Problem 1

(a) $A = \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}$

$det(A) = \begin{bmatrix} 6-\lambda & -2 \\ -2 & 3-\lambda \end{bmatrix} = (6-\lambda)(3-\lambda) - 4 = \lambda^2 - 9\lambda + 14$

$\lambda_1 = 2 \quad \lambda_2 = 7$

$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} => \left(\begin{array}{cc|c} 4 & -2 & 0 \\ -2 & 1 & 0 \end{array}\right) => \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \vec{v_1} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$

$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} => \left(\begin{array}{cc|c} -1 & -2 & 0 \\ -2 & -4 & 0 \end{array}\right) => \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \vec{v_2} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} \frac{1}{2} & -2 \\ 1 & 1 \end{pmatrix}$

$P^{-1} = \begin{pmatrix} \frac{2}{5} & \frac{4}{5} \\ \frac{-2}{5} & \frac{1}{5} \end{pmatrix}$

$P^{-1}AP = \begin{pmatrix} \frac{2}{5} & \frac{4}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix} \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} \begin{pmatrix} \frac{1}{2} & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$

(b) $A = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 6 \end{pmatrix}$

$det(A) = (3-\lambda) \begin{bmatrix} 3-\lambda & 2 \\ 2 & 6-\lambda \end{bmatrix} = (3-\lambda)((3-\lambda)(6-\lambda) - 4) = (3-\lambda)(\lambda^2 - 9\lambda + 14)$

$\lambda_1 = 3 \quad \lambda_2 = 2 \quad \lambda_3 = 7$

$\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 3 \end{pmatrix} => \left(\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{array}\right) \quad \vec{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 4 \end{pmatrix} => \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \vec{v_2} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} -4 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & -1 \end{pmatrix} => \left(\begin{array}{ccc|c} -4 & 0 & 2 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \vec{v_3} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$

$$P = \begin{pmatrix} 0 & -2 & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{-2}{5} & 0 & \frac{1}{5} \\ \frac{2}{5} & 0 & \frac{4}{5} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

(c) $A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 5 \end{pmatrix}$

$det(A) = (3-\lambda) \begin{bmatrix} 4-\lambda & 1 \\ 1 & 5-\lambda \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & 5-\lambda \end{bmatrix} = (3-\lambda)((4-\lambda)(5-\lambda)-1)+(-5+\lambda) = (\lambda-4)(\lambda^2-8\lambda+13)$

$\lambda_1 = 4 \quad \lambda_2 = 4+\sqrt{3} \quad \lambda_3 = 4-\sqrt{3}$

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} => \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \vec{v_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1+\sqrt{3} & -1 & 0 \\ -1 & \sqrt{3} & 1 \\ 0 & 1 & 1+\sqrt{3} \end{pmatrix} => \left(\begin{array}{ccc|c} 1 & 0 & 2-\sqrt{3} & 0 \\ 0 & 1 & 1-\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \vec{v_2} = \begin{pmatrix} -2+\sqrt{3} \\ 1 \\ -1+\sqrt{3} \end{pmatrix}$$

$$\begin{pmatrix} -1+\sqrt{3} & -1 & 0 \\ -1 & \sqrt{3} & 1 \\ 0 & 1 & 1+\sqrt{3} \end{pmatrix} => \left(\begin{array}{ccc|c} 1 & 0 & 2+\sqrt{3} & 0 \\ 0 & 1 & 1+\sqrt{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \vec{v_3} = \begin{pmatrix} 2+\sqrt{3} \\ 1 \\ 1+\sqrt{3} \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & -2+\sqrt{3} & 2+\sqrt{3} \\ -1 & 1 & 1 \\ 1 & -1+\sqrt{3} & 1+\sqrt{3} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4-\sqrt{3} & 0 \\ 0 & 0 & 4+\sqrt{3} \end{pmatrix}$$

Problem 2

(a) Eigenvalues are $\lambda_1 = -1\lambda_2 = 3\lambda_3 = 7$

$\vec{v_1} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \vec{v_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{v_3} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$

$det(A) = -21 \quad tr(A) = 9$

By spectral decomposition theorem (c):

$\vec{v_1}^T\vec{v_2} = 0 + 1 - 1 = 0$

$\vec{v_1}^T\vec{v_3} = 0 + b - 0 = 0 => b = 0$

$\vec{v_2}^T\vec{v_3} = a + b + 0 = a + b => 0$

Thus we could say that the only possible vector is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, but since the matrix P formed by these vectors doesn't span the dimension of 3, there are no such a and b.

(b) Eigenvalues are $\lambda_1 = -1\lambda_2 = 3\lambda_3 = 3$

By spectral decomposition theorem (b):

Eigenvalue of 3 should span the eigenspace with $dim = 2$, in other words it has 2 independent eigenvectors.

From prev point: $\vec{v_1}^T\vec{v_3} = 0 + b - 0 = 0 => b = 0$

$\begin{bmatrix} 1 & a \\ 1 & b \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$ In order for these two vectors to be independent a must be anything, but 0.

(c) $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T$

$$A = -\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \end{pmatrix} + 3\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3+3a & 3 & -3 \\ 3 & 2 & -4 \\ -3 & -4 & 4 \end{pmatrix}$$

$tr(A) = \lambda_1 + \lambda_2 + \lambda_3 = -1 + 3 + 3 = 5$

$tr(A) = 3 + 3a + 2 + 4 = 9 + 3a = 9 + 3(-4/3) = 5$

$$A = \begin{pmatrix} 3 + 3(-\frac{4}{3}) & 3 & -3 \\ 3 & 2 & -4 \\ -3 & -4 & 4 \end{pmatrix}$$

$$A = \begin{pmatrix} -1 & 3 & -3 \\ 3 & 2 & -4 \\ -3 & -4 & 4 \end{pmatrix}$$

Problem 3

(a) $\lambda_1 = 1 \lambda_2 = 2 \lambda_3 = \lambda$

$\vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \vec{v_2} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad \vec{v_3} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$

$\vec{v_1}^T \vec{v_2} = 1 + 0 - 1 = 0$

$\vec{v_1}^T \vec{v_3} = a = 0$

$\vec{v_2}^T \vec{v_3} = a + b + 0 = a + b => 0$

Thus we could state that with $v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ we will not span the eigenspace with $dim = 3$.

$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & b \\ 1 & -1 & 0 \end{pmatrix}$

$a = 0, b =$ anything, except 0

In order to span the space we must take the vector $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. This means we have only 2 eigenvalues and 3-rd eigenvector is generalized one.

Thus, lets solve the task both for case when $\lambda_3 = 1$ or when $\lambda_3 = 2$

$\lambda_3 = 1$

$$A = P^{-1} D P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

$\lambda_3 = 2$

$$A = P^{-1} D P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & \frac{-1}{2} & 0 \\ \frac{-1}{2} & \frac{3}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 2 \end{pmatrix}$$

Problem 4

$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T$

$\lambda_1, \lambda_2 \dots \lambda_n$ - real numbers

$\vec{v_1}, \vec{v_2} \dots \vec{v_n}$ - orthogonal basis of R^n

Vector multiplication on its transpose always leads to symmetric matrix, so by all means A is symmetric. In order to find eigenvectors and eigenvalues of matrix A we could decompose it into PD^T where matrix P will consist of eigenvectors and matrix D will contain all the eigenvalues as its diagonal entries.

Problem 5

(a) vv^T is always symmetric $(I_n - vv^T)$ - doesn't affect the fact the matrix is symmetric. So matrix is orthogonally diagonalizable.

$v^T = (v_1, v_2, v_3)$

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$A = I - \begin{pmatrix} v_1^2 & v_1v_2 & v_1v_3 \\ v_2v_1 & v_2^2 & v_2v_3 \\ v_1v_3 & v_2v_3 & v_3^3 \end{pmatrix}$$

$tr(A) = (1 - v_1^2) + (1 - v_2^2) + (1 - v_3^2)$

$$A = I - \begin{pmatrix} 1 - v_1^2 - \lambda & v_1v_2 & v_1v_3 \\ v_2v_1 & 1 - v_2^2 - \lambda & v_2v_3 \\ v_1v_3 & v_2v_3 & 1 - v_3^3 - \lambda \end{pmatrix}$$

(b) $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$tr(A) = 1 \quad \det(A) = -1$

$\lambda_1 = 1 \quad \lambda_2 = -1 \quad \lambda_3 = 1$

Skip writing calculation of eigenvectors...

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$D = P^{-1}AP = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 6

(a) if $A = A^{-T}$ than matrix is Hermetian.

$A = \begin{pmatrix} 1 & i & 0 \\ i & 2 & -i \\ 0 & -i & 1 \end{pmatrix} \quad A^{-T} = \begin{pmatrix} 1 & -i & 0 \\ i & 2 & i \\ 0 & -i & 1 \end{pmatrix}$

A is not equals to A^{-T}

Matrix is not Hermetian.

(b) $A = \begin{pmatrix} 1 & i & 0 \\ -i & 2 & -i \\ 0 & i & 1 \end{pmatrix} \quad A^{-T} = \begin{pmatrix} 1 & -i & 0 \\ i & 2 & i \\ 0 & -i & 1 \end{pmatrix}$

A is equals to A^{-T}

Matrix is Hermetian.

Problem 7

(a) $\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$ Sum in all row $= 3$ $tr(A) = 10 \quad \lambda_1 = 3 \lambda_2 = 7$

All eigenvalues are positive. This means that matrix is positive definite.

(b) $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 5 \end{pmatrix}$

This matrix is the same as in problem 1(c). lets take values from there.

$\lambda_1 = 4 \quad \lambda_2 = 4 + \sqrt{3} \quad \lambda_3 = 4 - \sqrt{3}$

Positive definite.

Problem 8

(a) $B^T = \begin{pmatrix} 1 & -1 & 2 \\ -3 & 3 & c \end{pmatrix} \quad B = \begin{pmatrix} 1 & -3 \\ -1 & 3 \\ 2 & c \end{pmatrix}$

$B^T B = \begin{pmatrix} 1+1+4 & -3-3+2c \\ -3-3+2c & 9+9+c^2 \end{pmatrix} = \begin{pmatrix} 6 & -6+2c \\ -6+2c & 18+c^2 \end{pmatrix}$

$det(B^T B) = 6(18+c^2) - (-6+2c)(-6+2c) = c^2 + 12c + 36$

Roots are $c_{1,2} = -6$

Answer: For all values except -6

(b) Since the only case matrix has no inverse is if $c = -6$, leads us to the fact that for all other values of c matrix is invertible, thus matrix is definite, because:

If matrix is positive definite, than all of eigenvalues are positive and thus, 0 is not an eigenvalue. If 0 is an eigenvalue it shows that matrix is non invertible.

We could state that matrix is positive definite if all entries on pivot positions are positive. So,

$$A = \begin{pmatrix} 6 & -6+2c \\ -6+2c & 18+c^2 \end{pmatrix} = \begin{pmatrix} 1 & -1+\frac{1}{3}c \\ -1+\frac{1}{3}c & 3+\frac{c^2}{6} \end{pmatrix} = \begin{pmatrix} 1 & -1+\frac{1}{3}c \\ 0 & \frac{3+\frac{c^2}{6}}{-1+\frac{c}{3}} - (-1+\frac{c}{3}) \end{pmatrix}$$

Now we need to find, when the expression at position x_{22} will be grater than 0

$$\frac{(3+\frac{c^2}{6}) - (1+\frac{c^2}{9})}{-1+\frac{c}{3}} = \frac{-\frac{c^2}{9}+\frac{c^2}{6}+2}{-1+\frac{c}{3}}$$

Now we could see that nominator always positive.

$-1 + \frac{c}{3} > 0$

$\frac{c}{3} > 1$

Matrix A is positive definite $\iff c > 3$

Problem 9

(a) $S = \begin{pmatrix} s & -14 & -4 \\ -4 & s & 4 \\ -4 & 4 & s \end{pmatrix} \quad S^T = S$

$det(S - I\lambda) = \begin{pmatrix} s-\lambda & -14 & -4 \\ -4 & s-\lambda & 4 \\ -4 & 4 & s-\lambda \end{pmatrix} = (S-\lambda)^3 - 16(S-\lambda) + 4(-4(S-\lambda)+16) - 4(-16+4(S-\lambda)) = (S-\lambda)^3 - 48(S-\lambda) + 128$

Lets make a substitution: $S - \lambda = x$

So, we will have: $x^3 - 48x + 128$

$x_1 = 4 \quad x_2 = -8 \quad x_3 = 4$

$S - \lambda_1 = 4 \quad \lambda_1 = S - 4$

$S - \lambda_2 = -8 \quad \lambda_2 = S + 8$

$S - \lambda_3 = 4 \quad \lambda_3 = S - 4$

Now we see, that matrix will be positive definite $S > 4$ and negative definite $S < -8$

(b) $T = \begin{pmatrix} t & -3 & 0 \\ -3 & t & 4 \\ 0 & 4 & t \end{pmatrix} \quad T^T = T$

$det(T - I\lambda) = \begin{pmatrix} t-\lambda & -3 & 0 \\ -3 & t-\lambda & 4 \\ 0 & 4 & t-\lambda \end{pmatrix} = (t-\lambda)^3 - 16(t-\lambda) + 3(-3(t-\lambda) = (t-\lambda)^3 - 25(t-\lambda)$

$t - \lambda = x$

$x^3 - 25x = 0$

$x_1 = 5 \quad x_2 = -5 \quad x_3 = 0$

$t - \lambda_1 = 5 \quad \lambda_1 = t - 5$

$t - \lambda_2 = -5 \quad \lambda_2 = t + 5$

$t - \lambda_3 = 0 \quad \lambda_3 = t$

Now we see, that matrix will be positive definite $t > 5$ and negative definite $t < -5$

Problem 10

(a) $Q(x_1, x_2) = 2x_1^2 + 2x_2^2 - 2x_1x_2$

$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$\lambda_1 = 1 \quad \lambda_2 = tr(A) - \lambda_1 = 3$

$D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$B_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \vec{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$B_2 = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \quad \vec{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad A = PDP^{-1} \quad D = P^{-1}AP$

$x = Py \quad x_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$y = P^{-1}x = P^T x$

$y^T D y = y_1^2 + 3y_2^2$

(b) $Q(x_1, x_2, x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 - 4x_1x_2 - 4x_2x_3$

$det(A) = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 3-\lambda & 2 & 0 \\ 2 & 4-\lambda & -2 \\ 0 & 2 & 5-\lambda \end{pmatrix} = (3-\lambda)((4-\lambda)(5-\lambda)-4) - 2(10-2\lambda) = \lambda^3 - 12\lambda^2 + 39\lambda + 28 = 0$

$\lambda_1 = 1 \quad \lambda_2 = 4 \quad \lambda_3 = 7$

$Q(y) = y^T D y = y^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix} y = y_1^2 + 4y_2^2 + 7y_3^2$

Problem 11

$3x^2 + 4xy + 6y^2 = 14$

$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$

$tr(A) = 9 \quad det(A) = 14$

$x^2 - 9x + 14 = 0$

$\lambda_1 = 7 \quad \lambda_2 = 2$

$B_1 = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \quad \vec{v_1} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$

$B_2 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \quad \vec{v_2} = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$

$P = \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & -\frac{1}{2} \end{pmatrix}$

Length $= \sqrt{\frac{1}{4} + 1} = \sqrt{\frac{5}{4}}$

Problem 12

(a) $A = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 3 \end{pmatrix} \quad \vec{v_1} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_1 = 2$

$tr(A) = 12 \quad det(A) = 32$

$\lambda_2\lambda_3 = 16 \quad \lambda_2 + \lambda_3 = 10$

$\lambda_2 = 8 \quad \lambda_3 = 2$

$$B_3 = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} \quad \vec{v_3} = \begin{pmatrix} 2x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

As we can see, $\vec{v_1}$ could be formed from solution for B_3

$$B_2 = \begin{pmatrix} -5 & -2 & 1 \\ -2 & -2 & -2 \\ 1 & -2 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \vec{v_2} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -\frac{1}{6} & \frac{2}{3} & \frac{7}{6} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 & 3 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(b)