Problem 2

a) $\left(\begin{array}{cc|c} 1 & k & 4 \\ 3 & 6 & 8 \end{array}\right) = \left(\begin{array}{cc|c} 1 & k & 4 \\ 0 & 6-3k & -4 \end{array}\right)$

$\begin{cases} x_1 + kx_2 = 4 \\ (6-3k)x_2 = -4 \end{cases}$ $\begin{cases} x_1 = 4 + \dfrac{4k}{6-3k} \\ x_2 = -\dfrac{4}{6-3k} \end{cases}$

For all values of k, except $k = 2$.

b) $\left(\begin{array}{cc|c} 1 & 4 & -2 \\ 3 & k & -6 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 4 & -2 \\ 0 & -12 & 0 \end{array}\right)$ For any k

c) $\left(\begin{array}{cc|c} -4 & 12 & k \\ 2 & -6 & -3 \end{array}\right) = \left(\begin{array}{cc|c} 2 & -6 & -3 \\ -4 & 12 & k \end{array}\right) = \left(\begin{array}{cc|c} 2 & -6 & -3 \\ 0 & 0 & -6k \end{array}\right)$

Inconsistent for any k.

Problem 2

$$\left(\begin{array}{ccc|c} a & a & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right) = \left(\begin{array}{ccc|c} a & a & 4 & 4 \\ a & 0 & b & 2 \\ 0 & a & 2 & b \end{array}\right) = \left(\begin{array}{ccc|c} a & a & 4 & 4 \\ 0 & a & 2 & b \\ a & 0 & b & 2 \end{array}\right) =$$

$$= \left(\begin{array}{ccc|c} a & a & 4 & 4 \\ 0 & a & 2 & b \\ 0 & -a & b-4 & -2 \end{array}\right) = \left(\begin{array}{ccc|c} a & a & 4 & 4 \\ 0 & a & 2 & b \\ 0 & 0 & b-2 & b-2 \end{array}\right)$$

if $b=2$ and $a=0$

$$\left(\begin{array}{ccc|c} 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & b \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{cases} 4x_3 = 4 \\ 2x_3 = b \end{cases} \quad x_3 = 1 \Rightarrow \begin{cases} x_1 = C \\ x_2 = C \\ x_3 = 1 \end{cases}$$

we obtain 2 parameter solution set

if $b=2$ an $a \neq 0$

$$\left(\begin{array}{ccc|c} a & a & 4 & 4 \\ 0 & a & 2 & b \\ 0 & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} a & 0 & 2 & 4-b \\ 0 & a & 2 & b \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{cases} x_1 = \dfrac{-b-2x_3+4}{a} \\ x_2 = \dfrac{b-2x_3}{a} \\ x_3 = C \end{cases}$$

we obtain 1 parameter solution set

if $b \neq 2$

$$\left(\begin{array}{ccc|c} a & a & 4 & 4 \\ 0 & a & 2 & b \\ 0 & 0 & b-2 & b-2 \end{array}\right) = \left(\begin{array}{ccc|c} a & a & 4 & 4 \\ 0 & a & 2 & b \\ 0 & 0 & 1 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} a & a & 0 & 0 \\ 0 & a & 0 & b-2 \\ 0 & 0 & 1 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} a & 0 & 0 & -b+2 \\ 0 & a & 0 & b-2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$a = 0$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & -b+2 \\ 0 & 0 & 0 & b-2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

we obtain system with no solutions

$a \neq 0$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \dfrac{-b+2}{a} \\ 0 & 1 & 0 & \dfrac{b-2}{a} \\ 0 & 0 & 1 & 1 \end{array}\right)$$

unique solution

Problem 3

$m = n = 3$

a) no solution

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1^* \\ 0 & 1 & 0 & b_2^* \\ 0 & 0 & 0 & b_3^* \end{array}\right) \quad b_3^* \neq 0$$

b) exactly one solution

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1^* \\ 0 & 1 & 0 & b_2^* \\ 0 & 0 & 1 & b_3^* \end{array}\right)$$

c) infinitely many solutions

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1^* \\ 0 & 1 & 1 & b_2^* \\ 1 & 0 & 0 & b_3^* \end{array}\right) \quad b_1^* = b_3^*$$

$m = 3 \quad n = 2$

$$\left(\begin{array}{cc|c} 1 & 0 & b_1^* \\ 0 & 1 & b_2^* \\ 0 & 0 & b_3^* \end{array}\right) \quad b_3^* \neq 0$$

$$\left(\begin{array}{cc|c} 1 & 0 & b_1^* \\ 0 & 1 & b_2^* \\ 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 0 & b_1^* \\ 1 & 0 & b_2^* \\ 0 & 0 & 0 \end{array}\right)$$

$m = 2 \quad n = 3$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & b_1^* \\ 0 & 0 & 0 & b_2^* \end{array}\right) \quad b_2^* \neq 0$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1^* \\ 0 & 0 & 1 & b_2^* \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1^* \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Problem 4

a) $\begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 7 \end{pmatrix}$ unique solution $\longrightarrow$ unique for generic rhs

b) $\begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ 0 & -7 & -6 \end{pmatrix}$ infinitely many solutions $\longrightarrow$ infinitely many solution for generic RHS

c) $\begin{pmatrix} 2 & 1 \\ 1 & 4 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & -7 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 3 \\ 0 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & -3 \\ 0 & -1 \end{pmatrix} =$

$= \begin{pmatrix} 1 & 4 \\ 0 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$ unique solution for homogeneous system.

For generic RHS no solutions.

d) $\begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 4 \end{pmatrix}$ unique solution

Unique solution for generic RHS.

Problem 5

First of all we can be sure that n+1 vectors are linearly dependent from the fact that when obtained echelon form we will see that one non-pivot column is presented. Thus we are free to throw it away because it is just redundant.

We could also conclude this fact from thinking about basis. If n vectors formed a basis and spanned $\mathbb{R}^n$, then n+1 vector could be obtained as a linear combination of n vectors.

Problem 6

Determinant shows us what value of parallelepiped we have. If volume is equal to 0, than the matrix is singular and do not span $\mathbb{R}^3$. So we could possibly have a plane or a line to be spanned by matrix, formed from vectors.

a) $A = \begin{vmatrix} 1 & 2 & 6 \\ 2 & -3 & k \\ -1 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 6 \\ 0 & -7 & k-12 \\ -1 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 6 \\ 0 & -7 & k-12 \\ 0 & 7 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 6 \\ 0 & -7 & k-12 \\ 0 & 0 & 7+k(-12) \end{vmatrix}$

$\det(A) = abc = tr(A)$

$$1(-7)(7+(k-12)) = 0$$

$$-7(k-5) = 0$$

$$-7k = -35$$

$$k = 5$$

b) $A = \begin{vmatrix} -1 & 2 & k \\ 2 & -4 & 3 \\ 1 & -2 & -3 \end{vmatrix}$ it could be seen straight forward that to make A linearly dependent $k = 3$ is fine.

But overall solution here is any k because first and second columns are already linearly dependent.

Problem 7

a) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $A \neq O_n$

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$$

if we set c or b as only non-zero entry, we will obtain $A \neq 0$ and $A^2 = O_n$.

$$\begin{array}{cc} A & A^2 \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{cc} A & A^2 \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rightarrow & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

b is the only non-zero entry c is the only non-zero entry

This assumption could be used for $n \times n$ matrix:

- leftmost bottom entry has to be the only non-zero entry
- rightmost top entry has to be the only non-zero entry

b) if we set a or d as only non-zero entry, we will obtain $A \neq O_n\ I_n$ and $A^2 = A$.

$$\begin{array}{cc} A & A^2 \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{cc} A & A^2 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

a is the only non-zero entry d is the only non-zero entry

This assumption could be used for $n \times n$ matrix:

- leftmost top entry has to be the only non-zero entry
- rightmost bottom entry has to be the only non-zero entry

Problem 7 (continue)

c) if we set whole anti-diagonal of matrix to non-zero entries we will obtain $A \neq I_n$ and $A^2 \neq I_n$.

$$A \qquad\qquad A^2$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This assumption could be used for $n \times n$ matrix. All anti-diagonal matrixes squared will give us the same results described above.

d) $A \qquad B \qquad AB \qquad BA$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}$$

Each part of sums laying on anti-diagonal are unique, so to get $A \neq O_n$, $B \neq O_n$, $AB \neq O_n$, $BA = O_n$ we have to choose any part of sum and set its entries to non-zero values. Other entries set to 0.

Example:

$bh \neq 0 \qquad a = c = d = 0$

$h \neq 0 \qquad c = f = g = 0$

$b \neq 0$

$$\begin{array}{cc} A & B \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{cc} AB & BA \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

This assumption could be used for $n \times n$ case. Just find the combination of sum laying on anti-diagonal and set its entries to non-zero values. Other entries set to zeros.

Problem 8

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 2(4-2)-(2-1)+(1-2)=4$$ Volume of parallelepiped.

$$\begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = i(1-2)-j(2-1)+k(4-1) = -j-j+3k = \sqrt{(-1)^2+(-1)^2+3^2} = \sqrt{11}$$

$$\begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = i(2-1)-j(4-1)+k(2-1) = i-3j+k = \sqrt{1^2+(-3)^2+1^2} = \sqrt{11}$$

$$\begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = i(4-1)-j(2-1)+k(1-2) = \sqrt{9+1+1} = \sqrt{11}$$

Answer: Volume is equal to 4. All faces are equal to $\sqrt{11}$.

Problem 9

$$A = \begin{pmatrix} row\ 1 \\ row\ 2 \\ row\ 3 \end{pmatrix} \quad det(A) = 5 \quad B = \begin{pmatrix} row1 + row2 \\ row2 + row3 \\ row3 + row1 \end{pmatrix} \quad C = \begin{pmatrix} row1 - row2 \\ row2 - row3 \\ row3 - row1 \end{pmatrix}$$

$det(B) - ?$ $\quad det(C) - ?$

$$\begin{pmatrix} row\ 1 \\ row\ 2 \\ row\ 3 \end{pmatrix} \longrightarrow \begin{pmatrix} row\ 2 \\ row\ 1 \\ row\ 3 \end{pmatrix} \longrightarrow \begin{pmatrix} row\ 2 \\ row\ 3 \\ row\ 1 \end{pmatrix}$$

$det(A) = 5 \quad det(A') = -5 \quad det(A'') = 5$

If A is a triangular matrix, then $det\ A$ is the product of the entries on the main diagonal of A.

$$A = \begin{pmatrix} row\ 1 \\ row\ 2 \\ row\ 3 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

Simplified form of A to show the idea

$det(A) = a + b + c$

$$B = \begin{pmatrix} row\ 1 + row\ 2 \\ row\ 2 + row\ 3 \\ row\ 3 + row\ 1 \end{pmatrix} = \begin{pmatrix} a+b & 0 & 0 \\ 0 & b+c & 0 \\ 0 & 0 & c+a \end{pmatrix}$$

$det(B) = (a+b) + (b+c) + (c+a)$
$= 2a + 2b + 2c = 10$

$$C = \begin{pmatrix} row\ 1 - row\ 2 \\ row2 - row3 \\ row3 - row1 \end{pmatrix} = \begin{pmatrix} a-b & 0 & 0 \\ 0 & b-c & 0 \\ 0 & 0 & c-a \end{pmatrix}$$

$det(C) = (a-b) + (b-c) + (c-a) =$
$= (a-a) + (b-b) + (c-c) = 0$

Answer: $det(B) = 10$
$det(C) = 0$

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Problem 10

$$A - \lambda I = \begin{pmatrix} a-\lambda & b & c & d \\ a & b-\lambda & c & d \\ a & b & c-\lambda & d \\ a & b & c & d-\lambda \end{pmatrix}$$

Rank of matrix A is equal to 1, thus it has only one eigenvalue that is non-zero. Thus:

$$\begin{pmatrix} a-\lambda & b & c & d \\ a & b & c & d \\ a & b & c & d \\ a & b & c & d \end{pmatrix}$$

$$(a-\lambda) + b + c + d = 0$$

$$a + b + c + d = \lambda$$

The matrix below is singular for $\lambda = a+b+c+d$.

Problem 11

a) Rank shows us number of independent columns in matrix.

$rank(A) = \dim A \Rightarrow A$ spans $R^{rank(A)}$

$rank(B) = \dim B \Rightarrow B$ spans $R^{rank(B)}$

By multiplying two matrixes we could possibly get matrix that could span the space $R^{\min(Rank(A), rank(B))}$, because other columns of this matrix are just a linear combinations of basis cols., so $rank(AB) \leq rank(A)$.

The same approach could be applied to $rank(AB) \leq rank(B)$,

b) $A = (m \times n)$ $B = (n \times m)$

$AB = (m \times m)$

assume $m > n$

$BA = (n \times n)$

Rank of A and B could be only less or equal than n.

From explanation in part a we can state that maximum space size spanned by AB or BA is less or equal to n.

$R^{\min(rank(A), rank(B))}$

As far as $m > n$, we could state that AB is singular because maximum rank possible is n.

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Problem 12

$$tr(AB) = tr(BA)$$

$$tr(A \pm B) = tr(A) \pm tr(B)$$

$$tr(I_n) = n$$

$$AB - BA = I_n$$

$$tr(AB - BA) = tr(I_n)$$

$$tr(AB) - tr(BA) = tr(I_n)$$

$$0 \neq n$$

Answer: there is no $n \times n$ matrixes A and B such that $AB - BA = I_n$.

Problem 13

a) $\begin{vmatrix} c & 1 & 3 \\ 1 & -1 & 4 \\ 1 & 2 & -1 \end{vmatrix} = c(1-8)-(-1-6)+(4+3) = 7c+7+7 = 7c+14$

$7c = -14$

$c = -2$

For all real numbers, except -2 form basis in $\mathbb{R}^3$.

b) $\begin{pmatrix} c & 1 & -2 \\ 1 & -1 & 2 \\ 1 & 2 & -4 \end{pmatrix}$ Second and third columns are linearly dependent. Independently from value c these vectors cant form basis in $\mathbb{R}^3$.

c) $\begin{pmatrix} c & 1 & 0 & 3 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & -1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 3 \\ 0 & 2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 5 \end{pmatrix} \Rightarrow$ 3 pivot columns. Thus idependently from forth vector and value of c the basis for $\mathbb{R}^3$ is set.

d) $\begin{pmatrix} c & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \Rightarrow$ these two vectors have not enough info to form basis in R_3 independently from value of c.

Problem 14

$B = \{v_1, v_2, v_3\}$ $B' = \{v_1', v_2', v_3'\}$ for $\mathbb{R}^3$

$$v_1 = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix} \quad v_2 \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix}$$

$$v_1' = \begin{pmatrix} -6 \\ -6 \\ 0 \end{pmatrix} \quad v_2' = \begin{pmatrix} -2 \\ -6 \\ 4 \end{pmatrix} \quad v_3' = \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix}$$

a) To find transition matrix we need to solve $Bx = B'$

$Bx = B'$

$x = B'B^{-1}$

$$B^{-1} = \begin{pmatrix} -\frac{1}{12} & -\frac{1}{12} & -\frac{5}{12} \\ -\frac{3}{8} & \frac{1}{8} & \frac{3}{8} \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{pmatrix} \qquad T(B \to B') = \begin{pmatrix} 0 & -\frac{4}{3} & -\frac{17}{6} \\ \frac{3}{2} & \frac{3}{2} & 3 \\ -\frac{3}{2} & -\frac{3}{2} & -\frac{3}{2} \end{pmatrix}$$

b) Coordinate vector $(u)_B$ for $u = (-5; 8; -5)^T$ can be found by solving equation $Bx = u$

$x = B^{-1}u$

$$x = \begin{pmatrix} -\frac{1}{12} & -\frac{1}{12} & -\frac{5}{12} \\ -\frac{3}{8} & \frac{1}{8} & \frac{3}{8} \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{pmatrix} \begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Coordinate vector $(u)_B$

Problem III (continue)

c)

$$x = (BT)x = u$$

$$x = (BT)^{-1} u$$

$$x = \begin{pmatrix} -6 & -2 & -2 \\ -6 & -6 & -3 \\ 0 & 4 & 7 \end{pmatrix} \begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 19 \\ -43 \\ 16 \end{pmatrix}$$

d) $x = (B')^{-1} u$

$$(B')^{-1} = \begin{pmatrix} -\frac{5}{24} & \frac{1}{24} & -\frac{1}{24} \\ \frac{8}{24} & -\frac{7}{24} & -\frac{1}{24} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

$$x = \frac{1}{12} \begin{pmatrix} 19 \\ -43 \\ 16 \end{pmatrix}$$