda)$

The formula above holds for every $\lambda$, so assume $\lambda = 0$, thus $det(A) = \lambda_1\lambda_2\dots\lambda_n$

**Trace**:

In matrix $D$ that is the matrix consisting of all the eigenvalues of matrix $A$ the $tr(D)$ is the sum of all eigenvalues.

$A = P^{-1}DP$

$D = tr(P^{-1}AP) = tr((AP)) = tr((AP)P^{-1}) = tr(A)$

(d) **Product**:

The formula holds as is.

**Sum**: Using the fact that Jordan form of matrix $A$ and matrix $A$ itself share the same characteristic polynomial.

$0 = \prod_{i=1}^{n}(\lambda - \lambda_i) = \lambda^n - \lambda^{n-1}\sum_{i=1}^{n}\lambda_i + \dots + (-1)^n\prod_{i=1}^{n}\lambda_i$

There are $n$ terms containing a power $\lambda_{n-1}$ in the determinant expansion: $A_{11}\lambda_{n1}\dots A_{11}\lambda_{n-1}$. Collecting these terms, we get that the coefficient associated with $\lambda_{n-1}$ in the characteristic polynomial. We get $trace(A) = \sum_{i=1}^{n}\lambda_i$.